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Author(s):
Marmulla, Gordana; Brandes, Ulrik

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Influence, inertia, and independence: a diffusion model for temporal social networks

Gordana Marmulla and Ulrik Brandes

Department of Humanities, Social and Political Sciences, Social Networks Lab, ETH Zürich, Zurich, Switzerland

ABSTRACT

In this work, we propose a diffusion model for temporal social networks and relate it to other well-known models of social influence by investigating its formal properties. The model establishes dyadic influence weights based on two antagonistic components: the susceptibility to be influenced (or, conversely, inertia with respect to the status quo) and becoming independent of prior influence. The proposed model generalizes the Friedkin-Johnsen model by the inertia with respect to the current influence relationships. We show that this generalization is an over-parameterization for static but not for dynamic influence networks. These findings suggest that the model at hand expands the set of existing social influence models in a non-trivial way.

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1. Introduction

Social influence is the process through which individuals change their thoughts, feelings, attitudes, or behavior by interacting with other individuals or groups (Rashotte, 2007). Theories of social psychology explain why there is social influence through conformity (Asch, 1956), cohesion (Coleman et al., 1957), power (French & Raven, 1959), and leadership (Katz & Lazarsfeld, 2017), among others. One way how influence can change attitudes, behavior, etc., is through assimilation: the target of influence becomes more similar to the source of influence. Assimilative change is supported by social psychological theories such as cognitive dissonance theory (Festinger, 1957), the laws of imitation (Tarde, 1903), or social learning theory (Bandura, 1977).

Any observable behavior is not only a response to stimuli but also a stimulus perceived by others (Newcomb, 1951), such that each individual can act as both the source and target of influence. In an influence network, where nodes represent individuals and (directed) edges represent (directed) influence, this

CONTACT Gordana Marmulla gmarmulla@ethz.ch Department of Humanities, Social and Political Sciences, Social Networks Lab, ETH Zürich, Weinbergstrasse 109, Zürich 8092, Switzerland

dual function enables influence to propagate through the network. The influence network is the infrastructure for the dynamics (Holme, 2015) of the behavioral states, and it can be static or itself dynamic. Both the influence network and the behavioral states exhibit a characteristic timescale whose relationship to each other is decisive for modeling. When the behavioral states evolve sufficiently fast compared to the dynamics of the interactions, there is no need to model the network as dynamic (Holme & Saramäki, 2012), and it is acceptable to assume the network to be static (Porter & Gleeson, 2016). On the other hand, if both timescales are similar, it is necessary to model the network and the behavioral states as concurrent, where the behavioral states could also feed back the network dynamics (so-called adaptive or coevolutionary networks (Gross & Blasius, 2008)). Models of social influence processes where influence evolves over time are expected to yield important insights (Mason et al., 2007). We adopt the perspective of a dynamic influence process on a *dynamic* influence network, but without a feedback mechanism from the behavioral states back to the influence network, i.e., without a selection process.

In this paper, we propose a diffusion model for temporal social networks which generalizes the Friedkin-Johnsen model (Friedkin & Johnsen, 1990). The model was recently used in a pilot study to model influence propagation in a temporal network of coaching relationships in the Australian Football League to identify the most influential coaches in terms of their influence on other coaches (Marmulla et al., 2023). The proposed model consists of two antagonistic components: the susceptibility of a node to be influenced or, conversely, the inertia with respect to the status quo (as a player), and becoming independent from prior influences (as a coach). Given these two components, the process establishes influence relationships between any two nodes based on a given temporal influence network and independence rates.

While the pilot study in Marmulla et al. (2023) is focused on an application, this paper focuses on formal properties of the model and relates it to the literature on social influence models. In this way, the paper aims to avoid the addition of yet another influence model without investigating its relation to other models, which is a central problem in the literature on social influence (Flache et al., 2017). The investigation is based on the evolution of *dyadic* influence relationships. This perspective is different from the default focus in the existing literature, which is predominantly on the evolution of the individual behaviors. For temporal networks in particular, the focus of the field has been more on what is diffusing through the network rather than the temporal networks themselves (Holme, 2015). However, as Friedkin himself noted after introducing the Friedkin-Johnsen model, it does not only describe a process of opinion formation but is actually an “elemental process model of social influence” (Friedkin, 1991, p. 1478).

Our main contributions are twofold. First, we propose a diffusion model that generalizes the model of Friedkin and Johnsen (1990) for opinion pooling with stubborn actors. The generalization comes from the inertia with respect to the current influence relationships, which is not present in the Friedkin-Johnsen model. This inertia can be considered as an equivalent of the "status quo bias" from the field of decision-making (Samuelson & Zeckhauser, 1988): the tendency of individuals to stick with the status quo. Second, we show that the found generalization reduces to an over-parametrization at equilibrium on a static influence structure. This finding means that under a static influence network, for almost every instance of the Friedkin-Johnsen model, there is a corresponding instance of the proposed model such that both coincide in the limit. However, under a dynamic influence structure, it is in general not possible to capture the inertia by a suitable instance of the Friedkin-Johnsen model.

The remainder of the paper is organized as follows. In Section 2, the diffusion model is outlined. Relations to the existing literature on the influence relationships are presented in Section 3. We conclude that the proposed model expands the set of social influence models and gives suggestions for future work in Section 4.

2. Model

The proposed diffusion process models how influence relationships between individuals are evolving. In contrast to the common focus of diffusion models on changes in actual behaviors, opinions, etc. (commonly referred to as states), we rather model the change in influence weights between individuals. Specifically, we make two general assumptions: the influence of one individual on another is independent of their states, and the influence relationships determine the state of each individual. Given these assumptions, the behavioral states can be neglected since their values do not affect the diffusion process itself, and it is sufficient to know the influence relationships. Having said this, the values of the states and their evolution are not the subject of this paper. However, the way the influence weights are modeled has consequences for the requirements regarding the specification of the states. These requirements are, in turn, relevant whenever the model is used to represent actual behavior, and are addressed at the end of the section.

In the model, the dyadic influence relationships evolve over discrete time steps through two antagonistic components: the individuals' susceptibility to influence (or, conversely, inertia with respect to the status quo), and becoming independent of past influence of others. At any moment, every node in the network is subject to accumulated influence from others expressed as a *distribution of influence weights*. The distribution associated with a node describes the share of influence received from the respective others and the share of its own independence. The extent to which influence and independence change the relationships of a node depends on the one hand on global parameters at the macro level and on the other hand on dyadic and individual weights at the micro level.

2.1. Influence and inertia

Influence is modeled as an assimilative influence. The influence relationships of an influenced node i become more similar to the influence relationships of an influencing node j. This means that node i is becoming more similar to j in terms of who it is influenced by. The assimilation is implemented as a weighted average of both distributions. The extent of assimilation is regulated by the product of two values: the general susceptibility to influence γ and a time-dependent dyadic weight $W_{ij}^{(t)}$. The parameter γ can range from zero (i.e., not susceptible at all) to one (i.e., totally susceptible). The quantity $1 - \gamma$ describes the general level of inertia to stick to the status quo, the current influence relationships.

2.2. Independence

Following the APA Dictionary of Psychology (American Psychological Association, 2023), we refer to independence as the freedom from the influence of other individuals. Becoming independent is modeled as decreasing the influence of others while increasing an individual's own independence. The decay of the prior influence is implemented by downscaling the distribution of a node i, where the amount lost due to the scaling is then added to the share of i's independence. The extent of independence is regulated by the product of two values: the general independence parameter $1 - \alpha$ and a time-dependent individual weight $E_{ii}^{(t)}$. The parameter $1 - \alpha$ can range from zero (i.e., no independence) to one (i.e., total independence).

2.3. Formalization

The novel diffusion process can be expressed as a recursive formula of operations on a directed and weighted adjacency matrix $\mathcal{I}^{(t)}$, which represents the influence relationships at time t:

$$\mathcal{I}^{(t)} = \underbrace{\left[I_n - (1-\alpha)E^{(t)}\right]}_{\substack{\text{decay of} \\ \text{influence}}} \underbrace{\left(\gamma W^{(t)} + (1-\gamma)I_n\right)\mathcal{I}^{(t-1)}}_{\substack{\text{influence propagation and} \\ \text{inertia to change the status quo}}} + \underbrace{(1-\alpha)E^{(t)}}_{\text{independence}} \tag{1}$$

with a row stochastic $W^{(t)}, \mathcal{I}^{(0)} \in [0,1]^{n \times n}$, diagonal $E^{(t)} \in [0,1]^{n \times n}$, parameters $\gamma, \alpha \in [0,1]$, and I_n as the identity matrix of size n, where n is the number of nodes. A single entry $\mathcal{I}_{ij}^{(t)}$ describes the share of influence node j has on i at time t. The ith row $\mathcal{I}^{(t)}_{i\star}$ represents the influence distribution of i. As a result, $\mathcal{I}^{(t)}$ is row stochastic.

The matrix $W^{(t)}$ contains the time-dependent dyadic influence weights. An entry $W^{(t)}_{ij}$ is larger than zero if j influences i at time t. The value $W^{(t)}_{ij} \in [0, 1]$ describes the relative weight with which j influences i. The matrix $W^{(t)}$ is assumed to be row stochastic, i.e., the influence weights on an individual follow a convex combination. The matrix

$$P^{(t)} = \gamma W^{(t)} + (1 - \gamma)I_n, \tag{2}$$

contains influence and the inertia toward that influence. As a weighted average of two row stochastic matrices ($W^{(t)}$ and I_n), $P^{(t)}$ is also row stochastic. There are three different kinds of non-zero entries in $P^{(t)}$:

$$P^{(t)}_{ij} = \begin{cases} \gamma W^{(t)}_{ij}, & \text{if } i \neq j & \text{influence} \\ (1-\gamma)\left[1 + \frac{\gamma}{1-\gamma} W^{(t)}_{ii}\right], & \text{if } i = j \text{ and } W^{(t)}_{ii} < 1 & \text{inertia} \\ 1, & \text{if } i = j \text{ and } W^{(t)}_{ii} = 1 & \text{preservation} \\ 0, & \text{otherwise} & \end{cases}. \tag{3}$$

The entries are, first, the influence of others. Second, the node's inertia toward this influence. Third, preservation entries ensuring that a node's influence distribution does not change, if no influence has happened. The maximum extent of influence is bounded from above by γ, which is reached when the dyadic weight is $W^{(t)}_{ij} = 1$. A weight of $W^{(t)}_{ij} = 1$ also means j is the only source of influence on i at time t. The minimum extent of i's inertia toward influence is bounded from below by $1 - \gamma$, which is reached when $W^{(t)}_{ii} = 0$.

The diagonal matrix $E^{(t)} \in [0, 1]^{n \times n}$ contains the time-dependent individual independence rates. An entry $E^{(t)}_{ii}$ is larger than zero if i becomes independent at time t. Therefore, the matrix $(1 - \alpha)E^{(t)}$ has only two different kinds of entries on its diagonal:

$$\left[(1-\alpha)E^{(t)}\right]_{ii} = \begin{cases} 0, & \text{if } E^{(t)}_{ii} = 0 \text{ or } \alpha = 1 & \text{preservation} \\ (1-\alpha)E^{(t)}_{ii} > 0, & \text{if } E^{(t)}_{ii} > 0 \text{ and } \alpha < 1 & \text{independence} \end{cases}. \tag{4}$$

The entries are, first, preservation such that a node's influence distribution does not change if the node has not become independent. Second, independence entries where a node i's influence distribution (i.e., row i of $P^{(t)}\mathcal{I}^{(t-1)}$) is decayed with factor $1 - (1 - \alpha)E^{(t)}_{ii}$ while $(1 - \alpha)E^{(t)}_{ii}$ is added to i's share of independence. The maximum growth of independence is bounded above by $1 - \alpha$, which is reached when $E^{(t)}_{ii} = 1$.

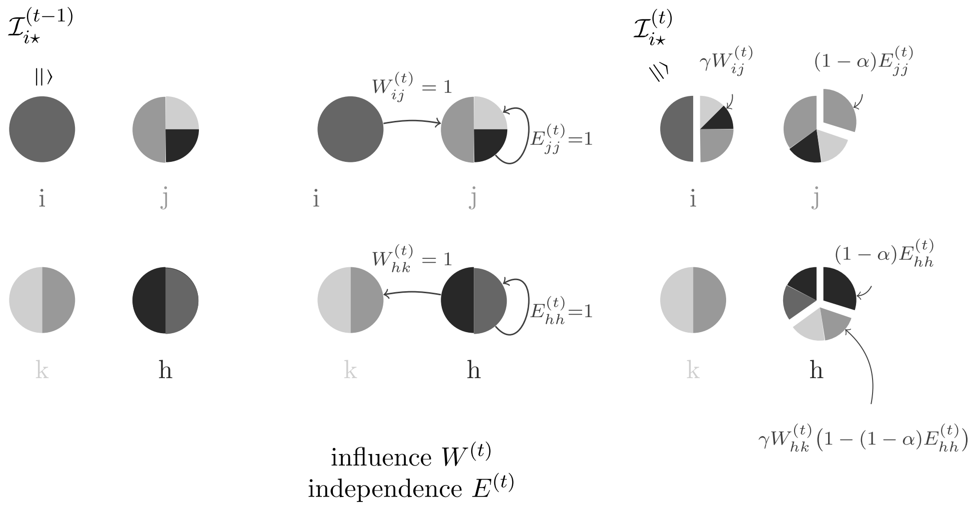


Figure 1. Visualization of exemplary influence relationship changes. The nodes' distributions are depicted by pie charts, i.e., one pie equals one row in $\mathcal{I}^{(\bullet)}$. Each node is associated with a unique color which is recognizable from the node's label. A pie slice in a particular color then represents the share of influence from the node with this individual color on the node associated with that pie. The distribution changes are caused by node j influencing i, j becoming independent, and h being influenced by k while simultaneously becoming independent. In this example, the global susceptibility is set to $\gamma = 0.5$, and the independence rate to $1 - \alpha = 0.3$.

A visualization of exemplary distribution changes when a node gets influenced (node i), becomes independent (node j), or both (node h) is illustrated in Figure 1.

2.4. Generalization

The proposed diffusion process can be generalized. So far, in the model, decreasing the influence of others on a node was synonymous with increasing the node's own independence. We can generalize the model by allowing not only for an increase in the own independence but also for an increase in the influence from any other node. At the formal level, this generalization can be described by

$$\mathcal{I}^{(t)} = \underbrace{\left[I_n - (1-\alpha)E^{(t)}\right]}_{\text{decay of influence}} \underbrace{P^{(t)}\mathcal{I}^{(t-1)}}_{\substack{\text{influence}\\ \text{propagation}}} + \underbrace{(1-\alpha)E^{(t)}D^{(t)}}_{\substack{\text{exogenous adjustment of}\\ \text{influence relationships}}} \tag{5}$$

with a diagonal $E^{(t)} \in [0,1]^{n\times n}$ and a row stochastic, not necessarily diagonal $D^{(t)}$. If $D^{(t)} = I_n$, we get the proposed model (1). At the interpretative level, the generalization now reflects a process in which influence relationships that have been developed by endogenous influence propagation within the network can be adjusted exogenously.

2.5. Behavioral state variable

Although we do not directly model the behavioral states, their values can be easily calculated under the assumptions we have made. The behavioral state variable represents, for example, the nodes' opinions or behaviors. We assume that the influence relationships determine the nodes' states. That is, the state of each node at time t equals the weighted average of the given initial states $S^{(0)}$ of its influencing nodes at time t:

$$S^{(t)} = \mathcal{I}^{(t)} S^{(0)}. \tag{6}$$

This assumption then requires the well definedness of convex combinations of the behavioral states, which are elements of a set $\mathcal{C}$. In general, it is possible for $\mathcal{C}$ to be a special barycentric algebra that allows convex combinations without referring to a vector space (Romanowska & Smith, 1985). In practice, however, it is more common for $\mathcal{C}$ to be a convex set, a subset of a vector space. For example, $\mathcal{C}$ could be the real numbers, $\mathcal{C} = \mathbb{R}$.

3. Relationship to existing work

The proposed diffusion model shares aspects with several well-known and fundamental influence models. In this section, we construct the proposed model by adding these joint aspects one by one to variable $\mathcal{I}^{(t)}$, which describes the evolution of the influence relationships. The stepwise addition is done until we arrive at the model presented in Equation 7. While adding, we refer to the fundamental models sharing that aspect. In variable $\mathcal{I}^{(t)}_{\mathrm{XX}}$, the evolution of the influence relationships is described for model XX (where XX is an abbreviation of the model's name). Original notations of the referenced models were adapted to allow for easier visibility of similarities and differences between the models – given the same influence structure W or $W^{(t)}$ for the static or dynamic structure, respectively.

By the end of this section, the following three findings are made. First, the proposed model includes the Friedkin-Johnsen model (Friedkin & Johnsen, 1990) as a special case. Second, given a static influence structure, this seeming generalization reduces to an over-parametrization when the influence process is supposed to run forever. The long-term behavior of influence models is one of the most frequently raised questions: whether the process reaches an equilibrium, i.e., whether the limit

$$\lim_{t\to\infty} \mathcal{I}^{(t)} = \mathcal{I}^{(\infty)} \tag{7}$$

exists and under what conditions it is reached. The found over-parametrization implies that the model's inertia with respect to the current influence relationships does not expand the outcomes of the Friedkin-Johnsen model in the limit. But, third, when having a dynamic influence structure, the inertia generally makes a difference. The difference we demonstrate does not relate to the equilibrium but to single-time steps. More broadly, we are not focused on or intend to tackle the question of equilibrium under a dynamic influence structure. We construct the model in seven steps as follows.

Influence is a distributed force of a fixed and finite amount, which is implicitly assumed by all subsequent models. It is a result of using a non-negative stochastic matrix $\mathcal{I}^{(t)}$ to represent the interpersonal influence relationships. By appropriate modifications on the matrix throughout the processes, the stochasticity and the constant total amount of influence equivalent to the number of nodes are maintained.

Influence is assumed to be assimilative, reducing differences and making the target of influence more similar to the source of influence (Flache et al., 2017). The assimilation is implemented by weighted averaging as given by a static influence network $W \in [0,1]^{n\times n}$, which is row stochastic:

$$\mathcal{I}^{(t)} = W\mathcal{I}^{(t-1)}$$

with a given row stochastic initial $\mathcal{I}^{(0)} \in [0,1]^{n\times n}$. The classical model of Degroot (1974), based on French (1956) and Harary (1959), uses the same implementation

$$\mathcal{I}_{\mathrm{DG}}^{(t)} = W\mathcal{I}_{\mathrm{DG}}^{(t-1)} \tag{8}$$

but with the specific initial $\mathcal{I}_{\mathrm{DG}}^{(0)} = I_n$.

The initial influence relationships are freely selectable. An initial state must be assumed from which the influence relationships develop. All of the referenced models, either implicitly or explicitly, assume that the nodes start entirely independent, i.e., $\mathcal{I}^{(0)} = I_n$. This assumption is in contrast to the free choice of $\mathcal{I}^{(0)}$ that we propose.

Incorporating independence allows separating from received influence. This separation is implemented by relatively weighting the propagated influences against pure independence with factors α and $1-\alpha$, respectively:

$$\mathcal{I}^{(t)} = \alpha W\mathcal{I}^{(t-1)} + (1-\alpha)I_n.$$

The independence is also included in the model of Friedkin and Johnsen (1990). Therein, Friedkin and Johnsen proposed that opinions (the behavioral state) evolve as an average of one's own opinion and the opinion of influencers, but with an anchorage on the initial opinion, reflecting some stubbornness. For the Friedkin-Johnsen model, the opinions' evolution can be written in the form of Equation (6) meaning that the opinions are completely dependent on the evolution of the influence relationships. From that equation, it can be seen that the anchoring at the initial behavioral state (opinion) corresponds to the independence in the influence relationships. The influence relationships of the static Friedkin-Johnsen model evolve dependent on the parameter α, which they introduced to represent a homogeneous "susceptibility to interpersonal influence":

$$\mathcal{I}_{\mathrm{FJ}}^{(t)} = \alpha W \mathcal{I}_{\mathrm{FJ}}^{(t-1)} + (1-\alpha) I_n \tag{9}$$

with $\mathcal{I}_{\mathrm{FJ}}^{(0)} = I_n$.

An inertia with respect to the current influence relationships is assumed. The extent of inertia is controlled by the size of $(1-\gamma)$:

$$\mathcal{I}^{(t)} = \alpha[\gamma W + (1-\gamma) I_n]\mathcal{I}^{(t-1)} + (1-\alpha) I_n. \tag{10}$$

While we assume the inertia parameter to be static, Demarzo et al. (2003) included a time-dependent parameter $\gamma_t \in (0,1]$ that controls the weight with which agents "listen" to others relative to themselves by

$$\mathcal{I}_{\mathrm{DM}}^{(t)} = [\gamma_t W + (1-\gamma_t) I_n]\mathcal{I}_{\mathrm{DM}}^{(t-1)} \tag{11}$$

with $\mathcal{I}_{\mathrm{DM}}^{(0)} = I_n$.

Remark 1. The Friedkin-Johnsen model (9) is included in model (10) as a special case: choosing $\gamma = 1$ and $\mathcal{I}^{(0)} = I_n$ in $\mathcal{I}^{(t)}$ given by (10) makes the dynamics of both models coincide, i.e., $\mathcal{I}^{(t)} = \mathcal{I}_{\mathrm{FJ}}^{(t)}$ for all t. When the dynamics coincide, the limits do so as well. However, as the subsequent Lemma 3.1 shows, $\mathcal{I}^{(t)}$ at equilibrium is simply an over-parameterized version of $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ under the same static influence structure.

Lemma 3.1. *Let $\mathcal{I}^{(t)}$ be as given by (10) with $\alpha \in [0,1)$, $\gamma \in [0,1]$, and a row stochastic $W \in [0,1]^{n\times n}$. Then, $\mathcal{I}^{(t)}$ converges, and there is an $\alpha' \in [0,1)$ such that $\lim_{t\to\infty} \mathcal{I}^{(t)} = \lim_{t\to\infty} \mathcal{I}_{\mathrm{FJ}}^{(t)}$ for $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ as given by (9) under the same W, independent on the chosen $\mathcal{I}^{(0)}$. The α' is given by*

$$m : [0, 1) \times [0, 1] \to [0, 1)$$

$$(\alpha, \gamma) \mapsto \alpha' := \frac{\alpha\gamma}{\alpha\gamma + (1 - \alpha)}. \tag{12}$$

The matrix series $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ converges faster than $\mathcal{I}^{(t)}$ to the joint limit.

Remark 2. Lemma 3.1 cannot be extended for the entire parameter space. If $\alpha = 1$, the model $\mathcal{I}^{(t)}$ reduces to an instance of the Degroot model $\mathcal{I}_{\mathrm{DG}}^{(t)}$, which is isomorphic to a finite time-homogeneous Markov chain. Consider the influence network of two nodes defined by an irreducible and periodic Markov chain

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, the model's inertia makes $\mathcal{I}^{(t)}$ an irreducible and aperiodic Markov chain for any $\gamma \in (0, 1)$, which will converge to the stationary distribution π:

$$\lim_{t \to \infty} \mathcal{I}^{(t)} = (\gamma W + (1 - \gamma) I_n)^t = \mathbf{1}_n \pi^T$$

with $\mathbf{1}_n$ as a vector of size n consisting of ones. The matrix $\mathbf{1}_n \pi^T$, as the product of a column and row vector, has rank one. However, there is no α' for the Friedkin-Johnsen model $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ which would result in the same limit: for $\alpha' = 1$, $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ does not converge due to the periodicity of W; for $\alpha' < 1$, the limit $\mathcal{I}_{\mathrm{FJ}}^{(t)} = (I_n - \alpha' W)^{-1}(1 - \alpha')$ is a matrix of full rank and cannot be reduced to rank one by any choice of $\alpha' \in [0, 1)$.

The extent of independence is assumed to consist of a homogeneous and a heterogeneous level. Additional to the homogeneous parameter α, a heterogeneous level that allows for individual differences is added by scaling with a diagonal matrix $E \in [0, 1]^{n \times n}$:

$$\mathcal{I}^{(t)} = [I_n - (1 - \alpha)E](\gamma W + (1 - \gamma) I_n)\mathcal{I}^{(t-1)} + (1 - \alpha)E. \tag{13}$$

For the Friedkin-Johnsen model, there is a (totally) heterogeneous version (Friedkin & Johnsen, 1999) given by

$$\mathcal{I}_{\mathrm{FJ}}^{(t)} = AW\mathcal{I}_{\mathrm{FJ}}^{(t-1)} + (I_n - A) \tag{14}$$

with a diagonal $A \in [0, 1]^{n \times n}$ and $\mathcal{I}_{\mathrm{FJ}}^{(0)} = I_n$. We refer here to the version without any coupling constraint between A and W.

Remark 3. As for the homogeneous case, choosing $\gamma = 1, A = I_n - (1-\alpha)E$, and $\mathcal{I}^{(0)} = I_n$ makes the heterogeneous Friedkin-Johnsen model (14) a special case of (13). As we will see in Lemma 3.4, adding a heterogeneous level of independence does not change the fact that model $\mathcal{I}^{(t)}$ as given by (13) at equilibrium is simply an over-parametrization of the Friedkin-Johnsen model $\mathcal{I}_{\text{FJ}}^{(t)}$ as given by (14). But the heterogeneity sophisticates the conditions of equilibrium. Parsegov et al. (2017) provide a specific block form of a substochastic matrix M through a partition and permutation of the nodes, which allowed them to derive a necessary and sufficient convergence condition for $\mathcal{I}_{\text{FJ}}^{(t)}$. The partition is defined as follows.

Definition 3.2. Consider a substochastic matrix $M = DR$ with $D, R \in [0,1]^{n\times n}$, D diagonal, and R row stochastic. We define the $\mathcal{S}$-partition of the set of nodes $V = \{1,\ldots,n\}$ into $V = V_1 \cup V_2$ by

$$V_1 := \left\{ i \in V : D_{ii} < 1 \vee \left(D_{ii} = 1 \wedge \left[\exists j \in V_1, k \in \mathbb{N} : D_{jj} < j \wedge [R^k]_{ij} > 0 \right] \right) \right\}$$

$$V_2 := \left\{ i \in D : D_{ii} = 1 \wedge \left[\nexists j \in V_1, k \in \mathbb{N} : D_{jj} < 1 \wedge [R^k]_{ij} > 0 \right] \right\} \qquad (15)$$

Any substochastic M can be written as required from Definition 3.2 by setting $D_{ii} = \sum_{j=1}^{n} M_{ij}$; $R_{ij} := M_{ij}/D_{ii}$ for $D_{ii} > 0$ and $R_{i\star}$ as any stochastic vector if $D_{ii} = 0$. The $\mathcal{S}$-partition (15) splits the nodes into two sets. The set V_1 contains "stubborn" ($D_{ii} < 1$) nodes and nodes that are directly or indirectly influenced by a stubborn node ($[R^k]_{ij} > 0$), while there is not a single stubborn node contained in V_2. Besides potential influence between nodes of the same set, nodes in V_2 can influence nodes in V_1 but not the other way around. Therefore, permuting the nodes such that V_1 precedes V_2 results in an upper triangular block form $M = DR$

$$M = DR = \begin{bmatrix} D_{11} & \mathbf{0} \\ \mathbf{0} & I_{n_2} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ \mathbf{0} & R_{22} \end{bmatrix} = \begin{bmatrix} D_{11}R_{11} & D_{11}R_{12} \\ \mathbf{0} & R_{22} \end{bmatrix} \qquad (16)$$

with $\mathbf{0}$ representing zero matrices of appropriate sizes and n_2 as the cardinality of set V_2. By using this block form, Parsegov et al. were able to identify criteria for substochastic matrices M with spectral radius $\rho(M) = 1$ whose powers nevertheless converge (Parsegov et al., 2017, Lemma 1). In turn, they used this characterization to prove the following theorem (the notation has been adapted).

Theorem 3.3 *(Theorem 1, Corollary 2, Parsegov et al. (2017)). Let DR be a substochastic matrix with $D, R \in [0,1]^{n\times n}$, D diagonal, and R row stochastic. Assume DR of the block form (16) and let V_1, V_2 denote the $\mathcal{S}$-partition defined by*

(15) with n_1, n_2 as their respective cardinality. Then, $\rho(D_{11}R_{11}) < 1$. It holds $\rho(DR) < 1$ if and only if $V_2 = \emptyset$. If $V_2 \neq \emptyset$, the limit $\lim_{t\to\infty}(DR)^t$ exists if and only if $\lim_{t\to\infty}(R_{22})^t = R_{22}^\infty$ also exists. If existent, the limit of powers is given by

$$\lim_{t\to\infty}(DR)^t = \begin{bmatrix} \mathbf{0} & (I_{n_1} - D_{11}R_{11})^{-1} D_{11}R_{12}R_{22}^\infty \\ \mathbf{0} & R_{22}^\infty \end{bmatrix} \tag{17}$$

and the series converges

$$\sum_{k=0}^{\infty}(DR)^k(I_n - D) = \begin{bmatrix} (I_{n_1} - D_{11}R_{11})^{-1}(I_{n_1} - D_{11}) & \mathbf{0} \\ 0 & \mathbf{0} \end{bmatrix}. \tag{18}$$

Theorem 3.3 states that the influence relationships of all nodes converge if and only if those of V_2 do so. We use the $\mathcal{S}$ -partition and the resulting block form to derive whether models $\mathcal{I}^{(t)}$ and $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ can coincide in converged equilibrium and, if so, what conditions are required for this to occur. We find in the following Lemma that the over-parametrization, as found in Lemma (3.1) for homogeneous independence, also applies to the more general case with an additional level of heterogeneous independence.

Lemma 3.4. *Let $\mathcal{I}^{(t)}$ be as given by (13) with $\gamma \in (0,1]$, $\alpha \in [0,1)$, a row stochastic $W \in [0,1]^{n\times n}$, and a diagonal $E \in [0,1]^{n\times n}$. Let V_1, V_2 denote the $\mathcal{S}$-partition induced by (15) for $D = I_n - (1-\alpha)E$ and $R = \gamma W + (1-\gamma)I_n$. Let n_1, n_2 denote the respective cardinality of V_1, V_2. If either $V_2 = \emptyset$, or if $\gamma W_{22} + (1-\gamma)I_{n_2}$ is irreducible and aperiodic when $V_2 \neq \emptyset$, then $\mathcal{I}^{(t)}$ converges. Let $A \in [0,1]^{n\times n}$ be the diagonal matrix defined by*

$$\mathcal{M} : [0,1) \times [0,1]^n \times (0,1] \to [0,1]^n$$

$$(\alpha, E_{ii}, \gamma) \mapsto A_{ii} := \frac{\gamma[1-(1-\alpha)E_{ii}]}{\gamma[1-(1-\alpha)E_{ii}] + (1-\alpha)E_{ii}} \tag{19}$$

for $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ as given by (14). Then, if W_{22} is aperiodic and $\mathcal{I}_{22}^{(0)} = I_{n_2}$ when $V_2 \neq \emptyset$, equality applies for the limits $\lim_{t\to\infty}\mathcal{I}^{(t)} = \lim_{t\to\infty}\mathcal{I}_{\mathrm{FJ}}^{(t)}$ under the same W and independent of $\mathcal{I}_{11}^{(0)}, \mathcal{I}_{12}^{(0)}$.

Remark 4. The mapping $\mathcal{M}$ in (19) maintains the $\mathcal{S}$-partition of V: nodes in V_2 w.r.t. $(I_n - (1-\alpha)E)(\gamma W + (1-\gamma)I_n)$ will be in V_2 w.r.t. AW after applying $\mathcal{M}$. For specific cases, there can be alternative possibilities to define an appropriate A such that $\mathcal{I}^{(\infty)} = \mathcal{I}_{\mathrm{FJ}}^{(\infty)}$, which are not covered by Lemma 3.4. However, we do not elaborate on these specific cases here.

Remark 5. From both mappings, m and $\mathcal{M}$, it can be concluded that with matching limits emerging from a static influence structure, a larger inertia $1-\gamma$ in $\mathcal{I}^{(t)}$ reduces the "susceptibility to interpersonal influence" α' and A in $\mathcal{I}_{\mathrm{FJ}}^{(t)}$, or conversely a reinforcement of independence $1-\alpha'$ and $I_n - A$.

Influence and independence are assumed to change over time. Incorporating the dynamic perspective as the final step, we arrive at the model that has been proposed in the previous section:

$$\mathcal{I}^{(t)} = \left[I_n - (1-\alpha)E^{(t)}\right]\left(\gamma W^{(t)} + (1-\gamma)I_n\right)\mathcal{I}^{(t-1)} + (1-\alpha)E^{(t)}. \tag{20}$$

Taking time into account is an intuitive choice for generalizing models and making them more realistic. Chatterjee and Seneta (1977) proposed a time-dependent variant for the DeGroot model in which the influence network $W^{(t)}$ varies over time. Friedkin and Johnsen already presented a relaxed version of their model in their original paper (Friedkin & Johnsen, 1990), where all constructs may vary over time. The temporal Friedkin-Johnsen model is described by

$$\mathcal{I}_{\mathrm{FJ}}^{(t)} = A^{(t)} W^{(t)} \mathcal{I}_{\mathrm{FJ}}^{(t-1)} + \left(I_n - A^{(t)}\right) \tag{21}$$

with $A^{(t)}, W^{(t)} \in [0,1]^{n\times n}$, whereby $A^{(t)}$ is diagonal and $W^{(t)}$ is row stochastic, and $\mathcal{I}_{\mathrm{FJ}}^{(0)} = I_n$. In particular, the temporal models (20) and (21) allow nodes to be only influenced (no independence) at one time and become exclusively independent (receiving no influence) at another time.

The transition from static to dynamic influence and independence changes the associated timescales: from two separate timescales in the static case, where the behavioral states' dynamics are faster than influence and independence dynamics, to a single timescale in the dynamic case, where the behavioral states' dynamics take place on the same timescale as the influence and independence dynamics. We do not impose any restrictions on how much $W^{(t)}$ and $E^{(t)}$ can change in one single time step t. Between the two extremes of no change and any change, there can be intermediary stages. For example, changes are only permitted in the influence weights that are not zero and at a point in time that is either after the convergence of the behavioral dynamics under the previous influence network (Friedkin, 2011; Jia et al., 2015; Tian et al., 2022), or that is already after each update of the behavioral state (Jia et al., 2020; Tian et al., 2022).

For the time-dependent influence structure as well, one might ask for conditions under which the influence process converges and, more importantly, whether the proposed model is an over-parametrization of the temporal Friedkin-Johnsen model, as found in the static case. Even for the time-varying DeGroot model, which is a special case of both models $\mathcal{I}^{(t)}, \mathcal{I}_{\text{FJ}}^{(t)}$, the convergence conditions remain a challenge, of which only sufficient ones have been found so far (Proskurnikov & Tempo, 2018). For the temporal Friedkin-Johnsen model (21), Proskurnikov et al. (2017) provide sufficient conditions for its asymptotic stability, i.e., sufficient conditions such that

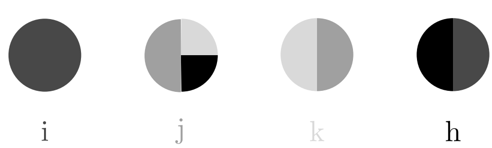
$$\lim_{t\to\infty} \prod_{b=0}^{t-1} A^{(t-b)} W^{(t-b)} = 0 \tag{22}$$

vanishes as part of $\mathcal{I}_{\text{FJ}}^{(t)}$. They use asymptotic stability to show that in the very special case where all nodes are anchored at the same value, the opinions then converge to a consensus (the anchored value). However, our focus is on the dyadic influence relationships and not on the individual opinions, and unlike in the static case, asymptotic stability does *not* imply the convergence of the influence relationships $\mathcal{I}_{\text{FJ}}^{(t)}$ (for an example see (Proskurnikov et al., 2017, p. 11899)). Therefore, instead of investigating whether $\mathcal{I}^{(t)}$ and $\mathcal{I}_{\text{FJ}}^{(t)}$ might be congruent in the limit, we show in the following Lemma that equality cannot be maintained for a single-time step in general. The proposed model's inertia causes this lack of maintenance.

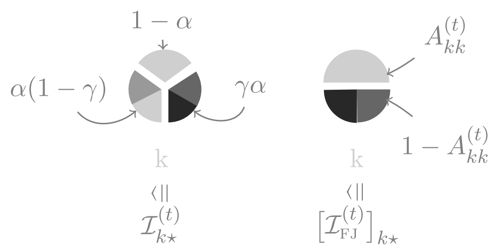
Lemma 3.5. *Let $\mathcal{I}^{(t-1)}$ and $\mathcal{I}_{\text{FJ}}^{(t-1)}$ be as given by* (20) *and* (21), *respectively. Let $\gamma < 1, \alpha > 0$ and $\mathcal{I}^{(t-1)} = \mathcal{I}_{\text{FJ}}^{(t-1)}$. If there is an i with $W_{ii}^{(t)}, \mathcal{I}_{ii}^{(t-1)} \neq 1$, then there is no diagonal $A^{(t)} \in [0,1]^{n\times n}$ such that $\mathcal{I}^{(t)} = \mathcal{I}_{\text{FJ}}^{(t)}$.*

Remark 6. For the Friedkin-Johnsen model, in the extreme case where a node k exhibits the entry $W_{kk}^{(t)} = 0$, k's current influence distribution $\mathcal{I}_{k\star}^{(t-1)}$ is entirely overwritten and replaced by a weighted average of the distribution(s) of k's influencing node(s) and, potentially, the point distribution due to independence. If there are nodes that have had influence on k but not on any of k's influencing nodes, they will completely lose their influence, regardless of what is in $A_{kk}^{(t)}$. This loss is prevented by the inertia in the proposed model. An exemplary visualization of this extreme case is shown in the following example.

Example 3.6. Assume the following influence distributions $\mathcal{I}^{(t-1)} = \mathcal{I}_{\text{FJ}}^{(t-1)}$ as represented by pie charts from the exemplary distribution changes in Figure 1:



Let k be influenced by h with weight $W_{kh}^{(t)} = 1$, i.e., $W_{kk}^{(t)} = 0$ in particular, while becoming independent with weight $E_{kk}^{(t)} = 1$. Then, the updated influence relationships of k consist of a parametrically weighted average of three different distributions for the proposed model (as long as the inertia to change the current relationships is present, $1 - \gamma > 0$, and independence is not totalitarian, $1 - \alpha < 1$). Whereas for the temporal Friedkin-Johnsen model, the average is taken over one distribution less, i.e., excluding $[\mathcal{I}_{\text{FJ}}^{(t-1)}]_{k\star}$:



Remark 7. In the dynamic case, the inertia acquires its raison d'être since the outcome of the proposed model cannot be achieved by simply adjusting the independence in the Friedkin-Johnsen model. Inertia enables an impact of past influence, which is obsolete in the static case because the same influence dynamics are repeated over and over again.

4. Conclusion

This paper proposes a diffusion model for temporal social networks that generalizes the Friedkin-Johnsen model. Influence relationships between every two nodes are established based on becoming influenced but sticking to the status quo due to inertia, and becoming independent from previously received influence. Relating to existing models, we find that the proposed model is an over-parametrization of the well-known Friedkin-Johnsen model on the same static influence structure. On a dynamic influence structure, however, the model cannot be substituted by an appropriate choice of the Friedkin-Johnsen model, because of interaction effects between inertia and network dynamics. This finding suggests that the proposed model non-trivially contributes to the set of social influence models currently present. In addition to comparing different models with the same timescale, a future step could be to compare different time scales for the proposed model. For example, whether there are instances of the dynamic model that can be approximated by an instance of the static model. We believe the model exhibits the potential to reflect various influence processes of dynamic social relationships, and its scope needs to be further examined. The model is deterministic but parametric, so a direction for future work might consider how much the given temporal network already determines the influence relationships. In other words, how much do the influence relationships vary depending on the chosen parameters? Ideally, an answer to that question is also valuable for developing a method for estimating the model's parameters when the most influential nodes are known.

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Appendix

Proof of Lemma 3.1. First, it is shown that (i) under the given assumptions, $\mathcal{I}^{(t)}$ does converge. Then, (ii) $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ is proved to be equal to $\mathcal{I}^{(\infty)}$ in the limit for the stated α'. Finally, it is shown that (iii) $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ converges faster than $\mathcal{I}^{(t)}$ to the joint limit.

(i) The non-recursive influence relationships of $\mathcal{I}^{(t)}$ equals

$$\mathcal{I}^{(t)} = [\alpha(\gamma W + (1-\gamma)I_n]^t \mathcal{I}^{(0)} + (1-\alpha)\sum_{k=0}^{t-1}[\alpha(\gamma W + (1-\gamma)I_n)]^k.$$

The first part converges to zero since $\alpha < 1$. Because $\alpha < 1$, the matrix $[\alpha(\gamma W + (1-\gamma)I_n)]$ is *strictly* substochastic, so its spectral radius is strictly smaller than one. For a matrix M with spectral radius $\rho(M) < 1$, the Neumann series gives

$$\sum_{k=0}^{\infty} M^k = (I_n - M)^{-1}. \tag{23}$$

With (23), the limit constitutes as

$$\mathcal{I}^{(\infty)} = \lim_{t\to\infty}\mathcal{I}^{(t)} = (1-\alpha)(I_n - \alpha[\gamma W + (1-\gamma)I_n])^{-1}. \tag{24}$$

(ii) Let α' be as defined by (12). Since $\alpha' < 1$, the series $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ converges for the same reasons as outlined above, and its limit is

$$\mathcal{I}_{\mathrm{FJ}}^{(\infty)} = \lim_{t\to\infty}\mathcal{I}_{\mathrm{FJ}}^{(t)} = (1-\alpha')(I_n - \alpha' W)^{-1}. \tag{25}$$

Plugging in $\alpha' = \frac{\alpha\gamma}{\alpha\gamma+(1-\alpha)}$ into $\mathcal{I}_{\mathrm{FJ}}^{(\infty)}$ yields equality of both limits:

$$\mathcal{I}_{\mathrm{FJ}}^{(\infty)} = \underbrace{\left(1 - \frac{\alpha\gamma}{\alpha\gamma + (1-\alpha)}\right)}_{=\frac{1-\alpha}{\alpha\gamma+1-\alpha}}\left[I_n - \frac{\alpha\gamma}{\alpha\gamma+(1-\alpha)}W\right]^{-1}$$

$$= (1-\alpha)\left[\frac{\alpha\gamma+(1-\alpha)}{\alpha\gamma+(1-\alpha)}((\alpha\gamma+1-\alpha)I_n - \alpha\gamma W)\right]^{-1}$$

$$= (1-\alpha)[I_n - \alpha(\gamma W + (1-\gamma)I_n)]^{-1} = \mathcal{I}^{(\infty)}.$$

(iii) The speed of convergence of the matrix series $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ and $\mathcal{I}^{(t)}$ depends on how fast the terms $(\alpha' W)^t$ and $[\alpha(\gamma W + (1-\gamma)I_n)]^t$, respectively, converge to zero. This, in turn, depends on the spectral radius of these matrices. The bounds on the spectral radius of a nonnegative matrix A (see Lemma 3.1.1., Bapat and Raghavan (1997))

$$\min_{1\le i\le n}\sum_{j=1}^{n} A_{ij} \le \rho(A) \le \max_{1\le i\le n}\sum_{j=1}^{n} A_{ij} \tag{26}$$

gives in our specific case

$$\rho(\alpha' W) = \alpha' \quad \text{and} \quad \rho(\alpha[\gamma W + (1-\gamma)I_n]) = \alpha.$$

With $\alpha \in [0, 1)$ and $\gamma \in [0, 1]$, it is then

$$0 \le \alpha(1-\gamma)(1-\alpha) \quad \Leftrightarrow$$

$$\rho(\alpha' W) = \alpha' = \frac{\alpha\gamma}{\alpha\gamma + (1-\alpha)} \le \alpha = \rho(\alpha[\gamma W + (1-\gamma)I_n])$$

which proves the claim.

Proof of Lemma 3.4. First, it is shown that (i) under the given assumptions, $\mathcal{I}^{(t)}$ does converge, and an expression for $\mathcal{I}^{(\infty)}$ is provided. Then, (ii) $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ is proved to coincide with $\mathcal{I}^{(\infty)}$ in the limit for the stated A.

(i) The convergence of $\mathcal{I}^{(t)}$ directly follows from Theorem 3.3 given the irreducibility and aperiodicity of $\gamma W_{22} + (1-\gamma)I_{n_2}$. The limit $\mathcal{I}^{(\infty)}$ constitutes as the sum of (17) and (18) but requires an adaption for the initial influence relationships:

$$\lim_{t\to\infty}\left[(DR)^t I^{(0)} + \sum_{k=0}^{t-1}(DR)^k(I_n - D)\right] \tag{27}$$

$$= \begin{pmatrix} \mathbf{0} & (I_{n_1} - D_{11}R_{11})^{-1}D_{11}R_{12}R_{22}^{\infty} \\ \mathbf{0} & R_{22}^{\infty} \end{pmatrix} I^{(0)} + \begin{pmatrix} (I_{n_1} - D_{11}R_{11})^{-1}(I_{n_1} - D_{11}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} (I_{n_1} - D_{11}R_{11})^{-1}\left[D_{11}R_{12}R_{22}^{\infty}I_{21}^{(0)} + I_{n_1} - D_{11}\right] & (I_{n_1} - D_{11}R_{11})^{-1}D_{11}R_{12}R_{22}^{\infty}I_{22}^{(0)} \\ R_{22}^{\infty}I_{21}^{(0)} & R_{22}^{\infty}I_{22}^{(0)} \end{pmatrix}$$

with a row stochastic $I^{(0)} \in [0, 1]^{n\times n}$. Then, both limits $\mathcal{I}^{(\infty)}$ and $\mathcal{I}_{\mathrm{FJ}}^{(\infty)}$ can be obtained by (28) inserting D, R, and $I^{(0)}$ accordingly.

(ii) Let A be as defined by (19). Then, V_1 and V_2 are closed under $\mathcal{M}$ such that it is also a partition w.r.t. $D = A, R = W$. This follows from $\mathcal{M}(\alpha, 0, \gamma) = 1$, $\mathcal{M}(\alpha, 1, \gamma) = 0$, and the fact that loops do not alter reachability properties. Because V_1, V_2 are closed under $\mathcal{M}$, we have the same block structure. Given that W_{22} is aperiodic, $\mathcal{I}_{\mathrm{FJ}}^{(t)}$ is also converging. The limits for $\mathcal{I}^{(\infty)}$ and $\mathcal{I}_{\mathrm{FJ}}^{(\infty)}$ are obtained by setting $D = I_n - (1-\alpha)E, R = \gamma W + (1-\gamma)I_n, I^{(0)} = \mathcal{I}^{(0)}$ and $D = A, R = W, I^{(0)} = I_n$, respectively, in (27). The limits coincide if each of the blocks does so. Given $\mathcal{I}_{22}^{(0)} = I_{n_2}$, the equality of the limits applies if the four equations are fulfilled

$$\textbf{I} \quad (I_{n_1} - A_{11}W_{11})^{-1}(I_{n_1} - A_{11}) = [I_{n_1} - (I_{n_1} - (1-\alpha)E_{11})(\gamma W_{11} + (1-\gamma)I_{n_1})]^{-1}(1-\alpha)E_{11},$$

$$\textbf{II} \quad (I_{n_1} - A_{11}W_{11})^{-1}A_{11}W_{12}W_{22}^{\infty} = [I_{n_1} - (I_{n_1} - (1-\alpha)E_{11})(\gamma W_{11} + (1-\gamma)I_{n_1})]^{-1} (I_{n_1} - (1-\alpha)E_{11})\gamma W_{12}(\gamma W_{22} + (1-\gamma)I_{n_2})^{\infty},$$

$$\textbf{III} \quad \mathbf{0} = \mathbf{0},$$

$$\textbf{IV} \quad W_{22}^{\infty} = (\gamma W_{22} + (1-\gamma)I_{n_2})^{\infty}.$$

Equation **III** trivially applies. The rows of W_{22}^{∞} consist of the stationary distribution π, which is the left stochastic eigenvector of W_{22} for eigenvalue one. Then,

$$\pi(\gamma W_{22} + (1-\gamma)I_{n_2}) = \gamma\pi + (1-\gamma)\pi = \pi,$$

i.e., row vector π is also a left eigenvector of $\gamma W_{22} + (1-\gamma)I_{n_2}$ for eigenvalue one. Thus, π is also the stationary distribution of the Markov chain described by matrix $\gamma W_{22} + (1-\gamma)I_{n_2}$. Therefore, the rows of $(\gamma W_{22} + (1-\gamma)I_{n_2})^{\infty}$ consist of π, and equation **IV** holds. It remains to show that the defined A solves **I** and **II**. Since **IV** holds, equation **II** holds if

$$\textbf{II}' \quad (I_{n_1} - A_{11}W_{11})^{-1}A_{11} = [I_{n_1} - (I_{n_1} - (1-\alpha)E_{11})(\gamma W_{11} + (1-\gamma)I_{n_1})]^{-1}(I_{n_1} - (1-\alpha)E_{11})\gamma.$$

We can write matrix A, given by (19), as

$$A_{11} = \gamma(I_{n_1} - (1-\alpha)E_{11})[\gamma(I_{n_1} - (1-\alpha)E_{11}) + (1-\alpha)E_{11}]^{-1}. \tag{28}$$

The inverse, which is part of A_{11}, is a diagonal matrix and is well-defined: this matrix would not have full rank if there is a single i with

$$0 = \gamma(1 - (1-\alpha)E_{ii}) + (1-\alpha)E_{ii} \quad \Leftrightarrow$$

$$1 = \underbrace{(1-\gamma)}_{<1}\underbrace{(1-(1-\alpha)E_{ii})}_{\leq 1}$$

which can never happen as $\gamma > 0$. From (28),

$$I_{n_1} - (1-\alpha)E_{11} = [\gamma I_{n_1} - (\gamma - 1)A_{11}]^{-1}A_{11} \tag{29}$$

is derived and inserted in **II'**. Using $M_2^{-1}M_1^{-1} = (M_1M_2)^{-1}$ it can be verified that **II'** and, therefore, also **II** holds. The verification of **I** is achieved by inserting

$$I_{n_1} - A_{11} = (1-\alpha)E_{11}[\gamma(I_{n_1} - (1-\alpha)E_{11}) + (1-\alpha)E_{11}]^{-1} \tag{30}$$

and A_{11} from (28) into its left-hand side, from which then the right-hand side can be derived using the commutativity of diagonal matrices, adding and subtracting $[I_{n_1} - (1-\alpha)E_{11}](1-\gamma)$ inside the inverse, and, again, using $M_2^{-1}M_1^{-1} = (M_1M_2)^{-1}$.

Proof of Lemma 3.5. The entries of the matrix describing the difference $\mathcal{I}^{(t)} - \mathcal{I}^{(t)}_{\mathrm{FJ}}$ vary between the on- and off-diagonal elements. Given $\mathcal{I}^{(t-1)} = \mathcal{I}^{(t-1)}_{\mathrm{FJ}}$, the difference of the updated entries amount to

$$\mathcal{I}^{(t)}_{ii} - \left[\mathcal{I}^{(t)}_{\mathrm{FJ}}\right]_{ii} = \left[\gamma\left(1-(1-\alpha)E^{(t)}_{ii}\right) - A^{(t)}_{ii}\right]\sum_{k=1}^{n} W^{(t)}_{ik}\mathcal{I}^{(t-1)}_{ki} \tag{31}$$

$$+\left(1-(1-\alpha)E^{(t)}_{ii}\right)(1-\gamma)\mathcal{I}^{(t-1)}_{ii} + (1-\alpha)E^{(t)}_{ii} - \left(1-A^{(t)}_{ii}\right),$$

$$\mathcal{I}^{(t)}_{ij} - \left[\mathcal{I}^{(t)}_{\mathrm{FJ}}\right]_{ij} = \left[\gamma\left(1-(1-\alpha)E^{(t)}_{ii}\right) - A^{(t)}_{ii}\right]\sum_{k=1}^{n} W^{(t)}_{ik}\mathcal{I}^{(t-1)}_{kj} \tag{32}$$

We consider the nodes that influence i at time t, serving thus as 'brokers' for propagating influence, and partition them according to i's accumulated influence on them:

$$\mathcal{B}_{<} := \left\{k \in V \ : \ W^{(t)}_{ik} > 0 \ \wedge \ \mathcal{I}^{(t-1)}_{ki} < 1\right\},$$

$$\mathcal{B}_{=} := \left\{k \in V \ : \ W^{(t)}_{ik} > 0 \ \wedge \ \mathcal{I}^{(t-1)}_{ki} = 1\right\}.$$

Because $W^{(t)}$ is row stochastic, the union $\mathcal{B}_{<} \cup \mathcal{B}_{=}$ is not empty. Note that i itself can be an element in $\mathcal{B}_{<} \cup \mathcal{B}_{=}$. We distinguish two cases and show that in neither of them an appropriate $A^{(t)}_{ii}$ can be defined such that $\mathcal{I}^{(t)} = \mathcal{I}^{(t)}_{\mathrm{FJ}}$ holds.

(i) $\mathcal{B}_{<} = \emptyset$: In this case, all nodes influencing i at time t are themselves completely influenced by i regarding the accumulated influence. Together with the row stochasticity of $W^{(t)}$, it follows for the sum $\sum_{k=1}^{n} W^{(t)}_{ik}\mathcal{I}^{(t-1)}_{ki} = 1$. Then, in the difference of the diagonal entries (31), the two occurrences of $A^{(t)}_{ii}$ cancel each other out such that under the given assumptions $\alpha > 0, \gamma < 1$, and $\mathcal{I}^{(t-1)}_{ii} \neq 1$ it is

$$\mathcal{I}^{(t)}_{ii} - \left[\mathcal{I}^{(t)}_{\mathrm{FJ}}\right]_{ii} = \left(1 - \underbrace{(1-\alpha)}_{<1} E^{(t)}_{ii}\right)\underbrace{(1-\gamma)}_{>0}\underbrace{\left(\mathcal{I}^{(t-1)}_{ii} - 1\right)}_{<0} \neq 0$$

independent of the choice of $A^{(t)}_{ii}$.

(ii) $\mathcal{B}_{<} \neq \emptyset$: There is a node $k \in \mathcal{B}_{<}$ with $W^{(t)}_{ik} > 0$ and $\mathcal{I}^{(t-1)}_{ki} < 1$. Because $\mathcal{I}^{(t)}$ is row stochastic, there is another node $j \neq i$ with $\mathcal{I}^{(t-1)}_{kj} > 0$. Then, it is $W^{(t)}_{ik}\mathcal{I}^{(t-1)}_{kj} > 0$, such that the difference of the off-diagonal entry (32) is zero if and only if

$$A_{ii}^{(t)} = \gamma\left[1 - (1-\alpha)E_{ii}^{(t)}\right].$$

However, with this $A_{ii}^{(t)}$, the on-diagonal difference (31) equals

$$\mathcal{I}_{ii}^{(t)} - \left[\mathcal{I}_{\mathrm{FJ}}^{(t)}\right]_{ii} = \left(1 - \underbrace{(1-\alpha)}_{<1} E_{ii}^{(t)}\right) \underbrace{(1-\gamma)}_{>0} \underbrace{\left(\mathcal{I}_{ii}^{(t-1)} - 1\right)}_{<0} \neq 0$$

and is never equal to zero under the given assumptions of $\alpha > 0$, $\gamma < 1$, and $\mathcal{I}_{ii}^{(t-1)} \neq 1$.