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The Symplectic Hitchin Component for Triangle Groups: dimension formula and parameters for $\Delta(3, 4, 4)$

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Abstract

The Hitchin component is a special connected component of the space of representations of a finitely generated group into a split real simple Lie group G, such as $\mathrm{PSL}(n,\mathbb{R}), \mathrm{PGSp}(2n,\mathbb{R}), \mathrm{PO}(n,n+1)$ and $\mathrm{PO}(n,n)$. It was first studied by Hitchin [19] in the case of representations of the fundamental group $\pi_1(S)$ of a closed surface S of genus $g \geq 2$ into $\mathrm{PSL}(n,\mathbb{R})$: when $n = 2$ the Hitchin component coincides with the Teichmüller space of the surface and Hitchin showed that for general n it is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. Later work of Choi and Goldman [8] for $n = 3$ and Labourie [22] for general n shows that Hitchin components share many properties with Teichmüller space. In particular they consist of discrete and faithful representations and so they are part of the interesting family of spaces called *higher Teichmüller spaces*.

In this thesis we study the Hitchin component of hyperbolic triangle groups $\Delta(p,q,r)$ into $\mathrm{GSp}(2n,\mathbb{R})$, where by *hyperbolic triangle group* we mean the group generated by the reflections in the sides of a geodesic triangle in the upper half plane. The notion of Hitchin component was extended from surface groups to fundamental groups of 2-dimensional orbifolds by Alessandrini, Lee and Schaffhauser [2], who show that it is homeomorphic to an open ball and they compute its dimension explicitly. The class of finitely generated groups studied in [2] covers fundamental groups of all surfaces of finite type, as well as the triangle groups considered in this work and their 2-index subgroups, which were already treated by Long and Thistlethwaite [25] and Weir [40].

In the first part of this work we show that the Hitchin component is a higher Teichmüller space (Proposition 2.4.3), that all representations in it are smooth points of the representation variety (Proposition 4.5.1) and we compute its dimension (Theorem 4.5.4). These results stem from the identification of the Zariski tangent space to the representation variety with the space of 1-cocycles twisted by the adjoint representation.

The second part of the thesis is devoted to give a parametrisation of the Hitchin component of the triangle group $\Delta(3,4,4)$ into $\mathrm{PGSp}(4,\mathbb{R})$, which is one-dimensional, in the same spirit as the work of Cooper, Long and Thistlethwaite [12] for fundamental groups of hyperbolic 3-manifolds.

Riassunto

La componente di Hitchin è una particolare componente connessa dello spazio delle rappresentazioni di un gruppo finitamente generato in un gruppo di Lie semplice e split, come ad esempio $\mathrm{PSL}(n, \mathbb{R}), \mathrm{PGSp}(2n, \mathbb{R}), \mathrm{PO}(n, n+1)$ e $\mathrm{PO}(n, n)$. Il primo a studiarla fu Hitchin [19] per rappresentazioni del gruppo fondamentale $\pi_1(S)$ di una superficie chiusa di geno $g \geq 2$ in $\mathrm{PSL}(n, \mathbb{R})$: per $n = 2$ la componente di Hitchin coincide con lo spazio di Teichmüller della superficie e Hitchin ha dimostrato che per n qualiasi è omeomorfa a $\mathbb{R}^{(2g-2)(n^2-1)}$. In seguito, il lavoro di Choi e Goldman [8] per $n = 3$ e di Labourie [22] per n qualsiasi ha mostrato che la componente di Hitchin possiede molte proprietà in comune con lo spazio di Teichmüller. In particolare, fa parte della famiglia di spazi chiamati *higher Teichmüller spaces*, in quanto tutte le rappresentazioni al suo interno sono discrete e iniettive.

Questa tesi si occupa dello studio della componente di Hitchin per rappresentazioni di gruppi triangolari iperbolici $\Delta(p, q, r)$ in $\mathrm{GSp}(2n, \mathbb{R})$, dove con *gruppo triangolare iperbolico* si indica il gruppo generato dalle riflessioni nei lati di un triangolo geodetico nel piano iperbolico $\mathbb{H}^2$. La nozione di componente di Hitchin è stata generalizzata da gruppi di superficie a gruppi di orbifold 2-dimensionali da Alessandrini, Lee e Schaffhauser [2], i quali hanno dimostrato che è omeomorfa a una palla aperta e ne hanno calcolato esplicitamente la dimensione. La classe di gruppi finitamente generati trattata in [2] include tutti i gruppi di superficie di tipo finito, come anche i gruppi triangolari di questa tesi e i loro sottogruppi di indice 2 già studiati da Long e Thistlethwaite [25] e Weir [40].

Nella prima parte della tesi dimostriamo che la componente di Hitchin è un *higher Teichmüller space* (Proposition 2.4.3), che tutte le rappresentazioni al suo interno sono punti regolari della varietà (Proposition 4.5.1) e ne calcoliamo la dimensione (Theorem 4.5.4). Questi risulati sono ottenuti tramite l'identificazione dello spazio tangente di Zariski alla varietà delle rappresentazioni con lo spazio dei cocicli 1-dimensionali su cui si agisce tramite la rappresentazione aggiunta.

La seconda parte della tesi è dedicata a dare una parametrizzazione della componente di Hitchin del gruppo triangolare $\Delta(3, 4, 4)$ in $\mathrm{PGSp}(4, \mathbb{R})$, che è 1-dimensionale, analogamente al lavoro di Cooper, Long e Thistlethwaite [12] per i gruppi fondamentali di 3-varietà.

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Contents

1. Introduction

1.1 Teichmüller space: classical and higher rank

Let S be a closed connected oriented topological surface of genus $g \geq 2$. The *Teichmüller space* $\mathrm{Teich}(S)$ of S is the space of equivalence classes of *marked hyperbolic structures*, i.e. pairs (X, f) where X is a hyperbolic surface and $f : S \to X$ is a homeomorphism. Two such pairs (X, f) and (Y, g) are equivalent if there exists an isometry $\alpha : X \to Y$ such that $\alpha \circ f$ is isotopic to g. The universal cover $\widetilde{X}$ of a hyperbolic surface X can be identified with the hyperbolic plane $\mathbb{H}^2$ on which the fundamental group $\pi_1(X)$ acts as deck transformations by orientation-preserving isometries, thus $\pi_1(X)$ is a subgroup of $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R})$. If (X, f) is a marked hyperbolic structure, once a base point is fixed, f induces an isomorphism on fundamental groups $f_* : \pi_1(S) \to \pi_1(X) \leq \mathrm{PSL}(2,\mathbb{R})$, which is called the *holonomy*. The holonomies of two equivalent hyperbolic structures are conjugated by an element of $\mathrm{PSL}(2,\mathbb{R})$, thus the Teichmüller space can be identified as a subset of the *character variety* $\chi(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$, which is the space of all group homomorphism from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ up to $\mathrm{PSL}(2,\mathbb{R})$-conjugation. Since $\pi_1(S)$ is finitely generated, the character variety carries a natural topology induced by the topology of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{Teich}(S)$ forms a connected component of $\chi(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ that is homeomorphic to $\mathbb{R}^{6g-6}$ and consists entirely of discrete and faithful representations [14].

This algebraic realization of the Teichmüller space as a space of representations is the starting point of higher Teichmüller theory: instead of studying representations of surface groups into $\mathrm{PSL}(2,\mathbb{R})$, one replaces $\mathrm{PSL}(2,\mathbb{R})$ by a simple Lie group G of higher rank, such as $\mathrm{PSL}(n,\mathbb{R})$, $n \geq 3$ or $\mathrm{PSp}(2n,\mathbb{R})$, $n \geq 2$. The object of interest is then the representation variety $\chi(\pi_1(S), G) = \mathrm{Hom}(\pi_1(S), G)/G$, or more precisely its connected components that consist entirely of discrete and faithful representations. A union of such connected components is called a *higher Teichmüller space* [41]. There are two main known families of higher Teichmüller spaces: Hitchin components $\mathrm{Hit}(\pi_1(S), G)$, which are defined when G is a split real simple Lie group, and spaces of maximal representations $\mathcal{M}(\pi_1(S), G)$ defined when G is a non-compact simple Lie group of Hermitian type. The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is both split and of Hermitian type and for this group Hitchin representations are maximal, that is $\mathrm{Hit}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R})) \subset \mathcal{M}(\pi_1(S), \mathrm{Sp}(2n,\mathbb{R}))$ [6] and the inclusion is proper [18]. Real split Lie groups and Lie groups of Hermitian type now fit in the broader class of Lie groups which admit a Θ-positive structure as defined by Guichard and Wienhard [18]. All known higher Teichmüller spaces consist of Θ-positive representations, and Guichard, Labourie and Wienhard [16] proved that Θ-positive representations are discrete and faithful. It is conjectured that all higher Teichmüller spaces arise as such [17].

1.2 The Hitchin component

The existence of a component of the character variety analogous to Teichmüller space for representations into a Lie group of higher rank was first demonstrated by Hitchin in 1992. In [19] he considers surface group representations into $G = \mathrm{PSL}(n, \mathbb{R})$ (and more generally into a split real simple Lie group) and singles out a special component of $\chi(\pi_1(S), G)$ that is homeomorphic to $\mathbb{R}^{(2g-2)\dim G}$ and which we now call the *Hitchin component*. Any split real simple Lie group G admits an embedding $\pi : \mathrm{PSL}(2, \mathbb{R}) \to G$ which is unique up to conjugation. For the classical Lie groups $\mathrm{PSL}(n, \mathbb{R})$, $\mathrm{PSp}(2n, \mathbb{R})$ and $\mathrm{PSO}(n, n+1)$ this is the irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ in the appropriate dimension. The Hitchin component $\mathrm{Hit}(\pi_1(S), G)$ is defined as the connected component of the character variety $\chi(\pi_1(S), G)$ containing the composition $\pi \circ i$, where $i : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ is a holonomy of S. Since the Teichmüller space is connected, any holonomy defines the same component.

The fact that the Hitchin component is an example of a higher Teichmüller space was proven for $n = 3$ by Choi and Goldman [8] and for general n by Labourie [22].

The geometric significance of representations in the Hitchin component or in other higher Teichmüller spaces leads to the natural study of the situation for surfaces with boundary and more generally for 2-dimensional compact orbifolds. Analogously to surfaces, a 2-dimensional closed orbifold Y admits hyperbolic structures if and only if it has negative orbifold Euler characteristic, see e.g. [38]. The space of hyperbolic structures, i.e. the Teichmüller space of the orbifold, was studied by Thurston [38] who showed that it is a connected component of $\chi(\pi_1(Y), \mathrm{PSL}(2, \mathbb{R}))$ consisting of discrete and faithful representations, which is homeomorphic to some Euclidean space. In [9] Choi and Goldman considered the Hitchin component of $\pi_1(Y)$ in $\mathrm{PGL}(3, \mathbb{R})$ and in [2] Alessandrini, Lee and Schaffhauser introduced $\mathrm{PGL}(n, \mathbb{R})$-Hitchin components for all $n \geq 2$. In fact, they consider any split real Lie group G and prove that the Hitchin component gives new examples of higher Teichmüller spaces. They also show that it is homeomorphic to an open Euclidean ball and compute its dimension explicitely. We remark that they consider the larger family of *compact* 2-dimensional orbifolds with negative orbifold Euler characteristic, thus allowing non-orientable surfaces as well as boundary components. The $\mathrm{PGL}(n, \mathbb{R})$-Hitchin component for orientable surfaces with boundary was already introduced by Labourie and McShane in [23] and the dimension of the Hitchin component for the orbifold given by the 2-sphere with three cone points was computed by Long and Thistlethwaite [25] for $\mathrm{PSL}(n, \mathbb{R})$ and by Weir [40] for $\mathrm{PSp}(2n, \mathbb{R})$. In this thesis we focus on the Hitchin component of hyperbolic triangle groups, which are fundamental groups of some closed 2-dimensional orbifolds with negative orbifold Euler characteristic.

Definition 1.2.1. A *triangle group* $\Delta(p,q,r)$ is a group with presentation

$$\Delta(p,q,r) = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1\rangle.$$

The group is called *hyperbolic* if $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. In this case the generators can be realized as reflections in the sides of a triangle Δ in the hyperbolic plane with internal angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, which is unique up to congruence. This action of $\Delta(p,q,r)$ gives a tiling of the hyperbolic plane $\mathbb{H}^2$ and the stabilizer of Δ is trivial [29, Thereom 2.8 and references therehein]. The quotient space is a triangle in which the boundary points (except the vertices) have a neighborhood isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \setminus \mathbb{R}^2$ (where $(\mathbb{Z}/2\mathbb{Z})$ acts on $\mathbb{R}^2$ via reflections through a line) and each vertex has a neighborhood isomorphic to $D_k \setminus \mathbb{R}^2$, where $k = p, q$ or r and D_k is the dihedral group generated by the reflections through two lines with an angle π/k between them. This is to say that the quotient space is a closed 2-dimensional orbifold with mirror singularities (the boundary of the triangle) and three cone points with cone angle $2\pi/k$ (the vertices). Notice that the orbifold has no boundary, although the underlying topological space does. For background on orbifolds we refer to [38] and [35]. The universal cover of an orbifold is defined in the same way as for surfaces and the orbifold fundamental group is the group of its deck transformations. For orbifolds arising from hyperbolic triangle groups, the universal cover is the hyperbolic plane $\mathbb{H}^2$ and it is not difficult to show that the orbifold fundamental group is isomorphic to $\Delta(p,q,r)$ [35, Section 2]. We conclude the discussion on orbifolds by recalling that one can define the orbifold Euler characteristic $\chi(Y)$ of an orbifold Y as the sum of the Euler characteristic of the underlying topological space and some constants depending on the number and order of the singularities. For a hyperbolic triangle group $\Delta(p,q,r)$ this is given by

$$\chi(\Delta) = 1 - \tfrac{1}{2}(1 - \tfrac{1}{p} + 1 - \tfrac{1}{q} + 1 - \tfrac{1}{r}) = \tfrac{1}{2}(\tfrac{1}{p} + \tfrac{1}{q} + \tfrac{1}{r} - 1) < 0.$$

In Section 2.2 we give an explicit description of the realisation of $\Delta(p,q,r)$ as a subgroup of the isometry group of $\mathbb{H}^2$, that is a discrete and faithful representation $\rho_0 : \Delta(p,q,r) \to \mathrm{PGL}(2,\mathbb{R})$. Since a triangle in the hyperbolic plane is determined by its angles, up to congruence, the representation ρ_0 is unique up to conjugation and we call it the *geometric representation* of $\Delta(p,q,r)$.

The image of the irreducible representation π_{2n} of $\mathrm{PGL}(2,\mathbb{R})$ into $\mathrm{PGL}(2n,\mathbb{R})$ is contained in the general symplectic group $\mathrm{PGSp}(2n,\mathbb{R})$, which is the group of linear transformations of $\mathbb{R}^{2n}$ that preserve up to sign a non-degenerate skew-symmetric bilinear form (see Section 2.3). Having illustrated how hyperbolic triangle groups fit in the context of orbifolds, we now define the Hitchin component of the character variety $\chi(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))$.

Definition 1.2.2. The *Hitchin component* $\mathrm{Hit}(\Delta(p,q,r), \mathrm{PGSp}(2n,\mathbb{R}))$ is the connected component of $\chi(\Delta(p,q,r), \mathrm{PGSp}(2n,\mathbb{R}))$ which contains the representation $[\pi_{2n} \circ \rho_0]$.

We call a homomorphism $\phi : \Delta(p, q, r) \to \mathrm{PGSp}(2n, \mathbb{R})$ a *Hitchin representation* if its class $[\phi]$ belongs to the Hitchin component. It follows from the analogous statement for surface groups that triangle group Hitchin representations are discrete and faithful (Proposition 2.4.3).

Before stating our results, we give some additional context and motivation for the interest in triangle groups. Every hyperbolic triangle group $\Delta(p, q, r)$ contains a subgroup of index 2 generated by the rotations about the vertices through angles $2\pi/p$, $2\pi/q$, $2\pi/r$, which in the presentation of Definition 1.2.1 is the orientation-preserving subgroup generated by the products ab, bc and ac. We denote such subgroups by $T(p, q, r)$ and call them *rotation triangle groups*. In the literature they also appear under the name of *von Dyck groups* or simply of triangle groups.

Much work has been done in the past years to study representations of $T(p, q, r)$ partly motivated by the fact that they contain surface subgroups of finite index. Long, Reid and Thistlethwaite [27] find a one-parameter family of representations of $T(3, 3, 4)$ into $\mathrm{SL}(3, \mathbb{Z})$, the image of which gives a non-conjugate family of subgroups of $\mathrm{SL}(3, \mathbb{Z})$ which are Zariski dense in $\mathrm{SL}(3, \mathbb{R})$. This family of representations lies in the Hitchin component. In the same vein Long and Thistlethwaite [26] find infinite families of Zariski dense surface groups of fixed genus inside $\mathrm{SL}(4, \mathbb{Z})$ and $\mathrm{SL}(5, \mathbb{Z})$ (as images of representations of $T(3, 3, 4)$) and $\mathrm{Sp}(4, \mathbb{Z})$ (as images of representations of $T(2, 4, 5)$). Again, these families consist of discrete and faithful representations and lie in the Hitchin component of the respective representation varieties. These families of representations do not parametrize the whole Hitchin component, which in all the above cases is 2-dimensional. Formulas for the dimension of the Hitchin component of $T(p, q, r)$ have been computed by Long and Thistlethwaite [25] when the target Lie group is $\mathrm{PSL}(n, \mathbb{R})$ and by Weir [40] when the target Lie group is $\mathrm{PSp}(2n, \mathbb{R})$ or $\mathrm{PSO}(n, n+1)$, $n \geq 1$. As we mentioned above, the dimension of the Hitchin component for orbifold fundamental groups has been computed in great generality by Alessandrini, Lee and Schaffhauser [2].

The first part of this thesis is devoted to the proof of a formula for the dimension of the Hitchin component $\mathrm{Hit}(\Delta(p, q, r), \mathrm{PGSp}(2n, \mathbb{R}))$. The result of [2] covers the $\Delta(p, q, r)$-case, but we give an independent more elementary proof. The result is the following.

Theorem 1.2.3. *Let $\Delta = \Delta(p, q, r)$ be a hyperbolic triangle group with generators a, b, c. Let $G = \mathrm{PGSp}(2n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. The dimension of the Hitchin component of $\chi(\Delta, \mathrm{PGSp}(2n, \mathbb{R}))$ is*

$$\dim \mathrm{Hit}(\Delta, G) = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(bc))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(ac))}), \quad (1.1)$$

where $\phi_0 : \Delta \to \mathrm{PGSp}(2n, \mathbb{R})$ is the base representation.

In fact, the group $\mathrm{GSp}(2n, \mathbb{R})$ can be replaced by $\mathrm{GL}(n, \mathbb{R})$.

We remark that (1.1) is not as explicit as the formulas in [2],[25] and [40], which are of combinatorial nature in p, q, r and n. To get a number from (1.1) one still needs to compute the dimension of $\mathfrak{g}^{\mathrm{Ad}\,\phi_0(xy)}$ for every product of two generators $x \neq y \in \{a, b, c\}$. This is the same as the dimension of the Lie algebra of the centralizer in $\mathrm{GSp}(2n, \mathbb{R})$ of $\widetilde{x}\widetilde{y}$, where $\widetilde{x}, \widetilde{y}$ are lifts of $\phi_0(x), \phi_0(y)$ (Lemma A.2.1). We carry out this computation for the group $\mathrm{GSp}(4, \mathbb{R})$ (Proposition 4.6.1) and obtain

Corollary 1.2.4. *The dimension of the* $\mathrm{PGSp}(4, \mathbb{R})$*-Hitchin component is*

$$\dim \mathrm{Hit}(\Delta(p,q,r), \mathrm{PGSp}(4,\mathbb{R})) = \begin{cases} 0 & \text{if } p = q = 3,\ r \geq 4 \text{ or } p = 2, q = 3, r \geq 7, \\ 1 & \text{if } p = 2, 3,\ r \geq q \geq 4, \\ 2 & \text{if } p, q, r \geq 4. \end{cases}$$

Computing the dimension of the centralizer $Z_{\mathrm{GSp}(2n,\mathbb{R})}(\widetilde{x}\widetilde{y})$ for general n is essentially the content of [40]. Even though (1.1) requires some additional work to be applied, the methods used to obtain it are interesting on their own. They are based on the identification of the Zariski tangent space to the representation variety with the space of 1-cocycles twisted by the adjoint representation. A byproduct is that the Hitchin component consists of smooth points of the character variety (Proposition 4.5.1). We give an overview of the method and of the necessary tools in the following section.

1.3 The deformation space

A triangle in the hyperbolic plane is determined up to congruence by its angles, therefore the geometric representation $\rho_0 : \Delta(p, q, r) \to \mathrm{PSL}(2, \mathbb{R})$ is *rigid* in the sense that any deformation is obtained by conjugation. However, composing ρ_0 with the irreducible representation of $\mathrm{PSL}(2, \mathbb{R})$ into $\mathrm{PGSp}(2n, \mathbb{R})$ might give sufficient space to obtain deformations that do not come from conjugation by $\mathrm{PGSp}(2n, \mathbb{R})$ and the space of deformations is the Hitchin component $\mathrm{Hit}(\Delta(p, q, r), \mathrm{PGSp}(2n, \mathbb{R}))$.

For ease of notation let $G = \mathrm{PGSp}(2n, \mathbb{R})$ and $\phi_0 = \pi_{2n} \circ \rho_0$ denote the base representation of the triangle group $\Delta = \Delta(p, q, r)$. As done by Goldman [14] for the classical case of surface groups, we study the deformation space of the representation ϕ_0 by looking at the local structure of $\mathrm{Hom}(\Delta, G)$ and of $\chi(\Delta, G)$ near ϕ_0.

We start with the tangent space to $\mathrm{Hom}(\Delta, G)$ at ϕ_0. Since Δ is finitely generated (with three generators and set of relations R) the space $\mathrm{Hom}(\Delta, G)$ is homeomorphic to $\bigcap_{r\in R} f_r^{-1}(e) \subset G^3$, where each map $f_r : G^3 \to G$ is defined by the evaluation of the relation $r \in R$ on G. Since G is a linear algebraic group this identification induces an algebraic structure on the representation variety. The *Zariski tangent space* at a homomorphism ϕ consists of all tangent vectors to a smooth path $t \mapsto \phi_t$ inside G^3 starting at ϕ which satisfies the relations $f_r = e$ *up to first order*, see Definition 3.1.2. It can be identified with the space of 1-cocycles $\Delta \to \mathfrak{g}$:

$$Z^1(\Delta, \mathfrak{g})_\phi = \{u : \Delta \to \mathfrak{g} \mid u(\gamma\delta) = u(\gamma) + \mathrm{Ad}(\phi(\gamma))u(\delta) \text{ for all } \gamma, \delta \in \Delta\}.$$

Tangent vectors to paths in the G-orbit of ϕ correspond to 1-coboundaries

$$B^1(\Delta, \mathfrak{g})_\phi = \{v : \Delta \to \mathfrak{g} \mid \exists X \in \mathfrak{g} \text{ such that } v(\gamma) = \mathrm{Ad}(\phi(\gamma))X - X\}.$$

A representation ϕ is a *smooth point* of $\mathrm{Hom}(\Delta, G)$ if the dimension of the Zariski tangent space at ϕ is minimal. If this is the case, then the first cohomology group $H^1(\Delta, \mathfrak{g})_{\phi_0}$ is the tangent space to the character variety at $[\phi]$. To prove Theorem 1.2.3 we show that every Hitchin representation ϕ is a smooth point and compute the dimension of $H^1(\Delta, \mathfrak{g})_{\phi}$. In fact, we find a formula for both the dimension of $H^1(\Delta, \mathfrak{g})_{\phi}$ and $Z^1(\Delta, \mathfrak{g})_{\phi}$ and argue that they are constant on Hitchin representations (Proposition 4.4.5 and Proposition 4.4.6). This shows that Hitchin representations are smooth points of the representation variety and that the dimension of the Hitchin component is $H^1(\Delta, \mathfrak{g})_{\phi_0}$.

We add some remarks on how we obtained formula (1.1) for the dimension of $H^1(\Delta, \mathfrak{g})_{\phi}$. Evaluation on the generators embeds 1-cocycles in $\mathfrak{g}^3$ and the image can be characterized as the intersection of the kernels of some linear maps $\mathfrak{g}^3 \to \mathfrak{g}$ (Lemma 3.1.5). Then we exploit the triangle group relations to compute the dimension of these spaces, in a similar way as one would do to compute the dimension of the Zariski tangent space to the representation variety of a surface group [14]. We can not obtain such a neat formula for the dimension (and consequent characterization of smooth points of the representation variety) as Goldman does, since contrary to surface groups triangle groups have more than one relation.

1.4 The Hitchin component of $\Delta(3, 4, 4)$ in $\mathrm{PGSp}(4, \mathbb{R})$

The second part of this thesis is devoted to explicitly parametrize the Hitchin component of the $(3, 4, 4)$-triangle group in $\mathrm{PGSp}(4, \mathbb{R})$. The choice of $(3, 4, 4)$ was dictated by the fact that according to Corollary 1.2.4, it is the triple of smallest possible numbers for which the $\mathrm{PGSp}(4, \mathbb{R})$-Hitchin component is not trivial. In fact, it is 1 dimensional, which means that to describe it it suffices to find a one-parameter family of deformations of the base representation ϕ_0. To this end we apply the method introduced by Cooper, Long and Thistlethwaite [12] for the exact computation of character varieties of fundamental groups of low-dimensional manifolds and orbifolds. The method has been applied in [27] and in [26] to compute the families of representations mentioned above, as well as by Weir [40] to find additional one-parameter families of representations in the Hitchin component of $\chi(T(3, 3, 4), \mathrm{PSL}(4, \mathbb{R}))$, and by Paupert and Thistlethwaite [33] to describe the deformation space of the Bianchi group Bi(3). We thoroughly describe the method in Section 5.2 and its implementation in Section 5.3, so that the interested reader could follow it verbatim to parametrize the $\mathrm{PGSp}(4, \mathbb{R})$-Hitchin component of any triangle group $\Delta(p, q, r)$ with $p = 2, 3$ and $r \geq q \geq 4$ (if $p = 2$ and $q = 4$, then $r > 4$). We are also inclined to think that computations of one-parameter families - or of the entire two-dimensional component - for the case of $p, q, r \geq 4$ can be performed analogously.

On the other hand if one wishes to substitute the target group $\mathrm{PGSp}(4, \mathbb{R})$ with one of higher dimension there are adjustments to be done. One of the crucial steps of the method consists in finding a “normal form” to which (the generators of) each representation can be conjugated. This is usually done by finding an ad-hoc basis of $\mathbb{R}^4$ using eigenvectors of the generators. Since triangle groups have three generators (and not only two as for the groups considered in [12],[26] and [33]) we find an $\mathbb{R}^4$ basis exploiting the fact that dihedral representations are locally rigid and the symplectic features of representations of triangle groups into $\mathrm{GSp}(2n, \mathbb{R})$. More precisely, let $\phi : \Delta(p, q, r) \to \mathrm{GSp}(2n, \mathbb{R})$ be a homomorphism. The images under ϕ of the generators a, b, c of the triangle group $\Delta(p, q, r)$ are involutions of $\mathbb{R}^{2n}$ whose eigenspaces are n-dimensional Lagrangian subspaces. These six eigenspaces (two for each generator) determine the representation ϕ. The conjugation action of the symplectic group $\mathrm{GSp}(2n, \mathbb{R})$ on $\mathrm{Hom}(\Delta(p, q, r), \mathrm{GSp}(2n, \mathbb{R}))$ becomes an action on the set of 6-tuples of Lagrangian subspaces of $\mathbb{R}^{2n}$. The restriction of the representation ϕ to the dihedral subgroup generated by a and c is given by a 4-tuple of (pairwise transverse) Lagrangian subspaces, the orbit of which is determined by the *crossratio* of the 4-tuple together with the *Maslov index* of a sub 3-tuple. For the exact definition of these two invariants we refer to Appendix B, as for the purpose of this introduction it suffices to say that the crossratio is the conjugacy class of an invertible $n \times n$ matrix whose eigenvalues - in this dihedral group setting - belong to a finite set. In particular, if the crossratio is diagonalizable one can determine the orbit of the 4-tuple of Lagrangians and find a standard form for the images of a and c under the representation ϕ. When $n = 2$ the crossratio is diagonalizable, but we do not know if this still holds true for larger n and if it is not the case one needs another strategy.

Before describing the parametrization of the Hitchin component that we found, we remark that by looking at the space of 6-tuples of pairwise Lagrangian subspaces and their crossratios Burelle [5] proved that representations of triangle groups $\Delta(p, q, r)$ into $\mathrm{PGL}(2n, \mathbb{R})$ and $\mathrm{PGSp}(2n, \mathbb{R})$ which factor through the (possibly twisted) diagonal embedding of $\mathrm{PSL}(2, \mathbb{R})$ into $\mathrm{PGL}(2n, \mathbb{R})$ do not admit deformations.

We express the entire Hitchin component by means of a tautological representation, in that for each generator we give a single matrix, whose entries are algebraic expression in a single parameter u. Assigning values to u determines a specific representation.

Theorem 1.4.1. *The 1-dimensional* $\mathrm{PGSp}(4, \mathbb{R})$*-Hitchin component of the triangle group* $\Delta(3, 4, 4)$ *is given by a tautological representation* Ψ_u *whose entries lie in the field* $\mathbb{Q}(u)(\tau, \sigma, \sqrt{2})$*, where* τ *is a real root of the cubic polynomial*

$$\tfrac{1}{3}u^2(32 + 86u^2 - 5u^4) + u^2(-20 + \tfrac{13}{3}u^2)\tau + (2 - \tfrac{11}{3}u^2)\tau^2 + \tau^3 \tag{1.2}$$

and $\sigma = \sqrt{\frac{3}{2}(u^2 + \tau + 2)}$*. The images of the generators of* $\Delta(3, 4, 4)$ *are represented by the matrices* $\Psi_u(a), \Psi_u(b)$ *and* $\Psi_u(c)$ *given in Appendix C. The base representation* ϕ_0 *is obtained for* $u = 5\sqrt{2}$*.*

We remark that any choice of a root of (1.2) and of sign for σ gives a homomorphism of $\Delta(p, q, r)$, meaning that the images of a, b and c satisfy the triangle group relations (up to sign). However, only one choice describes the Hitchin component. In particular, for some values of u not all the roots of the cubic polynomial are real, this means that the representation maps into $\mathrm{PGSp}(4, \mathbb{C})$ and not $\mathrm{PGSp}(4, \mathbb{R})$. A description of the situation is given in Section 5.4. The images of a and c have been conjugated to standard form, so no parameter appears there.

1.5 Organization

The thesis is organized as follows. In Chapter 2 we define the representation variety, describe the geometric representation of a triangle group and precisely define the $\mathrm{PGSp}(2n, \mathbb{R})$-Hitchin component and Hitchin representations.

In Chapter 3 we define the Zariski tangent space and smooth points of the representation variety and we summarize the cohomological theory of deformations, covering also the particular cases of surface groups and triangle groups. Appendix A contains the necessary background on group cohomology with twisted coefficients.

Chapter 4 is devoted to the computation of the dimension formula of Theorem 1.2.3. We describe the space of 1-cocycles for triangle groups and the first cohomology group. Then we compute the dimension of the Zariski tangent space to a Hitchin representation, discuss the smoothness of the Hitchin component and give a formula for its dimension. In the last section of the chapter we explicitly compute the dimension of the $\mathrm{PGSp}(4, \mathbb{R})$-Hitchin component obtaining Corollary 1.2.4.

In Chapter 5 we compute the entire Hitchin component for $\Delta(3, 4, 4)$ in $\mathrm{PGSp}(4, \mathbb{R})$. In Section 5.1 we determine a normal form for Hitchin representations up to conjugation, as well as their trace field and matrix entry field. The necessary background about the symplectic action on Lagrangian spaces is given in Appendix B. In Section 5.2 we give an overview of the computation methodology. More details and the technical aspects of the implementation are given in Section 5.3.

2. The Hitchin component for triangle groups

In this chapter we define the representation and the character varieties of a finitely generated group Γ into a Lie group G. Then we define a geometric representation for triangle groups, review the irreducible representation into the general symplectic group and finally define Hitchin representations and the Hitchin component for triangle groups.

2.1 The representation variety

Let Γ be a finitely generated group and G be a Lie group.

Definition 2.1.1. The *representation variety* is the set of homomorphisms $\mathrm{Hom}(\Gamma, G)$ endowed with the compact-open topology.

Since Γ is discrete the compact-open topology is the topology of pointwise convergence. Given a finite set S of generators of Γ we can identify $\mathrm{Hom}(\Gamma, G)$ as a closed subset of the product G^S. To see this, let R be a set of relations for the generating set S. Every relation $r \in R$ can be written in the form

$$r = s_1^{\epsilon_1} \dots s_m^{\epsilon_m}$$

with $s_i \in S$, $\epsilon_i = \pm 1$. We fix one such expression once and for all and denote by $f_r : G^S \to G$ the map

$$f_r((g_s)_{s\in S}) = g_{s_1}^{\epsilon_1} \cdot \ldots \cdot g_{s_m}^{\epsilon_m}.$$

For every homomorphism $\varphi : \Gamma \to G$ it holds $f_r((\varphi(s))_{s\in S}) = \varphi(s_1)^{\epsilon_1} \cdot \ldots \cdot \varphi(s_m)^{\epsilon_m} = \varphi(r) = e$.

Lemma 2.1.2. *The space* $\mathrm{Hom}(\Gamma, G)$ *is homeomorphic to the closed subset* $\bigcap_{r\in R} f_r^{-1}(e)$ *of* G^S *via*

$$\begin{aligned} \Theta : \mathrm{Hom}(\Gamma, G) &\to G^S \\ \varphi &\mapsto (\varphi(s))_{s\in S}. \end{aligned}$$

Proof. The map Θ is injective because every homomorphism is determined on the generators of Γ and the image is precisely $\bigcap_{r\in R} f_r^{-1}(e)$. Recall that a subbasis of the compact-open topology on $\mathrm{Hom}(\Gamma, G)$ is given by the sets of the form

$$S(C,U) = \{f : \Gamma \to G \mid f(C) \subset U\} \cap \mathrm{Hom}(\Gamma, G).$$

To see continuity, consider an open set $\prod_{s\in S} U_s$ of G^S, then the preimage

$$\Theta^{-1}\left(\bigcap f_r^{-1}(e) \cap \prod_{s\in S} U_s\right) = \bigcap_{s\in S} C(s, U_s)$$

is open. To see that Φ is open, notice that every element $\gamma \in \Gamma$ defines a continuous map $f_\gamma : G^S \to G$, in the same way as we saw above for the relations r. Then for any finite set $C \subset \Gamma$ and any open set $U \subset G$ the image of $S(C,U)$ is

$$\begin{aligned}\Theta(S(C,U)) = \Theta\left(\bigcap_{\gamma\in C} S(\gamma, U)\right) &= \bigcap_{\gamma\in C} \Phi(S(\gamma,U)) = \bigcap_{\gamma\in C}\{(\varphi(s))_{s\in S} \mid \varphi(\gamma) \in U\} \\ &= \bigcap_{\gamma\in C}\{(\varphi(s))_{s\in S} \mid f_\gamma((\varphi(s))_s) \in U\} = \bigcap_{\gamma\in C} f_\gamma^{-1}(U).\end{aligned}$$

Hence open and this concludes the proof. □

There is a continuous action of G by conjugation on $\mathrm{Hom}(\Gamma, G)$: for $g \in G$ and $\varphi : \Gamma \to G$ we define $g \cdot \varphi : \Gamma \to G$ by

$$(g \cdot \varphi)(\gamma) = g\varphi(\gamma)g^{-1}.$$

We denote the quotient, called the *character variety*, by

$$\chi(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/G.$$

The image in $\chi(\Gamma, G)$ of a homomorphism ϕ is denoted by $[\phi]$.

2.2 Triangle groups

Recall from the introduction that a hyperbolic triangle group $\Delta(p, q, r)$ is a group with a presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1\rangle,$$

with $1/p + 1/q + 1/r < 1$. It can be realized as a subgroup of the isometry group of the upper half plane $\mathbb{H}^2$ by fixing a geodesic triangle with angles π/p, π/q, π/r in $\mathbb{H}^2$ and letting the generators of $\Delta(p,q,r)$ act as reflections in its sides. The resulting representation is what we call a *geometric representation* and it is unique up to conjugacy since triangles in $\mathbb{H}^2$ are determined up to congruence by their angles. In this section we fix one explicit geometric representation. We need to recall some hyperbolic geometry.

The Lie group $\mathrm{PGL}(2,\mathbb{R}) = \{g \in \mathrm{GL}(2,\mathbb{R}) \mid \det(g) = \pm 1\}/\{\pm \mathrm{id}\}$ acts on the hyperbolic plane as follows: a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}(2,\mathbb{R})$ with determinant 1 determines the Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}$$

while one with determinant -1 gives the Möbius anti-transformation

$$z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}.$$

This action identifies the isometry group of $\mathbb{H}^2$ and $\mathrm{PGL}(2,\mathbb{R})$. Indeed, the former is generated by reflections along Euclidean vertical half-lines (line reflections) and along half-circles orthogonal to $\partial\mathbb{H}^2$ (circle inversions) [30, Proposition 2.1.11]. Every circle inversion can be obtained by conjugating the inversion $z \mapsto \frac{1}{\overline{z}}$ along the unit circle with translations $z \mapsto z+b$ and dilations $z \mapsto az$. Conjugation by translations also transform every line reflection into $z \mapsto \overline{z}$. This embeds the isometry group of $\mathbb{H}^2$ as a subgroup of the group of all Möbius transformations and anti-transformations, and in fact they are isomorphic [30, Proposition 2.3.8].

As hyperbolic triangle groups are realized by reflections along geodesics, which are vertical lines and half-circles orthogonal to $\partial\mathbb{H}^2$, we need to describe line reflections and circle inversions explicitly. Denote by r_x^∞ the reflection along the vertical line at $x \in \mathbb{R}$. As the line at x can be translated to the vertical line at 0 by the map $z \mapsto z - x$ and $r_0^\infty(z) = -\overline{z}$, it follows that $r_x^\infty(z) = -\overline{z} + 2x$. This is given by the matrix

$$r_x^\infty = \begin{pmatrix} -1 & 2x \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{R}).$$

Denote by $r_{C_r(x)}$ the circle inversion along a half circle $C_r(x)$ of radius r with center in $x \in \mathbb{R}$. The half circle can be translated and scaled to the unit half circle by the map $z \mapsto \frac{z-x}{r}$. Therefore $r_{C_r(x)}$ is given by $z \mapsto \frac{r^2}{\overline{z}-x} + x$ which corresponds to the matrix $\frac{1}{r}\left(\begin{smallmatrix} x & r^2-x^2 \\ 1 & -x \end{smallmatrix}\right) \in \mathrm{PGL}(2,\mathbb{R})$. The reflection along a half circle with endpoints $x < y \in \mathbb{R}$ is described by the matrix

$$r_x^y = \frac{1}{y-x}\begin{pmatrix} x+y & -2xy \\ 2 & -(x+y) \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{R}),$$

since the radius of the half circle is $\frac{y-x}{2}$ and it is centered at $\frac{x+y}{2}$.

To fix a geometric representation $\rho_0 : \Delta(p,q,r) \to \mathrm{PGL}(2,\mathbb{R})$ it suffices to choose a triangle in the hyperbolic plane with geodesic sides and desired internal angles. Let one side be the half circle between 0 and 1, the second side be the vertical line at $k \in \mathbb{R} \subseteq \partial\mathbb{H}^2$ and the third side be the half circle with endpoints $m < l \in \mathbb{R} \subseteq \partial\mathbb{H}^2$.

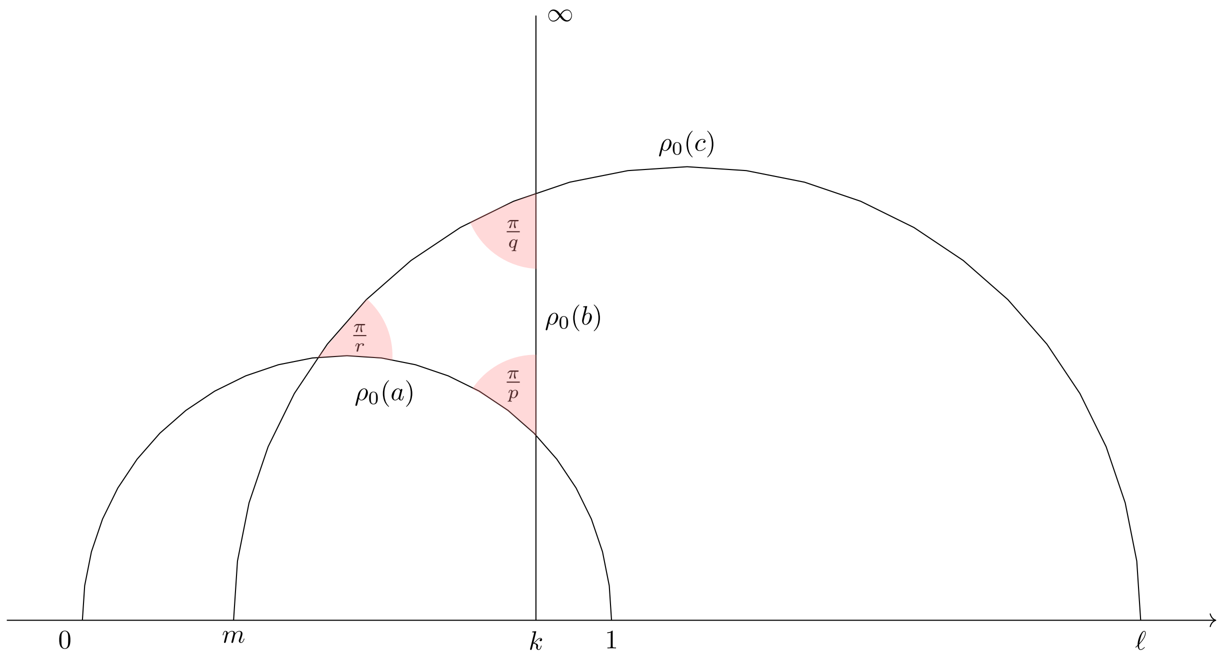


Figure 2.1: The geometric representation ρ_0 of $\Delta(p,q,r)$.

The representation ρ_0 maps the generators a, b, c of the triangle group $\Delta(p,q,r)$ to the reflections in the sides, as in Figure 2.1. By the above discussion we have matrix representatives

$$\begin{aligned} \rho_0(a) &= r_0^1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \\ \rho_0(b) &= r_\infty^k = \begin{pmatrix} -1 & 2k \\ 0 & 1 \end{pmatrix} \\ \rho_0(c) &= r_m^\ell = \frac{1}{\ell - m} \begin{pmatrix} m+\ell & -2m\ell \\ 2 & -(m+\ell) \end{pmatrix} \end{aligned} \tag{2.1}$$

The endpoints m, k, ℓ are determined by the internal angles of the triangle. Indeed, suppose that $(x,y,z,w) \in (\partial\mathbb{H}^2)^4$ are so oriented that the geodesic $\widehat{xz}$ with endpoints (x,z) and the geodesic $\widehat{yw}$ with endpoints (y,w) intersect. Let $v = \widehat{xz} \cap \widehat{yw}$ be the intersection point and θ be the angle between the arcs $\widehat{vx}$ and $\widehat{vy}$ (cfr. Figure 2.2). Then $\cos^2\left(\frac{\theta}{2}\right) = \frac{(x-w)(y-z)}{(x-z)(y-w)}$ (see [42, Example 2]). In our setting we have

$$\cos^2(\pi/2p) = k, \quad \cos^2(\pi/2q) = \frac{\ell - k}{\ell - m}, \quad \cos^2(\pi/2r) = \frac{\ell(1-m)}{\ell - m}. \tag{2.2}$$

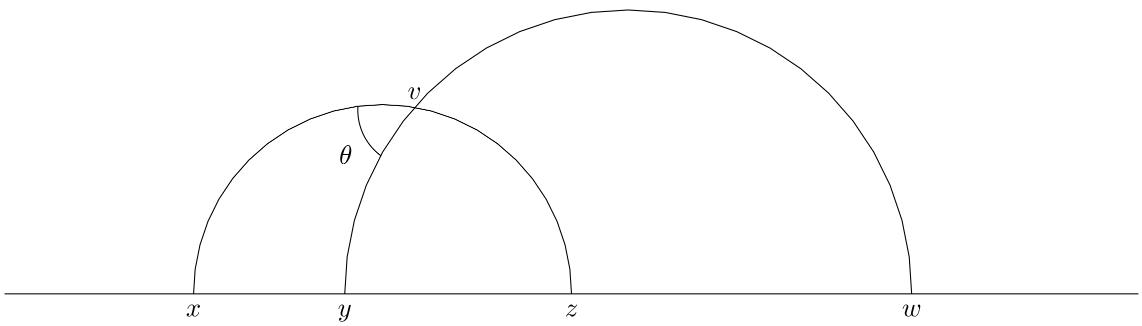


Figure 2.2: The angle between intersecting hyperbolic geodesics.

Solving for k, ℓ, m gives

$$\begin{aligned} k &= c_p \\ \ell &= \frac{mc_q - c_p}{c_q - 1} \\ m &= \frac{c_p + c_q - c_r \pm \sqrt{(c_p + c_q - c_r)^2 + 4c_pc_q(c_r - 1))}}{2c_q}. \end{aligned} \tag{2.3}$$

where $c_p = \cos^2(\pi/2p)$, $c_q = \cos^2(\pi/2q)$, $c_r = \cos^2(\pi/2r)$. The two solutions correspond to the triangles lying on the left (solution with the minus) and on the right (solution with the plus) of the vertical line at k.

Remark 2.2.1. For the matrix representatives of (2.1) it holds $\rho_0(a)^2 = \rho_0(b)^2 = \rho_0(c)^2 = \mathrm{id}$ and $(\rho_0(a)\rho_0(b))^p = (\rho_0(a)\rho_0(c))^r = -\mathrm{id}$, while $(\rho_0(b)\rho_0(c))^q = (-1)^{q+1}\mathrm{id}$. We outline the argument for $\rho_0(a)\rho_0(c)$, the other cases being analogous. An explicit computation shows that $\rho_0(a)\rho_0(c)$ has the two distinct eigenvalues $e^{i\pi/r}, e^{-i\pi/r}$. Therefore it is diagonalizable over $\mathbb{C}$ and so is $(\rho_0(a)\rho_0(c))^r$ with eigenvalues $(e^{\pm i\pi/r})^r = -1$. The alternating sign for $\rho_0(b)\rho_0(c)$ is due to the fact that its eigenvalues are $-e^{i\pi/q}, -e^{-i\pi/q}$.

2.3 The irreducible representation

Let $\mathrm{SL}^{\pm}(2,\mathbb{R}) = \{g \in \mathrm{GL}(2,\mathbb{R}) \mid \det(g) = \pm 1\}$. For any $m \geq 2$ denote by $\pi_m : \mathrm{SL}^{\pm}(2,\mathbb{R}) \to \mathrm{GL}(m,\mathbb{R})$ the representation where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}^{\pm}(2,\mathbb{R})$$

acts on the space $P_{m-1}[X,Y]$ of homogeneous polynomials of degree $m-1$ in the two variables X, Y by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X^{m-1-i}Y^i = (aX + cY)^{m-1-i}(bX + dY)^i$$

for every $0 \leq i \leq m-1$. We call π_m the (m-dimensional) *irreducible representation*. Let $m = 2n$ be even, then the skew-symmetric $2n \times 2n$-matrix

$$F_n = \begin{pmatrix} & & & \binom{2n-1}{0}^{-1} \\ & & -\binom{2n-1}{1}^{-1} & \\ & \iddots & & \\ -\binom{2n-1}{2n-1}^{-1} & & & \end{pmatrix}$$

describes a non-degenerate symplectic form. With respect to the basis $\{X^{2n-1}, X^{2n-2}Y, \ldots, Y^{2n-1}\}$ of $P_{2n-1}[X,Y]$ the irreducible representation preserves F_n up to sign, more precisely for all $g \in \mathrm{SL}^{\pm}(2,\mathbb{R})$ it holds

$$\pi_{2n}(g)^T F_n \pi_{2n}(g) = \det(g) F_n. \tag{2.4}$$

That is, $\pi_{2n}(g)$ is (anti)symplectic. We denote the set of symplectic and antisymplectic matrices with respect to the form F_n by $\mathrm{GSp}(F_n,\mathbb{R})$. By a change of basis we can conjugate $\mathrm{GSp}(F_n,\mathbb{R})$ into $\mathrm{GSp}(2n,\mathbb{R}) = \{g \in \mathrm{GL}(2n,\mathbb{R}) \mid g^T \Omega_{2n} g = \pm \Omega_{2n}\}$, where Ω_{2n} is the *standard* symplectic form:

$$\Omega_{2n} = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}. \tag{2.5}$$

The homomorphism that we get after conjugation is still called and denoted the *irreducible representation* $\pi_{2n} : \mathrm{SL}^{\pm}(2,\mathbb{R}) \to \mathrm{GSp}(2n,\mathbb{R})$. Denote also $\pi_{2n} : \mathrm{PGL}(2,\mathbb{R}) \to \mathrm{PGSp}(2n,\mathbb{R})$ the induced representation.

Example 2.3.1. When $n = 2$ the irreducible action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $P_3[X,Y]$ with respect to the basis $\{X^3, X^2Y, XY^2, Y^3\}$ is given by

$$\pi_4(g) = \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & 2abc + a^2d & b^2c + 2abd & 3b^2d \\ 3ac^2 & bc^2 + 2acd & 2bcd + ad^2 & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{pmatrix}.$$

It preserves (up to sign) the symplectic form

$$F_2 = \begin{pmatrix} & & & 1 \\ & & -1/3 & \\ & 1/3 & & \\ -1 & & & \end{pmatrix}.$$

After conjugating by the 4×4 matrix $x = \left(\begin{smallmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1/3 & 0 \end{smallmatrix}\right)$ it holds $\pi_4(g) \in \mathrm{GSp}(4,\mathbb{R})$ and the irreducible representation is

$$\pi_4 : \mathrm{SL}^{\pm}(2,\mathbb{R}) \to \mathrm{GSp}(4,\mathbb{R})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^3 & -a^2b & -b^3 & -3ab^2 \\ -3a^2c & 2abc+a^2d & 3b^2d & 3(b^2c+2abd) \\ -c^3 & c^2d & d^3 & 3cd^2 \\ -ac^2 & (bc^2+2acd)/3 & bd^2 & 2bcd+ad^2 \end{pmatrix}.$$

2.4 Hitchin representations

A group homomorphism $\phi : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ is called a *Fuchsian representation* if there is a geometric representation $\rho : \Delta(p,q,r) \to \mathrm{PGL}(2,\mathbb{R})$ such that $\phi = \pi_{2n} \circ \rho$ where π_{2n} is the irreducible representation $\pi_{2n} : \mathrm{PGL}(2,\mathbb{R}) \to \mathrm{PGSp}(2n,\mathbb{R})$ of Section 2.3. When $\rho = \rho_0$ is the geometric representation of (2.1) we call

$$\phi_0 = \pi_{2n} \circ \rho_0 : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R}). \tag{2.6}$$

the *base representation*. Every Fuchsian representation is conjugate to the base representation.

Definition 2.4.1. The *Hitchin component* $\mathrm{Hit}(\Delta(p,q,r),\mathrm{PGSp}(2n,\mathbb{R}))$ is the connected component of the character variety $\chi(\Delta(p,q,r),\mathrm{PGSp}(2n,\mathbb{R}))$ which contains $[\phi_0]$.

A homomorphism $\phi : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ is called a *Hitchin representation* if its conjugacy class $[\phi]$ is an element of $\mathrm{Hit}(\Delta(p,q,r),\mathrm{PGSp}(2n,\mathbb{R}))$. We denote the set of Hitchin representations by $\mathcal{H}it(\Delta(p,q,r),\mathrm{PGSp}(2n,\mathbb{R}))$.

This is analogous as Hitchin representations in the context of surface groups (see Section 1.2). The fact that every triangle group contains a finite index surface group [30, Prop 3.1.14], allows one to deduce results for $\Delta(p,q,r)$-Hitchin representations from the analogous statements for surface groups Hitchin representations.

Lemma 2.4.2. *Let $\iota : \pi_1(S) \to \Delta(p,q,r)$ be a finite index surface group in $\Delta(p,q,r)$, $G \coloneqq \mathrm{PGSp}(2n,\mathbb{R})$, $H \coloneqq \mathrm{PGL}(2n,\mathbb{R})$ and let $\jmath : G \hookrightarrow H$ be the inclusion map. If $\phi : \Delta(p,q,r) \to G$ is a Hitchin representation, then also $\phi \circ \iota : \pi_1(S) \to G$ and $\jmath \circ \phi \circ \iota : \pi_1(S) \to H$ are Hitchin representations in the corresponding representation varieties.*

Proof. Let $\Delta = \Delta(p,q,r)$. Let $\phi_0 : \Delta \to G$ be the base representation, then $\jmath \circ \phi_0 : \Delta \to H$, $\jmath \circ \phi_0 \circ \iota : \pi_1(S) \to H$ and $\phi_0 \circ \iota : \pi_1(S) \to G$ are Fuchsian representations in the respective spaces and $[\jmath \circ \phi_0] \in \mathrm{Hit}(\Delta, H)$, $[\jmath \circ \phi_0 \circ \iota] \in \mathrm{Hit}(\pi_1(S), H)$, $[\phi_0 \circ \iota] \in \mathrm{Hit}(\pi_1(S), G)$.

We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(\Delta, G) & \xrightarrow{\iota_*} & \mathrm{Hom}(\pi_1(S), G) & \xrightarrow{\jmath_*} & \mathrm{Hom}(\pi_1(S), H) \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\
\mathrm{Hom}(\Delta, G)/G & \xrightarrow{\overline{\iota_*}} & \mathrm{Hom}(\pi_1(S), G)/G & \xrightarrow{\overline{\jmath_*}} & \mathrm{Hom}(\pi_1(S), H)/H
\end{array}$$

thus the set

$$p(\jmath_*\iota_*\mathcal{H}it(\Delta, G)) = \overline{\jmath_*}\,\overline{\iota_*}(p(\mathcal{H}it(\Delta, G))) = \overline{\jmath_*}\,\overline{\iota_*}(\mathrm{Hit}(\Delta, G))$$

is a connected subset of $\mathrm{Hom}(\pi_1(S), H)/H$, which contains $[\jmath \circ \phi_0 \circ \iota] = \overline{\jmath_*}\,\overline{\iota_*}([\phi_0])$. Therefore it is contained in $\mathrm{Hit}(\pi_1(S), H)$, that is, $p(\jmath_*\iota_*\mathcal{H}it(\Delta, G)) \subseteq \mathrm{Hit}(\pi_1(S), H)$. Which means that $\jmath_*\iota_*\mathcal{H}it(\Delta, G) \subseteq \mathcal{H}it(\pi_1(S), H)$.

Analogously,

$$p(\iota_*\mathcal{H}it(\Delta, G)) = \overline{\iota_*}(p(\mathcal{H}it(\Delta, G))) = \overline{\iota_*}(\mathrm{Hit}(\Delta, G))$$

is connected, contains the Fuchsian representation $[\phi_0 \circ \iota]$ of $\mathrm{Hom}(\pi_1(S), G)/G$ and therefore it is contained in $\mathrm{Hit}(\pi_1(S), G)$. That is, $p(\iota_*\mathcal{H}it(\pi_1(S), G)) \subset \mathrm{Hit}(\pi_1(S), G)$ and $\iota_*\mathcal{H}it(\Delta, G) \subset \mathcal{H}it(\pi_1(S), G)$. □

Proposition 2.4.3. *Hitchin representations* $\phi : \Delta(p, q, r) \to \mathrm{PGSp}(2n, \mathbb{R})$ *are faithful and have discrete image.*

Proof. Let $\iota : \pi_1(S) \hookrightarrow \Delta(p, q, r)$ be a finite index surface group and let $\jmath : \mathrm{PGSp}(2n, \mathbb{R}) \to \mathrm{PGL}(2n, \mathbb{R})$ be the inclusion. Let $\phi : \Delta \to \mathrm{PGSp}(2n, \mathbb{R})$ be a Hitchin representation. Then $\jmath \circ \phi \circ \iota : \pi_1(S) \to \mathrm{PGL}(2n, \mathbb{R})$ is a Hitchin representation and therefore discrete and faithful by results of Labourie [22, Theorem 1.5]. Thus also $\phi \circ \iota : \pi_1(S) \to \mathrm{PGSp}(2n, \mathbb{R})$ is discrete and faithful. We show that the same holds for ϕ. For ease of notation set $\Delta = \Delta(p, q, r)$ and $\Gamma = \pi_1(S)$.

First, we claim that ϕ is injective. Let $1 \neq \delta \in \Delta$. Then there is $q \in \mathbb{N}$ such that $\delta^q \in \Gamma$. For, since Γ is of finite index in Δ, the sequence $(\delta^i\Gamma)_{i=1}^{\infty}$ is finite and there are $i < j$ such that $\delta^i\Gamma = \delta^j\Gamma$. Thus with $q = j - i$ it holds $\delta^q \in \Gamma$. If $\delta^q \neq 1$, then since $\phi\big|_{\Gamma}$ is injective it holds $e \neq \phi(\delta^q) = \phi(\delta)^q$ and thus $\phi(\delta) \neq e$. If $\delta^q = 1$, then $\phi(\delta)$ is of finite order and since there are finitely many conjugacy classes of finite order elements in $\mathrm{PGSp}(2n, \mathbb{R})$ (see Proposition 4.4.8) it follows that $\phi(\delta)$ is conjugate to $\phi_0(\delta)$, where ϕ_0 denotes the base representation (cfr. Proposition 4.4.9) which is injective. In particular, $\phi_0(\delta) \neq e$ and thus $\phi(\delta) \neq e$ as well.

To see that ϕ has discrete image, notice first that since ϕ is injective, $\phi(\Gamma)$ is of finite index in $\phi(\Delta)$. Let $1 \neq g_1, \ldots, g_k \in \phi(\Delta)$ be such that $\phi(\Delta) = \phi(\Gamma) \sqcup \bigsqcup_{i=1}^{k} g_i\phi(\Gamma)$. Since $\phi(\Gamma)$ is discrete, there is an open subset $W \subset \mathrm{PGSp}(2n, \mathbb{R})$ such that $\phi(\Gamma) \cap W = \{e\}$, then $\widetilde{W} = W \setminus \bigsqcup_{i=1}^{k} g_i\phi(\Gamma)$ is open and $\widetilde{W} \cap \phi(\Delta) = W \cap \phi(\Gamma) = \{e\}$, which shows that $\phi(\Gamma)$ is discrete as well. □

3. Deformation theory

When G is an algebraic group the notions of Zariski tangent space and of smooth point of the representation variety play an important role in the discussion about the structure of the spaces $\mathrm{Hom}(\Gamma, G)$ and $\chi(\Gamma, G)$. In this chapter we review these notions and their relation with the (infinitesimal) deformation theory of representations. We follow and expand on Section 8 of [10], which gives a good short exposition of the material. Other references are [39], [21] and Chapter VI of [34]. An important role in the discussion is played by the group cohomology of Γ twisted by the adjoint representation of G. Background on the subject is given in Appendix A.

Throughout this chapter Γ denotes a finitely generated and finitely presented group with generating set $S = \{\gamma_1, \ldots, \gamma_k\}$ and relations R.

3.1 The Zariski tangent space

Let G be linear algebraic group defined over $\mathbb{R}$ such as $\mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{R}), \mathrm{O}(n, \mathbb{R})$, $\mathrm{GSp}(2n, \mathbb{R})$ or its quotient. In particular G is both a Lie group and an affine algebraic group. Then evaluation on the generating set of Γ (see Lemma 2.1.2) induces a structure of affine variety on $\mathrm{Hom}(\Gamma, G)$, that is $\mathrm{Hom}(\Gamma, G)$ is the zero locus of a set of polynomials with real coefficients in a number of variables which depends on the group G and the chosen generating set of Γ.

Example 3.1.1. Let $G = \mathrm{GSp}(2n, \mathbb{R}) = \{g \in M_{2n\times 2n}(\mathbb{R}) \mid g^T \Omega_{2n} g = \pm\Omega_{2n}\}$ where Ω_{2n} is the standard symplectic form on $\mathbb{R}^{2n}$ given in (2.5). The set $M_{2n\times 2n}(\mathbb{R})$ of $2n \times 2n$ matrices over $\mathbb{R}$ can be idenfied with the $(2n)^2$-dimensional affine space $\mathbb{R}^{(2n)^2}$ with coordinates x_{ij} such that $x_{ij}(g)$ is the i, j-entry of the matrix g. The condition $g^T \Omega_{2n} g = \pm\Omega_{2n}$ can be reformulated as $\left(g^T \Omega_{2n} g - \Omega_{2n}\right)\left(g^T \Omega_{2n} g + \Omega_{2n}\right) = 0$. Since each matrix entry of the left-hand-side of the equation is a polynomial in the entries of g we conclude that $\mathrm{GSp}(2n, \mathbb{R})$ is the zero locus of a set of $(2n)^2$ polynomials.

Let Γ be a finitely generated group with generating set S and relations R. Then G^S is an affine variety in $\mathbb{R}^{(2n)^2|S|}$. As in the discussion before Lemma 2.1.2 each relation r defines a map $f_r : \mathbb{R}^{(2n)^2|S|} \to \mathbb{R}^{(2n)^2}$ which is polynomial in the $(2n)^2|S|$ variables and since $\mathrm{Hom}(\Gamma, G) \cong \bigcap_{r\in R} f_r^{-1}(0)$, we conclude that $\mathrm{Hom}(\Gamma, G)$ is the zero locus of a set of polynomials. When $\Gamma = \Delta(p, q, r)$ there are 6 relations, each of which gives $(2n)^2$ polynomials. However some conditions might be redundant.

The affine algebraic structure on $\mathrm{Hom}(\Gamma, G)$ allows one to study the *tangent space* to a representation: the space $\mathrm{Hom}(\Gamma, G)$ is closed in G^S, so tangent vectors in the usual sense are not defined, but instead there is a notion of *Zariski tangent vector*. Namely, let $X \subseteq \mathbb{R}^N$ be an affine variety described by $X = \{F_1 = \ldots = F_m = 0\}$, $F_i : \mathbb{R}^N \to \mathbb{R}$, then the *Zariski* tangent space at $x \in X$ is the kernel of the Jacobian of $F = (F_1, \ldots, F_m)$, that is

$$T_x^{Zar} X = \ker d_x F = \ker \big(\frac{\partial F_i}{\partial x_j}\big)_{ij}.$$

Since we are considering representation varieties it is more convenient to work in G^S instead of $\mathbb{R}^N$.

Definition 3.1.2. For $r \in R$ let $f_r : G^S \to G$ be the maps defined by the relations of Γ, so that $\mathrm{Hom}(\Gamma, G) \cong \bigcap_{r \in R} f_r^{-1}(e)$ (see Lemma 2.1.2). Let $F = (f_r)_{r \in R} : G^S \to G^R$. The *Zariski tangent space* at $\phi \in \mathrm{Hom}(\Gamma, G)$ is:

$$T_\phi^{Zar} \mathrm{Hom}(\Gamma, G) = \ker d_\phi F \subset T_e G^S.$$

Equivalently, the Zariski tangent space at ϕ consists of all tangent vectors $\frac{d}{dt}\big|_{t=0} \phi_t$ tangent to a smooth path $t \mapsto \phi_t$ inside G^S with $\phi_0 = \phi$ and which satisfy the relations $f_r = 0$ *up to first order*, i.e. $\frac{d}{dt}\big|_{t=0} f_r(\phi_t) = 0$ for all $r \in R$.

The Zariski tangent space can be reformulated in terms of group cohomology (see Appendix A for background material on group cohomlogy). Since G is an affine algebraic group, it is in particular a Lie group and we denote its Lie algebra by $\mathfrak{g}$ and the adjoint representation by $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

Given a representation $\phi : \Gamma \to G$ the composition $\mathrm{Ad} \circ \phi : \Gamma \to \mathrm{GL}(\mathfrak{g})$ defines a homomorphism and the space of 1-cocycles $\Gamma \to \mathfrak{g}$ is

$$Z^1(\Gamma, \mathfrak{g})_\phi = \{u : \Gamma \to \mathfrak{g} \mid u(\gamma\eta) = u(\gamma) + \mathrm{Ad}(\phi(\gamma))u(\eta), \text{ for all } \gamma, \eta \in \Gamma\}.$$

Lemma 3.1.3. *Let $\phi \in \mathrm{Hom}(\Gamma, G)$. Then*

$$T_\phi^{Zar} \mathrm{Hom}(\Gamma, G) \cong Z^1(\Gamma, \mathfrak{g})_\phi.$$

Proof. Let $S = \{\gamma_1, \ldots, \gamma_k\}$ be a generating set for Γ with relations R and set $g_i = \phi(\gamma_i) \in G$, so that $\phi \in \mathrm{Hom}(\Gamma, G)$ corresponds to $(g_1, \ldots, g_k) \in G^k$ (and we will write ϕ also when we are in G^k).

By definition $T_\phi^{Zar} \mathrm{Hom}(\Gamma, G) = \bigcap_{r \in R} \ker d_\phi f_r \subset T_\phi G^k$, where $f_r : G^k \to G$. Every $g \in G$ induces an isomorphism $d_e R_g : \mathfrak{g} \to T_g G$ by right-multiplication, so that a vector $v \in \mathfrak{g}$ corresponds to $d_e R_g v = \frac{d}{dt}\big|_{t=0} \exp(tv) g$. In particular we can identify $\mathfrak{g}^k \cong T_\phi G^k$ via

$$\Phi \coloneqq d_e R_{g_1} \times \ldots \times d_e R_{g_k} : \mathfrak{g}^k \to \prod_i^k T_{g_i} G \cong T_\phi G^k. \tag{3.1}$$

In this way the Zariski tangent space $\bigcap_{r\in R}\ker d_\phi f_r$ becomes a subspace of $\mathfrak{g}^k$:

$$\bigcap_{r\in R}\ker d_\phi f_r \cong \bigcap_{r\in R}\ker\left(d_\phi f_r\circ\Phi\right)\subseteq \mathfrak{g}^k.$$

For every $r\in R$ we set

$$L_r := \ker(d_\phi f_r\circ\Phi) = \ker d_{(e,\dots,e)}f_r\circ(R_{g_1}\times\ldots\times R_{g_k}).$$

Evaluation on the generators of Γ embeds the space of 1-cocycles on Γ as a subspace of $\mathfrak{g}^k$ and we claim that

$$\bigcap_{r\in R} L_r = Z^1(\Gamma,\mathfrak{g})_\phi \text{ in } \mathfrak{g}^k.$$

Let F be the free group on S and $q: F\to\Gamma$ be the canonical map induced by the inclusion $S\hookrightarrow\Gamma$. Then $\phi\circ q$ is a homomorphism $F\to G$ and since there are no relations on F it holds $Z^1(F,\mathfrak{g})_{\phi\circ q}\cong\mathfrak{g}^k$ via $v\mapsto(v(\gamma_i))_{i=1}^k$.

Suppose that for every $v=(v(\gamma_i))_i\in\prod_i\mathfrak{g}=Z^1(F,\mathfrak{g})_{\phi\circ q}$ and every relation $r\in R$ it holds

$$d_{(e\times\ldots\times e)}f_r\circ(R_{g_1}\times\ldots\times R_{g_k})(v(\gamma_i))_{i=1}^k = v(r). \tag{3.2}$$

Then $\bigcap_{r\in R} L_r=\{v\in Z^1(F,\mathfrak{g})_{\phi\circ q}\mid v(r)=0 \text{ for all } r\in R\}$. Every 1-cocycle v on F which vanishes on R satisfies

$$v(zr)=v(z)+\operatorname{Ad}\phi(z)v(r)=v(z)$$

for every $z\in F$ and $r\in R$, hence it is of the form $v=u_1\circ q$ where u_1 is a 1-cocycle on Γ. This shows that

$$\bigcap_{r\in R} L_r=\{v\in Z^1(F,\mathfrak{g})_{\phi\circ q}\mid v(r)=0 \text{ for all } r\in R\}=Z^1(\Gamma,\mathfrak{g})_\phi,$$

as claimed. It remains to prove (3.2). To this end, let $r=s_1^{\epsilon_1}\ldots s_m^{\epsilon_m}$ with $s_i\in S$, $\epsilon_i=\pm1$. Since the curve $(\exp(tv(s_1)),\ldots,\exp(tv(s_k)))$ in G is tangent to $(v(s_i))_i$, the left-hand-side of (3.2) is the initial velocity of the curve

$$g(t):=f_r(\exp(tv(s_1))g_1,\ldots,\exp(tv(s_k))g_k)$$

and it suffices to show that $\frac{d}{dt}\big|_{t=0}g(t)=v(r)$. The proof is straightforward but lengthy. We write

$$\begin{aligned} g(t) &= \prod_{i=1}^m (e^{tv(s_i)}g_i)^{\epsilon_i}\\ &= (e^{tv(s_1)}g_1)^{\epsilon_1}\prod_{i=2}^m(e^{tv(s_i)}g_i)^{\epsilon_i}\underbrace{\phi(s_1^{\epsilon_1}\ldots s_m^{\epsilon_m})^{-1}}_{=\mathrm{id}}\\ &= \underbrace{(e^{tv(s_1)}g_1)^{\epsilon_1}\phi(s_1^{\epsilon_1})^{-1}}_{=:g_1(t),\text{ path through } e}\prod_{i=1}^{m-1}\underbrace{\phi(s_1^{\epsilon_1}\ldots s_i^{\epsilon_i})(e^{tv(s_{i+1})}g_{i+1})^{\epsilon_{i+1}}\phi(s_1^{\epsilon_1}\ldots s_{i+1}^{\epsilon_{i+1}})^{-1}}_{=:g_{i+1}(t),\text{ path through } e} \end{aligned}$$

and differentiating at $t=0$ we get $\frac{d}{dt}\big|_{t=0} g(t) = \frac{d}{dt}\big|_{t=0} g_1(t) + \sum_{i=1}^{m-1} \frac{d}{dt}\big|_{t=0} g_{i+1}(t)$.

It holds $\frac{d}{dt}\big|_{t=0} g_1(t) = v(s_1^{\epsilon_1})$. Indeed, if $\epsilon_1 = 1$,

$$\frac{d}{dt}\Big|_{t=0} g_1(t) = \frac{d}{dt}\Big|_{t=0} e^{tv(s_1)} = v(s_1) = v(s_1^{\epsilon_1}),$$

and if $\epsilon_1 = -1$,

$$\frac{d}{dt}\Big|_{t=0} g_1(t) = \frac{d}{dt}\Big|_{t=0} g_1^{-1} e^{-tv(s_1)} g_1 = \mathrm{Ad}(g_1^{-1})(-v(s_1)) = v(s_1^{-1}) = v(s_1^{\epsilon_1}),$$

where in the second-to-last step we used that v is a cocycle.

For $i \geq 1$ it holds $\frac{d}{dt}\big|_{t=0} g_{i+1}(t) = \mathrm{Ad}(g_1^{\epsilon_1} \dots g_i^{\epsilon_i}) v(s_{i+1}^{\epsilon_{i+1}})$. Indeed,

$$\begin{aligned}
\mathrm{Ad}(g_1^{\epsilon_1} \dots g_i^{\epsilon_i}) v(s_{i+1}^{\epsilon_{i+1}}) &= \begin{cases} \mathrm{Ad}(g_1^{\epsilon_1} \dots g_i^{\epsilon_i}) v(s_{i+1}) & \text{if } \epsilon_{i+1} = 1 \\ \mathrm{Ad}(g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}})(\epsilon_{i+1} v(s_{i+1})) & \text{if } \epsilon_{i+1} = -1 \end{cases} \\
&= \begin{cases} \frac{d}{dt}\big|_{t=0} g_1^{\epsilon_1} \dots g_i^{\epsilon_i} e^{t\epsilon_{i+1} v(s_{i+1})} (g_1^{\epsilon_1} \dots g_i^{\epsilon_i})^{-1} & \text{if } \epsilon_{i+1} = 1 \\ \frac{d}{dt}\big|_{t=0} g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}} e^{t\epsilon_{i+1} v(s_{i+1})} (g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}}) & \text{if } \epsilon_{i+1} = -1 \end{cases} \\
&= \begin{cases} \frac{d}{dt}\big|_{t=0} g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}} (g_{i+1}^{\epsilon_{i+1}})^{-1} \underbrace{e^{tv(s_{i+1})} g_{i+1}^{\epsilon_{i+1}}}_{(e^{tv(s_{i+1})} g_{i+1})^{\epsilon_{i+1}}} (g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}})^{-1} & \text{if } \epsilon_{i+1} = 1 \\ \frac{d}{dt}\big|_{t=0} \underbrace{g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}} e^{t\epsilon_{i+1} v(s_{i+1})}}_{g_1^{\epsilon_1} \dots g_i^{\epsilon_i} (e^{tv(s_{i+1})} g_{i+1})^{\epsilon_{i+1}}} (g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}})^{-1} & \text{if } \epsilon_{i+1} = -1 \end{cases} \\
&= \begin{cases} \frac{d}{dt}\big|_{t=0} g_1^{\epsilon_1} \dots g_i^{\epsilon_i} (e^{tv(s_{i+1})} g_{i+1})^{\epsilon_{i+1}} (g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}})^{-1} & \text{if } \epsilon_{i+1} = 1 \\ \frac{d}{dt}\big|_{t=0} g_1^{\epsilon_1} \dots g_i^{\epsilon_i} (e^{tv(s_{i+1})} g_{i+1})^{\epsilon_{i+1}} (g_1^{\epsilon_1} \dots g_{i+1}^{\epsilon_{i+1}})^{-1} & \text{if } \epsilon_{i+1} = -1 \end{cases} \\
&= \frac{d}{dt}\Big|_{t=0} g_{i+1}(t).
\end{aligned}$$

Therefore we conclude

$$\frac{d}{dt}\Big|_{t=0} g(t) = v(s_1^{\epsilon_1}) + \sum_{i=1}^{m-1} \mathrm{Ad}(g_1^{\epsilon_1} \dots g_i^{\epsilon_i}) v(s_{i+1}^{\epsilon_{i+1}}).$$

On the other hand since v is a 1-cocyle by Lemma A.1.1 it holds

$$v(r) = v(s_1^{\epsilon_1} \dots s_m^{\epsilon_m} = v(s_1^{\epsilon_1}) + \sum_{i=1}^{m-1} \mathrm{Ad}(g_1^{\epsilon_1} \dots g_i^{\epsilon_i}) v(s_{i+1}^{\epsilon_{i+1}}),$$

which concludes the proof.

□

The space of 1-cocyles also describes the initial velocities of paths $t \mapsto \phi_t$ of maps $\phi_t : \Gamma \to G$ which are homomorphisms *up to the first order*, that is such that $\frac{d}{dt}\big|_{t=0} \phi_t(\gamma)\phi_t(\eta)\phi_t(\gamma\eta)^{-1} = 0$ for all $\gamma, \eta \in \Gamma$.

Proposition 3.1.4. *Given $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ the family $\phi_t := \exp(tu(\cdot))\phi(\cdot)$, $t \in \mathbb{R}$, defines a path of maps $\phi_t : \Gamma \to G$ with*

$$\phi_0 = \phi \text{ and } \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi_t(\eta)\phi_t(\gamma\eta)^{-1} = 0 \text{ for all } \gamma, \eta \in \Gamma. \tag{3.3}$$

Conversely, every path $t \mapsto \phi_t$ satisfying (3.3) defines a 1-cocycle $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ by

$$u(\gamma) = \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi(\gamma)^{-1}.$$

Proof. Let $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ be a 1-cocyle and for each $\gamma \in \Gamma$ consider the path $t \mapsto \phi_t(\gamma) = \exp(tu(\gamma))\phi(\gamma)$ through $\phi(\gamma)$. Then since ϕ *is* an homomorphism, for $\gamma, \eta \in \Gamma$ we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi_t(\eta)\phi_t(\gamma\eta)^{-1} &= \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi(\gamma)^{-1}\phi(\gamma)\phi_t(\eta)\phi(\gamma\eta)^{-1}(\phi_t(\gamma\eta)\phi(\gamma\eta)^{-1})^{-1} \\
&= \frac{d}{dt}\Big|_{t=0}\exp(tu(\gamma))c_{\phi(\gamma)}(\exp(tu(\eta)))(\exp(tu(\gamma\eta)))^{-1} \\
&= \frac{d}{dt}\Big|_{t=0}\exp(tu(\gamma)) + \frac{d}{dt}\Big|_{t=0}c_{\phi(\gamma)}(\exp(tu(\eta))) \\
&\quad + \frac{d}{dt}\Big|_{t=0}(\exp(tu(\gamma\eta)))^{-1} \\
&= u(\gamma) + \mathrm{Ad}(\phi(\gamma))u(\eta) - u(\gamma\eta) = 0.
\end{aligned}$$

Conversely, suppose that $t \mapsto \phi_t$ satisfies (3.3) and for every $\gamma \in \Gamma$ set $u(\gamma) = \frac{d}{dt}\big|_{t=0}\phi_t(\gamma)\phi(\gamma)^{-1}$. Then similary as above for every $\gamma, \eta \in \Gamma$ it holds

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi_t(\eta)\phi_t(\gamma\eta)^{-1} \\
&= \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi(\gamma)^{-1}\phi(\gamma)\phi_t(\eta)\phi(\gamma\eta)^{-1}\left(\phi_t(\gamma\eta)\phi(\gamma\eta)^{-1}\right)^{-1} \\
&= \frac{d}{dt}\Big|_{t=0}\phi_t(\gamma)\phi(\gamma)^{-1} + \frac{d}{dt}\Big|_{t=0}\phi(\gamma)\phi_t(\eta)\phi(\gamma\eta)^{-1} + \frac{d}{dt}\Big|_{t=0}(\phi_t(\gamma\eta)\phi(\gamma\eta)^{-1})^{-1} \\
&= u(\gamma) + \mathrm{Ad}(\phi(\gamma))u(\eta) - u(\gamma\eta).
\end{aligned}$$

□

Another useful characterization of the Zariski tangent space is the following

Lemma 3.1.5. *Let Γ be finitely generated with generating set $S = \{\gamma_1, \ldots, \gamma_k\}$ and relations R. Then*

$$Z^1(\Gamma, \mathfrak{g})_\phi \cong \{(u_1, \ldots, u_k) \in \mathfrak{g}^k \mid \sum_{i=1}^{k} \mathrm{Ad}\,\phi(\partial_i r)u_i = 0 \text{ for all } r \in R\}$$

where $\partial_i = \frac{\partial}{\partial\gamma_i}$ is a derivation defined by $\partial_i(\gamma_j) := \delta_{ij}$ and satisfying $\partial_i(xy) = \partial_i(x) + x\partial_i(y)$ and $\mathrm{Ad}\,\phi(\partial_i r)$ is defined by linearity.

For the proof, which uses Fox derivatives, we refer to [28, Proposition 3.5] and [14, Section 3.6].

The space of 1-coboundaries

$$B^1(\Gamma, \mathfrak{g})_\phi = \{v : \Gamma \to \mathfrak{g} \mid \exists X \in \mathfrak{g} \text{ such that } v(\gamma) = \mathrm{Ad}(\phi(\gamma))X - X\}$$

also plays a role in this context as it can be identified with the tangent space to orbit of ϕ under the G-action by post-conjugation. We denote the orbit by

$$\mathcal{O}_\phi = \{g\phi(\cdot)g^{-1} \mid g \in G\} \subset \mathrm{Hom}(\Gamma, G).$$

Notice that $\mathcal{O}_\phi$ is isomorphic to the quotient of G by the stabilizer of ϕ for the conjugation action, which is the centralizer of the subgroup $\phi(\Gamma)$ in G, so it is closed. Therefore the orbit is a smooth manifold and the Zariski tangent space coincides with the usual notion of tangent space.

Lemma 3.1.6. *The isomorphism $\Phi : \mathfrak{g}^k \to T_\phi G^k$ of (3.1) restricts to an isomorphism*

$$B^1(\Gamma, \mathfrak{g})_\phi \cong T_\phi \mathcal{O}_\phi,$$

where the the 1-coboundaries $B^1(\Gamma, \mathfrak{g})_\phi$ are a subspace of $\mathfrak{g}^k$ via evaluation at the generators of Γ:

$$B^1(\Gamma, \mathfrak{g})_\phi = \{(\mathrm{Ad}(\phi(\gamma_1))X - X, \ldots, \mathrm{Ad}(\phi(\gamma_k))X - X) \mid X \in \mathfrak{g}\} \subset \mathfrak{g}^k.$$

Proof. For every γ_i in the generating set S of Γ let $g_i = \phi(\gamma_i) \in G$. As a subspace of G^k the orbit of ϕ is given by $\mathcal{O}_\phi = \{(gg_1g^{-1}, \ldots, gg_kg^{-1}) \mid g \in G\}$.

Let $X \in \mathfrak{g}$ and $t \mapsto g_t$ a path in G with $\frac{d}{dt}\big|_{t=0} g_t = X$. Then for every $g \in G$

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} g_t g g_t^{-1} g^{-1} &= \frac{d}{dt}\Big|_{t=0} g_t + \frac{d}{dt}\Big|_{t=0} g g_t^{-1} g^{-1} = X + \frac{d}{dt}\Big|_{t=0} c_g(g_t^{-1}) \\ &= X + d_e(c_g \circ i)X = X - d_e c_g X \\ &= X - \mathrm{Ad}(g)X,\end{aligned} \tag{3.4}$$

and thus $d_e R_{g_i}(X - \mathrm{Ad}(g_i)X) = \frac{d}{dt}\big|_{t=0} g_t g_i g_t^{-1}$. This shows that $\Phi(B^1(\Gamma, \mathfrak{g})_\phi) \subseteq T_\phi \mathcal{O}_\phi$. Equality follows since every tangent vector to the orbit is the inivial velocity of a path $t \mapsto (g_t g_1 g_t^{-1}, \ldots, g_t g_k g_t^{-1})$. □

Definition 3.1.7. The first cohomology group with coefficients in $\mathfrak{g}$ twisted by the adjoint action of ϕ is

$$H^1(\Gamma, \mathfrak{g})_\phi = Z^1(\Gamma, \mathfrak{g})_\phi / B^1(\Gamma, \mathfrak{g})_\phi.$$

3.2 Deformations and smooth points

The space $Z^1(\Gamma, G)_\phi$ of 1-cocyles is sometimes called the space of *infinitesimal deformations* of ϕ [21]. This is because by Proposition 3.1.4 a 1-cocycle is the tangent vector to a deformation $\phi_t : \Gamma \to G$ which consists of homomorphisms only *up to the first order*. It is natural to ask when there exist tangent paths consisting of actual homomorphisms.

Definition 3.2.1. (i) A 1-cocycle $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ is *integrable* if it is a tangent vector to a smooth deformation $t \mapsto \phi_t$ in $\mathrm{Hom}(\Gamma, G)$.

(ii) A homomorphism $\phi \in \mathrm{Hom}(\Gamma, G)$ is said to be an *integrably smooth point* if every 1-cocyle $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ is integrable.

Following [10] we investigate when the 1-cocycle is integrable *up to second order*, meaning it is the tangent vector to a path $t \to \phi_t$ of maps which satisfies the group relations up to the second derivative, i.e. for all $\gamma, \eta \in \Gamma$

$$\frac{d^2}{dt^2}\Big|_{t=0} \phi_t(\gamma)\phi_t(\eta)\phi_t(\gamma\eta)^{-1} = 0.$$

Let $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ and let $t \mapsto \phi_t$ be a path of homomorphisms up to first order with $\phi_0 = \phi$ and such that $\frac{d}{dt}\big|_{t=0} \phi_t(\gamma)\phi(\gamma)^{-1} = u(\gamma)$. Such a path exists by Lemma 3.1.3 and we first notice that there is an infinite sequence of maps $u_i : \Gamma \to \mathfrak{g}$, $i \in \mathbb{N}$, such that for all $\gamma \in \Gamma$ it holds

$$\phi_t(\gamma) = \exp\big(tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma)\big)\phi(\gamma).$$

To see this, fix $\gamma \in \Gamma$. Then $t \mapsto \phi_t(\gamma)\phi(\gamma)^{-1}$ is a smooth path in G through e and since the Lie group exponential of G is a local diffeomorphism, there is a smooth path $t \mapsto U_\gamma(t) \in \mathfrak{g}$ with $U_\gamma(0) = 0$ and $\phi_t(\gamma)\phi(\gamma)^{-1} = \exp(U_\gamma(t))$ for t small. By means of the Taylor series we can write $U_\gamma(t) = \sum_{i=1}^{\infty} t^i u_i(\gamma)$, where $u_i(\gamma) \in \mathfrak{g}$ for all $i \geq 1$. Notice that

$$u(\gamma) = \frac{d}{dt}\Big|_{t=0} \phi_t(\gamma)\phi(\gamma)^{-1} = \frac{d}{dt}\Big|_{t=0} \exp(\sum_{i=1}^{\infty} t^i u_i(\gamma)) = u_1(\gamma).$$

Thus $\phi_t(\gamma) = \exp\big(tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma)\big)\phi(\gamma)$, as desired. The first obstruction to u being integrable is the 2-cocycle $\sigma_2(u)$ defined by

$$\sigma_2(u) := [u, u] : (\gamma, \eta) \mapsto \frac{1}{2}[u(\gamma), \mathrm{Ad}(\phi(\gamma))u(\eta)],$$

where $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the Lie bracket. The fact that $\sigma_2(u)$ is a 2-cocycle is a straightforward computation (using that u is a 1-cocycle).

Lemma 3.2.2. *Let $u \in Z^1(\Gamma, \mathfrak{g})_\phi$ and let $\phi_t : \Gamma \to G$ be given by*

$$\phi_t(\gamma) = \exp\big(tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma)\big)\phi(\gamma).$$

Then ϕ_t is a homomorphism up to second order if and only if the 2-cocyle $[u, u]$ is a 2-coboundary.

Proof. For any $\gamma \in \Gamma$ set $U_\gamma(t) = tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma)$. It holds

$$\begin{aligned}
\phi_t(\gamma)^{-1}\phi_t(\gamma\eta) &= \left(\exp(tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma))\phi(\gamma)\right)^{-1} \exp(tu(\gamma\eta) + \sum_{i=2}^{\infty} t^i u_i(\gamma\eta))\rho(\gamma\eta) \\
&= \phi(\gamma)^{-1} \exp(tu(\gamma) + \sum_{i=2}^{\infty} t^i u_i(\gamma))^{-1} \exp(tu(\gamma\eta) + \sum_{i=2}^{\infty} t^i u_i(\gamma\eta))\rho(\gamma\eta) \\
&= \phi(\gamma)^{-1} \exp(-tu(\gamma) - \sum_{i=2}^{\infty} t^i u_i(\gamma)) \exp(tu(\gamma\eta) + \sum_{i=2}^{\infty} t^i u_i(\gamma\eta))\phi(\gamma)\phi(\eta) \\
&= \phi(\gamma)^{-1} \exp(-U_\gamma(t)) \exp(U_{\gamma\eta}(t))\phi(\gamma)\phi(\eta).
\end{aligned}$$

Thus $\phi_t(\gamma\eta) = \phi_t(\gamma)\phi_t(\eta)$ if and only if

$$\exp\left(-U_\gamma(t)\right) \exp\left(U_{\gamma\eta}(t)\right) = \phi(\gamma) \exp\left(U_\eta(t)\right) \phi(\gamma)^{-1}. \tag{3.5}$$

Let $0 \in V \subset \mathfrak{g}$ be a symmetric neighborhood of 0 such that the exponential map $\exp : V \to \exp(V)$ is a diffeomorphism, and let t be small such that $U_\gamma(t), U_\eta(t), U_{\gamma\eta}(t) \in V$.

Then with the Baker-Campbell-Hausdorff formula we can express the left-hand-side of (3.5) as

$$\begin{aligned}
\exp\left(-U_\gamma(t)\right) \exp\left(U_{\gamma\eta}(t)\right) = \exp(&-U_\gamma(t) + U_{\gamma\eta}(t) + \frac{1}{2}[-U_\gamma(t), U_{\gamma\eta}(t)] \\
&+ \frac{1}{12}\left([-U_\gamma(t), [-U_\gamma(t), U_{\gamma\eta}(t)]] + [U_{\gamma\eta}(t), [U_{\gamma\eta}(t), -U_\gamma(t)]]\right) + \ldots)
\end{aligned}$$

Let $Z_{\gamma,\eta}(t)$ be the argument of the right-hand-side. It holds $Z_{\gamma,\eta}(t) \in V$. Therefore (3.5) becomes

$$\exp(Z_{\gamma,\eta}(t)) = \phi(\gamma) \exp(U_\eta(t))\phi(\gamma)^{-1}$$

and since the exponential is a diffeomorphism on V this holds if and only if

$$Z_{\gamma,\eta}(t) = \exp^{-1}(\phi(\gamma) \exp(U_\eta(t))\phi(\gamma)^{-1}) = \mathrm{Ad}(\phi(\gamma))(U_\eta(t)).$$

This is an equation in $V \subset \mathfrak{g}$ and $\phi_t(\gamma\eta) = \phi_t(\gamma)\phi_t(\eta)$ up to to order 2 if and only if

$$\frac{d}{dt}\Big|_{t=0} Z_{\gamma,\eta}(t) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\phi(\gamma))U_\eta(t) \quad \text{and} \quad \frac{d^2}{dt^2}\Big|_{t=0} Z_{\gamma,\eta}(t) = \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Ad}(\phi(\gamma))U_\eta(t).$$

The summands of $Z_{\gamma,\eta}(t)$ containing t are $-U_\gamma(t) + U_{\gamma\eta}(t)$ thus

$$\frac{d}{dt}\Big|_{t=0} Z_{\gamma,\eta}(t) = \frac{d}{dt}\Big|_{t=0} \operatorname{Ad}(\phi(\gamma))U_\eta(t) \Leftrightarrow \frac{d}{dt}\Big|_{t=0}(-U_\gamma(t) + U_{\gamma\eta}(t)) = \operatorname{Ad}(\phi(\gamma))\frac{d}{dt}\Big|_{t=0}U_\eta(t)$$
$$\Leftrightarrow -u(\gamma) + u(\gamma\eta) = \operatorname{Ad}(\phi(\gamma))\frac{d}{dt}\Big|_{t=0}U_\eta(t)$$
$$\Leftrightarrow -u(\gamma) + u(\gamma\eta) = \operatorname{Ad}(\phi(\gamma))u(\eta).$$

which is the condition of u being a 1-cocycle. The summands of $Z_{\gamma,\eta}(t)$ containing t^2 are $-U_\gamma(t) + U_{\gamma\eta}(t) + \frac{1}{2}[-U_\gamma(t), U_{\gamma\eta}(t)]$ thus

$$\begin{aligned}
& \frac{d^2}{dt^2}\Big|_{t=0} Z_{\gamma,\eta}(t) = \frac{d^2}{dt^2}\Big|_{t=0} \operatorname{Ad}(\phi(\gamma))U_\eta(t) \\
\Leftrightarrow \quad & \frac{d^2}{dt^2}\Big|_{t=0}\left(-U_\gamma(t) + U_{\gamma\eta}(t) + \frac{1}{2}[-U_\gamma(t), U_{\gamma\eta}(t)]\right) = \operatorname{Ad}(\phi(\gamma))\frac{d^2}{dt^2}\Big|_{t=0}U_\eta(t) \\
\Leftrightarrow \quad & - u_2(\gamma) + u_2(\gamma\eta) - \frac{1}{2}[u(\gamma), u(\gamma\eta)] = \operatorname{Ad}(\phi(\gamma))u_2(\eta) \\
\Leftrightarrow \quad & \frac{1}{2}[u(\gamma), u(\gamma\eta)] = -u_2(\gamma) + u_2(\gamma\eta) - \operatorname{Ad}(\phi(\gamma))u_2(\eta) \\
\Leftrightarrow \quad & [u(\gamma), u(\gamma)] + [u(\gamma), \operatorname{Ad}(\phi(\gamma))u(\eta)] = 2(-u_2(\gamma) + u_2(\gamma\eta) - \operatorname{Ad}(\phi(\gamma))u_2(\eta)) \\
\Leftrightarrow \quad & [u(\gamma), \operatorname{Ad}(\phi(\gamma))u(\eta)] = -2\overline{\partial}^1(u_2)(\gamma,\eta).
\end{aligned}$$

This shows that the homomorphism condition is satisfied up to order 2 if and only if $[u,u] = -2\overline{\partial}^1(u_2)$, that is the obstruction $\sigma_2(u) = [u,u]$ is a 1-coboundary. □

If one keeps asking the same question for third order and so on, one finds a sequence of obstructions $\sigma_k(u) \in H^2(\Gamma, \mathfrak{g})_\phi$, $k \geq 2$, each defined in terms of the preceeding solutions. This is to say that the 1-cocycle u is integrable up to order n if and only if $\sigma_k(u) = 0$ for all $2 \leq k \leq n$ (see [4], [14]).

Definition 3.2.3. A 1-cocycle $u \in Z^1(\Gamma, G)_\phi$ is *formally integrable* if the obstructions $\sigma_n(u) = 0$ for all $n \geq 2$.

Being formally integrable a priori does not imply being tangent to a smooth deformation, but it turns out that this is the case.

Theorem 3.2.4 ([3])**.** *If a 1-cocycle is formally integrable, then it is integrable.*

Corollary 3.2.5. *If $H^2(\Gamma, \mathfrak{g})_\phi = 0$, then every 1-cocycle is the tangent vector to a smooth deformation and ϕ is an integrably smooth point of* $\operatorname{Hom}(\Gamma, G)$.

For surface groups we have the following result, due to Goldman [14, Section 1.4].

Theorem 3.2.6. *Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed surface, let G be a reductive*[1] *Lie group and $\phi : \Gamma \to G$. Then*

$$\dim H^2(\Gamma, \mathfrak{g})_\phi = \dim H^0(\Gamma, \mathfrak{g})_\phi.$$

[1]Here one needs a non-degenerate symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which is $\operatorname{Ad}(G)$-invariant. For the Lie group $\operatorname{PGSp}(2n, \mathbb{R})$ we can take the killing form.

In general, for any finitely generated group Γ and Lie group G with Lie algebra $\mathfrak{g}$ the space $H^0(\Gamma, \mathfrak{g})_\phi$ coincides with the Lie algebra of the centralizer $Z_G(\phi) = \{g \in G \mid g\phi(\gamma) = \phi(\gamma)g \text{ for all } \gamma \in \Gamma\}$ of the representation ϕ (Proposition A.2.2), that is,

$$H^0(\Gamma, \mathfrak{g})_\phi = \operatorname{Lie} Z_G(\phi).$$

From Goldman's theorem it follows that surface group Hitchin representations (i.e. homomorphisms of $\pi_1(S)$ into $\mathrm{PGSp}(2n, \mathbb{R})$ such that their conjugacy class is in the $\pi_1(S)$-Hitchin component) are integrably smooth points:

Corollary 3.2.7. *Let S be a closed surface and $\rho : \pi_1(S) \to \mathrm{PGSp}(2n, \mathbb{R})$ be a Hitchin representation. Then $H^2(\pi_1(S), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_\rho = 0$.*

Proof. By Lemma 10.1 in [22] the representation ρ is absolutely irreducible, that is $\rho^{\mathbb{C}} : \pi_1(S) \to \mathrm{PGSp}(2n, \mathbb{C})$ is irreducible. Therefore by Schur's Lemma $\rho^{\mathbb{C}}$ has trivial centralizer in $\mathrm{PGSp}(2n, \mathbb{C})$ and hence ρ has trivial centralizer in $\mathrm{PGSp}(2n, \mathbb{R})$. Since the Lie algebra of the centralizer coincides with $H^0(\pi_1(S), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_\phi$, the conclusion follows from Goldman's Theorem 3.2.6. □

We can deduce that Hitchin representations of triangle groups are integrably smooth as well.

Corollary 3.2.8. *Let $\phi : \Delta(p, q, r) \to \mathrm{PGSp}(2n, \mathbb{R})$ be a Hitchin representation. Then*

$$H^2(\Delta(p, q, r), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_\phi = H^0(\Delta(p, q, r), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_\phi = (0).$$

In particular, Hitchin representations are integrably smooth.

Proof. Let $\iota : \pi_1(S) \hookrightarrow \Delta(p, q, r)$ be a finite index surface group [30, Prop 3.1.14]. Let $\phi : \Delta(p, q, r) \to \mathrm{PGSp}(2n, \mathbb{R})$ be a Hitchin representation, then by Lemma 2.4.2 the composition $\phi \circ \iota : \pi_1(S) \to \mathrm{PGSp}(2n, \mathbb{R})$ is a surface group Hitchin representation and thus by Corollary 3.2.7

$$H^2(\pi_1(S), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_{\phi \circ \iota} = H^0(\pi_1(S), \operatorname{Lie} \mathrm{PGSp}(2n, \mathbb{R}))_{\phi \circ \iota} = (0).$$

The statement now follows from the fact that the map $H^n(\Delta(p, q, r), \mathfrak{g})_\phi \to H^n(\pi_1(S), \mathfrak{g})_{\phi \circ \iota}$ induced in cohomology by the inclusion ι is injective in every degree n (Proposition A.4.1). □

Remark 3.2.9. Corollary 3.2.7 and Corollary 3.2.8 hold also if we replace the group $\mathrm{PGSp}(2n, \mathbb{R})$ by $\mathrm{PGL}(n, \mathbb{R})$.

We go back to the general setting. There is a general notion of smoothness for points of an affine variety which in the context of representation varieties is stronger than being integrably smooth.

Definition 3.2.10. A point x of an affine variety $X \subset \mathbb{R}^n$ is a *smooth point* if there is an open neighbourhood $U \subset X$ of x such that U is an embedded submanifold of $\mathbb{R}^n$.

A useful criterion to deduct smoothness of a represention ϕ is to look at the dimension of the Zariski tangent space and show that it is minimal at ϕ (over a neighborhood of ϕ).

Lemma 3.2.11. *Let $X = F^{-1}(0) \subset \mathbb{R}^N$ be an affine variety and $x \in X$, $F : \mathbb{R}^N \to \mathbb{R}^m$. Suppose that $\operatorname{rank} d_xF \geq \operatorname{rank} d_yF$ for all y in an open set containing x. Then x is a smooth point of X.*

Proof. Let $\mu := \operatorname{rank} d_xF$. Since $F : \mathbb{R}^N \to \mathbb{R}^m$ is a smooth map and the rank of a smooth map is a lower semicontinuous map there is an open subset $V \subseteq \mathbb{R}^N$ with $x \in V$ such that $\operatorname{rank} d_yF \geq \mu$ for all $y \in V$ (i.e. the set $\{y \in \mathbb{R}^N \mid \operatorname{rank} d_yF \geq \mu\}$ is open, which is true for any μ). By assumption (up to restricting V) then it holds $\operatorname{rank} d_yF = \mu$ for all $y \in V$, so the rank is constant on V. Thus $F\big|_V : V \to \mathbb{R}^m$ is a smooth map of constant rank. Thus $F\big|_V^{-1}(0) = V \cap R^{-1}(0) = V \cap X$ is a submanifold of V and hence (V open) it is a submanifold of $\mathbb{R}^N$. $\square$

3.3 Local rigidity

If the second cohomology group tells something about the existence of smooth deformations, the first cohomology group can be used to determine when a deformation is trivial, that is, to study the local rigity of a representation.

Definition 3.3.1. A representation $\phi : \Gamma \to G$ is *locally rigid* if the G-orbit of ϕ is open in $\operatorname{Hom}(\Gamma, G)$.

Definition 3.3.2. A representation $\phi : \Gamma \to G$ is *infinitesimally rigid* if $H^1(\Gamma, \mathfrak{g})_\phi = (0)$.

The following theorem motivates the terminology.

Theorem 3.3.3 (Weil's rigidity theorem [39])**.** *If ϕ is infinitesimally rigid, then it is locally rigid.*

A nice presentation of Theorem 3.3.3 can be found in [34, Theorem 6.7]. Another proof is given in Theorem C of [31].

Since all cohomology groups of a dihedral group D_r of order $2r$ vanish (Lemma A.3.1) we obtain the following.

Corollary 3.3.4. *Let $\varphi_t : D_r \to \mathrm{PGSp}(2n, \mathbb{R})$ be a continuous path of homomorphisms defined on an interval $I \ni 0$. Then there is $T > 0$ such that for all $|t| \leq T$ φ_t is in the orbit of φ_0.*

4. The dimension of the Hitchin component

As before let $\Delta(p,q,r) = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1\rangle$ be an hyperbolic triangle group. We denote by $T(p,q,r)$ the 2-index subgroup generated by the products ab, bc and ca. In this chapter we find a formula for the dimension of the Zariski tangent space at any representation $\phi : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ whose restriction to $T(p,q,r)$ has trivial centralizer. Then we show that the dimension is constant on the set of Hitchin representations. In particular, Hitchin representations are smooth points of the representation variety and we give a formula for the dimension of the Hitchin component. We end the chapter by explicitly computing the dimension for $\mathrm{PGSp}(4,\mathbb{R})$.

Throughout this chapter, unless otherwise specified, G is a *simple*[1] Lie group with Lie algebra $\mathfrak{g}$.

4.1 Preliminaries

We set some notation and prove a few technical facts that we will need. Given a subset $U \subset \mathrm{GL}(\mathfrak{g})$ we denote the space of *U-invariant vectors* by

$$\mathfrak{g}^U = \{v \in \mathfrak{g} \mid u(v) = v \text{ for all } u \in U\}.$$

Given a subset $S \subseteq G$, then $\mathrm{Ad}_G(S)$ is a subset of $\mathrm{GL}(\mathfrak{g})$ and by Lemma A.2.1 it holds

$$\mathfrak{g}^{\mathrm{Ad}_G(S)} = \mathrm{Lie}\, Z_G(S),$$

where $Z_G(S)$ is the centralizer of S in G. In particular when $S = \{g\}$ is a single-point set, we have

$$\mathfrak{g}^{\mathrm{Ad}_G(g)} = \ker(\mathrm{Ad}_G(g) - \mathrm{id}_{\mathfrak{g}}) = \mathrm{Lie}\, Z_G(g). \tag{4.1}$$

Since G is a simple Lie group the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ is a non-degenerate bilinear form on $\mathfrak{g} \times \mathfrak{g}$. Recall that it is also Ad_G-invariant.

[1] The results in Section 4.2 and Section 4.3 hold for any Lie group G. The assumption is needed for Lemma 4.1.1.

Lemma 4.1.1. *Let $g \in G$. Then* $\operatorname{im}(\operatorname{Ad}_G(g) - \operatorname{id})^{\perp} = \ker(\operatorname{Ad}_G(g) - \operatorname{id})$, *where the orthogonal complement is taken with respect to the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$.*

Proof. We show that the kernel of $\operatorname{Ad}_G(g) - \operatorname{id}$ is contained in the orthogonal complement of the image. Let $\xi \in \ker(\operatorname{Ad}_G(g) - \operatorname{id})$. Then for all $\eta \in \mathfrak{g}$

$$\begin{aligned} B_{\mathfrak{g}}(\xi, (\operatorname{Ad}_G(g) - \operatorname{id})\eta) &= B_{\mathfrak{g}}(\xi, \operatorname{Ad}_G(g)\eta) - B_{\mathfrak{g}}(\xi, \eta) \\ &\overset{\xi \in \ker(\operatorname{Ad}_G(g) - \operatorname{id})}{=} B_{\mathfrak{g}}(\operatorname{Ad}_G(g)\xi, \operatorname{Ad}_G(g)\eta) - B_{\mathfrak{g}}(\xi, \eta) \\ &\overset{\operatorname{Ad}_G-\text{inv}}{=} B_{\mathfrak{g}}(\xi, \eta) - B_{\mathfrak{g}}(\xi, \eta) = 0. \end{aligned}$$

For the other inclusion, suppose that $\eta \in \operatorname{im}(\operatorname{Ad}_G(g) - \operatorname{id})^{\perp}$, that is $B_{\mathfrak{g}}(\eta, (\operatorname{Ad}_G(g) - \operatorname{id})\xi) = 0$ for all $\xi \in \mathfrak{g}$. Let $\zeta \in \mathfrak{g}$ be arbitrary and let $\xi \coloneqq \operatorname{Ad}_G(g)^{-1}\zeta$. Then

$$\begin{aligned} B_{\mathfrak{g}}((\operatorname{Ad}_G(g) - \operatorname{id})\eta, \zeta) &= B_{\mathfrak{g}}(\operatorname{Ad}_G(g)\eta, \zeta) - B_{\mathfrak{g}}(\eta, \zeta) \\ &\overset{\operatorname{Ad}_G-\text{inv}}{=} B_{\mathfrak{g}}(\eta, \operatorname{Ad}_G(g)^{-1}\zeta) - B_{\mathfrak{g}}(\eta, \zeta) \\ &= B_{\mathfrak{g}}(\eta, \xi) - B_{\mathfrak{g}}(\eta, \operatorname{Ad}_G(g)\xi) \\ &= B_{\mathfrak{g}}(\eta, (\operatorname{id} - \operatorname{Ad}_G(g))\xi) \\ &= -B_{\mathfrak{g}}(\eta, (\operatorname{Ad}_G(g) - \operatorname{id})\xi) = 0. \end{aligned}$$

Since $\zeta \in \mathfrak{g}$ was arbitrary and $B_{\mathfrak{g}}$ is non-degenerate, it follows that $(\operatorname{Ad}_G(g) - \operatorname{id})\eta = 0$. ∎

Lemma 4.1.2. *Let V be a vector space and $T : V \to V$ be linear with $T^k = \operatorname{id}$. then*

$$\ker \sum_{i=0}^{k-1} T^i = \operatorname{im}(\operatorname{id} - T).$$

Proof. Let $S \coloneqq \sum_{i=0}^{k-1} T^i$. We want to show $\ker(S) = \operatorname{im}(\operatorname{id} - T)$.

$\supseteq$: It holds $S \circ T = \sum_{i=1}^{k} T^i$, thus $S \circ (\operatorname{id} - T) = \sum_{i=0}^{k-1} T^i - \sum_{i=1}^{k} T^i = T^0 - T^k = 0$.

$\subseteq$: Let $X \in \ker S$, so that $X = -\sum_{i=1}^{k-1} T^i X$ and set $Y \coloneqq \frac{1}{k} \sum_{i=1}^{k-1} (k - i) T^{i-1} X$.

Then

$$
\begin{aligned}
(1-T)(kY) &= \sum_{i=1}^{k-1}(k-i)T^{i-1}X - \sum_{i=1}^{k-1}(k-i)T^{i}X \\
&= \sum_{i=0}^{k-2}(k-(i+1))T^{i}X - \sum_{i=1}^{k-1}(k-i)T^{i}X \\
&= (k-1)X + \sum_{i=1}^{k-2}(k-i-1-k+i)T^{i}X - (k-(k-1))T^{k-1}X \\
&= (k-1)X - \sum_{i=1}^{k-2}T^{i}X - T^{k-1}X \\
&= (k-1)X - \sum_{i=1}^{k-1}T^{i}X \\
&= kX.
\end{aligned}
$$

□

4.2 1-cocycles of triangle groups

Let $\phi : \Delta(p, q, r) \to G$ be a homomorphism. We describe the Zariski tangent space at ϕ both to the representation variety and to the orbit $\mathcal{O}_\phi$ using the descriptions given in Section 3.1 (Lemma 3.1.3 and Lemma 3.1.5).

Corollary 4.2.1. *Let $\gamma_1, \gamma_2, \gamma_3$ be the generators of Δ and let $\phi : \Delta \to G$ be a homomorphism. Let $g_i \coloneqq \phi(\gamma_i)$ for $i = 1, 2, 3$. Then*

$$
\begin{aligned}
Z^1(\Delta, \mathfrak{g})_\phi \cong \big\{(u_1, u_2, u_3) \in \mathfrak{g}^3 \mid\ & \mathrm{Ad}(g_1)u_1 + u_1 = 0,\ \mathrm{Ad}(g_2)u_2 + u_2 = 0,\ \mathrm{Ad}(g_3)u_3 + u_3 = 0, \\
& \sum_{i=0}^{p-1} \mathrm{Ad}(g_1g_2)^i \left(u_1 + \mathrm{Ad}(g_1)u_2\right) = 0, \\
& \sum_{i=0}^{q-1} \mathrm{Ad}(g_2g_3)^i \left(u_2 + \mathrm{Ad}(g_2)u_3\right) = 0, \\
& \sum_{i=0}^{r-1} \mathrm{Ad}(g_3g_1)^i \left(u_3 + \mathrm{Ad}(g_3)u_1\right) = 0\big\}.
\end{aligned} \tag{4.2}
$$

We give two proofs of this result, one using Lemma 3.1.5 and one using the isomorphism Φ of Lemma 3.1.3.

Proof 1 of Corollary 4.2.1. Lemma 3.1.5 asserts that if $\Gamma = \langle \gamma_1, \dots, \gamma_k \mid R \rangle$, then

$$Z^1(\Gamma, \mathfrak{g})_\phi \cong \{(u_1, \dots, u_k) \in \mathfrak{g}^k \mid \sum_{i=1}^{k} \operatorname{Ad}\phi(\partial_i r) u_i = 0 \text{ for all } r \in R\}$$

where $\partial_i = \frac{\partial}{\partial \gamma_i}$ is a derivation defined by $\partial_i(\gamma_j) \coloneqq \delta_{ij}$ and satisfying $\partial_i(xy) = \partial_i(x) + x\partial_i(y)$ and $\operatorname{Ad}\phi(\partial_i r)$ is defined by linearity. In our case $\Delta(p,q,r)$ is generated by $\gamma_1, \gamma_2, \gamma_3$ with defining relations

$$r_a = \gamma_1^2,\ r_b = \gamma_2^2,\ r_c = \gamma_3^2,\ r_p = (\gamma_1\gamma_2)^p,\ r_q = (\gamma_2\gamma_3)^q,\ r_r = (\gamma_3\gamma_1)^r.$$

We compute the term $\sum_{i=1}^{k} \operatorname{Ad}\phi(\partial_i r) u_i$ for all relations r.

Let $r = r_a$, then $\partial_1 r_a = \partial_1 \gamma_1 \gamma_1 = \partial_1 \gamma_1 + \gamma_1 \partial_1 \gamma_1 = 1 + \gamma_1$ and $\partial_2 r_a = \partial_3 r_a = 0$. Thus

$$\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_a) u_i = \operatorname{Ad}\phi(1) u_1 + \operatorname{Ad}\phi(\gamma_1) u_1 = u_1 + \operatorname{Ad}\phi(\gamma_1) u_1.$$

Analogously,

$$\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_b) u_i = u_2 + \operatorname{Ad}\phi(\gamma_2) u_2,$$
$$\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_c) u_i = u_3 + \operatorname{Ad}\phi(\gamma_3) u_3.$$

For $r = r_p$ it holds

$$\begin{aligned}
\partial_1 r_p &= \partial_1 (\gamma_1\gamma_2)^{p-1} + (\gamma_1\gamma_2)^{p-1} \partial_1(\gamma_1\gamma_2) \\
&= \partial_1 (\gamma_1\gamma_2)^{p-2} + (\gamma_1\gamma_2)^{p-2} \partial_1(\gamma_1\gamma_2) + (\gamma_1\gamma_2)^{p-1} \partial_1(\gamma_1\gamma_2) \\
&\vdots \\
&= \partial_1(\gamma_1\gamma_2) + \gamma_1\gamma_2 \partial_1(\gamma_1\gamma_2) + \ldots + (\gamma_1\gamma_2)^{p-1} \partial_1(\gamma_1\gamma_2) \\
&= \sum_{i=0}^{p-1} (\gamma_1\gamma_2)^i \partial_1(\gamma_1\gamma_2) \\
&= \sum_{i=0}^{p-1} (\gamma_1\gamma_2)^i
\end{aligned}$$

and

$$\partial_2 r_p = \sum_{i=0}^{p-1} (\gamma_1\gamma_2)^i \partial_2(\gamma_1\gamma_2) = \sum_{i=0}^{p-1} (\gamma_1\gamma_2)^i \gamma_1,$$
$$\partial_3 r_p = \sum_{i=0}^{p-1} (\gamma_1\gamma_2)^i \partial_3(\gamma_1\gamma_2) = 0.$$

Thus $\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_p)u_i = \sum_{i=0}^{p-1} \operatorname{Ad}\phi(\gamma_1\gamma_2)^i u_1 + \sum_{i=0}^{p-1} \operatorname{Ad}\phi((\gamma_1\gamma_2)^i\gamma_1)u_2$.

Analogously,

$$\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_q)u_i = \sum_{i=0}^{q-1} \operatorname{Ad}\phi(\gamma_2\gamma_3)^i u_2 + \sum_{i=0}^{q-1} \operatorname{Ad}\phi((\gamma_2\gamma_3)^i\gamma_2)u_3,$$
$$\sum_{i=1}^{3} \operatorname{Ad}\phi(\partial_i r_r)u_i = \sum_{i=0}^{r-1} \operatorname{Ad}\phi(\gamma_3\gamma_1)^i u_3 + \sum_{i=0}^{r-1} \operatorname{Ad}\phi((\gamma_3\gamma_1)^i\gamma_3)u_1.$$

□

Proof 2 of Corollary 4.2.1. In the proof of Lemma 3.1.3 we saw that if $\Gamma = \langle \gamma_1, \ldots, \gamma_k \mid R\rangle$, the space of 1-cocyles is isomorphic to the intersection over all relations $r \in R$ of the subspaces

$$L_r = \ker d_{(e,\ldots,e)} f_r \circ (R_{g_1} \times \ldots \times R_{g_k}) \subset \mathfrak{g}^k,$$

where $f_r : G^k \to G$ are such that $\operatorname{Hom}(\Gamma, G) = \bigcap_{r\in R} f_r^{-1}(e)$, $g_i = \phi(\gamma_i)$ and R_{g_i} denotes right multiplication by g_i. In our case $\Delta(p, q, r)$ is generated by $\gamma_1, \gamma_2, \gamma_3$ with defining relations

$$r_a = \gamma_1^2,\ r_b = \gamma_2^2,\ r_c = \gamma_3^2,\ r_p = (\gamma_1\gamma_2)^p,\ r_q = (\gamma_2\gamma_3)^q,\ r_r = (\gamma_3\gamma_1)^r.$$

We compute the differential $d_{e,e,e}\left(r(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right) : \mathfrak{g}^3 \to \mathfrak{g}$ of every relation $r \in \mathcal{R}$.

- For r_a we have:

$$\begin{aligned}
d_{e,e,e}\left(r_a(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1, u_2, u_3) &= \frac{d}{dt}\Big|_{t=0} r_a(\exp(tu_1)g_1, \exp(tu_2)g_2, \exp(tu_3)g_3) \\
&= \frac{d}{dt}\Big|_{t=0} \exp(tu_1)g_1 \exp(tu_1)g_1 \\
&\overset{g_1^2=e}{=} \frac{d}{dt}\Big|_{t=0} \exp(tu_1)g_1 \exp(tu_1)g_1^{-1} \\
&= \frac{d}{dt}\Big|_{t=0} \exp(tu_1)c_{g_1}(\exp tu_1) \\
&= \frac{d}{dt}\Big|_{t=0} \exp tu_1 + \frac{d}{dt}\Big|_{t=0} c_{g_1}(\exp tu_1) \\
&= u_1 + \operatorname{Ad}(g_1)u_1.
\end{aligned}$$

- Analogously for r_b and r_c:

$$d_{e,e,e}\left(r_b(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1, u_2, u_3) = u_2 + \operatorname{Ad}(g_2)u_2,$$
$$d_{e,e,e}\left(r_c(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1, u_2, u_3) = u_3 + \operatorname{Ad}(g_3)u_3.$$

- For r_p we have to compute

$$d_{e,e,e}\left(r_p(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1,u_2,u_3) = \frac{d}{dt}\Big|_{t=0}(\exp(tu_1)g_1\exp(tu_2)g_2)^p.$$

For ease of notation let $x = \exp(tu_1)g_1\exp(tu_2)g_2$. For $i = 0,\dots,p$ it holds

$$\begin{aligned}(g_1g_2)^i x^{p-i} &= (g_1g_2)^i x x^{p-(i+1)} = (g_1g_2)^i x (g_1g_2)^{-(i+1)}(g_1g_2)^{i+1}x^{p-(i+1)}\\ &= (g_1g_2)^i \exp(tu_1)g_1\exp(tu_2)g_2(g_1g_2)^{-(i+1)}(g_1g_2)^{i+1}x^{p-(i+1)}\\ &= (g_1g_2)^i\exp(tu_1)(g_1g_2)^{-i}(g_1g_2)^i g_1\exp(tu_2)\underbrace{g_2(g_1g_2)^{-1}(g_1g_2)^{-i}}_{=((g_1g_2)^i g_1)^{-1}}(g_1g_2)^{i+1}x^{p-(i+1)}\\ &= c_{(g_1g_2)^i}(\exp(tu_1))c_{(g_1g_2)^i g_1}(\exp(tu_2))(g_1g_2)^{i+1}x^{p-(i+1)}\end{aligned}$$

Thus recursively we get

$$\begin{aligned}x^p = (g_1g_2)^0x^{p-0} &= \prod_{i=0}^{p-1}\left(c_{(g_1g_2)^i}(\exp(tu_1))c_{(g_1g_2)^ig_1}(\exp(tu_2))\right)(g_1g_2)^p x^0\\ &= \prod_{i=0}^{p-1}\left(c_{(g_1g_2)^i}(\exp(tu_1))c_{(g_1g_2)^ig_1}(\exp(tu_2))\right).\end{aligned}$$

Differentiating at $t=0$ we get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(\exp(tu_1)g_1\exp(tu_2)g_2)^p &= \sum_{i=0}^{p-1}\frac{d}{dt}\Big|_{t=0}c_{(g_1g_2)^i}(\exp(tu_1)) + \frac{d}{dt}\Big|_{t=0}c_{(g_1g_2)^ig_1}(\exp(tu_2))\\ &= \sum_{i=0}^{p-1}\mathrm{Ad}((g_1g_2)^i)u_1 + \mathrm{Ad}((g_1g_2)^ig_1)u_2\\ &= \sum_{i=0}^{p-1}\mathrm{Ad}(g_1g_2)^i\left(u_1 + \mathrm{Ad}(g_1)u_2\right).\end{aligned}$$

- Analogously, for r_q and r_r we have:

$$\begin{aligned}d_{e,e,e}\left(r_q(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1,u_2,u_3) &= \sum_{i=0}^{q-1}\mathrm{Ad}(g_2g_3)^i\left(u_2+\mathrm{Ad}(g_2)u_3\right)\\ d_{e,e,e}\left(r_r(R_{g_1}(\cdot), R_{g_2}(\cdot), R_{g_3}(\cdot))\right)(u_1,u_2,u_3) &= \sum_{i=0}^{r-1}\mathrm{Ad}(g_3g_1)^i\left(u_3+\mathrm{Ad}(g_3)u_1\right).\end{aligned}$$

□

4.3 First cohomology group of triangle groups

The restriction of a representation $\phi : \Delta(p,q,r) \to G$ to the dihedral group generated by any two of the generators of $\Delta(p,q,r)$ is infinitesimally rigid and this allows us to characterize the first cohomology group of ϕ in a simpler way. Throughout this section for ease of notation we denote $\Delta = \Delta(p,q,r)$. We refer to Appendix A for the notation on group cohomology.

Recall from Section 3.1 that

$$\begin{aligned}
&Z^1(\Delta,\mathfrak{g})_\phi = \{u : \Delta \to \mathfrak{g} \mid u(\gamma\delta) = u(\gamma) + \mathrm{Ad}(\phi(\gamma))u(\delta) \text{ for all } \gamma,\delta \in \Delta\},\\
&B^1(\Delta,\mathfrak{g})_\phi = \{v : \Delta \to \mathfrak{g} \mid \exists X \in \mathfrak{g} \text{ s.t. } v(\gamma) = \mathrm{Ad}(\gamma)X - X \text{ for all } \gamma \in \Delta\} \subseteq Z^1(\Delta,\mathfrak{g})_\phi,\\
&H^1(\Delta,\mathfrak{g})_\phi = Z^1(\Delta,\mathfrak{g})_\phi / B^1(\Delta,\mathfrak{g})_\phi.
\end{aligned}$$

We define

$$\begin{aligned}
\widetilde{Z^1_\phi} &\coloneqq \{u \in Z^1(\Delta,\mathfrak{g})_\phi \mid u(a) = u(c) = 0\}\\
\widetilde{B^1_\phi} &\coloneqq \widetilde{Z^1_\phi} \cap B^1(\Delta,\mathfrak{g})_\phi.
\end{aligned}$$

Proposition 4.3.1. *Every cohomology class in $H^1(\Delta,\mathfrak{g})_\phi$ has a representative in $\widetilde{Z^1_\phi}$. Therefore*

$$H^1(\Delta,\mathfrak{g})_\phi \cong \widetilde{Z^1_\phi}/\widetilde{B^1_\phi}.$$

Proof. Let $u \in Z^1(\Delta,\mathfrak{g})_\phi$. The two generators a, c of $\Delta(p,q,r)$ generate the dihedral subgroup $D_r = \langle a, c \mid a^2 = c^2 = (ac)^r = 1\rangle$ of order $2r$, and we denote by $i : D_r \hookrightarrow \Delta$ the inclusion. Since D_r is finite every 1-cocyle in $Z^1(D_r,\mathfrak{g})_{\phi\circ i}$ is a 1-coboundary (Lemma A.3.1) and thus the restriction of u to D_r is an element of $i^*\big(Z^1(\Delta,\mathfrak{g})_\phi\big) = Z^1(D_r,\mathfrak{g})_{\phi\circ i} = B^1(D_r,\mathfrak{g})_{\phi\circ i}$. Thus there is $v \in \mathfrak{g}$ such that for all $\gamma \in D_r$ it holds $u(\gamma) = \overline{\partial}^0_{D_r}(v)(\gamma)$. Let $\tilde{u} \coloneqq u - \overline{\partial}^0_\Delta(v) \in Z^1(\Delta,\mathfrak{g})_\phi$. Then $[u] = [\tilde{u}] \in H^1(\Delta,\mathfrak{g})_\phi$ and for all $\gamma \in D_r$ it holds $\overline{\partial}^0_\Delta(v)(\gamma) = \overline{\partial}^0_{D_r}(v)(\gamma)$ therefore $\tilde{u}(\gamma) = u(\gamma) - \overline{\partial}^0_\Delta(v)(\gamma) = u(\gamma) - \overline{\partial}^0_{D_r}(v)(\gamma) = 0$. □

Our next goal is to make these spaces more explicit.

Proposition 4.3.2. *The map $\bar{C}^1(\Delta,\mathfrak{g}) \to \mathfrak{g}^3$, $u \mapsto (u(a), u(b), u(c))$ identifies*

$$\begin{aligned}
&\widetilde{Z^1_\phi} \cong \mathrm{im}(\mathrm{id} - \mathrm{Ad}_G(\phi(b))) \cap \mathrm{im}(\mathrm{id} - \mathrm{Ad}_G(\phi(ab)) \cap \mathrm{im}(\mathrm{id} - \mathrm{Ad}_G(\phi(bc))), \text{ and}\\
&\widetilde{B^1_\phi} \cong (\mathrm{id} - \mathrm{Ad}_G(\phi(b)))\left((\ker(\mathrm{Ad}_G(\phi(a)) - \mathrm{id}) \cap \ker(\mathrm{Ad}_G(\phi(c)) - \mathrm{id})\right).
\end{aligned}$$

Proof. 1-cocycles in $Z^1(\Delta,\mathfrak{g})_\phi$ are determined by their values on the generators of Δ

as long as they respect the group relations and by Corollary 4.2.1:

$$\begin{aligned}Z^1(\Delta,\mathfrak{g})_\phi \cong \big\{(u_1,u_2,u_3)\in\mathfrak{g}^3 \mid\ & \mathrm{Ad}(\phi(a))u_1+u_1=0,\ \mathrm{Ad}(\phi(b))u_2+u_2=0,\ \mathrm{Ad}(\phi(c))u_3+u_3=0,\\ & \sum_{i=0}^{p-1}\mathrm{Ad}(\phi(ab))^i(u_1+\mathrm{Ad}(\phi(a))u_2)=0,\\ & \sum_{i=0}^{q-1}\mathrm{Ad}(\phi(bc))^i(u_2+\mathrm{Ad}(\phi(b))u_3)=0,\\ & \sum_{i=0}^{r-1}\mathrm{Ad}(\phi(ca))^i(u_3+\mathrm{Ad}(\phi(c))u_1)=0\big\}.\end{aligned}$$

By definition, then space $\widetilde{Z^1_\phi}$ consists of those cocycles in $Z^1(\Delta,\mathfrak{g})_\phi$ which vanish on a and c, which means

$$\widetilde{Z^1_\phi}\cong\{u_2\in\mathfrak{g}\mid \mathrm{Ad}_G(\phi(b))u_2+u_2=0,\ \sum_{i=0}^{p-1}\mathrm{Ad}_G(\phi(ab)^i\phi(a))u_2=0,\ \sum_{i=0}^{q-1}\mathrm{Ad}_G(\phi(bc))^iu_2=0\}.$$

Notice that if $\mathrm{Ad}_G(\phi(b))u_2+u_2=0$, then the condition $\sum_{i=0}^{p-1}\mathrm{Ad}_G(\phi(ab)^i\phi(a))u_2=0$ is equivalent to $\sum_{i=0}^{p-1}\mathrm{Ad}_G(\phi(ab))^iu_2=0$. Therefore we conclude that

$$\begin{aligned}\widetilde{Z^1_\phi}&\cong\{u_2\in\mathfrak{g}:\mathrm{Ad}_G(\phi(b))u_2+u_2=0,\ \sum_{i=0}^{p-1}\mathrm{Ad}_G(\phi(ab))^iu_2=0,\ \sum_{i=0}^{q-1}\mathrm{Ad}_G(\phi(bc))^iu_2=0\}\\ &=\ker\sum_{i=0}^{1}\mathrm{Ad}_G(\phi(b))^i\cap\ker\sum_{i=0}^{p-1}\mathrm{Ad}_G(\phi(ab))^i\cap\ker\sum_{i=0}^{q-1}\mathrm{Ad}_G(\phi(bc))^i\\ &\overset{\text{Lemma 4.1.2}}{=}\ \mathrm{im}(\mathrm{id}-\mathrm{Ad}_G(\phi(b)))\cap\mathrm{im}(\mathrm{id}-\mathrm{Ad}_G(\phi(ab))\cap\mathrm{im}(\mathrm{id}-\mathrm{Ad}_G(\phi(bc))).\end{aligned}$$

We now describe $\widetilde{B^1_\phi}$. Recall (Section 3.1) that in $\mathfrak{g}^3$ the space $B^1(\Delta,\mathfrak{g})_\phi$ is given by

$$B^1(\Delta,\mathfrak{g})_\phi=\{(\mathrm{Ad}_G(\phi(a))X)-X,\mathrm{Ad}_G(\phi(b))X-X,\mathrm{Ad}_G(\phi(c))X-X)\mid X\in\mathfrak{g}\}.$$

Thus

$$\begin{aligned}\widetilde{B^1_\phi}&=B^1(\Delta,\mathfrak{g})_\phi\cap\widetilde{Z^1_\phi}\\ &\cong\{\mathrm{Ad}_G(\phi(b))X-X\mid X\in\mathfrak{g}\text{ with }\ \mathrm{Ad}_G(\phi(a))X-X=\mathrm{Ad}_G(\phi(c))X-X=0\}\\ &=(\mathrm{Ad}_G(\phi(b))-\mathrm{id})\big((\ker(\mathrm{Ad}_G(\phi(a))-\mathrm{id})\cap\ker(\mathrm{Ad}_G(\phi(c))-\mathrm{id})\big).\end{aligned}$$

□

4.4 The dimension of the Zariski tangent space at ϕ

In this section we find a formula for the dimension of the Zariski tangent space and deduce from it that it is constant on Hitchin representations. In fact we prove more generally that it is constant on connected components consisting of representations whose restriction to the 2-index subgroup $T(p,q,r)$ has finite centralizer.

The strategy is to first compute the dimension of the first cohomology group $H^1(\Delta,\mathfrak{g})_\phi$ using the simplification of the previous section and then deduce from it the dimension of the tangent space. The computations are elementary, but require some algebraic manipulations. Let $\phi:\Delta(p,q,r)\to G$ be a homomorphism. By Proposition 4.3.1 it holds $\dim H^1(\Delta,\mathfrak{g})_\phi=\dim \widetilde{Z^1_\phi}-\dim\widetilde{B^1_\phi}$. We set

$$X:=\operatorname{im}(\operatorname{Ad}_G(\phi(b))-\operatorname{id}),\ Y:=\operatorname{im}(\operatorname{Ad}_G(\phi(ab))-\operatorname{id}),\ Z:=\operatorname{im}(\operatorname{Ad}_G(\phi(bc))-\operatorname{id}).$$

By Proposition 4.3.2 and using the notation of (4.1)

$$\begin{aligned}\widetilde{Z^1_\phi}&\cong X\cap Y\cap Z\\ \widetilde{B^1_\phi}&\cong(\operatorname{id}-\operatorname{Ad}_G(\phi(b)))\big(\mathfrak{g}^{\operatorname{Ad}(\phi(a))}\cap\mathfrak{g}^{\operatorname{Ad}(\phi(c))}\big).\end{aligned}$$

We want to compute the dimension of $\widetilde{Z^1_\phi}$ and $\widetilde{B^1_\phi}$ and we notice that by Lemma 4.1.1 it holds

$$\begin{aligned}X^\perp&=\ker(\operatorname{Ad}_G(\phi(b))-\operatorname{id})=\mathfrak{g}^{\operatorname{Ad}(\phi(b))},\\ Y^\perp&=\ker(\operatorname{Ad}_G(\phi(ab))-\operatorname{id})=\mathfrak{g}^{\operatorname{Ad}(\phi(ab))},\\ Z^\perp&=\ker(\operatorname{Ad}_G(\phi(bc))-\operatorname{id})=\mathfrak{g}^{\operatorname{Ad}(\phi(bc))}.\end{aligned}$$

We will use the following two lemmas.

Lemma 4.4.1. *Let $x,y\in G$ with $x^2=y^2=\operatorname{id}$. Then $\mathfrak{g}^{\langle\operatorname{Ad}_G(x),\operatorname{Ad}_G(y)\rangle}=\mathfrak{g}^{\operatorname{Ad}_G(x)}\cap\mathfrak{g}^{\operatorname{Ad}_G(y)}$ and*

$$\dim\mathfrak{g}^{\langle\operatorname{Ad}_G(x),\operatorname{Ad}_G(y)\rangle}=\tfrac{1}{2}\big(\dim\mathfrak{g}^{\operatorname{Ad}_G(x)}+\dim\mathfrak{g}^{\operatorname{Ad}_G(y)}+\dim\mathfrak{g}^{\operatorname{Ad}_G(xy)}-\dim\mathfrak{g}\big).$$

Proof. We first claim that for $X,Y\in\operatorname{GL}(\mathfrak{g})$ with $X^2=Y^2=\operatorname{id}$ it holds:

$$(\mathfrak{g}^X\cap\mathfrak{g}^Y)+(\mathfrak{g}^{-X}\cap\mathfrak{g}^{-Y})=\mathfrak{g}^{XY}. \tag{4.3}$$

The $\subseteq$-inclusion is obvious: if v is both X- and Y-invariant, then $XYv=Xv=v$ and if it is $(-X)$- and $(-Y)$-invariant, then $XYv=(-X)(-Y)v=-Xv=v$. For the other inclusion, suppose that v is XY-invariant, that is $XYv=v$. This implies $YXv=YXXYv=YYv=v$, as well as $Yv=X^2Yv=Xv$. We decompose

$$v=\frac{1}{2}(v+Xv)+\frac{1}{2}(v-Xv).$$

The first summand is both X- and Y-invariant. Indeed, $X(v+Xv) = Xv + X^2v = Xv + v$ and $Y(v+Xv) = Yv + YXv = Xv + v$. The second summand is both $(-X)$- and $(-Y)$-invariant. Indeed, $(-X)(v - Xv) = -Xv + X^2v = -Xv + v$ and $(-Y)(v - Xv) = -Yv + YXv = -Xv + v$. This shows (4.3).

Now, for a subspace $V \subset \mathfrak{g}$ we denote by $V^\perp$ its orthogonal complement with respect to the killing form $B_\mathfrak{g}$. By Lemma 4.1.1 and Lemma 4.1.2 it holds

$$(\mathfrak{g}^{\mathrm{Ad}(x)})^\perp = (\ker(\mathrm{id} - \mathrm{Ad}_G(x)))^\perp = \mathrm{im}(\mathrm{id} - \mathrm{Ad}_G(x)) = \ker(\mathrm{id} + \mathrm{Ad}_G(x)) = \mathfrak{g}^{-\mathrm{Ad}(x)}.$$

Analogously $(\mathfrak{g}^{\mathrm{Ad}(y)})^\perp = \mathfrak{g}^{-\mathrm{Ad}(y)}$. Therefore by (4.3) it holds

$$(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}) + ((\mathfrak{g}^{\mathrm{Ad}(x)})^\perp \cap (\mathfrak{g}^{\mathrm{Ad}(y)})^\perp) = \mathfrak{g}^{\mathrm{Ad}(xy)}.$$

Notice that the intersection of the two summands is trivial and thus the sum is direct and

$$\dim((\mathfrak{g}^{\mathrm{Ad}(x)})^\perp \cap (\mathfrak{g}^{\mathrm{Ad}(y)})^\perp) = \dim \mathfrak{g}^{\mathrm{Ad}(xy)} - \dim(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}).$$

On the other hand, using that for two subspaces $U, V \subset \mathfrak{g}$ it holds $(U \cap V)^\perp = U^\perp + V^\perp$, $\dim U^\perp = \dim \mathfrak{g} - \dim U$ and $\dim(U \cap V) = \dim U + \dim V - \dim(U+V)$, we get:

$$\begin{aligned}
\dim((\mathfrak{g}^{\mathrm{Ad}(x)})^\perp \cap (\mathfrak{g}^{\mathrm{Ad}(y)})^\perp) &= \dim(\mathfrak{g}^{\mathrm{Ad}(x)})^\perp + \dim(\mathfrak{g}^{\mathrm{Ad}(y)})^\perp - \dim((\mathfrak{g}^{\mathrm{Ad}(x)})^\perp + (\mathfrak{g}^{\mathrm{Ad}(y)})^\perp) \\
&= \dim(\mathfrak{g}^{\mathrm{Ad}(x)})^\perp + \dim(\mathfrak{g}^{\mathrm{Ad}(y)})^\perp - \dim(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)})^\perp \\
&= \dim(\mathfrak{g}^{\mathrm{Ad}(x)})^\perp + \dim(\mathfrak{g}^{\mathrm{Ad}(y)})^\perp - \big(\dim \mathfrak{g} - \dim(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}\big) \\
&= \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(x)} - \dim \mathfrak{g}^{\mathrm{Ad}(y)} + \dim(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}\big).
\end{aligned}$$

Therefore we conclude:

$$2 \dim(\mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}) = \dim \mathfrak{g}^{\mathrm{Ad}(xy)} - \dim \mathfrak{g} + \dim \mathfrak{g}^{\mathrm{Ad}(x)} + \dim \mathfrak{g}^{\mathrm{Ad}(y)}.$$

The lemma follows from $\mathfrak{g}^{\langle \mathrm{Ad}(x), \mathrm{Ad}(y) \rangle} = \mathfrak{g}^{\mathrm{Ad}(x)} \cap \mathfrak{g}^{\mathrm{Ad}(y)}$. ◻

Let $T := T(p,q,r)$ the 2-index subgroup of $\Delta(p,q,r)$ generated by the rotations ab, bc and ca.

Lemma 4.4.2. *Let $\phi : \Delta(p,q,r) \to G$ be a representation such that the restriction $\phi|_T$ has finite centralizer. Then*

$$\mathfrak{g}^{\mathrm{Ad}(\phi(ab))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} = (0).$$

Proof. Since $ac = (ab)(bc)$ the two elements ab and bc suffice to generate the subgroup T and using Lemma A.2.1

$$\mathfrak{g}^{\mathrm{Ad}(\phi(ab))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} = \mathfrak{g}^{\langle \mathrm{Ad}(\phi(ab)), \mathrm{Ad}\,\phi(bc) \rangle} = \bigcap_{\gamma \in T} \ker(\mathrm{id} - \mathrm{Ad}_G \phi(\gamma)) = \mathrm{Lie}\, Z_G(\phi\big|_T) = (0).$$

◻

Lemma 4.4.3. *Let $\phi : \Delta(p,q,r) \to G$ be a homomorphism such that $\phi|_T$ has finite centralizer. Let x, y, z be the generators of $\Delta(p,q,r)$. Then*

$$\mathfrak{g}^{\operatorname{Ad}\phi(x)} \cap \left(\mathfrak{g}^{\operatorname{Ad}\phi(xy)} + \mathfrak{g}^{\operatorname{Ad}\phi(xz)}\right) = \mathfrak{g}^{\langle \operatorname{Ad}\phi(x), \operatorname{Ad}\phi(y)\rangle} \oplus \mathfrak{g}^{\langle \operatorname{Ad}\phi(x), \operatorname{Ad}\phi(z)\rangle}$$

Proof. We fix some notation. Let $X \coloneqq \operatorname{Ad}\phi(x)$, $Y \coloneqq \operatorname{Ad}\phi(y)$ and $Z \coloneqq \operatorname{Ad}\phi(z)$. Let

$f_x \coloneqq \mathrm{id} - \operatorname{Ad}(\phi(x)) = \mathrm{id} - X$, $f_y \coloneqq \mathrm{id} - \operatorname{Ad}(\phi(y)) = \mathrm{id} - Y$, $f_z \coloneqq \mathrm{id} - \operatorname{Ad}(\phi(z)) = \mathrm{id} - Z$,

so that $\mathfrak{g}^{\operatorname{Ad}\phi(x)} = \mathfrak{g}^X = \ker f_x$ and analogously for y and z. In this notation we have to show

$$\mathfrak{g}^X \cap \left(\mathfrak{g}^{XY} + \mathfrak{g}^{XZ}\right) = \mathfrak{g}^{\langle X,Y\rangle} + \mathfrak{g}^{\langle X,Z\rangle}.$$

Notice that the sum on the right-hand-side is direct because by Lemma A.2.1 it holds

$$\mathfrak{g}^{\langle X,Y\rangle} \cap \mathfrak{g}^{\langle X,Z\rangle} = \mathfrak{g}^{\langle X,Y,Z\rangle} = \operatorname{Lie} Z_G(\phi) = (0).$$

Moreover, $\mathfrak{g}^{\langle X,Y\rangle} = \mathfrak{g}^{\langle X,XY\rangle} = \mathfrak{g}^X \cap \mathfrak{g}^{XY}$. Analogously, $\mathfrak{g}^{\langle X,Z\rangle} = \mathfrak{g}^X \cap \mathfrak{g}^{XZ}$. Thus using the above notation what we have to show is

$$\ker f_x \cap (\ker f_{xy} + \ker f_{xz}) = \big(\ker f_x \cap \ker f_{xy}\big) + \big(\ker f_x \cap \ker f_{xz}\big).$$

The $\supseteq$ inclusion is straightforward since $\ker f_x$ is a vector space.

To show the $\subseteq$ inclusion, we first claim that

$$f_x(\ker f_{xy}) \cap f_x(\ker f_{xz}) = (0).$$

Indeed, let $v \in f_x(\ker f_{xy}) \cap f_x(\ker f_{xz})$, so that there are $u \in \ker f_{xy}, w \in \ker f_{xz}$ with $f_x(u) = v = f_x(w)$. Since $XYu = u$ and $X^2 = \mathrm{id}$ we have

$$Xu = XXYu = Yu$$

and since $XZw = w$ and $X^2 = \mathrm{id}$

$$Xw = XXZw = Zw.$$

Thus

$$\begin{aligned} v &= f_x(u) = u - Xu = u - Yu = f_y(u) \\ v &= f_x(w) = w - Xw = w - Zw = f_z(w), \end{aligned}$$

which shows that (using Lemma 4.1.2)

$$\begin{aligned} v \in \operatorname{im}(f_y) \cap \operatorname{im}(f_z) \cap \operatorname{im}(f_x) &= \ker(\mathrm{id} + Y) \cap \ker(\mathrm{id} + Z) \cap \ker(\mathrm{id} + X) \\ &= \mathfrak{g}^{-X} \cap \mathfrak{g}^{-Y} \cap \mathfrak{g}^{-Z} \\ &\subseteq \mathfrak{g}^{\langle XY, YZ, ZX\rangle} = \operatorname{Lie} Z_G(\phi\big|_T) = (0). \end{aligned}$$

Thus $v = 0$ and proves the claim.

Finally let $v = u + w \in \ker f_x$ with $u \in \ker f_{xy}$, $w \in \ker f_{xz}$. Since $f_x(u + w) = 0$ it follows by the claim that

$$f_x(u) = f_x(-w) \in f_x(\ker f_{xy}) \cap f_x(\ker f_{xz}) = (0).$$

Thus

$$u \in \ker f_x \cap \ker f_{xy}$$

and

$$w \in \ker f_x \cap \ker f_{xz}.$$

□

Lemma 4.4.4. *Let $\phi : \Delta(p,q,r) \to G$ be a representation such that $\phi|_T$ has finite centralizer. Then*

(i) $\dim \widetilde{Z^1_\phi} = \frac{1}{2}(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(a))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(c))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))})$.

(ii) $\dim \widetilde{B^1_\phi} = \frac{1}{2}(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(a))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(c))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ac))} - \dim \mathfrak{g})$.

Proof. (i) Let X, Y and Z be as in the beginning of the section so that $\widetilde{Z^1_\phi} = X \cap Y \cap Z$. Then

$$\begin{aligned}
\dim \widetilde{Z^1_\phi} &= \dim X + \dim(Y \cap Z) - \dim(X + (Y \cap Z)) \\
&= \dim X + \dim Y + \dim Z - \dim(Y + Z) - \dim(X + (Y \cap Z)) \\
&= \dim \mathfrak{g} - \dim X^\perp + \dim \mathfrak{g} - \dim Y^\perp + \dim \mathfrak{g} - \dim Z^\perp \\
&\quad - (\dim \mathfrak{g} - \dim(Y + Z)^\perp) - (\dim \mathfrak{g} - \dim(X + (Y \cap Z))^\perp) \\
&= \dim \mathfrak{g} - \dim X^\perp - \dim Y^\perp - \dim Z^\perp + \dim(Y^\perp \cap Z^\perp) \\
&\quad + \dim(X^\perp \cap (Y^\perp + Z^\perp)) \\
&= \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(b))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} \\
&\quad + \dim(\mathfrak{g}^{\mathrm{Ad}(\phi(ab))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(bc))}) + \dim\left(\mathfrak{g}^{\mathrm{Ad}(\phi(b))} \cap (\mathfrak{g}^{\mathrm{Ad}(\phi(ab))} + \mathfrak{g}^{\mathrm{Ad}(\phi(bc))})\right) \\
&= \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(b))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} \\
&\quad + \dim \mathfrak{g}^{\langle \mathrm{Ad}(\phi(a)), \mathrm{Ad}(\phi(b)) \rangle} + \dim(\mathfrak{g}^{\langle \mathrm{Ad}(\phi(b)), \mathrm{Ad}(\phi(c)) \rangle}.
\end{aligned}$$

In the last equality we used Lemma 4.4.2 and Lemma 4.4.3. Moreover by Lemma 4.4.1

$$\begin{aligned}
\dim \mathfrak{g}^{\langle \mathrm{Ad}(\phi(a)), \mathrm{Ad}(\phi(b)) \rangle} &= \frac{1}{2}\big(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(a))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(b))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}\big), \\
\dim \mathfrak{g}^{\langle \mathrm{Ad}(\phi(b)), \mathrm{Ad}(\phi(c)) \rangle} &= \frac{1}{2}\big(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(b))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(c))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} - \dim \mathfrak{g}\big).
\end{aligned}$$

Therefore

$$\dim \widetilde{Z^1_\phi} = \frac{1}{2}(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(a))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(c))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))}). \quad (4.4)$$

(ii) It holds $\widetilde{B^1_\phi} \cong (\mathrm{id} - \mathrm{Ad}_G(\phi(b)))\big(\mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))}\big)$. Let

$$F \coloneqq (\mathrm{id} - \mathrm{Ad}_G(\phi(b)))|_{\mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))}} : \mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))} \to \mathfrak{g}.$$

Then

$$\ker(F) = \ker(\mathrm{id} - \mathrm{Ad}_G(\phi(b))) \cap \mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))} = \mathfrak{g}^{\langle \phi(a), \phi(b), \phi(c) \rangle} = \mathrm{Lie}\, Z_G(\phi) = (0).$$

Together with Lemma 4.4.1 we conclude

$$\begin{aligned}\dim \widetilde{B^1_\phi} &= \dim(\mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))}) - \dim \ker(F) = \dim(\mathfrak{g}^{\mathrm{Ad}(\phi(a))} \cap \mathfrak{g}^{\mathrm{Ad}(\phi(c))}) \\ &= \frac{1}{2}(\dim \mathfrak{g}^{\mathrm{Ad}(\phi(a))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(c))} + \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ac))} - \dim \mathfrak{g}).\end{aligned}$$

□

We can finally compute the dimension of the first cohomology group.

Proposition 4.4.5. *Let $\phi : \Delta(p, q, r) \to G$ be a representation such that $\phi\big|_T$ has finite centralizer. Then*

$$\dim H^1(\Delta, \mathfrak{g})_\phi = \frac{1}{2}\big(\dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ac))}\big).$$

Proposition 4.4.6. *Let $G = \mathrm{PGSp}(2n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ and let $\phi : \Delta(p, q, r) \to G$ be a Hitchin representation. The dimension of $T^{Zar}_\phi \operatorname{Hom}(\Delta(p, q, r), G)$ is*

$$\dim T^{Zar}_\phi \operatorname{Hom}(\Delta(p, q, r), G) = \frac{1}{2}\big(3 \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ac))}\big)$$

and it is constant on Hitchin representations.

Proof. The Zariski tangent space is isomorphic to $Z^1(\Delta, \mathfrak{g})_\phi$ and

$$\begin{aligned}\dim Z^1(\Delta, \mathfrak{g})_\phi &= \dim H^1(\Delta, \mathfrak{g})_\phi + \dim B^1(\Delta, \mathfrak{g})_\phi \\ &= \dim H^1(\Delta, \mathfrak{g})_\phi + \dim \mathfrak{g} - \dim H^0(\Delta, \mathfrak{g})_\phi.\end{aligned}$$

We used that $H^0(\Delta, \mathfrak{g})_\phi$ and $B^1(\Delta, \mathfrak{g})_\phi$ are respectively the kernel and the image of the linear boundary map $\overline{\partial}^0 : \mathfrak{g} \to \overline{C}^1(\Delta, \mathfrak{g})_\phi$ (see Appendix A). By Corollary 3.2.8 it holds $\dim H^0(\Delta, \mathfrak{g})_\phi = 0$ for every Hitchin representation, so the formula for the dimension follows from Proposition 4.4.5.

It is constant on connected components because there are finitely many conjugacy classes of finite order elements in $\mathrm{PSp}(2n, \mathbb{R})$, which implies that if $\gamma \in \Delta$ is of finite order and ϕ_1, ϕ_2 are any two representations in the same connected component of $\operatorname{Hom}(\Delta, \mathrm{PSp}(2n, \mathbb{R}))$, then two group elements $\phi_1(\gamma)$ and $\phi_2(\gamma)$ are conjugate in $\mathrm{PSp}(2n, \mathbb{R})$ (A precise proof is given in Proposition 4.4.9 below). □

Remark 4.4.7. Proposition 4.4.6 holds also if we replace $\mathrm{PGSp}(2n, \mathbb{R})$ by $\mathrm{PGL}(n, \mathbb{R})$, since the 0th cohomology group remains trivial (Remark 3.2.9) and an argument analogous of the one of Proposition 4.4.8 below shows that there are finitely many conjugacy classes of finite order elements in $\mathrm{GL}(n, \mathbb{R})$.

We end the section by showing that there are finitely many conjugacy classes of finite order elements in the symplectic group.

Proposition 4.4.8. *There are finitely many conjugacy classes of finite order elements in* $\mathrm{Sp}(2n, \mathbb{R})$.

Proof. Let $g \in \mathrm{Sp}(2n, \mathbb{R})$ be of finite order, that is $g^p = \mathrm{id}$ for some $p < \infty$. Then the eigenvalues of g are contained in the finite set $\Lambda_p = \{\omega^k \mid k = 1, \dots, p\}$ with $\omega = e^{2\pi i/p}$.

The group generated by g is a finite subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, in particular it is contained in a maximal compact subgroup. All maximal compact subgroups are conjugated, and the map $\iota : A + iB \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$ embeds $U(n)$ as a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. Thus there is $u \in U(n)$ such that g is conjugated to $\iota(u)$. Unitary matrices are unitarily diagonalizable, so we might assume that $u = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ for some $\lambda_i \in \mathbb{C}$. One can show that the eigenvalues of $\iota(u)$, and hence of g are $\lambda_i, \overline{\lambda_i}$, $i = 1, \dots n$, and since belong to the set Λ_p it holds

$$\{\lambda_i \mid i = 1, \dots n\} \subseteq \{\omega, \overline{\omega} \mid \omega \in \Lambda_p\} =: \overline{\Lambda}_p.$$

So there is a finite set $\overline{\Lambda}_p \subset \mathbb{C}$ such that for all $g \in \mathrm{Sp}(2n, \mathbb{R})$ of order p there are $\lambda_1, \dots, \lambda_n \in \overline{\Lambda}_p$ such that g is in the conjugacy class of $\iota(\mathrm{diag}(\lambda_1, \dots, \lambda_n))$. □

Proposition 4.4.9. *Let* $G = \mathrm{PGSp}(2n, \mathbb{R})$ *with Lie algebra* $\mathfrak{g}$. *For all* $\gamma \in \Delta$ *of finite order* $\dim \mathfrak{g}^{\mathrm{Ad}(\phi(\gamma))}$ *is constant on connected components of* $\mathrm{Hom}(\Delta, G)$ *and on* G*-orbits of connected components of* $\mathrm{Hom}(\Delta, G)/G$.

Proof. We have to show that for all $\gamma \in \Delta(p, q, r)$ of *finite order* and all ϕ_1, ϕ_2 in the same connected component of $\mathrm{Hom}(\Delta, G)$ it holds $\dim \mathfrak{g}^{\mathrm{Ad}\,\phi_1(\gamma)} = \dim \mathfrak{g}^{\mathrm{Ad}\,\phi_2(\gamma)}$.
Since for $g, h \in G$ it holds $\mathfrak{g}^{\mathrm{Ad}(ghg^{-1})} = \mathrm{Ad}_G(g^{-1})\mathfrak{g}^{\mathrm{Ad}(h)}$, it suffices to show that $\phi_1(\gamma)$ and $\phi_2(\gamma)$ are conjugate in G. To this end, let $\gamma \in \Delta$ of order k and consider the continuous map

$$\begin{aligned} \Phi_\gamma : \mathrm{Hom}(\Delta, G) &\longrightarrow G \longrightarrow G/\sim \\ \varphi &\longmapsto \varphi(\gamma) \longmapsto [\varphi(\gamma)], \end{aligned}$$

where $g_1 \sim g_2$ if and only if they are conjugate. It induces the continuous map

$$\begin{aligned} \overline{\Phi}_\gamma : \mathrm{Hom}(\Delta, G)/G &\to G/\sim \\ [\varphi] &\mapsto [\varphi(\gamma)] \end{aligned}$$

Since there are only finitely many conjugacy classes of elements of finite order k in G it holds

$$\Phi_\gamma(\mathrm{Hom}(\Delta, G)) \subseteq \{[g_1], \ldots, [g_{r_k}]\}.$$

and

$$\overline{\Phi}_\gamma(\mathrm{Hom}(\Delta, G)/G) \subseteq \{[g_1], \ldots, [g_{r_k}]\}.$$

If $C \subseteq \mathrm{Hom}(\Delta, G)$ is a connected component, then $\Phi_\gamma(C)$ is connected and a finite (disjoint) uonion of $[g_i]$'s each of which is closed, hence $\Phi_\gamma(C) = [g_{i_0}]$, which is shows that the images of γ under two representations in C are conjugate.

The argument for the orbit in $\mathrm{Hom}(\Delta, G)$ of a connected component $\overline{C}$ of $\mathrm{Hom}(\Delta, G)/G$ is analogous. Namely, let $C \subseteq \mathrm{Hom}(\Delta, G)$ be the orbit of $\overline{C}$, that is the preimage of C under the map $\mathrm{Hom}(\Delta, G) \to \mathrm{Hom}(\Delta, G)/G$. Then

$$\Phi_\gamma(C) = \overline{\Phi}_\gamma(\overline{C}) \subseteq \{[g_1], \ldots, [g_{r_k}]\},$$

and we conclude as above. □

4.5 Smoothness of the Hitchin component

In this section we show that the Hitchin component $\mathrm{Hit}(\Delta(p, q, r), \mathrm{PGSp}(2n, \mathbb{R}))$ is an analytic manifold and obtain a formula for its dimension from the results of the previous sections.

Proposition 4.5.1. *Hitchin representations are smooth points of the representation variety* $\mathrm{Hom}(\Delta(p, q, r), \mathrm{PGSp}(2n, \mathbb{R}))$.

Proof. Let $\phi : \Delta(p, q, r) \to \mathrm{PGSp}(2n, \mathbb{R})$ be a Hitchin representation. Let $F : G^N \to G^m$ be such that $\mathrm{Hom}(\Delta, G)$ is the zero locus of F. Then $\dim T_\phi^{Zar} \mathrm{Hom}(\Delta, G) = \dim \ker d_\phi F = N - \mathrm{rank}\, d_\phi F$. By Proposition 4.4.6 the dimension of the Zariski tangent space $T_\phi^{Zar} \mathrm{Hom}(\Delta, \mathrm{PGSp}(2n, \mathbb{R}))$ is the same for all Hitchin representations. Thus $\mathrm{rank}\, d_\phi F$ is constant as well and the conclusion follows from Lemma 3.2.11. □

Next we argue that the conjugation action of $\mathrm{PGSp}(2n, \mathbb{R})$ on Hitchin representations of triangle groups is free and proper.

Since a triangle group is the orbifold fundamental group of a closed 2-orbifold (compare with the introduction) by Lemma 2.9 of [2] every Hitchin representation is $\mathrm{PGSp}(2n, \mathbb{C})$-irreducible and has trivial centralizer in $\mathrm{PGSp}(2n, \mathbb{R})$ and in $\mathrm{PGSp}(2n, \mathbb{C})$. In particular, it follows by Theorem 1.1 of [21] that Hitchin representations are stable[2] and by Proposition 1.1 of the same [21] that the $\mathrm{PGSp}(2n, \mathbb{R})$-action on Hitchin representation is proper.

Moreover, since the action of a Lie group G on set of representations $\Delta \to G$ with trivial centralizer is free, it follows also that the $\mathrm{PGSp}(2n, \mathbb{R})$-action on Hitchin representations is free. It follows that the Hitchin component is an analytic manifold.

[2]A representation $\rho : \Delta \to \mathrm{PGSp}(2n, \mathbb{C})$ is stable if its conjugation-orbit is closed in $\mathrm{Hom}(\Delta, \mathrm{PGSp}(2n, \mathbb{C})$ and if it has finite centralizer.

Corollary 4.5.2. *Let $\Delta = \Delta(p,q,r)$ be an hyperbolic triangle group. The space of Hitchin representations $\mathcal{H}it(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))$ is an analytic manifold on which the $\mathrm{PGSp}(2n,\mathbb{R})$-action by conjugation is free and proper. The Hitchin component*

$$\mathrm{Hit}(\Delta, \mathrm{PGSp}(2n,\mathbb{R})) = \mathcal{H}it(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))/\mathrm{PGSp}(2n,\mathbb{R})$$

is an analytic manifold as well.

Proposition 4.5.1 implies that for a Hitchin representation ϕ the first cohomology group computes the tangent space to the character variety.

Lemma 4.5.3. *Let ϕ be a Hitchin representation. Then $T_{[\phi]}\chi(\Delta, \mathrm{PGSp}(2n,\mathbb{R})) \cong H^1(\Delta, \mathfrak{g})_\phi$.*

Proof. Let $\phi \in \mathrm{Hom}(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))$ be a Hitchin representation and denote by $\mathcal{O}_\phi$ its $\mathrm{PGSp}(2n,\mathbb{R})$-orbit. The projection $\pi : \mathcal{H}it \to \mathrm{Hit}$ is a submersion and it holds $\ker d_\phi \pi = T_\phi \mathcal{O}_\phi \cong B^1(\Delta, \mathfrak{g})_\phi$ by Lemma 3.1.6. Moreover, by Lemma 3.1.3 the tangent space to the representation variety at ϕ is given by the space of 1-cocycles. We conclude that

$$\begin{aligned} T_{[\phi]}\chi(\Delta, \mathrm{PGSp}(2n,\mathbb{R})) &\cong T_\phi \mathrm{Hom}(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))/\ker d_\phi \pi \\ &\cong Z^1(\Delta, \mathfrak{g})_\phi / B^1(\Delta, \mathfrak{g})_\phi = H^1(\Delta, \mathfrak{g})_\phi. \end{aligned}$$

□

Using Proposition 4.4.5 we finally deduce Theorem 1.2.3.

Theorem 4.5.4. *Let $\Delta = \Delta(p,q,r)$ be a hyperbolic triangle group with generators a, b, c. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{PGSp}(2n,\mathbb{R})$ and let $\mathrm{Ad} : \mathrm{PGSp}(2n,\mathbb{R}) \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. The dimension of the Hitchin component of $\chi(\Delta, \mathrm{PGSp}(2n,\mathbb{R}))$ is*

$$\dim \mathrm{Hit}(\Delta, \mathrm{PGSp}(2n,\mathbb{R})) = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(bc))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(ac))}), \tag{4.5}$$

where $\phi_0 : \Delta \to \mathrm{PGSp}(2n,\mathbb{R})$ is the base representation.

4.6 The dimension of the $\mathrm{PGSp}(4,\mathbb{R})$-Hitchin component

In this section we explicitly compute the dimension of the $\mathrm{PGSp}(4,\mathbb{R})$-Hitchin component for the hyperbolic triangle group $\Delta(p,q,r)$.

Proposition 4.6.1. *Let $x, y \in \{a,b,c\}$ be two of the generators of $\Delta(p,q,r)$, so that $x^2 = y^2 = (xy)^k = 1$ for some $k \in \{p,q,r\}$. Let $\phi_0 : \Delta(p,q,r) \to \mathrm{PGSp}(4,\mathbb{R})$ be the base representation. Then*

$$\dim \mathfrak{g}^{\mathrm{Ad}(\phi_0(xy))} = \begin{cases} 4 & \text{if } k = 2,3 \\ 2 & \text{if } k \geq 4. \end{cases}$$

Proof. The proof is essentially as in [40, Section 2.4]. It is more convenient to work in $\mathrm{GSp}(4,\mathbb{R})$ instead of in the quotient, so that we don't have to deal with representatives. This is possible because the differential of the quotient map $p : \mathrm{GSp}(4,\mathbb{R}) \to \mathrm{PGSp}(4,\mathbb{R})$ is a Lie algebra isomorphism, which for any $x \in \mathrm{PGSp}(4,\mathbb{R})$ restricts to an isomorphism

$$d_e p : \mathfrak{g}^{\mathrm{Ad}(\widetilde{x})} = \ker(\mathrm{Ad}_G(\tilde{x}) - \mathrm{id}) \overset{\cong}{\to} \ker(\mathrm{Ad}_{PG}(x) - \mathrm{id}) = \mathrm{Lie}(PG)^{\mathrm{Ad}(x)},$$

where $\tilde{x}$ is any lift of x. Let $A, B, C \in \mathrm{GSp}(4,\mathbb{R})$ be the lifts of $\phi_0(a), \phi_0(b), \phi_0(c)$ given by the image of (2.1) under the irreducible representation π_2. We have to show that for $\widetilde{x}, \widetilde{y} \in \{A, B, C\}$

$$\dim \big(\mathrm{Lie}\,\mathrm{GSp}(4,\mathbb{R})\big)^{\mathrm{Ad}(\widetilde{x}\widetilde{y})} = \begin{cases} 4 & \text{if } k = 2, 3 \\ 2 & \text{if } k \geq 4. \end{cases}$$

Notice that $\widetilde{x}^2 = \widetilde{y}^2 = \mathrm{id} \in \mathrm{GSp}(4,\mathbb{R})$ and $(\widetilde{x}\widetilde{y})^k = \pm\mathrm{id}$ (depending on k). As before, let Ω_4 be the standard symplectic form of (2.5) and set $G \coloneqq \mathrm{GSp}(4,\mathbb{R})$ and $G_{\mathbb{C}} \coloneqq \mathrm{GSp}(4,\mathbb{C})$. We denote by $c_h : G_{\mathbb{C}} \to G_{\mathbb{C}}, x \mapsto hxh^{-1}$ conjugation by $h \in G_{\mathbb{C}}$, and we let

$$Z_G(g) \coloneqq \{h \in G : hgh^{-1} = g\}$$
$$Z_{G_{\mathbb{C}}}(g) \coloneqq \{h \in G_{\mathbb{C}} : hgh^{-1} = g\}$$

be the centralizer of g in the respective groups. The relation between the centralizer of one element and its conjugate is given by the following proposition.

Proposition 4.6.2. *Let $G \leq \mathrm{GL}(4,\mathbb{C})$, $g \in G$ and $Y \in G_{\mathbb{C}}$. Then*

$$c_Y\big(Z_G(g)\big) = Z_{\mathrm{GL}(4,\mathbb{C})}(c_Y(g)) \cap c_Y(G).$$

Proof.

$$\begin{aligned} h \in Z_G(g) &\Leftrightarrow h \in G \text{ and } hgh^{-1} = g \\ &\Leftrightarrow h \in G \text{ and } hY^{-1}c_Y(g)Yh^{-1} = Y^{-1}c_Y(k)Y \\ &\Leftrightarrow h \in G \text{ and } (YhY^{-1})c_Y(g)(YhY^{-1})^{-1} = c_Y(k) \\ &\Leftrightarrow h \in G \text{ and } c_Y(h) \in Z_{\mathrm{GL}(4,\mathbb{C})}(c_Y(g)) \\ &\Leftrightarrow c_Y(h) \in c_Y(G) \cap Z_{\mathrm{GL}(4,\mathbb{C})}(c_Y(g)). \end{aligned}$$

□

By Lemma A.2.1 it holds $\dim \mathfrak{g}^{\mathrm{Ad}(\widetilde{x}\widetilde{y})} = \dim \mathrm{Lie}\, Z_G(\widetilde{x}\widetilde{y})$ and this is what we want to compute. So let $r \in \mathrm{SL}(2,\mathbb{R})$ with $\widetilde{x}\widetilde{y} = \pi_2(r)$. Then r has the two distinct complex conjugate eigenvalues $e^{i\frac{\pi}{k}}, e^{-i\frac{\pi}{k}}$ or $-e^{i\frac{\pi}{k}}, -e^{-i\frac{\pi}{k}}$. Let $\zeta \coloneqq \pm e^{i\frac{\pi}{k}}$. In particular r is diagonalizable over $\mathbb{C}$ and there is $\tau \in \mathrm{SL}(2,\mathbb{C})$ such that $\tau r \tau^{-1} = \mathrm{diag}(\zeta, \zeta^{-1})$. Then

$$c_{\pi_2(\tau)}(\widetilde{x}\widetilde{y}) = \pi_2(\tau r \tau^{-1}) = \pi_2(\mathrm{diag}(\zeta, \zeta^{-1})) = \begin{pmatrix} \zeta^3 & & & \\ & \zeta & & \\ & & \zeta^{-3} & \\ & & & \zeta^{-1} \end{pmatrix} =: D.$$

It holds $Z_G(\widetilde{x}\widetilde{y}) = Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) \cap G$ and $\operatorname{Lie} Z_{G_\mathbb{C}}(\widetilde{x}y)$ is a complex Lie subalgebra of $\operatorname{Lie} G_\mathbb{C}$, which, as complex vector space, has defining equations with coefficients in $\mathbb{R}$ because $\widetilde{x}\widetilde{y} \in \mathrm{GSp}(4,\mathbb{R})$. Thus

$$\dim_\mathbb{C} \operatorname{Lie} Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) = \dim_\mathbb{R} \operatorname{Lie} (Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) \cap G) = \dim_\mathbb{R} \operatorname{Lie} Z_G(\widetilde{x}\widetilde{y}).$$

Since $\pi_2(\tau) \in \mathrm{Sp}(4,\mathbb{C})$, $c_{\pi_2(\tau)}$ is an isomorphism of $G_\mathbb{C}$ and we have $c_{\pi_2(\tau)} (Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y})) = Z_{G_\mathbb{C}}(c_{\pi_2(\tau)}(\widetilde{x}\widetilde{y})) = Z_{G_\mathbb{C}}(D)$ as well as

$$\operatorname{Lie} Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) \cong d_e c_{\pi_2(\tau)} \operatorname{Lie} Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) = \operatorname{Lie} c_{\pi_2(\tau)} (Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y})) = \operatorname{Lie} Z_{G_\mathbb{C}}(D).$$

Putting everything together

$$\dim_\mathbb{R} \operatorname{Lie} Z_G(\widetilde{x}\widetilde{y}) = \dim_\mathbb{C} \operatorname{Lie} Z_{G_\mathbb{C}}(\widetilde{x}\widetilde{y}) = \dim_\mathbb{C} \operatorname{Lie} Z_{G_\mathbb{C}}(D),$$

and we compute the latter. We consider the three distinct cases $k > 3$, $k = 3$ and $k = 2$ separately.

$\underline{k > 3}$ In this case we remark the following:

Claim. For all $k > 3$ the diagonal entries of D are distinct and are all $\neq \pm 1$.

Proof. The diagonal entries of D are $\zeta, \zeta^{-1}, \zeta^3, \zeta^{-3}$ with $\zeta = \pm e^{i\frac{\pi}{k}}$ and they lie on the unit circle. Therefore they are equal to ± 1 if and only if they are real, which holds if and only if $\zeta = \zeta^{-1}$ or $\zeta^3 = \zeta^{-3}$. Since $k > 1$ the first case is not possible. The second case holds if and only if $\zeta^6 = 1$ which holds if and only if $\frac{6}{k} \in 2\mathbb{N}$ which is never the case if $k > 3$.

To show that they are all distinct it suffices to show that $\zeta^3 \neq \zeta$ and $\zeta^3 \neq \zeta^{-1}$.

- $\zeta^3 = \zeta \Leftrightarrow \zeta^2 = 1 \Leftrightarrow \frac{2\pi}{k} = 2\pi$, which is never the case if $k > 3$.
- $\zeta^3 = \zeta^{-1} \Leftrightarrow \zeta^4 = 1 \Leftrightarrow \frac{4\pi}{k} = 2\pi$, which is never the case if $k > 3$.

□

We have $Z_{G_\mathbb{C}}(D) = Z_{\mathrm{GL}(4,\mathbb{C})}(D) \cap G_\mathbb{C}$, so we first look at matrices with commute with D and then determine the ones which are in $G_\mathbb{C}$.

For $h \in \mathrm{GL}(4,\mathbb{C})$ it holds

$$\begin{aligned} hD = Dh &\Leftrightarrow (hD)_{ij} = (Dh)_{ij} \text{ for all } 1 \leq i,j \leq 4 \\ &\Leftrightarrow (D_{ii} - D_{jj})h_{ij} = 0 \text{ for all } 1 \leq i,j \leq 4, \end{aligned} \tag{4.6}$$

since

$$(hD)_{ij} = \sum_{k=1}^{4} h_{ik} \underbrace{D_{kj}}_{\neq 0 \text{ iff } k=j} = h_{ij} D_{jj}$$

$$(Dh)_{ij} = \sum_{k=1}^{4} D_{ik} h_{kj} = D_{ii} h_{ij}.$$

Since $k > 3$ we have $D_{ii} \neq D_{jj}$ for all $i \neq j$, thus h commutes with D if and only if $h = \mathrm{diag}(h_1, h_2, h_3, h_4)$ with $h_i \neq 0$.

Let now $h = \mathrm{diag}(h_1, h_2, h_3, h_4)$, then $h \in G_{\mathbb{C}}$ if and only if $h^T \Omega_4 h = \pm\Omega_4$, which holds if and only if

$$\begin{pmatrix} & & h_1h_3 & \\ & & & h_2h_4 \\ -h_1h_3 & & & \\ & -h_2h_4 & & \end{pmatrix} = \pm \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix},$$

which is equivalent to $h_1h_3 = \pm 1 = h_2h_4$.

We conclude that when $k > 3$ the centralizer is

$$Z_{G_{\mathbb{C}}}(D) = \{ \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \pm h_1^{-1} & \\ & & & \pm h_2^{-1} \end{pmatrix} : h_1, h_2 \in \mathbb{C}^* \},$$

and therefore $\mathrm{Lie}\, Z_{G_{\mathbb{C}}}(D)$ is 2-dimensional over $\mathbb{C}$.

<u>$k = 3$:</u> When $k = 3$ it holds $\zeta = \pm e^{i\frac{\pi}{3}}$ and $D = \begin{pmatrix} \mp 1 & & & \\ & \pm e^{i\frac{\pi}{3}} & & \\ & & \mp 1 & \\ & & & \pm e^{-i\frac{\pi}{3}} \end{pmatrix}$, in particular $D_{11} = D_{33} \neq D_{22}, D_{44}$ and $D_{22} \neq D_{44}$. By (4.6) a matrix $h \in \mathrm{GL}(4, \mathbb{C})$ commutes with D if and only if it is of the form

$$h = \begin{pmatrix} h_{11} & 0 & h_{13} & 0 \\ 0 & h_{22} & 0 & 0 \\ h_{31} & 0 & h_{33} & 0 \\ 0 & 0 & 0 & h_{44} \end{pmatrix}$$

with $\det(h) = h_{22}h_{44}(h_{11}h_{33} - h_{13}h_{31}) \neq 0$.
For such an h it holds

$$h^T \Omega_4 h = \begin{pmatrix} & & h_{11}h_{33} - h_{13}h_{31} & \\ & & & h_{22}h_{44} \\ h_{13}h_{31} - h_{11}h_{33} & & & \\ & -h_{22}h_{44} & & \end{pmatrix}$$

and thus $h^T \Omega_4 h = \pm\Omega_4$ if and only if $h_{11}h_{33} - h_{13}h_{31} + = \pm 1$ and $h_{22}h_{33} = \pm 1$. Thus the entries of h depend on six variables with two independent equations, which means that $Z_{G_{\mathbb{C}}}(D)$ is 4-dimensional.

$\underline{k=2}$: When $k=2$ it holds $\zeta = \pm i$ and $D = \pm \begin{pmatrix} -i & & & \\ & i & & \\ & & i & \\ & & & -i \end{pmatrix}$. In particular $D_{11} = D_{44}$ and $D_{22} = D_{33}$ and $D_{11} \neq D_{22}$. By (4.6) a matrix $h \in \mathrm{GL}(4, \mathbb{C})$ commutes with D if and only if it is of the form

$$h = \begin{pmatrix} h_{11} & 0 & 0 & h_{14} \\ 0 & h_{22} & h_{23} & 0 \\ 0 & h_{32} & 0 & h_{33} \\ h_{41} & 0 & 0 & h_{44} \end{pmatrix}.$$

For such an h it holds

$$h^T \Omega_4 h = \begin{pmatrix} 0 & h_{11}h_{32} - h_{22}h_{41} & h_{11}h_{33} - h_{23}h_{41} & 0 \\ -h_{11}h_{32} + h_{22}h_{41} & 0 & 0 & -h_{14}h_{32} + h_{22}h_{44} \\ -h_{11}h_{33} + h_{23}h_{41} & 0 & 0 & -h_{14}h_{33} + h_{23}h_{44} \\ 0 & h_{14}h_{32} - h_{22}h_{44} & h_{14}h_{33} - h_{23}h_{44} & 0 \end{pmatrix}$$

and thus $h^T \Omega_4 h = \pm\Omega_4$ if and only the eight non-zero entries of h satisfy four independent equations. Thus $Z_{G_\mathbb{C}}(D)$ is 4-dimensional.

□

Theorem 4.6.3. *The dimension of the Hitchin component of* $\chi(\Delta(p,q,r), \mathrm{PGSp}(4,\mathbb{R}))$ *is*

$$\dim \mathrm{Hit}(\Delta(p,q,r), \mathrm{PGSp}(4,\mathbb{R})) = \begin{cases} 0 & \text{if } p=q=3,\ r \geq 4 \text{ or } p=2, q=3, r \geq 7 \\ 1 & \text{if } p=2,3,\ r \geq q \geq 4 \\ 2 & \text{if } p,q,r \geq 4. \end{cases}$$

Proof. By Theorem 4.5.4 and Proposition 4.6.1 the dimension of $\mathrm{Hit}(\Delta(p,q,r), \mathrm{PGSp}(4,\mathbb{R}))$ is

$$\begin{aligned} &\frac{1}{2}\big(\dim \mathfrak{g} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ab))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(bc))} - \dim \mathfrak{g}^{\mathrm{Ad}(\phi(ac))}\big) \\ &= \begin{cases} \frac{1}{2}(10-4-4-2) = 0 & p=q=3\, r \geq 4, \text{ or } p=2, q=3, r \geq 7 \\ \frac{1}{2}(10-4-2-2) = 1 & p=2,3,\ r \geq q \geq 4, \\ \frac{1}{2}(10-2-2-2) = 2 & r \geq q \geq p \geq 4. \end{cases} \end{aligned}$$

□

5. Parameters for the Hitchin component of $\chi(\Delta(3,4,4))$

Cooper, Long and Thistlethwaite introduced in [12] a method for the exact computation of character varieties of fundamental groups of hyperbolic 3-manifolds, which has been applied also for character varieties of Bianchi groups [33] and extensively for fundamental groups of some orientable 2-dimensional orbifolds [26], [40], [11]. The groups considered in [26], [40], [11] are triangle groups $T(p,q,r)$ generated by rotations about the vertices of a hyperbolic triangle, hence subgroups of index 2 of the triangle groups treated in this work. To our knowledge no work has been done to compute varieties of *full* triangle groups, which is the subject of this chapter.

The method is described in some detail in [12] and [26] and often works well when the variety has dimension at most 2. Here we illustrate thoroughly how the technique applies to the 1-dimensional Hitchin component of the full triangle group $\Delta(3,4,4)$. The choice of $\Delta(3,4,4)$ was dictated by the fact that according to Theorem 4.6.3 it is the triangle group whose Hitchin component has smallest positive dimension with smallest parameters p,q,r (also $(2,4,5)$ would have been a possibility).

Let $\mathcal{H} = \mathrm{Hit}(\Delta(3,4,4), \mathrm{PGSp}(4,\mathbb{R}))$ be the Hitchin component. Our aim is to find a 1-parameter family of deformations of the base representation $\phi_0 : \Delta(3,4,4) \to \mathrm{PGSp}(4,\mathbb{R})$ which describes $\mathcal{H}$. Since it is a 1-dimensional algebraic variety, the image of each generator of $\Delta(3,4,4)$ is a matrix whose entries are algebraic functions of one parameter u. Therefore the matrix entries at a generic point of $\mathcal{H}$ can be considered to lie in a field F of transcendence degree[1] 1 over $\mathbb{R}$ and $\mathcal{H}$ is specified by a single *tautological* representation Ψ into $\mathrm{PGSp}(4,F)$. Individual representations are obtained from Ψ by evaluating at specific point u in the parameter space.

Theorem 5.0.1. *The 1-dimensional* $\mathrm{PGSp}(4,\mathbb{R})$*-Hitchin component of the triangle group* $\Delta(3,4,4)$ *is given by a tautological representation* Ψ_u *whose entries lie in the field* $\mathbb{Q}(u)(\tau,\sigma,\sqrt{2})$*, where* τ *is a real root of the cubic polynomial*

$$\tfrac{1}{3}u^2(32+86u^2-5u^4)+u^2(-20+\tfrac{13}{3}u^2)\tau+(2-\tfrac{11}{3}u^2)\tau^2+\tau^3$$

[1] Recall that if K is a field extension of F, then an element k of K is called *algebraic over* F if there exists some non-zero polynomial $g \in F[x]$ such that $g(k)=0$. Elements of K which are not algebraic are called *transcendental*. The *transcendence degree* of an extension field K over a field F is the smallest number of elements of K which are not algebraic over F but are needed to generate K.

and $\sigma = \sqrt{\frac{3}{2}(u^2 + \tau + 2)}$. *The images of the generators of* $\Delta(3,4,4)$ *are represented by the matrices* $\Psi_u(a), \Psi_u(b)$ *and* $\Psi_u(c)$ *given in Appendix C. The base representation* ϕ_0 *is obtained for* $u = 5\sqrt{2}$.

Recall that $\Delta(3,4,4) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^4 = 1\rangle$, and using Example 2.3.1 and (2.1) we can get representatives $a_0, b_0, c_0 \in \mathrm{GSp}(4, \mathbb{R})$ of the images of the generators under the base representation ϕ_0. The values of the parameters k, l and m in (2.1) are $k = \cos^2(\pi/8)$, $l = \frac{3}{(12-4\sqrt{3})\sin^2(\pi/8)}$ and $m = (3 - \sqrt{3})\sin^2(\pi/8)$. They satisfy

$$a_0^2 = b_0^2 = c_0^2 = \mathrm{id} \text{ and } (a_0b_0)^3 = (b_0c_0)^4 = (c_0a_0)^4 = -\mathrm{id}. \tag{5.1}$$

It suffices to find a family $(\phi_u(a), \phi_u(b), \phi_u(c))$ of triples in $\mathrm{GSp}(4, \mathbb{R})^3$ which satisfy (5.1) for every value of the parameter u. Then the map into $\mathrm{PGSp}(2n, \mathbb{R})^3$

$$u \mapsto ([\phi_u(a)], [\phi_u(b)], [\phi_u(c)]) \tag{5.2}$$

defines a deformation of ϕ (by homomorphisms).

For a matrix $x \in M_{4\times 4}(\mathbb{R})$ let $Symp(x) = (x^T\Omega_4 x - 1)(x^T\Omega_4 x + 1)$ be the relation which describes being a symplectic or antisymplectic matrix: $x \in \mathrm{GSp}(4, \mathbb{R})$ if and only if $Symp(x) = 0$. Consider the polynomial map

$$\begin{aligned} Rel : (\mathbb{R}^{16})^3 &\to (\mathbb{R}^{16})^9 \\ (a, b, c) &\mapsto (a^2 - 1, b^2 - 1, c^2 - 1, (ab)^3 + 1, (bc)^4 + 1, (ca)^4 + 1, \\ &\qquad Symp(a), Symp(b), Symp(c)). \end{aligned}$$

We are looking for a family $(\phi_u(a), \phi_u(b), \phi_u(c))$ in $Rel^{-1}(\{(0, \dots, 0)\})$ (which induces the tautological representation $\Psi : \Delta(p, q, r) \to \mathrm{PGSp}(4, F)$).

5.1 Trace field and matrix entry field

There are two fields that will play an important role in finding the tautological representation Ψ. Suppose that $(a, b, c) \in Rel^{-1}(0)$ represents a homomorphism ϕ. The first field under consideration is the field K generated by the *entries* of image matrices of ϕ, that is by the matrix entries of (a, b, c). The second field of interest is the subfield T of K generated by the *traces* of image matrices. We call T the *trace field*, notice that it is independent of conjugation.

We prove that, after a sensible conjugation, the matrix entry field K can be obtained from T by adding one generator and the proof of this fact will actually show that both T as well as K are determined by a finite number of traces $\mathrm{tr}(\phi(\gamma_i))$, $i = 1, \dots, 8$ (Remark 5.1.6).

The following in an analogue of Proposition 3.1 in [12] adapted to our setting.

Proposition 5.1.1. *Let G be the subgroup of $\mathrm{GL}(4,\mathbb{R})$ generated by matrices a, b, c where*

$$a = \begin{pmatrix} -1 & & & \\ & -1 & & \\ -2 & 0 & 1 & \\ 0 & 2 & & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & & -2k & 0 \\ & 1 & 0 & 2(1-k) \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad b = (b_{ij})_{i,j}. \tag{5.3}$$

Let T be the trace field of G, and let K be the field obtained by adjoining k to T.

Then the matrix entries

$$b_{11}, b_{22}, b_{33}, b_{44}, b_{13}, b_{24}, b_{31}, b_{42}$$

and the products of matrix entries

$$b_{12}b_{21},\ b_{12}b_{23},\ b_{12}b_{41},\ b_{12}b_{43}, \tag{5.4}$$
$$b_{14}b_{21},\ b_{14}b_{23},\ b_{14}b_{41},\ b_{14}b_{43}, \tag{5.5}$$
$$b_{21}b_{32},\ b_{21}b_{34},\ b_{23}b_{32},\ b_{23}b_{34}, \tag{5.6}$$
$$b_{32}b_{41},\ b_{32}b_{43},\ b_{34}b_{41}, b_{34}b_{43} \tag{5.7}$$

are in K.

Proof. It is easily checked that the eight traces

$$\mathrm{tr}(b), \mathrm{tr}(cb), \mathrm{tr}(ab), \mathrm{tr}(abc), \mathrm{tr}(acb), \mathrm{tr}(abac), \mathrm{tr}(cacb), \mathrm{tr}(abcac)$$

in which b occurs only once are all linear expressions (with coefficients in $\mathbb{Q}(k)$) in the eight entries

$$b_{11}, b_{22}, b_{33}, b_{44}, b_{13}, b_{24}, b_{31}, b_{42}.$$

Therefore we have a linear system for the entries $b_{11}, \dots, b_{42}$ over K and one can verify (e.g. using Mathematica) that the determinant of the matrix of coefficients is -256 . Therefore the system has a unique solution and $b_{11}, \dots, b_{42}$ lie in K.

Consider the sixteen traces

$$\mathrm{tr}(b^2), \mathrm{tr}(ab^2), \mathrm{tr}(acb^2), \mathrm{tr}(acab^2), \mathrm{tr}((ac)^2b^2), \mathrm{tr}((ac)^2ab^2), \mathrm{tr}((ac)^3ab^2),$$
$$\mathrm{tr}(acbcb), \mathrm{tr}(acabcb), \mathrm{tr}((ac)^2bcb), \mathrm{tr}((ac)^2abcb), \mathrm{tr}((ac)^3abcb),$$
$$\mathrm{tr}(acbab), \mathrm{tr}((ac)^2bab), \mathrm{tr}((ac)^2abab), \mathrm{tr}(acbacb).$$

in which b occurs exactly twice. Using Mathematica one checks that they are all linear expressions in the sixteen products

$$b_{12}b_{21},\ b_{12}b_{23},\ b_{12}b_{41},\ b_{12}b_{43},$$
$$b_{14}b_{21},\ b_{14}b_{23},\ b_{14}b_{41},\ b_{14}b_{43},$$
$$b_{21}b_{32},\ b_{21}b_{34},\ b_{23}b_{32},\ b_{23}b_{34},$$
$$b_{32}b_{41},\ b_{32}b_{43},\ b_{34}b_{41},\ b_{34}b_{43}$$

with coefficients in $T(k, b_{11}, \dots, b_{42})$ which by the above is equal to K. The matrix of coefficients has full rank. $\square$

Corollary 5.1.2. *Let G, K as in Proposition 5.1.1 and assume that the $(1,2)$-entry of b is $b_{12} \neq 0$. Let*

$$q = \begin{pmatrix} 1 & & & \\ & b_{12} & & \\ & & 1 & \\ & & & b_{12} \end{pmatrix}.$$

Then qGq^{-1} is generated by a, b, c as in (5.3) with $b_{12} = 1$ and the field generated by the matrix entries of qGq^{-1} is K.

Proof. The group $G' = qGq^{-1}$ is generated by $a' = qaq^{-1}$, $c' = qcq^{-1}$, $b' = qbq^{-1}$. Let K' be the field generated by the matrix entries of G'. Since conjugation does not affect traces the tracefield of G' is T. Let $q = \operatorname{diag}(q_1, q_2, q_3, q_4)$, then

$$qaq^{-1} = q \begin{pmatrix} -1 & & & \\ & -1 & & \\ -2 & & 1 & \\ & 2 & & 1 \end{pmatrix} q^{-1} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ -2\frac{q_3}{q_1} & & 1 & \\ & 2\frac{q_4}{q_2} & & 1 \end{pmatrix} = a',$$

and

$$qcq^{-1} = q \begin{pmatrix} 1 & -2k & 0 & \\ & 1 & 0 & 2(1-k) \\ & & -1 & \\ & & & -1 \end{pmatrix} q^{-1} = \begin{pmatrix} 1 & -2k\frac{q_1}{q_3} & 0 & \\ & 1 & 0 & 2(1-k)\frac{q_2}{q_4} \\ & & -1 & \\ & & & -1 \end{pmatrix} = c',$$

and

$$qbq^{-1} = q \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} q^{-1} = \begin{pmatrix} b_{11} & \frac{q_1}{q_2}b_{12} & \frac{q_1}{q_3}b_{13} & \frac{q_1}{q_4}b_{14} \\ \frac{q_2}{q_1}b_{21} & b_{22} & \frac{q_2}{q_3}b_{23} & \frac{q_2}{q_4}b_{24} \\ \frac{q_3}{q_1}b_{31} & \frac{q_3}{q_2}b_{32} & b_{33} & \frac{q_3}{q_4}b_{34} \\ \frac{q_4}{q_1}b_{41} & \frac{q_4}{q_2}b_{42} & \frac{q_4}{q_3}b_{43} & b_{44} \end{pmatrix} = b'$$

For all q in the statement of the corollary it holds $q_1 = q_3$ and $q_2 = q_4$, thus $a' = a$ and $c' = c$. In particular $k \in K'$ so $K = T(k) = T'(k) \subseteq K'$. Moreover, for the other inclusion it suffices to show that the entries of b' are in $K = T(k)$. Let $q = \operatorname{diag}(1, b_{12}, 1, b_{12})$ and $b'_{12} = \frac{q_1}{q_2}b_{12} = 1$. Thus by Proposition 5.1.1 applied to G' the following entries of b'

$$\begin{pmatrix} b'_{11} & 1 & b'_{13} & \\ & b'_{22} & & b'_{24} \\ b'_{31} & & b'_{33} & \\ & b'_{42} & & b'_{44} \end{pmatrix}$$

as well as $1 \cdot b'_{21}$, $1 \cdot b'_{23}$, $1 \cdot b'_{41}, b'_{43}$ (by (5.4)) are in K. Again by the proposition using (5.5) and (5.6) this in turn implies that b'_{14}, b'_{32}, b'_{34} are in K. □

Remark 5.1.3. (i) The matrix q of Corollary 5.1.2 is not symplectic, and there does not exist a symplectic matrix s which (under conjugation) preserves a and c and sends b to a matrix which has one entry equal to 1.

(ii) When $b_{12} = 0$ one can obtain the same result conjugating by

$$q = \begin{cases} \mathrm{id} & \text{if } b_{12} = b_{14} = b_{23} = b_{34} = b_{21} = b_{41} = b_{32} = b_{43} = 0, \\ \begin{pmatrix} 1 & & & \\ & b & & \\ & & 1 & \\ & & & b \end{pmatrix} & \text{with } b = \text{ any of } b_{14}, b_{32}, b_{34} \text{ which is } \neq 0, \\ \begin{pmatrix} 1 & & & \\ & b^{-1} & & \\ & & 1 & \\ & & & b^{-1} \end{pmatrix} & \text{with } b = \text{ any of } b_{21}, b_{41}, b_{23}, b_{43} \text{ which is } \neq 0. \end{cases}$$

Given a deformation ϕ_t of the base representation ϕ_0, we can always conjugate within the symplectic group so that the generators are as in Proposition 5.1.1.

Lemma 5.1.4. *Let $\phi_t : \Delta(p,q,r) \to \mathrm{PGSp}(4,\mathbb{R})$ be a continuous deformation of the base representation ϕ_0 and let $a_t, c_t \in \mathrm{GSp}(4,\mathbb{R})$ be the images of the two generators a, c. Then $a_t^2 = c_t^2 = \mathrm{id}$ and $(a_t c_t)^r = -\mathrm{id}$ and moreover for each t there is $g_t \in \mathrm{GSp}(4,\mathbb{R})$ such that*

$$g_t a_t g_t^{-1} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ -2 & & 1 & \\ & 2 & & 1 \end{pmatrix}, \; g_t c_t g_t^{-1} = \begin{pmatrix} 1 & & -2k & \\ & 1 & & 2(1-k) \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

where $k = \cos^2(\pi/(2r))$.

This result is a consequence of the fact that representations of dihedral groups are locally rigid and that $\mathrm{Sp}(4,\mathbb{R})$-orbits in the set of quadruples of pairwise transverse Lagrangian subspaces of $\mathbb{R}^4$ can be characterized by their crossratio (under some finite order assumption). We prove the lemma and give all the necessary background in Appendix B.

Remark 5.1.5. In Lemma 5.1.4 one can take the matrix g_t to vary continuously in t. Since after conjugation by g_0 the image of the generator b under the base representation has non vanishing $(1,2)$-entry, the same holds for every $g_t b_t g_t^{-1}$ when t is sufficiently small. In particular, the matrix entry field of $g_t \langle a_t, b_t, c_t \rangle g_t^{-1}$ is $T_t(k)$, where T_t is the trace field of $\langle a_t, b_t, c_t \rangle$.

Remark 5.1.6 (Finitely many traces suffice). By Corollary 5.1.2 and the proof of Propositon 5.1.1 after conjugating the matrix entry field is determined by the twenty-four traces

$$\begin{aligned}
&\operatorname{tr}(b), \operatorname{tr}(cb), \operatorname{tr}(ab), \operatorname{tr}(abc), \operatorname{tr}(acb), \operatorname{tr}(abac), \operatorname{tr}(cacb), \operatorname{tr}(abcac),\\
&\operatorname{tr}(b^2), \operatorname{tr}(ab^2), \operatorname{tr}(acb^2), \operatorname{tr}(acab^2), \operatorname{tr}((ac)^2b^2), \operatorname{tr}((ac)^2ab^2), \operatorname{tr}((ac)^3ab^2),\\
&\operatorname{tr}(acbcb), \operatorname{tr}(acabcb), \operatorname{tr}((ac)^2bcb), \operatorname{tr}((ac)^2abcb), \operatorname{tr}((ac)^3abcb),\\
&\operatorname{tr}(acbab), \operatorname{tr}((ac)^2bab), \operatorname{tr}((ac)^2abab), \operatorname{tr}(acbacb).
\end{aligned}$$

Since conjugation does not change the trace field, these traces determine the "original" trace field as well. In our specific (3,4,4)-triangle group setting and for representations close to the base representation most of these traces are constant or coincide with each other. In fact we can use the following properties to simplify the expressions of certain traces:

(i) If $g^T\Omega g = \Omega$, then $\Omega^{-1}g^T\Omega = g^{-1}$ and
$$\operatorname{tr}(g) = \operatorname{tr}(g^T) = \operatorname{tr}(\Omega^{-1}g^T\Omega) = \operatorname{tr}(g^{-1});$$
and if $g^T\Omega g = -\Omega$, then $\Omega^{-1}g^T\Omega = -g^{-1}$ and
$$\operatorname{tr}(g) = \operatorname{tr}(g^T) = \operatorname{tr}(\Omega^{-1}g^T\Omega) = \operatorname{tr}(-g^{-1}) = -\operatorname{tr}(g^{-1}).$$
In particular, the images a, b, c of the generators of $\Delta(3,4,4)$ are antisymplectic of order 2, therefore their trace is zero.

(ii) The (lifts of) images of the products ab, bc, ca satisfy $(ab)^3 = (bc)^4 = (ca)^4 = -\mathrm{id}$. Hence it holds $(ac)^2 = -(ac)^{-2} = -(ca)^2$ and analogous identities.

(iii) The (lifts of) images of the products ab, bc, ca are of finite order, and there are finitely many conjugacy classes of elements of finite order in $\mathrm{Sp}(2n,\mathbb{R})$ (Proposition 4.4.8). Therefore for representations a, b, c nearby the base representation a_0, b_0, c_0 it holds $\operatorname{tr}(ab) = \operatorname{tr}(a_0b_0)$, $\operatorname{tr}(bc) = \operatorname{tr}(b_0c_0)$ and $\operatorname{tr}(ca) = \operatorname{tr}(c_0a_0)$.

(iv) If an antisymplectic element x satisfies $x = x^{-1}$ then
$$\operatorname{tr}(abx) \overset{\text{(i)}}{=} -\operatorname{tr}((abx)^{-1}) = -\operatorname{tr}(xba) = -\operatorname{tr}(bax).$$

We obtain:

$\operatorname{tr}(b) = 0$

$\operatorname{tr}(cb) = 0$

$\operatorname{tr}(ab) = -1$

$\operatorname{tr}(acb) \overset{\text{(i)}}{=} -\operatorname{tr}(abc)$

$\operatorname{tr}(b^2) = 4$

$\operatorname{tr}(ab^2) = \operatorname{tr}(a) = 0$

$\operatorname{tr}(acb^2) = tr(ac) = 0$

$\operatorname{tr}((ac)^2bab) = \operatorname{tr}(cacbaba) = -\operatorname{tr}(cacab) = -\operatorname{tr}(abcac)$

$\operatorname{tr}(acbacb) \overset{\text{(i)}}{=} \operatorname{tr}((acbacb)^{-1}) = \operatorname{tr}(bcabca) = \operatorname{tr}(abcabc)$

$\operatorname{tr}((ac)^2abcb) = \operatorname{tr}(-(ca)^2abcb) = -\operatorname{tr}(cacbcb) = \operatorname{tr}(cabcbc) = \operatorname{tr}(abcb) = -\operatorname{tr}(babc) = -\operatorname{tr}(abac)$

$\operatorname{tr}(acab^2) = \operatorname{tr}(c) = 0$

$\operatorname{tr}((ac)^2b^2) = \operatorname{tr}(acac) = 0$

$\operatorname{tr}((ac)^2ab^2) = -\operatorname{tr}(a) = 0$

$\operatorname{tr}((ac)^3ab^2) = -\operatorname{tr}(c) = 0$

$\operatorname{tr}((ac)^3abcb) = -\operatorname{tr}(caabcb) = -\operatorname{tr}(cbcb) = 0$

$\operatorname{tr}(acbab) = \operatorname{tr}(cbaba) = -\operatorname{tr}(cab) = -\operatorname{tr}(abc)$

$\operatorname{tr}((ac)^2abab) = -\operatorname{tr}((ac)^2ba) = -\operatorname{tr}(cacb)$

Thus the only traces one needs to consider are

$$\mathrm{tr}(abc), \mathrm{tr}(abac), \mathrm{tr}(cacb), \mathrm{tr}(abcac), \mathrm{tr}(acbcb), \mathrm{tr}(acabcb), \mathrm{tr}((ac)^2bcb), \mathrm{tr}(abcabc). \quad (5.8)$$

5.2 Overview of computational methodology

In this section we describe the strategy, without giving much detail on the technical aspects (Newton algorithm, polynomial interpolation, integer relation finding algorithms) needed to put it into practice, which will be the subject of section 5.3. This overview follows closely and expands on Sections 6 and 7 of [12].

Step 1. Starting from the base representation we apply arbitrarily small perturbations to the (matrix coordinates of the) generating matrices a_0, b_0, c_0. This gives three new generating matrices, which have no reason to satisfy the group relations *Rel* anymore, but we can use the Newton process to converge to matrices (a_1, b_1, c_1) which *are* in the zero set of *Rel* and define a homomorphism ϕ_1. We check that ϕ_1 is not conjugate to ϕ_0, for example by looking at the traces of some $\gamma \in \langle a_1, b_1, c_1\rangle$.

Step 2. By Proposition 5.1.1 the matrix entry field is determined by the trace field, so the next step is to find a parameter for the trace field. We choose some $\gamma \in \Delta$ such that $\mathrm{tr}(\phi_0(\gamma)) \neq \mathrm{tr}(\phi_1(\gamma))$ and check whether $u := \mathrm{tr}(\phi(\gamma))$ it is a suitable parameter for the trace field. By "suitable" we mean that all other traces $\mathrm{tr}(\phi(\eta))$, $\eta \in \Delta$, should be of small degree over $\mathbb{Q}(u)$. Notice that if u is suitable in this sense and it gives representations ϕ_u, then when assigning to u a rational value the traces $\mathrm{tr}(\phi_u(\eta))$ are of small degree over $\mathbb{Q}$. Therefore to decide whether the chosen parameter is a good candidate, one chooses some rational values $u_1, \ldots, u_n \in \mathbb{Q}$ close to $u_0 = \mathrm{tr}(\phi_0(\gamma))$ and runs the Newton process again n times, each time adding the constraint that it has to converge to a representation ϕ_i such that $\mathrm{tr}(\phi_i(\gamma)) = u_i \in \mathbb{Q}$. Then one checks that for each $\eta \in \Delta$ and each i the traces $\mathrm{tr}(\phi_i(\eta))$ are algebraic over $\mathbb{Q}$ and of which degree. The proof of Proposition 5.1.1 provides a finite list of traces that is sufficient for this purpose, which can actually be reduced to the eight traces (5.8) of Remark 5.1.6.

In practice (cfr. Section 5.3) we chose $\gamma = abc \in \Delta(p, q, r)$, so that for the base representation $u_0 = \mathrm{tr}(\phi_0(abc)) = \mathrm{tr}(A_0B_0C_0) = 5\sqrt{3} \approx 8.66025$ and we set $u_i = 86/10 + 1/(100+i)$ for $i = 1, \ldots, 20$. Then all the traces are of degree 3 or 6 over $\mathbb{Q}$, and when they are of degree 6, their minimal polynomial is even.

Step 3. Guess the trace field T as an extension of finite degree over the field $\mathbb{Q}(u)$. By the previous step all traces seem to be algebraic of degree 3 or 6 over $\mathbb{Q}(u)$, and we suspect that the trace field has degree 6 over $\mathbb{Q}(u)$. If so, we can determine it using two traces. Indeed, let t_1 be a trace of degree 3, then

$$6 = [T : \mathbb{Q}(u)] = [T : \mathbb{Q}(u)(t_1)][\mathbb{Q}(u)(t_1) : \mathbb{Q}(u)] = 3[T : \mathbb{Q}(u)(t_1)],$$

therefore $[T : \mathbb{Q}(u)(t_1)] = 2$ and any $t_2 \in T \setminus \mathbb{Q}(u)(t_1)$ is algebraic of degree 2 over $\mathbb{Q}(u)(t_1)$ and it holds $T = \mathbb{Q}(u)(t_1, t_2)$.

If we find an expression for the minimal polynomial $p_\tau(x)$ of a degree 3 trace τ as an element of $\mathbb{Q}(u)[x]$, then Cardano's formulas describe τ in terms of u. Then finding the degree 2 minimal polynomial in $\mathbb{Q}(u)(\tau)$ of a second trace σ gives an expression $\sigma(u,\tau)$. The result are two parameters $\tau = \tau(u)$ and $\sigma = \sigma(u)$ such that the trace field is $\mathbb{Q}(u)(\tau,\sigma)$.

So Step 3 consists in finding an expression for the minimal polynomial $p_\tau(x)$ of a degree 3 trace τ as an element of $\mathbb{Q}(u)[x]$. The implementation is explained in Section 5.3 but roughly speaking at the end of Step 2 one has a sequence of 20 representations ϕ_i, and for the trace $\tau = \mathrm{tr}(\phi(\gamma))$ a corresponding sequence of minimal polynomials $p^i_\tau(x) \in \mathbb{Q}[x]$ such that $p^i_\tau(\tau_i) = 0$ (where $\tau_i = \mathrm{tr}(\phi_i(\gamma))$). The coefficients of each $p^i_\tau(x)$ are in $\mathbb{Q}$ and for fixed τ one interpolates each of them over u_i to find a polynomial in $\mathbb{Q}[u]$ which fits it. One ends up with a "tautological" minimal polynomial $p_\tau(x) \in \mathbb{Q}(u)[x]$ for the trace τ, the roots of which are elements of $\mathbb{Q}(u)$. We choose $\tau = \mathrm{tr}(abcabc)$.
Step 4. We compute the degree of other traces over $\mathbb{Q}(u)(\tau)$ to find one which is of degree 2 and then seek validation that the extension $\mathbb{Q}(u)(\tau,\sigma)$ is the trace field.

After some testing with different traces we notice that the trace of $abac$ is algebraic of degree 2 over $\mathbb{Q}(u)(\tau)$ with minimal polynomial

$$2x^2 - (3\tau + 3u^2 + 6).$$

We infer that the trace field is $\mathbb{Q}(u)(\tau)(\sigma)$ where $\sigma = +\sqrt{\frac{3}{2}(u^2+\tau+2)}$. A basis over $\mathbb{Q}(u)$ for the trace field would then be $\{1,\tau,\tau^2,\sigma,\tau\sigma,\tau^2\sigma\}$.

To check that a given basis $\{\rho_1,\ldots,\rho_k\}$ is the correct guess one has to verify that over a lattice in the parameter space every trace t in (5.8) can be expressed as a linear combination $\sum_{j=1}^k q_j(u)\rho_j$ where the $q_j(u)$ are rational functions of the parameter u. Let $\phi_i = \phi(u_i)$ be the representations obtained in Step 2 from rational values u_i. For each i the trace t corresponds to a trace t_i and by means of an integer relation finding algorithm (such as Mathematica's **FindIntegerNullVector**) one can identify integer numbers q^i_0 and $(q^i_j)^k_{j=1}$ such that

$$q^i_0 t_i = \sum_{j=1}^k q^i_j \rho^i_j,$$

where $\rho^i_j = \rho_j(\phi_i)$. Finally for every $j = 0,\ldots,k$ polynomial interpolation over u_i gives a polynomial $q_j(u) \in \mathbb{Z}[u]$ such that $q_j(u_i) = q^i_j$.

To summarize, upon completing this step we have a generic expression of the form $\mathrm{tr}(\phi(\gamma)) = \sum_{j=1}^k q_j(u)\rho_j$ for each trace, where the basis elements ρ_j depend on u as well.
Step 5. Determine an exact expression for the matrix entries in terms of the parameters. In Steps 1-4 we determined parameters u and $\tau = \tau(u)$, $\sigma = \sigma(u)$ such that for some rational values u_i we have generating matrices a_i, b_i, c_i close to a_0, b_0, c_0 whose trace field is $\mathbb{Q}(u_i)(\tau(u_i),\sigma(u_i))$. We can say that we have a *tautological trace field* $T(u)$, where the trace field to some individual representation close to ϕ_0 is obtained by evaluating at a specific rational point u in the parameter space. Let $G_i = \langle a_i, b_i, c_i\rangle$. By Lemma 5.1.4 and Corollary 5.1.2 after conjugation by some invertible matrix q_i we get

$\widetilde{G_i} = q_i G_i q_i^{-1} = \langle \widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i \rangle$, whose matrix entries are in $\mathbb{Q}(u_i)(\tau_i, \sigma_i, k) = \mathbb{Q}(\sqrt{2})(u_i)(\tau_i, \sigma_i)$ (since $k = \cos^2(\pi/8) = (2+\sqrt{2})/4$). The matrices $\widetilde{a}_i, \widetilde{c}_i$ are in the standard form:

$$\widetilde{a}_i = \begin{pmatrix} -1 & & & \\ & -1 & & \\ -2 & 0 & 1 & \\ 0 & 2 & & 1 \end{pmatrix}, \quad \widetilde{c}_i = \begin{pmatrix} 1 & & -2k & 0 \\ & 1 & 0 & 2(1-k) \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

(the same for all i), it holds $(\widetilde{b}_i)_{12} = 1$ and the other coefficients of $\widetilde{b}_i$ can be recovered by solving the two linear systems in the proof of Proposition 5.1.1. These two systems consist of linear equations over the trace field. After inserting the generic traces obtained in Step 4, the solution of the systems gives a tautological $\widetilde{b} \in \mathrm{GL}(4, \mathbb{Q}(\sqrt{2})(u)(\tau, \sigma))$ such that $\widetilde{b}(u_i, \tau_i, \sigma_i) = \widetilde{b}_i$.

Step 6. Use the formal algebra capabilities of Mathematica to verify that the generic matrices obtained in Step 5 satisfy the triangle group relations ($b^2 = \mathrm{id}$ and $(ab)^4 = (bc)^4 = -\mathrm{id}$).

Step 7. Conjugate back into GSp. The tautological expression that we found for the matrix $\widetilde{b}$ lies in $\mathrm{GL}(4, \mathbb{Q}(\sqrt{2})(u)(\tau, \sigma))$, but not in the symplectic group, cfr. Remark 5.1.3. We need a way to conjugate back into GSp and to explain how this is done we summarize the procedure so far. Starting with a deformation ϕ_t of the base representation in the first four steps we find the tracefield $T_t = \mathbb{Q}(u_t, \tau_t, \sigma_t)$. Now the theory tells us that we can conjugate ϕ_t in standard form (Lemma 5.1.4) and then conjugate again (Corollary 5.1.2) to obtain representations $\widetilde{\phi}_t$ so that the matrix entry field is $\widetilde{M}_t = \widetilde{T}_t(k) = T_t(k)$ since conjugation preserves traces. It holds $\widetilde{\phi}_t(a) = a_0$, $\widetilde{\phi}_t(c) = c_0$ and $\widetilde{\phi}_t(b)_{12} = 1$. The entries of $\widetilde{\phi}_t(b)$ are in $\widetilde{M}_t = T_t(k) = \mathbb{Q}(\sqrt{2})(u_t, \tau_t, \sigma_t)$ and they can be obtained by solving two systems of linear equations over $\mathbb{Q}(\sqrt{2})(u_t, \tau_t, \sigma_t)$. Step 5 consists in solving these systems with generic entries in $\mathbb{Q}(\sqrt{2})(u, \tau, \sigma)$ and get $\widetilde{b} = \widetilde{b}(u, \tau, \sigma)$ such that $\widetilde{b}(u_t, \tau_t, \sigma_t) = \widetilde{\phi}_t(b)$. This generic $\widetilde{b}$ satisfies the triangle group relations (Step 6), and therefore defines a tautological representation as desired, but it does not satisfy the symplectic condition. In fact, the conjugation of Corollary 5.1.2 maps $\phi_t(b)$ out of the symplectic group: it holds $\widetilde{\phi}_t(b) = q_{x_t} \phi_t(b) q_{x_t}^{-1}$, where $q_{x_t} = \mathrm{diag}(1, x_t, 1, x_t)$ does not act (by conjugation) on the standard symplectic group but it maps it into the group preserving the form

$$\Omega_{x_t^2} = \begin{pmatrix} & & 1 & \\ & & & 1/x_t^2 \\ -1 & & & \\ & -1/x_t^2 & & \end{pmatrix}.$$

That is $\widetilde{\phi}_t(b)^T \Omega_{x_t^2} \widetilde{\phi}_t(b) = -\Omega_{x_t^2}$. So we solve

$$\widetilde{b}^T \Omega_{x^2} \widetilde{b} = -\Omega_{x^2}$$

for x^2 to get a generic $x^2 = x^2(u, \tau, \sigma)$ such that $q_x^{-1} \widetilde{b} q_x$ is antisymplectic and together with a, c in normal form generates a triangle group which deforms the base representation.

The resulting tautological representation is given in Appendix C.

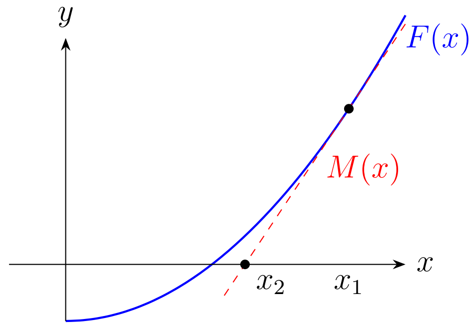


Figure 5.1: Newton method to approximate roots of $F : \mathbb{R} \to \mathbb{R}$.

5.3 Computation of the Hitchin component of $\Delta(3, 4, 4)$

We follow the steps outlined in Section 5.2 to compute the Hitchin component of the character variety $\chi(\Delta(3, 4, 4), \mathrm{PGSp}(4, \mathbb{R}))$.

The starting point is the base representation $\phi_0 : \Delta(3, 4, 4) \to \mathrm{PGSp}(4, \mathbb{R})$ and let a_0, b_0, c_0 be representatives in $\mathrm{GSp}(4, \mathbb{R})$ of the images of the triangle group generators. Recall that they do not define an actual representation but satisfy the relations $a_0^2 = b_0^2 = c_0^2 = \mathrm{id}$ and $(a_0b_0)^3 = (b_0c_0)^4 = (c_0a_0)^4 = -\mathrm{id}$. In the following when we talk about "representation" we mean a triple $(a, b, c) \in \mathrm{GSp}(4, \mathbb{R})$ which satisfies these relations.

Step 1. We perturb a_0, b_0, c_0 very slightly and then use Newton's method to try to converge to a representation a_1, b_1, c_1 close but not conjugate to ϕ_0. It is now fitting to illustrate how Newton's method is applied in this context. Using matrix coordinates we identify $\mathrm{GL}(4, \mathbb{R})$ as an open subset of $\mathbb{R}^{16}$, so that representations of $\Delta(3, 4, 4)$ into $\mathrm{GSp}(4, \mathbb{R})$ are characterized as the nullset of the polynomial map

$$\begin{aligned} Rel : (\mathbb{R}^{16})^3 &\to (\mathbb{R}^{16})^9 \\ (a, b, c) &\mapsto (a^2 - 1, b^2 - 1, c^2 - 1, (ab)^3 + 1, (bc)^4 + 1, (ca)^4 + 1, \\ &\qquad Symp(a), Symp(b), Symp(c)), \end{aligned}$$

where $Symp(x) = (x^T\Omega_4 x - \mathrm{id})(x^T\Omega_4 x + \mathrm{id})$ describes the general symplectic condition. Newton's method gives an algorithm to find roots of Rel which are close to the base representation, by means of approximating Rel with a linear function and looking for the roots of this function. To find roots of functions $F : \mathbb{R} \to \mathbb{R}$ of one variable, one starts with a point x_1 which should be close to the desired root of F, and considers the linear approximation of F at x_1, which is the tangent line $M(x) = F(x_1) + F'(x_1)(x - x_1)$ to F at x_1. If F satisfies certain conditions, the root of M is a better approximation of the real root than x_1 is. Thus one considers $x_2 = x_1 - F(x_1)/F'(x_1)$ and repeats the process until a sufficiently precise value is reached. The method is illustrated in Figure 5.1. This method can be generalized to functions $F : \mathbb{R}^k \to \mathbb{R}^k$, where at step n the tangent line is replaced by the linear approximation $M_n(x) = F(x_n) + J_n(x - x_n)$ in which J_n denotes the Jacobian of F at x_n. The sequence approximating the roots is then given by $x_{n+1} = x_n - J_n^{-1}F(x_n)$.

In practice, finding the inverse J_n^{-1} of the Jacobian can be computationally demanding and it is more convenient to solve the system of linear equations

$$J_n(x - x_n) = -F(x_n) \tag{5.9}$$

for the unknown $s_n = x - x_n$ and then set $x_{n+1} = x_n + s_n$. The solution s_n of (5.9) is called the *search direction*. Sometimes to reach convergence it might be necessary to control the *step length*, that is to set $x_{n+1} = x_n + \alpha_n s_n$ for some positive scalar α_n.

The same strategy is used to find roots of a function $F : \mathbb{R}^k \to \mathbb{R}^m$ with $m > k$, since the Jacobian is a $m \times k$ matrix. Notice that s_n solves (5.9) if and only if

$$\|J_n(s) + F(x_n)\|_2^2 = 0. \tag{5.10}$$

If the Jacobian has full-rank there is a unique solution s_n of $\|J_n(s) + F(x_n)\|_2^2 = 0$ and it is called the *Gauss-Newton direction*. This solution can be found using the QR-decomposition of the Jacobian J_n. For, let $Q_n \in \mathrm{O}(m)$ be orthogonal and $R_n \in M_{m\times k}(\mathbb{R})$ be upper triangular with $Q_n R_n = J_n$. Then

$$\|J_n(s) + F(x_n)\|_2^2 = \|Q_n R_n s + F(x_n)\|_2^2 = \|R_n s + F(x_n)\|_2^2,$$

thus s is a root if and only if $R_n s = -F(x_n)$, which can be solved easily by backward substitution since R_n is upper triangular.

When the Jacobian does not have full-rank or the rank is not known, the practical computation of the QR-factorization of J_n might be troublesome [13, Section 4.7.2].

A different approach to find the roots of $F : \mathbb{R}^k \to \mathbb{R}^m$, $k \geq m$, comes from optimization theory, where an overdetermined system of equations is not expected to have an exact solution and one looks for a "best-possible" approximation. Write $F = (f_1, \dots, f_m)$ where each $f_i : \mathbb{R}^k \to \mathbb{R}$ is real valued. Then the roots of F are the roots of its (squared) 2-norm $g(x) = \|F(x)\|_2^2 = \sum_{i=1}^m f_i(x)^2$, and one can look for points $x \in \mathbb{R}^m$ which minimize it. Such points are solutions of the *least squares problem*

$$\underset{x\in\mathbb{R}^k}{\text{minimize}} \sum_{i=1}^m f_i(x)^2, \tag{5.11}$$

which is also solved with iterative methods. In fact one often uses Newton methods: in each iteration x_n approximate the function g to be minimized with its quadratic Taylor expansion

$$g(x_n + s) \sim m_n(s) := g(x_n) + \nabla g(x_n)^T s + \frac{1}{2} s^T \nabla^2 g(x_n) s,$$

then look for points where the gradient $\nabla m_n(s)$ vanishes (which is a necessary condition for local minimizer of m_n [32, Theorem 2.2]), so the next iteration point will be $x_{n+1} = x_n + s_n$ where s_n solves the *Newton equation* $\nabla^2 g(x_n)s = -\nabla g(x_n)$. If the Hessian $\nabla^2 g(x_n)$ is positive definite, then s_n is a minimum of m_n (and not only a stationary point), otherwise one can use modified Newton methods [13, Section 4.4.2]. For least squares problems as (5.11), the gradient and Hessian matrix of g have a special structure which allows for ad-hoc methods. We shall remark two of these: the *Gauss-Newton* method and the *Levenberg-Marquardt* method, both based on an approximation of the Hessian of g. In fact, when the function g is a sum of squares $g(x) = \|F(x)\|_2^2 = \sum_{i=1}^m f_i(x)^2$, the Newton equation becomes

$$(J_n^T J_n + Q(x_n))s = -J_n^T F(x_n),$$

where J_n is the Jacobian of $F = (f_1, \dots, f_m)$ at x_n and $Q(x) = \sum_{i=1}^m f_i(x)\nabla^2 f_i(x)$. The Gauss-Newton method consists in approximating at each step the Hessian of g by $J_n^T J_n$, that is solving $J_n^T J_n s = -J_n^T F(x_n)$ (compare with equation (5.9)).

Alternatively, in the Levenberg-Marquardt method the search direction is defined as the solution of the equation

$$(J_n^T J_n + \lambda_n I)s = -J_n^T F(x_n),$$

where λ_n is a non-negative scalar. It can be shown [32, Lemma 10.2] that for some scalar Δ related to λ_n, the solution s coincides with the solution of the constrained problem

$$\min_{s\in\mathbb{R}^n} \|J_n F(s) + F(x_n)\|_2^2, \text{ subject to } \|s\|_2 \le \Delta.$$

(compare with (5.10)). We refer to Chapter 10 of [32] and Chapter 4 of [13] for more details on the matter.

Mathematica has the built-in function **FindMinimum**[], which given a 2-tuple $(g, \{\{x_1, X_1\}, \dots, \{x_k, X_k\}\})$ consisting of a function $g : \mathbb{R}^k \to \mathbb{R}$ on k variables $x_1, \dots, x_k$ and a point $(X_1, \dots, X_k) \in \mathbb{R}^k$, outputs both the minimum of the function g and the coordinate point at which this minimum is attained. When the function to be minimized is a sum of squares FindMinimum uses the Levenberg-Marquardt method.

The result is a representation $\phi_1 = (a_1, b_1, c_1)$ distinct and not conjugate to (a_0, b_0, c_0) (we check it by finding some $\gamma \in \Delta$ such that $\mathrm{tr}(\phi_1(\gamma)) \neq \mathrm{tr}(\phi_0(\gamma))$.

Step 2. Find a parameter u for the trace field. This parameter is chosen as the trace $\mathrm{tr}(\phi(\gamma)))$ of some $\gamma \in \Delta$ and must vary as the representations vary from ϕ_0, which can be checked by looking at the traces of the new representation found in Step 1. We see that the trace of $a_1b_1c_1$ varies. Moreover, if we add the constraint on the trace of abc to be a rational value close to $\mathrm{tr}(a_0b_0c_0)$, then the other traces *appear* to be roots of a cubic or a (even) sextic polynomials. We shall explain what we mean by "appear to be". We perform Newton's method of Step 1 again, this time forcing it to converge to a representation $\phi_1 = (a_1, b_1, c_1)$ for which $\mathrm{tr}(a_1b_1c_1)$ is a chosen rational value $u_1 \in \mathbb{Q}$ close to $\mathrm{tr}(a_0b_0c_0)$. This is achieved by adding the function $f_{u_1}(a, b, c) = \mathrm{tr}(abc) - u_1$ to the defining relations *Rel*. To see that all other traces of ϕ_1 are now algebraic, we use the Mathematica built-in function **MinimalPolynomial**[], which returns the minimal polynomial of any given algebraic number, in combination

i	$p_i(x)$
1	$-5181147932147786921984 + 222226846575612031040x - 2685096464057949250x^2 + 9951751411884375x^3$
2	$-5496249128085036664921 + 235748574930150517860x - 2848543233924582000x^2 + 10557772680600000x^3$
3	$-1942386854379992099232 + 83316361651585391520x - 1006732767909610250x^2 + 3731413426653125x^3$
4	$-6174510592732205598809 + 264855366748257496640x - 3200388189726208000x^2 + 11862365798400000x^3$
5	$-418492551608252416 + 17951693275434240x - 216924739463250x^2 + 804057384375x^3$
6	$-2307044317202712041643 + 98965735557248165580x - 1195908964386386000x^2 + 4432872225800000x^3$
7	$-7321756173859152813536 + 314090792399680657760x - 3795577727153980750x^2 + 14069347048584375x^3$
8	$-7741520509871087632081 + 332106236619655862160x - 4013367792174432000x^2 + 14876946777600000x^3$
9	$-2727050232352882956288 + 116991557137506444480x - 1413824577888749750x^2 + 5240937846378125x^3$
10	$-553049079128041409 + 23726598724572260x - 286738099318000x^2 + 1062936600000x^3$

Table 5.1: Minimal polynomials of $\mathrm{tr}(abcabc)$ at selected points u_i.

with **RootApproximant[]**, which approximates a high-accuracy numeric value by an algebraic number. So we feed a trace $t = \mathrm{tr}(\gamma_1)$, where γ_1 is word in the generators a_1, b_1, c_1, to MinimalPolynomial[RootApproximant[·]] and if t is (close to) an algebraic number, the output is a polynomial in $\mathbb{Z}[x]$ of which t is a root. As pointed out in [12], the input t is always rational if interpreted literally and one should consider unreliable any polynomial whose coefficients are excessively large. Concretely, the exact value of $\mathrm{tr}(a_0b_0c_0)$ is $5\sqrt{3} \approx 8.66025$ and we set $u_1 = 86/10 + 1/101$. After converging to a representation ϕ_1, the minimal polynomial of $\mathrm{tr}(a_1b_1c_1a_1b_1c_1)$ is

$$-5181147932147786921984 + 222226846575612031040x - 2685096464057949250x^2 + 9951751411884375 + x^3,$$

and all the traces of (5.8) also have plausible minimal polynomials of low (3 or 6) degree.

To further confirm our guess that $u = \mathrm{tr}(abc)$ is a suitable parameter, we repeat the above procedure for the sequence of rational values $u_i = 86/10 + 1/(100 + i)$, $i = 1, \dots, 20$. The first ten minimal polynomials for the trace of $\gamma = abcabc$ are displayed in Table 5.1.

Step 3. We proceed to interpolate the thirty polynomials. For each $i = 1, \dots, 20$ the trace $\mathrm{tr}(\phi_i(abcabc))$ has a minimal polynomial $p_i(x)$ of the form

$$p_i(x) = a_i + b_i x + c_i x^2 + x^3,$$

with $a_i, b_i, c_i \in \mathbb{Q}$ and the minimal polynomial of $\mathrm{tr}(\phi(abcabc))$ is expected to be in $\mathbb{Q}(u)[x]$, which means that the coefficients of $p_i(x)$ should each be the evaluation at u_i of a rational function in u (we normed the polynomials of Step 2 to have leading coefficient 1). We use polynomial interpolation over the u_i's to find $a, b, c \in \mathbb{Q}[u]$ with

$$a(u_i) = a_i,\ b(u_i) = b_i,\ c(u_i) = c_i.$$

The result is

$$\frac{1}{3}(32u^2 + 86u^4 - 5u^6) + (-20u^2 + \frac{13}{3}u^4)x + (2 - \frac{11}{3}u^2)x^2 + x^3.$$

Step 4. Find a basis for the trace field as an extension of finite degree over the field $\mathbb{Q}(u)$. Let τ be the trace of $abcabc$ (Cardano's formulas give an expression for $\tau = \tau(u)$, see discussion in Section 5.4) and let σ be the trace of $abac$. We check that σ is algebraic over $\mathbb{Q}(u)(\tau)$. A basis of $\mathbb{Q}(u)(\tau)$ over $\mathbb{Q}(u)$ is given by $\{1, \tau, \tau^2\}$, and we hope to find an expression[2]

$$\sigma^2 = a_1(u) + a_2(u)\tau + a_3(u)\tau^2,$$

with $a_1, a_2, a_3 \in \mathbb{Q}[u]$. For every $1 \leq i \leq 20$ the integer relation detector algorithm **FindIntegerNullVector** finds integers $a_0^i, a_1^i, a_2^i, a_3^i \in \mathbb{Z}$ such that

$$a_0^i\sigma_i^2 = a_1^i + a_2^i\tau_i + a_3^i\tau_i^2, \tag{5.12}$$

where $\sigma_i = \mathrm{tr}(a_ib_ia_ic_i)$ and $\tau_i = \mathrm{tr}(a_ib_ic_ia_ib_ic_i)$. If these integers are not *excessively* large one can reasonably believe to be on the correct path and proceed with the polynomial interpolation of the coefficients. For each $0 \leq j \leq 3$ polynomial interpolation of the data $(u_1, a_j^1), \ldots, (u_{20}, a_j^{20})$ results in a polynomial $a_j(u) \in \mathbb{Q}[u]$ such that $a_j(u_i) = a_j^i$ for all $1 \leq i \leq 20$. Some remarks are in order here. For every i the relation (5.12) is preserved under multiplication by any constant and in fact it is necessary to interpolate after an appropriate "normalization" k_i. To see why and how to determine the multipliers k_i, let us consider the coefficient a_0, which we denote by a. The goal is to approximate the integer values $a^i \in \mathbb{Z}$ with a polynomial in the rational variable u_i. This can almost surely not be done: suppose that there exists a polynomial $a(u) = \alpha_0 + \alpha_1 u + \ldots + \alpha_m u^m$ with $a(u_i) = a^i$. Write $u_i = n_i/d_i$ with $n_i, d_i \in \mathbb{Z}$, then it must hold

$$\mathbb{Z} \ni a^i = \alpha_0 + \alpha_1\frac{n_i}{d_i} + \ldots + \alpha_m\frac{n_i^m}{d_i^m},$$

which is only possible if the coefficients α_k are very large. Therefore it is more sensible to interpolate the values a^i/d_i^m, where $d_i \in \mathbb{Z}$ is the denominator of u_i and m is the expected degree of the polynomial (in practice one starts with $m = 1$ and increases it until the interpolation is successful). In the current example $m = 2$ seems to work.

Table 5.2 contains the output of FindIntegerNullVector for the coefficient $a = a_1$, divided by the denominator of u^2. We denote this value by $\widetilde{a}$. It is clear that no polynomial can fit these data, but many of them $(\widetilde{a}^1, \widetilde{a}^3, \widetilde{a}^4, \widetilde{a}^6, \widetilde{a}^7, \widetilde{a}^9, \ldots)$ seem to lie (up to sign) on a curve. We fit a polynomial $q(u)$ to these data points. The desired polynomial might have higher degree than $q(u)$, but we hope that it will be a close approximation. This is given support by the fact that $q(u_i)/\widetilde{a}^i$ is an integer for all data points, and we take these integers as our multipliers k_i. The list of the 20 multipliers k_i is given in Table 5.2, alongside the $\widetilde{a}^i$. Interpolating $(u_i, k_i \cdot \widetilde{a}^i)_{1\leq i\leq 20}$ gives the polynomial $a(u) = 3u^2 + 6$. The same procedure for all the coefficients gives

$$2\sigma^2 = 3u^2 + 6 + 3\tau.$$

We conjecture that a basis over $\mathbb{Q}(u)$ for the trace field is $\{1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2\}$ and we go ahead checking it. The strategy is analogous as the above. For each trace

[2]One might need to experiment with different traces.

i	$\widetilde{a}^i$	k_i
1	-228.391	1
2	76.1287	-3
3	228.381	-1
4	-228.376	1
5	-76.1239	3
6	-228.367	1
7	-228.363	1
8	76.1193	-3
9	228.354	-1
10	228.349	-1

i	$\widetilde{a}^i$	k_i
11	76.115	-3
12	-228.341	1
13	-228.337	1
14	-76.111	3
15	-228.329	1
16	-228.325	1
17	-76.1071	3
18	228.318	-1
19	228.314	-1
20	-76.1034	3

Table 5.2: Interpolation data for the coefficient a_0 of the trace $\sigma = \mathrm{tr}(abac)$.

$\mathrm{tr}(abc) = u$	$\mathrm{tr}(abcac) = -\frac{u}{4}\sigma + \frac{1}{4u}\tau\sigma$
$\mathrm{tr}(abcabc) = \tau$	$\mathrm{tr}(acbcb) = \frac{-u^2\sigma+\sigma\tau}{4u}$
$\mathrm{tr}(abac) = \sigma$	$\mathrm{tr}(acabcb) = -\frac{1}{2}(3u^2+2) + \frac{1}{2}\tau$
$\mathrm{tr}(cacb) = \frac{1}{2}(u^2-2) - \frac{1}{2}\tau$	$\mathrm{tr}((ac)^2bcb) = \frac{19}{4}u - \frac{5}{8}u^3 + (u - \frac{3}{4u})\tau - \frac{3}{8u}\tau^2$

Table 5.3: Traces of the words necessary to determine the trace field.

t of (5.8) we run the integer relation detector FindIntegerNullVector on the vector $(t, 1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2)$ and interpolate the resulting coefficients over the data points. In this way we obtain a generic relation $p_0(u)t + p_1(u) + p_2(u)\tau + \ldots + p_6(u)\sigma\tau^2 = 0$, where each p_i is a polynomial in u with integer coefficients, and we record that $t = -\frac{p_1(u)}{p_0(u)} - \frac{p_2(u)}{p_0(u)}\tau - \ldots - \frac{p_6(u)}{p_0(u)}\sigma\tau^2$. The results for the eight traces of (5.8) are presented in Table 5.3.

Step 5. By Lemma 5.1.4 we can conjugate our family of representations ϕ_t in the normal form of Proposition 5.1.1, so that

$$\phi_t(a) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \phi_t(c) = \begin{pmatrix} 1 & 0 & -2k & 0 \\ 0 & 1 & 0 & 2(1-k) \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

with $k = \cos^2(\pi/8) = \frac{1}{2}(2+\sqrt{2})$. By Corollary 5.1.2 we can conjugate again preserving the normal form of $\phi_t(a)$ and $\phi_t(c)$, so that the $(1,2)$-entry of $\phi_t(b)$ is equal to 1 and all other entries are in the trace field (extended by $\sqrt{2}$).

So we use the proof of Proposition 5.1.1 to compute each matrix $\phi_t(b)$ (after conjugation) by solving two linear systems of equations which involve only some traces for which we have generic expressions in $\mathbb{Q}(\sqrt{2})(u)(\tau,\sigma)$ by the previous steps. Inserting these generic expressions and solving the systems gives a generic expression for b with

$$
\begin{aligned}
b_{11} &= \left((-4+2\sqrt{2}-2\sqrt{2}u+u^2)-\tau+\sigma\tfrac{1}{2}(u-4)-\tau\sigma\tfrac{1}{2u}\right)\tfrac{\sqrt{2}+1}{8}\\
b_{13} &= \left((2+2\sqrt{2}u-u^2)+\tau+\sigma\tfrac{1}{2}(2\sqrt{2}-u)+\tau\sigma\tfrac{1}{2u}\right)\tfrac{1+\sqrt{2}}{8}\\
b_{22} &= \left((4+2\sqrt{2}-2\sqrt{2}u-u^2)+\tau-\sigma\tfrac{1}{2}(u-4)+\tau\sigma\tfrac{1}{2u}\right)\tfrac{\sqrt{2}-1}{8}\\
b_{24} &= \left((2-2\sqrt{2}u-u^2)+\tau-\sigma\tfrac{1}{2}(2\sqrt{2}+u)+\tau\sigma\tfrac{1}{2u}\right)\tfrac{\sqrt{2}-1}{8}\\
b_{31} &= \left(u(u-4)-\tau+\sigma\tfrac{u-4}{\sqrt{2}}-\tau\sigma\tfrac{1}{\sqrt{2}u}\right)\tfrac{1}{4}\\
b_{33} &= \left((4-2\sqrt{2}+2\sqrt{2}u-u^2)+\tau-\sigma\tfrac{1}{2}(u-4)+\tau\sigma\tfrac{1}{2u}\right)\tfrac{\sqrt{2}+1}{8} = -b_{11}\\
b_{42} &= \left(-u(u-4)+\tau+\sigma\tfrac{u-4}{\sqrt{2}}-\tau\sigma\tfrac{1}{\sqrt{2}u}\right)\tfrac{1}{4}\\
b_{44} &= \left((-4-2\sqrt{2}+2\sqrt{2}u+u^2)-\tau+\sigma\tfrac{1}{2}(u-4)-\tau\sigma\tfrac{1}{2u}\right)\tfrac{\sqrt{2}-1}{8} = -b_{22}
\end{aligned}
$$

Table 5.4: Generic entries of b obtained by solving the first linear system of Proposition 5.1.1.

coefficients in $\mathbb{Q}(\sqrt{2})(u)(\tau,\sigma)$. Going into more detail, we can find an invertible matrix $M_1 \in \mathrm{GL}(8,\mathbb{Q}(\sqrt{2}))$ such that

$$
M_1 \cdot \begin{pmatrix} b_{11}\\ b_{13}\\ b_{22}\\ b_{24}\\ b_{31}\\ b_{33}\\ b_{42}\\ b_{44} \end{pmatrix} = \begin{pmatrix} \operatorname{tr} b\\ \operatorname{tr} cb\\ \operatorname{tr} ab\\ \operatorname{tr} abc\\ \operatorname{tr} acb\\ \operatorname{tr} abac\\ \operatorname{tr} cacb\\ \operatorname{tr} abcac \end{pmatrix}.
$$

We express the traces on the right-hand-side as elements of $\mathbb{Q}(u)(\tau,\sigma)$, solve the system and get expression for $b_{11},\ldots,b_{44}$ in $\mathbb{Q}(\sqrt{2})(u)(\tau,\sigma)$. These are given in Table 5.4.

Then we find another invertible matrix $M_2 \in \mathrm{GL}(16, \mathbb{Q}(\sqrt{2}))$ such that

$$M_2 \cdot \begin{pmatrix} b_{12}b_{21} \\ b_{14}b_{21} \\ b_{12}b_{23} \\ b_{14}b_{23} \\ b_{21}b_{32} \\ b_{23}b_{32} \\ b_{21}b_{34} \\ b_{23}b_{34} \\ b_{12}b_{41} \\ b_{14}b_{41} \\ b_{32}b_{41} \\ b_{34}b_{41} \\ b_{12}b_{43} \\ b_{14}b_{43} \\ b_{32}b_{43} \\ b_{34}b_{43} \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(b^2) \\ \mathrm{tr}(ab^2) \\ \mathrm{tr}(acb^2) \\ \mathrm{tr}(acab^2) \\ \mathrm{tr}((ac)^2b^2) \\ \mathrm{tr}((ac)^2ab^2) \\ \mathrm{tr}((ac)^3ab^2) \\ \mathrm{tr}(acbcb) \\ \mathrm{tr}(acabcb) \\ \mathrm{tr}((ac)^2bcb) \\ \mathrm{tr}((ac)^2abcb) \\ \mathrm{tr}((ac)^3abcb) \\ \mathrm{tr}(acbab) \\ \mathrm{tr}((ac)^2bab) \\ \mathrm{tr}((ac)^2abab) \\ \mathrm{tr}(acbacb) \end{pmatrix} + \text{terms in } \mathbb{R}(u)(\tau, \sigma). \tag{5.13}$$

We express the traces on the right-hand-side as elements of $\mathbb{Q}(u)(\tau, \sigma)$, solve the system and get expression for the products $b_{12}b_{21}, \ldots, b_{34}b_{43}$ in $\mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$. Since we are looking for an expression of b after a conjugation for which $b_{12} = 1$ we can recursively find all the remaining coefficients of b from the above products: we get directly the entries b_{21}, b_{23}, b_{41} and b_{43} and the remaining unknown entries b_{14}, b_{32} and b_{34} can be obtained multiplying by the inverse of b_{21}. The field $\mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$, as vector field over $\mathbb{Q}(\sqrt{2})(u)$, has basis

$$\{1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2\}.$$

We find expressions in this basis for the inverse of b_{21} using the following algebraic methods (cfr. [15, Example 7.3.8]). Let K be a field. Then any two nonzero polynomials $f, g \in K[x]$ have a greatest common divisor $h \in K[x]$ of the form $h = af + bg$ with $a, b \in K[x]$ [15, Theorem 1.8.6]. From this we deduce

Proposition 5.3.1. *Let α be algebraic over K with minimal polynomial $f(x) \in K[x]$ and $\beta \in K(\alpha) = K[\alpha]$ with $g(x) \in K[x]$ such that $g(\alpha) = \beta$. Then there are $a, b \in K[x]$ such that $1 = af + bg$. In particular, $1 = a(\alpha)f(\alpha) + b(\alpha)g(\alpha) = b(\alpha)g(\alpha)$ and $b(\alpha) \in K[\alpha]$ is the inverse of $g(\alpha) = \beta$.*

Proof. Let $h \in K[x]$ be the greatest common divisor of f, g of the form $h = af + bg$ for some $a, b \in K[x]$ [15, Theorem 1.8.6]. Since f is monic and irreducible, either $h = 1$ or g is a multiple of f. But since $f(\alpha) = 0$ and $0 \neq \beta = g(\alpha)$, it must hold $h = 1$. It follows $1 = a(\alpha)f(\alpha) + b(\alpha)g(\alpha) = b(\alpha)g(\alpha)$ and $b(\alpha) \in K[\alpha]$ is the inverse of $g(\alpha) = \beta$. □

The polynomials a, b as in Proposition 5.3.1 can be computed with the *Euclidean algorithm* using polynomial division with remainder. Suppose that $\deg(f) \geq \deg(g)$, then do repeated division with remainder each time obtaining a remainder of smaller degree:

$$\begin{aligned} f &= q_1 g + g_1 \\ g &= q_2 g_1 + g_2 \\ g_1 &= q_3 g_2 + g_3 \\ &\vdots \\ g_{r-2} &= q_r g_{r-1}. \end{aligned}$$

One can show [15, Proposition 1.6.11] that $g_{r-1} \neq 0$ is the greatest common divisor of g and f. Moreover, for $1 \leq i \leq r$, let $Q_i = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix}$ and set $Q = Q_1 \cdot \ldots \cdot Q_r \in \mathrm{GL}(2, K[x])$. Then

$$Q_{11} f + Q_{21} g = g_{r-1}$$

and thus $g_{r-1} \in K$ and $b = Q_{21}/g_{r-1}$.

In our setting, we apply the above to $K = \mathbb{Q}(\sqrt{2})(u)(\tau)$, $\alpha = \sigma$ (with minimal polynomial f over K of degree 2) and $\beta = b_{21} \in K[\sigma]$. The polynomial $g \in K[x]$ with $g(\sigma) = b_{21}$ is (necessarily) of degree 1, so the Euclidean algorithm stops with $g_1 \in \mathbb{Q}(\sqrt{2})(u)(\tau), q_1, q_2 \in \mathbb{Q}(\sqrt{2})(u)(\tau)[x]$. It holds $(Q_1 Q_2)_{21} = -q_1$, thus the inverse of b_{21} is $-q_1(\sigma)/g_1$.

The element $1/g_1 \in \mathbb{Q}(\sqrt{2})(u)(\tau)$, i.e. the inverse of g_1 in $\mathbb{Q}(\sqrt{2})(u)(\tau)$, can be computed in the same way, this time over the field $K = \mathbb{Q}(\sqrt{2})(u)$.

The matrix entries $b_{21}, b_{23}, b_{41}, b_{43}, b_{14}, b_{32}, b_{34}$ are given in Appendix C.

Step 6. In Step 5 we produced a generic matrix b with entries in $\mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$ which together with the matrices a, c in the standard form of Proposition 5.1.1 should describe a triangle group representation. We check using the formal algebra capabilities of Mathematica that indeed a, b, c satisfy the triangle group relations $b^2 = 1$ and $(ab)^3 = -\mathrm{id}$, $(bc)^4 = -\mathrm{id}$.

Step 7. The matrix b is not (anti)symplectic, because it is obtained after a conjugation which maps outside of $\mathrm{GSp}(4, \mathbb{R})$. The final task is therefore to conjugate back. As explained in Step 7 of the previous Section 5.2 we find $y \in \mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$ such that $b^T \Omega_y b = -\Omega_y$, where

$$\Omega_y = \begin{pmatrix} & & 1 & \\ & & & 1/y \\ -1 & & & \\ & -1/y & & \end{pmatrix}.$$

Notice that $1/y \in \mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$ as well. The expressions for y and $1/y$ are given in Appendix C. With $x = \pm\sqrt{y}$ and $q_x = \mathrm{diag}(1, x, 1, x)$ the matrix $q_x^{-1} b q_x$ is antisymplectic with respect to the standard form and together with a, c in normal form it generates a triangle group. The base representation is obtained for $x = -\sqrt{y}$. Finally we get an expression

$$b = \begin{pmatrix} b_{11} & b_{12}x & b_{13} & b_{14}x \\ b_{21}/x & b_{22} & b_{23}/x & b_{24} \\ b_{31} & b_{32}x & b_{33} & b_{34}x \\ b_{41}/x & b_{42} & b_{43}/x & b_{44} \end{pmatrix}$$

where the entries b_{ij} are the one obtained in Step 5 and are given in Appendix C.

5.4 About the parameters

We have a description of representations close to (a conjugate of) the base representation ϕ_0 for which the images of a and c are fixed and b varies in the parameters u, τ, σ, where $\tau = \tau(u)$ and $\sigma = \sigma(\tau)$ are such that

$$f(\tau) = \frac{1}{3}(32u^2 + 86u^4 - 5u^6) + (-20u^2 + \frac{13}{3}u^4)\tau + (2 - \frac{11}{3}u^2)\tau^2 + \tau^3 = 0,$$
$$g(\sigma) = \sigma^2 - \tfrac{1}{2}(3u^2 + 6 + 3\tau) = 0.$$

Any cubic root τ of f and any square root σ of g define a triangle group representation. Cubic roots are given explicitly by Cardano's formulas. Since we are interested in describing representations close to the base representation ϕ_0, we look at which values we get when we insert $u_0 = \mathrm{tr}(\phi_0(abc))$, and we see that $\sigma = +\sqrt{\tfrac{1}{2}(3u^2 + 6 + 3\tau)}$.

Remark 5.4.1 (Analysis of the parameter τ)*.* The roots of the cubic polynomial

$$f(x) = \frac{1}{3}(32u^2 + 86u^4 - 5u^6) + (-20u^2 + \frac{13}{3}u^4)x + (2 - \frac{11}{3}u^2)x^2 + x^3$$

are given according to Cardano's formula [36, §I.5] by

$$t_1(u) = S + T - \tfrac{1}{3}a_1$$
$$t_2(u) = -\tfrac{1}{2}(S + T) - \tfrac{1}{3}a_1 + \tfrac{1}{2}i\sqrt{3}(S - T)$$
$$t_3(u) = -\tfrac{1}{2}(S + T) - \tfrac{1}{3}a_1 - \tfrac{1}{2}i\sqrt{3}(S - T)$$

where $a_1 = 2 - \frac{11}{3}u^2$, $a_2 = -20u^2 + \frac{13}{3}u^4$, $a_3 = \frac{1}{3}(32u^2 + 86u^4 - 5u^6)$ and

$$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54},$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}.$$

The expression $\Delta = Q^3 + R^2$ is the discriminant of $f(x)$, which in this case is given by

$$\Delta = -\tfrac{256}{2187}(u^2(-3 + u^2)^2(-3 - 74u^2 + u^4)).$$

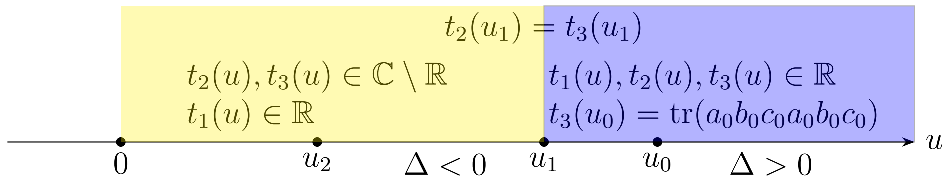


Figure 5.2: Description of the parameter $\tau(u)$.

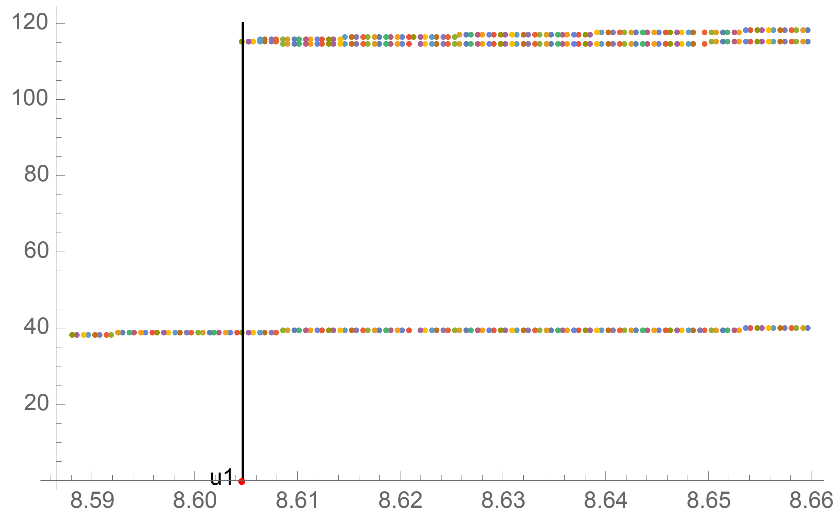
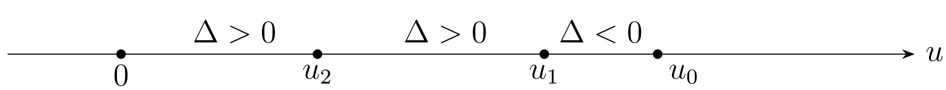


Figure 5.3: Roots of the minimal polynomial of the trace $\tau = \mathrm{tr}(abcabc)$.

Since for $u \in \mathbb{R}$ the cubic polynomial $f(x)$ has real coefficients it holds

$$\Delta = 0 \Leftrightarrow \text{ two roots of } f \text{ coincide,}$$
$$\Delta > 0 \Leftrightarrow f \text{ has one real root and two non-real conjugate roots,}$$
$$\Delta < 0 \Leftrightarrow f \text{ has three real roots.}$$

The real roots of Δ are $u_1 = \sqrt{37 + 14\sqrt{7}}$, $u_2 = \sqrt{3}$, 0, and $-u_2, -u_1$:



For $u \geq u_1$ all $t_1(u), t_2(u), t_3(u) \in \mathbb{R}$ and $t_3(u_0) = \mathrm{tr}(a_0 b_0 c_0 a_0 b_0 c_0)$. That is, the choice $\tau = t_3(u)$ describes the Hitchin component in b. When $-u_1 < u < u_1$ the only real root is $t_1(u)$ and $t_1(u_1) \neq t_3(u_1)$, which means that the function $t_3(u)$ describing the parameter τ can not be continuously continued for values smaller than u_1. The situation is represented in Figure 5.2. The three roots $t_1(u), t_2(u), t_3(u)$ nearby u_1 are printed out in Figure 5.3.

A. Group cohomology with twisted coefficients

Let G be a Lie group with Lie algebra $\mathfrak{g}$, Γ be a finitely generated group and $\rho : \Gamma \to G$ be a homomorphism. In this section we recall the definition of the cohomology groups $H^*(\Gamma, \mathfrak{g})_\rho$ with coefficients in $\mathfrak{g}$ twisted by the Adjoint action of ρ. Then we describe how these groups behave when we restrict ρ to a finite index subgroup $\Gamma' \leq \Gamma$ and when we project G onto $PG := G/C(G)$. That is, we express the relation between $H^*(\Gamma, \mathfrak{g})_\rho$ and $H^*(\Gamma', \mathfrak{g})_{\rho\circ i}$ and $H^*(\Gamma, \mathrm{Lie}(PG))_{\pi\circ\rho}$, where $i : \Gamma' \to \Gamma$ is the inclusion of the finite index subgroup and $\pi : G \to G/C(G)$ is the projection.

A.1 Group cohomology with coefficient in a Γ-module

We first recall the definition of the group cohomology of Γ with coefficients in a Γ-module V. Let Γ be a group which acts on a module V via a homomorphism $\rho : \Gamma \to \mathrm{Hom}(V)$. We denote the action by $\gamma \cdot v := \rho(\gamma)(v)$. Then $H^*(\Gamma, V)_\rho$ is the cohomology of the complex $\left(\bar{C}^n(\Gamma, V), \bar{\partial}^n\right)_n$, where

$$\begin{aligned}\bar{C}^0(\Gamma, V) &= V,\\ \bar{C}^n(\Gamma, V) &= \{f : \Gamma^n \to V\},\end{aligned}$$

and the differential $\bar{\partial}^n : \bar{C}^n(\Gamma, V) \to \bar{C}^{n+1}(\Gamma, V)$ is given by

$$\begin{aligned}\bar{\partial}^0(v)(\gamma) &= \gamma \cdot v - v,\\ \bar{\partial}^n(f)(\gamma_1, \ldots, \gamma_{n+1}) &= \gamma_1 \cdot (f(\gamma_2, \ldots, \gamma_{n+1})) + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_{n+1})\\ &\quad + (-1)^{n+1} f(\gamma_1, \ldots, \gamma_n).\end{aligned}$$

An element $f \in \bar{C}^n(\Gamma, V)$ is a *n-cocycle* if $\bar{\partial}^n(f) = 0$. The space of n-cocycles is

$$Z^n(\Gamma, V) = \ker \bar{\partial}^n \subset \bar{C}^n(\Gamma, V).$$

An element $f \in \bar{C}^n(\Gamma, V)$ is a *n-coboundary* if there exists $g \in \bar{C}^{n-1}(\Gamma, V)$ with $\bar{\partial}^{n-1}(g) = f$. The space of n-coboundaries is

$$B^n(\Gamma, V) = \mathrm{im}\, \bar{\partial}^{n-1} \subset \bar{C}^n(\Gamma, V).$$

Since $\bar{\partial}^n \circ \bar{\partial}^{n-1} = 0$ it holds $B^n \subset Z^n$ and the group cohomology of Γ with coefficients in V is

$$H^n(\Gamma, V)_\rho = \ker \bar{\partial}^n / \operatorname{im} \bar{\partial}^{n-1} = Z^n/B^n.$$

We remark that the chain complex $\left(\bar{C}^*(\Gamma, V), \bar{\partial}^*\right)$ is called the *bar resolution*. It is isomorphic to the complex $\left(C^*(\Gamma, V)^\Gamma, \partial^*\right)$ of Γ-invariant functions $f : \Gamma^{n+1} \to V$, where Γ acts on $C^n(\Gamma, V)$ by

$$(\gamma \cdot f)(x_0, \dots, x_n) \coloneqq \gamma \cdot \left(f(\gamma^{-1}x_0, \dots, \gamma^{-1}x_n)\right),$$

with differential $\partial^n(f)(x_0, \dots, x_{n+1}) \coloneqq \sum_{i=0}^{n+1} (-1)^i f(x_0, \dots, \widehat{x_i}, \dots, x_{n+1})$.

The isomorphism is given by

$$\begin{aligned} \bar{C}^n(\Gamma, V) &\to C^n(\Gamma, V)^\Gamma \\ f &\mapsto \left((x_0, \dots, x_{n+1}) \mapsto x_0 \cdot \left(f(x_0^{-1}x_1, \dots, x_{n-1}^{-1}x_n)\right)\right). \end{aligned}$$

We remark the following properties of 1-cocycles.

Lemma A.1.1. *For every 1-cocyle $f \in Z^1(\Gamma, V)$ it holds*

(i) $f(e) = 0$.

(ii) For every $\gamma \in \Gamma$*:* $f(\gamma^{-1}) = \gamma^{-1} \cdot (-f(\gamma))$.

(iii) For all $\gamma_1, \dots \gamma_k \in \Gamma$

$$f(\prod_{i=1}^{k} \gamma_i) = f(\gamma_1) + \sum_{i=1}^{k-1} (\gamma_1 \dots \gamma_i) \cdot f(\gamma_{i+1}).$$

Proof. (i) Since $\overline{\partial}^1 f = 0$, we have $0 = \overline{\partial}^1(f)(e, e) = e \cdot f(e) - f(e \cdot e) + f(e) = f(e)$.

(ii) Since $\overline{\partial}^1 f = 0$, using (i) we have $0 = \overline{\partial}^1(f)(\gamma, \gamma^{-1}) = \gamma \cdot f(\gamma^{-1}) - f(\gamma\gamma^{-1}) + f(\gamma) = \gamma \cdot f(\gamma^{-1}) + f(\gamma)$ and thus $f(\gamma^{-1}) = \gamma^{-1} \cdot (-f(\gamma))$.

(iii) We proceed by induction. The claim holds for $k = 2$ by definition of 1-cocycle. Thus suppose that it holds for k. Then by induction and since f is a 1-cocyle

$$\begin{aligned} f(\prod_{i=1}^{k+1} \gamma_i) &= f(\prod_{i=1}^{k} \gamma_i) \cdot f(\gamma_{k+1})\gamma_{i+1}) \\ &= (\prod_{i=1}^{k} \gamma_i) \cdot f(\gamma_{k+1}) + f(\prod_{i=1}^{k} \gamma_i) \\ &= (\prod_{i=1}^{k} \gamma_i) \cdot f(\gamma_{k+1}) + f(\gamma_1) + \sum_{i=1}^{k-1} (\gamma_1 \dots \gamma_i) \cdot f(\gamma_{i+1}) \\ &= f(\gamma_1) + \sum_{i=1}^{k} (\gamma_1 \dots \gamma_i) \cdot f(\gamma_{i+1}). \end{aligned}$$

□

A.2 Group cohomology with twisted coefficients

Let G be a Lie group with Lie algebra $\mathfrak{g}$, Γ be a finitely generated group and $\phi : \Gamma \to G$ be a representation. Then using the adjoint representation of G we obtain a homomorphism $\mathrm{Ad}_G \circ \phi : \Gamma \to \mathrm{GL}(\mathfrak{g})$. We denote the space of n-cocycles by

$$Z^n(\Gamma, \mathfrak{g})_\phi = \{u : \Gamma^n \to \mathfrak{g} \mid \overline{\partial}^n(u) \equiv 0\},$$

and the space of n-coboundaries by

$$B^n(\Gamma, \mathfrak{g})_\phi = \{u : \Gamma^n \to \mathfrak{g} \mid \exists v : \Gamma^{n-1} \to \mathfrak{g} \text{ with } \overline{\partial}^{n-1}(v) = u\}.$$

The resulting cohomology group is called the *cohomology of* Γ *with coefficients twisted by the Adjoint representation* and we denote it by

$$H^*(\Gamma, \mathfrak{g})_\phi \coloneqq H^*(\Gamma, \mathfrak{g})_{\mathrm{Ad}_G \circ \phi}.$$

We denote by $Z_G(\phi)$ the centralizer in G of the representation ϕ, that is,

$$Z_G(\phi) = \{g \in G \mid g\phi(\gamma) = \phi(\gamma)g \,\forall \gamma \in \Gamma\}.$$

The Lie algebra of $Z_G(\phi)$ can be characterized with the kernels of the linear maps $\mathrm{Ad}_G(\phi(\gamma)) - \mathrm{id}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$, $\gamma \in \Gamma$. In fact we have

Lemma A.2.1. *For a subset $S \subseteq G$ denote by $Z_G(S) = \{g \in G \mid gs = sg \,\forall s \in S\}$ its centralizer. Then*

$$\mathrm{Lie}\, Z_G(S) = \bigcap_{s \in S} \ker(\mathrm{Ad}_G(s) - \mathrm{id}_\mathfrak{g}).$$

Proof. Let $\exp : \mathfrak{g} \to G$ denote the Lie group exponential map. Then we have

$$\begin{aligned}
\mathrm{Lie}\, Z_G(S) &= \{X \in \mathfrak{g} \mid \exp tX \in Z_G(S) \,\forall t \in \mathbb{R}\} \\
&= \{X \in \mathfrak{g} \mid s \exp tX s^{-1} = \exp tX \,\forall s \in S, \forall t \in \mathbb{R}\} \\
&\subseteq \{X \in \mathfrak{g} : \frac{d}{dt}\Big|_{t=0} s \exp tX s^{-1} = \frac{d}{dt}\Big|_{t=0} \exp tX \,\forall s \in S\} \\
&= \{X \in \mathfrak{g} : d_e c_s X = X \,\forall s \in S\} \\
&= \{X \in \mathfrak{g} : \mathrm{Ad}_G(s) X = X \,\forall s \in S\}.
\end{aligned}$$

The inclusion is in fact an equality: Suppose that $\frac{d}{dt}\big|_{t=0} s \exp tX s^{-1} = \frac{d}{dt}\big|_{t=0} \exp tX$, this means that $t \mapsto \exp tX$ and $t \to s \exp tX s^{-1}$ have the same initial velocity. Since they are both one-parameter subgroups of G, they must agree. □

We deduce the following characterization of the 0th cohomology group.

Proposition A.2.2. *$H^0(\Gamma, \mathfrak{g})_\phi = \mathrm{Lie}\, Z_G(\phi)$.*

Proof. By definition the space $H^0(\Gamma, \mathfrak{g})_\phi$ is the kernel of the linear map $\bar{\partial}^0 : \mathfrak{g} \to \overline{C}^1(\Gamma, \mathfrak{g})$ which for $X \in \mathfrak{g}$, $\gamma \in \Gamma$ is defined by $\overline{\partial}^0(X)(\gamma) = \mathrm{Ad}\,\phi(\gamma)X - X$. The result follows by Lemma A.2.1 applied to the subgroup $S = \phi(\Gamma) \subseteq G$. □

A.3 Group cohomology of finite groups

Lemma A.3.1. *Let F be a finite group acting on a module V via a homomorphism $\rho : F \to \mathrm{Hom}(V)$. Then $H^k(F,V)_\rho = (0)$ for all $k \geq 1$.*

Proof. We first notice that the chain complex $(C^*(F,V),\partial^*)_*$ is exact. Indeed, let $f \in C^k(F,V)$ be a cocycle, that is $f : F^{k+1} \to V$ so that for any $(x_0,\dots,x_{k+1}) \in F^{k+1}$

$$\partial^k f(x_0,\dots x_{k+1}) = \sum_{i=0}^{k+1}(-1)^i f(x_0,\dots,\widehat{x_i},\dots,x_{k+1}) = 0.$$

Fix $x_0 \in F$ and define $g : F^k \to V$ by $g(x_1,\dots,x_k) \coloneqq f(x_0,x_1,\dots,x_k)$. Then

$$\begin{aligned}
\partial^{k-1} g(x_1,\dots,x_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1} g(x_1,\dots,\widehat{x_i},\dots,x_{k+1}) \\
&= \sum_{i=1}^{k+1}(-1)^{i-1} f(x_0,x_1,\dots,\widehat{x_i},\dots,x_{k+1}) \\
&= \sum_{i=0}^{k+1}(-1)^{i-1} f(x_0,\dots,\widehat{x_i},\dots,x_{k+1}) + f(x_1,\dots,x_{k+1}) \\
&= -\partial^k f(x_0,\dots,x_{k+1}) + f(x_1,\dots,x_{k+1}) \\
&= f(x_1,\dots,x_{k+1}).
\end{aligned}$$

The group cohomology $H^*(F,V)_\rho$ is computed by the cochain complex $C^*(F,V)^F$ of F-invariant cochains. Let $f \in C^k(F,V)^F$ be a F-invariant coycle, then by the above there is $g \in C^{k-1}(F,V)$ such that $\partial^{k-1}(g) = f$. Define $\widetilde{g} \coloneqq \frac{1}{|F|}\sum_{x\in F} x \cdot g$. Then $\widetilde{g}$ is F-invariant and

$$\partial^{k-1}\widetilde{g} = \frac{1}{|F|}\sum_{x\in F}\partial^{k-1}(x\cdot g) = \frac{1}{|F|}\sum_{x\in F} x\cdot\partial^{k-1}(g) = \frac{1}{|F|}\sum_{x\in F} x\cdot f = f.$$

□

A.4 Group cohomology of finite index subgroup

Let $\Gamma' \leq \Gamma$ be a finite index subgroup and denote by $i : \Gamma' \to \Gamma$ the inclusion and let $\rho : \Gamma \to \mathrm{Hom}(V)$ be a homomorphism. Then the cochain map

$$\begin{aligned}
i^n : \bar{C}^n(\Gamma,V) &\to \bar{C}^n(\Gamma',V) \\
f &\mapsto f \circ (i \times \ldots \times i) = f\big|_{(\Gamma')^n}
\end{aligned}$$

induces a map in cohomology

$$H^n(i^*) : H^n(\Gamma,V)_\rho \to H^n(\Gamma',V)_{\rho\circ i}.$$

Proposition A.4.1. *$H^n(i^*): H^n(\Gamma, V)_\rho \to H^n(\Gamma', V)_{\rho\circ i}$ is injective in every degree n.*

Proof. Let $i: \Gamma' \hookrightarrow \Gamma$ be a subgroup. We want to show that the cochain map

$$i^n : C^n(\Gamma, V)^\Gamma \to C^n(\Gamma' V)^{\Gamma'}$$

induces an injective map in cohomology in every degree n. We write i^n are the composition of the inclusion

$$j^n : C^n(\Gamma, V)^\Gamma \hookrightarrow C^n(\Gamma, V)^\Gamma$$

with the restriction map

$$\begin{aligned} res^n : C^n(\Gamma, V)^{\Gamma'} &\to C^n(\Gamma', V)^{\Gamma'} \\ f &\mapsto f\big|_{\Gamma' \times \ldots \Gamma'}. \end{aligned}$$

The latter induces an isomorphism in cohomology. So we consider the commutative diagram

$$\begin{array}{ccc} C^n(\Gamma, V)^\Gamma & \overset{j^n}{\hookrightarrow} & C^n(\Gamma, V)^{\Gamma'} \\ & \searrow^{i^n} & \downarrow res^n \\ & & C^n(\Gamma', V)^{\Gamma'} \end{array}$$

which in cohomology induces

$$\begin{array}{ccc} H^n(\Gamma, V)_\rho & \xrightarrow{H^n(j^*)} & H^n(C^*(\Gamma, V)^{\Gamma'}) \\ & \searrow^{H^n(i^*)} & \cong \downarrow H^n(res^*) \\ & & H^n(\Gamma', V)_{\rho\circ i}. \end{array}$$

To show that $H^n(i^*)$ is injective it suffices to define a left-inverse $L^n : H^n(\Gamma', V)_{\rho\circ i} \to H^n(\Gamma, V)_\rho$ of $H^n(i^*)$, so that $L^n \circ H^n(i^*) = \mathrm{id}_{H^n(\Gamma,V)_\rho}$. By the commutativity of the above diagram it suffices to define a cochain map

$$T^n : C^n(\Gamma, V)^{\Gamma'} \to C^n(\Gamma, V)^\Gamma$$

such that $T^n \circ j^n = \mathrm{id}_{C^n(\Gamma,V)^\Gamma}$. Indeed, given such T^n we set $L^n := H^n(T^*) \circ H^n(res^*)^{-1}$ to obtain

$$L^n \circ H^n(i^*) = H^n(T^*) \circ H^n(res^*)^{-1} \circ H^n(i^*) \overset{\circlearrowleft}{=} H^n(T^*) \circ H^n(j^*) = H^n(T^* \circ j^*) = \mathrm{id}_{H^n(\Gamma,V)_\rho}.$$

We now define $T^n : C^n(\Gamma, V)^{\Gamma'} \to C^n(\Gamma, V)^\Gamma$. Let $k < \infty$ be the index of Γ' in Γ and choose representatives $\Gamma'\gamma_1, \ldots, \Gamma'\gamma_k$ so that $\Gamma' \setminus \Gamma = \{\Gamma'\gamma_1, \ldots, \Gamma'\gamma_k\}$, $\gamma_i \in \Gamma$. Given a Γ'-invariant map $f : \Gamma^{n+1} \to V$ we define for $(x_0, \ldots, x_n) \in \Gamma^{n+1}$

$$\begin{aligned} T^n(f)(x_0, \ldots, x_n) &:= \frac{1}{k} \sum_{i=1}^{k} (\gamma_i^{-1} \cdot f)(x_0, \ldots, x_n) \\ &= \frac{1}{k} \sum_{i=1}^{k} \gamma_i^{-1} \cdot (f(\gamma_i x_0, \ldots, \gamma_i x_n)). \end{aligned}$$

We have to check that

1. $T^n(f)$ is Γ-invariant.

2. T^n is a cochain map (i.e. it commutes with the differentials)

3. $T^n \circ j = \mathrm{id}_{C^n(\Gamma,V)^\Gamma}$.

1. Let $\gamma \in \Gamma$, $(x_0, \ldots, x_n) \in \Gamma^{n+1}$. Then

$$\begin{aligned}(\gamma \cdot T^n(f))(x_0, \ldots, x_n) &= \gamma \cdot \big(T^n(f)(\gamma^{-1}x_0, \ldots, \gamma^{-1}x_n)\big) \\ &= \frac{1}{k}\sum_{i=1}^{k} \gamma \cdot \big(\gamma_i^{-1} \cdot \big(f(\gamma_i\gamma^{-1}x_0, \ldots, \gamma_i\gamma^{-1}x_n)\big)\big) \\ &= \frac{1}{k}\sum_{i=1}^{k} \gamma\gamma_i^{-1} \cdot f(\gamma_i\gamma^{-1}x_0, \ldots, \gamma_i\gamma^{-1}x_n).\end{aligned}$$

We have $\{\Gamma'\gamma_i\gamma^{-1}\}_{i=1}^k = \{\Gamma'\gamma_j\}_{j=1}^k$, so for each i there is a unique j such that $\Gamma'\gamma_i\gamma^{-1} = \Gamma'\gamma_j$. This implies $\gamma_j(\gamma_i\gamma^{-1})^{-1} = \gamma_j\gamma\gamma_i^{-1} \in \Gamma'$. In particular, $((\gamma_j\gamma\gamma_i^{-1})^{-1})_*f = f$. Thus

$$\begin{aligned}f(\gamma_i\gamma^{-1}x_0, \ldots, \gamma_i\gamma^{-1}x_n) &= ((\gamma_j\gamma\gamma_i^{-1})^{-1})_*f(\gamma_i\gamma^{-1}x_0, \ldots, \gamma_i\gamma^{-1}x_n) \\ &= (\gamma_j\gamma\gamma_i^{-1})^{-1} \cdot \big(f(\gamma_j\gamma\gamma_i^{-1}\gamma_i\gamma^{-1}x_0, \ldots, \gamma_j\gamma\gamma_i^{-1}\gamma_i\gamma^{-1}x_n)\big) \\ &= \gamma_i\gamma^{-1}\gamma_j^{-1} \cdot (f(\gamma_j x_0, \ldots, \gamma_j x_n)).\end{aligned}$$

Inserting this back in the above summation we get

$$\begin{aligned}(\gamma \cdot T^n(f))(x_0, \ldots, x_n) &= \frac{1}{k}\sum_{i=1}^{k} \gamma\gamma_i^{-1} \cdot \gamma_i\gamma^{-1}\gamma_{j(i)}^{-1} \cdot f(\gamma_{j(i)}x_0, \ldots, \gamma_{j(i)}x_n) \\ &= \frac{1}{k}\sum_{i=1}^{k} \gamma_{j(i)}^{-1} \cdot \gamma_i\gamma^{-1}\gamma_{j(i)}^{-1} \cdot f(\gamma_{j(i)}x_0, \ldots, \gamma_{j(i)}x_n) \\ &= \frac{1}{k}\sum_{j=1}^{k} \gamma_j^{-1} \cdot f(\gamma_j x_0, \ldots, \gamma_j x_n) \\ &= T^n(f)(x_0, \ldots, x_n).\end{aligned}$$

2. Let $f \in C^{n-1}(\Gamma, V)^{\Gamma'}$ and $(x_0, \dots, x_n) \in \Gamma^{n+1}$, then

$$\begin{aligned}
T^n \circ \partial^{n-1}(f)(x_0, \dots, x_n) &= T^n(\partial^{n-1} f)(x_0, \dots, x_n) \\
&= \frac{1}{k} \sum_{i=1}^{k} \gamma_i^{-1} \cdot \partial^{n-1}(f)(\gamma_i x_0, \dots, \gamma_i x_n) \\
&= \frac{1}{k} \sum_{i=1}^{k} \gamma_i^{-1} \cdot \sum_{j=0}^{n} (-1)^j f(\gamma_i x_0, \dots, \widehat{\gamma_i x_j}, \dots, \gamma_i x_n) \\
&= \sum_{j=0}^{n} (-1)^j \frac{1}{k} \sum_{i=1}^{k} \gamma_i^{-1} f(\gamma_i x_0, \dots, \widehat{\gamma_i x_j}, \dots, \gamma_i x_n) \\
&= \sum_{j=0}^{n} (-1)^j T^{n-1}(f)(x_0, \dots, \widehat{x_j}, \dots, x_n) \\
&= \partial^{n-1}(T^{n-1}(f))(x_0, \dots, x_n) \\
&= \partial^{n-1} \circ T^{n-1}(f)(x_0, \dots, x_n).
\end{aligned}$$

3. Let $f \in C^n(\Gamma, V)^{\Gamma}$, $(x_0, \dots, x_n) \in \Gamma^{n+1}$. Then $\gamma_i \cdot f = f$ for all $i = 1, \dots, k$ and therefore

$$\begin{aligned}
(T^n \circ j)(f)(x_0, \dots, x_n) = T^n(f)(x_0, \dots, x_n) &= \frac{1}{k} \sum_{i=1}^{k} (\gamma_i^{-1} \cdot f)(x_0, \dots, x_n) \\
&= \frac{1}{k} \sum_{i=1}^{k} f(x_0, \dots, x_n) \\
&= f(x_0, \dots, x_n).
\end{aligned}$$

□

Corollary A.4.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $\imath : \pi_1(S) \hookrightarrow \Delta(p, q, r)$ be a surface group of finite index in $\Delta(p, q, r)$ and $\phi : \Delta(p, q, r) \to G$ be a homomorphism. Then for every $n \geq 0$*

$$H^n(i^*) : H^n(\Delta(p, q, r), \mathfrak{g}))_\phi \to H^n(\pi_1(S), \mathfrak{g})_\phi$$

is injective.

B. Symplectic actions, Lagrangian spaces and crossratio

In the context of representations of triangle groups into $\mathrm{GSp}(2n, \mathbb{R})$, Lagrangian subspaces emerge as the eigenspaces of the images of the generators. Studying the symplectic action on the set of (pairs, triples, quadruples of) Lagrangians turns out to be a fruitful method to understand the local behaviour (rigidity) of representations. For example, Burelle [5] shows that diagonally embedded triangle groups are locally rigid by exploiting the relationship between the composition of two (Lagrangian) reflections and the crossratio of the associated reflection subspaces (Proposition B.3.3). Another rigidity result is Lemma 5.1.4, in which we give a normal form for representations of dihedral groups which are closed to the (restriction of the) irreducible representation.

We remark that Lagrangian subspaces (with the symplectic action and invariants thereof) come into play also in the context of surface groups representations, in particular when studying maximal and Hitchin components. Very roughly, maximal representations of surface groups are characterized by the existence of an equivariant boundary map $\xi : \partial\mathbb{H}^2 \to \check{S}$ satisfying a certain *maximality* condition [6, Theorem 8], where $\check{S}$ is the Shilov boundary of the symmetric space associated to $G = \mathrm{PSp}(2n, \mathbb{R})$ and coincides with the set of Lagrangian subspaces of $\mathbb{R}^{2n}$. A triple of pairwise transverse Lagrangian subspaces is called *maximal* if its Maslov index (which is an integer between $-n$ and n) attains the maximal possible value. A *maximal representation* is one such that the associated boundary map ξ maps positively oriented triples of points in $\partial\mathbb{H}^2$ to maximal triples of Lagrangians.

The analysis of the symplectic action on Lagrangian subspaces and the related invariants leads to further descriptions of (maximal) representations $\pi_1(S) \to \mathrm{PSp}(2n, \mathbb{R})$, for example Alessandrini, Guichard, Rogozinnikov and Wienhard [1] characterize maximal representations as the positive locus of *framed symplectic representations* , Strubel [37] gives Fenchel-Nielsen coordinates for maximal representations, and Burger and Pozzetti [7] study representations over a real closed field $\mathbb{F}$ which admit a *maximal framing*, obtaining a Collar Lemma for maximal representations.

In this chapter we review the symplectic action on tuples of pairwise transverse Lagrangians, their Maslov index and crossratio. We explain how one associates to the geometric representation of a triangle group a quadruple of Lagrangian subspaces and describe its crossratio. Finally we combine the above with the local rigidity of dihedral groups to prove Lemma 5.1.4.

B.1 Symplectic action on Lagrangian subspaces

Let $\mathbb{R}^{2n}$ be endowed with the standard symplectic form $\omega(x,y) = \sum_{i=1}^{n} x_i y_{n+i} - x_{n+i} y_i$ which with respect to the standard basis $e_1, \ldots, e_{2n}$ is given by the matrix $\Omega_{2n} = \left(\begin{smallmatrix} 0_n & \mathrm{id}_n \\ -\mathrm{id}_n & 0_n \end{smallmatrix}\right)$. We denote by $\mathrm{Sp}(2n,\mathbb{R})$ the symplectic group, the subgroup of $\mathrm{GL}(2n,\mathbb{R})$ preserving ω and by $\mathrm{GSp}(2n,\mathbb{R})$ the *general* symplectic group, the subgroup of $\mathrm{GL}(2n,\mathbb{R})$ preserving ω *up to sign*. It is useful to record that an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\mathrm{(G)Sp}(2n,\mathbb{R})$ if and only if

$$\begin{cases} A^T C - C^T A = 0 \\ B^T D - D^T B = 0 \\ A^T D - C^T B = (\pm)1. \end{cases}$$

A *Lagrangian subspace* of $\mathbb{R}^{2n}$ is an n-dimensional subspace $l \subset \mathbb{R}^{2n}$ such that $\omega\big|_{l\times l} \equiv 0$. We denote the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ by $\mathcal{L}(2n)$.

We can associate to any $2n \times n$ matrix $M = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ of maximal rank the n-dimensional subspace of $\mathbb{R}^{2n}$ spanned by its columns (X, Y are $n \times n$ matrices). Right-multiplication by $\mathrm{GL}(n,\mathbb{R})$ identifies matrices which span the same subspace and $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ is Lagrangian if and only if $X^T Y$ is symmetric.

Let l_∞ be the Lagrangian subspace spanned by $e_1, \ldots, e_n$ and l_0 be the Lagrangian subspace spanned by $e_{n+1}, \ldots, e_{2n}$; in the above notation they are respectively $l_\infty = \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)$ and $l_0 = \left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)$. A subspace $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ is transverse to l_∞ if and only the matrix $\left(\begin{smallmatrix} X & \mathrm{id} \\ Y & 0 \end{smallmatrix}\right)$ is non-singular, that is, if Y is non-singular. Thus subspaces transverse to l_∞ admit a basis of the form $\left(\begin{smallmatrix} X \\ \mathrm{id} \end{smallmatrix}\right)$ and they are Lagrangian if and only if X is symmetric. Moreover, they are transverse to l_0 if and only if X is non-singular.

The symplectic group $\mathrm{GSp}(2n,\mathbb{R})$ acts on the set of Lagrangian subspaces by left multiplication. It is well known that the action is transitive and it is also transitive on pairs of transverse Lagrangians. We give a proof of this fact for completeness.

Lemma B.1.1. *The symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *acts transitively on pairs of transverse Lagrangian subspaces.*

Proof. Given two transverse Lagrangians l_1 and l_2 any basis $(v_1, \ldots, v_n)$ of l_1 can be extended (canonically) to a *symplectic* basis[1] $(v_1, \ldots, v_n, w_1, \ldots, w_n)$ of $\mathbb{R}^{2n}$ such that $(w_1, \ldots, w_n)$ is a basis of l_2. Let g be the matrix whose columns are the vectors $v_1, \ldots, v_n, w_1, \ldots, w_n$. Then $gl_\infty = l_1$, $gl_0 = l_2$ and g is symplectic since

$$g^T \Omega_{2n} g = \begin{pmatrix} \omega(v_i,v_j) & \omega(v_i,w_j) \\ -\omega(v_i,w_j) & \omega(v_i,w_j) \end{pmatrix} = \Omega_{2n}.$$

□

[1]A basis $(v_1, \ldots, v_n, w_1, \ldots, w_n)$ is symplectic if $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$.

For $0 \leq p, q \leq n$ such that $p+q=n$ let $I_{p,q} = \begin{pmatrix} \mathrm{id}_p & 0_{p\times q} \\ 0_{q\times p} & -\mathrm{id}_q \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R})$ and denote by $l_{p,q}$ the Lagrangian subspace spanned by the columns of $\begin{pmatrix} I_{p,q} \\ \mathrm{id} \end{pmatrix}$.

Lemma B.1.2. *Every triple (l_1, l_2, l_3) of pairwise transverse Lagrangian subspaces of $\mathbb{R}^{2n}$ is in the $\mathrm{Sp}(2n,\mathbb{R})$-orbit of $(l_0, l_{p,q}, l_\infty)$ for some $p+q=n$.*

Proof. Let (l_1, l_2, l_3) be pairwise transverse Lagrangians. By Lemma B.1.1 we can assume that $l_1 = l_0$ and $l_3 = l_\infty$, so that we consider the (l_0, l, l_∞) with l transverse to both l_0 and l_∞. In particular l has a basis of the form $\begin{pmatrix} X \\ \mathrm{id} \end{pmatrix}$ where X is symmetric and invertible. By Sylverster's theorem there is $S \in \mathrm{GL}(n,\mathbb{R})$ such that $SXS^T = I_{p,q}$ for some $p+q=n$. Then $h = \begin{pmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{pmatrix} \in \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(l_0, l_\infty)$ and $h\begin{pmatrix} X \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} SX \\ (S^T)^{-1} \end{pmatrix} = \begin{pmatrix} I_{p,q} \\ \mathrm{id} \end{pmatrix}$. □

Corollary B.1.3. *There are $n+1$ orbits of $\mathrm{Sp}(2n,\mathbb{R})$ in the set of triples of pairwise transverse Lagrangians.*

B.2 Maslov index and crossratio

A fundamental tool in the study of Lagrangian subspaces is the Maslov index.

Definition B.2.1. The *Maslov index* of a triple (l_1, l_2, l_3) of (not necessarily transverse) Lagrangian subspaces of $\mathbb{R}^{2n}$ is the signature of the quadratic form Q on the direct sum $l_1 \oplus l_2 \oplus l_3$ defined by

$$Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1).$$

We denote it by $M(l_1, l_2, l_3)$.

The map $M : \mathcal{L}(2n)^3 \to \mathbb{Z}$, $(l_1, l_2, l_3) \mapsto M(l_1, l_2, l_3)$ is called the *Maslov* or *Kashiwara cocycle*. It follows directly from the definition that M is alternating

$$M(l_1, l_2, l_3) = -M(l_2, l_1, l_3) = -M(l_1, l_3, l_2),$$

and invariant under $\mathrm{Sp}(2n,\mathbb{R})$. Let l_1, l_3 be two transverse Lagrangian subspaces, that is $\mathbb{R}^{2n} = l_1 \oplus l_3$. We denote by $p_{l_1}^{\|l_3} : \mathbb{R}^{2n} \to l_1$ the projection onto l_1 parallel to l_3. Then we have

Lemma B.2.2 ([24, Lemma 1.5.4])**.** *If $l_1, l_2, l_3 \in \mathcal{L}(\mathbb{R}^{2n})$ are pairwise transverse, then $M(l_1, l_2, l_3)$ is the signature of the quadratic form on l_2 defined by*

$$Q_{l_1,l_2,l_3}(v) = \omega(p_{l_1}^{\|l_3}(v), v).$$

Example B.2.3. $M(l_\infty, l_{p,q}, l_0) = p - q$.

Proof. Any $v \in l_{p,q}$ is of the form $v = \left(\begin{smallmatrix} I_{p,q} \\ \mathrm{id} \end{smallmatrix}\right)\alpha$ for some $\alpha \in \mathbb{R}^n$ (i.e. in the basis given by the columns of $\left(\begin{smallmatrix} I_{p,q} \\ \mathrm{id} \end{smallmatrix}\right)$ the vector v is given by α). By the decomposition $\mathbb{R}^{2n} = l_0 \oplus l_\infty$ there are $\beta, \gamma \in \mathbb{R}^n$ such that

$$\left(\begin{smallmatrix} I_{p,q} \\ \mathrm{id} \end{smallmatrix}\right)\alpha = \underbrace{\left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)\beta}_{p_{l_0}^{\| l_\infty}(v)} + \underbrace{\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)\gamma}_{p_{l_\infty}^{\| l_0}(v)}.$$

It must hold $\gamma = I_{p,q}\alpha$ and $\beta = \alpha$, which means that $p_{l_\infty}^{\| l_0}(v) = \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right) I_{p,q}\alpha$. It follows

$$Q_{l_1,l_2,l_3}(v) = \left(\begin{smallmatrix} I_{p,q}\alpha \\ 0 \end{smallmatrix}\right)^T \Omega \left(\begin{smallmatrix} I_{p,q}\alpha \\ \alpha \end{smallmatrix}\right) = \alpha^T I_{p,q}\alpha,$$

which means that in the basis given by the columns of $\left(\begin{smallmatrix} I_{p,q} \\ \mathrm{id} \end{smallmatrix}\right)$ the quadratic form Q_{l_1,l_2,l_3} is given by $I_{p,q}$ and has signature $p - q$. □

Corollary B.2.4. *The Maslov index classifies* $\mathrm{Sp}(2n,\mathbb{R})$ *orbits of triples of pairwise transverse Lagrangians. In fact,* (l_1, l_2, l_3) *is in the* $\mathrm{Sp}(2n,\mathbb{R})$*-orbit of* $(l_0, l_{p,n-p}, l_\infty)$ *if and only if* $M(l_1, l_2, l_3) = n - 2p$.

Proof. Let $m \in \mathbb{Z}$ be the Maslov index of (l_1, l_2, l_3). By Lemma B.1.2 there are $g \in \mathrm{Sp}(2n,\mathbb{R})$ and $0 \leq p \leq n$ such that $g(l_1, l_2, l_3) = (l_0, l_{p,n-p}, l_\infty)$. Since the Maslov index is $\mathrm{Sp}(2n,\mathbb{R})$ invariant, it follows by Example B.2.3 that

$$m = M(l_1, l_2, l_3) = M(l_0, l_{p,n-p}, l_\infty) = -M(l_\infty, l_{p,n-p}, l_0) = -(p - (n-p)) = n - 2p.$$

Therefore $p = (n-m)/2$ and the triple is in the orbit of $(l_0, l_{\frac{n-m}{2}, \frac{n+m}{2}}, l_\infty)$. □

To describe quadruples of Lagrangian subspaces we need to introduce the *crossratio*. We follow the approach of [7]. Let $(l_1, l_2, l_3, l_4) \in \mathcal{L}(2n)^4$ be such that l_1 is transverse to l_2 and l_3 is transverse to l_4.

Definition B.2.5. The *crossatio* of (l_1, l_2, l_3, l_4) is the endomorphism of l_1 defined by

$$R(l_1, l_2, l_3, l_4) = p_{l_1}^{\| l_2} \circ p_{l_4}^{\| l_3}\big|_{l_1}.$$

The crossatio has the following equivariance property: for all $g \in \mathrm{GL}(2n,\mathbb{R})$, we have

$$R(gl_1, gl_2, gl_3, gl_4) = gR(l_1, l_2, l_3, l_4)g^{-1}.$$

Fixing a basis for l_1 we have an explicit expression for R.

Lemma B.2.6 ([7]). *(i) Suppose that the columns of* $\left(\begin{smallmatrix} X_i \\ \mathrm{id} \end{smallmatrix}\right)$ *form a basis* $\mathcal{B}_i$ *for* l_i*. Then the expression for* $R(l_1, l_2, l_3, l_4)$ *with respect to the basis* $\mathcal{B}_1$ *of* l_1 *is given by*

$$R(l_1, l_2, l_3, l_4) = (X_1 - X_2)^{-1}(X_4 - X_2)(X_4 - X_3)^{-1}(X_1 - X_3).$$

(ii) Suppose that l_0, l, m, l_∞ are pairwise transverse. Then l and m admit bases of the form $=\left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)$ and $m = \left(\begin{smallmatrix} M \\ \mathrm{id} \end{smallmatrix}\right)$, respectively, and

$$R(l_0, l, m, l_\infty) = L^{-1}M.$$

Proof. We prove the second assertion. Since m, l are transverse to l_0 and l_∞ they have bases of the stated form, with M and L symmetric and invertible. By definition $R(l_0, l, m, l_\infty) = p_{l_0}^{\|l}\big|_{l_\infty} \circ p_{l_\infty}^{\|m}\big|_{l_0}$, so we compute the two projections with respect to the bases $\mathcal{B}_0 = \left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)$ of l_0 and $\mathcal{B}_\infty = \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)$ of l_∞.

Let $v \in l_\infty$ with coordinates $\alpha \in \mathbb{R}^n$, so that $v = \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)\alpha$. Since $l_0 \oplus l = \mathbb{R}^{2n}$ there are $\beta, \gamma \in \mathbb{R}^n$ such that

$$\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)\alpha = \left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)\beta + \left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)\gamma.$$

It must hold $\alpha = L\gamma$ and $\beta = -\gamma = -L^{-1}\alpha$. Therefore

$$p_{l_0}^{\|l}(\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)\alpha) = -\left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)L^{-1}\alpha.$$

For the other projection, let $w \in l_0$ with coordinates $\alpha \in \mathbb{R}^n$ and $\beta, \gamma \in \mathbb{R}^n$ such that

$$\left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)\alpha = \left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)\beta + \left(\begin{smallmatrix} M \\ \mathrm{id} \end{smallmatrix}\right)\gamma.$$

Then $\gamma = \alpha$ and $\beta = -M\gamma = -M\alpha$. Therefore

$$p_{l_\infty}^{\|m}(\left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)\alpha) = -\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)M\alpha.$$

This gives

$$p_{l_0}^{\|l} \circ p_{l_\infty}^{\|m}(\left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)\alpha) = p_{l_0}^{\|l}(-\left(\begin{smallmatrix} \mathrm{id} \\ 0 \end{smallmatrix}\right)M\alpha) = \left(\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right)L^{-1}M\alpha$$

which shows that with respect to the basis $\mathcal{B}_0$ of l_0, R is given by $L^{-1}M$. □

We end this section by analyzing the orbits of 4-tuples of pairwise transverse Lagrangians. We need the following linear algebra fact.

Lemma B.2.7 ([20, Table 4.5.15T])**.** *Let $A, B \in \mathrm{GL}(n, \mathbb{R})$ be symmetric. Then $A^{-1}B$ is real diagonalizable if and only if there is $S \in \mathrm{GL}(n, \mathbb{R})$ and D_1, D_2 diagonal matrices with $SAS^T = D_1$ and $SBS^T = D_2$.*

Proposition B.2.8. *Let $(l_1, l_2, l_3, l_4) \in \mathcal{L}(2n)^4$ be pairwise transverse and suppose that the crossratio $R(l_1, l_2, l_3, l_4)$ is diagonalizable with real eigenvalues (not necessarily distinct). Then the 4-tuple (l_1, l_2, l_3, l_4) is in the $\mathrm{Sp}(2n, \mathbb{R})$-orbit of $(l_0, l_{p,n-p}, l, l_\infty)$, where l is the column space of $\left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)$ with L invertible and diagonal.*

Proof. The diagonal $\mathrm{Sp}(2n, \mathbb{R})$-action conjugates the crossratio and by Corollary B.2.4 we can assume that $(l_1, l_2, l_3, l_4) = (l_0, l_{p,n-p}, l, l_\infty)$ for some $0 \le p \le n$. Since l is transverse to both l_0 and l_∞ it admits a basis $\left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)$ with L invertible and symmetric. By Lemma B.2.6 the crossratio is $I_{p,n-p}^{-1}L$ and it is diagonalizable. So according to

Lemma B.2.7 there is $S \in \mathrm{GL}(n, \mathbb{R})$ such that $SI_{p,n-p}S^T$ and SLS^T are diagonal. Set $g = \begin{pmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{pmatrix}$ which is symplectic, fixes l_0 and l_∞ and it holds

$$gl_{n,n-p} = \begin{pmatrix} SI_{p,n-p}S^T \\ \mathrm{id} \end{pmatrix}, \; gl = \begin{pmatrix} SLS^T \\ \mathrm{id} \end{pmatrix}.$$

Thus to conclude the proof it suffices to show that we can choose S such that $SI_{p,n-p}S^T = I_{p,n-p}$. By Sylvester's theorem $SI_{p,n-p}S^T$ has p positive diagonal entries and the rest are negative. We can find a permutation matrix P which reorders them so that the first p diagonal entries $r_1, \dots, r_p$ are positive and the other ones are negative. Now let $R \coloneqq \mathrm{diag}(\frac{1}{\sqrt{r_1}}, \dots, \frac{1}{\sqrt{r_p}}, \frac{1}{\sqrt{|r_{p+1}|}}, \dots, \frac{1}{\sqrt{|r_n|}})$. Then

$$RPSI_{p,q}S^TP^TR^T = R\,\mathrm{diag}(r_1, \dots, r_n)R^T = I_{p,n-p}.$$

Since RP only permutes and multiplies the diagonal entries of SLS^T, the matrix $(RPS)L(RPS)^T$ is still diagonal and we can take RPS instead of S to get the desired result. $\square$

B.3 Eigenspaces of triangle group representations

Let $\Delta(p,q,r) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^r = 1 \rangle$ be a hyperbolic triangle group, so that $1/p + 1/q + 1/r < 1$. Let $a_0, b_0, c_0 \in \mathrm{SL}^{\pm}(2, \mathbb{R})$ be the images of a, b, c under the geometric representation $\rho_0 : \Delta(p,q,r) \to \mathrm{GL}(2, \mathbb{R})$ given in (2.1) and $\pi_{2n} : \mathrm{SL}^{\pm}(2, \mathbb{R}) \to \mathrm{GSp}(2n, \mathbb{R})$ be the irreducible representation (see Section 2.3). Then $A_0 = \pi_{2n}(a_0)$, $B_0 = \pi_{2n}(b_0)$ and $C_0 = \pi_{2n}(c_0)$ are lifts in $\mathrm{GSp}(2n, \mathbb{R})$ of $\phi_0(a), \phi_0(b)$ and $\phi_0(c)$ respectively, where $\phi_0 : \Delta(p,q,r) \to \mathrm{PGSp}(2n, \mathbb{R})$ denotes the base representation (2.6).

Proposition B.3.1. *(i) A_0, B_0 and C_0 are antisymplectic and satisfy $A_0^2 = B_0^2 = C_0^2 = \mathrm{id}$.*

(ii) The set of eigenvalues of A_0B_0 is $\{e^{\frac{\pi i}{p}(2n-(2j+1))} \mid j = 0, \dots 2n-1\}$.
The set of eigenvalues of B_0C_0 is $\{-e^{\frac{\pi i}{q}(2n-(2j+1))} \mid j = 0, \dots 2n-1\}$.
The set of eigenvalues of A_0C_0 is $\{e^{\frac{\pi i}{r}(2n-(2j+1))} \mid j = 0, \dots 2n-1\}$.

(iii) $(A_0B_0)^p = (A_0C_0)^r = -\mathrm{id}$ *and* $(B_0C_0)^q = \begin{cases} \mathrm{id} & \text{if } q \text{ is odd}, \\ -\mathrm{id} & \text{if } q \text{ is even}. \end{cases}$

Proof. It is easy to compute from (2.1) that $a_0^2 = b_0^2 = c_0^2 = \mathrm{id}$ and that a_0, b_0, c_0 have determinant -1. Since π_{2n} is a homomorphism it holds $A_0^2 = B_0^2 = C_0^2 = \mathrm{id}$ and (2.4) shows that A_0, B_0, C_0 are antisymplectic. This proves (i).

For (ii) and (iii), since the three elements a_0b_0, b_0c_0, c_0a_0 are diagonalizable (over $\mathbb{C}$), it suffices to study their eigenvalues and the image under the irreducible representation

π_{2n} of a diagonal element. Recall that the $2n$-irreducible representation is defined by the action of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ on the vector space of homogeneous polynomials $P_{2n-1}[X,Y]$ which for diagonal elements $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \mathrm{SL}^{\pm}(2,\mathbb{R})$ is given by

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} X^{2n-1-j}Y^j = (a^{2n-1-j}d^j)X^{2n-1-j}Y^j, \qquad \text{for } j = 0,\dots,2n-1.$$

Notice that as j varies the expression is symmetric in a and d. We prove the statements for A_0C_0, the argument for other products being analogous. We saw in Remark 2.2.1 that the eigenvalues of a_0c_0 are $e^{i\pi/r}, e^{-i\pi/r}$. Therefore the eigenvalues of $A_0C_0 = \pi_{2n}(a_0c_0)$ are

$$(e^{i\pi/r})^{2n-1-j}(e^{-i\pi/r})^j = e^{(i\pi/r)(2n-2j-1)}$$

for $j = 0,\dots,2n-1$. This implies that the eigenvalues of $(A_0C_0)^r$ are $e^{2\pi(n-j)i}e^{-i\pi} = -1$. □

Corollary B.3.2. *$A_0, B_0, C_0 \in \mathrm{GSp}(2n,\mathbb{R})$ have eigenvalues 1 and -1 and the corresponding eigenspaces are Lagrangian subspaces of $\mathbb{R}^{2n}$.*

Proof. Let x be any of A_0, B_0, C_0. By Proposition B.3.1 $x^2 = \mathrm{id}$ and x is antisymmetric. Let λ be an eigenvalue of x with eigenvector v. Then $\lambda = \pm 1$ because $v = x^2v = \lambda^2 v$. Let $E_1(x)$ and $E_{-1}(x)$ be the corresponding eigenspaces. We show that neither of them is trivial. Indeed, since $x \neq \pm\mathrm{id}$ there are $u, w \in \mathbb{R}^{2n}$ such that $xw - w \neq 0$ and $xu + u \neq 0$, and these are the desired eigenvectors because $x(xw - w) = x^2w - xw = -(xw - w)$ and $x(xu + u) = x^2u + xu = u + xu$. To see that the eigenspaces are Lagrangian subspaces, let $v, w \in E_1(x)$. Since x is antisymmetric it holds

$$\omega(v,w) = \omega(xv,xw) = -\omega(v,w),$$

thus $\omega(v,w) = 0$. The proof for $E_{-1}(x)$ is identical. □

Let $l, m \in \mathcal{L}(2n)$ be transverse, so that $l \oplus m = \mathbb{R}^{2n}$ and denote by $p_l^{\|m} : \mathbb{R}^{2n} \to l$ the projection into l parallel to m, as in Section B.2. The reflection in the pair l, m is the map

$$R_l^m = p_l^{\|m} - p_m^{\|l}.$$

That is, R_l^m is the linear map which is the identity on l and minus the identity on m.

Notice that the images A_0, B_0, C_0 under the representation $\phi_0 : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ of the generators a, b, c are reflections in their eigenspaces.

The following result due to [5] relates the composition of two reflections with the crossratio.

Proposition B.3.3 ([5, Proposition 3.1])**.** *Let $(l_1,l_2,l_3,l_4) \in \mathcal{L}(2n)^4$ be pairwise transverse. The characteristic polynomial $p_T(\lambda)$ of the composition $T := R_{l_1}^{l_2} \circ R_{l_4}^{l_3} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is related to the characteristic polynomial $p_R(\lambda)$ of the crossratio $R = R(l_1,l_2,l_3,l_4) : l_1 \to l_1$ by the following equation:*

$$p_T(\lambda) = (-4\lambda)^n p_R\left(\frac{(1+\lambda)^2}{4\lambda}\right).$$

Proof. Since the $\mathrm{Sp}(2n,\mathbb{R})$ action on pairs of transverse Lagrangians is transitive and $R_{gl}^{gm} = gR_l^m g^{-1}$ we might assume without loss of generality that $(l_1, l_2, l_3, l_4) = (l_0, l, m, l_\infty)$, where l, m have bases given by the columns of $\left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} M \\ \mathrm{id} \end{smallmatrix}\right)$, respectively. The matrices representing the linear maps $R_{l_0}^l$ and $R_{l_\infty}^m$ with respect to the standard basis of $\mathbb{R}^{2n}$ are

$$R_{l_0}^l = \left(\begin{smallmatrix} -\mathrm{id} & 0 \\ -2L^{-1} & \mathrm{id} \end{smallmatrix}\right), \; R_{l_\infty}^m = \left(\begin{smallmatrix} \mathrm{id} & -2M \\ 0 & -\mathrm{id} \end{smallmatrix}\right).$$

Thus $R_{l_0}^l \circ R_m^{l_\infty} = \left(\begin{smallmatrix} -\mathrm{id} & 2M \\ -2L^{-1} & 4L^{-1}M-\mathrm{id} \end{smallmatrix}\right)$. Its characteristic polynomial is

$$\begin{aligned} p_T(\lambda) &= \det(T - \lambda\mathrm{id}) = \det\begin{pmatrix} -(1+\lambda)\mathrm{id} & 2M \\ -2L^{-1} & 4L^{-1}M-(\lambda+1)\mathrm{id} \end{pmatrix} \\ &= \det\big(\big(4L^{-1}M - (\lambda+1)\mathrm{id}\big)\left(-(1+\lambda)\right)\mathrm{id} + 4L^{-1}M\big) \\ &= \det\big(-4(1+\lambda)L^{-1}M + (1+\lambda)^2\mathrm{id} + 4L^{-1}M\big) \\ &= \det\big((1+\lambda)^2\mathrm{id} - 4\lambda L^{-1}M\big) \\ &= (-4\lambda)^n \det\Big(-\tfrac{(1+\lambda)^2}{4\lambda}\mathrm{id} + L^{-1}M\Big) \\ &= (-4\lambda)^n p_{L^{-1}M}\big(\tfrac{(1+\lambda)^2}{4\lambda}\big) \end{aligned}$$

The assertion now follows from Lemma B.2.6. □

Since the compositions A_0B_0, B_0C_0 and C_0A_0 are of finite order, we can describe the set of eigenvalues of the crossratio of the associated subspaces. We illustrate this for the composition C_0A_0.

Example B.3.4 (Crossratio of irreducible representation)**.** Consider the eigenspaces of $A_0 = \pi_{2n}(a_0)$ and $C_0 = \pi_{2n}(c_0)$:

$$\begin{aligned} A_0^+ &= E_1(A_0), \qquad & C_0^+ &= E_1(C_0), \\ A_0^- &= E_{-1}(A_0), \qquad & C_0^- &= E_{-1}(C_0). \end{aligned}$$

They are Lagrangian subspaces and we denote by R_0 the crossratio $R_0 = R(A_0^+, A_0^-, C_0^-, C_0^+)$.

By Proposition B.3.1(ii) the eigenvalues of A_0C_0 are $e^{i\theta_j}$ with $\theta_j = \frac{\pi}{r}(2n - (2j+1))$ for
$j = 0, \ldots, 2n-1$ and therefore by Proposition B.3.3 the eigenvalues of R_0 are

$$\frac{(1+e^{i\theta_j})^2}{4e^{i\theta_j}} = \tfrac{1}{4}(e^{-i\theta_j} + 2 + e^{i\theta_j}) = \tfrac{1}{2}(1 + \cos(\theta_j)) = \cos^2(\tfrac{\theta_j}{2}).$$

Since $\theta_{n-1-k} = -\theta_{n+k}$ for $k = 0, \ldots, n-1$, the set of eigenvalues of R_0 is

$$\{\cos^2\left(\tfrac{\pi}{2r}\right), \cos^2\left(\tfrac{3\pi}{2r}\right), \ldots, \cos^2\left(\tfrac{\pi}{2r}(2n-3)\right), \cos^2\left(\tfrac{\pi}{2r}(2n-1)\right)\}.$$

B.4 Standard form for dihedral representations

The goal is to prove Lemma 5.1.4, which states that given any deformation of the base representation ϕ_0 the images of two of the generators of $\Delta(p,q,r)$ can be conjugated into some normal form.

As before, let $\phi_0 : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ be the base representation and let $A_0, C_0 \in \mathrm{GSp}(2n,\mathbb{R})$ be lifts of $\phi_0(a), \phi_0(c)$ as in the beginning of Section B.3. By Corollary B.3.2 they have eigenvalues ± 1 with corresponding Lagrangian eigenspaces which we denote by

$$\begin{aligned} A_0^+ &= E_1(A_0), & C_0^+ &= E_1(C_0), \\ A_0^- &= E_{-1}(A_0), & C_0^- &= E_{-1}(C_0). \end{aligned}$$

It holds $(A_0C_0)^r = -\mathrm{id}$ and when $r > 3$ the four eigenspaces are pairwise transverse. We denote by M_0 the Maslov index $M_0 = M(A_0^+, A_0^-, C_0^+)$ and by R_0 the crossratio $R_0 = R(A_0^+, A_0^-, C_0^-, C_0^+)$, which is invertible and diagonalizable for $n = 2$ (see Example B.3.4). We do not know whether it is diagonalizable for $n > 2$. The following is a reformulation of Lemma 5.1.4 for general n.

Lemma B.4.1. *Let $\phi_t : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ be a continuous deformation of the base representation ϕ_0 and let $a_t, c_t \in \mathrm{GSp}(2n,\mathbb{R})$ be lifts of $\phi_t(a), \phi_t(c)$ starting at $A_0 = \pi_{2n}(a_0)$, $C_0 = \pi_{2n}(c_0)$. Then $A_t^2 = C_t^2 = \mathrm{id}$ and $(A_tC_t)^r = -\mathrm{id}$ and moreover if the crossratio R_0 is diagonalizable, then for each t there exists $g_t \in \mathrm{GSp}(2n,\mathbb{R})$ such that*

$$g_tA_tg_t^{-1} = \begin{pmatrix} -\mathrm{id}_n & 0 \\ -2I_{p_0,n-p_0} & \mathrm{id}_n \end{pmatrix},\ g_tC_tg_t^{-1} = \begin{pmatrix} \mathrm{id}_n & -2L \\ 0 & -\mathrm{id}_n \end{pmatrix}, \tag{B.1}$$

where $p_0 = \frac{1}{2}(n - M_0)$ and L is a $n \times n$ diagonal matrix whose entries satisfy

$$\{L_{11}, \ldots, L_{p_0p_0}, -L_{p_0+1,p_0+1}, \ldots, -L_{nn}\} = \{\cos^2\left(\tfrac{\pi}{2r}(2(n-k)-1)\right) \mid k = 0, \ldots, n-1\}.$$

Remark B.4.2 (Proof of Lemma 5.1.4). When $n = 2$ an explicit computation shows that the Maslov index of (A_0^+, A_0^-, C_0^+) is 0, therefore $p_0 = 1$ and

$$\begin{cases} L_{11} = \cos^2(\pi/(2r)) \\ L_{22} = -\cos^2(3\pi/(2r)) \end{cases} \quad \text{or} \quad \begin{cases} L_{11} = \cos^2(3\pi/(2r)) \\ L_{22} = -\cos^2(\pi/(2r)). \end{cases}$$

One can explicitly conjugate the base representation in the standard form of (B.1) to see that it holds the first case.

Proof. Let $\phi_t : \Delta(p,q,r) \to \mathrm{PGSp}(2n,\mathbb{R})$ be a deformation of ϕ_0 and let $A_t, C_t \in \mathrm{GSp}(2n,\mathbb{R})$ be continuous lifts of $\phi_t(a), \phi_t(c)$ starting at $A_0 = \pi_{2n}(a_0)$, $C_0 = \pi_{2n}(c_0)$.

Let $\varphi_t = \phi_t\big|_{\langle a,c\rangle} : \langle a,c\rangle \to \mathrm{PSp}(2n,\mathbb{R})$ be the restriction of ϕ_t to the subgroup of $\Delta(p,q,r)$ generated by a and c, which is isomorphic to the finite dihedral group D_r of order $2r$. By Corollary 3.3.4 φ_0 is locally rigid, that is, there is a family $(g_t)_t \subset \mathrm{Sp}(2n,\mathbb{R})$

such that $g_t\varphi_t(\cdot)g_t^{-1} = \varphi_0(\cdot)$. So it suffices to show that we can conjugate φ_0 into the standard form of (B.1).

By assumption the crossatio R_0 is diagonalizable, so by Proposition B.2.8 the 4-tuple $(A_0^+, A_0^-, C_0^-, C_0^+)$ is in the $\mathrm{Sp}(2n,\mathbb{R})$-orbit of $(l_0, l_{p_0,n-p_0}, l, l_\infty)$, where $p_0 = \frac{1}{2}(n - M_0)$ and there is a basis for l given by the columns of $\left(\begin{smallmatrix} L \\ \mathrm{id} \end{smallmatrix}\right)$ with L diagonal and invertible. Thus φ_0 is conjugate to the homomorpshim $\varphi : \langle a, c\rangle \to \mathrm{GSp}(2n,\mathbb{R})$ given by the reflections $\varphi(a) = R_{l_0}^{l_{p_0,n-p_0}}$, $\varphi(c) = R_{l_\infty}^{l}$, whose matrix expression with respect to the standard basis is

$$\varphi(a) = \begin{pmatrix} -\mathrm{id}_n & 0 \\ -2I_{p_0,n-p_0} & \mathrm{id}_n \end{pmatrix},\ \varphi(c) = \begin{pmatrix} \mathrm{id}_n & -2L \\ 0 & -\mathrm{id}_n \end{pmatrix}.$$

Thus φ_0 and hence all the φ_t can be conjugated into (B.1). We conclude the proof by showing that there are finitely many possibilities for the diagonal entries $L_{11}, \dots, L_{nn}$ of L. Indeed, the crossratio R_0 is conjugate to

$$R(l_0, l_{p_0,n-p_0}, l, l_\infty) = I_{p_0,n-p_0}^{-1} L = \mathrm{diag}(L_{11}, \dots, L_{pp}, -L_{p+1,p+1}, \dots, -L_{nn}),$$

which has eigenvalues $\{L_{11}, \dots, L_{pp}, -L_{p+1,p+1}, \dots, -L_{nn}\}$. On the other hand, by Example B.3.4 the eigenvalues of the crossratio R_0 are:

$$\left\{\cos^2\left(\tfrac{\pi}{2s}(2(n-k)-1)\right) \mid k = 0, \dots, n-1\right\}.$$

□

C. The tautological representation

The tautological base representation Ψ_u obtained in Chapter 5 is given on the generators a, b, c of $\Delta(p, q, r)$ by

$$\Psi_u(a) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$\Psi_u(b) = \begin{pmatrix} b_{11} & b_{12}x & b_{13} & b_{14}x \\ b_{21}/x & b_{22} & b_{23}/x & b_{24} \\ b_{31} & b_{32}x & b_{33} & b_{34}x \\ b_{41}/x & b_{42} & b_{43}/x & b_{44} \end{pmatrix}$$

$$\Psi_u(c) = \begin{pmatrix} 1 & 0 & -1-\sqrt{2}/2 & 0 \\ 0 & 1 & 0 & 1-\sqrt{2}/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The entries b_{ij} are given in Section C.1 below. The elements x and $1/x$ are given in Section C.2 below.

C.1 The entries b_{ij}

The coefficient b_{ij} are obtained in **Step 5** of Section 5.3. The entries $b_{11}, b_{13}, b_{22}, b_{24}, b_{31}, b_{33}, b_{42}, b_{44}$ are given in Table C.1.

$b_{11} = \left((-4+2\sqrt{2}-2\sqrt{2}u+u^2) - \tau + \sigma\frac{1}{2}(u-4) - \tau\sigma\frac{1}{2u}\right)\frac{\sqrt{2}+1}{8}$
$b_{13} = \left((2+2\sqrt{2}u-u^2) + \tau + \sigma\frac{1}{2}(2\sqrt{2}-u) + \tau\sigma\frac{1}{2u}\right)\frac{1+\sqrt{2}}{8}$
$b_{22} = \left((4+2\sqrt{2}-2\sqrt{2}u-u^2) + \tau - \sigma\frac{1}{2}(u-4) + \tau\sigma\frac{1}{2u}\right)\frac{\sqrt{2}-1}{8}$
$b_{24} = \left((2-2\sqrt{2}u-u^2) + \tau - \sigma\frac{1}{2}(2\sqrt{2}+u) + \tau\sigma\frac{1}{2u}\right)\frac{\sqrt{2}-1}{8}$
$b_{31} = \left(u(u-4) - \tau + \sigma\frac{u-4}{\sqrt{2}} - \tau\sigma\frac{1}{\sqrt{2}u}\right)\frac{1}{4}$
$b_{33} = \left((4-2\sqrt{2}+2\sqrt{2}u-u^2) + \tau - \sigma\frac{1}{2}(u-4) + \tau\sigma\frac{1}{2u}\right)\frac{\sqrt{2}+1}{8} = -b_{11}$
$b_{42} = \left(-u(u-4) + \tau + \sigma\frac{u-4}{\sqrt{2}} - \tau\sigma\frac{1}{\sqrt{2}u}\right)\frac{1}{4}$
$b_{44} = \left((-4-2\sqrt{2}+2\sqrt{2}u+u^2) - \tau + \sigma\frac{1}{2}(u-4) - \tau\sigma\frac{1}{2u}\right)\frac{\sqrt{2}-1}{8} = -b_{22}$

Table C.1: Eight entries of $\Psi_u(b)$ obtained in Step 5.

The remaining entries $b_{14}, b_{21}, b_{23}, b_{32}, b_{34}, b_{41}, b_{43}$ have a slightly more complicated expression. Each of them is an element of $\mathbb{Q}(\sqrt{2})(\tau,\sigma)$ which as vector space over $\mathbb{Q}(\sqrt{2})$ has basis $1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2$. Thus we can write each b_{ij} as

$$b_{ij} = \frac{1}{D_{ij}}\left(b_{ij0} + b_{ij1}\tau + b_{ij2}\tau^2 + b_{ij3}\sigma + b_{ij4}\sigma\tau + b_{ij5}\sigma\tau^2\right).$$

The coefficients D_{ij} and $b_{ijk} \in \mathbb{Q}(\sqrt{2})(u)$, $k = 0, \ldots, 5$, are given in Table C.2 below.

$b_{140} = -168(-82+75\sqrt{2})u + (23508-7926\sqrt{2})u^2 + (5560+5508\sqrt{2})u^3 + (-5110+1521\sqrt{2})u^4 - 12(201+83\sqrt{2})u^5 + (308-48\sqrt{2})u^6 + 236u^7 - 15(-2+\sqrt{2})u^8$
$b_{141} = 6(-462+273\sqrt{2}-6(106+15\sqrt{2})u+(368-278\sqrt{2})u^2+(476-16\sqrt{2})u^3+(-10+37\sqrt{2})u^4+2(-36+5\sqrt{2})u^5+4(-2+\sqrt{2})u^6)$
$b_{142} = -9(154-91\sqrt{2}-4(-27+7\sqrt{2})u+2(-4+7\sqrt{2})u^2+4(-5+\sqrt{2})u^3+(-2+\sqrt{2})u^4)$
$b_{143} = 2u(168(5+7\sqrt{2})+2(-716+1483\sqrt{2})u+10(-161+61\sqrt{2})u^2+(97-677\sqrt{2})u^3+(365-209\sqrt{2})u^4+(49+5\sqrt{2})u^5+5(-1+\sqrt{2})u^6)$
$b_{144} = -4(63(-4+9\sqrt{2})+9(-29+25\sqrt{2})u+(41-259\sqrt{2})u^2+(107-99\sqrt{2})u^3+7(3+2\sqrt{2})u^4+4(-1+\sqrt{2})u^5)$ $b_{145} = 6(11-43\sqrt{2}+(11-15\sqrt{2})u+(3+5\sqrt{2})u^2+(-1+\sqrt{2})u^3)$
$D_{14} = 32u(-3+u^2)(-511-188u+66u^2+28u^3+u^4)$

$b_{210} = 48u+54u^2+4u^3+3u^4-2u^5$
$b_{211} = -6-4u+4u^3$
$b_{212} = -3-2u$
$b_{213} = -u(-16+4u-4u^2+u^3)$
$b_{214} = 2(2-2u+u^2)$
$b_{215} = -1$
$D_{21} = 64u$

$b_{230} = 24u + 54u^2 + (100 - 68\sqrt{2})u^3 + 3u^4 + 4(-2 + \sqrt{2})u^5$
$b_{231} = -6 + 4(-7 + 5\sqrt{2})u - 8(-2 + \sqrt{2})u^3$
$b_{232} = -3 + 4(-2 + \sqrt{2})u$
$b_{233} = 2u(8 + (20 - 17\sqrt{2})u + 2u^2 + (-2 + \sqrt{2})u^3)$
$b_{234} = -2(4 - 5\sqrt{2} + 2u + 2(-2 + \sqrt{2})u^2)$
$b_{235} = 2(-2 + \sqrt{2})$
$D_{23} = 128(-2 + \sqrt{2})u$

$b_{320} = 6u(560(1 - 57\sqrt{2}) - 4(-27403 + 29272\sqrt{2})u + (142560 - 83514\sqrt{2})u^2 + (48894 - 22322\sqrt{2})u^3 + (-26329 + 15663\sqrt{2})u^4 + (-22692 + 14831\sqrt{2})u^5 + 4(13 + 43\sqrt{2})u^6 + (2576 - 1782\sqrt{2})u^7 + (117 - 89\sqrt{2})u^8 + (-98 + 69\sqrt{2})u^9)$
$b_{321} = -12(9030 - 8820\sqrt{2} + (18512 - 12369\sqrt{2})u + (5340 - 1687\sqrt{2})u^2 + (-14273 + 9707\sqrt{2})u^3 + (-7658 + 5063\sqrt{2})u^4 + (2332 - 1587\sqrt{2})u^5 - 7(-214 + 149\sqrt{2})u^6 + (-91 + 61\sqrt{2})u^7 + (-78 + 55\sqrt{2})u^8)$
$b_{322} = 6(210(-43 + 42\sqrt{2}) + (-13177 + 9399\sqrt{2})u + (-2608 + 1319\sqrt{2})u^2 + (2884 - 2058\sqrt{2})u^3 + (964 - 656\sqrt{2})u^4 + (-139 + 99\sqrt{2})u^5 + (-58 + 41\sqrt{2})u^6)$
$b_{323} = u(1680(-87 + 68\sqrt{2}) + 4(-31781 + 31872\sqrt{2})u + 28(-5599 + 3852\sqrt{2})u^2 + (-27527 + 8463\sqrt{2})u^3 + (63972 - 45339\sqrt{2})u^4 + (12324 - 7764\sqrt{2})u^5 + (-7738 + 5520\sqrt{2})u^6 + (-873 + 597\sqrt{2})u^7 + (334 - 237\sqrt{2})u^8)$
$b_{324} = 2(-210(19 + 12\sqrt{2}) - 6(-7027 + 5048\sqrt{2})u + 9(-449 + 721\sqrt{2})u^2 + (-29188 + 20931\sqrt{2})u^3 + (442 - 876\sqrt{2})u^4 + (5136 - 3672\sqrt{2})u^5 + (61 - 21\sqrt{2})u^6 + 7(-38 + 27\sqrt{2})u^7)$
$b_{325} = -3(-7795 + 6315\sqrt{2} + (-5084 + 3641\sqrt{2})u - 4(-410 + 333\sqrt{2})u^2 + (1186 - 848\sqrt{2})u^3 + (-69 + 57\sqrt{2})u^4 + (-66 + 47\sqrt{2})u^5)$
$D_{32} = 8u(-3 + u^2)(-25 + 2u^2)(-511 + u(-188 + u(66 + \ u(28 + u))))$

$b_{340} = 96180u - 55440\sqrt{2}u - 6(-55589 + 50324\sqrt{2})u^2 + (238888 - 182538\sqrt{2})u^3 + (67283 - 28038\sqrt{2})u^4 + 4(-9521 + 7929\sqrt{2})u^5 + 4(-11261 + 7281\sqrt{2})u^6 + 2(-1211 + 507\sqrt{2})u^7 + (5429 - 3684\sqrt{2})u^8 + (386 - 228\sqrt{2})u^9 + 30(-7 + 5\sqrt{2})u^10$
$b_{341} = -6(7245 - 6720Sqrt\sqrt{2} + (10302 - 7611\sqrt{2})u + (3666 - 977\sqrt{2})u^2 + 32(-255 + 194\sqrt{2})u^3 + (-5605 + 3624\sqrt{2})u^4 + (1228 - 1011\sqrt{2})u^5 + (1086 - 747\sqrt{2})u^6 + 38(-1 + \sqrt{2})u^7 + 8(-7 + 5\sqrt{2})u^8)$
$b_{342} = -9(35(69 - 64\sqrt{2}) - 4(-766 + 563\sqrt{2})u + (586 - 308\sqrt{2})u^2 + (-650 + 496\sqrt{2})u^3 + (-231 + 158\sqrt{2})u^4 - 6(-5 + 4\sqrt{2})u^5 - 2(-7 + 5\sqrt{2})u^6)$
$b_{343} = u(420(-216 + 121\sqrt{2}) + (-22152 + 7061\sqrt{2})u + (-32568 + 24229\sqrt{2})u^2 + 15(-1237 + 1128\sqrt{2})u^3 - 2(-9300 + 6227\sqrt{2})u^4 + (5460 - 4161\sqrt{2})u^5 + 6(-428 + 287\sqrt{2})u^6 + (-369 + 254\sqrt{2})u^7 + (120 - 85\sqrt{2})u^8)$
$b_{344} = 315(-56 + 53\sqrt{2}) - 9(-3192 + 1997\sqrt{2})u + (5382 - 5740\sqrt{2})u^2 + 48(-425 + 274\sqrt{2})u^3 + (-912 + 787\sqrt{2})u^4 + (3672 - 2471\sqrt{2})u^5 + (114 - 70\sqrt{2})u^6 + 8(-24 + 17\sqrt{2})u^7$
$b_{345} = -3(5(-507 + 356\sqrt{2}) + 42(-44 + 29\sqrt{2})u + (540 - 382\sqrt{2})u^2 + (432 - 295\sqrt{2})u^3 + (-21 + 16\sqrt{2})u^4 + (-24 + 17\sqrt{2})u^5)$
$D_{34} = 4u(-3 + u^2)(-25 + 2u^2)(-511 + u(-188 + u(66 + \ u(28 + u))))$

$b_{410} = -32(6+\sqrt{2})u^2 + 96(1+3\sqrt{2})u^3 + (276+250\sqrt{2})u^4 - 48(3+2\sqrt{2})u^5 + (18+5\sqrt{2})u^6$
$b_{411} = -96(1+\sqrt{2})u - 12(6+7\sqrt{2})u^2 + 48(4+3\sqrt{2})u^3 + (-42-13\sqrt{2})u^4$
$b_{412} = -6(2+\sqrt{2}) - 48(1+\sqrt{2})u + (30+11\sqrt{2})u^2$
$b_{413} = 4u^2(-16+4u-4u^2+u^3)$
$b_{414} = -8u(2-2u+u^2)$
$b_{415} = 4u$
$D_{41} = 256u^2$

$b_{430} = 160(3+2\sqrt{2})u^2 - 384u^3 - 336\sqrt{2}u^3 - 32u^4 - 112\sqrt{2}u^4 + 24(4+3\sqrt{2})u^5 - 4(5+2\sqrt{2})u^6 + 3(3+2\sqrt{2})(-4+u)u^2(12+u) + 3/2(3+2\sqrt{2})(-4+u)u^4(12+u)$
$b_{431} = 48(4+3\sqrt{2})u - 3(3+2\sqrt{2})(-4+u)u + 320u^2 + 240\sqrt{2}u^2 - 48(4+3\sqrt{2})u^3 - 3/2(3+2\sqrt{2})(-4+u)u^3 + 40u^4 + 16\sqrt{2}u^4 - 3(3+2\sqrt{2})u(12+u) + 3/2(3+2\sqrt{2})(-4+u)u^2(12+u) - 3/2(3+2\sqrt{2})u^3(12+u)3(3+2\sqrt{2}) + 24(4+3\sqrt{2})u - 3/2(3+2\sqrt{2})(-4+u)u + 3/2(3+2\sqrt{2})u^2 - 4(5+2\sqrt{2})u^2 - 3/2(3+2\sqrt{2})u(12+u)$
$b_{432} = 3(3+2\sqrt{2}) + 24(4+3\sqrt{2})u - 3/2(3+2\sqrt{2})(-4+u)u + 3/2(3+2\sqrt{2})u^2 - 4(5+2\sqrt{2})u^2 - 3/2(3+2\sqrt{2})u(12+u)$
$b_{433} = 32(2+\sqrt{2})u^2 - 16u^3 - 56\sqrt{2}u^3 + 8(2+\sqrt{2})u^4 - 4u^5$
$b_{434} = 16u + 24\sqrt{2}u - 8(2+\sqrt{2})u^2 + 8u^3$
$b_{435} = -4u$
$D_{43} = 256u^2$

Table C.2: Coefficients of $b_{14}, b_{21}, b_{23}, b_{32}, b_{34}, b_{41}, b_{43}$.

C.2 y and $1/y$ in $\mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$

The elements x and $1/x$ appearing in the matrix entries of b in (5.3) are

$$x = -\sqrt{y},$$
$$1/x = -\sqrt{1/y}.$$

Both y and $1/y$ are elements of the field $\mathbb{Q}(\sqrt{2})(u)(\tau, \sigma)$, which as vector field over $\mathbb{Q}(\sqrt{2})$ has basis $1, \tau, \tau^2, \sigma, \sigma\tau, \sigma\tau^2$. It holds

$$y = \tfrac{1}{64u}(m_0 + m_1\tau + m_2\tau^2 + m_3\sigma + m_4\sigma\tau + m_5\sigma\tau^2)$$

and

$$1/y = \tfrac{1}{D}(q_0 + q_1\tau + q_2\tau^2 + q_3\sigma + q_4\sigma\tau + q_5\sigma\tau^2),$$

where the coefficients $m_0, \ldots, m_5 \in \mathbb{Q}(\sqrt{2})(u)$ are given in Table C.3 and $D, q_0, \ldots, q_5 \in \mathbb{Q}(\sqrt{2})(u)$ are given in Table C.4.

$m_0 = u^2(78 + 72\sqrt{2} + (92 + 84\sqrt{2})u - 3(11 + 8\sqrt{2})u^2 + 2u^3)$
$m_1 = -6(5 + 4\sqrt{2}) - 4(11 + 9\sqrt{2})u + 12(4 + 3\sqrt{2})u^2 - 4u^3$
$m_2 = -15 - 12\sqrt{2} + 2u$
$m_3 = u(-8(2 + \sqrt{2}) + 2(2 + 7\sqrt{2})u - 2(2 + \sqrt{2})u^2 + u^3)$
$m_4 = -4 - 6\sqrt{2} + 2(2 + \sqrt{2})u - 2u^2$
$m_5 = 1$

Table C.3: Coefficients of y.

$D = (u(-3 + u^2)^2(25(1089 + 680\sqrt{2}) + 70(310 + 223\sqrt{2})u + (628 + 940\sqrt{2})u^2 - 4(859 + 610\sqrt{2})u^3 - (643 + 484\sqrt{2})u^4 + 2(68 + 47\sqrt{2})u^5 + (34 + 24\sqrt{2})u^6))$
$q_0 = 42336u - 54(3907 + 3460\sqrt{2})u^2 + 4(-765 + 14134\sqrt{2})u^3 + (32735 + 26432\sqrt{2})u^4 + (16826 - 32066\sqrt{2})u^5 + 2(-323 + 868\sqrt{2})u^6 + 32(-296 + 195\sqrt{2})u^7 + (711 - 900\sqrt{2})u^8 - 18(-73 + 23\sqrt{2})u^9 + 90(-1 + 2\sqrt{2})u^{10}$
$q_1 = 378(125 + 68\sqrt{2}) - 36(111 + 922\sqrt{2})u - 108(398 + 143\sqrt{2})u^2 + 4(-1593 + 6412\sqrt{2})u^3 + 2(5869 - 1724\sqrt{2})u^4 + 2(3022 - 2992\sqrt{2})u^5 + 4(-499 + 533\sqrt{2})u^6 + 2(-542 + 204\sqrt{2})u^7 - 144(-1 + 2\sqrt{2})u^8$
$q_2 = 189(125 + 68\sqrt{2}) - 6(297 + 541\sqrt{2})u + 12(-595 + 32\sqrt{2})u^2 + 300(-1 + 4\sqrt{2})u^3 - 3(-311 + 304\sqrt{2})u^4 - 3(-62 + 30\sqrt{2})u^5 - 3(18 - 36\sqrt{2})u^6$
$q_3 = -u(72(550 + 43\sqrt{2}) - 2(29878 + 7349\sqrt{2})u + 2(2914 + 8471\sqrt{2})u^2 + (21443 - 6984\sqrt{2})u^3 - 2(3474 + 4925\sqrt{2})u^4 + (-5998 + 3206\sqrt{2})u^5 + 2(702 + 577\sqrt{2})u^6 + (907 - 272\sqrt{2})u^7 + 10(-10 + 7\sqrt{2})u^8)$
$q_4 = 2(63(-138 + 25\sqrt{2}) + 9(578 + 575\sqrt{2})u + (10735 - 1588\sqrt{2})u^2 - 2(2270 + 1489\sqrt{2})u^3 + (-3968 + 711\sqrt{2})u^4 + 9(122 + 27\sqrt{2})u^5 + (531 - 74\sqrt{2})u^6 + 8(-10 + 7\sqrt{2})u^7)$
$q_5 = -3(2473 + 800\sqrt{2} - 2(422 + 19\sqrt{2})u - 2(451 + 72\sqrt{2})u^2 + (264 - 36\sqrt{2})u^3 + (105 + 4\sqrt{2})u^4 + 2(-10 + 7\sqrt{2})u^5)$

Table C.4: Coefficients of $1/y$.

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