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Tropical Feynman integration in the physical region

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The software `feyntrop` for direct numerical evaluation of Feynman integrals is presented. We focus on the underlying combinatorics and polytopal geometries facilitating these methods. Especially matroids, generalized permutohedra and normality are discussed in detail.

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1. Introduction

The evaluation and understanding of Feynman integrals play an important role in many areas of modern physics, for example in particle accelerator phenomenology [1], gravitational wave physics [2, 3], calculations of the magnetic moment of the muon [4] and critical exponents in statistical field theory [5]. Many modern numerical methods uses the canonical differential equation approach [6], see e.g. `AMFlow` [7], `DiffExp` [8] and `SeaSyde` [9]. Deriving the canonical differential equation is a potential bottleneck in these calculations which can be sidestepped using direct integration. Analytic integration can be performed in `HyperInt` [10] while numerical Monte Carlo techniques are for example implemented in `pySecDec` [11]. The program `feyntrop` [12] is of the latter type and integrates dimensionally regulated quasi-finite integrals numerically using Monte Carlo methods. The relevant sectors of integration are not determined using sector decomposition but relies on special properties of the Newton polytope of the integrand. If these polytopes are *generalized permutohedra*, the decomposition into relevant sectors is greatly simplified. Integrable singularities of the integrand are regulated with the *tropical approximation* [13] which also makes the integrand bounded from both above and below. For details on the tropical approximation, see [12, 14].

The program `feyntrop` has recently been used in [15] to numerically verify the canonical differential equation result for a four-point three-loop process with one massive leg. It has also been used in [16] to calculate Feynman integrals in ϕ^4-theory to 13 loops and beyond. The tropical way of thinking also sheds light on infrared singularities [17] and how to calculate entire amplitudes directly without using Feynman integrals at all [18].

2. Feynman Integrals and Generalized Hypergeometry

Feynman integrals have many different equivalent representations, each with its own advantages and disadvantages. Consider one-particle irreducible Feynman graphs $G := (E, V)$ with the number of cycles (loops) given by $L = |E| - |V| + 1$. The vertex set V has the disjoint partition $V = V_{\text{ext}} \sqcup V_{\text{int}}$ where each $u \in V_{\text{ext}}$ is assigned an external incoming momenta $p_u \in \mathbb{R}^{1,D-1}$. Each edge $e \in E$ is assigned a non-negative mass m_e.

For the purpose of direct numerical evaluation in `feyntrop` the following projective representation is used:

$$\mathcal{I} = \Gamma(\omega) \int_{\mathbb{P}_+^E} \phi \quad \text{with} \quad \phi = \left(\prod_{e \in E} \frac{x_e^{\nu_e}}{\Gamma(\nu_e)} \right) \frac{1}{\mathcal{U}(\boldsymbol{x})^{D/2}} \left(\frac{1}{\mathcal{V}(\boldsymbol{x}) - i\varepsilon \sum_{e \in E} x_e} \right)^{\omega} \Omega \,. \tag{1}$$

Where the integration domain is over the *projective simplex* $\mathbb{P}_+^E = \{\boldsymbol{x} = [x_1 : \cdots : x_{|E|}] \in \mathbb{RP}^{E-1} : x_e > 0\}$ with respect to its canonical Kronecker form

$$\Omega = \sum_{e=1}^{|E|} (-1)^{|E|-e} \frac{dx_1}{x_1} \wedge \cdots \wedge \widehat{\frac{dx_e}{x_e}} \wedge \cdots \wedge \frac{dx_{|E|}}{x_{|E|}} \,. \tag{2}$$

The *superficial degree of divergence* of the graph G are given by $\omega = \sum_{e \in E} \nu_e - DL/2$. We write $\mathcal{V}(\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x})/\mathcal{U}(\boldsymbol{x})$ as a shorthand for the quotient of the two *Symanzik polynomials* that is defined from the underlying graph G by

$$\mathcal{U}(\boldsymbol{x}) := \sum_T \prod_{e \notin T} x_e\,, \qquad \mathcal{F}(\boldsymbol{x}) := \mathcal{F}_0 + \mathcal{F}_m = -\sum_F p(F)^2 \prod_{e \notin F} x_e + \mathcal{U}(\boldsymbol{x}) \sum_{e \in E} m_e^2 x_e\,, \tag{3}$$

where we sum over all spanning trees T and all spanning two-forests F of G, and $p(F)^2$ is the squared momentum running between the two-forest components. By this definition, $\mathcal{U}$ and $\mathcal{F}$ are homogeneous of degree L, resp. $L+1$, and hence $\mathcal{V}$ is a homogeneous rational function of degree 1.

2.1 Contour deformation

In order to define the integral on the correct analytic branch we need to implement Feynman's causal $i\varepsilon$ prescription. This is done using a finite contour deformation respecting the projective invariance [19].

The deformation is given by the embedding $\iota_\lambda : \mathbb{P}_+^E \hookrightarrow \mathbb{CP}^{|E|-1}$:

$$\iota_\lambda : x_e \mapsto X_e := x_e \exp\left(-i\lambda \frac{\partial \mathcal{V}}{\partial x_e}(\boldsymbol{x})\right). \tag{4}$$

Since the boundary is characterized by $x_e = 0$, ι_λ does not change the boundary. Using Cauchy's theorem, the integral is independent of ι_λ as long as the deformation does not cross any poles of ϕ. Set

$$\mathcal{I} = \Gamma(\omega) \int_{\iota_\lambda(\mathbb{P}_+^E)} \phi = \Gamma(\omega) \int_{\mathbb{P}_+^E} \iota_\lambda^* \phi \tag{5}$$

where $\iota_\lambda^*\phi$ is the pull-back, it can be written with the Jacobian $\iota_\lambda^*\Omega = \det(\mathcal{J}_\lambda(\boldsymbol{x}))\Omega$ where

$$\mathcal{J}_\lambda(\boldsymbol{x})^{e,h} = \delta_{e,h} - i\lambda x_e \frac{\partial^2 \mathcal{V}}{\partial x_e \partial x_h}(\boldsymbol{x}) \text{ for all } e, h \in E. \tag{6}$$

The deformed Feynman integral can thus be written as

$$\mathcal{I} = \Gamma(\omega) \int_{\mathbb{P}_+^E} \iota_\lambda^* \phi = \Gamma(\omega) \int_{\mathbb{P}_+^E} \left(\prod_{e \in E} \frac{X_e^{\nu_e}}{\Gamma(\nu_e)} \right) \frac{\det \mathcal{J}_\lambda(\boldsymbol{x})}{\mathcal{U}(\boldsymbol{X})^{D/2} \cdot \mathcal{V}(\boldsymbol{X})^{\omega}} \Omega \tag{7}$$

where $\boldsymbol{X} = \iota_\lambda(\boldsymbol{x})$.

2.2 Generalized hypergeometry

Another useful representation is due to Lee and Pomeransky [20]:

$$\mathcal{I} = \frac{\Gamma(D/2)}{\Gamma(D/2 - \omega)} \int_0^\infty \left(\prod_{e \in E} \frac{x_e^{\nu_e} dx_e}{x_e \Gamma(\nu_e)} \right) \frac{1}{\mathcal{G}^{D/2}} \qquad \text{where} \qquad \mathcal{G} = \mathcal{U} + \mathcal{F}. \tag{8}$$

In this form, it is a generalized hypergeometric integral (Mellin transform) [21, 22] of the type studied by Passare and collaborators [23, 24]. This means that it satisfies a generalized hypergeometric system of partial differential equations in the sense of Gel'fand, Graev, Kapranov and Zelevinskiĭ (GGKZ, commonly shortened to GKZ) [25–29].

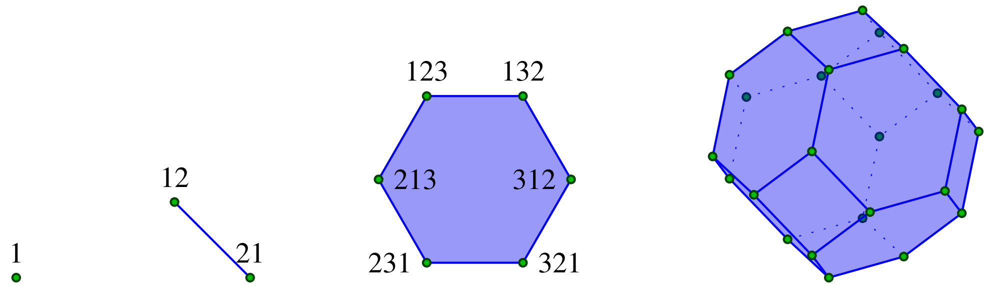


Figure 1: The first four permutohedra where it is easily seen from the explicit vertex coordinates (permutations) that all edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j$.

Using multi-index notation we may write the Lee-Pomeransky polynomial as $\mathcal{G} = \sum_{i=1}^{r} c_i x^{\alpha_i}$ with $c_i \neq 0$ and $\alpha_i \in \mathbb{Z}_{\geq 0}^{|E|}$ for all $i = 1, \ldots, r$. We define the two matrices

$$A := \{1\} \times A_- = \begin{pmatrix} 1 & 1 & \cdots & 1, \\ \alpha_1 & \alpha_2 & \cdots & \alpha_r \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{(|E|+1)\times r}, \text{ and} \tag{9}$$

$$\beta := \begin{pmatrix} -D/2, & -\nu_1 & , \ldots & , -\nu_{|E|} \end{pmatrix}^T \in \mathbb{C}^{|E|+1}, \tag{10}$$

from which we construct the GKZ hypergeometric system $H_A(\beta)$ as the sum of two ideals:

$$I_A := \left\langle \partial^u - \partial^v \mid u, v \in \mathbb{Z}_{\geq 0}^r \text{ s.t. } Au = Av \right\rangle, \quad \text{and} \tag{11}$$

$$Z_A(\beta) := \left\langle \Theta_i(c, \partial) \mid \Theta = A \cdot \begin{pmatrix} c_1\partial_1 \\ \vdots \\ c_r\partial_r \end{pmatrix} - \beta \right\rangle. \tag{12}$$

The ideal I_A is actually an ideal in the *commutative* polynomial ring $\mathbb{Q}[\partial_1, \cdots, \partial_r]$, and as such has a finite generating set $I_A = \langle h_1, \ldots h_\ell \rangle$ with $h_i \in \mathbb{Q}[\partial_1, \cdots, \partial_r]$. This ideal I_A is a *toric ideal* and it gives the defining equations of the projective *toric variety*

$$X_A = \{z \in \mathbb{P}^{r-1} \mid h_1(z) = \cdots = h_\ell(z) = 0\}$$

associated to the matrix A, see e.g. [30], [31, II, Chapter 5].

3. Generalized Permutohedra

The classical *perumtohedron* is a polytopal model of permutations, see Fig. 1. For n elements the permutohedron P_n is the $(n-1)$-dimensional polytope in $\mathbb{R}^n$ with vertices $(\sigma(1), \ldots, \sigma(n))$ where σ runs over all permutations of $[n] := \{1, 2, \ldots, n\}$. Every point in P_n satisfy $\sum_i x_i = n(n+1)/2$, meaning that P_n lies in a hyperplane and hence $\dim(P_n) = n - 1$. Note that every edge in P_n is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j$ where $\mathbf{e}_i$ denote the standard basis of $\mathbb{R}^n$.

In the theory of GKZ, the permutohedron appears as a *secondary polytope* $\Sigma(A)$, where A denotes the vertices of the triangular prism $Q = \Delta^1 \times \Delta^{n-1}$. The vertices (or equivalently, the top-dimensional normal cones) of $\Sigma(A)$ correspond to regular triangularizations of Q and also to

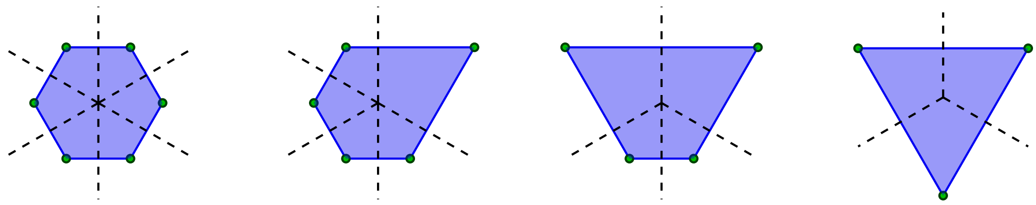


Figure 2: A permutohedron and the three deformations of it into generalized permutohedra, keeping the edge directions parallel to $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j$.

regions of convergence of series solutions to the generalized hypergeometric system defined by A [31].

As remarked above, the permutohedron has the property that all edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$, this is the defining property of a *generalized permutohedra* (GP) [32] (cf. Fig. 2) but can also be strengthened to define the *matroid polytope* [33, 34].

Definition 3.1 (Generalized permutohedron)**.** A polytope P is said to be a *generalized permutohedron* if all its edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j$.

Definition 3.2 (Matroid polytope)**.** A polytope P is said to be a *matroid polytope* if all its vertices lie in a hypersimplex and all edges are equal to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j$.

We remark that a matroid polytope is bijective to its associated matroid and that every matroid polytope is a generalized permutohedra. The geometry of toric varieties are closely connected to matroids [33, 34]:

Theorem 3.3 ([34, Lemma 1.4])**.** *The torus orbit of a point $p \in \mathbf{Gr}(k,n)$ is isomorphic to the toric variety defined by the matroid polytope of the representable matroid defined by the columns of the matrix of Steifel coordinates of p.*

Another important property that is more on the algebraic side is that of *normality*.

Definition 3.4. The semigroup $\mathbb{N}A$ is said to be *normal* if $\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_+ A$.

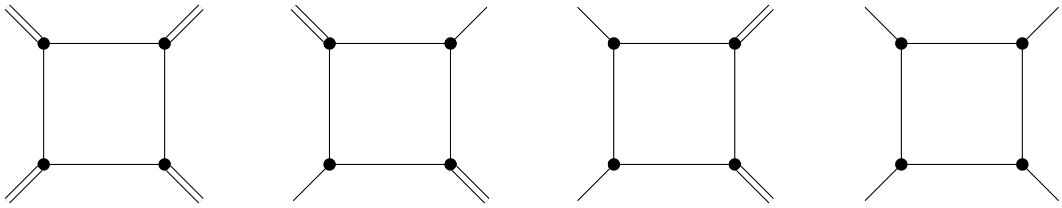
Normality is satisfied by all closed torus orbits in a Grassmannian [35]:

Corollary 3.5. *The closure of any torus orbit in a Grassmannian is projectively normal in its Plücker embedding.*

The importance of normality in the study of Feynman integrals comes in that it guarantees that characteristic variety and dimension of the solution space of the generalized hypergeometric system $H_A(\beta)$ are independent of the parameters β of the integral (that is, space-time dimension and propagator powers). Normality is in general stronger than this, what is actually of interest is that the toric ideal defined by A should be *Cohen-Macaulay* [29, 36], by a theorem of Hochster [37], normality implies the Cohen-Macaulay property.

Normality is connected to the GP property according to following theorem (cf. [38, Fig. 5]).

Theorem 3.6. *If P is a generalized permutohedron and $A = \mathbb{Z}^n \cap P$, then A is normal.*



(a) Off-shell box **(b)** Two-mass easy **(c)** Two-mass hard **(d)** On-shell

Figure 3: Box diagrams with all internal masses equal to zero and some external legs being off-shell ($p^2 \neq 0$, denoted by double lines) and some on-shell ($p^2 = 0$).

Example 3.7. We consider four different kinematic setups for the one-loop box integral with all internal masses equal to zero, see Fig. 3.

The A-matrix for the off-shell box is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{13}$$

which is precisely the vertices of the second hypersimplex $\Delta(2,5)$ and the toric variety $X_A = \mathbf{V}(I_A)$ is a Veronese-like embedding of $\mathbb{P}^4$. This variety is isomorphic to the orbit closure of a generic point in the Grassmannian $\mathbf{Gr}(2,5)$ under the natural mapping of $(\mathbb{C}^*)^5$. Also note that $\mathrm{conv}(A)$ is the basis polytope of the uniform matroid $U_{2,5}$. From this we know that $\mathbb{N}A$ is normal (hence I_A is Cohen-Macaulay) and $\mathrm{conv}(A)$ is a generalized permutohedron.

The three-mass box and two-mass easy, Fig. 3b, correspond to sub matroid strata and thus normality and the GP property follows directly.

However, the two-mass hard (Fig. 3c), one-mass and on-shell box (Fig. 3d) do not correspond to any matroid strata. This means that $\mathrm{conv}(A)$ is not a matroid polytope so neither normality nor the GP property follows directly. From the results in [38, Corollary 5.6] it follows that $\mathbb{N}A$ is normal, however, $\mathrm{conv}(A)$ is not a GP.

Below we summarize some of the known results for GP and normality, we always use $A = \{1\} \times \mathrm{Supp}(\mathcal{G})$ with $\mathcal{G} = \mathcal{U} + \mathcal{F}$ and $\mathbf{N}[f] = \mathrm{conv}(\mathrm{Supp}(f))$ as the Newton polytope of f.

- $\mathbf{N}[\mathcal{U}]$ is a matroid polytope, thus always a GP.
- For $m_e \neq 0$ for all $e \in E$, then $\mathbf{N}[\mathcal{F}] = \mathbf{N}[\mathcal{U}] + \Delta(1, |E|)$ and thus always a GP. Moreover, if no cancellation between $\mathcal{F}_0$ and $\mathcal{F}_m$ occurs, then A is normal [39].
- If no cancellation between $\mathcal{F}_0$ and $\mathcal{F}_m$ occurs, $p(V')^2 \neq 0$ for all $V' \subset V_{\mathrm{ext}}$ and every internal vertex is connected to an external vertex via a massive path, then $\mathbf{N}[\mathcal{F}]$ is a GP and A is normal [40] cf. [38].
- When $m_e = 0$ for all $e \in E$ and $V = V_{\mathrm{ext}}$, then $\mathbf{N}[\mathcal{F}]$ is a matroid polytope and thus GP and A is normal [39].

- When $m_e = 0$ for all $e \in E$ and $p(V')^2 \neq 0$ for all $V' \subset V_{\text{ext}}$, then $\mathbf{N}[\mathcal{F}]$ is a matroid polytope and thus GP and A is normal [40] cf. [38].

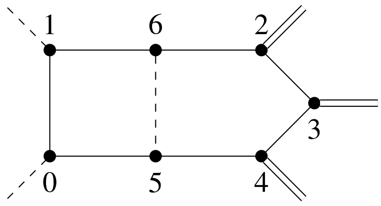
4. The program `feyntrop`

The program `feyntrop` is available at

https://github.com/michibo/feyntrop

It is a `C++` program with `Python` and `JSON` interface. It uses the contour deformation from section 2.1 and the sampling relies on the generalized permutohedron property, section 3.

Example 4.1 ([12, Section 6.5]). The following 5-point process with three massive external legs and a massive loop



can be evaluated to percent precision to five orders in the dimensional regulator ϵ on a laptop in two seconds.

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