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A GEOMETRIC APPROACH TO ISOPERIMETRIC PROBLEMS AND THE SEMIGEOSTROPHIC SYSTEM

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Alice laughed: "There's no use trying," she said; "one can't believe impossible things." "I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

Lewis Carroll, *Through the Looking Glass*, 1871

Sommario

Questa tesi tratta due argomenti distinti: il problema isoperimetrico iperbolico e il sistema semigeostrofico nel contesto sferico.

Nella prima parte, affrontiamo la Congettura di Gromov-Ros, che afferma che le sfere geodetiche sono le regioni che minimizzano il perimetro per un determinato volume negli spazi simmetrici di rango uno di tipo non compatto, quali gli spazi iperbolici reali, complessi, quaternionici e il piano Cayley ottionico (16-dimensionale). Nel Capitolo 1 dimostriamo innanzitutto che le sfere di qualsiasi raggio sono quantitativamente stabili rispetto a qualsiasi perturbazione C^1 abbastanza piccola. Di conseguenza, tramite un argomento di riscalamento e risultati di stabilità in $\mathbb{R}^n$, dimostriamo che le sfere sono ottimali nel regime di piccoli volumi. Nel Capitolo 2 dimostriamo la congettura nella classe di insiemi che condividono una simmetria radiale adeguata indotta dall'azione naturale correlata alla fibrazione di Hopf nello spazio ambiente. Questa ipotesi ci consente di ridurre il problema a un problema isoperimetrico pesato nello spazio iperbolico reale. La parte principale di questo capitolo consiste nel dimostrare la versione iperbolica della congettura di Brakke (recentemente dimostrata da Chambers nel contesto Euclideo): le sfere centrate sono isoperimetriche in $\mathbb{R}H^n$ con una densità log-convessa radiale sui funzionali di perimetro e volume.

Nella seconda parte, affrontiamo il sistema semigeostrofico su una sfera rotante. Nel Capitolo 3 dimostriamo l'esistenza locale nel tempo e l'unicità delle soluzioni lisce su qualsiasi dominio semplicemente connesso e conformemente piatto, con un termine di Coriolis non nullo abbastanza regolare. Nel Capitolo 4 ci occupiamo della singolarità che si forma intorno all'equatore a causa della degenerazione del termine di Coriolis. Dimostriamo la stabilità globale nel tempo dell'equazione linearizzata attorno a una nuova famiglia di soluzioni statiche assialmente simmetriche. L'argomento si basa nell'estendere il dominio alla sfera quadridimensionale S^4, assorbendo la singolarità all'interno del peso derivante dalla metrica indotta, e quindi effettuare un attento argomento di sezionamento.

Abstract

This thesis discusses two distinct topics: the hyperbolic isoperimetric problem and the well posedness of the semigeostrophic system in the spherical setting.

The first part adresses the Gromov-Ros Conjecture, which claims that geodesic balls are the regions that minimize the perimeter for any given volume in rank one symmetric spaces of non-compact type. These are the real, complex, quaternionic hyperbolic spaces, and 16-dimensional octonionic (Cayley) plane. In Chapter 1 we first prove that spheres of any radius are uniformly quantitatively stable with respect to any small enough C^1-perturbation. As a consequence, via a rescaling argument and deep stability results in $\mathbb{R}^n$, we show that spheres are optimal in the small volume regime. In Chapter 2 we prove the conjecture in the class of sets enjoying a suitable radial symmetry induced by the natural action related to the Hopf-fibration on the ambient space. This hypothesis allows us to reduce the problem to a weighted isoperimetric problem in the real hyperbolic space. The main part of this chapter is the hyperbolic version of Brakke's conjecture (recently proved by Chambers in the Euclidean setting): centered balls are isoperimetric in $\mathbb{R}H^n$ endowed with a radial log-convex density on the perimeter and volume functionals.

In the second part, we approach the well posedness of the semigeostrophic system over a rotating sphere. In Chapter 3 we prove local-in-time existence and uniqueness of smooth solutions over any simply connected and conformally flat domain, with regular enough non-vanishing Coriolis term. In Chapter 4 we deal with the singularity forming around the equator due to the degeneracy of the Coriolis term. We prove global-in-time stability of the linearized equation around a new axially symmetric family of static solutions. The argument relies on lifting the domain to the higher dimensional four sphere S^4, absorbing the singularity inside the weight coming from the lifted metric, and then perform a careful slicing argument.

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Contents

Organization of the thesis

The content presented has been separated in two distinctive parts so to reflect the main subjects of interest anticipated in the title: the isoperimetric problem and the semigeostrophic system. Each part contains a specific introduction that highlights the historical origins, its developments, relevance in the contemporary scientific panorama, and the state of the art at the present day. The main part of the thesis consists in two subsequent chapters following each introductory parts, in which we present a total of four original articles written during the author's doctorate.

The topics addressed in the two parts are quite far from each other: the isoperimetric problem comes naturally as one of the earliest instance of geometric optimization, and the semigeostrophic system is a set of equations arising in fluid mechanics in particular meteorological regimes. For this reason, the choice of providing two separate introductions was made with the intention of increasing the clarity of exposition, and accentuate the different goals and methods specific to each partThat said, as it will be apparent in the respective introductions, both topics are incredibly rich and can be seen through a variety of different lenses, involving Geometric Analysis, Calculus of Variations, Riemannian Geometry, Spectral Theory, Functional Analysis, Optimal Transport, and Geometric Group Theory, just to name a few. The connections and incursions with unexpected topics therefore provides a great freedom in conceiving the problems arising in both thematic parts. The presentation of the content has been carried out with the hope of bringing out the common geometric approach employed in both domains.

Part I

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The isoperimetric problem

Introduction

> Devenere locos, ubi nunc ingentia cernis moenia
> surgentemque novae Karthaginis arcem,
> mercatique solum, facti de nomine Byrsam,
> taurino quantum possent circumdare tergo.
>
> Virgilius, Aeneis, Liber 1, versus 365-376

Historical Roots

The interested reader is invited to consult the very enjoyable and detailed historical introduction by Blåsjö [18], together with Ros [100], and the introduction of the book by Capogna, Danielli, Paul, and Tyson [27].

The isoperimetric problem has its mythological roots in the tragic legend of queen Dido, and more specifically in her presumed foundation of the city of Carthage in 814 BC. The more ancient reference in this regard is attributed to Timaeus in 300 BC, and the myth is further revised in Virgil's Aeneid, written around 20 BC. Dido, first-born of Belus, king of the Phoenician city-state of Tyre, flees her homeland after the murder of her husband Sychaeus perpetrated secretly by her brother Pygmalion. After long wandering, passing from Cyprus and Malta, queen Dido and her court dock their ships at the Libyan coast. Jarba, the local king, allows the queen to settle his people on a land that could be enclosed by an ox-hide. The mockery of this proposal does not frighten the intelligent queen, who asks her servants to cut the skin in thin stripes, knot them together in a rope long enough to enclose an entire hill on the shore. In this semicircular land Dido founded the city of Carthage (originally Brisia, ox-hide in greek). Unfortunately, according to the ancient version of the myth, there is no happy ending for the poor queen, who kills herself with a dagger invoking the name of her late husband, Sicheus, after Jarba forced her into marriage.

In mathematical terms, Jarba's offer, astutely addressed by Dido, reads as follows:

Problem. Given $L > 0$, find $\Omega \subset \mathbb{R}^2$ with perimeter L and maximal area A.

With isoperimetric (*is-perimetros*, having the same perimeter), we denote all mathematical questions of this type. Notice that this problem has a natural and equivalent dual formulation: given $A > 0$ find $\Omega \subset \mathbb{R}^2$ with area A and minimal perimeter L. For Dido the morphological constraint on the coast forced the walls of her city to draw a semicircle on the land, and the accuracy of this solution can be empirically confirmed looking at any historical map of European cities. Intuition tells us that the optimal set of the unconstrained problem must be the circle of diameter $d = L/\pi$ (or $d = 2\sqrt{A/\pi}$ in its dual reformulation), and this is in fact the case. Indeed, the following elegant expression, called *isoperimetric inequality in the plane*, summarized remarkably this fact.

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be of perimeter L and area A. Then $L^2 - 4\pi A \geq 0$.*

Notice that equality is attained when Ω is a circle, and this is indeed the only possible case, implying that circles are uniquely optimal. This fact was known to the Greek mathematicians, and the first demonstration attributed to Zenodorus in the case of polygonal shapes was transmitted to us by Pappus and Theon of Alexandria.

Over the centuries, an impressive variety of alternative proofs have been provided, spacing from purely geometric arguments, integral rearrangements, convex geometry, Fourier series, probability, calculus of variations, and optimal transport. Without claiming to be exhaustive (or excessively rigorous), we sketch here three radically different and elegant solutions to the isoperimetric problem in the plane, with the hope of convincing the reader of how fruitful this elementary question has been in the history of mathematics.

Probabilistic proof by Santaló [103]. Let Ω be smooth, with perimeter $L > 0$, and area A. By reflecting every 'valley' of its boundary to the outside, we can assume Ω convex. For $r = L/2\pi$ define

$$X : \mathbb{R}^2 \to [0, +\infty], \quad X(x, y) = \#\{\partial B_r(x, y) \cap \partial\Omega\},$$

the map that counts how many intersection points does the circle centered at (x, y) and radius r have with the boundary of Ω. We want to give two estimates of the integral

$$E(X) = \int_{\mathbb{R}^2} X \, dx \, dy,$$

that is the (unaveraged) expected value of intersection points when casting random circles on Ω. Letting Ω_r be the r-thickening of Ω, and κ the curvature of $\partial\Omega$, we have that

$$E(X) = \int_{\Omega_r} X \, dx \, dy \geq \int_{\Omega_r} 2 \, dx \, dy = 2\Big(A + \int_0^r \int_{\partial\Omega} (1 + \kappa\tau) \, d\ell \, d\tau\Big) = 2\Big(A + rL + r^2\pi\Big),$$

where we took advantage of the Gauss-Bonnet Theorem to compute the area of Ω_r. Notice now that a circle or radius r crosses any given segment $d\ell$ if and only if its center lies in the union of all circles or radius r centered in $d\ell$. Moreover, the area of this region is at the first order of size $4r d\ell$ (being essentially the symmetric difference of two intersected disks of radius r). Hence, conceiving $d\ell$ as an infinitesimal portion of $\partial\Omega$ we can compute

$$E(X) = \int_{\partial\Omega} 4r \, d\ell = 4Lr.$$

Combining the two estimates, we get $A + r^2\pi \leq Lr$. Recalling that $r = L/2\pi$, so that the perimeter of ∂B_r is the same as Ω, we obtain the isoperimetric inequality in the plane as wished. □

Optimal Transport proof by Gromov. Let Ω be smooth, with perimeter $L > 0$, and area A. Since the isoperimetric inequality is scaling invariant, let us suppose that $A = 1$, and let $r = 1/\sqrt{\pi}$ so that $A(B_r) = 1$. Let $\mu = \chi_\Omega \, dx$ and $\nu = \chi_{B_r} dx$ be the characteristic probability measures of Ω and B_r, respectively. By Brenier Theorem [1], see [25], there exists an optimal transport map T, which is characterized by being the gradient of a convex function φ pushing μ into ν: $T_\# \mu = \nu$. Notice that the latter property implies $\det(D^2\varphi) = 1$. We now perform a chain of sharp inequalities:

$$\begin{aligned} rL &= \int_{\partial\Omega} r \, d\ell \geq \int_{\partial\Omega} |T| \, d\ell \geq \int_{\partial\Omega} T \cdot n \, d\ell = \int_\Omega \operatorname{div}(T) \, dx = \int_\Omega \Delta\varphi \, dx \\ &\geq 2 \int_\Omega \det(D^2\varphi)^{1/2} \, dx = 2, \end{aligned}$$

where we used in order: T maps Ω in B_r, Cauchy-Schwarz, the Divergence Theorem, $\operatorname{div}(T) = \operatorname{div}(\nabla\varphi) = \Delta\varphi$, and lastly the arithmetic-geometric mean inequality applied to the positive eigenvalues of $D^2\varphi$. It follows that $L^2 \geq 4/r^2 = 4\pi = 4\pi A$ as wished. □

Symmetrization proof by Steiner [113]. Let Ω be smooth, with perimeter $P(\Omega)$, and area $A(\Omega)$. Let $\operatorname{pr} : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate, that is $\operatorname{pr}(x, y) = x$. As illustrated in Figure 1, we construct a new axially symmetric set Ω^* by replacing at every $x \in \mathbb{R}$ the intersection $\operatorname{pr}^{-1}\{x\} \cap \Omega$ with the centered segment ℓ_x with same length. By Fubini, this procedure is area preserving, meaning that $A(\Omega) = A(\Omega^*)$. We claim that

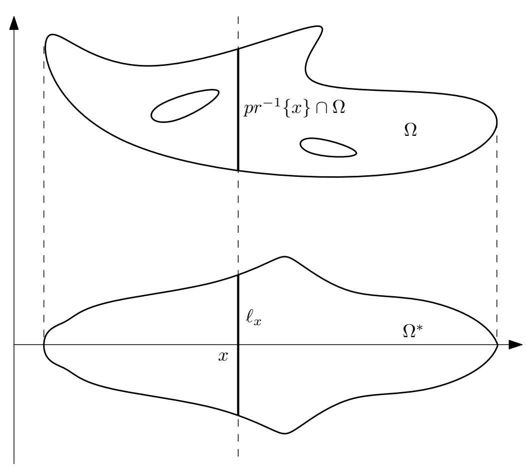

Figure 1: Steiner symmetrization.

$P(\Omega^*) \leq P(\Omega)$. In fact, letting β and β^* be the angles that the normal of $\partial\Omega$ and $\partial\Omega^*$ make

[1] We decided to take T optimal for simplicity of exposition. It is worth saying that the original proof employs an earlier class of transport maps developed by Knothe in the context of monotone rearrangements, see for instance [49, Section 1.5]

with $\frac{\partial}{\partial y}$, by the coarea formula and Jensen inequality we have that

$$\begin{aligned} P(\Omega) &= \int_{\mathbb{R}} \int_{\mathrm{pr}^{-1}\{x\}\cap\partial\Omega} \sqrt{1+\tan(\beta)^2}\, d\mathscr{H}^0\, dx \\ &\geq \int_{\mathbb{R}} \mathscr{H}^0(\mathrm{pr}^{-1}\{x\}\cap\partial\Omega) \sqrt{1+\Big(⨍_{\mathrm{pr}^{-1}\{x\}\cap\partial\Omega} \tan(\beta)\, d\mathscr{H}^0\Big)^2}\, dx \\ &\geq \int_{\mathbb{R}} 2\sqrt{1+\tan(\beta^*)^2}\, dx \\ &= P(\Omega^*), \end{aligned}$$

a wished. Clearly, this procedure can be performed not only with respect to the abscissa, but to any line $L \subset \mathbb{R}^2$. We will denote the associated axially symmetric set with $S_L(\Omega)$. Iterating this operation on Ω with respect to suitable linearly independent lines $L_1, L_2, \dots$ intuition [2] tells us that the resulting sequence of sets $S_{L_1}(\Omega), S_{L_2}(S_{L_1}(\Omega)), \dots$ should converge to some shape that is invariant (up to rigid motion) under *any* symmetrization with respect to *any* line, hence a circle. Since we know that $S_L(\cdot)$ decreases the perimeter and preserves the area, if perimeter and area pass to the limit continuously, we deduce that the circle is in fact an isoperimetric region of the plane for any given area. We sketch this compactness argument: suppose without loss of generality that Ω convex, and notice that $S_L(\cdot)$ preserves convexity. Define the space

$$\mathcal{S}(\Omega) := \{S_{L_1}(S_{L_2}(\dots S_{L_k}(\Omega))) : L_1, \dots, L_k \subset \mathbb{R}^2 \text{ linearly independent}\},$$

endowed with the Gromov-Hausdorff distance. Then, arguing by monotonicity of the circumradius under subsequent symmetrizations (see Chavel [32, Chapter VI.4]), it is possible to show that there always exists a sequence $(\Omega_j) \subset \mathcal{S}(\Omega)$ converging to a disk B_r with same area $A(B_r) = A(\Omega)$. Consider now a minimizing sequence of compact and smooth sets (K_n) such that

$$\lim_{n\to+\infty} P(K_n) = \inf\Big\{P(K) : K \text{ compact, convex, smooth, } A(K) = A(\Omega)\Big\}.$$

By taking subsequent symmetrizations of K_n in $\mathcal{S}(K_n)$, one can modify the sequence so that

$$K_n \subset B_{r+1/n},$$

and hence, by the Blaschke selection principle, up to extracting a subsequence $K_n \to K_\infty$ to some convex set K_∞ in the Gromov-Hausdorff metric. One concludes the argument by continuity of area and perimeter in the class of compact sets. □

[2] The original proof by Steiner was completely formalized by Schwarz [108], and Carathéodory-Study [28], closing a 70-year-old diatribe on the lack of its rigorous foundation, due to Steiner's complete rejection of adopting analytical arguments.

The Modern approach

After the quick historical introduction in the previous section, let us outline how the isoperimetric problem is conceived from a more general point of view. Given some ambient space M and a class of subsets $\mathscr{X}$ for which it makes sense to speak about perimeter $P(\cdot)$ and volume $V(\cdot)$ (for instance $M = \mathbb{R}^n$ and $\mathscr{X}$ the space of smooth sets or more generally (M, g) a Riemannian manifold and $\mathscr{X}$ the collection of Caccioppoli sets) approaching the isoperimetric problem signifies essentially one of the following four things:

i. Characterize the minimal perimeter for a given volume v, that is investigate the *isoperimetric profile*

$$I(v) := \inf\Big\{P(E) : E \in \mathscr{X}, V(E) = v\Big\}.$$

ii. Determine whether the above infimum is attained by some optimal set, that is prove the *existence* of *isoperimetric regions*.

iii. Establish qualitative properties of the isoperimetric regions, such as *uniqueness*, and ultimately an *explicit description*.

iv. Obtain quantitative properties in the form of *stability*, meaning: if a set is very close (in some suitable topology) to an isoperimetric region, is it also going to be almost optimal?

The most popular instance regarding the first point is controlling the isoperimetric profile I from below by some geometrically interesting explicit function. This type of equations are called today isoperimetric inequalities à la Levy-Gromov, in honor of the seminal work of Levy [71] and Gromov [57] in the class of spaces with positive Ricci curvature, and are extensively studied at present. Regarding the second point, existence is usually proven by mean of compactness arguments in the frame of Calculus of Variations as first proven (and never published) in $\mathbb{R}^n$ by Weierstrass during a lecture (collected after in [120]). If M is a compact manifold, existence is always ensured. The non-compact case is very subtle, and existence may fail (see the recent work by Glaudo and Antonelli [7]). If the isometry group acts cocompactly[3] on M however, existence is ensured by an ingenious 'volume trapping' argument by Morgan [83], later generalized by Galli and Ritoré [54].

The explicit description of the isoperimetric sets remains however an extremely hard question since it requires most of the times ad-hoc methods tailored to solve very specific 'well behaved' situations. The symmetrization method by Steiner introduced in the previous section extends to the Euclidean space $\mathbb{R}^n$, the sphere S^{n-1}, and the hyperbolic plane $\mathbb{R}H^n$ proving that geodesic spheres are the unique isoperimetric regions. Apart from the model spaces, we report here a list of spaces in which the isoperimetric sets are known:

[3] Meaning that every point on M can be displaced in a fixed compact subset of M via an isometry.

Authors	M
Steiner 1842, Schwarz 1890, Schmidt 1930s, De Giorgi 1958 [37, 105, 106, 107, 108, 113]	$\mathbb{R}^n$, S^{n-1}, $\mathbb{R}H^n$
Hsiang-Hsiang 1989 [65]	$\mathbb{R} \times \mathbb{R}H^2$
Ritoré, Ros 1992-1996 [97, 98]	$\mathbb{R}P^3$, $T^2 \times \mathbb{R}$
Howards, Hutchings, Morgan 1999 [63]	$S^1 \times \mathbb{R}$, flat $\mathbb{T}^2$ and Klein bottle
Pedrosa, Ritoré 1999 [94]	$S^1 \times \mathbb{R}^n$, $S^1 \times \mathbb{R}H^n$, $S^1 \times S^n$, $2 \leq n \leq 7$
Benjamini, Cao, Howards, Hutchings, Morgan, Pansu, Topping, Ritoré 1990s [14, 85, 92, 96, 118]	Certain surfaces of revolution
Pedrosa 2004 [93]	$\mathbb{R} \times S^2$
Viana 2018 [119]	Some lens spaces

Many elementary spaces are missing: what about $S^2 \times \mathbb{R}^2$, the complex hyperbolic and projective planes $\mathbb{C}H^2$ and $\mathbb{C}P^2$, Lie groups like the smooth Heisenberg group $\mathcal{N}^3$? Even allowing some (pretty sure) forgetfulness, this list shows that we are awfully far from classifying isoperimetric sets in general[4].

Establishing stability results of analytic-geometric inequalities is today an active area of research. In our context, the question of stability reads generally as follows: given a region E and an isoperimetric region B of the same volume, is it possible to estimate the *isoperimetric deficit* $\delta(E) = P(E) - P(B)$ by some appropriate distance between E and B? Fuglede [52] did a first result in this direction showing that in $\mathbb{R}^n$ there exists $c(n) > 0$, such that for every convex $E \subset \mathbb{R}^n$ with barycenter o the following holds:

$$\frac{P(E) - P(B(o))}{P(B(o))} \geq c(n)\Big(\frac{V(E \triangle B(o))}{V(B(o))}\Big)^2,$$

where $B(o)$ is the ball centered in o with same volume as E, and $\triangle$ denotes the symmetric difference. Thirteen years later, Fusco, Maggi, and Pratelli [53] showed the sharp stability result for general Caccioppoli sets in $\mathbb{R}^n$ with the Fraenkel asymmetry distance in the right hand side:

$$\frac{P(E) - P(B)}{P(B)} \geq c(n)\Big(\inf_{x \in \mathbb{R}^n} \frac{V(E \triangle B(x))}{V(B(x))}\Big)^2.$$

The proof, that relies on a delicate symmetrization technique, was further simplified through a penalization argument by Cicalese and Leonardi in [34], and extended to the real hyperbolic space by Bögelein, Duzaar and Scheven in [19].

The results presented in the following two chapters concern stability and optimality of geodesic spheres in a class of manifolds called *rank one symmetric spaces of non-compact type*. Despite the cumbersome name, they constitute a natural generalization of the more familiar real hyperbolic space $\mathbb{R}H^n$, which is one of its simplest members. In the next section we introduce those spaces, discussing their main properties and finally formulating the conjecture of interest.

[4] It is not surprising that allowing boundaries on M has wild consequences: a complete classification of the isoperimetric regions in the cube $[0,1] \times [0,1] \times [0,1]$ is still missing!

Symmetric Spaces and Gromov-Ros Conjecture

As references on symmetric spaces we cite the books of Eberlein [40] and Helgason [61].

Symmetric spaces are a rich class of manifolds characterized by the following property: every geodesic symmetry extends to a global isometry, or said otherwise, they are invariant with respect to every geodesic reflection (for instance consider $\mathbb{R}^n$ or S^{n-1}). Locally, this property is equivalent to the parallel nature of the Riemann tensor $\nabla R = 0$. Their complete classification is due to the monumental work of Élie Cartan in the '30s and Marcel Berger in the late '50s. Surprisingly, symmetric spaces can be realized in an elegant algebraic fashion as the quotient of a *symmetric pair* (G, K), where G is a semisimple Lie group acting transitively on M, and K represents the isotropy group, that is defined as all elements in G fixing an arbitrarily chosen point in M. The classification follows a duality between two categories: *compact* and *non-compact* symmetric spaces, and is delineated by a parameter known as the *rank*, representing the maximum dimension of a subspace within the tangent space at any given point, where the sectional curvature is zero.

We are interested in the class of *rank one symmetric spaces of non-compact type*, which are the real $\mathbb{R}H^n$, complex $\mathbb{C}H^m$, quaternionic $\mathbb{H}H^m$ hyperbolic spaces, and the Cayley plane $\mathbb{O}H^2$. Algebraically, they can be realized as the following quotients:

$M = G/K$	G	K
$\mathbb{R}H^m$	$\mathrm{SO}(m,1)$	$\mathrm{SO}(m)$
$\mathbb{C}H^m$	$\mathrm{SU}(m,1)$	$\mathrm{SU}(m)$
$\mathbb{H}H^m$	$\mathrm{Sp}(m,1)$	$\mathrm{Sp}(m)\,\mathrm{Sp}(1)$
$\mathbb{O}H^2$	F_4^{-20}	$\mathrm{Spin}(9)$

where F_4^{-20} is the real form of rank one of the exceptional Lie group F_4. There is however a more geometric construction, that extends the usual projective model of the real hyperbolic space (Mostow [87, Chapter 19], Bridson and Haefliger [26, p. 300]): for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ a real division algebra of dimension $d \in \{1, 2, 4\}$, we endow the space $\mathbb{K}^{m+1}$ with the pseudo-Hermitian product

$$\langle z, w\rangle := -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k\bar{w}_k.$$

The space $\mathbb{K}H^m$ is obtained by projectivizing the subspace

$$\{z \in \mathbb{K}^{m+1} : \langle z, z\rangle < 1\}$$

via $\pi : (z_0, \ldots, z_m) \mapsto (z_1 z_0^{-1}, \ldots, z_m z_0^{-1})$, endowing the quotient with the Bergmann metric d defined as

$$\cosh^2\Big(d(\pi z, \pi w)\Big) := \frac{\langle z, w\rangle\langle w, z\rangle}{\langle z, z\rangle\langle w, w\rangle}.$$

The construction of the Cayley plane is more subtle due to the lack of associativity of the octonions. We invite the interested reader to consult [87, p. 139].

In general, rank one symmetric spaces enjoy some additionally properties: we know that up to renormalization the sectional curvature lies in $[-4, -1]$, and they are simply connected (hence diffeomorphic to $\mathbb{R}^n$). Moreover, they are two-point homogeneous and harmonic (they allow a non-trivial radial solution of the Laplacian in small punctured balls [70]). For this reason they are often considered as the natural generalization of space forms ($\mathbb{R}^n$, S^{n-1}, $\mathbb{R}H^n$) in the context of non-constant curvature manifolds.

Drawing inspiration from the classical symmetrization procedures by Steiner and Schwarz that we have seen at the beginning of this introduction, a rich isometry group of the ambient space should imply several symmetric invariances of the optimal sets. Following this principle, balls are good candidates, since the isotropy group of their center acts transitively on their boundary. From this philosophical reasoning, the conjecture of interest is

Conjecture. (Gromov-Ros, [58]) Geodesic spheres are isoperimetric regions in all rank one symmetric spaces on non-compact type.

Before giving some additional background in the next section, it is worth spending some words on the differential anatomy of $\mathbb{K}H^m$ and its spheres. Let us describe the complex hyperbolic space $M = \mathbb{C}H^m$, which is in particular a Kähler manifold with constant holomorphic negative curvature. Let J be the complex endomorphism, and $v \in T_oM$ any vector in M. Then, the sectional curvature of M is distributed according to the underlying complex structure, meaning $\sec(v, Jv) = -4$ and $\sec(v, w) = -1$ for every w orthogonal to v and Jv, see Figure 2.

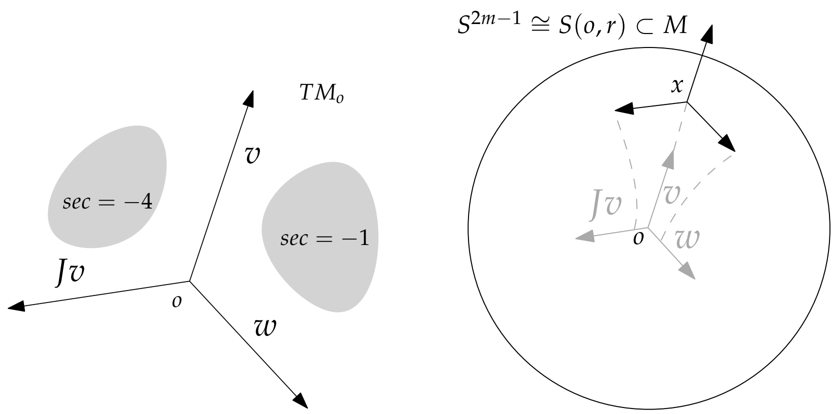

Figure 2: The anatomy of $\mathbb{C}H^m$.

Since $\nabla R = 0$, the arrangement of curvatures is preserved when transporting v, Jv, and w along the geodesic determined by v by any distance $r > 0$, so that on the surface of the sphere centered at o and of radius r there are exactly two principal curvatures: $\coth(r)$ of multiplicity 1 along Jv, and $2\coth(2r)$ with multiplicity $2m-2$ along all possible w's. Hence, we infer that spheres in $\mathbb{C}H^m$ are isometric to *Berger's spheres*, which are round Euclidean spheres of radius $\sinh(r)$, with the metric rescaled by a $\cosh(r)$ factor along the characteristic direction given by $J\nu$, where ν is the normal to the sphere. More generally, a sphere of radius r in $\mathbb{K}H^m$ is isometric to a Euclidean sphere of radius $\sinh(r)$ with the metric rescaled by a $\cosh(r)$ factor along the horizontal distribution induced by the *Hopf-fibration*:

$$S^{d-1} \to S^{n-1}(r) \to \mathbb{K}P^{m-1}.$$

State of the Art

Ensured by the cocompact action of the isometry group on $\mathbb{K}H^m$ (recall that rank one symmetric spaces are two-point homogeneous), existence of isoperimetric sets for all volumes is a classic fact, as we have seen in the previous section. It is also known that spheres are stable under infinitesimal volume preserving perturbations, that is

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} P(B+\varepsilon u) = \delta^2 P(B)[u] \geq 0, \tag{1}$$

for every smooth normal volume preserving perturbation u. This was shown for $\mathbb{C}H^m$ by Barbosa, do Carmo and Eschenberg in their celebrated article [10] (see [99] for the general case). Their proof is a natural consequence of the following variational principle: let Σ be an oriented, closed and immersed submanifold of $\mathbb{K}H^m$, and let Ric_Σ and $\mathrm{I\!I}_\Sigma$ be the induced Ricci curvature tensor and second fundamental form of Σ. If

$$\mathrm{Ric}_\Sigma + \|\mathrm{I\!I}_\Sigma\|^2 = \text{ constant } = \lambda, \tag{2}$$

then Σ is stable if and only if $\lambda = \lambda_1$, the first eigenvalue of the induced Laplacian Δ_Σ. Thanks to a peculiar characterization of the Laplace spectrum over fibre bundles with totally geodesic immersed fibres (about this the work by Bergery and Bourguignon [15]), and the Riemannian structure of the spheres in $\mathbb{K}H^m$ described by the Hopf-fibration

$$S^{d-1} \to S^{n-1}(r) \to \mathbb{K}P^{m-1},$$

the first eigenvalue of the Laplacian is explicitly computable, reducing the proof of stability to a direct check of Equation (2).

In [29], Carron extended this stability result to all geometrically stable hypersurfaces (surfaces that realize the strict inequality in (1) whenever u does not represent an isometry) within the general framework of Riemannian manifolds. The proof, which is based on an implicit argument, applies in particular to space forms, and complex hyperbolic and projective spaces.

It is known that for all Riemannian manifolds with cocompact isometry group, all isoperimetric regions with sufficiently small volume are invariant under the action of the stabilizer of their center of mass. This result was mentioned first by Tomter in the context of the Heisenberg group in [117], referring to an unpublished article by Kleiner. Later, we can find the complete proof as a corollary of a more general result in the article by Nardulli introducing the concept of pseudo-bubbles, see [88]. Since spheres are the only surfaces preserved by the action of the isotropy group in $\mathbb{K}H^m$, we have as a direct implication that they are the unique isoperimetric regions in the small volume regime. This proof relies as well on an implicit argument.

It is worth saying that since optimality at the first order implies constant mean curvature at regular points, this problem is close to the celebrated Alexandrov Theorem [3], which in $\mathbb{C}H^m$ is only conjectured (about this [11, 16, 21, 82]).

Contributions

For clarity we restrict to the space $\mathbb{C}H^m$ the presentation of the results contained in the following two chapters. Similar results hold for all rank one symmetric space of non-compact type, as we will see in detail later on.

In Chapter 1 we prove the Gromov-Ros conjecture in the small volume regime via C^1-quantitative stability of spheres [111]. In contrast with [29, 89], the proof does not rely on an implicit argument.

Theorem (see Theorem 1.1)**.** *For every $R_0 > 0$ there exists $\varepsilon_0 > 0$, such that $\forall R \in (0, R_0]$ if in normal coordinates $\partial E = \{R(1+\rho(\omega))\omega : \omega \in S^{2m-1}\}$ is a volume preserving perturbation of $B(R)$ and $\|\rho\|_{C^1(S^{2m-1})} \leq \varepsilon_0$, then*

$$P(E) - P(B(R)) \geq C\|\rho\|_{W^{1,2}(S^{2m-1})}, \quad C = C(R, R_0) > 0.$$

The key is using the characterization of the Laplace spectrum on fiber bundles with immersed geodesic fibers (Bergery and Bourguignon [15]), and effectively managing the weights arising in the perimeter seen as an anisotropic functional in normal coordinates. The uniformicity in ε_0 allowed the following consequence.

Theorem (see Theorem 1.3)**.** *There exists a possibly computable $v_0 > 0$ such that geodesic balls with volume less than v_0 are uniquely isoperimetric in $\mathbb{C}H^m$.*

The proof is made by rescaling lifted isoperimetric sets on a tangent plane. The theory of almost-minimizing currents enters then into play to prove first L^1, then L^∞-proximity with a centered ball. Finally, regularity theory as developed by Figalli in [48] and deep stability results in $\mathbb{R}^n$ [53] made the conclusion of the argument possible.

In Chapter 2 we prove the Gromov-Ross conjecture in a new class of sets called *Hopf-symmetric* [109]: subsets of $\mathbb{C}H^m \cong \mathrm{SU}(m,1)/\,\mathrm{U}(m)$ invariant under the action of $S^1 \cong e^{i\theta}\mathrm{id}_m \leqslant \mathrm{U}(m)$.

Theorem (see Theorem 2.1)**.** *Balls are uniquely isoperimetric in the class of Hopf-symmetric sets.*

The proof relies on a new comparison argument between $\mathbb{C}H^m$ and $\mathbb{R}H^{2m}$ that makes it a consequence of the following *weighted* isoperimetric result in the real hyperbolic space.

Theorem (see Theorem 2.5)**.** *Centered geodesic balls are uniquely isoperimetric in $\mathbb{R}H^n$ with respect to the* weighted *volume and perimeter*

$$V_f(E) = \int_E f\, d\mathscr{H}^n, \qquad P_f(E) = \int_{\partial E} f\, d\mathscr{H}^{n-1},$$

if f is radial and strictly log-convex.

This result, which is of independent interest, is the hyperbolic version of Brakke's conjecture in the Euclidean space (see notably [51, 69, 86, 101]) which was recently proved by Chambers in [30]. For a partial contribution in $\mathbb{R}H^n$, see [72]. We overcome major geometric obstructions in the application of Chamber's strategy by taking advantage of the hyperbolicity of the underlying space. Notably, we develop a new curvature comparison result for curves in $\mathbb{R}H^2$.

Chapter 1

❊

C^1-Stability and small volume regime

1.1 Main results

The goal of this chapter is to show the following results.

Theorem 1.1. *Let $M = \mathbb{K}H^m$ be a rank one symmetric space of non-compact type and $R_0 > 0$ any fixed radius. Let $E \subset M$ be a volume preserving perturbation of the ball $\mathbf{B}^n(R)$, $0 < R \leq R_0$, with boundary of the form*

$$\partial E = \{\exp_o(R(1+\rho(\varphi))\varphi) : \varphi \in S^{n-1} \subset T_oM\},$$

where $o \in M$ is a fixed base-point, and $\rho \in C^1(S^{n-1}, (-1, +\infty))$. Denote with $\boldsymbol{\rho} : \mathbf{S}^{n-1}(R) \to (-1, +\infty)$ the perturbation ρ viewed as a function from the geodesic sphere in M, given in normal coordinates as

$$\boldsymbol{\rho}(R\varphi) = \rho(\varphi), \quad \varphi \in S^{n-1}.$$

Then, there exist $\varepsilon = \varepsilon(M, R_0) > 0$ and $C = C(M, R_0, R) > 0$, such that

$$\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R)) \geq C\big(\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} + \|\nabla^g \boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}\big),$$

provided $\|\rho\|_{C^1} \leq \varepsilon$. In particular, if E is isoperimetric, then $E = \mathbf{B}^n(R)$.

To establish this result, we will demonstrate the following explicit lower bound under the technical assumption that ρ is barycenter preserving. To obtain Theorem 1.1 we can compensate for this assumption with a small transvection (that amounts to a translation obtained via a composition of central symmetries) of the perturbed set.

Theorem 1.2. *Under the same assumptions of Theorem 1.1, suppose additionally that E has barycenter in $o \in M$. Denote with λ_2^R the second eigenvalue of the Laplacian over $\mathbf{S}^{n-1}(R)$. Then, there exists $\varepsilon = \varepsilon(M, R_0) > 0$ such that*

$$\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R)) \geq \frac{R^2\lambda_2^R}{48}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} + \frac{R^2}{32}\|\nabla^g \boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))},$$

provided $\|\rho\|_{C^1} \leq \varepsilon$.

As an application of Theorem 1.2, we will give a new quantitative proof of the isoperimetric theorem in the small volume regime.

Theorem 1.3. *Let $M = \mathbb{K}H^m$ be a rank one symmetric space of non-compact type. Then, there exists a possibly computable volume $\bar{v} = \bar{v}(M) > 0$ such that all geodesic balls $\mathbf{B}^n(R) \subset M$ with volume $\mathrm{Vol}_g(B(R)) < \bar{v}$ are uniquely isoperimetric in M.*

Remark 1.4. *With exactly the same arguments, the results of Theorem 1.1 and Theorem 1.2 hold true under the weaker assumption of ρ belonging to the Sobolev space $W^{1,\infty}(S^1, (-1, +\infty))$.*

1.1.1 Distributions and spectral decomposition on spheres

Recall that geodesic spheres

$$\mathbf{S}^{n-1}(o, R) := \{x \in M : \mathrm{dist}_g(x, o) = R\},$$

centered at $o \in M$ with radius $R > 0$, are homogeneous submanifolds (that is admitting a transitive action of the isometry group) of constant mean curvature. Since all spheres with the same radius are isometric, we will often denote with $\mathbf{S}^{n-1}(R)$ a generic geodesic sphere in M of radius $R > 0$ with induced metric that we will keep calling g. Analogously, we denote with

$$\mathbf{B}^n(o, R) := \{x \in M : \mathrm{dist}_g(x, o) < R\},$$

the open geodesic ball centered in $o \in M$ with radius $R > 0$, and with $\mathbf{B}^n(R)$ a generic open geodesic ball of radius $R > 0$ in M.

For every non-zero vector N_x at $x \in M$, the Jacobi operator

$$\mathbf{R}(\cdot, N_x)N_x : T_xM \to T_xM,$$

has exactly three eigenvalues: $\{-4, -1, 0\}$. Denoting with $\mathcal{H}_x$ and $\mathcal{V}_x$ the eigenspaces associated to the eigenvalues -4 and -1 respectively, the tangent plane T_xM splits orthogonally as

$$T_xM = \mathcal{H}_x \oplus \mathcal{V}_x \oplus \mathbb{R}N_x, \tag{1.1}$$

where $\dim_{\mathbb{R}}(\mathcal{H}_x) = d - 1$, and $\dim_{\mathbb{R}}(\mathcal{V}_x) = d(n-1)$. Hence, for every non-vanishing vector field $N \in \Gamma(TU)$ defined on an open set $U \subset M$, the maps $x \mapsto \mathcal{H}_x$ and $x \mapsto \mathcal{V}_x$ induce two well defined distributions $\mathcal{H}$ and $\mathcal{V}$ on U. We will denote the orthogonal projections with

$$\begin{aligned}(\cdot)^h :& TU \to \mathcal{H},\\ (\cdot)^v :& TU \to \mathcal{V},\\ (\cdot)^n :& TU \to \mathbb{R}N.\end{aligned}$$

Notice that when $\mathbb{K} = \mathbb{R}$, then $\mathcal{H} = \emptyset$. In this exceptional case we set $(\cdot)^v \equiv 0$. In particular, when N is a radial vector field emanating from a base point $o \in M$ (that is the vector field $\frac{\partial}{\partial r}$ in normal coordinates (r, φ) centered at o), then the orthogonality of (1.1) implies that the tangent bundle of any sphere $\mathbf{S}^{n-1}(o, R)$ splits orthogonally with respect to g as the direct sum of $\mathcal{H}$ and $\mathcal{V}$ restricted to $\mathbf{S}^{n-1}(o, R)$. It turns out that this splitting also arises from the

vertical and horizontal distribution associated to the celebrated Hopf fibration of Euclidean spheres
$$S^{d-1} \to S^{m-1}(R) \to \mathbb{K}P^{m-1},$$
where $\mathbb{K}P^{m-1}$ is the real, complex, quaternionic and octonionic projective space of complex dimension $\dim_{\mathbb{K}}(\mathbb{K}P^{m-1}) = m-1$, respectively. This particular structure allows computations about the spectral decomposition of $L^2(\mathbf{S}^{n-1}(R), g)$ with respect to the induced Riemannian Laplacian, see [10, 15, 99] and very recently [17]. In our case, it will be sufficient to know that the associated eigenvalues $\{\lambda_i^R\}_{i\in\mathbb{N}}$ satisfy the bound

$$\lambda_j^R \geq \frac{j(j+d-2) + j(n-d)\cosh^2(R)}{\sinh^2(R)\cosh^2(R)}, \tag{1.2}$$

with equality when $j = 1$. We will denote with

$$\{f_{j,k}^R \in L^2(\mathbf{S}^{n-1}(R), g) : 1 \leq k \leq n_j\}_{j\geq 0}$$

the spherical harmonics with muliplicity $n_j \geq 1$ constituting an orthogonal basis of $L^2(\mathbf{S}^{n-1}(R), g)$, so that
$$\|\nabla^g f_{j,k}^R\|^2_{L^2(\mathbf{S}^{n-1}(R),g)} = \lambda_j^R \|f_{j,k}^R\|^2_{L^2(\mathbf{S}^{n-1}(R),g)},$$
where ∇^g denotes the Riemannian gradient with respect to g on $\mathbf{S}^{n-1}(R)$.

1.1.2 Useful geometric identities by comparison

Denote with $g_e = \langle\cdot,\cdot\rangle$ and $|\cdot|$ the usual Euclidean metric and norm on $\mathbb{R}^n$, and with $S^{n-1}(x,r)$ and $B^n(x,r)$ the Euclidean spheres and open balls centered in $x \in \mathbb{R}^n$ with radius $r > 0$. As usual, S^{n-1} and B^n denote generic unit spheres and balls. In order to do computations in (M,g) we decided to work in normal coordinates $(r,\varphi) \in (0,+\infty) \times S^{n-1}$. Let $\mathrm{P}(\cdot)$ and $\mathrm{V}(\cdot)$ be the perimeter and volume operators in $\mathbb{R}^n$ with respect to the Euclidean metric. Set
$$\boldsymbol{\omega}_n := \mathrm{V}(B^n).$$
From now on, $o \in M$ will be an arbitrarily fixed base-point, if not stated otherwise. Taking the pullback metric $\exp_o^* g$ we can identify isometrically M with $\mathbb{R}^n$. Let $N \in \Gamma(M \setminus \{o\})$ be the radial, unit vector field emanating from o. Thanks to the previous discussion, we can find an explicit formula relating g_e with g.

Lemma 1.5. *For every $x = (r,\varphi) \in M \setminus \{o\}$, the splitting*

$$T_xM = \mathcal{H}_x \oplus \mathcal{V}_x \oplus \mathbb{R}N_x,$$

is orthogonal with respect to g_e. In particular, one has that

$$g(X,Y) = \langle X^n, Y^n\rangle + \frac{\cosh^2(r)\sinh^2(r)}{r^2}\langle X^h, Y^h\rangle + \frac{\sinh^2(r)}{r^2}\langle X^v, Y^v\rangle, \tag{1.3}$$

for all $X, Y \in T_xM$.

Proof. Fix an arbitrary unit direction $N_o \in T_oM$, and let $V_o \in T_oM$ be any vector orthogonal to it with respect to $g|_o = g_e|_o$. Since the radial geodesics emanating from o with respect to g are the same as the Euclidean ones, the Jacobi field $Y(t)$ along the geodesic $\sigma : t \mapsto tN_o$, determined by the initial conditions $Y(0) = 0$, $\dot{Y}(0) = V_o$ is the same for both metrics. Let $V(t)$ and $V_e(t)$ be the parallel transport of V_o along σ with respect to g and g_e, respectively. By the very definition of symmetric spaces, the curvature tensor $\mathbf{R}$ is itself parallel along geodesics. This implies that

$$tV_e(t) = Y(t) = \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} V(t),$$

provided V_o belongs to the κ-eigenspace of the Jacobi operator $\mathbf{R}(\cdot, N_o)N_o$. Therefore, parallel vector fields in the eigenspaces are collinear for the two metrics. Hence, for $t > 0$ the linear subspaces $\mathcal{H}_{\sigma(t)}$ and $\mathcal{V}_{\sigma(t)}$ are nothing else than the parallel transport of the corresponding eigenspaces of $\mathbf{R}(\cdot, N_o)N_o$ along σ. It follows that the splitting $T_xM = \mathcal{H}_x \oplus \mathcal{V}_x$ is orthogonal not only with respect to g, but also with respect to the Euclidean metric g_e. Equation (1.3) is a direct consequence of this fact and the definition of the distribution $\mathcal{H}$ and $\mathcal{V}$. □

In particular, the volume density on M associated to the metric g is radial and given by

$$\omega_g(r, \varphi) = \omega_g(r) = \frac{\sinh^{n-1}(r)\cosh^{d-1}(r)}{r^{n-1}}. \tag{1.4}$$

Let $\mathrm{Per}_g(\cdot)$ and $\mathrm{Vol}_g(\cdot)$ be the perimeter and volume operators in M. As a consequence of the previous Lemma we have the following formulae.

Lemma 1.6. *Let E be a subset of M with smooth boundary. Then, in normal coordinates we have that*

$$\mathrm{Vol}_g(E) = \int_E \omega_g(r)\, d\mathscr{H}^n, \tag{1.5}$$

and

$$\mathrm{Per}_g(E) = \int_{\partial E} \omega_g(r) \Big(|\nu^n|^2 + \frac{r^2}{\cosh^2(r)\sinh^2(r)} |\nu^h|^2 + \frac{r^2}{\sinh^2(r)} |\nu^v|^2 \Big)^{1/2} d\mathscr{H}^{n-1}, \tag{1.6}$$

where ν denotes the normal vector field to ∂E with respect to g_e.

Proof. Equation (1.5) is tautological. We prove Equation (1.6). Denoting with

$$\mathrm{vol}_g := \omega_g(r)dx = \omega_g(r)dx^1 \wedge \cdots \wedge dx^n,$$

the volume form in M, and with ν_g and ν the normal vector field of ∂E with respect to g and g_e respectively, we have that

$$\mathrm{Per}_g(E) = \int_{\partial E} \iota_{\nu_g} \mathrm{vol}_g = \int_{\partial E} \omega_g(r) \iota_{\nu_g} \mathrm{vol},$$

where we denote the interior product $(\iota_{\nu_g}\operatorname{vol}_g)(\cdot) = \operatorname{vol}_g(\nu_g,\cdot) \in \Omega^{n-1}(\partial E)$. Now, for a fixed $x \in \partial E$, choose an orthonormal basis $\{v_2,\dots,v_n\}$ of $T_x\partial E$ orthogonal to ν with respect to g_e. Then,

$$\begin{aligned}(\iota_{\nu_g}\operatorname{vol})_x(v_2,\dots,v_n) &= \operatorname{vol}_x(\nu_g,v_2,\dots,v_n) = \langle \nu_g,\nu\rangle \operatorname{vol}_x(\nu,v_2,\dots,v_n)\\ &= \langle \nu,\nu_g\rangle(\iota_\nu \operatorname{vol})_x(v_2,\dots,v_n),\end{aligned}$$

showing that

$$\operatorname{Per}_g(E) = \int_{\partial E}\omega_g(r)\langle\nu,\nu_g\rangle\iota_\nu\operatorname{vol} = \int_{\partial E}\omega_g(r)\langle\nu,\nu_g\rangle\, d\mathscr{H}^{n-1}.$$

We are left to compute $\langle\nu,\nu_g\rangle$. By Lemma 1.5 we have that

$$\tilde\nu_g := \nu^n + \frac{r^2}{\cosh^2(r)\sinh^2(r)}\nu^h + \frac{r^2}{\sinh^2(r)}\nu^v,$$

realizes $g(\tilde\nu_g, v_i) = \langle\nu, v_i\rangle = 0$ for all $i = 2,\dots,n$, implying that $\tilde\nu_g$ is collinear to ν_g. Since $g(\tilde\nu_g,\tilde\nu_g) = \langle\nu,\tilde\nu_g\rangle$ we get that

$$\begin{aligned}\langle\nu,\nu_g\rangle &= \langle\nu,\tilde\nu_g\rangle g(\tilde\nu_g,\tilde\nu_g)^{-1/2} = \langle\nu,\tilde\nu_g\rangle^{1/2}\\ &= \Big(|\nu^n|^2 + \frac{r^2}{\cosh^2(r)\sinh^2(r)}|\nu^h|^2 + \frac{r^2}{\sinh^2(r)}|\nu^v|^2\Big)^{1/2},\end{aligned}$$

concluding the proof of the lemma. □

Define the barycenter of E as

$$\operatorname{Bar}_g(E) := \operatorname{argmin}_{p\in M}\Big\{\int_E \operatorname{dist}_g^2(x,p)\, d\operatorname{vol}_g(x)\Big\} \in M.$$

It is always unique and well defined since the negative curvature of M implies that the above functional is strictly convex in $p \in M$, see [19, Section 2.5]. Differentiating, we have that $p = \operatorname{Bar}_g(E)$ if and only if

$$-2\int_E \exp_p^{-1}(x)\, d\operatorname{vol}_g(x) = 0. \tag{1.7}$$

In the normal coordinates pointed at $\operatorname{Bar}_g(E)$, this reads as

$$0 = \int_E x\, d\operatorname{vol}_g(x) = \int_E r\varphi\omega_g(r)\, d\mathscr{H}^n. \tag{1.8}$$

In the particular case in which ∂E is a C^1-radial perturbation of $\mathbf{S}^{n-1}(R)$

$$\partial E = \{\exp_o(R(1+\rho(\varphi))) : \varphi\in S^{n-1}\} = \{(R(1+\rho(\varphi)),\varphi) : \varphi \in S^{n-1}\}, \tag{1.9}$$

for some C^1-function $\rho : S^{n-1} \to (-1,+\infty)$, then the normal ν with respect to g_e is given by

$$\nu = \Big(\varphi - \frac{\nabla\rho}{1+\rho}\Big)\Big(1 + \frac{|\nabla\rho|^2}{(1+\rho)^2}\Big)^{-1/2},$$

where ∇ denotes the gradient with respect to the round metric on S^{n-1}. Applying Equation (1.6) of Lemma 1.6 one gets that

$$\operatorname{Per}_g(E) = \int_{S^{n-1}} \omega_g(r) r^{n-1} \Big(1 + R^2 \frac{|\nabla^h \rho|^2 + \cosh^2(r)|\nabla^v \rho|^2}{\sinh^2(r)\cosh^2(r)}\Big)^{1/2}\Big|_{r=R(1+\rho(\varphi))} d\varphi, \tag{1.10}$$

where $\nabla^h \rho$ and $\nabla^v \rho$ are the projections of the vector $\nabla \rho \in T_{(R(1+\rho(\varphi)),\varphi)}M$ on $\mathcal{H}$ and $\mathcal{V}$ respectively. To simplify the exposition, define

$$\phi(r) := \int_0^r \tau^{n-1} \omega_g(\tau)\, d\tau, \tag{1.11}$$

and

$$\psi(r) := \int_0^r \tau^{n} \omega_g(\tau)\, d\tau, \tag{1.12}$$

where we recall that $\omega_g(r)$ is the volume density defined in (1.4). Then, we obtain the formula

$$\operatorname{Vol}_g(E) = \int_{S^{n-1}} \phi(R(1+\rho))\, d\varphi, \tag{1.13}$$

and when the barycenter is at zero

$$0 = \int_{S^{n-1}} \varphi \psi(R(1+\rho(\varphi))) d\varphi. \tag{1.14}$$

Setting $\rho \equiv 0$, we recover the volume and perimeter of the geodesic ball $\mathbf{B}^n(R)$:

$$\operatorname{Vol}_g(\mathbf{B}^n(R)) = n\boldsymbol{\omega}_n \phi(R), \quad \operatorname{Per}_g(\mathbf{B}^n(R)) = n\boldsymbol{\omega}_n \phi'(R).$$

For example, when $\mathbb{K} = \mathbb{C}$, we can compute

$$\operatorname{Vol}_g(\mathbf{B}^n(R)) = \boldsymbol{\omega}_n \sinh^n(R), \quad \operatorname{Per}_g(\mathbf{B}^n(R)) = n\boldsymbol{\omega}_n \sinh^{n-1}(R)\cosh(R).$$

1.1.3 Finite perimeter sets

We recall the definition and some properties of finite perimeter sets in a general Riemannian manifold. We refer to [76] for a detailed presentation in the Euclidean space.

Definition 1.7 (Sets with finite perimeter)**.** Let (M, g) be a smooth Riemannian manifold with volume element $d\,\mathrm{vol}$, and $E \subset M$ be a measurable subset. For any open subset $O \subset M$ we will denote with $\Gamma_c(TO)$ the set of smooth vector fields on M compactly supported in O. We define the relative perimeter of E in O as

$$\operatorname{Per}_g(E, O) := \sup\Big\{ \int_O \operatorname{div}^g(\xi)\, d\,\mathrm{vol}_g, : \xi \in \Gamma_c(TO),\ \sup_{x \in O} g(\xi, \xi) \leq 1 \Big\}.$$

If $\operatorname{Per}_g(E, O) < +\infty$ for all $O \subset\subset M$ we say that E is a set with locally finite perimeter, and if $\operatorname{Per}_g(E) := \operatorname{Per}_g(E, M) < +\infty$ we say that E is a set with finite perimeter.

Letting $D\chi_E$ be the distributional gradient of the characteristic function χ_E, then

$$\mathrm{Per}_g(E,O) = |D\chi_E|(O),$$

where $|D\chi_E|$ denotes the total variation of the measure $D\chi_E$.

Definition 1.8. Let $E \subset M$ be a set of locally finite perimeter. We define its reduced boundary as

$$\partial^* E := \Big\{ x \in \mathrm{spt}(|D\chi_E|) : \exists\, \nu_g(x) := \lim_{r\to 0^+} -\frac{D\chi_E(B_x(r))}{|D\chi_E(B_x(r))|} \text{ unit tangent vector at } x \Big\}.$$

The next theorem allows us to express $\mathrm{Per}_g(E)$ as an integration over the reduced boundary, where $\nu_g(x)$, the measure theoretic outwards unit normal to E, is well defined. For the proof we refer to [6].

Theorem 1.9 (De Giorgi structure theorem)**.** *Let $E \subset M$ be a set with locally finite perimeter. Then,*

$$D\chi_E = \nu_g(x) d\mathscr{H}^{n-1}|_{\partial^* E}, \text{ and } P(E) = |D\chi_E|(M) = \mathscr{H}^{n-1}(\partial^* E).$$

This characterization allows us to generalize Equation (1.6) of Lemma 1.6 for the class of finite perimeter sets.

Lemma 1.10. *Let $E \subset M$ be a finite perimeter set. Then, for all open subset O of M we have that*

$$\mathrm{Per}_g(E,O) = \int_{\partial^* E \cap O} \omega_g(r) \Big(|\nu^n|^2 + \frac{r^2}{\cosh^2(r)\sinh^2(r)} |\nu^h|^2 + \frac{r^2}{\sinh^2(r)} |\nu^v|^2 \Big)^{1/2} d\mathscr{H}^{n-1}, \quad (1.15)$$

where ν is the measure theoretic outwards unit normal to E with respect to the flat metric g_e.

1.2 Uniform C^1-strong stability

Fix any upper bound $R_0 > 0$ and a radius $R \in (0, R_0]$ for the perturbed sphere given in normal coordinates

$$\partial E = \{ (R(1+\rho(\varphi)), \varphi) : \varphi \in S^{n-1} \} \subset M,$$

where $\rho \in C^1(S^{n-1}, (-1, +\infty))$ is a volume and barycentric preserving perturbation, in the sense that

$$\mathrm{Vol}_g(E) = \mathrm{Vol}_g(\mathbf{B}^n(R)),$$

and

$$\mathrm{Bar}_g(E) = \mathrm{Bar}_g(\mathbf{B}^n(R)) = 0.$$

We suppose

$$\|\rho\|_{C^1} := \|\rho\|_{C^1(S^{n-1})} = \sup_{\varphi \in S^{n-1}} \Big(|\rho(\varphi)| + |\nabla \rho(\varphi)| \Big) \le \varepsilon,$$

for some small $\varepsilon \in (0, \frac{1}{2})$ yet to define. To simplify the exposition, denote

$$\bar{\rho}(\varphi) := R(1 + \rho(\varphi)),$$

and with $\boldsymbol{\rho} : \mathbf{S}^{n-1}(R) \to (-1, +\infty)$ the perturbation viewed as a function from the geodesic sphere in M, given in normal coordinates as

$$\boldsymbol{\rho}(R\varphi) = \rho(\varphi), \quad \varphi \in S^{n-1}.$$

Notice that

$$\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} := \|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R),g)} = \int_{S^{n-1}} |\rho(\varphi)|^2 \phi'(R)\, d\varphi,$$

and

$$\|\nabla^g \boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} := \|\nabla^g \boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R),g)} = \int_{S^{n-1}} \frac{|\nabla^h \rho|^2 + \cosh^2(R)|\nabla^v \rho|^2}{\sinh^2(R)\cosh^2(R)} \phi'(R)\, d\varphi. \quad (1.16)$$

For $k \in \{1, 2, 3\}$, define the auxiliary functions $\omega_k : \mathbb{R} \to \mathbb{R}$ by

$$\omega_k(r) := \frac{\phi^{(k)}(r)}{r^{n-k}}, \quad (1.17)$$

where $\phi(r)$ is defined as in (1.11). Notice that $\omega_1 = \omega_g$. We need the next three lemmas to start with the estimates.

Lemma 1.11. *For $k \in \{1, 2, 3\}$, the function ω_k defined in (1.17) is positive, even and strictly convex with removable singularity at zero equal to*

$$\lim_{r \to 0} \omega_k(r) = \begin{cases} 1, & k = 1, \\ (n-1), & k = 2, \\ (n-1)(n-2), & k = 3. \end{cases}$$

In particular, $\min \omega_k = \omega_k(0)$, which is strictly positive, unless $(n, k) = (2, 3)$. The constants

$$A_k := (n-k) + \max_{r \in [0, R_0]} r \frac{\omega_k'(r)}{\omega_k(r)},$$
$$B_k := 2^n \max_{r \in [0, R_0]} \frac{\omega_k(2r)}{\omega_k(r)},$$
$$C_k := \omega_k(2R_0),$$

are finite, depend only on (n, d, k, R_0), and realize the following inequalities

$$\phi^{(k)}(R(1+\tau)) \geq (1 - A_k|\tau|)\phi^{(k)}(R), \quad (1.18)$$
$$\phi^{(k)}(R(1+\tau)) \leq B_k \phi^{(k)}(R), \quad (1.19)$$
$$\omega_k(R(1+\tau)) \leq C_k, \quad (1.20)$$

uniformly in $R \in [0, R_0]$ and $\tau \in \mathbb{R}$, $|\tau| \leq 1$, where $\phi(r)$ is as in (1.11).

Proof. We explicitly compute

$$\begin{aligned}
r^{n-1}\omega_1(r) &= \phi'(r) = \sinh^{n-1}(r)\cosh^{d-1}(r),\\
r^{n-2}\omega_2(r) &= \phi''(r) = (n-1)\sinh^{n-2}(r)\cosh^{d}(r) + (d-1)\sinh^{n}(r)\cosh^{d-2}(r),\\
r^{n-3}\omega_3(r) &= \phi'''(r) = (n-1)(n-2)\sinh^{n-3}(r)\cosh^{d+1}(r)\\
&\qquad + (2dn-d-n)\sinh^{n-1}(r)\cosh^{d-1}(r) + (d-1)(d-2)\sinh^{n+1}(r)\cosh^{d-3}(r).
\end{aligned}$$

Since the functions $r^{-1}\sinh(r)$ and $r\sinh(r)$ are even, strictly convex functions with positive (removable singularity) at zero, $\sinh(r)$ is odd and $\cosh(r)$ is even, we can infer that ω_k is itself convex, even and positive. Developing by Taylor we get that

$$\begin{aligned}
r^{n-1}\omega_1(r) &= r^{n-1} + o(r^n),\\
r^{n-2}\omega_2(r) &= (n-1)r^{n-2} + o(r^{n-1}),\\
r^{n-3}\omega_3(r) &= (n-1)(n-2)r^{n-3} + (2dn-d-n)r^{n-1} + o(r^n),
\end{aligned}$$

proving that ω_k can be extended at zero with value 1, $(n-1)$ and $(n-1)(n-2)$, according to k being equal to 1, 2 or 3 and that

$$\lim_{r\to 0} r\frac{\omega_k'(r)}{\omega_k(r)} = \begin{cases} 2, & \text{if } (n,k)=(2,3),\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad \lim_{r\to 0}\frac{\omega_k(2r)}{\omega_k(r)} = \begin{cases} 4, & \text{if } (n,k)=(2,3),\\ 1, & \text{otherwise.}\end{cases}$$

This shows that A_k and B_k are well defined finite constants. To prove Equation (1.18), since ω_k is convex and $(1+\tau)^l \geq 1 + l\tau$ for all $|\tau|\leq 1$ and $l\in\mathbb{Z}$, we can estimate

$$\begin{aligned}
\phi^{(k)}(R(1+\tau)) &= R^{n-k}(1+\tau)^{n-k}\omega_k(R(1+\tau))\\
&\geq (1+(n-k)\tau)R^{n-k}\big(\omega_k(R) + \omega_k'(R)R\tau\big)\\
&= (1+(n-k)\tau)\big(1 + R\frac{\omega_k'(R)}{\omega_k(R)}\tau\big)\phi^{(k)}(R)\\
&\geq (1-A_k|\tau|)\phi^{(k)}(R).
\end{aligned}$$

Equations (1.19) and (1.20) are immediate, given the nature of ω_k and the bound on τ. □

Lemma 1.12. *For every function $\rho\in C^1(S^1,(-1,+\infty))$ and $0<R\leq R_0$ there exists a constant $D>0$, depending only on R_0, such that*

$$|\nabla\rho|^2 \geq \bar\rho^2\frac{|\nabla^h\rho|^2 + \cosh^2(\bar\rho)|\nabla^v\rho|^2}{\sinh^2(\bar\rho)\cosh^2(\bar\rho)} \geq (1-D|\rho|)R^2\frac{|\nabla^h\rho|^2+\cosh^2(R)|\nabla^v\rho|^2}{\sinh^2(R)\cosh^2(R)}, \tag{1.21}$$

where $\bar\rho\in C^1(S^{n-1},(0,+\infty))$ is defined as $\bar\rho(\varphi) = R(1+\rho(\varphi))$.

Proof. Setting

$$\begin{aligned}
\omega_1^1(r) &:= \frac{\sinh^2(r)}{r^2},\\
\omega_1^2(r) &:= \frac{\sinh^2(r)\cosh^2(r)}{r^2},
\end{aligned}$$

and arguing as in Lemma 1.11, we notice that ω_1^1 and ω_1^2 are strictly convex, even and equal to one at zero. Setting

$$\xi(r) := \frac{|\nabla^h \rho|^2}{\omega_1^2(r)} + \frac{|\nabla^v \rho|^2}{\omega_1^1(r)},$$

we get that

$$\xi(r) \leq \frac{1}{\omega_1^1(r)}(|\nabla^h \rho|^2 + |\nabla^v \rho|^2) \leq |\nabla \rho|^2,$$

which is the first inequality of Equation (1.21) when $r = \bar{\rho}$. For the second inequality, for $i = 1, 2$ we have by convexity of ω_1^i that

$$\omega_1^i(R) \geq \omega_1^i(\bar{\rho}) - (\omega_1^i)'(\bar{\rho})R\rho,$$

implying that

$$\begin{aligned} \omega_1^i(\bar{\rho}) &\leq \omega_1^i(R) + (\omega_1^i)'(\bar{\rho})R\rho \leq \omega_1^i(R)\Big(1 + R\frac{(\omega_1^i)'(\bar{\rho})}{\omega_1^i(R)}|\rho|\Big) \\ &\leq \omega_1^i(R)(1 + R_0(\omega_1^i)'(2R_0)|\rho|), \end{aligned}$$

and

$$\frac{1}{\omega_1^i(\bar{\rho})} \geq \frac{1}{1 + R_0(\omega_1^i)'(2R_0)|\rho|}\frac{1}{\omega_1^i(R)} \geq (1 - R_0(\omega_1^i)'(2R_0)|\rho|)\frac{1}{\omega_1^i(R)}.$$

Hence, setting $D := \max_{i \in \{1,2\}}\{R_0(\omega_1^i)'(2R_0)\}$, we can estimate

$$\xi(\bar{\rho}) \geq (1 - D|\rho|)\xi(R),$$

completing the proof of the Lemma. □

Recall that we denote with λ_1^R the first non-zero eigenvalue of the Laplacian operator on the sphere $\mathbf{S}^{n-1}(R)$.

Lemma 1.13. *Let $\phi(r)$ as in (1.11). For all $r > 0$ we have the following identity*

$$\phi'''(r) = \phi'(r)\Big(\frac{\phi''(r)^2}{\phi'(r)^2} - \lambda_1^r\Big), \tag{1.22}$$

where λ_1^r is the first eigenvalue of the Laplacian on the sphere $\mathbf{S}^{n-1}(r)$.

Proof. Since $\phi'(r) = r^{n-1}\omega_g(r) = \sinh^{n-1}(r)\cosh^{d-1}(r)$ we can compute

$$\begin{aligned} \phi''(r) &= (n-1)\sinh^{n-2}(r)\cosh^d(r) + (d-1)\sinh^n(r)\cosh^{d-2}(r) \\ &= (n-1)\coth(r)\phi'(r) + (d-1)\tanh(r)\phi'(r), \end{aligned}$$

and

$$\begin{aligned}\phi'''(r) &= \phi''(r)\Big((n-1)\coth(r) + (d-1)\tanh(r)\Big) + \phi'(r)\Big(\frac{d-1}{\cosh^2(r)} - \frac{n-1}{\sinh^2(r)}\Big)\\ &= \frac{\phi''(r)^2}{\phi'(r)} + \phi'(r)\frac{(d-1)\sinh^2(r) - (n-1)\cosh^2(r)}{\cosh^2(r)\sinh^2(r)}\\ &= \frac{\phi''(r)^2}{\phi'(r)} + \phi'(r)\frac{(d-1) - (n-1-d+1)\cosh^2(r)}{\cosh^2(r)\sinh^2(r)} = \frac{\phi''(r)^2}{\phi'(r)} - \phi'(r)\lambda_1^r,\end{aligned}$$

as wished. □

We are now ready to prove a first estimate.

Proposition 1.14 (Intermediate estimate)**.** *Under the assumptions of Theorem 1.2 one has that*

$$\begin{aligned}\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R)) \geq -\Big(\Big(\frac{B_3C_2C_3}{3} + A_3C_2^2\Big)\|\rho\|_{C^0} + R^2\lambda_1^R\Big)\frac{1}{2}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}\\ + R^2(1-(D+2)\|\rho\|_{C^1})\frac{1}{2}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}.\end{aligned} \tag{1.23}$$

Here the constants A_k, B_k, C_k and D have been defined in Lemma 1.11 and Lemma 1.12.

Proof. Setting

$$\xi(r) := r^2\frac{|\nabla^h\rho|^2 + \cosh^2(r)|\nabla^v\rho|^2}{\sinh^2(r)\cosh^2(r)},$$

we have by Equation (1.10) that

$$\mathrm{Per}_g(E) = \int_{S^{n-1}}\Big(1 + \frac{\xi(\bar\rho)}{(1+\rho)^2}\Big)^{1/2}\phi'(\bar\rho)\,d\varphi.$$

By the elementary inequalities

$$\frac{1}{(1+t)^2} \geq (1-2t),\ \text{for all } t>-1,\ \text{and } (1+t)^{1/2} \geq 1 + \frac{t}{2} - \frac{t^2}{8} \text{ for all } t \geq 0,$$

we have that

$$\begin{aligned}\mathrm{Per}_g(E) &\geq \int_{S^{n-1}}(1+\xi(\bar\rho)(1-2\rho))^{1/2}\phi'(\bar\rho)\,d\varphi\\ &\geq \int_{S^{n-1}}\Big(1 + (1-2\rho)\Big(1-(1-2\rho)\frac{\xi(\bar\rho)}{4}\Big)\frac{\xi(\bar\rho)}{2}\Big)\phi'(\bar\rho)\,d\varphi\\ &\geq \int_{S^{n-1}}\phi'(\bar\rho)\,d\varphi + \int_{S^{n-1}}\Big(1-2\rho-\xi(\bar\rho)\Big)\frac{\xi(\bar\rho)}{2}\phi'(\bar\rho)\,d\varphi\\ &\geq \int_{S^{n-1}}\phi'(\bar\rho)\,d\varphi + (1-2\|\rho\|_{C^1})(1-D\|\rho\|_{C^0})\int_{S^{n-1}}\frac{\xi(R)}{2}\phi'(\bar\rho)\,d\varphi\\ &\geq \int_{S^{n-1}}\phi'(\bar\rho)\,d\varphi + (1-(D+2)\|\rho\|_{C^1})\int_{S^{n-1}}\frac{\xi(R)}{2}\phi'(\bar\rho)\,d\varphi\\ &= \int_{S^{n-1}}\phi'(\bar\rho)\,d\varphi + (1-(D+2)\|\rho\|_{C^1})\frac{1}{2}R^2\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))},\end{aligned} \tag{1.24}$$

where at the end we took advantage of Equations (1.16) and (1.21). We need to treat the first term

$$\int_{S^{n-1}} \phi'(\bar\rho)\, d\varphi.$$

Now, by Taylor expansion there exists $\eta : S^{n-1} \to [0,1]$ such that

$$\phi'(\bar\rho) = \phi'(R) + \phi''(R)R\rho + \frac{R^2\rho^2}{2}\phi'''(R(1+\eta\rho)).$$

On the other side, thanks to Lemma 1.11 and Lemma 1.13 we have that

$$\begin{aligned}\phi'''(R(1+\eta\rho)) &\geq (1 - A_3\|\rho\|_{C^0})\phi'''(R) = (1 - A_3\|\rho\|_{C^0})\Big(\frac{\phi''(R)^2}{\phi'(R)^2} - \lambda_1^R\Big)\phi'(R)\\ &\geq \frac{\phi''(R)^2}{\phi'(R)} - \Big(A_3\Big(\frac{R^{n-2}\omega_2(R)}{R^{n-1}\omega_1(R)}\Big)^2\|\rho\|_{C^0} + \lambda_1^R\Big)\phi'(R)\\ &\geq \frac{\phi''(R)^2}{\phi'(R)} - \Big(A_3C_2^2R^{-2}\|\rho\|_{C^0} + \lambda_1^R\Big)\phi'(R),\end{aligned}$$

which gives the following estimate

$$\begin{aligned}\int_{S^{n-1}} \phi'(\bar\rho)\, d\varphi \geq \mathrm{Per}_g(\mathbf{B}^n(R)) &+ \int_{S^{n-1}} \phi''(R)R\rho + \frac{\phi''(R)^2}{\phi'(R)}\frac{R^2\rho^2}{2}\, d\varphi\\ &- \Big(A_3C_2^2\|\rho\|_{C^0} + R^2\lambda_1^R\Big)\int_{S^{n-2}} \frac{\rho^2}{2}\phi'(R)\, d\varphi.\end{aligned} \tag{1.25}$$

Now, by the volume preserving constraint over ρ, we can integrate the Taylor expansion

$$\phi(\bar\rho) - \phi(R) = \phi'(R)R\rho + \frac{R^2\rho^2}{2}\phi''(R) + \frac{R^3\rho^3}{6}\phi'''(R(1+\bar\eta\rho)), \tag{1.26}$$

where $\bar\eta : S^{n-1} \to [0,1]$ is suitably chosen, to obtain

$$\begin{aligned}\Big|\int_{S^{n-1}} \phi'(R)R\rho + \frac{R^2\rho^2}{2}\phi''(R)\, d\varphi\Big| &= \Big|\int_{S^{n-1}} \frac{R^3\rho^3}{6}\phi'''(R(1+\bar\eta\rho))\, d\varphi\Big|\\ &\leq \|\rho\|_{C^0}B_3\frac{R}{3}\int_{S^{n-1}} \frac{\rho^2}{2}R^2\phi'''(R)\, d\varphi\\ &= \|\rho\|_{C^0}B_3\frac{R}{3}\int_{S^{n-1}} \frac{\rho^2}{2}R^2\frac{\phi'''(R)}{\phi'(R)}\phi'(R)\, d\varphi\\ &= \|\rho\|_{C^0}B_3\frac{R}{3}\int_{S^{n-1}} \frac{\rho^2}{2}R^2\frac{R^{n-3}\omega_3(R)}{R^{n-1}\omega_1(R)}\phi'(R)\, d\varphi\\ &\leq \|\rho\|_{C^0}B_3C_3\frac{R}{3}\int_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\, d\varphi.\end{aligned}$$

This precious estimate allows us to treat (1.25):

$$\begin{aligned}\int_{S^{n-1}} \phi''(R)R\rho + \frac{\phi''(R)^2}{\phi'(R)}\frac{R^2\rho^2}{2}\,d\varphi &\geq -\left|\frac{\phi''(R)}{\phi'(R)}\right|\left|\int_{S^{n-1}} \phi'(R)R\rho + \frac{R^2\rho^2}{2}\phi''(R)\,d\varphi\right| \\ &\geq -\|\rho\|_{C^0} B_3C_3\frac{R}{3}\left|\frac{\phi''(R)}{\phi'(R)}\right|\int_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,d\varphi, \\ &\geq -\|\rho\|_{C^0} B_3C_3\frac{R}{3}R^{-1}C_2\int_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,d\varphi, \\ &= -\|\rho\|_{C^0}\frac{B_3C_2C_3}{3}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))},\end{aligned}$$

which combined with (1.24) and (1.25) gives the desired inequality. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Recall from Section 1.1.1, that there exists an orthogonal decomposition of $L^2(\mathbf{S}^{n-1}(R), g)$ in spherical harmonics of the form

$$\{f^R_{j,k} \in L^2(\mathbf{S}^{n-1}(R), g) : 1 \leq k \leq n_j\}_{j\geq 0}.$$

Choosing the renormalization so that

$$\frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\|f^R_{j,k}\|^2_{L^2(\mathbf{S}^{n-1}(R))} = 1,$$

$f^R_0 \equiv 1$, and the eigenspace associated to λ^R_1 is spanned by restricting on $\mathbf{S}^{n-1}(R)$ the harmonic polynomials of degree one

$$f^R_{1,k}(x) := \sqrt{n}\frac{x^k}{R}, \quad k = 1, \ldots, n,$$

given in the cartesian coordinates chart $x = R\varphi$. We develop $\boldsymbol{\rho}$ on the spherical harmonics of $\mathbf{S}^{n-1}(R)$ by setting for all $j \geq 0$ and $1 \leq l \leq n_j$ the coefficients

$$c_{j,k} := \frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\langle \boldsymbol{\rho}, f^R_{j,k}\rangle_{L^2(\mathbf{S}^{n-1}(R))},$$

so that

$$\frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} = \sum_{j\geq 0}\sum_{k=1}^{n_j} c^2_{j,k},$$

and

$$\frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} = \sum_{j\geq 1}\sum_{k=1}^{n_j} \lambda^R_j c^2_{j,k}.$$

To simplify the exposition, we will write $\sum_{j,k}$ instead of the double sums, and $⨍_{S^{n-1}}$ the mean with respect to $(\mathbf{S}^{n-1}(R), g)$, that is $(\mathrm{Per}_g(\mathbf{B}^n(R)))^{-1}\int_{S^{n-1}}$. To take advantage of the spectral gap to control the negative term in (1.23), we have to estimate the zero and first harmonics c_0 and $c_{1,k}$. From Equation (1.26), the volume preservation implies the following estimate

$$\begin{aligned}\left|\int_{S^{n-1}} \rho\phi'(R)d\varphi\right| &= \left|\int_{S^{n-1}} \frac{R\rho^2}{2}\phi''(R) + \frac{R^2\rho^3}{6}\phi'''(R(1+\rho\eta))\,d\varphi\right| \\ &\leq (C_2+C_3)\int_{S^{n-1}} \frac{\rho^2}{2}\,\phi'(R)\,d\varphi,\end{aligned} \tag{1.27}$$

allowing us by Cauchy-Schwarz inequality to treat the first harmonic as

$$c_0^2 = \left|\fint_{S^{n-1}} \rho\phi'(R)\,d\varphi\right|^2 \leq (C_2+C_3)^2\|\rho\|_{C^0}^2 \fint_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,d\varphi. \tag{1.28}$$

The barycentric preservation and Equation (1.14) give us the analogue for the second harmonics: first, by Taylor approximation there exists $\theta : S^{n-1} \to [0,1]$ such that

$$\begin{aligned}\psi(\bar\rho) - \psi(R) &= \psi'(R)R\rho + \psi''(R(1+\theta\rho))\frac{R^2\rho^2}{2} \\ &= \phi'(R)R^2\rho + \big(\phi'(R(1+\theta\rho)) + R(1+\theta\rho)\phi''(R(1+\theta\rho))\big)\frac{R^2\rho^2}{2}.\end{aligned}$$

Then, for any $k \in \{1,\ldots,n\}$, we have that

$$\begin{aligned}\left|\int_{S^{n-1}} \frac{x^k}{R}\rho\phi'(R)\,dx\right| &= \left|\int_{S^{n-1}} \frac{x^k}{R}\frac{\rho^2}{2}\Big(\phi'(R(1+\theta\rho)) + R(1+\theta\rho)\phi''(R(1+\theta\rho))\Big)\,dx\right| \\ &\leq (B_1+2C_2)\int_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,dx,\end{aligned}$$

which implies by Cauchy-Schwarz inequality that

$$\begin{aligned}\sum_{k=1}^n c_{1,k}^2 &= \sum_{k=1}^n\Big(\fint_{S^{n-1}} \sqrt{n}\frac{x^k}{R}\rho\phi'(R)\,d\varphi\Big)^2 \leq n^2(B_1+2C_2)^2\Big(\fint_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,d\varphi\Big)^2 \\ &\leq \|\rho\|_{C^0}^2 n^2(B_1+2C_2)^2\Big(\fint_{S^{n-1}} \frac{|\rho|}{2}\phi'(R)\,d\varphi\Big)^2 \\ &\leq \|\rho\|_{C^0}^2 n^2(B_1+2C_2)^2\frac{1}{2}\fint_{S^{n-1}} \frac{\rho^2}{2}\phi'(R)\,d\varphi.\end{aligned} \tag{1.29}$$

Combining (1.27) and (1.29) we obtain that

$$c_0^2 + \sum_{k=1}^n c_{1,k}^2 \leq \Big((C_2+C_3)^2 + \frac{n^2}{2}(B_1+2C_2)^2\Big)\|\rho\|_{C^0}\sum_{j,k} c_{j,k}^2. \tag{1.30}$$

Set $K_1 := (C_2+C_3)^2 + \frac{n^2}{2}(B_1+2C_2)^2$. We have in particular that

$$\begin{aligned}\frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} = \sum_{j,k} c^2_{j,k} &\leq \frac{1}{1-K_1\|\rho\|_{C^0}}\sum_{j\geq 2,k} c^2_{j,k}\\ &\leq \frac{1}{\lambda_2^R(1-K_1\|\rho\|_{C^0})}\sum_{j\geq 2,k}\lambda_j^R c^2_{j,k}\\ &\leq \frac{1}{\lambda_2^R(1-K_1\|\rho\|_{C^0})}\frac{1}{\mathrm{Per}_g(\mathbf{B}^n(R))}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}.\end{aligned}$$

Plugging this last key estimate in Equation (1.23) of Proposition 1.14

$$\begin{aligned}\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R)) \geq -\Big(K_2\|\rho\|_{C^0} + R^2\lambda_1^R\Big)\frac{1}{2}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}\\ + R^2(1-K_3\|\rho\|_{C^1})\frac{1}{2}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))},\end{aligned} \tag{1.31}$$

where $K_2 = \frac{B_3C_2C_3}{3} + A_3C_2^2$ and $K_3 = D+2$, we are finally able to control the negative term taking advantage of the spectral gap between the two first harmonics. In fact, one can check that

$$\frac{\lambda_1^R}{\lambda_2^R} \leq \frac{(d-1)+(n-d)\cosh^2(R)}{2d+2(n-d)\cosh^2(R)} < \frac{1}{2},$$

uniformly in R, and therefore supposing

$$\|\rho\|_{C^0} \leq \frac{1}{3K_1},$$

so that

$$\frac{\lambda_1^R}{\lambda_2^R(1-K_1\|\rho\|_{C^1})} < \frac{1}{2}\frac{1}{(1-K_1\|\rho\|_{C^1})} \leq \frac{3}{4},$$

we can estimate

$$\begin{aligned}&\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R))\\ &\qquad\geq -K_2\|\rho\|_{C^0}\frac{1}{2}\|\boldsymbol{\rho}\|^2_{L^2} + R^2\Big(1 - \frac{\lambda_1^R}{\lambda_2^R(1-K_1\|\rho\|_{C^0})} - K_3\|\rho\|_{C^1}\Big)\frac{1}{2}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2}\\ &\qquad\geq -K_2\|\rho\|_{C^0}\frac{1}{2}\|\boldsymbol{\rho}\|^2_{L^2} + R^2\Big(\frac{1}{4} - K_3\|\rho\|_{C^1}\Big)\frac{1}{2}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2}\\ &\qquad\geq \Big(\frac{R^2\lambda_2^R}{12} - K_2\|\rho\|_{C^0}\Big)\frac{1}{2}\|\boldsymbol{\rho}\|^2_{L^2} + R^2\Big(\frac{1}{8} - K_3\|\rho\|_{C^1}\Big)\frac{1}{2}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2}.\end{aligned}$$

Finally, if

$$\|\rho\|_{C^1} < \varepsilon := \min\Big\{\frac{1}{2}, \frac{1}{3K_1}, \frac{R^2\lambda_2^R}{24K_2}, \frac{1}{16K_3}\Big\},$$

we obtain the desired inequality

$$\mathrm{Per}_g(E) - \mathrm{Per}_g(\mathbf{B}^n(R)) \geq \frac{R^2\lambda_2^R}{48}\|\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))} + \frac{R^2}{32}\|\nabla^g\boldsymbol{\rho}\|^2_{L^2(\mathbf{S}^{n-1}(R))}.$$

We are left to prove that $\varepsilon > 0$ can be chosen uniformly in R, that is

$$R \mapsto R^2 \lambda_2^R,$$

is uniformly bounded away from zero in $[0, R_0]$, being all other constants already depending only on (n, d, R_0). This is a consequence of the exact explicit form of the eigenvalues of the Laplacian on $\mathbf{S}^{n-1}(R)$, that can be expressed as

$$\frac{a \cosh^2(R) + b}{\sinh^2(R) \cosh^2(R)},$$

for some coefficients $a, b \in \mathbb{N}$, see [17, Theorem A]. In particular, $R \mapsto R^2 \lambda_2^R$ is uniformly bounded away from zero in $[0, R_0]$, as wished. □

1.3 Minimality of balls in small volume regime

This section is devoted to the proof of Theorem 1.3. We proceed in three steps: first we show that small isoperimetric sets are uniformly almost-area-minimizing. We recall that a set $E \subset M$ is almost-minimizing if it is optimal up to an error uniformly proportional to the size of the perturbation. In our case, given an isoperimetric region E with volume v, this translates to the existence of a universal constant $C > 0$ such that

$$\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) \leq \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + \frac{C}{v^{1/n}} \mathrm{Vol}_g(E \triangle F),$$

whenever $E \triangle F \subset \mathbf{B}^n(x,s)$ and $s \leq s_1 = s_1(v)$. Then, we prove that this condition combined with the strong stability results in the Euclidean space, imply the L^1 and L^∞-proximity to a geodesic ball with respect to the induced Euclidean metric g_e when o is the barycenter of E. Since almost-minimizing sets sufficiently close to a smooth surface are $C^{1,\alpha}$-normal perturbations of it, we conclude the argument by applying Theorem 1.2.

1.3.1 Almost-minimality

For any subset $G \subset M$, denote the dilation by $\tau > 0$ with respect to the normal coordinates (r, φ) pointed at $o \in M$ with

$$\tau G := \{(\tau r, \varphi) \in M : (r, \varphi) \in G\}.$$

Recall that we denote with $\mathrm{P}(\cdot)$ and $\mathrm{V}(\cdot)$ the perimeter and volume functionals with respect to the Euclidean metric g_e. We prove the following estimates.

Lemma 1.15. *Let $G \subset M$ be a finite perimeter set contained in $\mathbf{B}^n(o, R)$ for some $R > 0$. Then, there exists $C = C(n, d, R) > 0$ such that for all $t \in [0, 1]$ the following estimates on the volume and perimeter of its dilation by $(1+t)$ hold*

$$(1+t)^n \mathrm{Vol}_g(G) \leq \mathrm{Vol}_g((1+t)G) \leq (1 + Ct) \mathrm{Vol}_g(G), \tag{1.32}$$

and

$$\operatorname{Per}_g((1+t)G) \le (1+Ct)\operatorname{Per}_g(G). \tag{1.33}$$

Moreover, one has that

$$\mathrm{V}(G) \le \operatorname{Vol}_g(G) \le (1+\omega_g'(R)R)\,\mathrm{V}(G), \tag{1.34}$$

and

$$\mathrm{P}(G) \le \operatorname{Per}_g(G) \le (1+\omega_g'(R)R)\,\mathrm{P}(G). \tag{1.35}$$

Proof. Thanks to Equation 1.5 we can express

$$\operatorname{Vol}_g((1+t)G) = \int_{(1+t)G} \omega_g(r)\, d\mathscr{H}^n = (1+t)^n \int_G \omega_g((1+t)r)\, d\mathscr{H}^n.$$

The first inequality of Equation (1.32) is immediate from the fact that ω_g is monotone. On the other side, arguing as in Lemma 1.11 we have that by the convexity of ω_g we can estimate

$$\omega_g((1+t)r) \le \Big(1 + r\frac{\omega_g'(r(1+t))}{\omega_g(r)}t\Big)\omega_g(r) \le \Big(1 + R\omega_g'(2R)t\Big)\omega_g(r),$$

proving that

$$\begin{aligned}\operatorname{Vol}_g((1+t)G) &\le (1+t)^n(1+R\omega_g'(2R)t)\operatorname{Vol}_g(G)\\ &\le (1+(2^n-1)t)(1+R\omega_g'(2R)t)\operatorname{Vol}_g(G) \le (1+C_1t)\operatorname{Vol}_g(G),\end{aligned}$$

for $C = (2^n+2)(R\omega_g'(2R)+1)$, as wished. Taking advantage of the integral representation of the perimeter (1.15), we have that $\operatorname{Per}_g((1+t)G)$ is equal to

$$(1+t)^{n-1}\int_{\partial^* G} \omega_g(r(1+t))\Big(|\nu^n|^2 + \frac{1}{\omega_1^2(r(1+t))}|\nu^h|^2 + \frac{1}{\omega_1^1(r(1+t))}|\nu^v|^2\Big)^{1/2} d\mathscr{H}^{n-1},$$

where $\omega_1^1(r)^{-1} = \frac{r^2}{\sinh^2(r)}$ and $\omega_1^2(r)^{-1} = \frac{r^2}{\sinh^2(r)\cosh^2(r)}$, are decreasing functions. We conclude that

$$\operatorname{Per}_g((1+t)G) \le (1+t)^{n-1}\big(1+\omega_g'(2R)t\big)\operatorname{Per}_g(G) \le (1+Ct)\operatorname{Per}_g(G),$$

as wished. Equations (1.34) and (1.35) are obtained analogously. □

Before proving the almost-minimality of isoperimetric sets with small volume, we need to state two important results.

Proposition 1.16. *There exist $\bar v = \bar v(n,d) > 0$ and $\mu = \mu(n,d) > 0$ such that*

$$\operatorname{diam}(E) \le \mu \operatorname{Vol}_g(E)^{1/n},$$

whenever E is an isoperimetric set with volume $\operatorname{Vol}_g(E) \le \bar v$.

Proof. This result holds in all generality for manifolds with uniform bound on the Ricci curvature from below and positive injectivity radius. See [89, Lemma 4.9] and the recent paper [8, Proposition 4.23] for the very general case of RCD spaces. □

Lemma 1.17. *Fix $v_0 > 0$. Then, there exists $C_0 = C_0(v_0, n) > 0$ such that for any finite perimeter set E with $\mathrm{Vol}_g(E) \leq v_0$ one has that*

$$C_0 \,\mathrm{Per}_g(E) \geq \mathrm{Vol}_g(E)^{(n-1)/n}.$$

Proof. This result holds in all generality for manifolds with bounded Ricci curvature from below. See [59, Lemma 3.5], and [54, Lemma 3.10] for an alternative proof in the general setting of sub-Riemannian manifolds. □

From now on, we will fix $v_0 = \bar{v} > 0$, $\mu > 0$ and $C_0 > 0$ as in the statement of Proposition 1.16 and Lemma 1.17. We are now ready to prove that isoperimetric sets are almost-minimizers uniformly in $0 < v \leq \bar{v}$.

Proposition 1.18 (Almost-minimality in M)**.** *There exists $C_1 = C_1(\bar{v}, n, d) > 0$ such that if E is an isoperimetric set and $\mathrm{Vol}_g(E) = v \leq \bar{v}$, then*

$$\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) \leq \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + \frac{C_1}{v^{1/n}} \mathrm{Vol}_g(E \triangle F), \tag{1.36}$$

whenever

$$0 < s \leq s_1 = \min\Big\{1, \phi^{-1}\Big(\frac{v}{2n\omega_n}\Big)\Big\},$$

and $F \subset M$ is such that $F \triangle E \subset \mathbf{B}^n(x,s)$. In particular,

$$\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) \leq \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + \frac{C_1}{v^{1/n}} \mathrm{Per}_g(E \triangle F) \phi(s)^{1/n}. \tag{1.37}$$

Proof. Since the isoperimetric profile $I_M(v) := \min\{\mathrm{Per}_g(G) : \mathrm{Vol}_g(G) = v\}$ is increasing (see the article of Hsiang [66, Lemma 3]), we can suppose without loss of generality that

$$0 \leq \mathrm{Vol}_g(E) - \mathrm{Vol}_g(F) \leq \mathrm{Vol}_g(\mathbf{B}^n(s)) = n\boldsymbol{\omega}_n \phi(s).$$

In particular, imposing $s \leq \phi^{-1}(v/(2n\boldsymbol{\omega}_n))$ we have that $\mathrm{Vol}_g(F) \leq v/2$. Also, notice that we can suppose $\mathrm{Per}_g(F, \mathbf{B}^n(x,s)) \leq \mathrm{Per}_g(\mathbf{B}^n(x,s))$, because otherwise Equation (1.36) is satisfied since

$$\begin{aligned}\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) &\leq \mathrm{Per}_g(E \cup \mathbf{B}^n(x,s)) - \mathrm{Per}_g(E, M \setminus \mathbf{B}^n(x,s)) \\ &\leq \mathrm{Per}_g(\mathbf{B}^n(s)) \leq \mathrm{Per}_g(F, \mathbf{B}^n(x,s)).\end{aligned}$$

Let o be any point in E. Then, by Proposition 1.16, $E \subset \mathbf{B}^n(o, \mu v^{1/n})$. Therefore, for $s \leq 1$, we can suppose without loss of generality that $F \subset \mathbf{B}^n(o, \mu\bar{v}^{1/n} + 2)$, because otherwise $\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) = 0 \leq \mathrm{Per}_g(F, \mathbf{B}^n(x,s))$. By Lemma 1.15, Equation (1.32), there exists $t^* \in (0,1]$ such that

$$\mathrm{Vol}_g((1+t^*)F) = v,$$

where the dilation is taken with respect to the normal coordinates based at $o \in M$. On the other side, minimality of E and Equation (1.33) imply that

$$\mathrm{Per}_g(E) \le \mathrm{Per}_g((1+t^*)F) \le (1+Ct^*)\,\mathrm{Per}_g(F),$$

and therefore, for almost every $0 < s \le s_1 := \min\{1, \phi^{-1}(v/(2n\boldsymbol{\omega}_n))\}$ we have that

$$\begin{aligned}
&\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) \\
&\quad \le (1+Ct^*)\,\mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + Ct^*\,\mathrm{Per}_g(E, M \setminus \mathbf{B}^n(x,s)) \\
&\quad \le \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + Ct^*\big(\mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + I_M(v)\big) \\
&\quad \le \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + Ct^*\big(n\boldsymbol{\omega}_n\phi'(s) + I_M(v)\big).
\end{aligned}$$

We notice that Lemma 1.17 implies that $C_0 I_M(v) \le v^{(n-1)/n}$, and by monotonicity of ϕ', $\phi'(s) \le \phi'(\phi^{-1}(2v/(n\boldsymbol{\omega}_n))) \le \bar{C}v^{(n-1)/n}$, for some constant depending only on $(n, d, \bar{v})$ (to see this, look at Taylor expansions in the proof of Lemma 1.11). Hence

$$\mathrm{Per}_g(E, \mathbf{B}^n(x,s)) \le \mathrm{Per}_g(F, \mathbf{B}^n(x,s)) + Cv^{(n-1)/n}t^*.$$

We are left to find an upper bound for t^*. Since

$$\mathrm{Vol}_g(E) - \mathrm{Vol}_g(F) \le \mathrm{Vol}_g(E \triangle F),$$

and by Equation (1.33)

$$\mathrm{Vol}_g(E) - \mathrm{Vol}_g(F) = \mathrm{Vol}_g((1+t^*)F) - \mathrm{Vol}_g(F) \ge ((1+t^*)^n - 1)\,\mathrm{Vol}_g(F) \ge \frac{nv}{2}t^*,$$

we get that

$$t^* \le \frac{2\,\mathrm{Vol}_g(E \triangle F)}{nv},$$

proving Equation (1.36). Equation (1.37) follows from Lemma 1.17 observing that

$$\begin{aligned}
\mathrm{Vol}_g(E \triangle F) &= \mathrm{Vol}_g(E \triangle F)^{(n-1)/n}\,\mathrm{Vol}_g(E \triangle F)^{1/n} \le C_0\,\mathrm{Per}_g(E \triangle F)\,\mathrm{Vol}_g(\mathbf{B}^n(x,s)) \\
&= C_0\,\mathrm{Per}_g(E \triangle F) n\boldsymbol{\omega}_n\phi(s).
\end{aligned}$$

□

1.3.2 L^1 and L^∞-proximity to a geodesic ball

We prove first that for small enough volumes isoperimetric sets are L^1-close to geodesic balls with respect to the Euclidean metric g_e. Then, the almost-minimality of Proposition 1.18 improves this to L^∞ by rescaling. From now on, we will always suppose that

– $E \subset M$ is an isoperimetric set with small volume $\mathrm{Vol}_g(E) = v \le \bar{v}$, where $\bar{v} > 0$ is as in Proposition 1.16.

- There exists $o \in M$ so that $E \subset \mathbf{B}^n(o, \mu v^{1/n})$ in virtue of Proposition 1.16. We will say that a point $p \in M$ is *admissible* if $E \subset \mathbf{B}^n(p, \mu v^{1/n})$.
- The Euclidean metric g_e (and its associated geometric concepts $B^n(x,s)$, $\mathrm{V}(\cdot)$, $\mathrm{P}(\cdot)$, etc) is the one arising from normal coordinates pointed at $o \in M$.

Proposition 1.19. *Let $E \subset M$ be an isoperimetric set of volume $\mathrm{Vol}_g(E) = v \leq \bar{v}$. Consider g_e to be the Euclidean metric associated to the normal coordinates pointed at some point $o \in M$, so that $E \subset \mathbf{B}^n(o, \mu v^{1/n})$. Then, there exists a constant $C = C(n, d, \bar{v}) > 0$ and a point $\tilde{x} = \tilde{x}(o) \in M$ such that*

$$Cv^{1/n} \geq \Big(\frac{\mathrm{V}(E \triangle B^n(\tilde{x}, \tilde{t}))}{\mathrm{V}(E)}\Big)^2, \tag{1.38}$$

where $\tilde{t} > 0$ is so that $\mathrm{V}(E) = \mathrm{V}(B^n(\tilde{t}))$. In particular

$$V(E \triangle B^n(\tilde{x}, \tilde{t})) \leq Cv^{1+1/2n}. \tag{1.39}$$

Proof. We start by proving that

$$\phi'(s) \leq \cosh(s)^{d-1/n}(n\phi(s))^{(n-1)/n}.$$

In fact

$$\begin{aligned} \phi(s) = \int_0^s \phi'(\tau)\, d\tau = \int_0^s \sinh^{n-1}(\tau) \cosh^{d-1}(\tau)\, d\tau \geq \frac{1}{\cosh(s)} \int_0^s \sinh^{n-1}(\tau) \cosh(\tau)\, d\tau \\ = \frac{1}{n\cosh(s)} \sinh^n(s), \end{aligned}$$

implies that

$$\frac{\phi'(s)}{(n\phi(s))^{(n-1)/n}} \leq \cosh(s)^{d-1+(n-1)/n} = \cosh(s)^{d-1/n}.$$

Let $\tilde{s} > 0$ so that

$$\mathrm{Vol}_g(E) = \mathrm{Vol}_g(\mathbf{B}^n(\tilde{s})).$$

Then, by Lemma 1.15, Equations (1.34) and (1.35) we get that

$$\begin{aligned} \mathrm{P}(E) &\leq \mathrm{Per}_g(E) \leq \mathrm{Per}_g(\mathbf{B}^n(s)) = n\boldsymbol{\omega}_n \phi'(\tilde{s}) \\ &\leq n\boldsymbol{\omega}_n^{1/n} \cosh(\tilde{s})^{d-1/n}(n\boldsymbol{\omega}_n\phi(\tilde{s}))^{(n-1)/n} \\ &= n\boldsymbol{\omega}_n^{1/n} \cosh(\tilde{s})^{d-1/n} \mathrm{Vol}_g(E)^{(n-1)/n} \\ &\leq n\boldsymbol{\omega}_n^{1/n} \cosh(\tilde{s})^{d-1/n}(1 + \omega'(\mu\bar{v}^{1/n})\mu\bar{v}^{1/n})\, \mathrm{V}(E)^{(n-1)/n} \\ &\leq n\boldsymbol{\omega}_n^{1/n}(1 + Cv^{1/n})\, \mathrm{V}(E)^{(n-1)/n}, \end{aligned}$$

since $\tilde{s} \leq \mu v^{1/n}$. By the quantitative strong isoperimetric inequality in $\mathbb{R}^n$, see [50], Equation (1.38) follows immediately. Equation (1.39) is a consequence of the fact that $\mathrm{V}(E) \leq \mathrm{Vol}_g(E) \leq v$. □

We argue now by rescaling. Let

$$\lambda^n := \mathrm{V}(E) = \mathrm{V}(B^n(\tilde{t})) = \boldsymbol{\omega}_n \tilde{t}^n,$$

and define the rescaled volume and perimeter operators

$$\mathrm{Vol}_g^*(G^*) := \lambda^{-n}\,\mathrm{Vol}_g(\lambda G^*),$$
$$\mathrm{Per}_g^*(G^*) := \lambda^{-(n-1)}\,\mathrm{Per}_g(\lambda G^*).$$

Set $E^* := \lambda^{-1}E$. Then we have immediately that the set E^* is renormalized with respect to g_e, that is

$$\mathrm{V}(E^*) = 1.$$

Moreover, the L^1-proximity is uniformly given by

$$\mathrm{Vol}_g^*(E^* \triangle B^n(\lambda^{-1}\tilde{x}, \boldsymbol{\omega}_n^{-1/n})) \leq Cv^{1/2n}, \tag{1.40}$$

in virtue of Proposition 1.19 and Lemma 1.15, Equation (1.34). Finally, there exist $C^* > 0$ and $s^* > 0$ depending only on $(n, d, \bar{v}) > 0$ such that

$$\mathrm{Per}_g^*(E^*, B^n(x,s)) \leq \mathrm{Per}_g^*(F^*, B^n(x,s)) + C^*\,\mathrm{Vol}_g^*(E^* \triangle F^*), \tag{1.41}$$

whenever $E^* \triangle F^* \subset B^n(x,s)$ and $0 < s \leq s^*$. This is a consequence of Proposition 1.18 and the bounds on the sectional curvature, giving the existence of $\Lambda = \Lambda(n, d, \bar{v}) > 0$ such that $B^n(x,s) \subset \mathbf{B}^n(x, \Lambda s)$ provided $\mathrm{dist}_g(o,x) \leq 2\mu\bar{v}^{1/n} + 2$, $0 < s < 1$.

Proposition 1.20. *Let $E \subset M$ be an isoperimetric set of volume $\mathrm{Vol}_g(E) = v < \bar{v}$, and $\lambda > 0$ such that $E^* := \lambda^{-1}E$ has Euclidean volume equal to one. There exists $\tilde{x}(o) \in M$ and $c_1 = c_1(n, \bar{v}) > 0$ such that*

$$B^n(\lambda^{-1}\tilde{x}, (1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}) \subset E^* \subset B^n(\lambda^{-1}\tilde{x}, (1 + c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}).$$

Proof. Let $x \in \partial E^*$, and $h > 0$ be the Euclidean distance of x to $\partial B(\lambda^{-1}\tilde{x}, \boldsymbol{\omega}_n^{-1/n})$. For $0 < r < \min\{h/2, \tilde{s}\}$, define the function

$$W(r) := \mathrm{Vol}_g^*(E^* \cap B^n(x,r)).$$

Since $(E^* \cap B^n(x,r)) \subset (E^* \triangle B^n(\lambda^{-1}\tilde{x}, \boldsymbol{\omega}_n^{-1/n}))$, we have thanks to Equation (1.40) that

$$W(r) \leq Cv^{1/2n}.$$

On the other side, setting $F^* := E^* \setminus B^n(x,r)$, the uniform almost-minimality gives

$$\mathrm{Per}_g^*(E^*) \leq \mathrm{Per}_g^*(F^*) + C^* W(r).$$

This, with Lemma 1.17, imply that

$$\begin{aligned} C_0^{-1}W(r)^{(n-1)/n} &\le \operatorname{Per}_g^*(E^* \cap B^n(x,r)) \le \operatorname{Per}_g^*(E^*, B^n(x,r)) + \operatorname{Per}_g^*(E^* \cap \partial B^n(x,r)) \\ &\le C^* W(r) + 2\operatorname{Per}_g^*(E^* \cap \partial B^n(x,r)) \\ &= C^* W(r) + 2W'(r). \end{aligned}$$

This shows that there exists $\bar{C} > 0$ such that

$$\bar{C} r^n \le W(r) \le C v^{1/2n},$$

implying

$$0 < r < \min\{s^*, C v^{1/2n^2}/\bar{C}\},$$

showing, up to taking $\bar{v} > 0$ small enough, that $h \le 2Cv^{1/2n^2}/\bar{C}$. This proves that there exists $c_1 > 0$ such that ∂E^* is contained in the annulus

$$A := B^n(\lambda^{-1}\tilde{x}, (1 + c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}) \setminus B^n(\lambda^{-1}\tilde{x}, (1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}).$$

The L^1-proximity (1.40) implies that E^* contains the ball $B^n(\lambda^{-1}\tilde{x}, (1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n})$, because otherwise

$$\begin{aligned} C v^{1/2n} \ge \operatorname{Vol}_g^*(E^* \triangle B^n(\lambda^{-1}\tilde{x}, \boldsymbol{\omega}_n^{-1/n})) &\ge \operatorname{Vol}_g^*(B^n(\lambda^{-1}\tilde{x}, (1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n})) \\ &\ge \operatorname{Vol}_g^*(\mathbf{B}^n(\lambda^{-1}\tilde{x}, (1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n})) \\ &= \lambda^{-n} n \boldsymbol{\omega}_n \phi'((1 - c_1 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n})), \end{aligned}$$

leading to a contradiction when $v > 0$ is small enough, since $\lambda^n \sim v$. □

To complete the proof of L^∞-proximity for the rescaled isoperimetric set E^*, we need to prove that the center of the ball $\lambda^{-1}\tilde{x}$ goes to the origin as v goes to zero. This is possible is we impose the lifting point $o \in M$ to be the barycenter of E, that is $o = \operatorname{Bar}_g(E)$.

Lemma 1.21. *Let $E \subset M$ be as in Proposition 1.19. Then, the barycenter $\operatorname{Bar}_g(E)$ is admissible, in the sense that*

$$E \subset \mathbf{B}^n(\operatorname{Bar}_g(E), \mu v^{1/n}).$$

Proof. In virtue of Proposition 1.16, there exists $o \in M$ such that $E \subset \mathbf{B}^n(o, v^{1/n}\mu/2)$. If $p := \operatorname{Bar}_g(E) \in \mathbf{B}^n(o, v^{1/n}\mu/2)$ we are done. If this is not the case, then

$$g(\exp_p^{-1}(x), \exp_p^{-1}(o)) > 0,$$

for all $x \in E$, contradicting Equation (1.7). □

This allows us to chose $o = \operatorname{Bar}_g(E)$ in Proposition 1.19 and Proposition 1.20. We can prove that the associated center of the Euclidean ball $\lambda^{-1}\tilde{x}$ goes to the origin as v goes to zero.

Proposition 1.22. *Let $E \subset M$ be as in Proposition 1.19. Choosing the normal coordinates pointed at $o = \mathrm{Bar}_g(E) \in M$, we have that there exists $C = C(n, d, \bar{v}) > 0$ so that*

$$|\lambda^{-1}\tilde{x}| \leq C v^{1/2n^2}.$$

In particular, there exists $c_2 = c_2(n, d, \bar{v}) > 0$ such that

$$B^n(0, (1 - c_2 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}) \subset E^* \subset B^n(0, (1 + c_2 v^{1/2n^2})\boldsymbol{\omega}_n^{-1/n}). \tag{1.42}$$

Proof. Recall that by Equation (1.8), if the barycenter of E is at the origin with respect to the normal coordinates $x = r\varphi$, $(r, \varphi) \in (0, +\infty) \times S^{n-1}$, then

$$0 = \int_E x\omega_g(r)\, d\mathscr{H}^n.$$

Rescaling, we have that

$$0 = \int_{E^*} x\omega_g(\lambda r)\, d\mathscr{H}^n.$$

Therefore

$$\begin{aligned} |\lambda^{-1}\tilde{x}| &= \left| \int_{E^*} (\lambda^{-1}\tilde{x} - x) + x(\omega_g(\lambda r) - 1)\, d\mathscr{H}^n \right| \\ &\leq \left| \int_{E^* - \lambda^{-1}\tilde{x}} x\, d\mathscr{H}^n \right| + \int_{E^*} r(\omega_g(\lambda r) - 1)\, d\mathscr{H}^n. \end{aligned}$$

Thanks to Proposition 1.20, we can estimate the first integral as follows:

$$\begin{aligned} \left| \int_{E^* - \lambda^{-1}\tilde{x}} x\, d\mathscr{H}^n \right| &\leq \left| \int_{S^{n-1}} \int_{\boldsymbol{\omega}_n^{-1/n}(1 - c_1 v^{1/2n^2})}^{\boldsymbol{\omega}_n^{-1/n}(1 + c_1 v^{1/2n^2})} \chi_E(r\varphi) r^n \varphi\, dr\, d\varphi \right| \\ &\leq \mathrm{P}(\mathbf{B}^n) \int_{\boldsymbol{\omega}_n^{-1/n}(1 - c_1 v^{1/2n^2})}^{\boldsymbol{\omega}_n^{-1/n}(1 + c_1 v^{1/2n^2})} r^n\, dr \\ &= \frac{\mathrm{P}(\mathbf{B}^n)\boldsymbol{\omega}_n^{-(n+1)/n}}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (1 - (-1)^k) c_1^k v^{k/2n^2} \\ &\leq \frac{C}{2} v^{1/2n^2}, \end{aligned}$$

for some constant $C > 0$. Lemma 1.21 implies that $E^* \subset B^n(0, K)$, for some universal $K > 0$. Hence, we estimate the second integral as

$$\begin{aligned} \int_{E^*} r(\omega_g(\lambda r) - 1)\, d\mathscr{H}^n &= \int_{E^*} r \int_0^{\lambda r} \omega_g'(\tau)\, d\tau\, d\mathscr{H}^n \\ &\leq K^2 \lambda \omega_g'(\lambda K) \\ &\leq \frac{C}{2} v^{1/n}, \end{aligned}$$

showing that $|\lambda^{-1}\tilde{x}| \leq C v^{1/2n^2}$. Equation (1.42) is a consequence of this and Proposition 1.20. □

After stating the key regularity result, the proof of Theorem 1.3 will be a corollary of Theorem 1.2 and Proposition 1.22.

Theorem 1.23. *Let $(E_\varepsilon)_{\varepsilon>0}$ be a sequence of sets with finite perimeter in $\mathbb{R}^n$ and $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ a sequence of functionals of the form*

$$\mathcal{F}_\varepsilon(G) := \int_{\partial^* G} f_\varepsilon(x,\nu)\, d\mathscr{H}^{n-1}(x),$$

where G is a generic set of finite perimeter, ν its measure theoretic outwards unit normal and $(f_\varepsilon)_{\varepsilon>0}$ a family of C^2-functions, uniformly λ-elliptic, and with uniformly bounded Hessian in a fixed ball $B(0,2R)$, that is

$$\sup\Big\{|D^2_{\xi\xi} f_\varepsilon(x,\xi)| : (x,\xi) \in B(0,2R)\times S^{n-1}\Big\} \le \Lambda,$$

for a universal constant $\Lambda > 0$. If for all $\varepsilon > 0$

$$B^n(0,(1-\varepsilon)r) \subset E_\varepsilon \subset B^n(0,(1+\varepsilon)r),$$

and E_ε is uniformly almost-minimizing with respect to $\mathcal{F}_\varepsilon$, then there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \le \varepsilon_1$

$$\partial E_\varepsilon = \Big\{ r(1+\rho_\varepsilon(\varphi)) : \varphi \in S^{n-1}\Big\},$$

where $\rho_\varepsilon \in C^1(S^{n-1})$ and $\|\rho_\varepsilon\|_{C^1} \to 0$ as $\varepsilon \to 0$.

Proof. See [48, Theorem 2.2]. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Consider a sequence of decreasing volumes $\bar{v} \ge v_k \to 0$, and let E_k be one isoperimetric region with volume v_k in M. Let E_k^* be the rescaling of E_k pointed at $\mathrm{Bar}_g(E_k)$, so that $\mathrm{V}(E_k^*) = 1$. Then, looking at E_k^* as a sequence of sets with finite perimeter in $\mathbb{R}^n$, we can apply Proposition 1.22 and Theorem 1.23 to infer that there exists $\rho_k^* \in C^1(S^{n-1})$ so that

$$\partial E_k^* = \{\boldsymbol{\omega}_n^{-1/n}(1+\rho_k^*(\varphi)) : \varphi \in S^{n-1}\},$$

and $\|\rho_k^*\|_{C^1} \to 0$ as $k \to \infty$. Therefore, letting $R_k > 0$ be such that $\mathbf{B}^n(R_k) = v_k$, and $\lambda_k^n = \mathrm{V}(E_k)$, we have that

$$\partial E_k = \{R_k(1+\rho_k(\varphi)) : \varphi \in S^{n-1}\},$$

where $\rho_k := \frac{\lambda_k \boldsymbol{\omega}_n^{-1/n}}{R_k} - 1 + \frac{\lambda_k \boldsymbol{\omega}_n^{-1/n}}{R_k}\rho_k^* \to 0$ in C^1 as $k \to +\infty$. Applying Theorem 1.2, we conclude the proof. □

Chapter 2

❊

Hyperbolic log-convex densities and Hopf symmetries

2.1 Preliminaries

We denote by $(H^n_{\mathbb{R}}, g_H)$ the real hyperbolic space of dimension $n \in \mathbb{N}$ with constant sectional curvature equal to -1. Call d_H the induced Riemannian distance. Choose an arbitrary base point $o \in H^n_{\mathbb{R}}$. We say that a function $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$ is (strictly) radially log-convex if

$$\ln(f(x)) = h(d_H(o, x)),$$

for a smooth, (strictly) convex and even function $h : \mathbb{R} \to \mathbb{R}$. We define the weighted perimeter and volume of a set with finite perimeter $E \subset H^n_{\mathbb{R}}$ as

$$V_f(E) := \int_E f \, d\mathscr{H}^n, \quad \text{and} \quad P_f(E) = \int_{\partial^* E} f \, d\mathscr{H}^{n-1}.$$

Here, following the notation in [76], $\partial^* E$ denotes the reduced boundary of E. A set of finite perimeter E with volume $V_f(E) = v > 0$ is called isoperimetric if it solves the minimization problem

$$\mathfrak{J}(v) := \inf\Big\{ P_f(F) : V_f(F) = v, F \subset H^n_{\mathbb{R}} \text{ of finite perimeter} \Big\}. \tag{2.1}$$

The first goal of this Chapter is to show the following characterization of the isoperimetric sets, which will be developed in Section 2.2.

Theorem 2.1. *For any strictly radially log-convex density f, geodesic balls centered at $o \in H^n_{\mathbb{R}}$ uniquely minimize the weighted perimeter for any given weighted volume with respect to P_f and V_f.*

Our main motivation in proving such result is the tight relation of this problem with the (unweighted) isoperimetric problem in the complex hyperbolic spaces $H^m_{\mathbb{C}}$, the quaternionic spaces $H^m_{\mathbb{H}}$ and the Cayley plane $H^2_{\mathbb{O}}$ restricted to a family of sets sharing a particular symmetry that we define as follows.

Definition 2.2 (Hopf-symmetric sets)**.** Let $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $d = \dim(\mathbb{K}) \in \{2, 4, 8\}$ and $(M, g) = (H^m_{\mathbb{K}}, g)$ be the associated rank one symmetric space of non-compact type of real dimension $n = dm$, $m = 2$ if $\mathbb{K} = \mathbb{O}$. Fix an arbitrary point $o \in M$ and let N be the unit length radial vector field emanating from o. Then, up to renormalization of the metric, the Jacobi operator $R(\cdot, N)N$ arising from the Riemannian curvature tensor R is a self-adjoint operator of TM, and has exactly three eigenvalues: $\{0, -1, -4\}$. The (-4)-eigenspace defines at every point $x \neq o$ a distribution $\mathcal{H}_x$ of real dimension $d-1$. A C^1-set $E \subset M$ with normal vector field ν is said to be *Hopf-symmetric* if $\nu(x)$ is orthogonal to $\mathcal{H}_x$ at each point $x \in \partial E$, $o \notin \partial E$.

Remark 2.3. *Let $h : S^{n-1} \to \mathbb{K}P^{m-1}$ be the celebrated Hopf fibration. Then, for any C^1-profile $\rho : S^{n-1} \to (0, +\infty)$ so that ρ is constant along the fibres of h, the set with boundary*

$$\partial E := \{\exp_o(\rho(x)x) : x \in S^{n-1} \subset T_oM\},$$

is Hopf-symmetric, where $\exp_o$ is the exponential map of M at an arbitrary point $o \in M$.

Remark 2.4. *Being* Hopf-symmetric *has not to be confused with the standard notion of being* Hopf *in $H^m_{\mathbb{C}}$, that is a set with principal curvature along the characteristic directions $J\nu$, where J denotes the associated complex structure. It is worth saying that spheres are the only Hopf, compact, embedded constant mean curvature surfaces in $H^m_{\mathbb{C}}$, as it is proven by A. A. Borisenko in [21]. The natural generalization of this concept when $\mathbb{K} \in \{\mathbb{H}, \mathbb{O}\}$ is being a* curvature-adapted hypersurface*, that is, the normal Jacobi operator $R(\cdot, \nu)\nu$ commutes with the shape operator.*

We adopt the notation of Definition 2.2 for the rest of the Chapter. Let P and V be the perimeter and volume functionals induced by g in $H^m_{\mathbb{K}}$. Consider the (unweighted) isoperimetric problem

$$\inf\Big\{P(F) : V(F) = v, F \subset H^m_{\mathbb{K}} \text{ Hopf-symmetric}\Big\}. \tag{2.2}$$

We dedicate Section 2.3 to the proof of the following theorem.

Theorem 2.5. *If geodesic balls centered at $o \in H^n_{\mathbb{R}}$ are isoperimetric with respect to Problem (2.1) for the strictly radial log-convex density*

$$f(x) = \cosh(d_H(o, x)))^{d-1}, \quad d = \dim(\mathbb{K}),$$

then geodesic balls in $H^m_{\mathbb{K}}$ are optimal with respect to the isoperimetric Problem (2.2).

The explicit expression of the perimeter for Hopf-symmetric sets that we will develop in the proof of Theorem 2.5, and Theorem 2.1 will lead to the following consequence.

Corollary 2.6. *In the class of Hopf-symmetric sets, geodesic balls are the unique isoperimetric sets in $H^m_{\mathbb{K}}$.*

In the past two decades, numerous researchers have shown great interest in studying the isoperimetric problem within manifolds with positive densities on the perimeter and volume. In the context of radial weights on $\mathbb{R}^n$, C. Rosales, A. Cañete, V. Bayle and F. Morgan established the existence of isoperimetric sets by imposing certain growth conditions on the weight. Furthermore, they proved that spheres centered at the origin are stable if and only if the weight is log-convex, see [101, Theorem 3.10]. Under this last assumption, K. Brakke conjectured that balls centered at the origin are not only stable, but in fact global minimizers of the weighted perimeter.

In [86] A. Pratelli and F. Morgan provided several important new qualitative results and examples related to this topic.

Brakke's conjecture was proved in the large volume regime by A. Kolesnikov and R. Zhdanov in [69] through an ingenious application of the divergence theorem. The small volume regime was then proved by A. Figalli and F. Maggi in [51] via a rescaling argument taking advantage of deep quantitative stability estimates on the spheres.

Finally, the complete proof was given by G. R. Chambers in [30]. The analysis relies on a meticulous examination of the generating profile of spherical symmetrized sets. In fact, the first and main part of this chapter is an adaptation of the method to our negatively curved case.

It is worth saying that this strategy was moreover successfully employed by W. Boyer, B. Brown, G. R. Chambers, A. Loving and S. Tammen in [24] to show the surprising fact that for all volumes balls whose boundary passes through the origin are isoperimetric with respect to the radial polynomial weight $|x|^p$, for all $p > 0$.

For what concerns curved ambient spaces, various results have been obtained. In [85] F. Morgan, M. Hutchings, and H. Howards focused their attention on the plane enowed with a smooth, rotationally symmetric metric with radially increasing Gauss curvature, proving that an isoperimetric set in this case is either a circle, a complement of a circle or an annulus. In warped product manifolds a significant result is due to S. Howe, who in addition of generalizing the aforementioned result by A. Kolesnikov and R. Zhdanov, established several situations in which the fibres minimize the vertical volume, see [64]. For what concerns the hyperbolic setting, Brakke's conjecture in the two dimensional case was proved according to I. McGuillivray in [79]. The work of E. Bongiovanni, A. Diaz, A. Kakkar, and N. Sothanaphan in [20] provides an affermative answer to Brakke's conjecture for large volume sets containing the origin, in the general setting of two dimensional surfaces of revolution, in which the product of the metric factor with the given volume density is eventually log-convex. This applies for instance to the hyperbolic plane with density equal to $\exp(d_H(x,o)^2)$ for some fixed base point $o \in H^2_{\mathbb{R}}$, see [20, Corollary 5.10].

Finally, very recently in [72] H. Li and B. Xu showed sharp isoperimetric inequalities in $H^n_{\mathbb{R}}$ endowed with radial density of the form

$$\phi(\sinh(d_H(x,o))\cosh(d_H(x,0)),$$

for ϕ even, log-convex, and $o \in H^n_{\mathbb{R}}$ any base point. The proof, that generalizes the result by J. Scheuer and C. Xia in [104], cleverly applies the result of G. R. Chambers by projecting the hyperbolic space onto $\mathbb{R}^n$ and employing a comparison argument. This result proves Theorem 2.5 in the case of the complex hyperbolic space by simply taking $\phi \equiv 1$. Our contribution consists in a further generalization: observe that the density

$$f(x) := \phi(\sinh(d_H(x,o))\cosh(d_H(x,0))$$

is always strictly log-convex, but the converse is not true: for instance when $f(x) = \cosh(d_H(x,o))^{d-1}$ for $d > 2$ (like in Theorem 2.5), the associated function

$$\phi(R) := \frac{\cosh(\operatorname{arsinh}(R))^{d-1}}{\cosh(\operatorname{arsinh}(R))} = \cosh(\operatorname{arsinh}(R))^{d-2}$$

is not log-convex.

Remark 2.7. *In extending the proof of Brakke's conjecture from the Euclidean space to the hyperbolic space, we decided for simplicity to assume the weight to be strictly log-convex rather than simply log-convex. This choice was motivated by the technical difficulties arising from the presence of regions with constant weight. It is worth noting that this restriction has no bearing on the application being studied.*

In what follows, we will always assume $E \subset H^n_{\mathbb{R}}$ to be an isoperimetric set with respect to the weighted problem (2.1).

2.1.1 Qualitative properties of the isoperimetric sets

The main argument of this Chapter is grounded in the principles of existence and regularity of isoperimetric sets in manifolds with densities. We refer to the work of E. Milman [80, Section 2.2 and 2.3] as a very general reference. Existence, boundedness and mean-convexity of isoperimetric sets in $\mathbb{R}^n$ endowed with a various family of densities was the focus of the article by F. Morgan and A. Pratelli [86]. For completeness, the detailed application of their arguments to our hyperbolic setting can be found in the Appendix A.1, Theorems A.2, A.3, and A.4. Regularity of area minimizing surfaces has been the object of study of geometric measure theorists for many decades. The result ensuring smoothness away from a singular set of Hausdorff dimension at most $n-8$ is by now a well-established and widely acknowledged fact. For a presentation of the historical background, we recommend referring to [84, Chapter 8], and [80, Section 2.2] for numerous references on the subject. In analogy with the unweighted case, the last crucial property of the isoperimetric sets E is to have constant weighted mean curvature

$$\mathbf{H}_f := H + \partial_\nu \ln(f), \tag{2.3}$$

at each regular point of ∂E. Here, H denotes the unaveraged inward Riemannian mean curvature, and ν the outward pointing unit normal. The peculiar form of $\mathbf{H}_f$ is obtained via a direct computation of the volume preserving first variation of the perimeter, see [101, Section 3]. The next theorem summarizes all the before mentioned properties of isoperimetric sets.

Theorem 2.8 (Existence and regularity)**.** *For any volume $v > 0$ there exists a set $E \subset H^n_{\mathbb{R}}$ of finite perimeter and weighted volume $V_f(E) = v$ solving the isoperimetric problem* (2.1)*. Moreover, E enjoys the following properties:*

– *∂E is a bounded C^∞ embedded hypersurface outside a singular set of Hausdorff dimension at most $n-8$.*

– *There exists $\lambda \in \mathbb{R}$ such that at any regular point $x \in \partial E$, $\boldsymbol{H}_f(x) = \lambda$. As a consequence, ∂E is mean-convex at each regular point $y \in \partial E$, that is $H(y) \geq (n-1)$.*

– *If the tangent cone at $x \in \partial E$ lies in a halfspace, then it is a hyperplane, and therefore ∂E is regular at x. In particular, ∂E is regular at points $x^\star \in \partial E$ satisfying $d_H(x^\star, o) = \sup_{x\in\partial E} d_H(x,o)$.*

2.1.2 The Poincaré model of $H^n_{\mathbb{R}}$

Adopting the Poincaré model, $H^n_{\mathbb{R}}$ is conformal to the open Euclidean unit ball. At a point $x \in H^n_{\mathbb{R}}$ the metric is

$$g_H = \frac{4}{(1-r^2)^2} g_{\text{flat}},$$

where $r = |x|$ will always denote the Euclidean distance of x from the origin, and g_{flat} the usual Euclidean metric of $\mathbb{R}^n$. The hyperbolic distance from the origin is then given by

$$d_H(x,0) = 2\,\text{artanh}(r).$$

We define the boundary at infinity $\partial_\infty H^n_{\mathbb{R}}$ of $H^n_{\mathbb{R}}$ to be the Euclidean unit sphere $\partial B(0,1) = S^{n-1}$. We will identify the base point $o \in H^n_{\mathbb{R}}$ of the radial density f with the origin 0 in $B(0,1)$.

2.1.3 Isometries and special frames in $H^2_{\mathbb{R}}$

Denote by e_1 and e_2 the horizontal and vertical Cartesian axes in the two dimensional Poincaré disk model. Also, let $(H^2_{\mathbb{R}})_+$ be the intersection of $H^2_{\mathbb{R}}$ with the closed upper half-plane having e_1 as boundary. The isometry group of $(H^2_{\mathbb{R}}, g_H)$ is completely determined (up to orientation) by the Möbius transformations preserving the boundary $\partial_\infty H^2_{\mathbb{R}}$. Hence, geodesic circles coincide with Euclidean circles completely contained in $H^2_{\mathbb{R}}$. Their curvature lies in $(1,+\infty)$. Circles touching $\partial_\infty H^2_{\mathbb{R}}$ in a point are called horocycles, and have curvature equal to 1. Geodesics are arcs of (possibly degenerate) circles hitting $\partial_\infty H^2_{\mathbb{R}}$ perpendicularly in two points. Arcs of (possibly degenerate) circles that are not geodesics are called hypercycles, and have constant curvature in $(-1,1)\setminus\{0\}$. It will be convenient to work with a particular frame: define

$$S : (H^2_{\mathbb{R}})_+ \to \mathbb{R},$$

to be the hyperbolic distance of a point in $(H^2_{\mathbb{R}})_+$ from the horizontal axis e_1. Set $X = \nabla S$, where we naturally extend by continuity X at e_1. Then, denoting with $X^\perp$ the counter-clockwise rotation of X by $\frac{\pi}{2}$ radians, since the level sets of S are equidistant to each other, $\{X, X^\perp\}$ forms an orthonormal frame of $(H^2_{\mathbb{R}})_+$, see Figure 2.1. The integral curves of X are all geodesic rays hitting e_1 perpendicularly. For each $l \in [0,1)$, let δ_l be the integral curve of $X^\perp$ so that $\delta_l(0) = (0,l)$. Then, $(\delta_l)_{l\in[0,1)}$ is a family of equidistant hypercycles foliating $(H^2_{\mathbb{R}})_+$, crossing e_2 perpendicularly and with constant curvature which coincides with the

Eucliden one: $K_1 = \frac{2l}{1+l^2} = \frac{1}{R(l)}$, where $R(l) \in (0, +\infty]$ is the radius of the Euclidean circle representing the curve. Similarly, set $\{N, N^\perp\}$ to be the orthonormal frame on $H^2_{\mathbb{R}} \setminus \{0\}$ where N is the radial unit length vector field emanating from the origin. Then, integral curves of N are rays of geodesics, and integral curves of $N^\perp$ are concentric geodesic cycles. Notice that the frame $\{X, X^\perp\}$ is invariant under the one-parameter subgroup of hyperbolic isometries fixing e_1 ($X^\perp$ is the infinitesimal generator of the action by translations) and, up to reverse the orientation, under the reflections with respect to any geodesic integral curve of X. Finally, notice that on e_1 and e_2, $\{X, X^\perp\}$ is a positive rescaling of $\{(0,1), (-1,0)\}$.

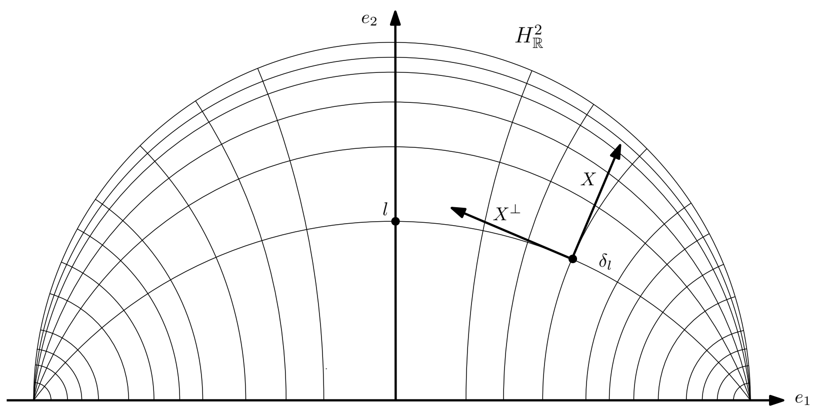

Figure 2.1: The special frame $\{X, X^\perp\}$.

For a regular curve parametrized by arc length η we denote with $\kappa_\eta(t)$ the inward signed curvature of η at $\eta(t)$. We recall the identity

$$\kappa_\eta \dot{\eta}^\perp = \nabla_{\dot{\eta}} \dot{\eta},$$

where here ∇ denotes the standard Levi-Civita connection associated to g_H.

2.1.4 Reduction to $H^2_{\mathbb{R}}$

From now on, let E be an isoperimetric set with arbitrary weighted volume. Since both the density f and the conformal term of g_H are radial, the coarea formula implies that spherical symmetrization pointed at the origin preserves the weighted volume and does not increase the weighted perimeter of E (see [86, Theorem 6.2]). For this reason, we will assume E spherically symmetric with respect to the e_1 axis. Now, intersecting E with the Euclidean plane spanned by $\{e_1, e_2\}$, we obtain a spherically symmetric profile $\Omega \subset H^2_{\mathbb{R}}$. Let $x^\star$ be the furthest point of Ω lying in the positive part of the e_1 axis (this is always possible by reflecting Ω with respect to the e_2 geodesic). Let $\gamma : [-a, a] \to H^2_{\mathbb{R}}$ be a counter-clockwise, arclength parametrization of the boundary of the connected component of Ω containing $x^\star$, so that $\gamma(0) = x^\star$, see Figure 2.2. The curve γ enjoys the following properties:

– γ is smooth on $(-a, a)$. Indeed, if there exists $a^* \in (-a, a)$ such that $\gamma(a^*)$ is not regular, then ∂E contains a singular set of Hausdorff dimension $n - 2$, but this is impossible because of Theorem 2.8.

- γ is symmetric with respect to the axis e_1.

- The curve γ forms a simple, closed curve.

- Writing $\gamma = (\gamma_1, \gamma_2)$ in cartesian coordinates, one has that $\operatorname{sgn}(\gamma_2(t)) = \operatorname{sgn}(t)$. In particular, $\gamma : [0, a) \to (H^2_{\mathbb{R}})_+$.

- $\dot{\gamma}(0) = X(\gamma(0))$.

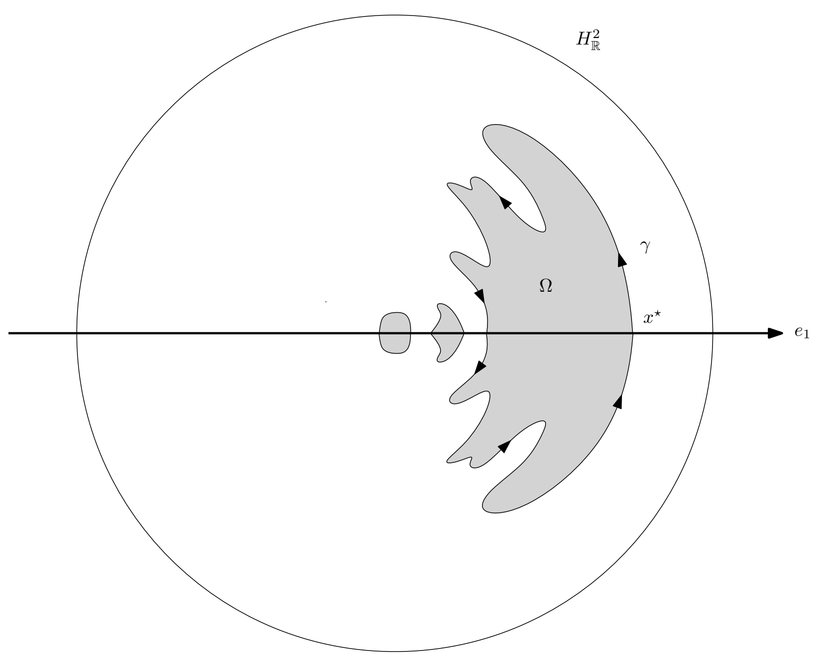

Figure 2.2: The spherical symmetrization.

To translate Equation (2.3) as a property of the profile γ, we need the following definition.

Definition 2.9. For any $t \in [0, a)$, denote by

- $C_t = C_t(s)$ the (possibly degenerated) oriented circle tangent to $\gamma(t)$, with center on e_1, parametrized by arclength and such that $C_t(0) = \gamma(t)$. Denote by $\kappa(C_t)$ its constant curvature.

- $c_t = c_t(s)$ the (possibly degenerated) oriented circle tangent to $\gamma(t)$, parametrized by arclength, such that $c_t(0) = \gamma(t)$ and $\kappa(c_t) = \kappa_\gamma(t)$.

- $x(C_t)$ and $x(c_t)$ the hyperbolic center of C_t and c_t respectively. Similarly, let $x_1(C_t)$ and $x_1(c_t)$ be the first Euclidean coordinate of $x(C_t)$ and $x(c_t)$ respectively.

Remark 2.10. *Let $F \subset B(0,1) \subset \mathbb{R}^n$. Then, at every regular point $x \in \partial F$, the mean curvature H is related with the Euclidean mean curvature H^{flat} by*

$$H(x) = \frac{1-r^2}{2} H^{flat}(x) + (n-1) g_{flat}(x, \tilde{\nu}),$$

where $\tilde{\nu}$ is the outward normal vector to ∂F with Euclidean norm equal to one. In particular, when $n = 2$, denoting with κ^{flat} the usual Euclidean curvature, one has that

$$\kappa_\eta = \frac{1-|\gamma(t)|^2}{2} \kappa_\eta^{flat} + g_{flat}(\eta, \tilde{\nu}).$$

Therefore, $\kappa^{flat}(c_t) = \kappa_\gamma^{flat}$, that is comparison circles c_t and C_t in the hyperbolic setting coincide with comparison circles with respect to the Euclidean metric. From this formula, we also deduce that for any Euclidean circle $\mathscr{C}$

$$\kappa_{\mathscr{C}} = \frac{1}{2}\Big(\frac{1-|x_0|^2}{\tau_0} + \tau_0\Big) = \coth(\tau),$$

where x_0 and τ_0 are the Euclidean center and radius, and τ is the hyperbolic radius.

Lemma 2.11. *On $t \in [0, a)$ it holds*

$$H(t) = \kappa_\gamma(t) + (n-2)\kappa(C_t).$$

In particular,

$$\boldsymbol{H}_f(t) = \kappa_\gamma(t) + (n-2)\kappa(C_t) + h'(d_H(o, \gamma(t))) g_H(\nu(t), N(\gamma(t))) = \lambda,$$

where $\nu = -\dot{\gamma}^\perp$.

We call $H_1(t) := \partial_\nu(\ln(f))(\gamma(t)) = h'(d_H(o, \gamma(t))) g_H(\nu(t), N(\gamma(t)))$ the term coming from the log-convex density.

Proof. In [30, Proposition 3.1] it is shown that the Euclidean mean curvature of the spherically symmetric set E can be computed as

$$H^{\text{flat}} = \kappa_\gamma^{\text{flat}} + (n-2)\kappa^{\text{flat}}(C_t).$$

Thanks to Remark 2.10 we have that

$$\begin{aligned} H(\gamma(t)) &= \frac{1-|\gamma(t)|^2}{2} H^{\text{flat}}(\gamma(t)) + (n-1) g_{\text{flat}}(\gamma(t), \nu) \\ &= \frac{1-|\gamma(t)|^2}{2} \Big(\kappa_\gamma^{\text{flat}} + (n-2)\kappa^{\text{flat}}(C_t)\Big) + (n-1) g_{\text{flat}}(\gamma(t), \nu) \\ &= \kappa_\gamma + (n-2)\kappa(C_t). \end{aligned}$$

□

2.2 The proof

We have seen that existence, boundedness, and regularity of isoperimetric sets (Theorems 2.8, A.2, and A.3) together with the radial nature of the density f allows us to assume the optimal set E to be bounded and spherically symmetric with generating curve smooth away from the axis of symmetrization. Consequently, the problem is reduced to a planar situation, in which the profile curve γ solves the ordinary differential equation induced by the constant weighted mean curvature $\mathbf{H}_f$ of the original isoperimetric set, as stated in Lemma 2.11. Adapting Chamber's analysis to our specific situation presents difficulties as the nonexistence of a natural choice of a frame on the tangent space as in the Euclidean plane. Consequently, to carry out a rigorous curvature-comparison analysis (for instance Lemma 2.26), it is crucial to carefully select a frame that appropriately accommodates the geometry of our particular case, as we did in Section 2.1.3. The proof of Theorem 2.1 relies on showing that γ represents a circumference centered at the origin. The argument presented shows that refuting this possibility leads to a surprising consequence: the curve γ must make a curl, as represented in Figure 2.3, contradicting the fact that γ is the parameterization of a spherically symmetric set. More rigorously, the contradiction arises as the combination of the next two lemmas.

Lemma 2.12. *For every $t \in (0, a)$*

$$g_H(N, \dot{\gamma}) \leq 0.$$

Proof. The fact that the set Ω is spherically symmetric implies that $t \mapsto g_{\text{flat}}(\gamma(t), \gamma(t))$ is non increasing. Differentiating in t gives the desired sign of the angle between N and $\dot{\gamma}$. □

Section 2.2.1 is devoted to the proof of the next lemma.

Lemma 2.13 (The tangent lemma)**.** *If γ is not a circle centered in the origin, there exist $0 < a_0 < a_1 < a_2 < a$ such that $\dot{\gamma}(a_0) = X^{\perp}(\gamma(a_0))$, $\dot{\gamma}(a_1) = -X(\gamma(a_1))$ and $\dot{\gamma}(a_2) = X(\gamma(a_2))$.*

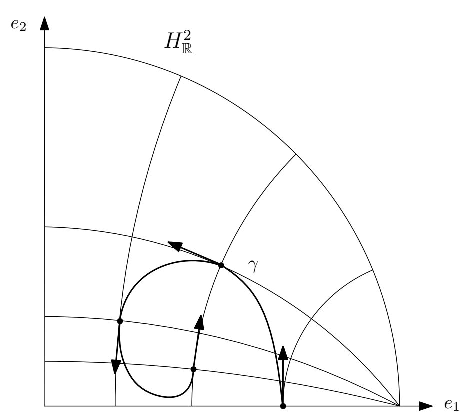

Figure 2.3: The curl described in Lemma 2.13.

Assuming now that Lemma 2.13 holds true, the proof of the main result goes as follows.

Proof of Theorem 2.1. If γ is a circle centered at the origin we are done. Otherwise, Lemma 2.13 ensures the existence of $0 < a_2 < a$ such that $\dot{\gamma}(a_2) = X(\gamma(a_2))$. This violates the inequality of Lemma 2.12, because at a_2

$$g_H(N, \dot{\gamma}) = g_H(N, X) > 0.$$

Therefore, the profile curve γ has to be a circumference centered in the origin. Uniqueness is established by observing that up to measure zero the only set which, when spherically symmetrized, results in a centered ball, is a centered ball itself. □

2.2.1 Proof of the tangent lemma

The proof is made by following the behaviour of γ step by step: first we show that γ has to arch upwards with curvature strictly greater than one. The endpoint of this arc will be $\gamma(a_0)$, where $\dot{\gamma}(a_0) = X^{\perp}(\gamma(a_0))$. Then, it goes down curving strictly faster than before, and this result about curvature is the tricky point to generalize in the hyperbolic setting. It turns out that the special frame given by the hypercyclical foliation $(\delta_l)_{l\in[0,1)}$ is the good one. Then, arguing by contradiction, we will show that this behaviour must end at a point $0 < a_0 < a_1 < a$, where $\dot{\gamma}(a_1) = -X(\gamma(a_1))$. Finally, we prove the existence of a_2 so that $\dot{\gamma}(a_2) = X(\gamma(a_2))$ taking advantage of the mean-curvature convexity of Ω. We start by looking at what happens at the starting point.

Lemma 2.14. *One has that $\dot{\gamma}(0) = X(\gamma(0))$, $\dot{\kappa}_\gamma(0) = 0$ and $\kappa_\gamma(0) \geq \kappa(C_0) > 1$.*

Proof. This is a consequence of the symmetry of γ with respect to the e_1 axis, and of the fact that $\gamma(0)$ represent the furthest point from the origin of Ω. □

Lemma 2.15. *If there exists $t^* \in [0, a)$ such that $x_1(C_{t^*}) = 0$ and $\kappa_\gamma(t^*) = \kappa(C_{t^*})$, then γ has to be a centered circle.*

Proof. In this case $\gamma(t)$ and $C_{t^*}(s)$ solve the same ODE, with same initial data. Therefore, they have to coincide locally, and hence globally. □

Definition 2.16. Call $\alpha : [0, a) \to [\pi, -\pi)$ the oriented angle made by $\dot{\gamma}$ with $X^{\perp}$. We say that $\dot{\gamma}(t)$ is in the I, II, III and IV quadrant if $\alpha(t)$ belongs to $[\pi/2, \pi]$, $[0, \pi/2]$, $[0, -\pi/2]$ and $[-\pi/2, -\pi]$ respectively. We add *strictly* if $\dot{\gamma}$ is not collinear to X and $X^{\perp}$.

Lemma 2.17. *If for some $t \in [0, a)$, $\dot{\gamma}(t)$ belongs to the II quadrant, then $x_1(C_t) \geq 0$.*

Proof. We first treat the case $t \in (0, a)$. Expressing $N(\gamma(t))$ in the $\{X, X^{\perp}\}$ frame, we have thanks to Lemma 2.12 that

$$\begin{aligned} 0 \geq g_H(N, \dot{\gamma}) &= g_H(X, N) g_H(X, \dot{\gamma}) + g_H(X^{\perp}, N) g_H(X^{\perp}, \dot{\gamma}) \\ &= g_H(X, N) \sin(\alpha) + g_H(X^{\perp}, N) \cos(\alpha). \end{aligned} \tag{2.4}$$

If $\alpha = \pi/2$, then

$$0 \geq g_H(X, N),$$

which is possible only when $\gamma_2(t) = 0$, that is $t \in \{0, a\}$. If $\alpha \in [0, \pi/2)$ then $\cos(\alpha) > 0$, implying that $g_H(X^\perp, N) \leq -g(X, N)\tan(\alpha) \leq 0$. Notice that this is possible only if $\gamma_1(t) \geq 0$. Calling $-\vartheta < 0$ the angle that N makes with X, we get by Equation (2.4) that

$$\tan(\alpha) \leq \tan(\vartheta). \tag{2.5}$$

Now, observe that the two geodesic rays σ_γ, σ_N starting at $\gamma(t)$ with initial velocities $\dot\sigma_\gamma(0) = \dot\gamma^\perp(t)$ and $\dot\sigma_N(0) = N^\perp(\gamma(t))$, together with the axis e_1 and the geodesic orthogonal to e_1 passing from $\gamma(t)$ bound two geodesic triangles $\triangle_\gamma$ and $\triangle_N$. Call d the distance between $\gamma(t)$ and e_1. Then, the length of the sides ℓ_γ and ℓ_N of $\triangle_\gamma$ and $\triangle_N$ respectively, lying on e_1 are given via hyperbolic trigonometric laws by

$$\tanh(\ell_\gamma) = \tan(\alpha)\sinh(d) \leq \tan(\vartheta)\sinh(d) = \tanh(\ell_N).$$

But this implies that $x(C_t)$, which is the intersection of σ_γ with e_1, has first coordinate positive, as claimed. If $t = 0$, then $C_0 = c_0$ and approximates $\gamma(0)$ up to the fourth order. Therefore, if $x_1(C_t) < 0$, then there exists $\varepsilon > 0$ such that $\gamma|_{(\varepsilon, 2\varepsilon)}$ lies outside the ball centered in the origin and with radius $d_H(\gamma(0), o)$. This is a contradiction because by construction $\gamma(0)$ is the furthest point of Ω from o. □

Our next goal is to show four important properties of the curve γ. The proof is made by comparison with the circles c_t and C_t, and the preservation of the weighted mean curvature $\mathbf{H}_f$. For this reason, we need the following preliminary lemma.

Lemma 2.18. *Let $\eta = \eta(s)$ be an arc-length, counter-clockwise parametrization of a circle centered at $(0, y)$ such that $\eta(0) = (\tau, y)$ and $\eta(L) = (0, y + \tau)$. Let $O = (-\tilde{o}, 0)$ be an arbitrary point lying on e_1 with $\tilde{o} \in [0, 1)$, and $\nu(s)$ the outward pointing normal to $\eta(s)$. Then, setting*

$$\tilde{H}_1(s) := \partial_\nu(h(d_H(O, x)))|_{x=\eta(s)},$$

if $y = 0$, then

$$\tilde{H}_1'(s) \leq 0, \ \text{in } (0, L], \tag{2.6}$$

and

$$\tilde{H}_1''(0) \leq 0. \tag{2.7}$$

Both inequalities are strict if $\tilde{o} \neq 0$. If $y \in (0, 1)$ and $\tilde{o} \neq 0$, then

$$\tilde{H}_1'(L) < 0. \tag{2.8}$$

Proof. Let $T : H^2_{\mathbb{R}} \to H^2_{\mathbb{R}}$ be the unique isometry fixing e_1 and sending the origin to O. Then,

$$\tilde{H}_1(s) = \partial_\nu(h(d_H(O, x)))|_{x=\eta(s)} = h'(d_H(O, \eta(s)))g_H(\nu(s), T_*N(\eta(s))).$$

For the sake of exposition, we will omit the arguments in the following computations. Differentiating one time in s we have that

$$\begin{aligned}\tilde{H}_1'(s) &= h'' g_H(T_*N, \dot\eta) g_H(\nu, T_*N) + h' \frac{d}{ds} g_H(\nu, T_*N)\\ &= h'' g_H(T_*N, \dot\eta) g_H(\nu, T_*N) - h'\Big(g_H(\nabla_{\dot\eta}\dot\eta^\perp, T_*N) + g_H(\dot\eta^\perp, \nabla_{\dot\eta} T_*N)\Big)\\ &= h'' g_H(T_*N, \dot\eta) g_H(\nu, T_*N) - h'\Big(-g_H(T_*N, \dot\eta)\kappa_\eta + g_H(T_*N, \dot\eta) g_H(T_*N^\perp, \dot\eta)\kappa_1\Big),\end{aligned}$$

where κ_1 is the curvature of the integral curve of $T_*N^\perp$ passing through $\eta(s)$, which is a geodesic sphere centered at O. Suppose first that $y = 0$ and $\tilde{o} \neq 0$. Then, $\dot\eta = N^\perp$, and

$$\tilde{H}_1'(s) = h'' g_H(T_*N, N^\perp) g_H(N, T_*N) - h' g_H(T_*N, N^\perp)(-\kappa_\eta + g_H(T_*N, N)\kappa_1) < 0,$$

because $h'' > 0$, $h' > 0$, $g_H(T_*N, N^\perp) < 0$, $g_H(T_*N, N) > 0$ and $\kappa_\eta > \kappa_1$ since $\tilde{o} \neq 0$. This proves Equation (2.6) when $\tilde{o} \neq 0$. The same holds in the context of Equation (2.8) since $\dot\eta(L) = N^\perp$. Up to relaxing the inequalities, the proof when $\tilde{o} = 0$ is exactly the same. To prove Equation (2.7), we differentiate $\tilde{H}_1$ one more time, obtaining

$$\begin{aligned}\tilde{H}_1''(s) = h''' g_H(T_*N, \dot\eta)^2 g_H(\nu, T_*N) &+ h'' \frac{d}{ds} g_H(T_*N, \dot\eta) g_H(\nu, T_*N)\\ &+ h'' g_H(T_*N, \dot\eta) \frac{d}{ds} g_H(\nu, T_*N) + h'' g_H(T_*N, \dot\eta) \frac{d}{ds} g_H(\nu, T_*N)\\ &+ h' \frac{d^2}{ds^2} g_H(\nu, T_*N).\end{aligned}$$

Observe that in zero $g_H(T_*N, \dot\eta) = 0$, hence only the second and last term survive

$$\tilde{H}_1''(0) = h'' \frac{d}{ds}\Big|_{s=0} g_H(T_*N, N^\perp) g_H(T_*N, N) + h' \frac{d^2}{ds^2}\Big|_{s=0} g_H(T_*N, N).$$

Taking advantage of the explicit expression for $\frac{d}{ds} g(T_*N, N)$ we obtained before, we get

$$\begin{aligned}\frac{d^2}{ds^2}\Big|_{s=0} g_H(T_*N, N) &= -\frac{d}{ds}\Big|_{s=0} \Big(g_H(T_*N, N^\perp)\big(-\kappa_\eta + g_H(T_*N, N)\kappa_1\big)\Big)\\ &= \big(\kappa_\eta - g_H(T_*N, N)\kappa_1\big) \frac{d}{ds}\Big|_{s=0} g_H(T_*N, N^\perp),\end{aligned}$$

which implies that

$$\tilde{H}_1''(0) = \underbrace{\big(h'' g_H(T_*N, N) + h'\kappa_\eta - h' g_H(T_*N, N)\kappa_1\big)}_{>0} \frac{d}{ds}\Big|_{s=0} g_H(T_*N, N^\perp).$$

Hence, we are left to show that $\frac{d}{ds}\Big|_{s=0} g_H(T_*N, N^\perp) < 0$. Developing again we get

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} g_H(T_*N, N^\perp) &= g_H(\nabla_{N^\perp} T_*N, N^\perp)|_{s=0} + g_H(T_*N, \nabla_{N^\perp} N^\perp)|_{s=0}\\ &= g_H(T_*N, N)^2 \kappa_1|_{s=0} - g_H(TN, N)\kappa_\eta|_{s=0}\\ &= \kappa_1 - \kappa_\eta < 0.\end{aligned}$$

□

We are now ready to prove the next result.

Lemma 2.19. *The following four points hold.*

i. If for $t \in (0,a)$ one has that $\kappa_\gamma(t) \geq \kappa(C_t) > 1$, then $t \mapsto x_1(C_t)$ is smooth and $\frac{d}{dt}x_1(C_t) \geq 0$.

ii. If γ is not a centered circle, then $\ddot{\kappa}_\gamma(0) > 0$.

iii. If for $t \in (0,a)$, $\dot{\gamma}(t)$ is in the II quadrant and $\kappa_\gamma(t) = \kappa(C_t) > 1$, then $\dot{\kappa}_\gamma(t) \geq 0$. Moreover, if $\dot{\gamma}(t) \neq X^\perp(\gamma(t))$ and C_t is not centered in the origin, then $\dot{\kappa}_\gamma(t) > 0$.

iv. If for $t \in (0,a)$ one has that $\dot{\gamma}(t) = X^\perp(\gamma(t))$, $\gamma_1(t) > 0$ and $\kappa_\gamma(t) \geq \kappa(C_t) > 1$, then $\dot{\kappa}_\gamma(t) > 0$.

Proof. We start with point i. Observe that since c_t approximates γ up to the third order around $\gamma(t)$, it suffices to prove $\frac{d}{dt}x_1(C_t) \geq 0$ replacing γ with c_t. Also, we can suppose $x(c_t)$ on e_2 by composing with the unique hyperbolic isometry translating $x(c_t)$ on e_2 and fixing e_1. The curvature condition $\kappa_\gamma(t) \geq \kappa(C_t) > 1$ ensures that $x(c_t) \in (H^2_{\mathbb{R}})_+$. By monotonicity of the function $\tanh(\cdot/2)$, it suffices to prove the claim for the Euclidean center of C_t. Thus, we have reduced the problem to an explicit computation in the Euclidean plane, that can be found in [30, Lemma 5.3]. Thanks to Lemma 2.18 the proofs of the other points go exactly as in [30, Lemma 3.4, Lemma 3.5 and Lemma 3.7]. We show point ii. Differentiating $\mathbf{H}_f$ twice, we get that

$$\ddot{\kappa}_\gamma(0) = -(n-2)\ddot{\kappa}(C_0) - H_1''.$$

By symmetry, $c_0 = C_0$. Moreover, since $\dot{\kappa}_\gamma(0) = 0$, we have that both c_0 and C_0 approximate γ up to the fourth order near zero. Hence, $\ddot{\kappa}(C_0) = 0$. Therefore, it suffices to determine the sign of H_1'' replacing γ with C_0. Let $T : H^2_{\mathbb{R}} \to H^2_{\mathbb{R}}$ be the unique isometry fixing e_1 that moves $x(C_0)$ to the origin. The result follows by Equation (2.7) of Lemma 2.18 setting $O = T(0)$, and noticing that $T(0) \neq 0$ by Lemma 2.15. The proofs of points iii. and iv. are similar: in the first case the condition $\kappa_\gamma(t) = \kappa(C_t)$ implies that $c_t = C_t$ approximates γ near t up to the third order, the same holds if $\dot{\gamma}(t) = X^\perp(\gamma(t))$ by symmetry. Hence, substituting γ with c_t and differentiating one time $\mathbf{H}_f$, we have to determine the sign of H_1' in the case of a circle, via Lemma 2.18. □

We are now ready to analyse the first behaviour of γ.

Definition 2.20 (Upper curve)**.** The upper curve is the (possibly empty) maximal connected interval $I_U \subset [0,a)$ such that $0 \in I_U$ and for all $t \in I_U$

a. $\dot{\gamma}(t)$ is in the II quadrant,

b. $\kappa_\gamma(t) \geq \kappa(C_t) > 1$,

c. $t \mapsto x_1(C_t)$ is smooth and $\frac{d}{dt}x_1(C_t) \geq 0$.

We set

$$a_0 := \sup I_U.$$

In the discussion, we will sometimes identify the upper curve with its image through γ.

Definition 2.21. We say that a curve η is graphical with respect to the hypercyclic foliation $(\delta_l)_{l\in[0,1)}$ if η meets each δ_l at most once.

Notice that the upper curve (if non empty) is graphical with respect to the hypercyclical foliation because $\dot{\gamma}$ is in the II quadrant

Proposition 2.22. *The upper curve is non empty and enjoys the following properties*

i. $0 < a_0 < a$*,*

ii. $a_0 \in I_U$*,*

iii. $\gamma_1(a_0) > 0$*,*

iv. $\dot{\gamma}(a_0) = X^{\perp}(\gamma(a_0))$*.*

Proof. Thanks to Lemma 2.19, the proof goes exactly as [30, Lemma 3.11 and Proposition 3.12]. We sketch for completeness the idea behind each point. We start by showing that the upper curve is non empty.

By Lemma 2.14 we know that $\dot{\gamma}(0) = X(\gamma(0))$, $\dot{\kappa}_\gamma(0) = 0$, and $\kappa_\gamma(0) \geq \kappa(C_0) > 1$. Moreover, by Lemma 2.19 point ii. since by assumption γ is not a centered circle, we get that $\ddot{\kappa}_\gamma(0) > 0$. By continuity, since $c_0 = C_0$ approximates γ up to the fourth order near zero, we have that there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$ points a. and b. in Definition 2.20 are satisfied. Finally, point c. of Definition 2.20 follows from Lemma 2.19 point i. which asserts that $\kappa_\gamma(t) \geq \kappa(C_t) > 1$ implies that $x_1(C_t)$ is smooth and $\frac{d}{dt}x_1(C_t) \geq 0$. Hence, $[0, \varepsilon) \subset I_U$, proving that the upper curve cannot be empty.

Notice that $0 < a_0$ cannot be equal to a since otherwise the curve γ does not close itself on e_1, simply because $\dot{\gamma}$ belongs to the II quadrant by definition of I_U. By the regularity of γ and that I_U is defined by three closed conditions, we have that $a_0 \in I_U$. By composing with the unique hyperbolic isometry sending $\gamma(a_0)$ on e_2 fixing e_1, we can see that $x_1(C_{a_0}) \leq 0$ because $\dot{\gamma}(a_0)$ belongs to the II quadrant. Lemma 2.15 and Lemma 2.17 imply that $x_1(C_0) > 0$ and since $\frac{d}{dt}x_1(C_t) \geq 0$ in I_U, one must have that $\gamma_1(a_0) > 0$. The last point is proved by contradiction: if $\dot{\gamma}(a_0) \neq X^{\perp}(\gamma(a_0))$, then $a_0 \in I_U$ implies that $\dot{\gamma}(a_0)$ is strictly in the II quadrant. If $\kappa_\gamma(a_0) = \kappa(C_{a_0}) > 1$, then $c_{a_0} = C_{a_0}$ approximates γ to the third order and Lemma 2.19 point iii. implies that there exists some $\delta > 0$ such that $\kappa_\gamma(t) \geq \kappa(C_t) > 1$ for $t \in [a_0, a_0 + \delta)$. The same holds if $\kappa_\gamma(a_0) > \kappa(C_{a_0}) > 1$ by continuity. This means that $[a_0, a_0 + \delta) \subset I_U$, which is not possible by the very definition of a_0. Hence, $\dot{\gamma}(a_0) = X^{\perp}(\gamma(a_0))$. □

Definition 2.23 (Lower curve)**.** The lower curve is the maximal connected interval $I_L \subset [a_0, a)$ such that for all $t \in I_L$

a. $a_0 \in I_L$,

b. $\dot{\gamma}(t)$ is in the III quadrant,

c. calling $\bar{t} \in I_U$ the unique time such that $S(\gamma(t)) = S(\gamma(\bar{t}))$ we have that $\kappa_\gamma(t) \geq \kappa_\gamma(\bar{t})$.

We set

$$a_1 := \sup I_L.$$

Notice that a_0 truly belongs to I_L, so $I_L \neq \emptyset$. Also, the lower curve is graphical with respect to the hypercyclical foliation because $\dot{\gamma}$ is in the III quadrant. Our next goal is to prove that $a_1 < a$. Again, we proceed by contradiction, and the intuition is the following: suppose that $a_1 = a$. If $\kappa_\gamma(t) = \kappa_\gamma(\bar{t})$ for all $t \in I_L$, then the lower curve is nothing else than the upper curve reflected with respect to the geodesic orthogonal to e_1 and passing through $\gamma(a_0)$. Hence, $\lim_{t\to a^+} \alpha(t) = -\frac{\pi}{2}$. Otherwise, if the $\gamma|_{I_L}$ curves strictly faster than the upper curve at some point, then the angle of incidence $\lim_{t\to a^+} \alpha(t) < -\frac{\pi}{2}$ (see Figure 2.4). But this cannot be true, because it contradicts the regularity of ∂E pointed out in Theorem 2.8. To prove that the lower curve curves strictly faster than the upper curve we need first to express the curvature with respect to the $\{X, X^\perp\}$ frame, and next prove three comparison lemmas.

Lemma 2.24. *Let η any regular curve parametrized by arclength such that $\dot{\eta}(t)$ is not collinear to $X(\eta(t))$. Then,*

$$-\kappa_\eta(t) = \dot{\beta}(t) - K_1(\eta(t))\cos(\beta(t)),$$

where $\beta(t)$ denotes the angle between $\dot{\eta}$ and $X^\perp$, and K_1 is the curvature of the leaf δ_l passing through $\eta(t)$.

Proof. Decompose $\dot{\eta} = AX + BX^\perp$. Then, since $\kappa_\eta \dot{\eta}^\perp = \nabla_{\dot{\eta}}\dot{\eta}$, we get that

$$-\cos(\beta)\kappa_\eta = g_H\Big(\nabla_{\dot{\eta}}\dot{\eta}, X\Big) = \partial_t\big(\sin(\beta)\big) - g_H\Big(\dot{\eta}, \nabla_{\dot{\eta}}X\Big).$$

Now, keeping in mind that $\nabla_X X = 0$ and $g_H\big(\nabla_{X^\perp}X^\perp, X\big) = -K_1(\eta(t))$, we get

$$g_H\Big(AX + BX^\perp, \nabla_{AX+BX^\perp}X\Big) = B^2 g_H\Big(X^\perp, \nabla_{X^\perp}X\Big) = \cos(\beta)^2 K_1(\eta(t)).$$

Dividing both sides by $\cos(\beta)$ we get the desired identity. □

We can prove our first curvature comparison lemma.

Lemma 2.25 (κ comparison lemma)**.** *Let $\eta_1 : (0, A_1] \to H^2_{\mathbb{R}}$ and $\eta_2 : (0, A_2] \to H^2_{\mathbb{R}}$ be two hypercyclical graphical curves parametrized by arclength and with velocity vectors in the II quadrant. Suppose that there exists $l_0 \in [0, 1)$ such that*

$$\lim_{t\to 0^+} \eta_1(t) \text{ and } \lim_{t\to 0^+} \eta_2(t),$$

exist and belong to the same leaf δ_{l_0}. *Also, suppose that* $\eta_1(A_1) = \eta_2(A_2)$, $\dot\eta_1(A_1) = \dot\eta_2(A_2)$, *and that if* $S(\eta_1(t)) = S(\eta_2(\tau))$ *then* $\kappa_{\eta_1}(t) \geq \kappa_{\eta_2}(\tau)$. *Then, calling* α_1 *and* α_2 *the angle made by* $\dot\eta_1$ *and* $\dot\eta_2$ *with* $X^\perp$ *we have that*

$$\lim_{t\to 0^+} \alpha_1(t) \geq \lim_{t\to 0^+} \alpha_2(t).$$

Moreovoer, if for some t *and* τ *such that* $S(\eta_1(t)) = S(\eta_2(\tau))$, *one has that* $\kappa_1(t) > \kappa_2(\tau)$, *then*

$$\lim_{t\to 0^+} \alpha_1(t) > \lim_{t\to 0^+} \alpha_2(t).$$

Proof. Since the curves are graphical with respect to the hypercyclical foliation we can operate a change of variable: we observe that the two height functions $s_1(t) := S(\eta_1(t))$ and $s_2(\tau) = S(\eta_2(\tau))$ are bijections with same image of the form $(l_0, L]$. By hypothesis $\kappa_{\eta_1}(s_1^{-1}(l)) \geq \kappa_{\eta_2}(s_2^{-1}(l))$ for every $l \in (l_0, L]$. Comparing the two curves in the $l \in (l_0, L]$ variable, since $s_i' = g_H(\nabla S, \dot\eta_i) = g_H(X, \dot\eta_i) = \sin(\alpha_i)$, $i = 1, 2$, we get by Lemma 2.24 that

$$0 \leq \kappa_{\eta_1}(l) - \kappa_{\eta_2}(l) = \dot\alpha_2(l)\sin(\alpha_2(l)) - \dot\alpha_1(l)\sin(\alpha_1(l)) - \frac{2l}{1+l^2}\big(\cos(\alpha_2(l)) - \cos(\alpha_1(l))\big).$$

Multiplying by $(1 + l^2)$ and integrating we finally get that

$$\begin{aligned} 0 &\leq \int_{l_0}^{L} (1+l^2)(\cos(\alpha_1)\cos(\alpha_2))' + 2l(\cos(\alpha_1) - \cos(\alpha_2))\, dl \\ &= \int_{l_0}^{L} \frac{d}{dl}\Big((1+l^2)(\cos(\alpha_1) - \cos(\alpha_2))\Big)\, dl = \lim_{l\to l_0^+}(1+l^2)(\cos(\alpha_2(l)) - \cos(\alpha_1(l))). \end{aligned}$$

If the two curvatures are different somewhere, then the inequality between the two angles is strict. □

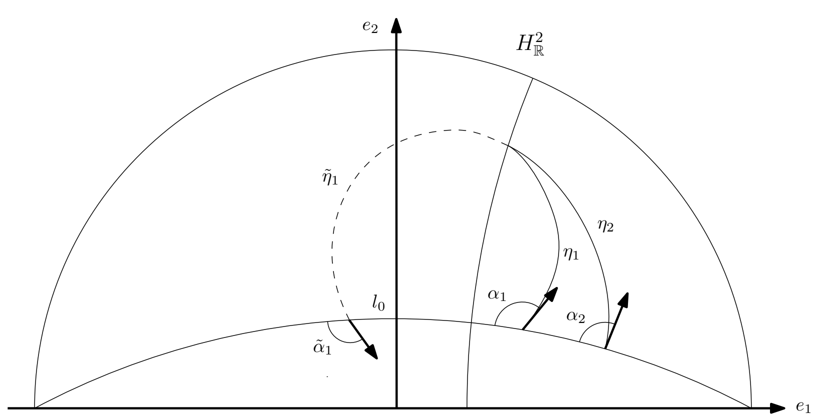

Figure 2.4: The curvature comparison.

Lemma 2.26 ($\kappa(C_t)$ comparison lemma)**.** *Let η_1, η_2 be as in Lemma 2.25. Then, for any two points $\eta_1(t_1)$ and $\eta_2(t_2)$ on the same leaf δ_l, calling C^1 and C^2 the comparison circles at $\eta_1(t_1)$ and $\eta_2(t_2)$ as in Definition 2.9, we have that*

$$\kappa(C^1) \leq \kappa(C^2).$$

Proof. For $i = 1, 2$, the hyperbolic radius of C^i together with e_1 and the geodesic starting from $\eta_i(t_i)$ and hitting e_1 perpendicularly bound a geodesic triangle $\triangle_i$. Let d_i^1 be the hyperbolic radius, d_i^2 be the side touching e_1 and d_i^3 the the remaining side of $\triangle_i$. Similarly, for $i = 1, 2$ and $j = 1, 2, 3$, call β_i^j the angle opposite to d_i^j, and ℓ_i^j the length of d_i^j. We refer to Figure 2.5. By construction $\beta_1^1 = \beta_2^1 = \frac{\pi}{2}$, $\beta_i^2 = \alpha_i$, and since $\eta_1(t_1)$ and $\eta_2(t_2)$ are in the same hypercycle by hypothesis, we get $\ell_1^3 = \ell_2^3$. Then, by the hyperbolic law of cosines and by Lemma 2.25 we get

$$\kappa(C^1) = \coth(\ell_1^1) = \frac{\cos(\alpha_1)}{\tanh(\ell_1^3)} \leq \frac{\cos(\alpha_2)}{\tanh(\ell_2^3)} = \coth(\ell_2^1) = \kappa(C^2). \tag{2.9}$$

□

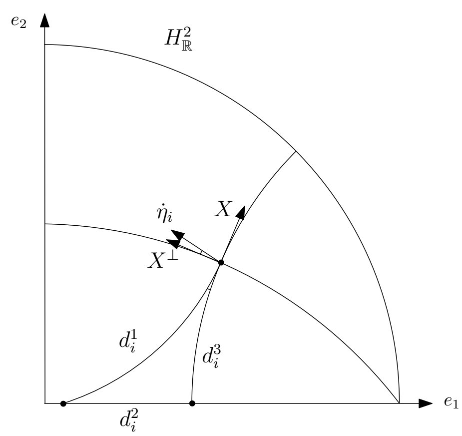

Figure 2.5: The curvature comparison for C_t.

Lemma 2.27 (H_1 comparison lemma)**.** *Let η_1, η_2 be as in Lemma 2.25 and let $\tilde\eta_1$ be the reflection of η_1 with respect to the geodesic passing through $\eta_1(A_1)$ and crossing e_1 perpendicularly. Reverse its parametrization, so that the angle that $\dot{\tilde\eta}_1$ makes with $X^\perp$ is equal to $\tilde\alpha_1 = -\alpha_1$. Moreover, suppose that*

$$g_H(N, \dot\eta_2) \leq 0.$$

Denote the unitary outward pointing normals to η_1 and η_2 by $\tilde\nu_1$ and ν_2. Then, for any two points $\tilde\eta_1(t_1)$ and $\eta_2(t_2)$ on the same leaf δ_l we have that

$$g_H(N(\tilde\eta_1(t_1)), \tilde\nu_1(t_1)) \leq g_H(N(\eta_2(t_2)), \nu_2(t_2)),$$

with equality if and only if $\dot\eta_2(t_2)$ and $\dot{\tilde\eta}_1(t_1)$ are tangent to the same circle centered in the origin and $\dot{\tilde\eta}_1(t_1) = -\dot\eta_2(t_2)$.

Proof. Parametrize $\delta_l : \mathbb{R} \to H^2_{\mathbb{R}}$ by arclength in the $X^\perp$ direction, so that δ_l intersects e_2 at $\delta_l(0)$. Let $\vartheta(s)$ be the angle that N makes with $X^\perp$ at $\delta_l(s)$ and $s_2 < s_1$ be such that $\delta_l(s_1) = \tilde{\eta}_1(t_1)$ and $\delta_l(s_2) = \eta_2(t_2)$. Let $\Theta_1, \Theta_2 \in [0, \frac{\pi}{2}]$ be the angles that N makes with $\nu_1(t_1)$ and $\nu_2(t_2)$ respectively. Then, $\Theta_1 := \vartheta(s_1) - \tilde{\alpha}_1 - \frac{\pi}{2}$, $\Theta_2 := \vartheta(s_2) - \alpha_2 - \frac{\pi}{2}$,

$$g_H(N(\tilde{\eta}_1(t_1)), \tilde{\nu}_1(t_1)) = \cos(\Theta_1),$$

and

$$g_H(N(\eta_2(t_2)), \nu_2(t_2)) = \cos(\Theta_2).$$

We need to investigate if $\Theta_2 \leq \Theta_1$, and when $\Theta_2 < \Theta_1$. Let $s_2^* \in \mathbb{R}$ be such that the unit vector at $\delta_l(s_2^*)$ that forms an angle of α_2 with $X^\perp$ is tangent to a circle centered in the origin. The value s_2^* exists in the interval $[s_2, 0)$ because by Lemma 2.17, Equation (2.5), we have that $\vartheta(s_2) - \frac{\pi}{2} \geq \alpha_2$ and, in the intersection of δ_l with e_2 we have that $\vartheta(0) - \frac{\pi}{2} = 0 \leq \alpha_2$. By continuity there must be a point $s_2 \leq s_2^* < 0$ such that $\vartheta(s_2^*) - \frac{\pi}{2} = \alpha_2$. Then

$$\Theta_2 = -\vartheta(s_2^*) + \vartheta(s_2) = -\int_{s_2}^{s_2^*} \dot{\vartheta}(s)\, ds.$$

Set $s_1^* := -s_2^*$, and notice that $\vartheta(-s) = \pi - \vartheta(s)$ for every $s \geq 0$. Then,

$$\begin{aligned}
\Theta_1 &= \vartheta(s_1) - \tilde{\alpha}_1 - \frac{\pi}{2} = \vartheta(s_1) + \alpha_2 - (\tilde{\alpha}_1 + \alpha_2) - \frac{\pi}{2} \\
&= \vartheta(s_1) + \vartheta(s_2^*) - \pi - (\tilde{\alpha}_1 + \alpha_2) \\
&= \vartheta(s_1) - \vartheta(s_1^*) - (\tilde{\alpha}_1 + \alpha_2) \\
&= -\int_{s_1}^{s_1^*} \dot{\vartheta}(s)\, ds - (\tilde{\alpha}_1 + \alpha_2).
\end{aligned}$$

Hence,

$$\Theta_2 - \Theta_1 = (\tilde{\alpha}_1 + \alpha_2) - \int_{s_2}^{s_2^*} \dot{\vartheta}(s)\, ds + \int_{s_1}^{s_1^*} \dot{\vartheta}(s)\, ds \leq -\int_{s_2}^{s_2^*} \dot{\vartheta}(s)\, ds + \int_{s_1}^{s_1^*} \dot{\vartheta}(s)\, ds,$$

since $\tilde{\alpha}_1 + \alpha_2 \leq 0$ by Lemma 2.25. Let $\ell = 2\,\mathrm{artanh}(l)$ be the hyperbolic distance of δ_l from e_1. We claim that

$$\dot{\vartheta}(s) = -\big(\sinh(\ell)\cosh(\ell)\cosh^2(\mathrm{sech}(\ell)s) + \coth(\ell)\sinh^2(\mathrm{sech}(\ell)s)\big)^{-1} < 0. \tag{2.10}$$

Assuming that this identity holds, letting $\delta := s_2^* - s_2$, we have that

$$\begin{aligned}
\Theta_2 - \Theta_1 &\leq -\int_{s_2^*-\delta}^{s_2^*} \dot{\vartheta}(s)\, ds + \int_{s_1^*-\delta}^{s_1^*} \dot{\vartheta}(s)\, ds + \int_{s_1}^{s_1^*-\delta} \dot{\vartheta}(s)\, ds \\
&= \int_0^{\delta} \dot{\vartheta}(s_2^* - \tau) - \dot{\vartheta}(s_2^* + \tau)\, d\tau + \int_{s_1}^{s_1^*-\delta} \dot{\vartheta}(s)\, ds \\
&\leq \int_{s_1}^{s_1^*-\delta} \dot{\vartheta}(s)\, ds \leq 0,
\end{aligned}$$

where we used that $s_1^* = -s_2^*$ and $\dot\vartheta$ is a strictly negative, even function increasing in $[0, +\infty)$. This implies that $\Theta_2 \leq \Theta_1$, with equality if and only if $\delta = 0$ and $\alpha_1 = \alpha_2$, that is when $\dot\eta_2$ and $\dot{\tilde\eta}_1$ are tangent to the same circle centered in the origin. We are left to prove Equation (2.10). Let $\beta(s)$ be the angle that X makes with N in $\delta_l(s)$. Since $\beta(s) + \frac{\pi}{2} = \vartheta(s)$, it suffices to compute $\dot\beta(s)$. The hypercycle $\delta_l(s)$ has curvature $\frac{2l}{1+l^2} = \tanh(\ell)$. The circle centered in the origin passing through $\delta_l(s)$ has curvature $\coth(d_H(0, \delta_l(s))) = \frac{\cos(\beta(s))}{\tanh(\ell)}$, by the hyperbolic trigonometric laws (as in Equation (2.9)). Now, we obtain an ODE for $\beta(s)$ arguing as in Lemma 2.24: at any time $s \in \mathbb{R}$ we have that

$$\begin{aligned}
-\tanh(\ell)\cos(\beta) &= -\tanh(\ell) g_H(\dot\delta_l, N^\perp) = g_H(\nabla_{\dot\delta_l}\dot\delta_l, N) \\
&= \frac{d}{ds}(g_H(\dot\delta_l, N)) - g_H(\dot\delta_l, \nabla_{\dot\delta_l} N) \\
&= -\cos(\beta)\dot\beta + g_H(\dot\delta_l, N^\perp)^2 g_H(\nabla_{N^\perp} N^\perp, N) \\
&= -\cos(\beta)\dot\beta - \frac{\cos(\beta)^3}{\tanh(\ell)}.
\end{aligned}$$

Dividing both sides by $\cos(\beta) \neq 0$ it follows that

$$\begin{cases} \dot\beta(s) = \tanh(\ell) - \frac{\cos(\beta(s))^2}{\tanh(\ell)}, & s \in \mathbb{R}, \\ \beta(0) = 0. \end{cases}$$

By integration, one can compute the explicit solution

$$\beta(s) = -\arctan\big(\operatorname{csch}(\ell)\tanh(\operatorname{sech}(\ell)s)\big),$$

that by differentiation gives

$$\dot\vartheta(s) = \dot\beta(s) = -\big(\sinh(\ell)\cosh(\ell)\cosh^2(\operatorname{sech}(\ell)s) + \coth(\ell)\sinh^2(\operatorname{sech}(\ell)s)\big)^{-1},$$

proving Equation (2.10). □

We can prove the main result about the lower curve.

Proposition 2.28. *It γ is not a circle centered in the origin, the lower curve is contained in $[a_0, a)$, that is $0 < a_1 < a$. Furthermore, $a_1 \in I_L$ and $\dot\gamma(a_1) = -X(\gamma(a_1))$.*

Proof. By property iv. of Lemma 2.19, $a_1 > a > 0$. Suppose by contradiction that $a_1 = a$. Set $\tilde\eta_1$ to be the (reparametrized) lower curve and η_2 the upper curve. Choose any point $t \in I_L$ with corresponding $\bar t \in I_U$. Applying Lemma 2.26 and Lemma 2.27 to $\tilde\eta_1(t)$ and $\eta_2(\bar t)$, and taking advantage of the expression for $\mathbf{H}_f$ given in Lemma 2.11, we get that

$$\begin{aligned}
\mathbf{H}_f(\gamma(\bar t)) = \mathbf{H}_f(\gamma(t)) &= \kappa_\gamma(t) + (n-2)\kappa(C_t) + h'(d_H(0,\gamma(t)))g_H(\nu(t), N) \\
&< \kappa_\gamma(t) + (n-2)\kappa(C_{\bar t}) + h'(d_H(0,\gamma(t)))g_H(\nu(t), N).
\end{aligned}$$

We have that

$$d_H(0, \gamma(t)) \leq d_H(0, \gamma(\bar{t})).$$

This can be verified again via the trigonometric rules for hyperbolic triangles: fix $t \in I_U$, and call β and $\bar{\beta}$ the angle that N makes with $X(\gamma(t))$ and $X(\gamma(\bar{t}))$ respectively. Notice that $0 \leq \beta \leq \bar{\beta}$. Then, calling d the distance of $\gamma(t)$ and $\gamma(\bar{t})$ from e_1, we get that

$$\tanh(d_H(0, \gamma(t))) = \frac{\tanh(d)}{\cos(\beta)} \leq \frac{\tanh(d)}{\cos(\bar{\beta})} = \tanh(d_H(0, \gamma(\bar{t}))).$$

Hence

$$\mathbf{H}_f(\gamma(\bar{t})) < \kappa_\gamma(t) + (n-2)\kappa(C_{\bar{t}}) + h'(d_H(0, \gamma(\bar{t})))g_H(\nu(\bar{t}), N),$$

implying

$$\kappa_\gamma(\bar{t}) < \kappa_\gamma(t),$$

since h is strictly convex. This is a contradiction because Lemma 2.25 tells us that the lower curve hits the e_1 axis with an angle strictly smaller than $-\frac{\pi}{2}$. Therefore, $a_1 < a$. Since γ is smooth in $a_1 < a$, and the conditions on I_L are closed, we deduce that $a_1 \in I_L$. Suppose now that $\alpha(a_1)$ is strictly in the III quadrant. Since $a_1 \in I_L$, we can apply again the comparison lemmas to $\gamma(a_1)$ and $\gamma(\bar{a}_1)$ to infer

$$\kappa_\gamma(\bar{a}_1) < \kappa_\gamma(a_1).$$

By continuity of κ_γ and $\dot{\gamma}$ around a_1, we get that there exists a neighbourhood of a_1 in which $\dot{\gamma}$ is in the III quadrant and the above inequality holds in the not strict sense. But this implies that a_1 is not the supremum of I_L. Therefore, the velocity vector of γ at a_1 has to be equal to $-X$. □

We prove the last part of the tangent lemma.

Proposition 2.29. *If γ is not a centered circle, then there exists $0 < a_1 < a_2$ such that $\dot{\gamma}(a_2) = X(\gamma(a_2))$.*

Proof. If a_2 exists we are done. Otherwise, we show that the non existence contradicts the mean-curvature convexity of Ω. Let

$$I_c := \{t \in [a_1, a) : \dot{\gamma} \text{ is in the I or IV quadrant}\}.$$

Here the index stands for *curl curve*. Set $\tilde{a}_2 := \sup I_c$. Since $\kappa(a_1) > 1$ we have that $a_1 < \tilde{a}_2$. If $\tilde{a}_2 < a$, then the mean convexity of Ω implies that

$$\kappa_\gamma(\tilde{a}_2) \geq (n-1) - (n-2)\kappa(C_{\tilde{a}_2}) > 0.$$

To see this, move $\gamma(\tilde{a}_2)$ on e_2 as in Lemma 2.19. Then, $C_{\tilde{a}_2}$ is oriented clockwise, and hence has negative curvature. But this implies that we can extend I_c after $\tilde{a}_2$, contradicting the definition of $\tilde{a}_2$. So, we need to rule out the situation in which $\tilde{a}_2 = a$. If it is the case, then again for mean-convexity one has that in I_c

$$\kappa_\gamma(t) > 1.$$

Moreover, for $t \in I_c \setminus \{a_1\}$ we have that $\dot{\gamma}$ lies in the IV quadrant, because otherwise this implies together with $\kappa_\gamma(t) > 0$ that γ cannot close at e_1. Therefore $\alpha(t)$ lies in the IV quadrant and it is strictly increasing, implying that

$$\lim_{t \to a^+} \alpha(t) < -\frac{\pi}{2}.$$

This cannot happen because of the before mentioned regularity properties of isoperimetric sets. □

The proof of the tangent lemma is then the collection of the results we showed in this section.

Proof of Lemma 2.13. The existence of the chain $0 < a_0 < a_1 < a_2 < a$ is ensured by Proposition 2.22, Proposition 2.28 and Proposition 2.29. □

2.3 Symmetric sets in $H^m_{\mathbb{K}}$

Consider any rank one symmetric space of non-compact type $(M^n, g) = (H^m_{\mathbb{K}}, g)$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Set $d = \dim(\mathbb{K}) \in \{2, 4, 8\}$ so that the real dimension of M is $n = md$. Recall that if $\mathbb{K} = \mathbb{O}$, we only have the Cayley plane $H^2_{\mathbb{O}}$. As classical references on symmetric spaces we cite the books of Eberlein [40] and Helgason [61]. Fix an arbitrary base point $o \in M$, and let N be the unit-length, radial vector field emanating from it. As in Definition 2.2, let $\mathcal{H}$ be the distribution on $M \setminus \{o\}$ induced by the (-4)-eigenspace of the Jacobi operator $R(\cdot, N)N$. Denote with $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ with respect to g. For every $x \in M \setminus \{o\}$, we have the orthogonal splitting

$$T_x M = \mathcal{H}_x \oplus \mathcal{V}_x,$$

with orthogonal projections $(\cdot)^{\mathcal{H}}$ and $(\cdot)^{\mathcal{V}}$. Let now $(\bar{M}^n, g_H) = (H^n_{\mathbb{R}}, g_H)$, and choose an arbitrary base point $\bar{o}$ in it. The isometric identification of $T_o M$ with $T_{\bar{o}} \bar{M}$ according to the flat metrics $(\exp^M_o)^* g|_0$ and $(\exp^{\bar{M}}_{\bar{o}})^* g_H|_0$, induces a well defined diffeomorphism

$$\Psi = \exp^{\bar{M}}_{\bar{o}} \circ (\exp^M_o)^{-1} : M \to \bar{M}.$$

With a slight abuse of notation, we still denote with g_H the metric $\Psi^* g_H$, that makes M isometric to $\bar{M}$. The following lemma allows us to compare g with g_H.

Lemma 2.30. *For every $x \in M \setminus \{o\}$, the splitting*

$$T_x M = \mathcal{H}_x \oplus \mathcal{V}_x,$$

is orthogonal with respect to g_H. In particular, letting d_H be the Riemannian distance induced by g_H on M, one has that

$$g(v,w) = \cosh^2(d_H(o,x))g_H(v^{\mathcal{H}}, w^{\mathcal{H}}) + g_H(v^{\mathcal{V}}, w^{\mathcal{V}}), \tag{2.11}$$

for all $v, w \in T_xM$.

Proof. Fix an arbitrary unit direction $N_o \in T_oM$, and let $V_o \in T_oM$ be any vector orthogonal to it with respect to $g|_o = g_H|_o$. Since the radial geodesics emanating from o are the same for g and g_H, the Jacobi field $Y(t)$ along the geodesic $\sigma : t \mapsto \exp_o^M(tN_o)$, determined by the initial conditions $Y(0) = 0$, $\dot{Y}(0) = V_o$ is the same for both metrics. Let $V(t)$ and $V_H(t)$ be the parallel transport of V_o along σ with respect to g and g_H, respectively. By the very definition of symmetric spaces, the curvature tensor R is itself parallel along geodesics. This implies that

$$\sinh(t)V_H(t) = Y(t) = \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}V(t),$$

provided V_o belongs to the κ-eigenspace of the Jacobi operator $R(\cdot, N_o)N_o$. Therefore, parallel vector fields in the eigenspaces are collinear for the two metrics. Hence, for $t > 0$ the linear subspaces $\mathcal{H}_{\sigma(t)}$ and $\mathcal{V}_{\sigma(t)}$ are nothing else than the parallel transport of the corresponding eigenspaces of $R(\cdot, N_o)N_o$ along σ. It follows that the splitting $T_xM = \mathcal{H}_x \oplus \mathcal{V}_x$ is orthogonal not only with respect to g, but also with respect to the hyperbolic metric g_H. Equation (2.11) is a direct consequence of this fact and the definition of the distribution $\mathcal{H}$. □

We can now prove Theorem 2.5.

Proof of Theorem 2.5. Let $E \subset M$ be a Hopf-symmetric set with outward pointing normal vector field ν with respect to g. By the very definition of Hopf-symmetry, $\nu^{\mathcal{H}} \equiv 0$. Therefore, thanks to Lemma 2.30, ν is orthonormal to ∂E also with respect to g_H. Let vol and vol_H the volume forms associated to g and g_H. We have that

$$P(E) = \int_{\partial E} \iota_\nu \,\mathrm{vol} = \int_{\partial E} \cosh^{d-1}(d_H(o,x))\iota_\nu \,\mathrm{vol}_H(x),$$

where $\iota : \Omega(M)^p \to \Omega(M)^{p-1}$ denotes the interior product $\iota_X\alpha(\cdot) = \alpha(X, \cdot)$. The volume of E is given by the formula

$$V(E) = \int_E \mathrm{vol} = \int_E \cosh^{d-1}(d_H(o,x))\,\mathrm{vol}_H(x).$$

Hence, the volume and perimeter of Hopf-symmetric sets in M correspond to the volume and perimeter of $\Psi(E)$ in $H^n_{\mathbb{R}}$ with density equal to $f(x) = \cosh^{d-1}(d_H(o,x))$, concluding the proof. □

Part II

✻

The semigeostrophic system

Introduction

> Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.
>
> L. F. Richardson, Weather Prediction by Numerical Processes.

Fluids, meteorology, and predictability

The foundation of modern fluid mechanics dates back to the 18th century, engaging eminent members of the scientific community of the time such as Newton, Pitot, Bernoulli, McLaurin, d'Alembert, and Euler (for a comprehensive and beautifully written history of mathematics we refer to [22]). It is due to the latter the formalization of the differential equations of motions, appeared first in *Mémoires de l'académie des sciences de Berlin* (1757), in an article [44] that still astonishes for its modern exposition, being the first instance of a system of partial differential equations (vectorial, in post-quaternionic terminology) ever written. An incompressible fluid under some external total force F_t in Eulerian coordinates is described via a velocity vector field u_t representing the magnitude and direction of the flow at every given point, a scalar pressure p_t, and density ρ_t, related as follows

$$\begin{cases} (\partial_t + u_t \cdot \nabla) u_t + \frac{1}{\rho_t} \nabla p_t = \frac{1}{\rho_t} F_t, \\ \operatorname{div}(u_t) = 0, \\ \partial_t \rho_t + \operatorname{div}(\rho_t u_u) = 0. \end{cases}$$

The first equation is nothing else that Newton second law $m \cdot a = F$, where the acceleration has to be taken along the flow. In fact, the differential operator $(\partial_t + u_t \cdot \nabla)$, called material or advective derivative, measures the rate of change of any vector field X_t along the integral lines

$$\begin{cases} \dot{\alpha}(x,t) = u_t(\alpha(x,t),t), \\ \alpha(x,0) = x, \end{cases}$$

since

$$\partial_t (X(\alpha(x,t),t)) = \big((\partial_t + u_t \cdot \nabla) X\big)(\alpha(x,t),t).$$

The second equation is an infinitesimal way of saying that the flow is volume preserving, and hence incompressible. The third equation, that we will soon forget assuming $\rho_t \equiv 1$ for simplicity, imposes conservation of mass by requiring that the density is transported along the flow: $\rho(\alpha(x,t),t) = \rho(x,0)$.

Whether this equation (or the famous Navier-Stokes equation, its viscous sibling [90, 114]) makes sense under suitable natural conditions (in mathematical jargon if it is well posed) or can develop wild physical nonsense (for instance vortices with infinite velocity) is a question that challenges the scientific community to this day, making it one of the most popular areas of research in mathematical physics, as already predicted by Euler himself in his article:

> *On comprend aisément que cette matiere est beaucoup plus difficile, & quelle renferme des recherches incontrollablements plus profondes (...).*

Tightly related with fluid mechanics, the foundation of meteorology[1], the branch of atmospherical sciences dealing with the prediction of weather, required a slow process punctuated by several technical discoveries essential to its development. The invention of the barometer by Torricelli (1643) and the mercury thermometer by Fahrenheit (1714) were crucial to obtain accurate measurements, and the construction of the first telegraph by Morse in the mid-1800 made the world smaller and the production of the first weather maps possible.

It is a fascinating fact that the possibility of predicting the weather by solving numerically the fundamental equations of fluid mechanics over some discretized cells (which is essentially what meteorologist still do at present) was already considered *before* the decisive inventions achieved under the technological urge of World War II: the radar, constructed by Bay in 1936, and the first modern computers (like the Z1, constructed by Zuse in 1941, and the famous Colossus, build in 1944 by Turing and Flowers). The author of this pioneering considerations was the mathematician and physicist Richardson[2], who published in 1922 a visionary book [95] that laid the foundation for the modern systematic method of weather forecasting of our contemporary times. The wish in his volume

> *Perhaps some day in the dim future it will be possible to advance the computations faster than the weather advances and at a cost less than the saving to mankind due to the information gained. But that is a dream.*

became reality exactly 40 years later, when TIROS I, the first meteorological satellite, was launched into space.

As explained with details in the introduction of Cullen's book [36], Richardson's method does not provide a direct understanding of the weather since it conceives the atmosphere as any other abstract mathematical fluids. Why do we have cyclones, depressions, and fronts in some part of the globe and not in others? A priori, weather forecasting via numerical simulations based on the fundamental equations of motions may lead to failure. Since this

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[1]About this we suggest the interested reader to consult the introductory article in National Geographic `https://education.nationalgeographic.org/resource/science-art-meteorology/`.

[2]The same person who formulated the Coast Paradox, a fundamental observation in the development of fractal geometry.

is clearly not the case, how can we explain the success of a science that has made enormous progress since its foundation? The complexity relies in the variety of atmospherical phenomena that happens at any time and length scales, classified in four types:

- *Microscale*: taking place within a few kilometers and in less than a day. For instance chemical and physical interactions happening on ground level, and their interplay with soil and vegetation.
- *Mesoscale*: happening on scales up to one thousand kilometers. They include all phenomena of convection and circulation of masses of air due to the heterogeneous distribution of temperature and density.
- *Synoptic scale*: even up to thousands of kilometers, this class deals with high and low pressure systems. Cyclones and hurricanes are typical examples of low pressure systems developing on tropical latitudes, and extremely cold temperatures and clear skies are the results of high pressure systems in the arctics.
- *Global scale*: are all remaining phenomena that describe global patterns evolving over time scales that can reach years: from winds travelling from low to high pressure systems, to oceanic currents, and thermal energy distributions.

As the scale decreases, prediction becomes increasingly challenging. In the 1960s, the foundation of chaos theory, developed, among others, by mathematician and meteorologist Lorenz [75], provided significant support to the unpredictability of weather. This theory highlighted the *butterfly effect*, where small-scale interactions between phenomena can lead to large-scale effects, underlining the complexity of weather forecasting.

While this intricate reality is undeniable, it is also true that weather over reasonably long periods is not completely chaotic, and it can be predicted most of the time with accuracy. The reason of this lies on the concept of *large scale control*, as developed a decade later in the 1970s. This notion is based on the observation that large-scale dynamics is topically insensible to small-scale phenomena. This principle was then enforced by the discovery that large-scale flow is essentially two dimensional, and hence more predictable than the three dimensional dynamics.

The importance of deriving new approximate models that take into account only large-scale dynamics is therefore motivated by the above considerations in the context of weather forecast predictability. In the next section, we will present and derive the semigeostrophic equations (which we also refer to as SG for short), as introduced by Hoskins in 1975 [62] based on the previous work of Eliassen [43]. This system was explicitly developed to model weather phenomena in synoptic scales under the particular assumption that the Coriolis force, and thus the dynamics induced by the earth's rotation, predominantly influences the flow. In contrast with the two dimensional Euler equation over a rotating sphere (see the article by Taylor [115]), the well posedness of the semigeostrophic system remains a delicate and fascinating subject of study with connections with unexpected branches of mathematics, as we shall see later.

The semigeostrophic approximation (SG)

We describe here the derivation of the semigeostrophic equation starting from the Euler equation as introduced in the previous section. Consider the two dimensional sphere rotating on itself with constant angular velocity ω. The canonical Riemannian metric is given in spherical coordinates (θ, φ) by

$$g = d\theta^2 + \sin^2(\theta) d\varphi^2,$$

where $\theta \in (0, \pi)$ represents the latitude and $\varphi \in (0, 2\pi)$ the longitude. The Euler equation in this curved setting takes the following form

$$\begin{cases} \partial_t u_t + \nabla_{u_t} u_t + f J u_t + \nabla p_t = 0, \\ \operatorname{div}(u_t) = 0, \end{cases}$$

where u_t is now a vector field on the sphere, and the advection derivative $(\partial_t + u_t \cdot \nabla)$ has to be replaced with the covariant derivative $(\partial_t + \nabla_{u_t})$ in order to ensure the image to take place in TS^2 by projection. The term fJu_t is the *Coriolis term*, induced by the earth rotation, where J denotes counter-clockwise rotation of $\pi/2$-radians (complex endomorphism of TS^2), and $f = 2\omega\cos(\theta)$, see Figure 2.6. Notice that neglecting the vertical components

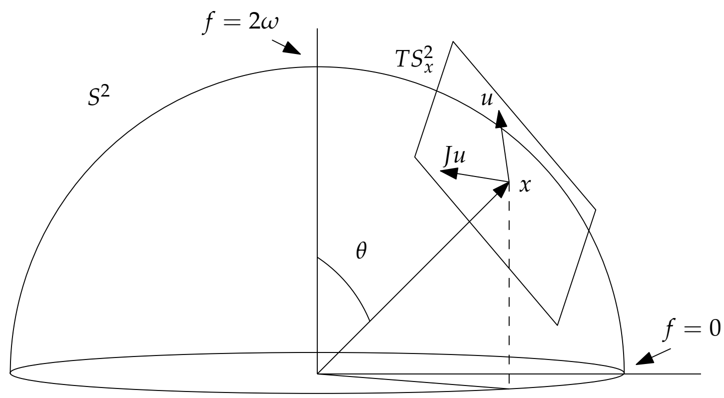

Figure 2.6: The J-action represented on a hemisphere.

by considering the equation on the surface of the sphere, we are already implementing the first principle described in the previous section, namely that large-scales fluids have a mostly two dimensional dynamics.

We assume now that we want to study the atmosphere in synoptic scales when the Coriolis force is the dominating term in the equation. This last requirement is quantified by the *Rossoby number*, an a-dimensional constant Ro that relates the length scale L with the horizontal velocity scale U^h, approximating typically the ratio between the advection term and the Coriolis force:

$$Ro := \frac{U^h}{Lf} \approx \frac{|\partial_t u_t + \nabla_{u_t} u_t|}{|f u_t|}.$$

We suppose that $Ro \ll 1$, that is we are in what is called a *geostrophic regime* (in contrast with the *cyclostrophic regime* when $Ro \gg 1$). We call u_t^G the *geostrophic wind*, the component velocity that represents the purely geostrophic balance in the Euler equation, when the advection term is completely negligible:

$$fJu_t^G + \nabla p_t = 0.$$

The semigeostrophic equation is then obtained by considering advected only the geostrophic wind in the Euler equation:

$$\begin{cases} \partial_t u_t^G + \nabla_{u_t} u_t^G + fJu_t + \nabla p_t = 0, \\ fJu_t^G + \nabla p_t = 0, \\ \mathrm{div}(u_t) = 0, \end{cases}$$

which is clearly equivalent to

$$\begin{cases} \partial_t u_t^G + \nabla_{u_t} u_t^G + fJ(u_t - u_t^G) = 0, \\ u_t^G = f^{-1} J \nabla p_t, \\ \mathrm{div}(u_t) = 0. \end{cases}$$

Two main difficulties emerge already from this formulation: the degeneracy of f^{-1} approaching the equator, and the implicit nature of u_t, which in this context has the role of a Lagrange multiplier that forces u_t^G to stay in the particular form $f^{-1}J\nabla p_t$. Hence, u_t in the SG system plays the same role as ∇p_t in the Euler equation, with the additional complexity of appearing non-linearly in the equation.

Before highlighting the main achievements in the theory of SG system, we present the most fascinating feature of this equation, as it was first sensed by Hoskins [62] and deeply investigated by Cullen [36]: in the flat case with constant Coriolis force (so replacing S^2 with $\mathbb{T}^2$, localizing the problem in an infinitesimally small neighbourhood of the arctics) it is possible to establish a dual reformulation consisiting in a fully non-linear version of the Euler vorticity equation where the Laplace is replaced with a Monge-Ampère equation. This can be done as follows: the SG system on $\mathbb{T}^2$ is

$$\begin{cases} \partial_t \nabla p_t + (u_t \cdot \nabla)\nabla p_t + (u_t - \nabla^{\perp} p_t) = 0, \\ \mathrm{div}(u_t) = 0, \end{cases}$$

so if we define the time dependent measure μ_t as[3]

$$\mu_t = (T_t)_{\#}\,\mathrm{vol},$$

where $T_t = \nabla p_t + x$, and vol is the Lebesgue measure restricted to $\mathbb{T}^2$, we get that for every

[3]Reall the *push forward* of a measure is defined by the identity $\mu_t(A) := \mathrm{vol}(T_t^{-1}(A))$, for all A Borel.

test function $\phi \in C^\infty(\mathbb{T}^2)$ the following holds:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^2}\phi\,d\mu_t &= \frac{d}{dt}\int_{\mathbb{T}^2}\phi\,d(T_t)_\#\,\mathrm{vol} = \frac{d}{dt}\int_{\mathbb{T}^2}\phi\circ T_t\,d\,\mathrm{vol}\\ &= \int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot\partial_t T_t\,d\,\mathrm{vol}\\ &= -\int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot((u_t\cdot\nabla)\nabla p_t + (u_t - \nabla^\perp p_t))\,d\,\mathrm{vol}\\ &= -\int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot((u_t\cdot\nabla)(\nabla p_t + x) - (\nabla p_t + x)^\perp + x^\perp)\,d\,\mathrm{vol}\\ &= -\int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot((u_t\cdot\nabla)T_t - T_t^\perp + x^\perp)\,d\,\mathrm{vol},\end{aligned}$$

where we used that $(v\cdot\nabla)x = v$ for every vector v. Now, since u_t is divergence free we have the following magic cancellation

$$-\int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot(u_t\cdot\nabla)T_t\,d\,\mathrm{vol} = -\int_{\mathbb{T}^2}\nabla(\phi\circ T_t)\cdot u_t\,d\,\mathrm{vol} = \int_{\mathbb{T}^2}(\phi\circ T_t)\mathrm{div}(u_t)\,d\,\mathrm{vol} = 0.$$

So, if moreover $T_t^{-1} = S_t$ is well defined, we get finally

$$\frac{d}{dt}\int_{\mathbb{T}^2}\phi\,d\mu_t = \int_{\mathbb{T}^2}(\nabla\phi\circ T_t)\cdot(T_t^\perp - x^\perp)\,d\,\mathrm{vol} = \int_{\mathbb{T}^2}\nabla\phi\cdot(y - S_t)^\perp\,d\mu_t,$$

showing that μ_t is *transported* along $U_t := (S_t - y)^\perp$, that is

$$\partial_t\mu_t + \mathrm{div}(\mu_t U_t) = 0.$$

The structure of T_t and its invertibility are guaranteed by Optimal Transport theory, creating an unexpected and elegant bridge between large-scale meteorology and optimization. Brenier's Theorem [25] (generalized to the periodic case by Cordero-Erausquin [35]), ensures the existence of a *unique convex* map P such that

$$\int_{\mathbb{T}^2}\phi\,d\mu = \int_{\mathbb{T}^2}\phi\circ\nabla P\,d\,\mathrm{vol},\quad \forall\phi\in C^\infty(\mathbb{T}^2),$$

for any given probability measure μ absolutely continuous with respect to vol. Moreover, the inverse transport map is given via Legendre transform ∇P^*. This result not only provides a unique candidate for the construction of the map $T_t = \nabla p_t + x = \nabla(p_t + |x|^2/2)$ and its inverse S_t, but it is surprisingly compatible with the physics of the problem: energy considerations made by Cullen on the L^2-norm of the geostrophic wind u_t^G shows that requiring $p_t + |x|^2/2$ *convex* is a necessary condition of stability. This is nowadays known as *Cullen stability principle*. So, the previous observations formally lead us to the *dual reformulation* the the SG system

$$\begin{cases}\partial_t\mu_t + \mathrm{div}(U_t\mu_t) = 0,\\ U_t = (\nabla P_t^* - y)^\perp,\\ \det(D^2 P_t^*) = \mu_t,\\ P_t \text{ convex},\end{cases}$$

where the Monge-Ampère equation derives from requiring $(\nabla P_t^*)_\# \mu = \text{vol}$. Comparing this system with the more familiar *vorticity formulation* of the Euler equation obtained by setting $\omega_t = \text{curl}(u_t)$

$$\begin{cases} \partial_t \omega_t + \text{div}(u_t \omega_t) = 0, \\ u_t = -\nabla^\perp \psi_t, \\ \Delta \psi_t = \text{curl}(-\nabla^\perp \psi_t) = \omega_t, \end{cases}$$

one discovers that essentially the SG system is the non-linear twin of the Euler equation, where the Laplace operator is replaced by the fully non-linear Monge-Ampère equation.

State of the art

In the context of the dual formulation presented in the previous section, Optimal Transport theory allows the construction of a solution, see Benamou and Brenier [13], Faria, Lopes Filho, and Nussenzveig Lopes [46], Feldman and Tudorascu [47], and recently the work of Bourne, Egan, Pelloni, Wilkinson [23] based on semi-discrete optimal transport techniques (see also [12, 41] for interesting numerical implementations).

The task of translating the solution from the dual reformulation to the original Eulerian coordinates is extremely delicate. In fact, it is not difficult to check that after some formal algebraic manipulations on the SG system, the velocity (disappeared in the dual system thanks to the magic cancellation explained before) takes the explicit form

$$u_t = \partial_t \nabla P_t^*(\nabla P_t) + D^2 P_t^*(\nabla P_t)(x - \nabla P_t)^\perp,$$

in terms of the corrected pressure $P_t = p_t + |x|^2/2$. A priori, however, the hessian $D^2 P_t^*$ is a measured-valued matrix, since P^* is simply convex, and thus any regularity result on u_t might appear compromised. The break-thought relies on deep regularity theory for bounded Monge-Ampère equations, as developed by De Philippis and Figalli, taking the elegant form

$$0 < \lambda \leq \det(D^2 P) \leq \Lambda \Rightarrow D^2 P \in L \log^k L \quad \forall k,$$

and further improved with Savin to the sharp regularity $D^2 P \in L^{1+\varepsilon}$ for some $\varepsilon = \varepsilon(\lambda, \Lambda) > 0$ (see [38, 39]). Thank to this general estimate, Ambrosio, Colombo, De Philippis, and Figalli established the global-in-time existence and uniqueness of weak solutions in [5], making the transition back to the original coordinates possible, proving in fact that $u_t \in L^{1+\varepsilon}$.

Local-in-time existence and uniqueness of smooth solutions was then solved in the dual framework by Loeper in [74], where the convergence of the dual reformulation of the SG system to the Euler vorticity equation was made rigorous via the linearization of the determinant

$$D^2 P_t = \mathbb{I} + \varepsilon D^2 \psi_t + o(\varepsilon) \Rightarrow \det(D^2 P_t) = 1 + \varepsilon \Delta \psi_t + o(\varepsilon),$$

through a suitable rescaling of the variables in terms of $\varepsilon > 0$. An additional proof of local-in-time existence of smooth solutions was provided in Lagrangian coordinates by Cheng, Cullen, and Feldman in [33], allowing any positive and *varying* Coriolis term over $\mathbb{T}^2$.

Contributions

In Chapter 3 we prove the local-in-time existence and uniqueness of solutions in subdomains of a rotating sphere [110].

Theorem (see Theorem 3.1)**.** *Let Ω be an open, smooth, and simply connected subset of S^2 such that $\bar{\Omega}$ is contained either in the upper or in the lower open hemisphere. Let $\nabla p_0 \in H^s(\Omega, \mathbb{R}^2)$, $s \geq 4$, and suppose that there exists $\mu_0 < 1$ such that the uniform ellipticity condition*

$$\mathcal{Q}_0 := \mathbb{I} + D^2 p_0 - \nabla p_0 \otimes \nabla \ln(f) - \nabla \ln(f) \otimes \nabla p_0 \geq (1 - \mu_0)\mathbb{I} > 0,$$

is satisfied in Ω. Then, there exists $t^* > 0$ such that for all $0 < t' < t^*$ there exists a unique pair

$$\nabla p_t \in C^1(0, t'; C^{s-3,\alpha}(\Omega, \mathbb{R}^2)) \cap L^\infty(0, t'; H^s(\Omega, \mathbb{R}^2)),$$

and

$$u_t = -\nabla^\perp \psi_t \in C(0, t'; C^{s-2,\alpha}(\Omega, \mathbb{R}^2)) \cap L^\infty(0, t'; H^s(\Omega, \mathbb{R}^2)),$$

solving the SG system in $[0, t'] \times \Omega$ with $\nabla p_t|_{t=0} = \nabla p_0$, and u_t tangent to $\partial\Omega$.

The proof is robust and overcomes the absence of a dual reformulation on the sphere, holding true in general bounded and conformally flat domains with nowhere vanishing and possibly varying f. We required Ω far from the equator since there $f = 2\omega\cos(\theta)$ vanishes, inducing a singularity (recall $u_t^G = f^{-1}\nabla^\perp p_t$). The argument is constructive. The delicate point relies in the implicit definition of u_t, which evolves in short time steps as a solution of a PDE in the form

$$\mathrm{div}(\mathrm{Cof}(\mathcal{Q}_t)\nabla\psi_t) + \mathsf{b} \cdot \nabla\psi_t = \mathrm{div}(\mathsf{F})$$

obtained by applying $\mathrm{div}(f\cdot)$ to the SG system. A careful energy estimate of the H^s-norm of ∇p_t uses, like for Euler, the incomprehensibility of the fluid [77]. The magic property of the cofactor matrix $\mathrm{div}(\mathrm{Cof}(D^2 p_t)) = 0$ induces an extra order of regularity crucial to close the whole argument. Finally, uniqueness follows via the Gronwall Lemma, showing in particular that solutions with L^2-close initial data remain quantitatively L^2-close for at least a short time.

While in Chapter 3 we addressed the problem of the curvature, in Chapter 4 we face the degeneracy of the Coriolis term approaching the equator. We first introduce a new family of axially symmetric solutions determined by a one dimesional differential equation. This is made possible by the particular property of the gradient of the coordiante function z when seeing S^2 embedded in $\mathbb{R}^3$, namely its hessian is just a rescaling of the identity: $\nabla^2 z = -z\mathrm{id}$. This particular vector fields (being a particular instance of a *concircular vector field*, in the Riemannian terminology) is particularly useful for our purpose since the Coriolis term is nothing else than $2\omega z$. We prove in Theorem 4.1 that axially symmetric solutions are globally stable with respect to the linearized SG system, provided that all initial data decay fast enough approaching the equator. In fact, the construction is based on a geometric strategy that compensates the severe singularity induced by the Coriolis term lifting the equation to the higher dimensional sphere S^4 and performing a slicing argument.

Chapter 3

❊

Local-in-time existence of SG on curved domains

3.1 Peliminaries

In this chapter we will have to distinguish the operators when are associated to a sphere or to the plane. For this reason we denote with D^g, div^g and ∇_g the Levi-Civita connection, the divergence and the gradient operator associated to some metric g. The SG system over the sphere looks then like this

$$\begin{cases} (\partial_t + D^g_u)u_G + f(u - u_G)^\perp = 0, \\ u_G = \frac{1}{f}\nabla_g^\perp p, \\ \mathrm{div}^g(u) = 0, \end{cases} \tag{3.1}$$

in its essential formulation. Operating a stereographic projection pointed at the South Pole, we can see (3.1) as taking place in the two dimensional plane endowed with the conformal metric and Coriolis term

$$g = \frac{4}{(1+|x|^2)^2}\big((dx^1)^2 + (dx^2)^2\big), \quad f = 2\omega\frac{1-|x|^2}{1+|x|^2}, \tag{3.2}$$

in canonical Cartesian coordinates (x^1, x^2).

We can give the general statement of this problem: let Ω be a sufficiently smooth, bounded and simply connected domain of $\mathbb{R}^2$, and let V, φ be two given smooth functions defined on $\bar{\Omega}$. Set

$$g := e^{-2V}\big((dx^1)^2 + (dx^2)^2\big), \quad \text{and } f = e^{-\varphi}, \tag{3.3}$$

and define the endomorphism of tangent bundle

$$J = (\cdot)^\perp : T\Omega \to T\Omega, \quad J = -dx^2 \otimes \frac{\partial}{\partial x^1} + dx^1 \otimes \frac{\partial}{\partial x^2},$$

to be the counter-clockwise rotation of $\pi/2$-radians. Given an initial pressure gradient $\nabla^g p_0$ we wonder whether it is possible to find a local-in-time smooth solution of the semigeostrophic system

$$\begin{cases} (\partial_t + D^g_u)\big(e^{\varphi}\nabla_g^{\perp} p\big) + e^{-\varphi}\big(u - e^{\varphi}\nabla_g^{\perp} p\big)^{\perp} = 0, & \text{in } \Omega, \\ \operatorname{div}^g(u) = 0, & \text{in } \Omega, \\ g(u,\nu) = 0, & \text{on } \partial\Omega, \\ \nabla^g p_t \mid_{t=0} = \nabla^g p_0, \end{cases} \tag{3.4}$$

where ν denotes the outer pointing normal vector to $\partial\Omega$. In particular, when φ and V are as in (3.2) and (3.3) we are in the spherical case, and when $V = \varphi = 0$, we are in the classic flat case. Here ∇^g, div^g and D^g denote the gradient, the divergence, and the covariant derivative induced by the conformal metric g. Whereas in the flat case for any two vectors v, w one has that

$$D^g_w v = (w \cdot \nabla^g) v = w^i \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j},$$

for a general metric one has to take into consideration the lower order terms arising from the curvature

$$D^g_w v = w^i \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j} + w^i v^j \Gamma^k_{ij}\frac{\partial}{\partial x^k},$$

where Γ^k_{ij} are the Christoffel symbols induced by g. See for instance [1, Chapter 8.2] for a clear and introductory exposition of fluid dynamics on manifolds.

3.1.1 Main result

For any vector field $\xi \in C^1(\Omega, \mathbb{R}^2)$ we define the *stability matrix* $\mathcal{Q}$ as

$$\mathcal{Q} = \mathcal{Q}[D\xi, \xi] := e^{2V+2\varphi}\Big(D\xi^T + (\nabla V + \nabla\varphi/2) \otimes \xi + \xi \otimes (\nabla V + \nabla\varphi/2) - \langle \xi, \nabla V\rangle \mathbb{I}\Big),$$

where $\mathbb{I}$ denotes the identity matrix. For matrices A and B, we will write

$$A \geq B, \text{ whenever } \langle A\xi, \xi\rangle \geq \langle B\xi, \xi\rangle \text{ for all } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

The main result of this Chapter is the following.

Theorem 3.1. *Let $k \geq 4$ be fixed, Ω be an open, simply connected and bounded subset of $\mathbb{R}^2$ with C^{k+1}-boundary, and V, φ be given functions in $C^{k+1}(\bar{\Omega})$. Let $\nabla p_0 \in H^k(\Omega, \mathbb{R}^2)$, and suppose that there exists $\mu_0 < 1$ such that*

$$\mathbb{I} + \mathcal{Q}[D^2 p_0, \nabla p_0] \geq (1 - \mu_0)\mathbb{I} > 0.$$

Then, there exists a constant $C = C(\Omega, V, \varphi, k) > 0$ such that, setting

$$t^* := C\left(\frac{1 - \mu_0}{\|\nabla p_0\|_{H^k(\Omega)} + 1}\right)^{(k+1)k+2},$$

for all $0 < t' < t^*$ and $\alpha \in (0,1)$ there exists a unique pair

$$\nabla p_t \in C(0, t'; C^{k-2,\alpha}(\Omega, \mathbb{R}^2)) \cap C^1(0, t'; C^{k-3,\alpha}(\Omega, \mathbb{R}^2)) \cap L^\infty(0, t'; H^k(\Omega, \mathbb{R}^2)),$$

and

$$u_t = -e^{2V}\nabla^\perp\psi_t \in C(0,t';C^{k-2,\alpha}(\Omega,\mathbb{R}^2)) \cap L^\infty(0,t';H^k(\Omega,\mathbb{R}^2)),$$

solving the semigeostrophic System (3.4) in $[0,t']$. Moreover, in $[0,t']$ the constant of uniform ellipticity of $\mathbb{I}+\mathcal{Q}[D^2p_t,\nabla p_t]$ is bounded away from zero.

3.1.2 Structure of the Chapter and strategy of the proof

In Section 3.2 we start by developing the estimates of general elliptic partial differential equations with Dirichlet boundary condition in the form

$$\begin{cases} \operatorname{div}(\mathsf{A}\nabla\phi) + \mathsf{b}\cdot\nabla\phi = \operatorname{div}(\mathsf{F}), & \text{in } \Omega \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $\mathsf{A} = \mathsf{A}(x)$ is supposed uniformly elliptic, that is $\mathsf{A}(x) \geq \lambda\mathbb{I}$ for some $\lambda > 0$ and all $x \in \Omega$. We will take advantage of the classic regularity theory in the Sobolev space $H^k(\Omega) = \{f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), |\alpha| \leq k\}$, $k \geq 4$, to find an explicit upper bound on the constant $C > 0$ realizing

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C\Big(\|\nabla\phi\|_{L^2(\Omega)} + \|\mathsf{F}\|_{H^k(\Omega)}\Big),$$

in terms of the H^{k-1}-norm of A, $\operatorname{div}(\mathsf{A})$, b and the elliptic constant λ. The key observation here is that if $\operatorname{div}(\mathsf{A})$ shares the same regularity as A, then we gain two derivatives for the solution ϕ instead of one.

Section 3.3 is devoted entirely to the construction of an approximate solution. We start by taking advantage of the conformal nature of the metric to "flatten" the Riemannian operators and see (3.1) as a lower order perturbation of the equation in $(\mathbb{R}^2, dx)$. Then, we formally obtain an elliptic partial differential equation for the potential of the velocity (recall that Ω is simply connected and the fluid is incompressible) of the form (3.5) "killing" the time derivative on the rotated gradient $\partial_t\nabla^\perp p$ by applying the divergence operator on both sides of the semigeostrophic equation. In particular A has the form $\mathbb{I}+\operatorname{Cof}(\mathcal{Q})$, and here is where the stability condition comes from as a necessary requirement of solvability. A very nice cancellation property of the cofactor matrix ensures $\|\mathcal{Q}\|_{H^{k-1}(\Omega)} \sim \|\operatorname{div}(\operatorname{Cof}(\mathcal{Q}))\|_{H^{k-1}(\Omega)}$, allowing us to take full advantage of the previous general elliptic estimates. We then construct a sequence of approximate solutions regularizing the semigeostrophic equation and discretizing the time in little steps.

In order to prove uniform existence of a sequence of regularized solutions, in Section 3.4 we operate an Energy Estimate on the Sobolev norm of the pressure gradient and the elliptic constant λ of $\mathbb{I}+\mathcal{Q}$. Here the elliptic regularity estimate on the velocity plays a role to prove that

$$\left|\frac{d}{dt}(-\lambda)\right| + \left|\frac{d}{dt}\|\nabla p\|_{H^k(\Omega)}\right| \lesssim \Big(\frac{\|\nabla p\|_{H^k(\Omega)+1}}{\lambda}\Big)^{M(k)},$$

for some exponent $M(k) > 0$. A Gronwall-type argument on a well chosen function completes the proof of uniform existence local-in-time of approximate solutions.

In Section 3.5 we extract a smooth solution of the semigeostrophic equations by applying a suitable argument of compactness. We complete the proof of Theorem 3.1 by showing via a Grönwall argument that the constructed solution is in fact unique in the given class of regularity.

3.2 Explicit elliptic estimates

We refer to [4] and [45] for the classical elliptic regularity methods that we will employ. We start by stating two useful interpolation results.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded and smooth domain. Then, for every $0 \leq k \leq m$ there exists a constant $c = c(k, m, \Omega) > 0$ such that*

$$\|v\|_{H^k(\Omega)} \leq c\|v\|_{L^2(\Omega)}^{1-\frac{k}{m}} \|v\|_{H^m(\Omega)}^{\frac{k}{m}},$$

for every $v \in H^m(\Omega)$.

Proof. The proof can be found in [2, Chapter 5]. □

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be smooth and bounded, and let v, w be functions in $H^r(\Omega) \cap L^\infty(\Omega)$ for some $r \geq 1$. Then, there exists $C = C(r, \Omega) > 0$ such that*

$$\|\partial^\alpha(vw)\|_{L^2(\Omega)} \leq C\Big(\|v\|_{L^\infty(\Omega)}\|w\|_{H^r(\Omega)} + \|w\|_{L^\infty(\Omega)}\|v\|_{H^r(\Omega)}\Big), \tag{3.6}$$

for all multi-index $|\alpha| = r$. In particular, the following inequalities

$$\|\partial^\alpha(vw) - v\partial^\alpha w\|_{L^2(\Omega)} \leq C_r\Big(\|\nabla v\|_{L^\infty(\Omega)}\|w\|_{H^{r-1}(\Omega)} + \|v\|_{H^r(\Omega)}\|w\|_{L^\infty(\Omega)}\Big), \tag{3.7}$$

and

$$\|\partial^\alpha(vw) - v\partial^\alpha w - w\partial^\alpha v\|_{L^2(\Omega)} \leq C_r\Big(\|\nabla v\|_{L^\infty(\Omega)}\|w\|_{H^{r-1}(\Omega)} + \|v\|_{H^{r-1}(\Omega)}\|\nabla w\|_{L^\infty(\Omega)}\Big), \tag{3.8}$$

hold.

Proof. The proof can be found in [77, Lemma 3.4]. □

3.2.1 Set-up

Let Ω be an open, bounded subset of $\mathbb{R}^2$, and suppose we are given a symmetric matrix $\mathsf{A} \in C^\infty(\bar{\Omega})^{2\times 2}$ and vector fields $\mathsf{b}, \mathsf{F} \in C^\infty(\bar{\Omega})^2$, such that there exists $\lambda > 0$ satisfying

$$0 < \lambda\mathbb{I} \leq \mathsf{A}.$$

Define $\operatorname{div}(\mathsf{A}) \in C^\infty(\bar{\Omega})^2$ as

$$\operatorname{div}(\mathsf{A})^j := \sum_{i=1}^{2} \partial_i \mathsf{A}_{ij},$$

such that

$$\operatorname{div}(\mathsf{A}\nabla\phi) = \operatorname{Tr}(\mathsf{A}D^2\phi) + \operatorname{div}(\mathsf{A})\cdot\nabla\phi, \quad \forall\phi\in C^2(\Omega).$$

Let $\phi\in C^\infty(\bar\Omega)$ be solution of

$$\begin{cases}\operatorname{div}(\mathsf{A}\nabla\phi) + \mathsf{b}\cdot\nabla\phi = \operatorname{div}(\mathsf{F}), & \text{in } \Omega,\\ \phi = 0, & \text{on } \partial\Omega.\end{cases} \tag{3.9}$$

The goal of this section is to prove the following global estimate.

Proposition 3.4 (Global estimates)**.** *Suppose that $\partial\Omega$ is of class C^{k+1} for some $k\geq 4$. Then, there exists a universal constant $C_{k,\Omega}>0$ such that*

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\Big(\lambda^{-(k+1)k}\mathbf{M}^{(k+1)k}\|\nabla\phi\|_{L^2(\Omega)} + \lambda^{-(k+1)}\mathbf{M}^k\|\mathsf{F}\|_{H^k(\Omega)}\Big), \tag{3.10}$$

where

$$\mathbf{M} := \Big(\|\mathsf{A}\|_{H^{k-1}(\Omega)} + \|\operatorname{div}(\mathsf{A})\|_{H^{k-1}(\Omega)} + \|\mathsf{b}\|_{H^{k-1}(\Omega)}\Big).$$

Moreover, if $\mathsf{b} = \nabla^\perp\mathsf{f}$ for some $\mathsf{f}\in C^\infty(\bar\Omega)$, then

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\lambda^{-(k+1)k-1}\mathbf{M}^{(k+1)k}\|\mathsf{F}\|_{H^k(\Omega)}. \tag{3.11}$$

Remark 3.5. *An important situation in which the particular case $\mathsf{b}=\nabla^\perp\mathsf{f}$ of Proposition 3.4 arises is when we symmetrize the elliptic matrix. In fact, suppose that the elliptic equation is of the form*

$$\operatorname{div}\big((\mathsf{A}+\tilde{\mathsf{A}})\nabla\phi\big) = \operatorname{div}(\mathsf{F}),$$

where $\tilde{\mathsf{A}}$ is an antisymmetric matrix. In this case we have that

$$\begin{aligned}\operatorname{div}\big(\tilde{\mathsf{A}}\nabla\phi\big) &= \operatorname{div}(\tilde{\mathsf{A}})\cdot\nabla\phi + \operatorname{Tr}(\tilde{\mathsf{A}}D^2\phi) = \partial_1\tilde{\mathsf{A}}_{12}\partial_2\phi + \partial_2\tilde{\mathsf{A}}_{21}\partial_1\phi\\ &= \partial_1\tilde{\mathsf{A}}_{12}\partial_2\phi - \partial_2\tilde{\mathsf{A}}_{12}\partial_1\phi = \nabla^\perp\tilde{\mathsf{A}}_{12}\cdot\nabla\phi,\end{aligned}$$

i.e. the coefficient b comes from the rotated potential $\mathsf{f}=\tilde{\mathsf{A}}_{12}$.

3.2.2 Rescaled elliptic estimates

Fix $k\geq 4$. To simplify the exposition of the following estimates, we will write

$$a\lesssim b, \text{ (or } a\lesssim_r b),$$

if there exists some constant $c=c(\Omega,k)>0$ (respectively $c=c(\Omega,k,r)>0$), such that

$$|a|\leq cb.$$

In this section, we will suppose that

$$\lambda, \|\mathsf{A}\|_{H^{k-1}(\Omega)}, \|\operatorname{div}(\mathsf{A})\|_{H^{k-1}(\Omega)}, \|\mathsf{b}\|_{H^{k-1}(\Omega)} \leq 1. \tag{3.12}$$

Consequently, by Sobolev embeddings, we also have that

$$\|\mathsf{A}\|_{W^{k-3,\infty}(\Omega)}, \|\mathsf{b}\|_{W^{k-3,\infty}(\Omega)} \lesssim 1.$$

We start by proving a local interior estimate.

Proposition 3.6 (Rescaled interior estimates)**.** *Fix $x_0 \in \Omega$ and $r > 0$ such that $B_r := B(x_0, r) \subset \Omega$. Then, the interior estimate*

$$\|\nabla\phi\|_{H^k(B_{r/2})} \lesssim_r \frac{1}{\lambda}\Big(\|\nabla\phi\|_{H^{k-1}(B_r)} + \|\mathsf{F}\|_{H^k(B_r)}\Big), \tag{3.13}$$

holds.

Proof. Let $|\alpha| = k$ be any multi-index. Then, differentiating α-times (3.9), we have that

$$0 = -\mathrm{div}\big(\partial_\alpha(\mathsf{A}\nabla\phi)\big) - \partial_\alpha(\mathsf{b}\nabla\phi) + \mathrm{div}(\partial_\alpha\mathsf{F}),$$

which implies, adding $\mathrm{div}(\mathsf{A}\partial_\alpha\nabla\phi)$ to both sides, that

$$\begin{aligned}\mathrm{div}(\mathsf{A}\partial_\alpha\nabla\phi) &= \mathrm{div}\big(\mathsf{A}\partial_\alpha\nabla\phi - \partial_\alpha(\mathsf{A}\nabla\phi)\big) - \partial_\alpha(\mathsf{b}\nabla\phi) + \mathrm{div}(\partial_\alpha\mathsf{F})\\ &= \mathrm{div}\big(\partial_\alpha\mathsf{A}\nabla\phi + \mathsf{A}\partial_\alpha\nabla\phi - \partial_\alpha(\mathsf{A}\nabla\phi)\big) - \partial_\alpha(\mathsf{b}\nabla\phi) + \mathrm{div}(\partial_\alpha\mathsf{F}) - \mathrm{div}(\partial_\alpha\mathsf{A}\nabla\phi),\end{aligned} \tag{3.14}$$

where in the second line we simply add and subtract $\mathrm{div}(\partial_\alpha\mathsf{A}\nabla\phi)$. Call

$$\mathsf{X} := \partial_\alpha\mathsf{A}\nabla\phi + \mathsf{A}\partial_\alpha\nabla\phi - \partial_\alpha(\mathsf{A}\nabla\phi).$$

Fix $x_0 \in \Omega$ and $r > 0$ such that $B_r := B(x_0, r) \subset \Omega$. Choose $\eta \in C_c^\infty(B_r)$ such that $\eta|_{B_{r/2}} \equiv 1$, $\eta|_{\mathbb{R}^2\setminus B_r} \equiv 0$ and $0 \le \eta \le 1$. Testing Equation (3.14) against $\xi := \eta^2\partial_\alpha\phi$ gives

$$\int \langle \mathsf{A}\partial_\alpha\nabla\phi, \nabla\xi\rangle\,dx = \int\langle \mathsf{X} + \partial_\alpha\mathsf{F}, \nabla\xi\rangle\,dx + \int \partial_\alpha(\mathsf{b}\nabla\phi)\xi\,dx + \int \mathrm{div}(\partial_\alpha\mathsf{A}\nabla\phi)\xi\,dx.$$

Since $\nabla\xi = \eta^2\nabla\partial_\alpha\phi + 2\partial_\alpha\phi\eta\nabla\eta$, taking advantage of the ellipticity of A we can estimate

$$\begin{aligned}\lambda\int \eta^2|\partial_\alpha\nabla\phi|^2\,dx \le& \underbrace{-\int\langle\mathsf{A}\partial_\alpha\nabla\phi, 2\partial_\alpha\phi\eta\nabla\eta\rangle\,dx}_{(I)} + \underbrace{\int\langle\mathsf{X} + \partial_\alpha\mathsf{F}, \nabla\xi\rangle\,dx}_{(II)}\\ &+ \underbrace{\int\partial_\alpha(\mathsf{b}\nabla\phi)\xi\,dx}_{(III)} + \underbrace{\int\mathrm{div}(\partial_\alpha\mathsf{A}\nabla\phi)\xi\,dx}_{(IV)}.\end{aligned} \tag{3.15}$$

We will now treat (I)-(IV) separately. By the Young inequality, since $\|\mathsf{A}\|_{L^\infty(\Omega)} \lesssim 1$, we have that

$$(I) \lesssim_r \frac{1}{\epsilon}\int_{B_r}|\partial_\alpha\phi|^2\,dx + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx,$$

for every $\epsilon > 0$. Now, observe that for every smooth function h and $0 < \epsilon \le 1$, it holds that

$$\begin{aligned}\int|h||\nabla\xi|\,dx &\le \int|h|\Big(\eta^2|\nabla\partial_\alpha\phi| + 2\eta|\nabla\eta||\partial_\alpha\phi|\Big)\,dx\\ &\lesssim \frac{1}{\epsilon}\int_{B_r}|h|^2\,dx + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx + \int|h|\eta|\nabla\eta||\partial_\alpha\phi|\,dx\\ &\lesssim_r \frac{1}{\epsilon}\int_{B_r}|h|^2\,dx + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx + \int_{B_r}|\partial_\alpha\phi|^2\,dx.\end{aligned} \tag{3.16}$$

Also, recalling that $k \geq 4$, by interpolation inequality (3.8) we can easily estimate

$$\|\mathsf{X}\|_{L^2(B_r)} \lesssim \Big(\|\mathsf{A}\|_{W^{1,\infty}(B_r)}\|\nabla\phi\|_{H^{k-1}(B_r)} + \|\mathsf{A}\|_{H^{k-1}(B_r)}\|D^2\phi\|_{L^\infty(B_r)}\Big) \lesssim \|\nabla\phi\|_{H^{k-1}(B_r)}.$$

Therefore, we obtain that

$$(II) \lesssim_r \frac{1}{\epsilon}\|\nabla\phi\|^2_{H^{k-1}(B_r)} + \frac{1}{\epsilon}\|\partial_\alpha \mathsf{F}\|^2_{L^2(B_r)} + \epsilon\int \eta^2|\nabla\partial_\alpha\phi|^2\,dx + \int_{B_r}|\partial_\alpha\phi|^2\,dx.$$

Finally, consider the terms (III) and (IV). Recall that in (3.12) we assumed only the $H^{k-1}(\Omega)$-norms of A, $\mathrm{div}(\mathsf{A})$ and b to be controlled by 1. This means that we need to integrate by parts in such a way that these terms are differentiated at most $(k-1)$-times. Choose $i \in \{1,2\}$ such that $\partial_\alpha = \partial_\beta\partial_i$, with $|\beta| = k-1$. Then

$$(III) = -\int \partial_\beta(\mathsf{b}\nabla\phi)\partial_i\xi\,dx = -\int\Big(\partial_\beta(\mathsf{b}\nabla\phi) - \partial_\beta\mathsf{b}\nabla\phi\Big)\partial_i\xi\,dx - \int(\partial_\beta\mathsf{b}\nabla\phi)\partial_i\xi\,dx,$$

which by (3.7) and (3.16) gives

$$(III) \lesssim_r \frac{1}{\epsilon}\|\nabla\phi\|^2_{H^{k-1}(B_r)} + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx.$$

Similarly we have that

$$\begin{aligned}
(IV) &= \int \partial_i\mathrm{div}(\partial_\beta\mathsf{A}\nabla\phi)\xi - \mathrm{div}(\partial_\beta\mathsf{A}\partial_i\nabla\phi)\xi\,dx \\
&= -\int \mathrm{div}(\partial_\beta\mathsf{A}\nabla\phi)\partial_i\xi - \langle\partial_\beta\mathsf{A}\partial_i\nabla\phi, \nabla\xi\rangle\,dx \\
&= -\int \mathrm{Tr}(\partial_\beta\mathsf{A}D^2\phi)\partial_i\xi + \mathrm{div}(\partial_\beta\mathsf{A})\cdot\nabla\phi\partial_i\xi - \langle\partial_\beta\mathsf{A}\partial_i\nabla\phi, \nabla\xi\rangle\,dx \\
&\leq \int\Big(|\mathrm{Tr}(\partial_\beta\mathsf{A}D^2\phi)| + |\partial_\beta\mathrm{div}(\mathsf{A})\cdot\nabla\phi| + |\partial_\beta\mathsf{A}\partial_i\nabla\phi|\Big)|\nabla\xi|\,dx \\
&\lesssim_r \frac{1}{\epsilon}\|\nabla\phi\|^2_{W^{1,\infty}(B_r)} + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx + \int_{B_r}|\partial_\alpha\phi|^2\,dx \\
&\lesssim_r \frac{1}{\epsilon}\|\nabla\phi\|^2_{H^{k-1}(B_r)} + \epsilon\int\eta^2|\nabla\partial_\alpha\phi|^2\,dx + \int_{B_r}|\partial_\alpha\phi|^2\,dx.
\end{aligned}$$

Letting $\epsilon = c_r\lambda$, for some small constant $c_r > 0$, Equation (3.15) gives

$$\|\partial_\alpha\nabla\phi\|^2_{L^2(B_{r/2})} \lesssim_r \frac{1}{\lambda^2}\Big(\|\nabla\phi\|^2_{H^{k-1}(B_r)} + \|\mathsf{F}\|^2_{H^k(B_r)}\Big),$$

as wished. □

Now, to obtain a similar estimate on the boundary, we start by treating the flat case.

Proposition 3.7 (Rescaled flat boundary estimates). *Let* $\Omega = B_r^+ := \{x^2 > 0\} \cap B(0,r)$. *Then,*

$$\|\nabla\phi\|_{H^k(B^+_{r/2})} \lesssim_r \frac{1}{\lambda^{k+1}}\Big(\|\nabla\phi\|_{H^{k-1}(B^+_r)} + \|\mathsf{F}\|_{H^k(B^+_r)}\Big), \tag{3.17}$$

Proof. We first start by estimating the norm of the tangential derivatives. Fix $k \geq 4$. Let $\eta \in C_c^\infty(\mathbb{R}^2)$ be a cutoff function such that $\eta\mid_{B_{r/2}} = 1$, $\eta \mid_{\mathbb{R}^2\setminus B_r} = 0$ and $0 \leq \eta \leq 1$. Then, since the test function $\xi := \eta^2\partial_1^k\phi$ vanishes on ∂B_r^+ (recall that we prescribed $\phi = 0$ on the segment $(-r,r)\times\{0\}$), we can repeat the proof of Proposition 3.6 for $\alpha = (k,0)$ obtaining the estimate

$$\|\nabla\partial_\alpha\phi\|^2_{L^2(B^+_{r/2})} = \|\nabla\partial_1^k\phi\|^2_{L^2(B^+_{r/2})} \lesssim_r \frac{1}{\lambda^2}\Big(\|\nabla\phi\|^2_{H^{k-1}(B^+_r)} + \|\mathsf{F}\|^2_{H^k(B^+_r)}\Big). \tag{3.18}$$

We now show that for all multi-index $\alpha = \alpha_l := (k-l,l)$ and $l = 0,\dots k$, we can estimate

$$\|\nabla\partial_{\alpha_l}\phi\|^2_{L^2(B^+_{r/2})} \lesssim_r \frac{1}{\lambda^{2(l+1)}}\Big(\|\nabla\phi\|^2_{H^{k-1}(B^+_r)} + \|\mathsf{F}\|^2_{H^k(B^+_r)}\Big).$$

We proceed by induction on l: we have already treated the case $l = 0$. Then, suppose the claim true for all $0 \leq l' \leq l$, for some fixed $0 \leq l < k$. We have to check the case $\alpha_{l+1} = (k-(l+1), l+1)$. Define $\gamma_1 = (k-(l+1), l)$ and $\gamma_2 = (k-l, l-1)$. In the following, suppose $s = 1$ if $l = 0$ and $s \in \{1,2\}$ if $l \geq 1$. We take advantage of Equation (3.9): after differentiation and suitable rearrangement we have that

$$\mathrm{Tr}(\mathsf{A}\partial_{\gamma_s}D^2\phi) = \Big(\mathrm{Tr}(\mathsf{A}\partial_{\gamma_s}D^2\phi) - \partial_{\gamma_s}\mathrm{Tr}(\mathsf{A}D^2\phi)\Big) - \partial_{\gamma_s}\Big(\big(\mathsf{b} + \mathrm{div}(\mathsf{A})\big)\cdot\nabla\phi\Big) - \mathrm{div}(\partial_{\gamma_s}\mathsf{F}),$$

which, developing the trace, becomes

$$\begin{aligned}\mathsf{A}_{22}\partial_{22}\partial_{\gamma_s}\phi = &\Big(\mathrm{Tr}(\mathsf{A}\partial_{\gamma_s}D^2\phi) - \partial_{\gamma_s}\mathrm{Tr}(\mathsf{A}D^2\phi)\Big) - \partial_{\gamma_s}\Big(\big(\mathsf{b} + \mathrm{div}(\mathsf{A})\big)\cdot\nabla\phi\Big) - \mathrm{div}(\partial_{\gamma_s}\mathsf{F})\\ &- \sum_{(i,j)\neq(2,2)}\mathsf{A}_{ij}\partial_{ij}\partial_{\gamma_s}\phi.\end{aligned}$$

Since A is elliptic, the coefficient A_{22} is controlled uniformly from below by the elliptic constant λ. Therefore, applying the L^2-norm over $B^+_{r/2}$ on both sides, and taking advantage of interpolation inequalities (3.6) and (3.7) we obtain the estimate

$$\begin{aligned}\lambda\|\partial_{22}\partial_{\gamma_s}\phi\|_{L^2(B^+_{r/2})} \lesssim_r\ &\|\mathsf{A}\|_{W^{1,\infty}(B^+_{r/2})}\|D^2\phi\|_{H^{k-2}(B^+_{r/2})} + \|\mathsf{A}\|_{H^{k-1}(B^+_{r/2})}\|D^2\phi\|_{L^\infty(B^+_{r/2})}\\ &+ \|\mathsf{b} + \mathrm{div}(\mathsf{A})\|_{L^\infty(B^+_{r/2})}\|\nabla\phi\|_{H^{k-1}(B^+_{r/2})} + \|\mathsf{b} + \mathrm{div}(\mathsf{A})\|_{H^{k-1}(B^+_{r/2})}\|\nabla\phi\|_{L^\infty(B^+_{r/2})}\\ &+ \|\mathsf{F}\|_{H^k(B^+_{r/2})} + \|\mathsf{A}\|_{L^\infty(B^+_{r/2})}\sum_{(i,j)\neq(2,2)}\|\partial_{ij}\partial_{\gamma_s}\phi\|_{L^2(B^+_{r/2})}\\ \lesssim_r\ &\|\nabla\phi\|_{H^{k-1}(B^+_{r/2})} + \|\mathsf{F}\|_{H^k(B^+_{r/2})} + \sum_{(i,j)\neq(2,2)}\|\partial_{ij}\partial_{\gamma_s}\phi\|_{L^2(B^+_{r/2})}.\end{aligned}$$

By the definition of the multi-indices γ_1 and γ_2, one can check that

$$\sum_{(i,j)\neq(2,2)} \|\partial_{ij}\partial_{\gamma_1}\phi\|_{L^2(B^+_{r/2})} \leq 3\|\nabla\partial_{\alpha_l}\phi\|_{L^2(B^+_{r/2})},$$

and

$$\sum_{(i,j)\neq(2,2)} \|\partial_{ij}\partial_{\gamma_2}\phi\|_{L^2(B^+_{r/2})} \leq 3\|\nabla\partial_{\alpha_{l-1}}\phi\|_{L^2(B^+_{r/2})}.$$

By induction, we obtain that

$$\lambda\|\partial_{22}\partial_{\gamma_s}\phi\|_{L^2(B^+_{r/2})} \lesssim_r \Big(1+\frac{1}{\lambda^l}+\frac{1}{\lambda^{l+1}}\Big)\big(\|\nabla\phi\|_{H^{k-1}(B_r)}+\|\mathsf{F}\|_{H^k(B_r)}\big),$$

and hence

$$\|\partial_{22}\partial_{\gamma_s}\phi\|_{L^2(B^+_{r/2})} \lesssim_r \frac{1}{\lambda^{l+2}}\Big(\|\nabla\phi\|_{H^{k-1}(B_r)}+\|\mathsf{F}\|_{H^k(B_r)}\Big).$$

When $l \geq 1$ it suffices to notice that $\nabla\partial_{\alpha_{l+1}} = \partial_{22}(\partial_{\gamma_2}, \partial_{\gamma_1})$ to infer that

$$\|\nabla\partial_{\alpha_{l+1}}\phi\|_{L^2(B^+_{r/2})} \lesssim_r \frac{1}{\lambda^{l+2}}\Big(\|\nabla\phi\|_{H^{k-1}(B_r)}+\|\mathsf{F}\|_{H^k(B_r)}\Big),$$

as wished. The same holds for $l = 0$ by noticing that $\nabla\partial_{\alpha_1} = (\partial_1\partial_{\alpha_0}, \partial_{22}\partial_{\gamma_1})$. □

Now we prove that we can recover the same estimate for a domain with curved boundary.

Proposition 3.8 (Rescaled curved boundary estimates)**.** *Let $\Omega \subset \mathbb{R}^2$ be any open domain with boundary of class C^{k+1}. Choose $x_0 \in \partial\Omega$ and $r > 0$ sufficiently small such that there exists a C^{k+1}-diffeomorphism*

$$\Phi : B(x_0, r) \cap \Omega \to B^+_r,$$

with inverse $\Psi = \Phi^{-1}$, such that $\Phi(\partial\Omega \cap B(x_0, r)) = (-r, r) \times \{0\}$, and $\det(D\Phi) = 1$. Then, calling $U^+_r := B(x_0, r) \cap \Omega$, we have that there exists $C_\Phi > 0$ such that

$$\|\nabla\phi\|_{H^k(U^+_{r/2})} \lesssim_r \frac{C_\Phi}{\lambda^{k+1}}\Big(\|\nabla\phi\|_{H^{k-1}(U^+_r)}+\|\mathsf{F}\|_{H^k(U^+_r)}\Big). \tag{3.19}$$

Proof. One can check directly that

$$\begin{aligned}
\phi'(y) &:= \phi(\Psi(y)),\\
\mathsf{A}'(y)_{rs} &:= \sum_{ij} \mathsf{A}(\Psi(y))_{ij}\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y)),\\
\mathsf{b}'(y)^r &:= \sum_i \mathsf{b}(\Psi(y))^i\partial_{x_i}\Phi^r(\Psi(y)),\\
\mathsf{F}'(y)^r &:= \sum_i \mathsf{F}^i(\Psi(y))\partial_{x_i}\Phi^r(\Psi(y)),
\end{aligned}$$

solves

$$\begin{cases} \operatorname{div}(\mathsf{A}'\nabla\phi') + \mathsf{b}'\cdot\nabla\phi' = \operatorname{div}(\mathsf{F}'), & \text{in } B_r^+, \\ \phi' = 0, & \text{on } (-r,r)\times\{0\}. \end{cases}$$

Moreover for every vector $\xi \neq 0$ we have that

$$\langle \mathsf{A}'(x)\xi,\xi\rangle = \langle \mathsf{A}(\Psi(y))D\Phi^t\xi, D\Phi^t\xi\rangle \geq \lambda|D\Phi^t\xi|^2 \geq \mu_\Phi\lambda|\xi|^2,$$

where $\mu_\Phi > 0$ is the infimum taken over $x \in U_r^+$ of the smallest eigenvalue of $D\Phi^t(x)D\Phi(x)$. It follows that $\mathsf{A}' \geq \mu_\Phi\lambda\mathbb{I}$. We have to compute $\operatorname{div}(\mathsf{A}')(y)^s = \sum_r \partial_{y_r}\mathsf{A}'(y)_{rs}$ in terms of A and $\operatorname{div}(\mathsf{A})$. Now,

$$\begin{aligned} \operatorname{div}(\mathsf{A}')(y)^s &= \sum_r \partial_{y_r}\mathsf{A}'(y)_{rs} = \sum_r \partial_{y_r}\Big(\sum_{ij}\mathsf{A}(\Psi(y))_{ij}\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y))\Big) \\ &= \sum_{ijrs}\partial_{x_s}\mathsf{A}(\Psi(y))_{ij}\partial_{y_r}\Psi^s(y)\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y)) \\ &\quad + \sum_{ijr}\mathsf{A}(\Psi(y))_{ij}\partial_{y_r}\Big(\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y))\Big), \end{aligned}$$

and since $\sum_r \partial_{y_r}\Psi^s(y)\partial_{x_i}\Phi^r(\Psi(y)) = \delta_{si}$, it follows that

$$\begin{aligned} &\operatorname{div}(\mathsf{A}')(y)^s \\ &\quad= \sum_{ijs}\partial_{x_i}\mathsf{A}(\Psi(y))_{ij}\partial_{x_j}\Phi^s(\Psi(y)) + \sum_{ijr}\mathsf{A}(\Psi(y))_{ij}\partial_{y_r}\Big(\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y))\Big) \\ &\quad= \sum_{js}\operatorname{div}(\mathsf{A})(\Psi(y))^j\partial_{x_j}\Phi^s(\Psi(y)) + \sum_{ijr}\mathsf{A}(\Psi(y))_{ij}\partial_{y_r}\Big(\partial_{x_i}\Phi^r(\Psi(y))\partial_{x_j}\Phi^s(\Psi(y))\Big). \end{aligned}$$

Therefore, there exists $C_\Phi > 0$ such that

$$\|\operatorname{div}(\mathsf{A}')\|_{H^{k-1}(B_r^+)} \leq C_\Phi\Big(\|\mathsf{A}\|_{H^{k-1}(U_r^+)} + \|\operatorname{div}(\mathsf{A})\|_{H^{k-1}(U_r^+)}\Big).$$

It suffices to apply Proposition 3.7 in order to complete the proof. □

By covering Ω with sufficiently small balls, we can prove a global estimate for the rescaled elliptic equation.

Proposition 3.9 (Rescaled global estimates)**.** *There exists $C_{k,\Omega} > 0$ such that*

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\Big(\lambda^{-(k+1)k}\|\nabla\phi\|_{L^2(\Omega)} + \lambda^{-(k+1)}\|\mathsf{F}\|_{H^k(\Omega)}\Big). \tag{3.20}$$

Moreover, if there exists $\mathsf{f} \in C^\infty(\bar{\Omega})$ such that $\mathsf{b} = \nabla^\perp\mathsf{f}$, then

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\lambda^{-(k+1)k-1}\|\mathsf{F}\|_{H^k(\Omega)}. \tag{3.21}$$

Proof. Covering Ω by sufficiently many balls, combining Propositions 3.6 and 3.8, it follows that for any $k \geq 4$ there exists $C_{k,\Omega} > 0$ such that

$$\|\nabla\phi\|_{H^k(\Omega)} \leq \frac{C_{k,\Omega}}{\lambda^{k+1}}\Big(\|\nabla\phi\|_{H^{k-1}(\Omega)} + \|\mathsf{F}\|_{H^k(\Omega)}\Big).$$

We distinguish two cases: if $\|\nabla\phi\|_{H^{k-1}(\Omega)} \leq \|\mathsf{F}\|_{H^k(\Omega)}$, then

$$\|\nabla\phi\|_{H^k(\Omega)} \leq \frac{2C_{k,\Omega}}{\lambda^{k+1}}\|\mathsf{F}\|_{H^k(\Omega)},$$

and we are done. Otherwise, since

$$\|\nabla\phi\|_{H^k(\Omega)} \leq \frac{2C_{k,\Omega}}{\lambda^{k+1}}\|\nabla\phi\|_{H^{k-1}(\Omega)},$$

the interpolation inequality of Theorem 3.2 implies that there exist $C'_{k,\Omega} > 0$ such that

$$\|\nabla\phi\|_{H^{k-1}(\Omega)} \leq C'_{k,\Omega}\|\nabla\phi\|_{L^2(\Omega)}^{1-\frac{k-1}{k}}\left(\frac{1}{\lambda^{k+1}}\|\nabla\phi\|_{H^{k-1}(\Omega)}\right)^{\frac{k-1}{k}},$$

and hence

$$\|\nabla\phi\|_{H^{k-1}(\Omega)} \leq ({C'}_{k,\Omega})^k\|\nabla\phi\|_{L^2(\Omega)}\lambda^{-(k+1)(k-1)}.$$

Finally, in both cases we have proven that there exists $C''_{k,\Omega} > 0$ such that

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C''_{k,\Omega}\Big(\lambda^{-(k+1)k}\|\nabla\phi\|_{L^2(\Omega)} + \lambda^{-(k+1)}\|\mathsf{F}\|_{H^k(\Omega)}\Big).$$

If $\mathsf{b} = \nabla^\perp\mathsf{f}$, then one can get rid of the L^2-norm of $\nabla\phi$ simply testing (3.9) against ϕ and computing

$$\begin{aligned}
\lambda\|\nabla\phi\|_{L^2(\Omega)}^2 &\leq \int(\nabla^\perp\mathsf{f}\cdot\nabla\phi)\phi\,dx + \int \mathsf{F}\cdot\nabla\phi\,dx \\
&= \frac{1}{2}\int \nabla^\perp\mathsf{f}\cdot\nabla(\phi^2)\,dx + \|\mathsf{F}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} \\
&= \frac{1}{2}\int_{\partial\Omega}\phi^2(\nabla^\perp\mathsf{f}\cdot\nu)\,dx - \frac{1}{2}\int \operatorname{div}(\nabla^\perp\mathsf{f})\phi^2\,dx + \|\mathsf{F}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} \\
&= \|\mathsf{F}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, plugging $\lambda\|\nabla\phi\|_{L^2(\Omega)} \leq \|\mathsf{F}\|_{L^2(\Omega)}$ in (3.20) we finally obtain that

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\Big(\lambda^{-(k+1)k-1}\|\mathsf{F}\|_{L^2(\Omega)} + \lambda^{-(k+1)}\|\mathsf{F}\|_{H^k(\Omega)}\Big),$$

finishing the proof of the proposition (recall that by hypothesis $\lambda \leq 1$). □

We can now easily prove the main result of this section.

Proof of Proposition 3.4. Renormalizing Equation (3.9) by dividing both sides by $\mathbf{M}$ we obtain, applying Proposition (3.9), that

$$\|\nabla\phi\|_{H^k(\Omega)} \leq C_{k,\Omega}\left(\left(\frac{\lambda}{\mathbf{M}}\right)^{-(k+1)k}\|\nabla\phi\|_{L^2(\Omega)} + \left(\frac{\lambda}{\mathbf{M}}\right)^{-(k+1)}\left\|\frac{\mathsf{F}}{\mathbf{M}}\right\|_{H^k(\Omega)}\right),$$

which gives (3.10). The same shows (3.11). □

3.3 Local-in-time existence of smooth solutions in Eulerian coordinates

3.3.1 Flattening

We would like to look at (3.4) as a perturbation of the semigeostrophic equation on the flat plane. Since g is conformal, we know that the gradient and the covariant derivative can be expressed as

$$\nabla_g h = e^{2V}\nabla h, \quad \text{for all } h \in C^1(\Omega),$$

and

$$D^g_X Y = (X\cdot\nabla)Y - dV(X)Y - dV(Y)X + \langle X, Y\rangle \nabla V = D^g Y \cdot X, \tag{3.22}$$

for all $X, Y \in C^1(\Omega, \mathbb{R}^2)$. Since by hypothesis u is divergence free, tangent to $\partial\Omega$ and Ω is simply connected, we can suppose that there exists some potential ψ such that

$$u = -\nabla_g^\perp \psi = -e^{2V}\nabla^\perp\psi =: e^{2V} v, \text{ and } \psi\mid_{\partial\Omega} = 0.$$

Converting all curved gradients into flat ones, substituting u with $e^{2V}v$ and multiplying Equation (3.4) by $e^{-\varphi-2V}$ we obtain that

$$\partial_t \nabla^\perp p + e^{-\varphi} D^g_v (e^{\varphi+2V}\nabla^\perp p) + e^{-2\varphi} v^\perp + e^{-\varphi}\nabla p = 0. \tag{3.23}$$

Thanks to Equation (3.22) we can write

$$\begin{aligned} e^{-\varphi} D^g_v (e^{\varphi+2V}\nabla^\perp p) &= e^{-\varphi} J\big(e^{\varphi+2V} D^g_v \nabla p + e^{\varphi+2V}\langle \nabla\varphi + 2\nabla V, v\rangle \nabla p\big) \\ &= e^{2V} J\big(D_g \nabla p + \nabla p \otimes (\nabla\varphi + 2\nabla V)\big) v \\ &= e^{2V}\mathrm{Cof}(D_g\nabla p + \nabla p\otimes(\nabla\varphi + 2\nabla V))\nabla\psi \\ &= e^{2V}\mathrm{Cof}(D^2 p + \mathsf{B}[\nabla p])\nabla\psi, \end{aligned}$$

where with $\mathrm{Cof}(\cdot)$ we denote the cofactor matrix, which in two dimensions is simply given by

$$\mathrm{Cof}(M) := -JMJ,$$

and for every vector field ξ we set

$$\mathsf{B}[\xi] := \nabla V \otimes \xi + \xi \otimes \nabla V - \langle \xi, \nabla V\rangle \mathbb{I} + \xi \otimes \nabla\varphi.$$

Plugging this into Equation (3.23) we finally obtain the semigeostrophic equation with flattened operators

$$\begin{cases} \partial_t \nabla^\perp p + e^{2V}\mathrm{Cof}(D^2 p + \mathsf{B}[\nabla p])\nabla\psi + e^{-2\varphi}\nabla\psi = -e^{-\varphi}\nabla p, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

3.3.2 An elliptic PDE for the velocity vector field

Applying the divergence operator on both sides of Equation (3.24), we remove the explicit dependencies on the time variable, obtaining

$$\operatorname{div}\Big(e^{2V}\operatorname{Cof}(D^2p + \mathsf{B}[\nabla p] + e^{-2\varphi-2V}\mathbb{I})\nabla\psi\Big) = -\operatorname{div}(e^{-\varphi}\nabla p).$$

In order to rewrite this as a classical elliptic equation in divergence form, we decompose B into its symmetric and antisymmetric part as

$$\mathsf{B}[\xi] = \underbrace{(\nabla V + \nabla\varphi/2)\otimes\xi + \xi\otimes(\nabla V + \nabla\varphi/2) - \langle\xi,\nabla V\rangle\mathbb{I}}_{=:\mathsf{B}^s[\xi]} + \underbrace{\frac{1}{2}(\xi\otimes\nabla\varphi - \nabla\varphi\otimes\xi)}_{=:\mathsf{B}^{as}[\xi]}.$$

Hence, we obtain the equation

$$\operatorname{div}\Big(e^{2V}\operatorname{Cof}(D^2p + \mathsf{B}^s[\nabla p] + e^{-2\varphi-2V}\mathbb{I})\nabla\psi\Big) + \nabla^\perp(e^{2V}\mathsf{B}^{as}_{12}[\nabla p])\cdot\nabla\psi = -\operatorname{div}(e^{-2\varphi}\nabla p),$$

(see Remark 3.5). Finally, to simplify the exposition, define

$$\begin{aligned} \mathsf{Q}[D\xi,\xi] &:= e^{2V+2\varphi}\operatorname{Cof}(D\xi^T + \mathsf{B}^s[\xi]),\\ \mathsf{f}[\xi] &:= e^{2V}\mathsf{B}^{as}_{12}[\xi],\\ \mathsf{F}[\xi] &:= -e^{-2\varphi}\xi, \end{aligned}$$

so that we can rewrite the equation as

$$\begin{cases} \operatorname{div}(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[D^2p,\nabla p])\nabla\psi) + \nabla^\perp(\mathsf{f}[\nabla p])\cdot\nabla\psi = \operatorname{div}(\mathsf{F}[\nabla p]), & \text{in } \Omega\\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Notice that in the definition of Q we decided to transpose the matrix $D\xi$. This has clearly no effect when $D\xi = D^2p$, but it will be important to obtain the suitable cancellation of terms in the following useful lemma.

Lemma 3.10 (Basic estimates on the coefficients)**.** *For every $k \geq 0$ there exists a constant $C = C(\varphi, V, k) > 0$ such that for every smooth vector field ξ on Ω the following estimates hold:*

$$\begin{aligned} &\|\mathsf{B}[\xi]\|_{H^k(\Omega)} + \|\mathsf{F}[\xi]\|_{H^k(\Omega)} + \|\mathsf{f}[\xi]\|_{H^k(\Omega)} \leq C\|\xi\|_{H^k(\Omega)},\\ &\|\mathsf{Q}[D\xi,\xi]\|_{H^k(\Omega)} + \|\operatorname{div}(\mathsf{Q}[D\xi,\xi])\|_{H^k(\Omega)} \leq C\|\xi\|_{H^{k+1}(\Omega)}. \end{aligned}$$

Proof. The first four inequalities follow immediately from the definition of $\mathsf{B}[\xi]$. To check the last one, simply observe that the only problematic term in $\mathsf{Q}[D\xi,\xi]$ is $\operatorname{Cof}(D\xi^T)$. Conclude by noticing that the cofactor matrix of the transpose jacobian matrix enjoys the following nice property

$$\operatorname{div}\Big(\operatorname{Cof}(D\xi^T)\Big) = \sum_{i,j}\partial_i(\operatorname{Cof}(D\xi^T))_{ij} = \partial^2_{12}\xi^2 - \partial^2_{12}\xi^1 - \partial^2_{21}\xi^2 + \partial^2_{21}\xi^1 = 0.$$

□

3.3.3 Discrete construction and local-in-time uniformly existence of regularized solutions

Before presenting the algorithm to construct an approximate solution, we need to fix some notation. For all vector field $X \in H^k(\Omega, \mathbb{R}^2)$, consider the unique Helmholtz-Hodge orthogonal decomposition

$$X = w + \nabla q,$$

where $\operatorname{div}(w) = 0$. From now on, we denote with

$$\mathfrak{H}(X) := \nabla q,$$

the orthogonal complement of the classical Leray projector. Explicitly, q solves the Neumann-type elliptic problem $\Delta q = \operatorname{div}(X)$ in Ω, $\partial_\nu q = X \cdot \nu$ on $\partial\Omega$. With $\mathfrak{I}_\epsilon$ we denote the standard mollification

$$\mathfrak{I}_\epsilon h := \eta_\epsilon * h, \, \forall h \in L^2(\Omega), L^2(\Omega, \mathbb{R}^2), L^2(\Omega, \mathbb{R}^{2\times 2}), \ldots$$

where η_ϵ is any smooth convolution kernel. We address the reader to [45, Appendix C] and [1, Chapter 7.5.5] for a brief recall of the principal properties and definitions of $\mathfrak{I}_\epsilon$ and $\mathfrak{H}$. Fix now $k \geq 4$ and suppose we are given $\nabla p_0 \in H^k(\Omega, \mathbb{R}^2)$ such that

$$\mathbb{I} + \mathsf{Q}[D^2 p_0, \nabla p_0] \geq (1 - \mu_0)\mathbb{I} > 0,$$

for some $\mu_0 < 1$. Choose a coefficient of mollification $\epsilon > 0$ and a time step $\tau > 0$. We set $\nabla p_0^0 := \nabla p_0$ and solve for $i = -1, 0, 1, \ldots$ and $s \in [0, \tau]$ the system

$$\begin{cases} \partial_s \nabla p_s^{i+1} = \mathcal{F}^\epsilon_{\psi^{i+1}}(\nabla p_s^{i+1}) \\ \quad := \mathfrak{H}\mathfrak{I}_\epsilon\Big(e^{2V}(\mathfrak{I}_\epsilon D^2 p_s^{i+1} + \mathsf{B}[\mathfrak{I}_\epsilon \nabla p_s^{i+1}] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1} + e^{-\varphi}\mathfrak{I}_\epsilon \nabla^\perp p_s^{i+1}\Big), \\ \nabla p_0^{i+1} = \nabla p_\tau^i, \end{cases} \tag{3.26}$$

where ψ^{i+1} is given by

$$\begin{cases} \operatorname{div}\Big(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_0^{i+1}, \mathfrak{I}_\epsilon \nabla p_0^{i+1}])\nabla \psi^{i+1}\Big) + \nabla^\perp(\mathsf{f}[\mathfrak{I}_\epsilon \nabla p_0^{i+1}])\nabla \psi^{i+1} & \\ \quad = \operatorname{div}\Big(\mathsf{F}[\mathfrak{I}_\epsilon \nabla p_0^{i+1}]\Big), & \text{in } \Omega \\ \psi^{i+1} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

Notice that (3.26) and (3.27) are nothing else than a regularized version of Equations (3.24) and (3.25), where ∇p_s^i evolves continuously on each time-step solving an ordinary differential equation of the form $\dot{y} = F(y)$ (we take the velocity constant on each interval $[i\tau, (i+1)\tau)$), and ψ^{i+1} evolves discretely as a solution of an elliptic equation. Our next goal is to prove that there exists a fixed interval of existence $[0, t^*)$ so that for every $\epsilon > 0$ and $\tau = t^*/N$, for $N \in \mathbb{N}$ big enough, the sequence $\{\nabla p_s^i, \nabla \psi^i\}_{i=0}^{N-1}$ exists. Solvability of System (3.26) is ensured by the following proposition.

Proposition 3.11. *Let $k \geq 2$, and $\epsilon > 0$. Then, for every ∇q_0 in $H^k(\Omega, \mathbb{R}^2)$ and $\nabla \phi \in L^\infty(\Omega, \mathbb{R}^2)$, there exists a global solution $\nabla q_s^\epsilon \in C^1(\mathbb{R}, H^k(\Omega, \mathbb{R}^2))$ of the following partial differential equation*

$$\begin{cases} \partial_s \nabla q_s^\epsilon = \mathcal{F}_\phi^\epsilon(\nabla q_s^\epsilon) = \mathfrak{H}\mathfrak{I}_\epsilon\Big(e^{2V}(\mathfrak{I}_\epsilon D^2 q_s^\epsilon + \mathsf{B}[\mathfrak{I}_\epsilon \nabla q_s^\epsilon] + e^{-2\varphi - 2V}\mathbb{I})\nabla^\perp \phi + e^{-\varphi}\mathfrak{I}_\epsilon \nabla^\perp q_s^\epsilon\Big), \\ \nabla q_0^\epsilon = \nabla q_0. \end{cases}$$

Proof. This is a direct application of the Cauchy-Lipschitz Theorem in the Banach space

$$\mathcal{X} := \Big\{\nabla q : q \in H^{k+1}(\Omega)\Big\} \subset H^k(\Omega, \mathbb{R}^2).$$

In fact, thanks to the Helmholtz-Hodge decomposition, it is clear that $\mathcal{F}_\phi^\epsilon$ maps $\mathcal{X}$ into itself. We just need to check that it is Lipschitz continuous. Let ∇q and ∇h elements in $\mathcal{X}$. Then, thanks to the properties of $\mathfrak{I}_\epsilon$ and $\mathfrak{H}$, we can estimate

$$\begin{aligned} &\|\mathcal{F}_\phi^\epsilon(\nabla q) - \mathcal{F}_\phi^\epsilon(\nabla h)\|_{H^k(\Omega)} \\ &\quad \leq \frac{C}{\epsilon^k}\|e^{2V}\big(\mathfrak{I}_\epsilon(D^2 q - D^2 h) + \mathsf{B}[\mathfrak{I}_\epsilon(\nabla q - \nabla h)]\big)\nabla^\perp \phi + e^{-\varphi}\mathfrak{I}_\epsilon(\nabla q - \nabla h)^\perp\|_{L^2(\Omega)} \\ &\quad \leq \frac{C}{\epsilon^k}\|e^{2V}\|_\infty \|\nabla \phi\|_{L^\infty(\Omega)}\Big(\|D^2 q - D^2 h\|_{L^2(\Omega)} + \|\mathsf{B}[\mathfrak{I}_\epsilon(\nabla q - \nabla h)]\|_{L^2(\Omega)}\Big) \\ &\qquad + \frac{C}{\epsilon^k}\|e^{-\varphi}\|_\infty \|\nabla q - \nabla h\|_{L^2(\Omega)}. \end{aligned}$$

Now, thanks to Lemma 3.10 we know that $\mathsf{B}[\cdot]$ is a continuous functional in $L^2(\Omega, \mathbb{R}^2)$ implying that there exists $C' = C'(V, \varphi, \Omega) > 0$ such that

$$\mathrm{Lip}(\mathcal{F}_\phi^\epsilon) \leq \frac{C'}{\epsilon^k}\Big(\|\nabla \phi\|_{L^\infty(\Omega)} + 1\Big) < +\infty,$$

as wished. □

System (3.27) is solvable at the step $(i+1)$ if the eigenvalue

$$-\mu_s^{i+1} := \inf_{|\xi|=1, x \in \Omega}\Big\{\langle \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_s^{i+1}, \mathfrak{I}_\epsilon \nabla p_s^{i+1}](x)\xi, \xi\rangle\Big\}, \tag{3.28}$$

is strictly greater than -1 at time $s = 0$. To analyse the behaviour of μ^{i+1}, define

$$-\mu_s^{i+1}(x) := \inf_{|\xi|=1}\Big\{\langle \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_s^{i+1}, \mathfrak{I}_\epsilon \nabla p_s^{i+1}](x)\xi, \xi\rangle\Big\}, \quad x \in \Omega,\ s \in [0, \tau].$$

Since $\nabla p_s^{i+1} \in C^1([0, \tau], H^k(\Omega, \mathbb{R}^2))$ we have that fixing x, $s \mapsto \mu_s^{i+1}(x)$ is a locally Lipschitz map, and therefore μ_s^{i+1}, being the infimum over $x \in \Omega$, is also locally Lipschitz and hence almost everywhere differentiable.

Lemma 3.12 (Dynamics of the elliptic constant)**.** *Let ∇p_s^i and $\nabla\psi^i$ be solutions of* (3.26) *and* (3.27)*, and let μ_s^i be defined as in* (3.28)*. There exists $C = C(V,\varphi,\Omega) > 0$ such that*

$$\frac{d}{ds}\Big|_{s=s_0}(1-\mu_s^{i+1}) \geq -C\Big(\|\nabla p_{s_0}^{i+1}\|_{H^4(\Omega)}+1\Big)\|\nabla\psi^{i+1}\|_{H^3(\Omega)} - C\|\nabla p_{s_0}^{i+1}\|_{H^3(\Omega)}, \tag{3.29}$$

for almost every s_0 in $(0,\tau)$.

Proof. Take $\delta \neq 0$ small, $x\in\Omega$ and $s_0\in(0,\tau)$. Then, let $\xi_\delta\in\mathbb{R}^2$ be the unit vector realizing

$$-\mu_{s_0+\delta}^{i+1}(x) = \langle \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_{s_0+\delta}^{i+1}(x), \mathfrak{I}_\epsilon\nabla p_{s_0+\delta}^{i+1}(x)]\xi_\delta,\xi_\delta\rangle.$$

Then,

$$\begin{aligned}
&\mu_{s_0}^{i+1}(x) - \mu_{s_0+\delta}^{i+1}(x)\\
&\qquad\geq \langle\Big(\mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_{s_0+\delta}^{i+1}(x), \mathfrak{I}_\epsilon\nabla p_{s_0+\delta}^{i+1}(x)] - \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_{s_0}^{i+1}(x), \mathfrak{I}_\epsilon\nabla p_{s_0}^{i+1}(x)]\Big)\xi_\delta,\xi_\delta\rangle\\
&\qquad= \langle\int_{s_0}^{s_0+\delta}\partial_t\mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_t^{i+1}(x), \mathfrak{I}_\epsilon\nabla p_t^{i+1}(x)]\,dt\cdot\xi_\delta,\xi_\delta\rangle\\
&\qquad\geq -\int_{s_0}^{s_0+\delta}\|\mathsf{Q}[\mathfrak{I}_\epsilon D^2\partial_t p_t^{i+1}(x), \mathfrak{I}_\epsilon\nabla\partial_t p_t^{i+1}(x)]\|_{L^\infty(\Omega)}.
\end{aligned}$$

By the Sobolev embedding of $L^\infty(\Omega)$ in $H^2(\Omega)$ and by Lemma 3.10, we obtain that there exists $C_1>0$ such that

$$\mu_{s_0}^{i+1}(x) - \mu_{s_0+\delta}^{i+1}(x) \geq -C_1\int_{s_0}^{s_0+\delta}\|\partial_t\nabla p_t^{i+1}\|_{H^3(\Omega)}\,dt.$$

Finally, thanks to the discrete construction of the pressure gradient given by Equation (3.26), the fact that $H^3(\Omega)$ is a Banach Algebra, we conclude that there exists $C>0$ such that

$$\mu_{s_0}^{i+1}(x) - \mu_{s_0+\delta}^{i+1}(x) \geq -C\int_{s_0}^{s_0+\delta}\Big(\|\nabla p_t^{i+1}\|_{H^4(\Omega)}+1\Big)\|\nabla\psi^{i+1}\|_{H^3(\Omega)} - C\|\nabla p_t^{i+1}\|_{H^3(\Omega)}\,dt.$$

The result follows by dividing everything by δ, and letting δ go to zero. ☐

3.4 Energy estimates

Proposition 3.13 (Energy estimates)**.** *Let ∇p_s^i and $\nabla\psi^i$ be solutions of* (3.26) *and* (3.27)*, and $k\geq 4$. Then, there exists $C = C(k,\Omega,V,\varphi)>0$ such that*

$$\frac{d}{ds}\|\nabla p_s^{i+1}\|_{H^k(\Omega)} \leq C\Big(\|\nabla p_s^{i+1}\|_{H^k(\Omega)}+1\Big)\|\nabla\psi^{i+1}\|_{H^k(\Omega)} + C\|\nabla p_s^{i+1}\|_{H^k(\Omega)}. \tag{3.30}$$

Proof. Fix any multi-index $|\alpha| \leq k$. Since the operators $\mathfrak{J}_\epsilon$ and $\mathfrak{H}$ commute and are self-adjoint with respect to the L^2-product, we can compute

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2}\int |\partial_\alpha \nabla p_s^{i+1}|^2\,dx &= \int \langle \partial_\alpha \nabla p_s^{i+1}, \partial_\alpha \partial_s \nabla p_s^{i+1}\rangle\,dx \\
&= \int \langle \mathfrak{J}_\epsilon \partial_\alpha \nabla p_s^{i+1}, \partial_\alpha \Big(e^{2V}(\mathfrak{J}_\epsilon D^2 p_s^{i+1} + \mathsf{B}[\mathfrak{J}_\epsilon \nabla p_s^{i+1}] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1}\Big)\rangle\,dx \\
&\quad + \int \langle \mathfrak{J}_\epsilon \partial_\alpha \nabla p_s^{i+1}, \partial_\alpha \Big(e^{-\varphi}\mathfrak{J}_\epsilon \nabla^\perp p_s^{i+1}\Big)\rangle\,dx.
\end{aligned}$$

Set $P_s := \mathfrak{J}_\epsilon \nabla p_s^{i+1}$. There exists $C_\varphi > 0$ such that

$$\begin{aligned}
&\frac{d}{ds}\frac{1}{2}\int |\partial_\alpha \nabla p_s^{i+1}|^2\,dx \\
&= \int \langle \partial_\alpha P_s, \partial_\alpha \Big(e^{2V}(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1} + e^{-\varphi}P_s^\perp\Big)\rangle\,dx \\
&\leq \int \langle \partial_\alpha P_s, \partial_\alpha \Big(e^{2V}(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1}\Big)\rangle\,dx \\
&\quad + C_\varphi \|\partial_\alpha P_s\|_{L^2(\Omega)} \|P_s\|_{H^{|\alpha|}(\Omega)}.
\end{aligned}$$

To estimate the remaining term, we argue by interpolation: subtracting and adding the term

$$R := \int \langle \partial_\alpha P_s, e^{2V}\partial_\alpha \Big(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp \psi^{i+1}\rangle\,dx,$$

to

$$\int \langle \partial_\alpha P_s, \partial_\alpha \Big(e^{2V}(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1}\Big)\rangle\,dx,$$

applying Cauchy-Schwarz and interpolation (3.7), we obtain that there exists $C_1 = C_1(\Omega) > 0$ such that

$$\begin{aligned}
&\int \langle \partial_\alpha P_s, \partial_\alpha \Big(e^{2V}(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp \psi^{i+1}\Big)\rangle\,dx - R + R \\
&\quad \leq C_1 \|\partial_\alpha P_s\|_{L^2(\Omega)} \Big(\|e^{2V}\nabla \psi^{i+1}\|_{W^{1,\infty}(\Omega)} \|DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I}\|_{H^{k-1}(\Omega)} \\
&\qquad + \|DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I}\|_{L^\infty(\Omega)} \|e^{2V}\nabla \psi^{i+1}\|_{H^k(\Omega)}\Big) + R.
\end{aligned}$$

Taking advantage once again of Lemma 3.10 and suitable Sobolev embeddings, we just proved that there exists $C_2 = C_2(\Omega, V, \varphi) > 0$ such that

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2}\int |\partial_\alpha \nabla p_s^{i+1}|^2\,dx &\leq C_2 \|\partial_\alpha P_s\|_{L^2(\Omega)} \big(\|P_s\|_{H^k(\Omega)} + 1\big) \|\nabla \psi^{i+1}\|_{H^k(\Omega)} \\
&\quad + C_\varphi \|\partial_\alpha P_s\|_{L^2(\Omega)} \|P_s\|_{H^k(\Omega)} + R.
\end{aligned} \tag{3.31}$$

We now estimate the contribution of R. First of all, it is easy to control the lower order terms simply by Cauchy-Schwarz and Lemma 3.10, obtaining that

$$
\begin{aligned}
R &= \int \langle \partial_\alpha P_s, e^{2V}\partial_\alpha\Big(DP_s + \mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp\psi^{i+1}\rangle\,dx \\
&\le \int \langle \partial_\alpha P_s, e^{2V}\partial_\alpha(DP_s)\nabla^\perp\psi^{i+1}\rangle\,dx \\
&\quad + \|\partial_\alpha P_s\|_{L^2(\Omega)}\|e^{2V}\nabla\psi^{i+1}\|_{L^\infty}\|\mathsf{B}[P_s] + e^{-2\varphi-2V}\mathbb{I}\|_{H^k(\Omega)} \\
&\le \int \langle \partial_\alpha P_s, e^{2V}\partial_\alpha(DP_s)\nabla^\perp\psi^{i+1}\rangle\,dx + C_3\|\partial_\alpha P_s\|_{L^2(\Omega)}\|\nabla\psi^{i+1}\|_{H^k(\Omega)}(\|P_s\|_{H^k(\Omega)}+1),
\end{aligned}
\tag{3.32}
$$

for some constant $C_3 = C_3(\Omega, V, \varphi) > 0$. Finally we get rid of the higher order term integrating by parts:

$$
\begin{aligned}
\int \langle \partial_\alpha P_s, e^{2V}\partial_\alpha(DP_s)\nabla^\perp\psi^{i+1}\rangle\,dx &= \int \langle \nabla\Big(\frac{|\partial_\alpha P_s|^2}{2}\Big), e^{2V}\nabla^\perp\psi^{i+1}\rangle\,dx \\
&= \int \mathrm{div}\Big(e^{2V}\nabla^\perp\psi^{i+1}\frac{|\partial_\alpha P_s|^2}{2}\Big) - \frac{|\partial_\alpha P_s|^2}{2}\mathrm{div}(e^{2V}\nabla^\perp\psi^{i+1})\,dx \\
&= \int_{\partial\Omega} e^{2V}\frac{|\partial_\alpha P_s|^2}{2}\nabla^\perp\psi^{i+1}\cdot\nu\,dx \\
&\quad - \int \frac{|\partial_\alpha P_s|^2}{2}\Big(e^{2V}\mathrm{div}(\nabla^\perp\psi^{i+1}) + e^{2V}\langle 2\nabla V, \nabla^\perp\psi^{i+1}\rangle\Big)\,dx \\
&\le C_4\|\partial_\alpha P_s\|^2_{L^2(\Omega)}\|e^{2V}\nabla\psi^{i+1}\|_{L^\infty(\Omega)}.
\end{aligned}
\tag{3.33}
$$

Combining (3.31), (3.32), (3.33), and summing over $|\alpha| = 0,\dots,k$ we obtain the desired result. □

Now that we have obtained a growth estimate on $\mu_s^{i+1}(x)$ and $\|\nabla p_s^{i+1}\|_{H^k(\Omega)}$, we need analyse the behaviour of the velocity vector field. This last estimate is a direct consequence of the explicit regularity results of Section 3.2.

Proposition 3.14 (Elliptic estimates on the velocity)**.** *Let ∇p_s^i and $\nabla\psi^i$ be solutions of (3.26) and (3.27), and let μ_s^i be defined as in (3.28). For any $k \ge 4$ there exists some constant $C = C(k, \Omega, V\varphi) > 0$ such that*

$$
\|\nabla\psi^{i+1}\|_{H^k(\Omega)} \le C(1-\mu_0^{i+1})^{-(k+1)k-1}\big(\|\nabla p_0^{i+1}\|_{H^k(\Omega)}+1\big)^{(k+1)k}\|\nabla p_0^{i+1}\|_{H^k(\Omega)}. \tag{3.34}
$$

Proof. It suffices to combine Proposition 3.9 and Lemma 3.10, recalling that in our case b comes from a rotated gradient by construction. □

Combining the estimates on the pressure gradient (3.30) and on the velocity vector field (3.34) we have that

$$
\begin{aligned}
&\frac{d}{ds}\|\nabla p_s^{i+1}\|_{H^k(\Omega)} \\
&\quad \le C(\|\nabla p_s^{i+1}\|_{H^k(\Omega)}+1)\frac{(\|\nabla p_0^{i+1}\|_{H^k(\Omega)}+1)^{k(k+1)}}{(1-\mu_0^{i+1})^{(k+1)k+1}}\|\nabla p_0^{i+1}\|_{H^k(\Omega)} + C\|\nabla p_s^{i+1}\|_{H^k(\Omega)},
\end{aligned}
\tag{3.35}
$$

and similarly by the estimate (3.29) on $1-\mu_s^{i+1}$ it holds that

$$\begin{aligned}\frac{d}{ds}&(1-\mu_s^{i+1})\\ &\geq -C(\|\nabla p_s^{i+1}\|_{H^k(\Omega)}+1)\frac{(\|\nabla p_0^{i+1}\|_{H^k(\Omega)}+1)^{k(k+1)}}{(1-\mu_0^{i+1})^{(k+1)k+1}}\|\nabla p_0^{i+1}\|_{H^k(\Omega)} - C\|\nabla p_s^{i+1}\|_{H^k(\Omega)},\end{aligned} \tag{3.36}$$

a.e. in $(0,\tau)$. Define

$$\Theta_s^{i+1} := \left(\frac{\|\nabla p_s^{i+1}\|_{H^k(\Omega)}+1}{1-\mu_s^{i+1}}\right)^{k(k+1)+2}, \tag{3.37}$$

together with the monotonically increasing Lipschitz function

$$\tilde{\Theta}_s^{i+1} := \max_{t\in[0,s]} \Theta_t^{i+1}. \tag{3.38}$$

The next lemma will constitute the crucial step in the proof of the main Theorem.

Lemma 3.15. *Let $\tilde{\Theta}_s^i$ be defined as in (3.38). There exits $C = C(\Omega, V, \varphi, k) > 0$ such that*

$$\frac{d}{ds}\tilde{\Theta}_s^{i+1} \leq C(\tilde{\Theta}_s^{i+1})^2, \tag{3.39}$$

almost everywhere in $[0,\tau]$.

Proof. In this proof we omit the $i+1$ index in our notation. Also, set $M = k(k+1)+2$. First of all, by Sobolev embedding we have that there exists $C_1 = C_1(\Omega) > 0$ such that

$$1 \leq \frac{\|\mathbb{I} + \mathsf{Q}[\mathfrak{I}_\epsilon D^2 p_s, \mathfrak{I}_\epsilon \nabla p_s]\|_{L^\infty(\Omega)}}{1-\mu_s} \leq \frac{1 + C_1\|\nabla p_s\|_{H^3(\Omega)}}{1-\mu_s},$$

hence, up to multiplying all the following estimates by $\max\{1, C_1\}$, we can suppose without loss of generality that

$$1 \leq \frac{1 + \|\nabla p_s\|_{H^k(\Omega)}}{1-\mu_s},$$

for all $s \in [0,\tau]$. In particular we have that

$$\begin{aligned}\frac{d}{ds}\|\nabla p_s\|_{H^k(\Omega)} &\leq C\max_{t\in[0,s]}\left\{\left(\left(\frac{\|\nabla p_t\|_{H^k(\Omega)}+1}{1-\mu_t}\right)^{k(k+1)+1}+1\right)\|\nabla p_t\|_{H^k(\Omega)}\right\}\\ &\leq 2C\max_{t\in[0,s]}\left\{\left(\frac{\|\nabla p_t\|_{H^k(\Omega)}+1}{1-\mu_t}\right)^{k(k+1)+1}\|\nabla p_t\|_{H^k(\Omega)}\right\}.\end{aligned}$$

The same bound clearly holds also for μ_s. Therefore, recalling (3.37) and that $M = k(k+1)+2$, we estimate

$$\begin{aligned}
&\frac{d}{ds}\Theta_s \\
&\quad = M\Big(\frac{\|\nabla p_s\|_{H^k(\Omega)}+1}{1-\mu_s}\Big)^{M-1}\Big(\frac{1}{1-\mu_s}\frac{d}{ds}\|\nabla p_s\|_{H^k(\Omega)} - \frac{\|\nabla p_s\|_{H^k(\Omega)}+1}{(1-\mu_s)^2}\frac{d}{ds}(1-\mu_s)\Big) \\
&\quad \le 2CM \max_{t\in[0,s]}\Big\{\Big(\frac{\|\nabla p_t\|_{H^k(\Omega)}+1}{1-\mu_t}\Big)^{M+k(k+1)}\frac{\|\nabla p_t\|_{H^k(\Omega)}}{1-\mu_t}\Big(1+\frac{\|\nabla p_t\|_{H^k(\Omega)}+1}{1-\mu_t}\Big)\Big\} \\
&\quad \le 4CM \max_{t\in[0,s]}\Big\{\Big(\frac{\|\nabla p_t\|_{H^k(\Omega)}+1}{1-\mu_t}\Big)^{M+k(k+1)+2}\Big\}.
\end{aligned}$$

This proves that

$$\frac{d}{ds}\Theta_s \le 4CM\tilde{\Theta}_s^2.$$

We distinguish two cases: if $\frac{d}{ds}\Theta_s \le 0$, then clearly

$$\frac{d}{ds}\tilde{\Theta}_s = 0,$$

and we are done. Otherwise

$$\frac{d}{ds}\tilde{\Theta}_s = \frac{d}{ds}\Theta_s \le 4CM\tilde{\Theta}_s^2,$$

completing the proof of the Lemma. □

We only need the following little observation before proving the main result of this section.

Lemma 3.16. *Let $(x_i)_{i\ge0}$ be any real sequence that satisfies for some $c>0$ the recursive relation*

$$x_{i+1} \le \frac{x_i}{1-cx_i}.$$

If there exists $N\in\mathbb{N}$ such that $x_0 \le \frac{1}{cN}$, then

$$x_{i+1} \le \frac{x_0}{1-c(i+1)x_0}, \quad \textit{for every } i\in\{-1,\dots,N-1\}. \tag{3.40}$$

Proof. The statement clearly holds for $i=-1$. Suppose (3.40) holds for $0\le i<N-1$. Since for every $C>0$ the map $x\mapsto \frac{x}{1-Cx}$ is monotonically increasing and continuous in $(-\infty,\frac{1}{C})$, we have in particular that

$$x_i \le \frac{x_0}{1-cix_0} \le \frac{1}{cN}\frac{N}{N-i} = \frac{1}{c(N-i)} < \frac{1}{c},$$

and therefore

$$x_{i+1} \le \frac{x_i}{1-cx_i} \le \frac{x_0}{1-cix_0}\frac{1-cix_0}{1-c(i+1)x_0} = \frac{x_0}{1-c(i+1)x_0},$$

completing the induction. □

We are now ready to prove uniform local-in-time existence for Systems (3.26) and (3.27). To simplify the statement, we glue together the piecewise approximated solution, naturally defining

$$\nabla p_t^{\tau,\epsilon} := \nabla p_s^i, \ \text{ if } t = i\tau + s,$$

and

$$\nabla \psi_t^{\tau,\epsilon} := \nabla \psi^i, \ \text{ if } t \in [i\tau, (i+1)\tau).$$

Theorem 3.17. *Let Ω be a bounded subset of $\mathbb{R}^2$ with boundary of class C^{k+1} and $\nabla p_0 \in H^k(\Omega, \mathbb{R}^2)$ such that*

$$\mathsf{Q}[D^2 p_0, \nabla p_0] \geq -\mu_0 \mathbb{I} > -\mathbb{I},$$

for some $\mu_0 < 1$ and $k \geq 4$. Then, there exist a constant $C = C(\Omega, V, \varphi, k) > 0$ and $t^* > 0$ such that for every $\tau = t^*/N$, $N \in \mathbb{N}$ big enough and $\epsilon > 0$, there exists an approximate solution

$$\{\nabla p_s^i, \nabla \psi^i\}_{i=0}^{N-1} \in C^1([0,\tau], H^k(\Omega, \mathbb{R}^2)) \times H^k(\Omega, \mathbb{R}^2)$$

of Systems (3.26) and (3.27), where t^* can be taken equal to

$$t^* := C \left(\frac{1 - \mu_0}{\|\nabla p_0\|_{H^k(\Omega)} + 1} \right)^{(k+1)k+2}.$$

In particular, for every $0 < t' < t^*$, there exists $C' = C'(\Omega, V, \varphi, k) > 0$ such that

$$\|\nabla p_t^{\tau,\epsilon}\|_{H^k(\Omega)}, \|\nabla \psi_t^{\tau,\epsilon}\|_{H^k(\Omega)} \leq C',$$

for all $t \in [0, t']$.

Proof. Integrating for $s \in [0, \tau]$ Equation(3.39) of Lemma 3.15 at time i, and recalling that $\tilde{\Theta}_0^{i+1} = \tilde{\Theta}_\tau^i$, we obtain the recursive relation

$$\tilde{\Theta}_0^{i+1} \leq \frac{\tilde{\Theta}_0^i}{1 - C\tau \tilde{\Theta}_0^i},$$

which, applying Lemma 3.16, gives the bound

$$\tilde{\Theta}_0^{i+1} \leq \frac{\tilde{\Theta}_0^0}{1 - C\tau(i+1)\tilde{\Theta}_0^0}, \tag{3.41}$$

for every $i = \{-1, 0, 1, \ldots, N-1\}$ provided

$$\tilde{\Theta}_0^0 = \Theta_0^0 \leq \frac{1}{C\tau N},$$

for some $N \in \mathbb{N}$. Hence, setting

$$t^* := \frac{1}{C\Theta_0^0},$$

we ensure the local existence of an approximate solution in $[0, t^*)$ uniformly in $\epsilon > 0$ and for every $\tau = t^*/N$, $N \in \mathbb{N}$ big enough. In particular, (3.41) implies that for any interval of time $[0, t']$ with $t' < t^*$ the uniform bound

$$\tilde{\Theta}_s^i \leq C',$$

holds, where $C' > 0$ can be taken such that

$$t' = \frac{1}{C}\Big(\frac{1}{\Theta_0^0} - \frac{1}{C'}\Big).$$

□

3.5 Compactness argument and proof of the main Theorem

Fix any $0 < t' < t^*$, and $N_0 \in \mathbb{N}$ large. For every $N \geq N_0$ define

$$\nabla p_t^N := \nabla p_t^{t'/2^N, t'/2^N},$$

and

$$\nabla \psi_t^N := \nabla \psi_t^{t'/2^N, t'/2^N}.$$

Then, by Theorem 3.17, the sequence $(\nabla p_t^N)_{N \geq N_0}$ is uniformly bounded in the space

$$\mathcal{W} := \Big\{\nabla q_t \in L^\infty(0, t'; H^k(\Omega, \mathbb{R}^2)),\ \text{and}\ \partial_t \nabla q_t \in L^\infty(0, t'; H^1(\Omega, \mathbb{R}^2)))\Big\}.$$

Since the embedding of $H^k(\Omega)$ in $C^{k-2,\alpha}(\Omega)$ is compact (see [2, Chapter 6]) and $C^{k-2,\alpha}(\Omega)$ embeds continuously in $H^1(\Omega)$, by Aubin-Lions-Simons Lemma we have that

$$\mathcal{W} \hookrightarrow C(0, t'; C^{k-2,\alpha}(\Omega, \mathbb{R}^2)),$$

is compact as well. Extracting a converging sub-sequence we obtain (after relabelling) that

$$\nabla p_t^N \to \nabla p_t\ \text{ in } C(0, t'; C^{k-2,\alpha}(\Omega, \mathbb{R}^2)),$$

for some $\nabla p_t \in C(0, t'; C^{k-2,\alpha}(\Omega, \mathbb{R}^2))$. Moreover, looking at $(\nabla p_t^N)_{N \geq N_0}$ as bounded subset of the space $L^2(0, t'; H^k(\Omega, \mathbb{R}^2))$, we can affirm that

$$\nabla p_t^N \rightharpoonup \nabla p_t \text{ in } L^2(0, t'; H^k(\Omega, \mathbb{R}^2)).$$

Let $\nabla \psi_t$ be solution of the System (3.25) associated to the limit ∇p_t, i.e.

$$\begin{cases} \operatorname{div}(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[D^2 p_t, \nabla p_t])\nabla \psi_t) + \nabla^\perp(\mathsf{f}[\nabla p_t]) \cdot \nabla \psi_t = \operatorname{div}(\mathsf{F}[\nabla p_t]), & \text{in } \Omega \\ \psi_t = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.42}$$

Observe that the lower bound on the uniform elliptic constants $1-\mu_s^i$ proved in Lemma 3.12 combined with the pressure and velocity bounds of Theorem 3.17 provide a uniform elliptic bound in the limit, that we will denote with

$$-\mu_t := \inf\Big\{\langle \mathsf{Q}[D^2 p_t, \nabla p_t](x)\xi, \xi\rangle : |\xi| = 1,\, x \in \Omega\Big\}.$$

By qualitative elliptic regularity, we can affirm that $\nabla\psi_t \in C(0,t';C^{k-2,\alpha}(\Omega,\mathbb{R}^2))$. Fix $t \in (0,t')$ and let

$$t_N := \min\{jt'/2^N \geq t : j = 0,\dots,N\},$$

and observe that the difference $\psi_t - \psi_{t_N}^N = \psi_t - \psi_t^N$ solves the equation

$$\begin{cases} \operatorname{div}(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[D^2 p_t, \nabla p_t])\nabla(\psi_t - \psi_t^N)) + \nabla^\perp(\mathsf{f}[\nabla p_t]) \cdot \nabla(\psi_t - \psi_t^N) = \mathsf{X}_t^N, & \text{in } \Omega \\ \psi_t - \psi_t^N = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \mathsf{X}_t^N := {}& \operatorname{div}\big(\mathsf{F}[\nabla p_t - \nabla p_t^N]\big) - \operatorname{div}\big(e^{-2\varphi}\mathsf{Q}[D^2(p_t - p_t^N), \nabla(p_t - p_t^N)]\nabla\psi_t^N\big) \\ & - \nabla^\perp(\mathsf{f}[\nabla(p_t - p_t^N)]) \cdot \nabla\psi_t^N. \end{aligned}$$

We can argue as at the end of Proposition 3.4, to estimate

$$\|\nabla\psi_t - \nabla\psi_t^N\|_{L^2(\Omega)} \leq \frac{1}{1-\mu_t}\|\mathsf{X}_t^N\|_{L^2(\Omega)} \to 0,$$

uniformly in $(0,t')$ thanks to the bounds given by Theorem 3.17. Moreover, by weak compactness of $L^2(0,t';H^k(\Omega,\mathbb{R}^2))$, we have that

$$\nabla\psi_t^N \rightharpoonup \nabla\psi_t \in L^2(0,t';H^k(\Omega,\mathbb{R}^2)).$$

To summarise, we have the following proposition.

Proposition 3.18. *Up to taking a subsequence of* $(\nabla p_t^N, \nabla\psi_t^N)_{N\geq N_0}$ *there exist*

$$\nabla p_t, \nabla\psi_t \in C(0,t';C^{k-2,\alpha}(\Omega,\mathbb{R}^2)) \cap L^2(0,t';H^k(\Omega,\mathbb{R}^2)),$$

such that

$$\nabla p_t^N \to \nabla p_t,$$

strongly in $C(0,t';C^{k-2,\alpha}(\Omega,\mathbb{R}^2))$ *and weakly in* $L^2(0,t';H^k(\Omega,\mathbb{R}^2))$*, and*

$$\nabla\psi_t^N \to \nabla\psi_t,$$

strongly in $L^\infty(0,t';C^{k-2,\alpha}(\Omega,\mathbb{R}^2))$ *and weakly in* $L^2(0,t';H^k(\Omega,\mathbb{R}^2))$.

We are now ready to prove the main result of this Chapter.

Proof of Theorem 3.1. We first show existence and then uniqueness.

Existence: We prove that $(\nabla p_t, \nabla\psi_t)$ is a weak solution, the conclusion follows from the additional regularity showed before. Let $\xi_t \in C^1_c([0,t'), C^\infty(\Omega,\mathbb{R}^2))$ be any test function, denote with $\{\cdot,\cdot\}$ the standard inner product of $L^2(0,t';L^2(\Omega,\mathbb{R}^2))$ and with $\mathfrak{I}^N := \mathfrak{I}_{t'/N}$. Then, testing (3.26) against ξ_t we have that

$$\begin{aligned}0 = &\{\nabla p^N_t, \partial_t\xi_t\} - \{\nabla p_0, \xi_0\}\\ &+ \{\mathfrak{H}\mathfrak{I}^N\xi_t, e^{2V}\Big(\mathfrak{I}^N D^2 p^N_t + \mathsf{B}[\mathfrak{I}^N\nabla p^N_t] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp\psi^N_t + e^{-\varphi}\mathfrak{I}^N\nabla^\perp p^N_t\}.\end{aligned}$$

Then, we can write

$$\begin{aligned}&\{\nabla p_t, \partial_t\xi_t\} - \{\nabla p_0, \xi_0\} + \{\mathfrak{H}\xi_t, e^{2V}\Big(D^2 p_t + \mathsf{B}[\nabla p_t] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp\psi_t + e^{-\varphi}\nabla^\perp p_t\} =\\ &\{\nabla(p_t - p^N_t), \partial_t\xi_t\} + \{\mathfrak{H}\xi_t - \mathfrak{I}^N\mathfrak{H}\xi_t, e^{2V}\Big(D^2 p_t + \mathsf{B}[\nabla p_t] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp\psi_t + e^{-\varphi}\nabla^\perp p_t\}\\ &\quad + \{\mathfrak{H}\mathfrak{I}^N\xi_t, e^{2V}\Big(D^2(p_t - \mathfrak{I}^N p^N_t) + \mathsf{B}[\nabla(p_t - \mathfrak{I}^N p_t)]\Big)\nabla^\perp\psi_t + e^{-\varphi}\nabla^\perp(p_t - \mathfrak{I}^N p^N_t)\}\\ &\quad + \{\mathfrak{H}\mathfrak{I}^N\xi_t, e^{2V}\Big(\mathfrak{I}^N D^2 p^N_t + \mathsf{B}[\mathfrak{I}^N\nabla p^N_t] + e^{-2\varphi-2V}\mathbb{I}\Big)\nabla^\perp(\psi^N_t - \psi_t)\},\end{aligned}$$

which goes to zero as N goes to $+\infty$, thanks to the uniforms bounds of Theorem 3.17 and Proposition 3.18. Therefore, we have that $(\nabla p_t, \nabla\psi_t)$ solves weakly

$$\partial_t\nabla p_t = \mathfrak{H}\Big(e^{2V}(D^2 p_t + \mathsf{B}[\nabla p_t] + e^{-2\varphi-2V}\mathbb{I})\nabla^\perp\psi_t + e^{-\varphi}\nabla^\perp p_t\Big) =: \mathfrak{H}(X_t).$$

We now take advantage of the elliptic equation solved by ψ_t in order to get rid of the Hodge-Helmholtz decomposition in the right-hand side. Here is the only point in the proof where we need to assume Ω simply connected (see Remark 3.19 for the periodic case $\Omega = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$). The orthogonal complementary of $\mathfrak{H}(\cdot)$

$$w_t := X_t - \mathfrak{H}(X_t),$$

is tangent to $\partial\Omega$ and divergence free by construction of $\mathfrak{H}(X_t)$. Moreover, since

$$\mathrm{curl}(w_t) = -\mathrm{div}(X^\perp_t) = -\mathrm{div}\Big(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[D^2 p_t, \nabla p_t])\nabla\psi_t + \nabla^\perp(\mathsf{f}[\nabla p_t])\cdot\nabla\psi_t - \mathsf{F}[\nabla p_t]\Big) = 0,$$

by construction of $\nabla\psi_t$, we conclude that w_t in a harmonic vector field, and hence equal to zero since Ω is simply connected. Therefore, $X_t = \mathfrak{H}(X_t)$.

Uniqueness: Suppose we are given another solution $(\nabla\bar{p}_t, \bar{u}_t)$ sharing the same regularity as $(\nabla p_t, u_t)$ and such that $\nabla\bar{p}_0 = \nabla p_0$ and $\bar{u}_t$ is tangent to $\partial\Omega$. Let $\bar{\psi}_t$ be such that

$-e^{2V}\nabla^\perp\bar{\psi}_t = \bar{u}_t$. Set $q_t := p_t - \bar{p}_t$. We can estimate

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|\nabla^\perp q_t\|^2_{L^2(\Omega)} \\
&\quad = -\int \langle \nabla^\perp q_t, e^{2V}\Big(\mathrm{Cof}(D^2 p_t + \mathsf{B}[\nabla p_t])\nabla\psi_t - \mathrm{Cof}(D^2\bar{p}_t + \mathsf{B}[\nabla\bar{p}_t])\nabla\bar{\psi}_t\Big) \\
&\qquad + e^{-2\varphi}(\nabla\psi_t - \bar{\nabla}\psi_t)\rangle\, dx \\
&\quad = -\int \langle \nabla^\perp q_t, e^{2V}\mathrm{Cof}(D^2 p_t + \mathsf{B}[\nabla p_t] + e^{-2V-2\varphi}\mathbb{I})(\nabla\psi_t - \nabla\bar{\psi}_t) \\
&\qquad + e^{2V}\mathrm{Cof}(D^2 q_t + \mathsf{B}[\nabla q_t])\nabla\bar{\psi}_t\rangle\, dx \\
&\quad \le C\Big(\|\nabla q_t\|_{L^2(\Omega)}\|\nabla\psi_t - \nabla\bar{\psi}_t\|_{L^2(\Omega)} + \|\nabla q_t\|^2_{L^2(\Omega)}\Big),
\end{aligned} \tag{3.43}$$

for some $C > 0$ depending uniquely on φ, V, the $L^\infty([0,t'];W^{1,\infty}(\Omega))$-norm of ∇p_t and the $L^\infty([0,t'];L^\infty(\Omega))$-norm of $\nabla\bar{\psi}_t$. Here, the term involving the Hessian of q_t was treated as follows:

$$\begin{aligned}
\int \langle \nabla^\perp q_t, e^{2V}\mathrm{Cof}(D^2 q_t)\nabla\bar{\psi}_t\rangle\, dx &= \int \langle \nabla q_t, e^{2V} D^2 q_t \nabla^\perp\bar{\psi}_t\rangle\, dx \\
&= \int \langle \nabla\Big(\frac{e^{2V}|\nabla q_t|^2}{2}\Big), \nabla^\perp\bar{\psi}_t\rangle\, dx - \int |\nabla q_t|^2 \langle \nabla\Big(\frac{e^{2V}}{2}\Big), \nabla^\perp\bar{\psi}_t\rangle\, dx \\
&= -\int |\nabla q_t|^2 \langle \nabla\Big(\frac{e^{2V}}{2}\Big), \nabla^\perp\bar{\psi}_t\rangle\, dx \\
&\le \|e^{2V}\|_{C^1(\Omega)}\|\nabla\bar{\psi}_t\|_{L^\infty(\Omega)}\|\nabla q_t\|^2_{L^2(\Omega)}.
\end{aligned}$$

Now, the difference $(\psi_t - \bar{\psi}_t)$ solves

$$\begin{cases}
\mathrm{div}(e^{-2\varphi}(\mathbb{I} + \mathsf{Q}[D^2 p_t, \nabla p_t])\nabla(\psi_t - \bar{\psi}_t)) + \nabla^\perp(\mathsf{f}[\nabla p_t])\cdot\nabla(\psi_t - \bar{\psi}_t) = \mathsf{X}_t, & \text{in } \Omega \\
\psi_t - \bar{\psi}_t = 0, & \text{on } \partial\Omega,
\end{cases} \tag{3.44}$$

where

$$\mathsf{X}_t := \mathrm{div}\big(\mathsf{F}[\nabla q_t]\big) - \mathrm{div}\big(e^{-2\varphi}\mathsf{Q}[D^2 q_t, \nabla q_t]\nabla\bar{\psi}_t\big) - \nabla^\perp(\mathsf{f}[\nabla q_t])\cdot\nabla\bar{\psi}_t.$$

Multiplying (3.44) by $(\psi_t - \bar{\psi}_t)$ and integrating by parts, one obtains that

$$\begin{aligned}
&(1-\mu_t)\|\nabla\psi_t - \nabla\bar{\psi}_t\|^2_{L^2(\Omega)} \\
&\quad \le C\|\nabla\psi_t - \nabla\bar{\psi}_t\|_{L^2(\Omega)}\|\nabla q_t\|_{L^2(\Omega)} + \int \mathrm{div}(e^{2V}\mathrm{Cof}(D^2 q_t)\nabla\bar{\psi}_t)(\psi_t - \bar{\psi}_t)\, dx,
\end{aligned}$$

again for some $C > 0$ depending on φ, V and the $L^\infty([0,t'];L^\infty(\Omega))$-norm of $\nabla\bar{\psi}_t$. Taking advantage again of the key identity $\mathrm{div}(\mathrm{Cof}(D^2 q_t)) = 0$, we have that

$$\begin{aligned}
&\int \mathrm{div}(e^{2V}\mathrm{Cof}(D^2 q_t)\nabla\bar{\psi}_t)(\psi_t - \bar{\psi}_t)\, dx \\
&\qquad = \int e^{2V}\mathrm{Tr}(\mathrm{Cof}(D^2 q_t)(D^2\bar{\psi}_t + 2\nabla V \otimes \nabla\bar{\psi}_t))(\psi_t - \bar{\psi}_t)\, dx.
\end{aligned}$$

Integrating by parts we can distribute one derivative of the Hessian of q_t over the remaining terms, obtaining

$$\int e^{2V}\operatorname{Tr}(\operatorname{Cof}(D^2 q_t)(D^2\bar{\psi}_t + 2\nabla V\otimes\nabla\bar{\psi}_t))(\psi_t - \bar{\psi}_t)\,dx \\ \leq C\|\nabla q_t\|_{L^2(\Omega)}\|\nabla\psi_t - \nabla\bar{\psi}_t\|_{L^2(\Omega)},$$

where $C>0$ depends on φ, V and the $L^\infty([0,t'];H^4(\Omega))$-norm of $\nabla\bar{\psi}_t$. This shows that

$$\|\nabla\psi_t - \nabla\bar{\psi}_t\|_{L^2(\Omega)} \leq C\frac{\|\nabla q_t\|_{L^2(\Omega)}}{1-\mu_t},$$

which combined with (3.43) and the Grönwall lemma implies that

$$\|\nabla p_t - \nabla\bar{p}_t\|_{L^2(\Omega)} = \|\nabla q_t\|_{L^2(\Omega)} \leq \|\nabla q_0\|e^{Ct} = 0.$$

Thus $\nabla p_t = \nabla\bar{p}_t$ in $[0,t']$. By uniqueness of solution of the elliptic equations without zero order terms we infer that $u_t = \bar{u}_t$ in $[0,t']$ (see [45]). Hence, the pair $(\nabla p_t, u_t)$ constitutes the unique solution in the class of regularity of Theorem 3.1. □

Remark 3.19. *With some minor adjustments, it is possible to include the non-simply connected flat periodic case $\Omega = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $V=\varphi=0$. We have to substitute in Equation (3.25) the boundary condition $\psi = 0$ on $\partial\Omega$ with $\int_{\mathbb{T}^2}\psi\,dx = 0$, and impose periodicity conditions on ψ_s^{i+1}, p_0 and p_s^{i+1}. We need also to adjust the operator $\mathfrak{H}(X) = \nabla q$, defined now to be the inverse operator of the problem*

$$\begin{cases}\Delta q = \operatorname{div}(X),\\ \int_{\mathbb{T}^2} q\,dx = 0.\end{cases}$$

Existence of an uniform regularized solution that converges on $[0,t']$ to $(\nabla p_t, \nabla\psi_t)$ still holds. The only problem to fix is that there exist non-trivial harmonic fields on $\mathbb{T}^2$. However, they do not play any role in our problem, and this can be showed with a direct computation: recall that we are in the situation

$$\partial_t\nabla p_t = \mathfrak{H}\Big((D^2 p_t + \mathbb{I})\cdot\nabla^\perp\psi_t + \nabla^\perp p_t\Big) = \mathfrak{H}(X_t),$$

and we want to get rid of $\mathfrak{H}$. Write

$$X_t = \mathfrak{H}(X_t) + w_t = \nabla q_t + w_t,$$

where $\operatorname{div}(w_t) = \operatorname{curl}(w_t) = 0$ thanks to the construction of $\nabla\psi_t$ and $\mathfrak{H}(X)$. Therefore, by duality we can see w_t as an element of the de Rahm Cohomology $H^1_{dR}(\mathbb{T}^2)\cong\mathbb{R}^2$, which is generated by the two covector fields dx^1 and dx^2, which are closed but not exact since $x\mapsto x^1$ and $x\mapsto x^2$ are not periodic functions. Hence, there exist $\alpha_t^1,\alpha_t^2\in\mathbb{R}$ such that

$$w_t = \alpha_t^1\frac{\partial}{\partial x^1} + \alpha_t^2\frac{\partial}{\partial x^2}.$$

Now, choose $k \in \{1,2\}$, and observe that

$$\int_{\mathbb{T}^2} \langle X_t, \frac{\partial}{\partial x^k} \rangle \, dx = \int_{\mathbb{T}^2} \langle \nabla q_t + w_t, \frac{\partial}{\partial x^k} \rangle \, dx = \alpha_t^k,$$

Hence, taking advantage of the explicit form of X_t and integrating by parts we conclude that

$$\begin{aligned}
\alpha_t^k &= \int_{\mathbb{T}^2} \langle (D^2 p_t + \mathbb{I}) \cdot \nabla^\perp \psi_t + \nabla^\perp p_t, \frac{\partial}{\partial x^k} \rangle \, dx = \int_{\mathbb{T}^2} \langle D^2 p_t \cdot \nabla^\perp \psi_t, \frac{\partial}{\partial x^k} \rangle \, dx \\
&= \int_{\mathbb{T}^2} \langle \nabla(\partial_k p_t), \nabla^\perp \psi_t \rangle \, dx \\
&= \int_{\mathbb{T}^2} \operatorname{div}(\partial_k p_t \nabla^\perp \psi_t) + \partial_k p_t \cdot \operatorname{div}(\nabla^\perp \psi_t) \, dx = 0.
\end{aligned}$$

This shows $\mathfrak{H}(X_t) = X_t$ as wished.

Chapter 4

❊

Global-in-time stability for linearized SG with degenerate Coriolis

4.1 Preliminaries

Recall that the two dimensional SG system on the upper hemisphere (S^2_+, g) reads as follows:

$$\begin{cases} (\partial_t + \nabla_{u_t})u_t^G + f(u_t - u_t^G)^\perp = 0, & \text{in } (0,+\infty) \times S^2_+, \\ \operatorname{div}_g(u_t) = 0 & \text{in } (0,+\infty) \times S^2_+, \\ p_t|_{t=0} = p_0, & \text{in } S^2_+, \\ u_t^G := f^{-1}\nabla^\perp p_t. \end{cases} \tag{4.1}$$

The Coriolis force, up to rescaling the dimensions, is equal to the height variable $z \in (0, 1]$ when considering S^2 embedded in $\mathbb{R}^3$. A stationary solution $(p_0, u_0) = (p_0, -\nabla^\perp\psi_0)$ is called axially symmetric if it depends uniquely on the height variable z. The linearization over the upper hemisphere $S^2_+ = \{x \in S^2 : z(x) > 0\}$ of the semigeostrophic equation around this particular solution is given by

$$\begin{cases} (\partial_t + \nabla_{u_0})(z^{-1}\nabla^\perp q_t) + \nabla q_t + \nabla_{v_t}(z^{-1}\nabla^\perp p_0) + zv_t^\perp = 0, & \text{in } I \times S^2_+, \\ \operatorname{div}_g(v_t) = 0, & \text{in } I \times S^2_+, \\ q_t|_{t=0} = q_0, & \text{in } S^2_+. \end{cases} \tag{4.2}$$

In this article we prove global-in-time existence of smooth solutions for this system. After presenting our main result, we discuss the employed terminology.

Theorem 4.1. *Let $k \geq 5$ be fixed, and let $(\nabla p_0, u_0) = (\nabla p_0, -\nabla^\perp\psi_0)$ be axially symmetric k-compatible stationary solutions of (4.1). Then, for every k-admissible initial perturbation ∇q_0, there exists a unique solution $(\nabla q_t, v_t) = (\nabla q_t, -\nabla^\perp\phi_t)$ of Equation (4.2) such that*

$$z^{-1}\nabla q_t, v_t \in C_{loc}([0,+\infty), C^{k-3,\alpha}_{loc}(S^2_+)) \cap C^1_{loc}([0,+\infty), C^{k-4,\alpha}_{loc}(S^2_+)),$$

and the potential ϕ_t is such that

$$\int_{S^2_+} z^2\phi_t \, d\,\text{vol}_g = 0,$$

for every $t \geq 0$.

We will see that being k-compatible and k-admissible, as we will define in a moment, is a natural requirement in order to perform a lift of the equation from S^2_+ to the four-sphere S^4. Doing so, we gain in the induced warped metric a density factor that compensates the degeneracy of the elliptic equation solved by the potential of the velocity v_t. More precisely, the lifting will be performed along the map

$$\pi : S^4 \setminus S^1 \to S^2_+,\ \pi : (x_1, \dots, x_5) \mapsto (x_1, x_2, z) = (x_1, x_2, \sqrt{x_3^2 + x_4^2 + x_5^2}),$$

where

$$S^4 = \{x \in \mathbb{R}^5 : |x|^2 = x_1^2 + \dots + x_5^2 = 1\}, \quad S^1 = \{x \in S^4 : x_1^2 + x_2^2 = 1\},$$

and

$$S^2_+ = \{(x_1, x_2, z) \in \mathbb{R}^3 : x_1^2 + x_2^2 + z^2 = 1, z > 0\}.$$

The before mentioned compatibility conditions are then described in the following definition.

Definition 4.2. For $k \in \mathbb{N}$ we say that the axially symmetric solution $(\nabla p_0, u_0) \in C^2(S^2_+) \times C^1(S^2_+)$ of Equation (4.1) on S^2_+ is *k-compatible* if

- The $(1,1)$-tensor $\mathsf{A} := \mathrm{id} - z^{-1} J \circ \mathrm{Hess}(\bar{p}_0) \circ J$ is uniformly elliptic. Here $\bar{p}_0$ is such that $z\nabla \bar{p}_0 = \nabla p_0$ and $J = (\cdot)^\perp$.
- $\pi^*(z^{-2}\,\mathrm{Hess}(\bar{p}_0)) \in H^k(S^4)$.
- $\pi^*(z^{-2}\,\mathrm{Hess}(\bar{\psi}_0)) \in H^k(S^4)$.

We will say that the initial perturbation q_0 is *k-admissible* if $\pi^*(z^{-1}\nabla q_0) \in H^k(S^4)$.

4.2 Riemannian geometry

Excellent references for this topic are for instance [1, 32, 55, 67]. Let M^n be a differentiable manifold with tangent bundle TM. For every vector bundle $\Pi : E \to M$ we call $E_x = \Pi^{-1}\{x\}$ the fibre of E over $x \in M$ and we denote with $\Gamma(E)$ the space of sections of E, that is the set of maps $S : M \to E$ so that $\Pi \circ S = \mathrm{id}$. In particular $C^\infty(M) := \Gamma(M \times \mathbb{R})$, $\mathfrak{X}(M) := \Gamma(TM)$, $\Omega^p(M) := \Gamma(\Lambda^p(TM))$, are respectively the set of functions, vector fields, and p-differential forms on M. Given a Riemannian metric $g \in \Gamma(TM^* \otimes TM)$, that is a symmetric and positive definite endomorphism of the tangent bundle, we will denote with $|v|_g := \sqrt{g(v,v)}$ the induced norm on TM, and with $\sqrt{|g|}$ its volume density, so that in local coordiantes (x^i) the volume form is given by $d\,\mathrm{vol}_g = \sqrt{|g|}dx^1 \wedge \dots \wedge dx^n$. Then, we let

$$\nabla : \Gamma(E) \to \Gamma(E \otimes TM^*), \tag{4.3}$$

be the Levi-Civita covariant derivative induced by g. We use the notations $\nabla_v S = \nabla S \cdot v := \nabla(S)(v) \in \Gamma(E)$ for $S \in \Gamma(E)$ and $v \in TM$. For instance, $\nabla_v f = df(v)$ when

$f \in C^\infty(M)$, and d denotes the exterior differential $d : \Omega^p(M) \to \Omega^{p+1}(M)$. We will denote with $R \in \Gamma(T^{4,0}M)$ the Riemann curvature tensor

$$R(X, Y, Z, T) := g(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, T), \tag{4.4}$$

and recall that if M has constant sectional curvature $\kappa \in \mathbb{R}$ then

$$R(X, Y, Z, T) = \kappa g(X, Z)g(Y, T) - \kappa g(X, T)g(Y, Z). \tag{4.5}$$

Example 4.3. Let $S^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere with induced round metric. Then, identifying for every $x \in S^{n-1}$ the tangent plane TS^{n-1}_x with $\{v \in \mathbb{R}^n : v \cdot x = 0\}$, the Levi-Civita connection on TS^{n-1} is given by

$$(\nabla_v w)_x = \sum_{i,j=1}^{n} v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} + (w \cdot v)x = (dw_x + x \otimes w)v.$$

The metric g induces naturally an isomorphism $(\cdot)^\sharp : TM^*_x \to TM_x$ by setting $g_x(\alpha^\sharp, v) := \alpha(v)$ for $\alpha \in TM^*_x$ and $v \in TM_x$, and $(\cdot)^\flat : TM_x \to TM^*_x$ by setting $v^\flat(w) := g(v, w)$ for $v, w \in TM_x$. In order to make our notation as compact as possible, we will denote the gradient $(df)^\sharp \in \Gamma(TM)$ of a function f again with ∇f. The hessian of f is the $(1, 1)$-tensor defined as $\mathrm{Hess}(f) : X \mapsto \nabla_X \nabla f$. The divergence operator div_g is the trace of the coraviant derivative with respect to g, and in local coordinates (x^i) is given by $\mathrm{div}_g(X) = |g|^{-1/2}\partial_i(|g|^{1/2}X^i)$. In particular, we call $f \mapsto \Delta f = \mathrm{div}_g(\nabla f)$ the Laplace-Beltrami operator, which is also equal to the trace of $\mathrm{Hess}(f)$.

A differentiable map between two manifolds $\pi : N \to B$ is called a submersion if its differential $d\pi_x$ is surjective at every $x \in N$. If moreover N is endowed with a Riemannian metric and π is itself surjective, we obtain the induced orthogonal splitting $TN = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}_x = \ker(d\pi_x)$ is called the vertical tangent boundle, and $\mathcal{H}_x = (\mathcal{V}_x)^\perp$ is called the horizontal tangent bundle. Finally, π is called a Riemannian submersion if its differential $d\pi_x$ maps isometrically $\mathcal{H}_x$ on $TB_{\pi(x)}$, that is

$$g_N(X_x, Y_x) = g_B(d\pi_x \cdot X_x, d\pi_x \cdot Y_x), \quad X_x, Y_x \in \mathcal{H}_x.$$

A vector is called vertical if it belongs to $\mathcal{V}$, and horizontal if it belongs to $\mathcal{H}$. An horizontal vector field $X \in \Gamma(\mathcal{H})$ is called basic if it is the lift of a vector field in B, that is there exists $\xi \in \Gamma(TB)$ such that

$$d\pi_x \cdot X = \xi_{\pi(x)}.$$

There is a one-to-one correspondence between basic horizontal vector fields and vector fields on B, and therefore we will denote with

$$X = \pi^* \xi \in \Gamma(\mathcal{H}),$$

the corresponding basic horizontal vector field of $\xi \in \Gamma(TB)$. The following identity relates the covariant derivative on N with the covariant derivaitve on B (see [91]):

$$\mathrm{pr}_{\mathcal{H}}\Big(\nabla^N_{\pi^*\xi} \pi^* \zeta\Big) = \pi^*(\nabla^B_\xi \zeta), \tag{4.6}$$

where $\mathrm{pr}_{\mathcal{H}} : TN \to \mathcal{H}$ denotes the orthogonal projection on the horizontal bundle. For a tensor $M \in T^{1,1}(B)$ we define its lift $\pi^* M \in T^{1,1}(N)$ via

$$g_N((\pi^* M)_x \cdot X_x, Y_x) := g_B(M_{\pi(x)} \cdot (d\pi_x) \cdot X_x, d\pi_x \cdot Y_x). \tag{4.7}$$

Finally we notice that the pullback on the horizontal bundle is consistent with the usual pullback for forms, in particular, for all function h on B, we have that

$$\nabla(\pi^* h) = \pi^*(\nabla h), \tag{4.8}$$

where $\pi^* h = h \circ \pi$.

4.3 Sobolev spaces, smoothing, and interpolations

We refer to [59, 60]. The duality induced by g between TM and TM^* allows naturally the extension of g as metric over the (p,q)-tensor bundle $T^{p,q}(M) := (\otimes^p TM) \otimes (\otimes^q TM^*)$ by setting $g(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp)$ for every $\alpha, \beta \in T^{0,1}(M)$, and defining inductively the product on simple tensors

$$g(S_1 \otimes T_1, S_2 \otimes T_2) := g(S_1, S_2) g(T_1, T_2), \tag{4.9}$$

and $|S|_g := \sqrt{g(S,S)}$. We say that a section $S \in \Gamma(T^{p,q}(M))$ belongs to $L^2(M, T^{p,q}(M))$ if

$$\|S\|_{L^2(M)} := \Big(\int_M |S|_g^2 \, d\,\mathrm{vol}_g \Big)^{1/2} < +\infty. \tag{4.10}$$

Notice that $L^2(M, T^{p,q}(M))$ is an Hilbert space with product

$$\langle S, T \rangle_{L^2(M)} := \int_M g(S,T) \, d\,\mathrm{vol}_g \,. \tag{4.11}$$

In this particular case, setting $\nabla^s := (\nabla \circ \nabla \circ \cdots \circ \nabla)$ s-times (with the convention $\nabla^0 := \mathrm{id}$), we notice that $\nabla^s S \in \Gamma(T^{p,q+s}(M))$. The Sobolev space $H^k(M, T^{p,q}(M))$ is defined as

$$H^k(M, T^{p,q}(M)) := \Big\{ S \in T^{p,q}(M) : \nabla^s S \in L^2(M, T^{p,q+s}(M)), s = 0, \ldots, k \Big\}. \tag{4.12}$$

Turns out that $H^k(M, T^{p,q}(M))$ is a Banach space if endowed with the natural norm

$$\|S\|_{H^k(M)} := \Big(\sum_{s=0}^{k} \|\nabla^k S\|^2_{L^2(M)} \Big)^{1/2}. \tag{4.13}$$

If clear from the context, we will write simply $L^2(M)$ and $H^k(M)$ and not $L^2(M, T^{p,q}(M))$ and $H^k(M, T^{p,q}(M))$. Notice that if M is compact (that is bounded and without boundary), then $H^k(M)$ does not depend on g anymore, and a tensor S belongs to $H^k(M)$ if and only if

$$\|\mathcal{L}_{X_1} \ldots \mathcal{L}_{X_s} S\|_{L^2(M)} < +\infty, \quad 0 \le s \le k, \tag{4.14}$$

for all collection of smooth vector fields $\{X_1,\dots,X_k\}$ belonging to any spanning subset of $\Gamma(TM)$. Here $\mathcal{L}_X$ denotes the Lie derivative in the direction X. In this setting an efficient way of smoothing is by the heat semi-group (see [31, Chapter VI and Appendix B], [102, Chapter 1], and [67, Chapter 3]): let α be any p-form. Then for $t>0$ we will denote with $e^{t\Delta}\alpha$ the solution of

$$\partial_t\alpha_t=\Delta_g\alpha_t, \qquad \alpha_0=\alpha. \tag{4.15}$$

where $\Delta_g=d\circ\delta+\delta\circ d$ is the Hodge-Laplacian on forms, and $\delta:\Omega^{p+1}(M)\to\Omega^p(M)$ is the codifferential. Formally we have that

$$\alpha_t=e^{t\Delta}\alpha=\sum_{j=0}^{+\infty}g(\alpha,\beta_j)e^{-t\lambda_j}\beta_j, \tag{4.16}$$

where (λ_j) and (β_j) are the eigenvalues and associated orthonormal basis of $L^2(\Omega^p(M))$ induced by $(\Delta_g)^{-1}$. For any small parameter $\varepsilon>0$, function f, and vector field X we denote their ε-mollification as

$$\mathfrak{J}f:=e^{\varepsilon\Delta}f, \qquad \mathfrak{J}X:=(e^{\varepsilon\Delta}X^\flat)^\sharp, \tag{4.17}$$

which amounts to run the heat flow for a short time span ε. The regularization properties of this are well known (it suffices to check this in the smooth category thanks to the density result in [42, Proposition 3.2]), and summarized in the following lemma.

Lemma 4.4 (Properties of mollifiers)**.** *Let (M^n,g) be a closed (compact and without boundary) Riemannian manifold. Then, for every $\varepsilon_0>0$ and $k\in\mathbb{N}\cup\{0\}$ and $0<\varepsilon<\varepsilon_0$:*

i. $\mathfrak{J}:L^2(M)\to L^2(M)$ is self-adjoint: $\langle\mathfrak{J}S,T\rangle_{L^2(M)}=\langle S,\mathfrak{J}T\rangle_{L^2(M)}$ for every L^2-functions (vector fields) S and T.

ii. $\mathfrak{J}:L^2(M)\to H^k(M)$ is continuous: there exists $C>0$ such that $\|\mathfrak{J}S\|_{H^k(M)}\le C\varepsilon^{-k}\|S\|_{L^2(M)}$ for every function (vector field) S.

iii. $\|\mathfrak{J}S-S\|_{H^k(M)}\to 0$ as $\varepsilon\to 0$ whenever S is a function (vector field) in $H^k(M)$.

iv. $\mathcal{L}_\xi(\mathfrak{J}S)=\mathfrak{J}(\mathcal{L}_\xi S)$ whenever $S\in H^1(M)$ and ξ is a Killing *vector field, that is $\mathcal{L}_\xi g=0$.*

Point iv. is really interesting for simplifying the exposition of our future computations: in fact, suppose that there exists a collection of Killing fields $\xi_1,\dots,\xi_{n+1}\in\Gamma(TM)$ spannig TM. This is the case for example when $M=S^n$ by choosing the vector fields representing the infinitesimal rotations around $(n+1)$ linearly independent directions. Then, one can compute the H^k-norm via Equation (4.14) with respect to this particular spanning family, gaining the useful commutativity property with $\mathfrak{J}$. For this reason, we adopt the following terminology.

Definition 4.5. Let (M^n,g) be a Riemannian manifold. We will call $\{\xi_1,\dots,\xi_{n+1}\}$ a *Killing spanning family* if ξ_j is a smooth Killing vector field for $j=1,\dots,n+1$ and

$$\operatorname{span}\{\xi_1,\dots,\xi_{n+1}\}=\Gamma(TM).$$

Finally, we state classical results about Sobolev embeddings and interpolation inequalities in the context of manifolds. We refer to [68] for the Euclidean setting, and in whole generality to [116, Chapter 3.6].

Proposition 4.6 (Kato-Ponce inequalities). *Let D a differential operator of degree $k \in \mathbb{N}$ on a Riemannian manifold (M^n, g). Then, there exists $C = C(\mathsf{D}, M) > 0$ such that for every function $f \in H^k(M) \cap C^1(M)$ and vector field $X \in H^k(M) \cap L^\infty(M)$ one can bound the commutator $[\mathsf{D}, f\cdot] : X \mapsto \mathsf{D}(fX) - f(\mathsf{D}X)$ as*

$$\|[\mathsf{D}, f\cdot]X\|_{L^2(M)} \leq C\|f\|_{C^1(M)}\|X\|_{H^{k-1}(M)} + C\|f\|_{H^k(M)}\|X\|_{L^\infty(M)}. \tag{4.18}$$

In particular, if $\mathbb{T}$ is an differential operator of degree one, by replacing X with $\mathbb{T}Y$, $Y \in H^{k+1}(M)$, one obtains that there exists $C = C(\mathsf{D}, \mathbb{T}, M) > 0$ such that

$$\|[\mathsf{D}, f\mathbb{T}]Y\|_{L^2(M)} \leq C\|f\|_{C^1(M)}\|Y\|_{H^k(M)} + C\|f\|_{H^k(M)}\|Y\|_{C^1(M)}. \tag{4.19}$$

It is also known that elements in $H^k(M)$ are regular when k is large enough, as we state precisely in the next proposition (see [59, Theorem 2.7, Theorem 2.8]).

Proposition 4.7 (Sobolev embeddings). *Let (M^n, g) be a compact Riemannian manifold, and $S \in H^k(M)$. Then, $S \in C^{s,\alpha}(M)$ provided $k \geq n/2 + s + \alpha$ for some $\alpha \in (0, 1)$.*

In particular, combining Propositions 4.6 and 4.7, we have that $H^k(M)$ is a Banach algebra (up to rescaling the norms) if $k > n/2 + 1$, which means that there exists $C = C(k, M) > 0$ such that

$$\|g(S, T)\|_{H^k(M)} \leq C\|S\|_{H^k(M)}\|T\|_{H^k(M)} \tag{4.20}$$

for every $S, T \in H^k(M)$ functions or vector fields.

4.4 Preliminary results

4.4.1 Stationary solutions and derivation of the linearized equation

The semigeostrophic equation in the open upper hemisphere (S^2_+, g) over a time interval containinig $t = 0$ reads

$$\begin{cases} (\partial_t + \nabla_{u_t})u_t^G + f(u_t - u_t^G)^\perp = 0, & \text{in } I \times S^2_+, \\ \operatorname{div}_g(u_t) = 0 & \text{in } I \times S^2_+, \\ p_t|_{t=0} = p_0, & \text{in } S^2_+, \\ u_t^G := f^{-1}\nabla^\perp p_t, \end{cases} \tag{4.21}$$

where $u_t \in \Gamma(TM)$ is the time dependent *velocity vector field*, $u_t^G = f^{-1}\nabla^\perp p_t$ is the *geostrophic velocity*, p_t represents the *pressure of the fluid*, and f is the *Coriolis term*, which in this case is equal to $f = 2\Omega z$. Here, $\Omega > 0$ is the angular velocity of the earth rotation, and $z \in (0,1]$ is the height in cylindrical coordinates aligned with the rotation axis. With $J = (\cdot)^\perp$ we denote the complex endomorphism of tangent bundle that rotates the vectors by $\pi/2$-radians in the counter-clockwise direction. The initial data is given in term of the pressure. An elementary computation shows that rescaling the dimensions as follows

$$p(t,\cdot) \mapsto p(2\Omega t,\cdot), \quad u(t,\cdot) \mapsto 2\Omega u(2\Omega t,\cdot)$$

allows us to set $\Omega = 1$ without loss of generality in Equation (4.21) giving

$$\begin{cases} (\partial_t + \nabla_{u_t})(z^{-1}\nabla^\perp p_t) + z(u_t - z^{-1}\nabla^\perp p_t)^\perp = 0, & \text{in } I \times S^2_+, \\ \operatorname{div}_g(u_t) = 0 & \text{in } I \times S^2_+, \\ p_t|_{t=0} = p_0, & \text{in } S^2_+, \\ u_t^G := f^{-1}\nabla^\perp p_t. \end{cases} \tag{4.22}$$

The next observation, which is a well known fact in Riemannian geometry, will give us the main ingredient to construct an axially symmetric family of stationary solutions of (4.22).

Lemma 4.8. *Let z be the height coordinate on S^2. Then, the gradient of z satisfies $\nabla_X \nabla z = -zX$ for every $X \in \Gamma(TM)$. Said otherwise,* $\operatorname{Hess}(z) = -z\operatorname{id}$.

Remark 4.9. *In fact, ∇z is what is called a* concircular vector field*, which are special elements $\zeta \in \Gamma(TM)$ satisfying for all $X \in \Gamma(TM)$ the identity $\nabla_X \zeta = -hX$ for some function h. For instance, the position vector field $x \mapsto x$ is a typical concircular vector field in $\mathbb{R}^n$,* $\operatorname{Hess}(x) = \operatorname{id}$.

Definition 4.10. We say that a function $h : S^2_+ \to \mathbb{R}$ is axially symmetric if in cylindrical coordinates $h(z,\varphi) = H(z)$ for some profile function $H : (0,1] \to \mathbb{R}$.

We are ready to construct a family of stationary solutions of Equation (4.22).

Lemma 4.11. *Let $p_0 \in C^2(S^2_+)$ and $\psi_0 \in C^1(S^2_+)$ be two time independent axially symmetric maps with profile $P_0 \in C^2((0,1])$ and $\Psi_0 \in C^1((0,1])$ respectively. If*

$$-\Psi_0'(z)P_0'(z) + z\Psi_0'(z) + P_0'(z) = 0, \quad \forall z \in (0,1), \tag{4.23}$$

and $u_0 := -\nabla^\perp \psi_0 \in \Gamma(TS^2_+)$, then (p_0, u_0) is a stationary solution of the semigeostrophic Equation (4.22).

Proof. This is a direct computation taking advantage of Proposition 4.8. Notice that since $u_0 = -\nabla^\perp \psi_0 = -\Psi_0'\nabla^\perp z$ and $\nabla p_0 = P_0'\nabla z$, we get that

$$\begin{aligned} (\partial_t + \nabla_{u_0})(z^{-1}\nabla^\perp p_0) + z(u - z^{-1}\nabla^\perp p_0)^\perp &= -\Psi_0'\nabla_{\nabla^\perp z}(z^{-1}P_0'\nabla^\perp z) + z\Psi_0'\nabla z + P_0'\nabla z \\ &= -\Psi_0' g(\nabla^\perp z, (z^{-1}P_0')'\nabla z)\nabla^\perp z - \Psi_0' z^{-1}P_0'\nabla_{\nabla^\perp z}\nabla^\perp z + z\Psi_0\nabla z + P_0'\nabla z \\ &= (-\Psi_0'P_0' + z\Psi_0' + P_0')\nabla z = 0, \end{aligned}$$

in virtue of Equation (4.23). Notice that in the last line we used Lemma 4.8 to compute

$$\nabla_{\nabla^\perp z}\nabla^\perp z = (\nabla_{\nabla^\perp z}\nabla z)^\perp = (-z\nabla^\perp z)^\perp = z\nabla z,$$

where $(\cdot)^\perp$ commutes with the covariant derivative since S^2 is a Kähler manifold, and hence in particular the complex structure is compatible with the Riemannian structure (meaning $\nabla J = 0$). □

Given a stationary solution (p_0, u_0) like in Lemma 4.11, we can now *linearize* the semi-geostrophic equation around a small perturbation of the initial conditions from (p_0, u_0) to $(p_0, u_0) + \varepsilon(q_0, v_0)$, obtaining

$$\begin{cases} (\partial_t + \nabla_{u_0})(z^{-1}\nabla^\perp q_t) + \nabla q_t + \nabla_{v_t}(z^{-1}\nabla^\perp p_0) + z v_t^\perp = 0, & \text{in } I \times S^2_+, \\ \operatorname{div}_g(v_t) = 0, & \text{in } I \times S^2_+, \\ q_t|_{t=0} = q_0, & \text{in } S^2_+. \end{cases} \tag{4.24}$$

Since $\operatorname{div}_g(v_t) = 0$ and S^2_+ is simply connected, there exists a scalar potential ϕ_t such that $v_t = -\nabla^\perp \phi_t$. The goal of this chapter is to show global-in-time existence of smooth solutions of Equation (4.24) overcoming the severe singularity in the proximity of the equator $\{z = 0\}$.

4.4.2 A degenerated PDE for the velocity and weighted Sobolev spaces

Applying $\operatorname{div}_g(z\cdot)$ to Equation (4.24) one obtains an autonomous equation for $v_t = -\nabla\phi_t$ since $\operatorname{div}_g(z\partial_t z^{-1}\nabla^\perp q_t) = \operatorname{div}_g(\nabla^\perp \partial_t q_t) = 0$, and hence

$$\operatorname{div}_g(z^2\nabla\phi_t + z\nabla_{v_t}(z^{-1}\nabla^\perp p_0)) = -\operatorname{div}_g(z\nabla_{u_0}(z^{-1}\nabla^\perp q_t) + z\nabla q_t). \tag{4.25}$$

Suppose now that there exists $\bar{P}_0 \in C^2((0,1])$ solution of

$$z\bar{P}_0' = P_0'.$$

Then, the associated axially symmetric map $\bar{p}_0(z,\varphi) = \bar{P}_0(z)$ satisfies

$$\nabla\bar{p}_0 = \bar{P}_0'\nabla z = z^{-1}P_0'\nabla z = z^{-1}\nabla p_0$$

showing that the left hand side of (4.25) is equal to

$$\operatorname{div}(z^2\mathsf{A}\cdot\nabla\phi_t),$$

where

$$\mathsf{A} := \mathrm{id} - z^{-1}J \circ \operatorname{Hess}(\bar{p}_0) \circ J \in \Gamma(T^{1,1}(S^2_+)).$$

To simplify the right hand side of (4.25), we take advantage of the following general identity.

Lemma 4.12. *Let M be a Riemannian manifold and $\xi \in \Gamma(TM)$ a Killing vector field. Then, for any vector $X \in \Gamma(TM)$ and function $h \in C^2(M)$ such that ∇h is everywhere orthogonal to ξ one has that*

$$\operatorname{div}(\nabla_{h\xi} X) = \operatorname{div}(\nabla_X (h\xi)) + \mathcal{L}_{h\xi}\operatorname{div}(X). \tag{4.26}$$

In particular, if X is divergence free we have that

$$\operatorname{div}(\nabla_{h\xi} X) = \operatorname{div}(\nabla_X (h\xi)). \tag{4.27}$$

Proof. Let us first prove this identity when $h \equiv 1$. Since ∇ is torsion free and $\mathcal{L}_\xi$ commutes with div because ξ is Killing, we have that

$$\operatorname{div}(\nabla_\xi X) = \operatorname{div}(\nabla_X \xi + [\xi, X]) = \operatorname{div}(\nabla_X \xi + \mathcal{L}_\xi X) = \operatorname{div}(\nabla_X \xi) + \mathcal{L}_\xi \operatorname{div}(X).$$

Now, for a general h we get

$$\begin{aligned}
\operatorname{div}(\nabla_{h\xi} X) &= \operatorname{div}(h\nabla_\xi X) = \nabla h \cdot \nabla_\xi X + h\operatorname{div}(\nabla_\xi X)\\
&= \nabla h \cdot \nabla_\xi X + h\operatorname{div}(\nabla_X \xi) + h\mathcal{L}_\xi \operatorname{div}(X)\\
&= \nabla h \cdot \nabla_\xi X - \nabla h \cdot \nabla_X \xi + \operatorname{div}(h\nabla_X \xi) + \mathcal{L}_{h\xi}\operatorname{div}(X)\\
&= [\xi, X](h) + \operatorname{div}(\nabla_X (h\xi)) - \operatorname{div}(X(h)\xi) + \mathcal{L}_{h\xi}\operatorname{div}(X)\\
&= \xi(X(h)) - X(\xi(h)) - \xi(X(h)) + \operatorname{div}(\nabla_X (h\xi)) + \mathcal{L}_{h\xi}\operatorname{div}(X)\\
&= \operatorname{div}(\nabla_X (h\xi)) + \mathcal{L}_{h\xi}\operatorname{div}(X),
\end{aligned}$$

where we used $\operatorname{div}(\xi) = 0$ and $\xi(h) = \nabla h \cdot \xi = 0$. □

This lemma applied to $M = S^2_+$ and $\xi = \nabla^\perp z$ allows us to make v_t and ∇q_t comparable in term of regularity in Equation (4.25): noticing that $z\nabla_{u_0}(z^{-1}\nabla^\perp q_t) = \nabla_{u_0}\nabla q_t$ since $u_0 \perp \nabla z$ we get that

$$\operatorname{div}_g(z\nabla_{u_0}(z^{-1}\nabla^\perp q_t) + z\nabla q_t) = \operatorname{div}_g(\nabla_{\nabla^\perp q_t} u_0 + z\nabla q_t),$$

which by introducing $T := z^{-1}\nabla q_t$ gives us the following expression for (4.25)

$$\operatorname{div}_g(z^2 \mathsf{A} \cdot \nabla \phi_t) = -\operatorname{div}_g(z^2 \mathsf{B} \cdot T),$$

where B has the similar structure as A

$$\mathsf{B} = \mathrm{id} - z^{-1} J \circ \operatorname{Hess}(\psi_0) \circ J \in \Gamma(T^{1,1}(S^2_+)).$$

We summarize the formal computations we did till now in the following proposition.

Proposition 4.13 (PDE for the velocity vector field)**.** *Suppose $(q_t, v_t = -\nabla^\perp \phi_t)$ is a C^1-solution of the linearized semigeostrophic Equation (4.24) around a stationary C^2-solution $(p_0, u_0 = -\nabla^\perp \psi_0)$ in S^2_+ as in Lemma 4.11. Then, setting $T_t = z^{-1}\nabla q_t$, we have that*

$$\operatorname{div}_g(z^2 \mathsf{A} \cdot \nabla \phi_t) = -\operatorname{div}_g(z^2 \mathsf{B} \cdot T_t), \tag{4.28}$$

where $\mathsf{A} = \mathrm{id} - z^{-1} J \circ \operatorname{Hess}(\bar{p}_0) \circ J$, $\mathsf{B} = \mathrm{id} - z^{-1} J \circ \operatorname{Hess}(\psi_0) \circ J$, and $z\nabla \bar{p}_0 = \nabla p_0$.

Following the arguments of Montero in [81], we prove that Equation (4.28) has a unique solution in a carefully chosen weighted Sobolev space that we define now.

Definition 4.14 (Weighted Sobolev)**.** Define the weighted Sobolev spaces

$$H_2^k(M, T^{p,q}(S_+^2)) := \Big\{ S \in T^{p,q}(S_+^2) : z\nabla^s S \in L^2(M, T^{p,q+s}(S_+^2)), s = 0, \dots, k \Big\},$$

endowed with the norm

$$\|S\|_{H_2^k(S_+^2)} := \Big(\sum_{s=0}^{k} \|z\nabla^k S\|^2_{L^2(S_+^2)} \Big)^{1/2}. \tag{4.29}$$

Again, when clear from the context, we will simply write $H_2^k(S_+^2)$ instead of $H_2^k(M, T^{p,q}(S_+^2))$. Also, we will denote $L_2^2(S_+^2) := H_2^0(S_+^2)$.

To prove existence we need the next suitable Poincaré-type inequality.

Proposition 4.15 (Poincaré inequality)**.** *Let $h \in H_2^1(S_+^2)$ be a function with zero weighted average, that is*

$$h \in Z_2 := \Big\{ f \in H_2^1(S_+^2) : \int_{S_+^2} z^2 f \, d\,\mathrm{vol}_g = 0 \Big\}. \tag{4.30}$$

Then, there exists a universal constant $C > 0$ such that

$$\int_{S_+^2} z^2 h^2 \, d\,\mathrm{vol}_g \leq C \int_{S_+^2} z^2 |\nabla h|_g^2 \, d\,\mathrm{vol}_g \,. \tag{4.31}$$

Proof. This follows via a direct rephrasing of [81, Theorem A1] in the curved case when $p = m = k = 2$. In fact, the proof relies on showing that $H_2^1(S_+^2)$ embeds compactly in $L_2^2(S_+^2)$, an it does not use at all the flatness of the domain. Equation (4.31) follows by a classical argument of contradiction. □

We have then the following crucial existence and uniqueness result.

Proposition 4.16. *Let $\mathsf{M} \in \Gamma(T^{1,1}(S_+^2))$ essentially bounded, self-adjoint, and uniformly elliptic, that is there exist $\lambda, \Lambda > 0$ such that*

$$\lambda |\xi|_g^2 \leq g(\mathsf{M} \cdot \xi, \xi) \leq \Lambda |\xi|_g^2, \tag{4.32}$$

for every $\xi \in TS_+^2$. Then, for every vector field $\mathsf{F} \in H_2^1(S_+^2)$ there exists $\psi \in H_2^1(S_+^2)$ unique up to a constant solving

$$\mathrm{div}(z^2 \mathsf{M} \cdot \nabla \psi) = \mathrm{div}(z^2 \mathsf{F}),$$

in the following weak sense:

$$\int_{S_+^2} z^2 g(\mathsf{M} \cdot \nabla\psi, \nabla\phi) \, d\,\mathrm{vol}_g = \int_{S_+^2} z^2 g(\mathsf{F}, \nabla\phi) \, d\,\mathrm{vol}_g, \qquad \forall \phi \in H_2^1(S_+^2). \tag{4.33}$$

In particular, the following estimate

$$\|\nabla\psi\|_{L_2^2(S_+^2)} \leq \frac{\|\mathsf{F}\|_{L_2^2(S_+^2)}}{\lambda} \tag{4.34}$$

holds.

Proof. The argument is classic: let $Z_2 \subset H_2^1(S_+^2)$ be the subspace of functions with zero weighted average, as defined in Proposition 4.15. Introduce the functional $\mathcal{I} : Z_2 \to \mathbb{R}$ as

$$\mathcal{I} : h \mapsto \int_{S_+^2} \frac{z^2}{2} g(\mathsf{M} \cdot \nabla h, \nabla h) - z^2 g(\mathsf{F}, \nabla h)\, d\,\mathrm{vol}_g\,.$$

Then, $\mathcal{I}$ is clearly linear and bounded, hence continuous. Also, it is coercive since thanks to the ellipticity of M and Proposition 4.15 we can estimate

$$\begin{aligned}\mathcal{I}(h) &\geq \frac{\lambda}{2}\int_{S_+^2} z^2|\nabla h|_g^2\, d\,\mathrm{vol}_g - \frac{1}{\lambda}\int_{S_+^2} z^2|\mathsf{F}|_g^2\, d\,\mathrm{vol} - \frac{\lambda}{4}\int_{S_+^2} z^2|\nabla h|_g^2\, d\,\mathrm{vol}_g \\ &\geq \frac{\lambda}{4}\|\nabla h\|_{L_2^2(S_+^2)}^2 - \frac{1}{\lambda}\|\mathsf{F}\|_{L_2^2(S_+^2)}^2 \geq C\lambda\|h\|_{H_2^1(S_+^2)}^2 - \frac{1}{\lambda}\|\mathsf{F}\|_{L_2^2(S_+^2)}^2,\end{aligned}$$

which goes to infinity as $\|h\|_{H_2^1(S_+^2)} \to +\infty$. Hence, there exists a stationary point $\psi \in Z_2$ of $\mathcal{I}$, meaning that

$$\delta\mathcal{I}(\psi)[\phi] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I(\psi + \varepsilon\phi) = \int_{S_+^2} z^2 g(\mathsf{M} \cdot \nabla\psi, \nabla\phi)\, d\,\mathrm{vol}_g - \int_{S_+^2} z^2 g(\mathsf{F}, \nabla\phi)\, d\,\mathrm{vol}_g = 0$$

for all $\phi \in Z_g$. In fact, since the above equation holds also replacing ϕ with $\phi + c$ for any $c \in \mathbb{R}$, we deduce that it holds more generally for all $\phi \in H_2^1(S_+^2)$ by subtracting the weighted average. Suppose now $\tilde{\psi} \in H_2^1(S_+^2)$ is another weak solution, then setting $c \in \mathbb{R}$ such that $\tilde{\psi} + c \in Z_2$ we get that $\psi - (\tilde{\psi} + c)$ solves

$$\int_{S_+^2} z^2 g(\mathsf{M} \cdot \nabla(\psi - (\tilde{\psi} + c)), \nabla\phi)\, d\,\mathrm{vol}_g = 0,$$

for all $\phi \in H_2^1(S_+^2)$, implying by the elliplicity of M that $\|\nabla\tilde{\psi} - \nabla\psi\|_{L_2^2(S_+^2)} = 0$, showing $z\nabla\psi = z\nabla\tilde{\psi}$ almost everywhere. This proves the first part of the proposition. The second Equation (4.34) is a direct consequence of Equation (4.33) taking $\phi = \psi$. □

As a consequence, if in Equation (4.28) $\mathsf{B} \cdot T_t \in H_2^1(S_+^2)$ and A is uniformly elliptic, then there exists a unique weak potential for the perturbed velocity $\nabla\phi_t \in L_2^2(S_+^2)$.

4.4.3 From S^2 to S^4: a geometric argument

To obtain higher regularity estimates on the velocity $v_t = -\nabla^\perp \phi_t$, we specialize the ingenious argument of Montero in [81] from the Euclidean setting to the spherical one. To fix the ideas, we briefly sketch the strategy when the domain is the half plane $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ with flat metric $g_e = dx_1^2 + dx_2^2$, and the equation of interest is of the form

$$\mathrm{div}_{g_e}(x_1 \nabla \psi) = \mathrm{div}_{g_e}(x_1 \mathsf{F}). \tag{4.35}$$

Consider the higher dimensional space obtained by the revolution of $\mathbb{R}^2_+$ around the x_1-axis, which in this case is $\mathbb{R}^2_+ \times S^1 := \{(x_1, x_2, \theta) : x_1 > 0, x_2 \in \mathbb{R}, \theta \in [0, 2\pi)\}$. Then, the cylindrical metric $\mathbf{g}_e = dx_1^2 + dx_2^2 + (x_1)^2 d\theta^2$ in $\mathbb{R}^3$ restricted to $\mathbb{R}^2_+ \times S^1$ makes the projection

$$\mathrm{pr} : (\mathbb{R}^2_+ \times S^1, \mathbf{g}_e) \to (\mathbb{R}^2_+, g_e), \qquad (x_1, x_2, \theta) \mapsto (x_1, x_2),$$

into a Riemannian submersion. The key observation is that for every vector field X on $\mathbb{R}^2_+$ the following holds

$$\mathrm{div}_{g_e}(x_1 X) = x_1 \mathrm{div}_{\mathbf{g}_e}(\mathrm{pr}^* X),$$

Hence, visualizing $\mathbb{R}^2_+$ inside $\mathbb{R}^3$ with suitable coordinates allows to get rid of the degenerate weight, and if ψ solves (4.35), then $\mathrm{pr}^* \psi$ solves the non-degenerate Laplace equation

$$\Delta(\mathrm{pr}^* \psi) = \mathrm{div}_{g_c}(\mathrm{pr}^* \mathsf{F}), \tag{4.36}$$

(notice that the pullback commutes with the gradient). The well established regularity theory for elliptic equations in $\mathbb{R}^3$ can be applied to $\mathrm{pr}^* \psi$, and then translated back to $\mathbb{R}^2_+$.

We follow this strategy when the starting domain is the half hemisphere (S^2_+, g) instead of the half plane $(\mathbb{R}^2_+, g_e)$. To obtain the right power of the weight in Equation (4.28) (now squared), we look at S^2_+ inside the 4-manifold $S^2_+ \times S^2$. To guess the suitable metric we notice that

$$S^2_+ \times S^2 \subset (\overline{S^2_+} \times S^2)/\sim,$$

where $\overline{S^2_+} = \{(z, \varphi) \in S^2 : z \in [0, 1], \varphi \in [0, 2\pi)\}$ and the equivalence relation $(0, \varphi, \sigma_1) \sim (0, \varphi, \sigma_2)$ for all $\varphi \in [0, 2\pi)$ and $\sigma_1, \sigma_2 \in S^2$ is made to collapse the spherical fibres along the equator ∂S^2_+. Topologically the space $(\overline{S^2_+} \times S^2)/\sim$ is a double lift of S^2, and hence isometric to $S^4 \subset \mathbb{R}^5$ if endowed with the round metric

$$\mathbf{g} = g + z^2 g_{S^2} = \frac{dz^2}{1 - z^2} + (1 - z^2) d\varphi^2 + z^2 \Big(\frac{dw^2}{1 - w^2} + (1 - w^2) d\vartheta^2 \Big),$$

with cylindrical coordiantes $(z, \varphi, w, \vartheta) \in S^2_+ \times S^2$. In fact, in the more familiar spherical coordinates $(\theta_1, \theta_2, \theta_3, \theta_4) \in [0, \pi)^3 \times [0, 2\pi)$ on $S^4 \subset \mathbb{R}^5$

$$\begin{aligned}
x_1 &= \cos(\theta_1), \\
x_2 &= \sin(\theta_1) \cos(\theta_2), \\
x_3 &= \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \\
x_4 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \cos(\theta_4), \\
x_5 &= \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \sin(\theta_4),
\end{aligned}$$

with round metric

$$\mathbf{g} = d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)(d\theta_3^2 + \sin^2(\theta_3)d\theta_4^2),$$

one can identify S^2_+ with $(\theta_1, \theta_2) \in [0, \pi)^2$ via the isometric embedding in $\mathbb{R}^3$

$$\begin{aligned} x' &= \cos(\theta_1) = x_1, \\ y' &= \sin(\theta_1)\cos(\theta_2) = x_2, \\ z &= \sin(\theta_1)\sin(\theta_2) = \sqrt{x_3^2 + x_4^2 + x_5^2}. \end{aligned}$$

Remark 4.17. *Notice the following elementary fact: $z\nabla z$ is smooth in S^4, but the vector field ∇z is* singular *approaching the gluing region ∂S^2_+ . On the other hand, the lifted vector field $\nabla^\perp z = \frac{\partial}{\partial \varphi}$ is equal to $x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$ which is globally smooth on S^4 since tangential to the gluing region ∂S^2_+. This shows that the operation $(\cdot)^\perp$ does not extend continuously from S^2_+ to S^4.*

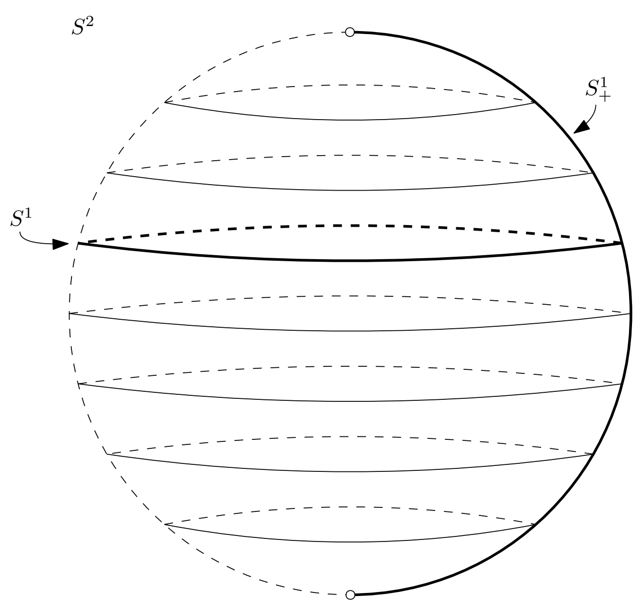

Figure 4.1: A low dimensional illustration of $S^2_+ \times S^2 = S^4 \setminus S^1$. Here $S^1_+ \times S^1 = S^2 \setminus S^0$.

Following this identification we can write

$$S^2_+ \times S^2 = S^4 \setminus S^1,$$

so that the projection

$$\pi : (S^4 \setminus S^1, \mathbf{g}) \to (S^2_+, g), \qquad (z, \varphi, w, \vartheta) \mapsto (z, \varphi) \tag{4.37}$$

is a Riemannian submersion. The goal of this section is to show that the elliptic equation (4.28) can be lifted on S^4. To so do we need to introduce the following terminology.

Definition 4.18. Let $\mathsf{M} \in \Gamma(T^{1,1}(S^2_+))$ be essentially bounded and uniformly elliptic like in Proposition 4.16, π as in (4.37), and $T(S^4 \setminus S^1) = \mathcal{H} \oplus \mathcal{V}$ its induced orthogonal splitting. We say that $\widetilde{\mathsf{M}} \in \Gamma(T^{1,1}(S^4))$ is a *elliptic extension* of M if for all $x \in S^4 \setminus S^1$

$$\widetilde{\mathsf{M}}_x = (\pi^*\mathsf{M})_x + \lambda(x)\,\mathrm{pr}_{\mathcal{V}}, \tag{4.38}$$

for some $\lambda : S^4 \to [\lambda, \Lambda]$ continuous. Here, $\mathrm{pr}_{\mathcal{V}}$ denotes the orthogonal projection of $T(S^4 \setminus S^1) = \mathcal{V} \oplus \mathcal{H}$ over $\mathcal{V}$.

Remark 4.19. *Observe that elliptic extensions are elliptic, since for all $X = \pi^*\xi \in \mathcal{H}$ one has*

$$\mathbf{g}(\widetilde{\mathsf{M}} \cdot X, X) = g(\mathsf{M} \cdot \xi, \xi) \geq \lambda|\xi|^2_g = \lambda|X|^2_{\mathbf{g}}$$

and for all $Y \in \mathcal{V}$, $\mathbf{g}(\widetilde{\mathsf{M}} \cdot Y, Y) = \mathbf{g}(\lambda Y, Y) \geq \lambda|Y|^2_{\mathbf{g}}$. Moreover, an elliptic extension always exists by setting $\lambda \equiv \lambda$.

Proposition 4.20 (Correspondence of weak solutions)**.** *Let $\pi : S^4 \setminus S^1 \to S^2_+$ be as in (4.37). Then, for $k \in \{0, 1\}$ and $\psi \in H^k_2(S^2_+)$ one has that*

$$\|\pi^*\psi\|_{H^k(S^4 \setminus S^1)} = 4\pi\|\psi\|_{H^k_2(S^2_+)}. \tag{4.39}$$

Let M, $\lambda > 0$, and F be like in Proposition 4.16. Then, any $\psi \in H^1_2(S^2_+)$ can be extended from $\pi^*\psi \in H^1(S^4 \setminus S^1)$ to $H^1(S^4)$ so that if ψ is the weak solution of

$$\begin{cases} \mathrm{div}_g(z^2\mathsf{M} \cdot \nabla\psi) = \mathrm{div}_g(z^2\mathsf{F}), \\ \psi \in H^1_2(S^2_+), \\ \int_{S^2_+} z^2\psi\, d\,\mathrm{vol}_g = 0 \end{cases} \tag{4.40}$$

in the sense of Proposition 4.16, then $\tilde{\psi} = \pi^*\psi$ is a weak solution of

$$\begin{cases} \mathrm{div}_{\mathbf{g}}(\widetilde{\mathsf{M}} \cdot \nabla\tilde{\psi}) = \mathrm{div}_{\mathbf{g}}(\pi^*\mathsf{F}), \\ \tilde{\psi} \in H^1(S^4), \\ \int_{S^4} \tilde{\psi}\, d\,\mathrm{vol}_{\mathbf{g}} = 0, \end{cases} \tag{4.41}$$

in $H^1(S^4)$, meaning

$$\int_{S^4} \mathbf{g}(\widetilde{\mathsf{M}} \cdot \nabla\tilde{\psi}, \nabla\eta)\, d\,\mathrm{vol}_{\mathbf{g}} = \int_{S^4} \mathbf{g}(\pi^*\mathsf{F}, \nabla\eta)\, d\,\mathrm{vol}_{\mathbf{g}}, \qquad \forall \eta \in H^1(S^4), \tag{4.42}$$

where $\widetilde{\mathsf{M}}$ is any elliptic extension of M as described in Definition 4.18. Conversely, by uniqueness of solution if $\tilde{\psi} \in H^1(S^4)$ solves (4.41) weakly, then $\tilde{\psi} = \pi^*\psi$ for $\psi \in H^1_2(S^2_+)$ solving weakly (4.40).

Proof. Equation (4.39) is a consequence of the coarea formula and the fact that π is a Riemannian submersion: for every measurable function h on $S^4 \setminus S^1$ we have that

$$\int_{S^4\setminus S^1} h\, d\,\mathrm{vol}_{\mathbf{g}} = \int_{S^2_+}\int_{S^2} hz^2\, d\,\mathrm{vol}_{S^2}\; d\,\mathrm{vol}_g,$$

and in particular

$$\int_{S^4\setminus S^1} |\pi^*\psi|^2_{\mathbf{g}}\, d\,\mathrm{vol}_{\mathbf{g}} = 4\pi \int_{S^2_+} z^2 |\psi|^2_g\, d\,\mathrm{vol}_g,$$

proving Equation (4.39) when $k = 0$. For $k = 1$, we do the same with $h = |\nabla(\pi^*\psi)|^2_{\mathbf{g}}$, noticing that

$$|\nabla(\pi^*\psi)|_{\mathbf{g}} = |d\pi^*\psi|_{\mathbf{g}} = |\pi^* d\psi|_{\mathbf{g}} = |d\psi|_g,$$

since the pullback commutes with the exterior differential and π is a Riemannian submersion. This proves that $\psi \in H^1_2(S^2_+)$ if and only if $\pi^*\psi \in H^2(S^4 \setminus S^1)$. Now, since the three dimensional Hausdorff measure of S^1 in S^4 is zero, we deduce that S^1 is a removable singularity, and hence $H^k(S^4/S^1, \mathbb{R}) = H^k(S^4, \mathbb{R})$ for every $k \in \mathbb{N}$, see [78, Section 1.1.18]. Therefore, given a measurable function ψ on S^2_+ we deduce that $\psi \in H^1_2(S^2_+)$ and $\int_{S^2_+} z^2\psi\, d\,\mathrm{vol}_g = 0$ if and only if $\pi^*\psi \in H^1(S^4)$ and $\int_{S^4} \pi^*\psi\, d\,\mathrm{vol}_{\mathbf{g}} = 0$ (again by the coarea formula). Let now ψ a solution of (4.40) and set $\tilde{\psi} = \pi^*\psi$. By density, it suffices to check Equation (4.42) for $\eta \in C^\infty(S^4)$. Notice that since $\nabla(\pi^*\psi) \in \Gamma(\mathcal{H})$ is a basic horizontal field, then

$$\widetilde{\mathsf{M}} \cdot \nabla(\pi^*\psi) = (\pi^*\mathsf{M} + \lambda\, \mathrm{pr}_{\mathcal{V}}) \cdot \nabla(\pi^*\psi) = \pi^*(\mathsf{M} \cdot \nabla\psi).$$

On the other side, defining $\bar{\eta} \in C^\infty(S^4 \setminus S^1)$ taking the mean of η on the fibres

$$\bar{\eta}(z, \varphi) := \int_{S^2} \eta(z, \varphi, w, \vartheta)\, d\,\mathrm{vol}_{S^2}(w, \vartheta),$$

we get by the coarea formula that

$$\int_{S^4} \mathbf{g}(\widetilde{\mathsf{M}} \cdot \nabla\tilde{\psi}, \nabla\eta)\, d\,\mathrm{vol}_{\mathbf{g}} = \int_{S^2_+} z^2 g(\mathsf{M} \cdot \nabla\psi, \nabla\bar{\eta})\, d\,\mathrm{vol}_g$$

which is equal to $\int_{S^2_+} z^2 g(\mathsf{F}, \nabla\bar{\eta})\, d\,\mathrm{vol}_g$, giving again by the same argument

$$\int_{S^4} \mathbf{g}(\widetilde{\mathsf{M}} \cdot \nabla\tilde{\psi}, \nabla\eta)\, d\,\mathrm{vol}_{\mathbf{g}} = \int_{S^2_+} z^2 g(\mathsf{F}, \nabla\bar{\eta}) = \int_{S^4} \mathbf{g}(\pi^*\mathsf{F}, \nabla\eta)\, d\,\mathrm{vol}_{\mathbf{g}},$$

as wished. Uniqueness of solutions for (4.41) follows by the uniform ellipticity of $\widetilde{\mathsf{M}}$ as in the proof of Proposition 4.16. □

4.5 Proof of the main result

4.5.1 The system in S^4

The idea now is to solve the problem lifted in S^4, since there the partial differential equation for the velocity is non degenerate and requires only interior estimates. Consequently, our next task is to pullback Equation (4.24) via the bundle projection map $\pi : S^4 \setminus S^1 \to S^2_+$, and check that the equation obtained makes sense in S^4. Recall that we need to avoid the problem of continuously extend the complex endomorphism J to S^4 (see Remark 4.17). Rewrite Equation (4.24) as

$$\begin{cases} (\partial_t + \nabla_{u_0})T_t - zT_t^\perp + \nabla_{v_t}(z^{-1}\nabla p_0) + zv_t = 0, \\ v_t = -\nabla^\perp \phi_t, \\ T_t = z^{-1}\nabla q_t, \\ T_0 = z^{-1}\nabla q_0. \end{cases} \tag{4.43}$$

by applying $(\cdot)^\perp$ to the equation. Now, notice that thanks to Lemma 4.8 we have that

$$-zT_t^\perp = \nabla_{T_t}\nabla^\perp z,$$

$$zv_t = (-zv_t^\perp)^\perp = (\nabla_{v_t^\perp}\nabla z)^\perp = \nabla_{v_t^\perp}\nabla^\perp z,$$

and

$$\nabla_{v_t}(z^{-1}\nabla p_0) + zv_t = \frac{\bar{P}_0''}{z} g(v_t, \nabla z) z\nabla z - (\bar{P}_0' - 1)\nabla_{v_t^\perp}\nabla^\perp z.$$

Lifting on on the horizontal bundle via π^* and recalling the identity (4.6), we obtain

$$(\partial_t + \mathrm{pr}_{\mathcal{H}}\nabla_{\pi^* u_0})\pi^* T_t + \mathrm{pr}_{\mathcal{H}}\nabla_{\pi^*(T_t + (1-\bar{P}_0')\nabla\phi_t)}\pi^*(\nabla^\perp z) + \frac{\bar{P}_0''}{z}\mathbf{g}(\pi^*(\nabla\phi_t), \pi^*(\nabla^\perp z))\pi^*(z\nabla z) = 0. \tag{4.44}$$

Define the lifts $W := \pi^*(\nabla^\perp z)$ and $Z := \pi^*(z\nabla z)$ (recall that they can be extended smoothly in S^4 by Remark 4.17), $S_t := \pi^* T_t$, $\tilde{\phi} := \pi^*\phi_t$, and $U_0 = \pi^* u_0$. Then, substituting $\mathrm{pr}_{\mathcal{H}}\nabla$ with the full covariant derivative on S^4, we end up with the lifted linearized semigeostrophic equation in the form

$$\begin{cases} (\partial_t + \nabla_{U_0})S_t + \nabla W \cdot (S_t + (1-\bar{P}_0')\nabla\tilde{\phi}_t) + \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}_t, W)Z = 0. \\ \mathrm{div}_{\mathbf{g}}(\widetilde{\mathsf{A}} \cdot \nabla\tilde{\phi}_t) = \mathrm{div}_{\mathbf{g}}(\pi^*\mathsf{B} \cdot S_t), \\ \int_{S^4} \tilde{\phi}_t \, d\,\mathrm{vol}_{\mathbf{g}} = 0, \\ S_0 = \pi^* T_0. \end{cases} \tag{4.45}$$

where we coupled the evolution equation of S_t with the lifted partial differential equation for the velocity in the same spirit of Equation (4.41) with $\widetilde{\mathsf{A}} = \pi^*\mathsf{A} + \mathrm{pr}_{\mathcal{V}}$.

4.5.2 Construction of approximate solutions

Fix $k \geq 5$, and let (p_0, u_0) a k-compatible solution and q_0 a k-admissible perturbation as introduced in Definition 4.2. To construct a solution of the lifted linearized semigeostrophic system on S^4, we proceed a follows: for $\varepsilon > 0$ we let S_t evolve continuously and $\nabla\tilde{\phi}$ discretely over small time intervals $[i\varepsilon, (i+1)\varepsilon)$ with $i \in \mathbb{N}_0$. Our goal is to construct two sequences $(S^i_s)_{i\in\mathbb{N}_0} \subset C^1([0,\varepsilon), H^k(S^4))$ and $(\nabla\tilde{\phi}^i)_{i\in\mathbb{N}_0} \subset H^k(S^4)$ iteratively as solutions of the following systems:

$$\begin{cases} \partial_s S^{i+1}_s = -\mathfrak{J}\Big(\nabla_{U_0}(\mathfrak{J}S^{i+1}_s) + \nabla W \cdot (\mathfrak{J}S^{i+1}_s + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}) + \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}^{i+1}, W)Z\Big), \\ S^{i+1}_0 = S^i_\tau, \\ S^0_0 = S_0 = \pi^* T_0, \end{cases} \tag{4.46}$$

and

$$\begin{cases} \operatorname{div}_{\mathbf{g}}(\widetilde{\mathsf{A}} \cdot \nabla\tilde{\phi}^{i+1}) = \operatorname{div}_{\mathbf{g}}(\pi^*\mathsf{B} \cdot \mathfrak{J}S^{i+1}_0), \\ \tilde{\phi}^{i+1} \in H^1(S^4), \\ \int_{S^4} \tilde{\phi}^{i+1}\, d\operatorname{vol}_{\mathbf{g}} = 0, \end{cases} \tag{4.47}$$

Solvability of system (4.46) is ensured by the following proposition, whereas solvability of Equation (4.47) is classical since we suppose $\widetilde{\mathsf{A}}$ elliptic.

Proposition 4.21. *Let $k \geq 1$ be fixed and $X_0 \in H^k(S^4)$, $U \in L^\infty(S^4)$ and $Y, Y' \in L^1(S^4)$ be given vector fields. Then, the following ordinary differential equation*

$$\begin{cases} \partial_s X_s = -\mathfrak{J}\Big(\nabla_U(\mathfrak{J}X_s) + \nabla W \cdot (\mathfrak{J}X_s + Y) + Y'\Big), \\ X|_{s=0} = X_0. \end{cases}$$

admits a unique global solution in $C^1_{loc}([0,+\infty), H^k(S^4))$.

Proof. By the Cauchy-Lipschitz Theorem applied in the Banach space $H^k(S^4)$, we need to prove that the map

$$\mathcal{F} : X \mapsto -\mathfrak{J}\Big(\nabla_U(\mathfrak{J}X_s) + \nabla W \cdot (\mathfrak{J}X_s + Y) + Y'\Big),$$

is globally Lipschitz. Let X, X' be in $H^k(S^4)$. Then, thanks to the properties of the mollifier $\mathfrak{J}$ as listed in Lemma 4.4, we can estimate

$$\begin{aligned} \|\mathcal{F}(X) - \mathcal{F}(X')\|_{H^k(S^4)} &= \|\mathfrak{J}((\nabla_U + \nabla W) \cdot \mathfrak{J}(X - X'))\|_{H^k(S^4)} \\ &\leq C\varepsilon^{-k}\|(\nabla_U + \nabla W) \cdot \mathfrak{J}(X - X')\|_{L^2(S^4)} \\ &\leq C\varepsilon^{-k}(1 + \|U\|_{L^\infty})\|X - X'\|_{H^1(S^4)}, \end{aligned}$$

proving that the map $\mathcal{F}$ is Lipschitz of constant $C\varepsilon^{-k}(1 + \|U\|_{L^\infty(S^4)})$, as wished. □

After existence, we are now ready to perform a energy estimate to obtain a priori uniform bounds.

Proposition 4.22. *Let $k \geq 4$ and $i \in \mathbb{N}_0$. If S_s^{i+1} is a solution of system (4.46) in $C^1([0,+\infty), H^k(S^k))$ with $\nabla\tilde{\phi}^{i+1} \in H^k(S^4)$, then there exists $C = C(k) > 0$ such that*

$$\begin{aligned}&\frac{d}{ds}\|S_s^{i+1}\|_{H^k(S^4)}\\ &\quad\leq C(\|U_0\|_{H^k(S^k)}+1)\|S_s^{i+1}\|_{H^k(S^4)} + C\Big(\|1-\bar{P}_0'\|_{H^k(S^4)} + \Big\|\frac{\bar{P}_0''}{z}\Big\|_{H^k(S^4)}\Big)\|\nabla\tilde{\phi}^{i+1}\|_{H^k(S^4)}.\end{aligned} \tag{4.48}$$

Proof. Let $\{\xi_1,\dots,\xi_5\}$ be a Killing spanning family of S^4 as in Definition 4.5. Let $\alpha \in \mathbb{N}_0^5$ be a multi-index of order $|\alpha| \leq k$, and set $\mathsf{D}^\alpha := \mathcal{L}_{\xi_1}^{\alpha_1} \circ \cdots \circ \mathcal{L}_{\xi_5}^{\alpha_5}$, where the power on $\mathcal{L}$ denotes the repeated application of the operator. Thanks to the commutative properties of the Killing fields with the self-adjoint mollifier $\mathfrak{J}$, we get that

$$\frac{d}{ds}\frac{1}{2}\|\mathsf{D}^\alpha S_s^{i+1}\|_{L^2(S^4)}^2 = -\Big\langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), \mathsf{D}^\alpha\Big(A_1+A_2+A_3\Big)\Big\rangle_{L^2(S^4)}$$

where

$$\begin{aligned}A_1 &= \nabla_{U_0}(\mathfrak{J}S_s^{i+1}),\\ A_2 &= \nabla W\cdot(\mathfrak{J}S_s^{i+1} + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}),\\ A_3 &= \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}^{i+1}, W)Z.\end{aligned}$$

We treat the three terms separately. For A_1 observe that

$$\begin{aligned}&\langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), \mathsf{D}^\alpha(\nabla_{U_0}(\mathfrak{J}S_s^{i+1}))\rangle_{L^2(S^4)}\\ &\quad= \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), [\mathsf{D}^\alpha,\nabla_{U_0}](\mathfrak{J}S_s^{i+1})) + \nabla_{U_0}(\mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}))\rangle_{L^2(S^4)}\\ &\quad= \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), [\mathsf{D}^\alpha,\nabla_{U_0}](\mathfrak{J}S_s^{i+1}))\rangle_{L^2(S^4)} + \frac{1}{2}\int_{S^4}\mathbf{g}(\nabla|\mathsf{D}^\alpha\mathfrak{J}S_s^{i+1}|_{\mathbf{g}}^2, U_0)\,d\,\mathrm{vol}_{\mathbf{g}}\\ &\quad= \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), [\mathsf{D}^\alpha,\nabla_{U_0}](\mathfrak{J}S_s^{i+1}))\rangle_{L^2(S^4)} - \frac{1}{2}\int_{S^4}\mathrm{div}_{\mathbf{g}}(U_0)|\mathsf{D}^\alpha\mathfrak{J}S_s^{i+1}|_{\mathbf{g}}^2\,d\,\mathrm{vol}_{\mathbf{g}}\\ &\quad= \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), [\mathsf{D}^\alpha,\nabla_{U_0}](\mathfrak{J}S_s^{i+1}))\rangle_{L^2(S^4)}\\ &\quad\leq \|\mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1})\|_{L^2(S^4)}\|[\mathsf{D}^\alpha,\nabla_{U_0}](\mathfrak{J}S_s^{i+1}))\|_{L^2(S^4)}\\ &\quad\leq C\|\mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1})\|_{L^2(S^4)}\Big(\|U_0\|_{C^1(S^4)}\|\mathfrak{J}S_s^{i+1}\|_{H^{|\alpha|}(S^4)} + \|U_0\|_{H^{|\alpha|}(S^4)}\|\mathfrak{J}S_s^{i+1}\|_{C^1(S^4)}\Big),\end{aligned}$$

where in the integration by parts we took advantage of $\partial S^4 = \varnothing$, and in the last inequality, after Cauchy-Schwarz we applied the interpolation inequalities of Proposition 4.6. For the second term, applying once again Cauchy-Schwarz and the suitable interpolation inequality gives us that

$$
\begin{aligned}
&\langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), \mathsf{D}^\alpha(\nabla W \cdot (\mathfrak{J}S_s^{i+1} + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}))\rangle_{L^2(S^4)} \\
&\quad = \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), ([\mathsf{D}^\alpha, \nabla W] + \nabla W \circ \mathsf{D}^\alpha)(\mathfrak{J}S_s^{i+1} + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}))\rangle_{L^2(S^4)} \\
&\quad \leq C\|\mathsf{D}^\alpha \mathfrak{J}S_s^{i+1}\|_{L^2(S^4)}\Big(\|\mathfrak{J}S_s^{i+1} + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}\|_{H^{|\alpha|}(S^4)} \\
&\qquad + \|\mathfrak{J}S_s^{i+1} + (1-\bar{P}_0')\nabla\tilde{\phi}^{i+1}\|_{C^1(S^4)}\Big).
\end{aligned}
$$

The last term is also treated similarly

$$
\begin{aligned}
&\langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), \mathsf{D}^\alpha(\frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}^{i+1}, W)Z)\rangle_{L^2(S^4)} \\
&\quad = \langle \mathsf{D}^\alpha(\mathfrak{J}S_s^{i+1}), \Big([\mathsf{D}^\alpha, \frac{\bar{P}_0''}{z}\cdot] + \frac{\bar{P}_0''}{z}\mathsf{D}^\alpha\Big)(\mathbf{g}(\nabla\tilde{\phi}^{i+1}, W)Z)\rangle_{L^2(S^4)} \\
&\quad \leq C\|\mathsf{D}^\alpha \mathfrak{J}S_s^{i+1}\|_{L^2(S^4)}\Big(\Big\|\frac{\bar{P}_0''}{z}\Big\|_{C^1(S^4)}\|\nabla\tilde{\phi}^{i+1}\|_{H^{|\alpha|}(S^4)} + \Big\|\frac{\bar{P}_0''}{z}\Big\|_{H^{|\alpha|}(S^4)}\|\nabla\tilde{\phi}^{i+1}\|_{L^\infty(S^4)}\Big).
\end{aligned}
$$

Summing over all $|\alpha| \leq k$, and recalling that if $k \geq 4$ by Proposition 4.7 $H^k(S^4)$ is a Banach algebra and $H^k(S^4) \hookrightarrow C^1(S^4) \subset L^\infty(S^4)$, we conclude that there exists a constant $C = C(k) > 0$ such that

$$
\begin{aligned}
&\frac{d}{ds}\|S_s^{i+1}\|_{H^k(S^4)} \\
&\quad \leq C(\|U_0\|_{H^k(S^k)} + 1)\|S_s^{i+1}\|_{H^k(S^4)} + C\Big(\|1-\bar{P}_0'\|_{H^k(S^4)} + \Big\|\frac{\bar{P}_0''}{z}\Big\|_{H^k(S^4)}\Big)\|\nabla\tilde{\phi}^{i+1}\|_{H^k(S^4)},
\end{aligned}
$$

as wished. □

To perform a Gronwall argument, we are left to estimate the growth of $\|\nabla\tilde{\phi}^{i+1}\|_{H^k(S^4)}$.

Proposition 4.23. *Let $\tilde{\phi}^{i+1}$ be solution of the elliptic Equation (4.47). Then, for every $k \geq 5$ there exists $C = C(k) > 0$ such that*

$$
\|\nabla\tilde{\phi}^{i+1}\|_{H^k(S^4)} \leq C\Big(\|\widetilde{\mathsf{A}}\|_{H^k(S^4)} + \|\pi^*\mathsf{B}\|_{H^k(S^4)}\Big)^k \lambda_0^{-k}\|S_0^{i+1}\|_{H^k(S^4)}. \tag{4.49}
$$

Proof. We suppose $\|\pi^*\mathsf{B}\|_{H^k}$ and $\|\widetilde{\mathsf{A}}\|_{H^k}$ less than one. The general result follows by rescaling Equation (4.47). Let $\{\xi_1, \ldots, \xi_5\}$ be a Killing spanning family of S^4 as in Definition 4.5. Let $\alpha \in \mathbb{N}_0^5$ be a multi-index of order $0 \leq |\alpha| \leq k$, and set $\mathsf{D}^\alpha := \mathcal{L}_{\xi_1}^{\alpha_1} \circ \cdots \circ \mathcal{L}_{\xi_5}^{\alpha_5}$, where the power on $\mathcal{L}$ denotes the repeated application of the operator. Since the divergence operator is invariant under isometries, we deduce that the Lie derivative along any $\xi_i's$ commutes with it. By the classical theory of linear elliptic equations, we know that $\tilde{\phi}^{i+1} \in C^\infty(S^4)$. Therefore, differetiating Equation (4.47) we get that

$$
\mathrm{div}_{\mathbf{g}}(\mathsf{D}^\alpha(\widetilde{\mathsf{A}} \cdot \nabla\tilde{\phi}^{i+1})) = \mathrm{div}_{\mathbf{g}}(\mathsf{D}^\alpha(\mathsf{B}_\varepsilon \cdot \mathfrak{J}S_0^{i+1})).
$$

When $\alpha = 0$, multiplying this by $\mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1}$ and integrating by parts (here again the fact that S^4 has no boundary simplifies a lot the argument) we obtain that

$$\lambda_0 \|\nabla \tilde{\phi}^{i+1}\|_{L^2(S^4)} \leq \|\widetilde{\mathsf{B}}\|_{L^\infty(S^4)} \|S_0^{i+1}\|_{L^2(S^4)} \leq \|S_0^{i+1}\|_{L^2(S^4)}.$$

More generally, when $|\alpha| \geq 1$ we can estimate as follows

$$\begin{aligned}\lambda_0 \|\mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1}\|^2_{L^2(S^4)} &\leq \langle \widetilde{\mathsf{A}} \cdot \mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1}, \mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1} \rangle_{L^2(S^4)} \\ &= \langle [\widetilde{\mathsf{A}} \cdot, \mathsf{D}^\alpha] \nabla \tilde{\phi}^{i+1}, \mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1} \rangle_{L^2(S^4)} + \langle \mathsf{D}^\alpha (\pi^* \mathsf{B} \cdot \mathfrak{J} S_0^{i+1}), \mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1} \rangle_{L^2(S^4)}.\end{aligned}$$

By Cauchy-Schwarz and interpolation inequalities of Proposition 4.6, we deduce that

$$\begin{aligned}\lambda_0 \|\mathsf{D}^\alpha \nabla \tilde{\phi}^{i+1}\|_{L^2(S^4)} &\leq \|[\widetilde{\mathsf{A}} \cdot, \mathsf{D}^\alpha] \nabla \tilde{\phi}^{i+1}\|_{L^2(S^4)} + \|\mathsf{D}^\alpha (\pi^* \mathsf{B} \cdot \mathfrak{J} S_0^{i+1})\|_{L^2(S^4)} \\ &\leq C \Big(\|\widetilde{\mathsf{A}}\|_{C^1(S^4)} \|\nabla \tilde{\phi}^{i+1}\|_{H^{|\alpha|-1}(S^4)} + \|\widetilde{\mathsf{A}}\|_{H^{|\alpha|}(S^4)} \|\nabla \tilde{\phi}^{i+1}\|_{L^\infty(S^4)} \\ &\quad + \|\pi^* \mathsf{B}\|_{C^1(S^4)} \|\mathfrak{J} S_0^{i+1}\|_{H^{|\alpha|}(S^4)} + \|\pi^* \mathsf{B}\|_{H^{|\alpha|}(S^4)} \|\mathfrak{J} S_0^{i+1}\|_{L^\infty(S^4)} \Big) \\ &\leq C \Big(\|\nabla \tilde{\phi}^{i+1}\|_{H^{|\alpha|-1}(S^4)} + \|\nabla \tilde{\phi}^{i+1}\|_{L^\infty(S^4)} + \|\mathfrak{J} S_0^{i+1}\|_{H^{|\alpha|}(S^4)} + \|\mathfrak{J} S_0^{i+1}\|_{L^\infty(S^4)} \Big).\end{aligned}$$

Summing over $0 \leq |\alpha| \leq k$ we get again by Proposition 4.7 that

$$\lambda_0 \|\nabla \tilde{\phi}^{i+1}\|_{H^k(S^4)} \leq C \Big(\|S_0^{i+1}\|_{H^k(S^4)} + \|\nabla \tilde{\phi}^{i+1}\|_{H^{k-1}(S^4)} \Big).$$

If $\|\nabla \tilde{\phi}^{i+1}\|_{H^{k-1}(S^4)} \leq \|S_0^{i+1}\|_{H^k(S^4)}$ then we are done. Otherwise, from interpolation we get that

$$\begin{aligned}\lambda_0 \|\nabla \tilde{\phi}^{i+1}\|_{H^k(S^4)} &\leq C \|\nabla \tilde{\phi}^{i+1}\|_{H^{k-1}(S^4)} \\ &\leq C \|\nabla \tilde{\phi}^{i+1}\|^{1/k}_{L^2(S^4)} \|\nabla \tilde{\phi}^{i+1}\|^{1-1/k}_{H^k(S^4)},\end{aligned}$$

implying that

$$(\lambda_0)^k \|\nabla \tilde{\phi}^{i+1}\|_{H^k(S^4)} \leq \|\nabla \tilde{\phi}^{i+1}\|_{L^2(S^4)} \leq \|S_0^{i+1}\|_{L^2(S^4)},$$

as wished. □

Proposition 4.24. *Let $k \geq 5$ and $(S_s^i, \tilde{\phi}^i)_{i \in \mathbb{N}}$ be solution of Equation (4.46) and Equation (4.47). Define for every $i \in \mathbb{N}_0$ and $s \in [0, \varepsilon)$ the monotonically increasing Lipschitz function*

$$\Theta_s^i := \sup_{\tau \in [0,s)} \|S_\tau^i\|_{H^k(S^4)}.$$

Then, there exists $C = C(k) > 0$ such that for every $s \in [0, \varepsilon]$

$$\Theta_s^i \leq \Theta_0^i e^{CC_0 s},$$

where $C_0 > 0$ is a constant depending only on the initial data that can be taken equal to

$$C_0 = \|U_0\|_{H^k(S^4)} + 1 + \Big(\|1 - \bar{P}_0'\|_{H^k(S^4)} + \|\bar{P}_0''/z\|_{H^k(S^4)} \Big) \big(\|\widetilde{\mathsf{A}}\|_{H^k(S^4)} + \|\widetilde{\mathsf{B}}\|_{H^k(S^4)} \big)^k \lambda_0^{-k}.$$

Proof. Combine Propostion 4.22 and Propostion 4.23. The result follows from Gronwall Lemma. □

4.5.3 Compactness argument

Define the approximate solutions

$$S_t^\varepsilon \in \mathrm{Lip}_{\mathrm{loc}}([0,+\infty), H^k(S^4))$$

and

$$\nabla\tilde{\phi}_t^\varepsilon \in L^\infty_{\mathrm{loc}}([0,+\infty), H^k(S^4))$$

by gluing over time intervals the sequences obtained in the previous section in the following way

$$S_t^\varepsilon := S_s^i, \text{ and } \nabla\tilde{\phi}_t^\varepsilon = \nabla\tilde{\phi}^i \text{ if } t = i\varepsilon + s \in [i\varepsilon, (i+1)\varepsilon).$$

Similarly, we extend the definition of $\Theta_s^i := \sup_{t\in[0,s)} \|S_\tau^i\|_{H^k(S^4)}$ to the function

$$\Theta_t^\varepsilon := \sup_{\tau\in[0,t)} \|S_\tau^\varepsilon\|_{H^k(S^4)}.$$

By construction and Proposition 4.24 we have that for every $t > 0$ it holds

$$\Theta_t^\varepsilon \le \Theta_0^\varepsilon e^{CC_0 t} = \|\pi^* T_0\|_{H^k(S^4)} e^{CC_0 t}, \tag{4.50}$$

uniformly in ε. We are now ready to prove existence of solution in S^4.

Theorem 4.25. *Let $k \ge 5$ be given, and let (p_0, u_0) be k-compatible and q_0 a k-admissible perturbation, in the sense of Definition 4.2. Then, there exist a unique pair*

$$S_t, \nabla\tilde{\phi}_t \in C_{loc}([0,+\infty), H^k(S^4)) \cap C^1_{loc}([0,+\infty), H^{k-1}(S^4)),$$

solving Equation (4.45).

Proof. In this proof all convergences have to be understood up to extraction of subsequences as $\varepsilon \to 0$. Let $I = [0, t^*]$ be an arbitrarily large time interval. By Proposition 4.23, Proposition 4.24, and Equation (4.50) we know that S_t^ε and $\nabla\tilde{\phi}_t^\varepsilon$ are uniformly bounded in $L^\infty(I, H^k(S^4))$ by some constant $C^* > 0$. In particular, by Banach-Alaoglu

$$S_t^\varepsilon \rightharpoonup S_t, \text{ in } L^2(I, H^k(S^4)),$$

and

$$\nabla\tilde{\phi}_t^\varepsilon \rightharpoonup \nabla\tilde{\phi}_t, \text{ in } L^2(I, H^k(S^4)),$$

for S_t and $\nabla\tilde{\phi}_t$ in $L^\infty(I, H^k(S^4))$. Taking advantage of Equation 4.46 and the fact that $H^k(S^4)$ is a Banach algebra, we see that also $\partial_t S_t^\varepsilon$ is uniformly bounded in $L^\infty(I, H^{k-1}(S^4))$, and therefore $S_t \in \mathrm{Lip}(I, H^{k-1}(S^4))$ and

$$\partial_t S_t^\varepsilon \rightharpoonup \partial_t S_t, \text{ in } L^2(I, H^{k-1}(S^4)).$$

Moreover, since $H^{k-1}(S^4)$ embeds compactly in $H^k(S^4)$ ([59, Theorem 2.9]), and $H^{k-1}(S^4)$ embeds continuously in itself, we infer by Aubin-Lions-Simon Lemma [9, 73, 112] that S_t^ε

converges strongly in $C(I, H^{k-1}(S^4))$. Following the same argument in [110, Theorem 1.3.4], one can check that S_t is a weak solution of Equation (4.45) in duality with $L^2(I, H^{k-1}(S^4))$. In particular, since

$$\operatorname{div}_{\mathbf{g}}(\widetilde{\mathsf{A}} \cdot \nabla\tilde{\phi}_t) = -\operatorname{div}_{\mathbf{g}}(\pi^*\mathsf{B} \cdot S_t),$$

by standard elliptic regularity (like in Proposition 4.23) we infer that $\nabla\tilde{\phi}_t$ inherits the same regularity as S_t, that is

$$\nabla\tilde{\phi}_t \in L^\infty(I, H^k(S^4)) \cap \operatorname{Lip}(I, H^{k-1}(S^4)).$$

We can improve this regularity adapting the argument in [77, Theorem 3.5]. We first show that $S_t \in C_w(I, H^k(S^4))$, the space of weakly continuous maps, that is all $Q : I \to H^k(S^4)$ such that for every $\Phi \in (H^k(S^4))^*$

$$t \mapsto \Phi(Q_t),$$

is continuous. Since $S_t^\varepsilon \to S_t$ in $C(I, H^{k-1}(S^4))$, in particular the same must hold for $C_w(I, H^{k-1}(S^4))$. Let $t \in I$, $\delta > 0$ and Φ as above. By density of $(H^{k-1}(S^4))^*$ in $(H^k(S^4))^*$, there exists $\Phi' \in (H^{k-1}(S^4))^*$ such that $\|\Phi - \Phi'\|_{(H^k(S^4))^*} < \delta/(4C^*)$. Hence,

$$\begin{aligned}|\Phi(S_t) - \Phi(S_s)| = |\Phi(S_t - S_s)| &\le |\Phi'(S_t - S_s)| + \|\Phi - \Phi'\|_{(H^k(S^4))^*}\|S_t - S_s\|_{H^k(S^4)} \\ &\le |\Phi'(S_t - S_s)| + \delta/2 < \delta,\end{aligned}$$

for $s \in I$ close enough to t. To prove that $S_t \in C(I, H^k(S^4))$ we are left to show that the map

$$t \mapsto \|S_t\|_{H^k(S^4)},$$

is continuous. By the weak lower-semi continuity of the norm we have that

$$\liminf_{t\to 0^+} \|S_t\|_{H^k(S^4)} \ge \|S_0\|_{H^k(S^4)},$$

and by Equation (4.50)

$$\begin{aligned}\limsup_{t\to 0^+} \|S_t\|_{H^k(S^4)} &= \inf_{\delta>0} \sup\{\|S_t\|_{H^k(S^4)} : t \in [0, \delta)\} \\ &\le \|S_0\|_{H^k(S^4)} \inf_{\delta>0} e^{CC_0\delta} = \|S_0\|_{H^k(S^4)},\end{aligned}$$

proving that $\lim_{t\to 0^+} \|S_t\|_{H^k(S^4)} = \|S_0\|_{H^k(S^4)}$. System (4.45) is time reversible and, by elementary L^2-energy estimates, admits a unique solution for every initial given data $S_0 \in H^k(S^4)$. We deduce the continuity of the H^k-norm for all time $t \in I$. For more details about this argument, we refer to [77, Theorem 3.5]. Hence, we proved that $S_t, \nabla\tilde{\phi}_t \in C(I, H^k(S^4)) \cap C^1(I, H^{k-1}(S^4))$, as wished. □

4.5.4 Slicing

Now we need to prove that it is possible to go back from the lifted equation to the original problem in S^2_+. It will be convenient to work with special horizontal and vertical vector fields, that we describe in the following lemma.

Lemma 4.26. *Let*

$$\pi : S^4 \setminus S^1 \to S^2_+,$$

be defined as $\pi : (x_1, \dots, x_5) \mapsto (x_1, x_2, z) = (x_1, x_2, \sqrt{x_3^2 + x_4^2 + x_5^2})$*, where*

$$S^4 = \{x \in \mathbb{R}^5 : |x|^2 = x_1^2 + \cdots + x_5^2 = 1\},$$

$$S^1 = \{x \in S^4 : x_1^2 + x_2^2 = 1\} = \{z = 0\} \subset S^4,$$

and

$$S^2_+ = \{(x_1, x_2, z) \in \mathbb{R}^3 : x_1^2 + x_2^2 + z^2 = 1, z > 0\}.$$

Then, the vector fields

$$W = \pi^*(\nabla^\perp z) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},$$
$$Z = \pi^*(z\nabla z) = x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} + x_5 \frac{\partial}{\partial x_5} - z^2 x,$$

are smooth and globally defined on S^4 *and* $\{W_x, Z_x\}$ *form an orthogonal basis of the horizontal distribution* $\mathcal{H}_x$*, provided* $x \notin S^1$*. Moreover,* W *is Killing, the tensor* $\nabla W \in T^{1,1}(S^4)$ *maps vectors from* $T(S^4 \setminus S^1)$ *to* $\mathcal{H}$*, and satisfies the identity*

$$(\nabla W \circ \nabla W)\xi = -z^2 \operatorname{pr}_{\mathcal{H}}(\xi), \quad \forall \xi \in T(S^4 \setminus S^1). \tag{4.51}$$

In particular, $\operatorname{pr}_{\mathcal{H}}(\xi) = 0$ *if and only if* $\nabla W \cdot \xi = 0$ *since*

$$|\nabla W \cdot \xi|_{\mathbf{g}} = |z^2 \operatorname{pr}_{\mathcal{H}}(\xi)|_{\mathbf{g}}. \tag{4.52}$$

Finally, at every $y \in S^4 \setminus S^1$ *there exist two globally defined vertical Killing vector fields* V_1 *and* V_2 *so that* $\{(V_1)_y, (V_2)_y\}$ *spans* $\mathcal{V}_y$*, and such that* $\nabla_W V_1 = \nabla_W V_2 = 0$*, and* ∇V_i *maps vectors from* $T(S^4 \setminus S^2)$ *to* $\mathcal{V}$*,* $i = 1, 2$*.*

Proof. It is convenient to start by defining V_1 and V_2 according to y. Notice that $y \in S^4 \setminus S^1$ if and only if y_3, y_4, y_5 are not simultaneously equal to zero. We define the vector fields $\{V_1, V_2\}$ as

$$\begin{cases} V_1 = x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4},\ V_2 = x_3 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_3}, & \text{if } y_3 \neq 0, \\ V_1 = x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4},\ V_2 = x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4}, & \text{if } y_4 \neq 0, \\ V_1 = x_3 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_3},\ V_2 = x_4 \frac{\partial}{\partial x_5} - x_5 \frac{\partial}{\partial x_4}, & \text{if } y_5 \neq 0. \end{cases}$$

We notice that V_1 and V_2 are always Killing since they represent a rotation of S^4, and they are vertical since

$$d\pi \cdot V_i = z^{-1}(0, 0, x_3, x_4, x_5) \cdot V_i = 0.$$

Moreover, $\{(V_1)_y, (V_2)_y\}$ are linearly independent and not trivial, and therefore form a base of $\mathcal{V}_y$. By the explicit form of the covariant derivative on S^4 one has that $\nabla_{V_i} W = \nabla_W V_i = 0$ for $i = 1, 2$, proving in particular that ∇W maps general vectors on horizontal vectors. To prove Equation (4.51) it is sufficient to compute $\nabla_W W = Z$ and $\nabla_Z W = -z^2 W$. Since W is itself Killing, we prove Equation (4.52) as follows:

$$\begin{aligned} |\nabla W \cdot \xi|^2_{\mathbf{g}} &= \mathbf{g}(\nabla W \cdot \xi, \nabla W \cdot \xi) = -\mathbf{g}((\nabla W \circ \nabla W) \cdot \xi, \xi) \\ &= \mathbf{g}(z^2 \operatorname{pr}_{\mathcal{H}}(\xi), \xi) = |z^2 \operatorname{pr}_{\mathcal{H}}(\xi)|^2_{\mathbf{g}}. \end{aligned}$$

To prove that ∇V_i sends general vectors to vertical vectors we are left to compute $\nabla V_i \cdot Z$ since we already know that $\nabla V_i \cdot W = 0$. This can be achieved by taking advantage of the Killing nature of V_i, by noticing that

$$\mathbf{g}(\nabla V_i \cdot Z, W) = -\mathbf{g}(Z, \nabla V_i \cdot W) = 0,$$

and

$$\mathbf{g}(\nabla V_i \cdot Z, Z) = -\mathbf{g}(Z, \nabla V_i \cdot Z) \Rightarrow \mathbf{g}(\nabla V_i \cdot Z, Z) = 0,$$

proving that $\nabla V_i \cdot Z$ is orthogonal to $\mathcal{H}$, and therefore vertical. □

Proposition 4.27. *For $k \geq 3$ let*

$$S_t \in C^1([0, +\infty), H^k(S^4)),$$

and

$$\nabla \tilde{\phi}_t \in L^\infty_{loc}([0, \infty), H^k(S^4)),$$

be solution of the lifted linearized semigeostrophic Equation (4.45). If $S_0(x) \in \mathcal{H}$ for all $x \in S^4 \setminus S^1$, then $S_t(x) \in \mathcal{H}$ for all $t > 0$ and $x \in S^4 \setminus S^1$.

Proof. Let $\{V_1, V_2\}$ like in Lemma 4.26, and define $h^i_t := \mathbf{g}(S_t, V_i)$, for $i = 1, 2$. Testing (4.45) against V_i we get that

$$\begin{aligned} 0 &= \partial_t h^i_t + \mathbf{g}(\nabla_{U_0} S_t, V_i) + \mathbf{g}(\nabla W \cdot (S_t + (1 - \bar{P}_0')\nabla\tilde{\phi}_t, V_i) + \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}_t, W)\mathbf{g}(Z, V_i) \\ &= \partial_t h^i_t + \mathbf{g}(U_0, \nabla h^i_t) - \mathbf{g}(S_t, \nabla_{U_0} V_i) = \partial_t h^i_t + \mathbf{g}(U_0, \nabla h^i_t) - \Psi_0' \mathbf{g}(S_t, \nabla_W V_i) \\ &= \partial_t h^i_t + \mathbf{g}(U_0, \nabla h^i_t), \end{aligned}$$

proving that the projection h^i_t is transported along U_0. We conclude that if h^1_0 and h^2_0 are identically zero, the same must hold also for $t > 0$, proving the claim by smoothness of S_t. □

Now that we know that S_t is an horizontal vector field, we need to prove that it is actually a basic one, that is there exists $T_t \in \Gamma(TS^2_+)$ such that $S_t = \pi^* T_t$. This could be achieved by showing that $\operatorname{pr}_{\mathcal{H}}(\mathcal{L}_V S_t) = 0$ for any $V \in \mathcal{V}$ and $t \geq 0$. We argue by first showing that if $(S_t, \nabla\tilde{\phi}_t)$ solves Equations (4.45), then for every Killing vector field $V \in \Gamma(\mathcal{V})$ the pair $(\mathcal{L}_V S_t, \mathcal{L}_V \nabla\tilde{\phi}_t)$ solves the same equations up to vertical terms. The argument is completed by performing an L^2-energy estimate on the vector field $(\nabla W \cdot \mathcal{L}_V S_t)$.

Proposition 4.28. *For $k \geq 4$ let*

$$S_t \in C^1([0,+\infty), H^k(S^4)),$$

and

$$\nabla\tilde{\phi}_t \in L^\infty_{loc}([0,\infty), H^k(S^4)),$$

be solution of the lifted linearized semigeostrophic Equation (4.45), and suppose that $S_t(x) \in \mathcal{H}_x$ for every $x \in S^4 \setminus S^1$. Let V be a vertical Killing vector field as defined in Lemma 4.26. Then, the pair $(S_t^V, \tilde{\phi}_t^V) := (\mathcal{L}_V S_t, \mathcal{L}_V\tilde{\phi}_t)$ is also a solution of (4.45) up to vertical terms, that is there exists $V_t' \in L^\infty_{loc}([0,\infty), H^{k-1}(S^4))$ vertical such that

$$\begin{cases}(\partial_t + \nabla_{U_0})S_t^V + \nabla W \cdot (S_t^V + (1-\bar{P}_0')\nabla\tilde{\phi}_t^V) + \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}_t^V, W)Z = V_t', \\ \mathrm{div}_{\mathbf{g}}(\widetilde{\mathsf{A}}\cdot\nabla\tilde{\phi}_t^V) = -\mathrm{div}_{\mathbf{g}}(\pi^*\mathsf{B}\cdot S_t^V), \\ \int_{S^4}\tilde{\phi}_t^V\, d\,\mathrm{vol}_{\mathbf{g}} = 0.\end{cases} \tag{4.53}$$

As a consequence, the following estimate holds for every $t \geq 0$:

$$\|\nabla W\cdot\mathcal{L}_V S_t\|_{L^2(S^4)} \leq \|\nabla W\cdot\mathcal{L}_V S_0\|_{L^2(S^4)}e^{C_0 t}, \tag{4.54}$$

for some $C_0 > 0$ depending only on the initial data Ψ_0 and $\bar{P}_0$.

Proof. We start by showing that $\tilde{\phi}_t^V$ solves the elliptic equation in (4.45) replacing S_t with S_t^V. Since V is Killing we have that $\mathcal{L}_V\nabla\tilde{\phi}_t = \nabla\mathcal{L}_V\tilde{\phi}_t$, because for every vector ξ the following holds:

$$\begin{aligned}\mathbf{g}(\mathcal{L}_V\nabla\tilde{\phi}_t, \xi) &= V(\mathbf{g}(\nabla\tilde{\phi}_t, \xi)) - \mathbf{g}(\nabla\tilde{\phi}_t, \mathcal{L}_V\xi) = V(d\tilde{\phi}_t(\xi)) - d\tilde{\phi}_t(\mathcal{L}_V\xi) \\ &= (\mathcal{L}_V d\tilde{\phi}_t)(\xi) = (d\mathcal{L}_V\tilde{\phi}_t)(\xi) = \mathbf{g}(\nabla(\mathcal{L}_V\tilde{\phi}_t), \xi),\end{aligned}$$

where we took advantage of the commutativity property of the Lie derivative with the exterior derivative. Moreover, from $\mathrm{div}_{\mathbf{g}}(V) = 0$ we get that

$$\int_{S^4}\tilde{\phi}_t^V\, d\,\mathrm{vol}_g = \int_{S^4}\mathcal{L}_V\tilde{\phi}_t\, d\,\mathrm{vol}_g = \int_{S^4}\mathbf{g}(\nabla\tilde{\phi}_t, V)\, d\,\mathrm{vol}_g = \int_{S^4}\tilde{\phi}_t\mathrm{div}_{\mathbf{g}}(V)\, d\,\mathrm{vol}_{\mathbf{g}} = 0.$$

Since $[\mathcal{L}_V, \mathrm{div}_{\mathbf{g}}] = 0$ we are left to prove that

$$[\mathcal{L}_V, \widetilde{\mathsf{A}}\cdot] = [\mathcal{L}_V, \pi^*\mathsf{B}\cdot] = 0.$$

Explicitly for a vector field ξ one has that

$$\mathcal{L}_V(\widetilde{\mathsf{A}}\cdot\xi) = \mathcal{L}_V\Big(\xi - \bar{P}_0'\,\mathrm{pr}_{\mathcal{H}}(\xi) + \frac{\bar{P}_0''}{z}\mathbf{g}(\xi, W)W\Big)$$

Since $\mathcal{L}_V W = \mathcal{L}_V Z = 0$ and z is constant along V, we have that

$$\mathcal{L}_V(\widetilde{\mathsf{A}}\cdot\xi) = \mathcal{L}_V\xi - \bar{P}_0'\,\mathrm{pr}_{\mathcal{H}}(\mathcal{L}_V\xi) + \frac{\bar{P}_0''}{z}\mathbf{g}(\mathcal{L}_V\xi, W)W = \widetilde{\mathsf{A}}\cdot\mathcal{L}_V\xi,$$

as wished. The same holds for $\pi^*\mathsf{B}$ since it shares a similar structure with $\widetilde{\mathsf{A}}$. Let us focus now on the equation for S_t. Thanks to Proposition 4.27 we know that S_t must be horizontal. In the following chains of identities we will symbolically add $\mathcal{V}$ when the equation holds up to vertical terms. We start by noticing that $\nabla V : \mathcal{H} \to \mathcal{V}$, and hence

$$\begin{aligned}
\mathcal{L}_V \nabla_W S_t &= [V, \nabla_W S_t] = \nabla_V \nabla_W S_t - \nabla V \cdot \nabla_W S_t = \nabla_V \nabla_W S_t + \mathcal{V} \\
&= \nabla_W \nabla_V S_t + R(W,V)S_t + \nabla_{[V,W]} S_t + \mathcal{V} = \nabla_W \nabla_V S_t + R(W,V)S_t + \mathcal{V} \\
&= \nabla_W \nabla_V S_t + \mathbf{g}(W, S_t)V - \mathbf{g}(V, S_t)W + \mathcal{V} \\
&= \nabla_W \nabla_V S_t + \mathcal{V},
\end{aligned}$$

where we used that ∇ is torsion free and hence $[V,W] = \nabla_V W - \nabla_W V = 0$. Notice that from

$$\nabla_W(\nabla V \cdot S_t) = 0 + \mathcal{V},$$

we get that

$$\mathcal{L}_V \nabla_W S_t = \nabla_W(\nabla_V S_t - \nabla_{S_t} V) + \mathcal{V} = \nabla_W \mathcal{L}_V S_t + \mathcal{V}.$$

Since Ψ_0' is constant along the integral lines of V, we conclude that

$$\mathcal{L}_V(\nabla_{U_0} S_t) = \Psi_0' \mathcal{L}_V(\nabla_W S_t) = \Psi_0' \nabla_W(\mathcal{L}_V S_t) + \mathcal{V} = \nabla_{U_0} S_t^V + \mathcal{V}.$$

In a similar way the term $\mathcal{L}_V(\nabla W \cdot S_t)$ can be treated as follows:

$$\begin{aligned}
\mathcal{L}_V(\nabla W \cdot S_t) &= [V, \nabla_{S_t} W] = \nabla_V \nabla_{S_t} W - \nabla V \cdot \nabla_{S_t} W = \nabla_V \nabla_{S_t} W + \mathcal{V} \\
&= \nabla_{S_t} \nabla_V W + R(S_t, V)W + \nabla_{[V,S_t]} W + \mathcal{V} \\
&= \nabla_{[V,S_t]} W + \mathbf{g}(S_t, W)V - \mathbf{g}(V, W)S_t + \mathcal{V} \\
&= \nabla_{[V,S_t]} W + \mathcal{V} \\
&= \nabla W \cdot \mathcal{L}_V S_t + \mathcal{V} = \nabla W \cdot S_t^V + \mathcal{V}.
\end{aligned}$$

It suffices now to apply $\mathcal{L}_V$ to Equation (4.45) and take advantage of the latter commutative properties up to vertical components to obtain Equation (4.53). The associated energy estimate goes as follows:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|\nabla W \cdot S_t^V\|^2_{L^2(S^4)} = \langle \nabla W \cdot S_t^V, \nabla W \cdot \partial_t S_t^V \rangle_{L^2(S^4)} \\
&\quad = -\left\langle \nabla W \cdot S_t^V, \nabla W \cdot \left(\nabla_{U_0} S_t^V + \nabla W \cdot \left(S_t^V + (1 - \bar{P}_0') \nabla \tilde{\phi}_t^V \right) + \frac{\bar{P}_0''}{z} \mathbf{g}(\nabla \tilde{\phi}_t^V, W) Z \right) \right\rangle_{L^2(S^4)}.
\end{aligned}$$

In particular

$$\begin{aligned}
\langle \nabla W \cdot S_t^V, \nabla W \cdot \nabla_{U_0} S_t^V \rangle_{L^2(S^4)} &= \langle \nabla W \cdot S_t^V, \nabla_{U_0}(\nabla W \cdot S_t^V) \rangle_{L^2(S^4)} \\
&= \int_{S^4} \frac{|S_t^V|^2_{\mathbf{g}}}{2} \mathrm{div}_{\mathbf{g}}(U_0)\, d\,\mathrm{vol}_g = 0,
\end{aligned}$$

and

$$\langle \nabla W \cdot S_t^V, \nabla W \cdot (\nabla W \cdot S_t^V) \rangle_{L^2(S^4)} = -\langle \nabla W \cdot (\nabla W \cdot S_t^V), \nabla W \cdot S_t^V \rangle_{L^2(S^4)},$$

since W is itself Killing. From the identity $\nabla W \circ \nabla W = -z^2 \operatorname{pr}_{\mathcal{H}}$ and the elliptic estimate

$$\begin{aligned}\|\nabla \tilde{\phi}_t^V\|_{L^2(S^4)} &\leq \lambda_0^{-1} \|\pi^* B \cdot S_t^V\|_{L^2(S^4)} \leq \lambda_0^{-1} \|\pi^* B\|_{L^\infty(S^4)} \|\operatorname{pr}_{\mathcal{H}}(S_t^V)\|_{L^2(S^4)} \\ &= \lambda_0^{-1} \|\pi^* B\|_{L^\infty(S^4)} \|\nabla W \cdot S_t^V\|_{L^2(S^4)}\end{aligned}$$

we finally obtain that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\nabla W \cdot S_t^V\|^2_{L^2(S^4)} &= -\left\langle \nabla W \cdot S_t^V, \nabla W \cdot \left(\nabla W \cdot \left((1-\bar{P}_0')\nabla\tilde{\phi}_t^V\right) + \frac{\bar{P}_0''}{z}\mathbf{g}(\nabla\tilde{\phi}_t^V, W)Z\right)\right\rangle_{L^2(S^4)} \\ &= \left\langle \nabla W \cdot S_t^V, z^2(\pi^* \mathsf{A} \cdot \nabla \tilde{\phi}_t^V)\right\rangle_{L^2(S^4)} = \left\langle \nabla W \cdot S_t^V, z^2(\widetilde{\mathsf{A}} \cdot \nabla \tilde{\phi}_t^V)\right\rangle_{L^2(S^4)} \\ &\leq \|\nabla W \cdot S_t^V\|_{L^2(S^4)} \|z^2 \widetilde{\mathsf{A}}\|_{L^\infty(S^4)} \|\nabla \tilde{\phi}_t^V\|_{L^2(S^4)} \\ &\leq \|\nabla W \cdot S_t^V\|_{L^2(S^4)} \|z^2 \widetilde{\mathsf{A}}\|_{L^\infty(S^4)} \lambda_0^{-1} \|\pi^* B\|_{L^\infty(S^4)} \|\nabla W \cdot S_t^V\|_{L^2(S^4)},\end{aligned}$$

proving Equation (4.54) via Gronwall Lemma. □

Proof of Theorem 4.1. Let

$$S_t, \nabla\tilde{\phi}_t \in C_{\mathrm{loc}}([0,+\infty), H^k(S^4)) \cap C^1_{\mathrm{loc}}([0,+\infty), H^{k-1}(S^4)),$$

be the solution constructed in Theorem 4.25. Combining Proposition 4.27 and Proposition 4.28 we know that S_t is a basic horizontal vector field for all $t > 0$, since by Equation 4.54 we have that for all Killing vertical vector field V the following holds

$$\begin{aligned}\|\nabla W \cdot \mathcal{L}_V S_t\|_{L^2(S^4)} &= \|\nabla W \cdot \mathcal{L}_V S_0\|_{L^2(S^4)} + \int_0^t \frac{d}{d\tau}\|\nabla W \cdot \mathcal{L}_V S_\tau\|_{L^2(S^4)}\, d\tau \\ &\leq \|\nabla W \cdot \mathcal{L}_V S_0\|_{L^2(S^4)} + \int_0^t \|\nabla W \cdot \mathcal{L}_V S_0\|_{L^2(S^4)} e^{C_0 \tau}\, d\tau = 0,\end{aligned}$$

because $\mathcal{L}_V S_0 = \mathcal{L}_V(\pi^* T_0)$ is purely vertical. Let $T_t \in \Gamma(TS^2_+)$ be such that $S_t = \pi^* T_t$. Consequently, Proposition 4.20 implies that $\tilde{\phi}_t$ has to be a lift of a function ϕ_t defined on S^2_+. By the classical Sobolev embeddings of Proposition 4.7 we have that

$$S_t, \nabla\tilde{\phi}_t \in C_{\mathrm{loc}}([0,+\infty), C^{k-3,\alpha}(S^4)) \cap C^1_{\mathrm{loc}}([0,+\infty), C^{k-4,\alpha}(S^4)),$$

for any $\alpha \in [0,1)$. Consequently, thanks to Proposition 4.20 we obtain

$$T_t, \nabla\phi_t \in C_{\mathrm{loc}}([0,+\infty), C^{k-3,\alpha}_{\mathrm{loc}}(S^2_+)) \cap C^1_{\mathrm{loc}}([0,+\infty), C^{k-4,\alpha}_{\mathrm{loc}}(S^2_+)) \cap C^1_{\mathrm{loc}}([0,+\infty), H^1_2(S^2_+)).$$

The pair (T_t, ϕ_t) solves Equation (4.43) since

$$\nabla_{U_0} S_t = \nabla_{\pi^* u_0} \pi^* T_t = \operatorname{pr}_{\mathcal{H}}(\nabla_{\pi^* u_0} \pi^* T_t) = \pi^*(\nabla_{u_0} T_t),$$

taking advantage of the fact that $\nabla_W : \mathcal{H} \to \mathcal{H}$. We prove that T_t is in the form $T_t = z^{-1}\nabla q_t$ for some potential q_t. To do so, we define $\omega_t := \mathrm{div}_g(zT^\perp)$, apply $\mathrm{div}_{\mathbf{g}}(zJ\cdot)$ to the equation and obtain

$$\partial_t\omega_t + \mathrm{div}_g(\mathsf{A}\cdot\nabla\tilde{\phi}_t) + \mathrm{div}_g(z\nabla_{u_0}T_t^\perp) + \mathrm{div}_g(z^2T_t) = 0, \tag{4.55}$$

that becomes taking advantage of the equation solved by $\nabla\tilde{\phi}_t$:

$$\partial_t\omega_t + \mathrm{div}_g(zJ\circ\mathrm{Hess}(\psi_0)\circ J\cdot T_t) + \mathrm{div}_g(z\nabla_{u_0}T_t^\perp) = 0, \tag{4.56}$$

which thanks to Equation (4.27) gives us

$$\partial_t\omega_t + \mathrm{div}_g(\omega_t u_0) = 0, \tag{4.57}$$

showing that $\omega_t \equiv 0$ for all $t > 0$, proving that there exists a potential q_t satisfying $\nabla^\perp q_t = zT_t^\perp$, as wished. Last step is uniqueness. We perform an Energy estimate: suppose that (T'_t, v'_t) is another solution of (4.24) in in the same class as (T_t, v_t), then

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|T_t - T'_t\|^2_{L^2_2(S^2_+)} \\
&\quad = -\int_{S^2_+}\langle T_t - T'_t, \nabla_{u_0}(T_t - T'_t) - z(T_t - T'_t)^\perp + \nabla_{v_t - v'_t}(z^{-1}\nabla p_0) + z(v_t - v'_t)\rangle z^2\, d\,\mathrm{vol}_g \\
&\quad = -\frac{1}{2}\int_{S^2_+} u_0\Big(z^2|T - T'|^2_g\Big)\, d\,\mathrm{vol}_g - \int_{S^2_+}\langle T_t - T'_t, \nabla_{v_t - v'_t}(z^{-1}\nabla p_0) + z(v_t - v'_t)\rangle z^2\, d\,\mathrm{vol}_g \\
&\quad \le \|T_t - T'_t\|_{L^2_2(S^2_+)}\|\nabla_{v_t - v'_t}(z^{-1}\nabla p_0) + z(v_t - v'_t)\|_{L^2_2(S^2_+)} \\
&\quad \le C\|T_t - T'_t\|_{L^2_2(S^2_+)}\|v_t - v'_t\|_{L^2(S^2_+)} \\
&\quad \le C\|T_t - T'_t\|^2_{L^2_2(S^2_+)},
\end{aligned}$$

for some $C > 0$ depending only on the initial data (p_0, u_0). The last inequality follows from the equation solved by the difference of the potentials for v_t and v'_t and Equation (4.34). By Gronwall, we conclude that if $T_0 = T'_0$, then the same must hold almost everywhere for all $t > 0$. Uniqueness follows from the additional regularity of T_t and T'_t. □

Appendix A

❊

A.1 Existence, boundedness and mean convexity

The objective of this section is to establish the existence, boundedness, and mean convexity of the isoperimetric sets in the hyperbolic space $H^n_{\mathbb{R}}$ when equipped with a radial density function $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$. Expressing

$$\ln(f(x)) = h(d_H(o, x))$$

for some $h : \mathbb{R} \to \mathbb{R}$, it will be sufficient to assume h lower-semicontinuous and divergent to infinity to ensure existence, and non-decreasing to ensure boundedness. We will take advantage of the log-convexity to establish the mean-convexity of the isoperimetric sets. The proof is a direct application of the arguments employed by Morgan and Pratelli in the flat case [86, Theorem 3.3, Theorem 4.3, Theorem 5.9, Theorem 6.5]. We recall that we work in the Poincaré model, that makes $H^n_{\mathbb{R}}$ conformal to the unit ball in $\mathbb{R}^n$. The metric at a point $x \in H^n_{\mathbb{R}}$ is given by

$$g_H = \frac{4}{(1 - r^2)^2} g_{\text{flat}},$$

where $r = |x|$ will always denote the Euclidean distance of x from the origin. In this coordinate system, the hyperbolic distance from the origin is given by

$$d_H(x, 0) = 2 \operatorname{artanh}(r).$$

We will denote with $\tilde{f}(r) := \exp(h(2 \operatorname{artanh}(r)))$ the profile of the radial weight in Poincaré coordinates. Then, one can check that for $k \in \{n - 1, n\}$ the k-dimensional weighted Hausdorff measures associated to g_H and f can be expressed as

$$d\mathscr{H}^k_f := f \, d\mathscr{H}^k = \tilde{f}(r) \Big(\frac{2}{1 - r^2} \Big)^k d\mathscr{H}^k_{\text{flat}}, \tag{A.1}$$

where we denote with the flat index the Hausdorff measures associated to g_{flat} in the Poincaré model. To simplify the exposition, let us define the function

$$\omega(r) := \frac{2}{1 - r^2}.$$

Finally we will denote with $B(r)$ the ball centered at the origin in $H^n_{\mathbb{R}}$ with Euclidean radius $r \in (0,1)$, and with $S^{n-1}(r)$ its boundary.

We start by proving that the isoperimetric profile is monotone. This step will be important to show boundedness later on.

Theorem A.1 (Monotonicity of the isoperimetric profile)**.** *Let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a lower-semicontinuous non-decreasing function and let $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$ be defined through $f(x) = \exp(h(d_H(x,o)))$ for some base point $o \in H^n_{\mathbb{R}}$. Then, the isoperimetric profile $\mathfrak{J}$ defined in* (2.1) *as*

$$\mathfrak{J}(v) := \inf\Big\{P_f(F) : V_f(F) = v, F \subset H^n_{\mathbb{R}} \text{ of finite perimeter}\Big\}$$

is non-decreasing in $v \in [0,+\infty)$. Moreover, $\mathfrak{J}$ is strictly increasing if there exist isoperimetric sets for all volumes.

Proof. Let E be any set of finite perimeter with finite volume $V_f(E) = v$. We claim that for all $r > 0$ such that $E(r) := E \cap B(r) \subsetneq E$ one has that

$$P_f(E(r)) < P_f(E). \tag{A.2}$$

If Equation (A.2) holds, then it suffices to notice that for every $0 < v' < v$ there exists $r' \in (0,1)$ such that

$$V_f(E(r')) = v',$$

which implies that

$$\mathfrak{J}(v') \leq P_f(E(r')) < P_f(E).$$

If E is isoperimetric, then we have immediately that $\mathfrak{J}(v') < \mathfrak{J}(v)$ for all $0 < v < v'$. Otherwise, for every $\varepsilon > 0$ let E_ε be a set of finite perimeter such that $V_f(E) = v$ and $P_f(E_\varepsilon) \leq \mathfrak{J}(v) + \varepsilon$. From the inequality

$$\mathfrak{J}(v') < P_f(E) \leq \mathfrak{J}(v) + \varepsilon$$

we infer that $\mathfrak{J}(v) \leq \mathfrak{J}(v')$ for all $0 < v < v'$. We are left to prove Equation (A.2). Let $\pi : \partial E \setminus B(r) \to S^{n-1}(r)$ be the normal projection on the sphere of radius r. Notice that π is strictly 1-Lipschitz with respect to the Euclidean distance. Then,

$$\pi(\partial E \setminus B(r)) \supseteq \partial E(r) \setminus \partial E. \tag{A.3}$$

In fact, the set E contains the (possibly empty) cone

$$C = \{\lambda x : \lambda \in [1, r^{-1}), x \in H\},$$

where $H = (\partial E(r) \setminus \partial E) \setminus \pi(\partial E \setminus B(r))$, and the dilation λx is to be understood with respect to the Euclidean structure in the Poincaré model. Since the density is non-decreasing, it follows that $V_f(C) = +\infty$ unless $\mathscr{H}^{n-1}_{\text{flat}}(H) = 0$. By assumption, the volume of E is finite, and therefore Equation (A.3) must hold up to a set of measure zero. By the coarea formula (see for instance [76, Chapter 13]) we finally get that

$$\begin{aligned} P_f(E(r)) &= \int_{\partial E \cap B(r)} \tilde{f}(r)\omega(r)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} + \int_{\partial E(r)\setminus \partial E} \tilde{f}(r)\omega(r)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} \\ &\leq \int_{\partial E \cap B(r)} \tilde{f}(r)\omega(r)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} + \int_{\pi(\partial E\setminus \partial B(r))} \tilde{f}(r)\omega(r)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} \\ &< \int_{\partial E \cap B(r)} \tilde{f}(r)\omega(r)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} + \int_{\partial E\setminus \partial B(r)} \tilde{f}(|\pi(x)|)\omega(|\pi(x)|)^{n-1}\, d\mathscr{H}^{n-1}_{\text{flat}} \\ &\leq P_f(E), \end{aligned}$$

where $\tilde{f}(r) = \exp(h(2\operatorname{artanh}(r)))$. □

We are now ready to establish existence.

Theorem A.2 (Existence of isoperimetric sets)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be a lower-semicontinuous function that diverges to infinity and let $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$ be defined through $f(x) = \exp(h(d_H(x, o)))$ for some base point $o \in H^n_{\mathbb{R}}$. Then, for all volumes there exists a set attaining the isoperimetric infimum in Equation (2.1).*

Proof. Fix $v > 0$ and let $(E_j)_{j\geq 1} \subset H^n_{\mathbb{R}}$ be a sequence of smooth sets of weighted volume v whose perimeter converges to the infimum of Equation (2.1). Without loss of generality, we can suppose $P_f(E_j) < \mathfrak{J}(v) + 1$. Intersecting this sequence with balls of growing radii $r_j \to 1$, the sequence splits into

$$E_j = (E_j \cap B(r_j)) \cup (E_j \setminus B(r_j)) = E_j^C \cup E_j^D.$$

Up to taking a subsequence, a standard argument of compactness (see [56, Theorem 1.19] and [84, Theorem 13.4]) shows that E_j^C converges to an isoperimetric set, whose volume is equal to v if and only if there is no volume escaping to infinity, that is

$$\lim_{R\to 1} \limsup_{j\to+\infty} V_f(E_j \setminus B(R)) = 0.$$

To establish our argument, we will proceed by contradiction. Let us assume that, after selecting a subsequence if necessary, there exists a positive value $\varepsilon > 0$ such that for every $R > 0$, there exists an index $j = j(R)$ satisfying the inequality

$$V_f(E_j \setminus B(R)) \geq \varepsilon. \tag{A.4}$$

Fix $0 < R < 1$ a number very close to 1 yet to define, and $j = j(R)$. Thanks to (A.1) we can rewrite Equation (A.4) as

$$\int_R^1 \omega(r)^n \tilde{f}(r) S_j(r)\, dr \geq \varepsilon, \tag{A.5}$$

where $S_j(r) := \mathscr{H}^{n-1}_{\text{flat}}(\partial E \cap S^{n-1}(r))$ and $\tilde{f}(r) = \exp(h(2\,\text{artanh}(r)))$. Then, denoting $M_j(R) := \sup_{r\in[R,1)} \omega(r)^{n-1} S_j(r)$ and $m(R) = \inf_{r\in[R,1)} \tilde{f}(r)$ we gave that

$$P_f(E_j) \geq M_j(R) m(R). \tag{A.6}$$

In particular, since $P_f(E_j)$ is uniformly bounded, up to taking R close enough to 1, we can suppose $m(R)$ large enough, so that

$$S_j(r) \leq \frac{\mathscr{H}^{n-1}_{\text{flat}}(S^{n-1}(r))}{2}$$

for all $r \in [R, 1)$. By the classical isoperimetric inequality on the sphere, there exists a dimensional constant $c_n > 0$ such that

$$\mathscr{H}^{n-2}_{\text{flat}}(\partial(E_j \cap S^{n-1}(r))) \geq c_n S_j(r)^{\frac{n-2}{n-1}},$$

for all $r \in [R, 1)$. By Vol'pert theorem (see [6, Theorem 3.108]), for almost every $r \in (0, 1)$ one has that

$$\partial(E_j \cap S^{n-1}(r))) = \partial E_j \cap S^{n-1}(r).$$

Therefore, the coarea formula (see for instance [76, Chapter 13]) allows us to obtain the following estimate on the weighted perimeter:

$$\begin{aligned}
P_f(E_j) &\geq \int_R^1 \omega(r)^{n-1} \tilde{f}(r) \mathscr{H}^{n-2}_{\text{flat}}(\partial E_j \cap S^{n-1}(r))\, dr \\
&= \int_R^1 \omega(r)^{n-1} \tilde{f}(r) \mathscr{H}^{n-2}_{\text{flat}}(\partial(E_j \cap S^{n-1}(r)))\, dr \\
&\geq c_n \int_R^1 \omega(r)^{n-1} \tilde{f}(r) S_j(r)^{\frac{n-2}{n-1}}\, dr \\
&\geq c_n \int_R^1 \omega(r)^{n} \tilde{f}(r) S_j(r) (S_j(R)\omega(r)^{n-1})^{-\frac{1}{n-1}}\, dr \\
&\geq c_n M_j^{-\frac{1}{n-1}} \varepsilon,
\end{aligned}$$

where in the last line we used assumption (A.5). On the other hand, thanks to (A.6) we get that

$$(\mathfrak{I}(v) + 1)^{\frac{n}{n-1}} \geq P_f(E_j) P_f(E_j)^{\frac{1}{n-1}} \geq c_n \varepsilon m(R)^{\frac{1}{n-1}}.$$

But this is impossible because $m(R)$ diverges to infinity as $R \to 1$. □

After existence, we prove boundedness of the isoperimetric set.

Theorem A.3 (Boundedness of the isoperimetric sets)**.** *Let $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a lower-semicontinuous non-decreasing function and let $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$ be defined through $f(x) = \exp(h(d_H(x, o)))$ for some base point $o \in H^n_{\mathbb{R}}$. Then, every set attaining the isoperimetric infimum in Equation (2.1) is bounded.*

Proof. We proceed by contradiction. Let E be an unbounded isoperimetric set. Let $r \in (0,1)$ be close enough to 1 so that
$$E(r) := E \cap B(r) \subsetneq E.$$
Define $E_r := E \cap S^{n-1}(r)$, and the two functions
$$V_f(r) := V_f(E \setminus B(r)), \quad P_f(r) := \mathscr{H}_f^{n-1}(\partial E \setminus B(r)).$$
Notice that $V_f(r)$ and $P_f(r)$ tend to zero as r tends to 1. Thanks to Theorems A.1 and A.2, we have that
$$P_f(E) > P_f(E(r)) = P_f(E) - P_f(r) + \mathscr{H}_f^{n-1}(E_r),$$
implying that
$$P_f(r) > \mathscr{H}_f^{n-1}(E_r). \tag{A.7}$$
Up to taking r closer to 1, we can assume that
$$\mathscr{H}_{\text{flat}}^{n-1}(E_r) \leq \frac{1}{2}\mathscr{H}_{\text{flat}}^{n-2}(\partial E_r),$$
where the boundary of E_r is taken inside the sphere $S^{n-1}(r)$. Therefore, by classic isoperimetric inequality on the sphere there exists a dimensional constant $c_n > 0$ such that
$$\mathscr{H}_{\text{flat}}^{n-2}(\partial E_r) \geq c_n \mathscr{H}_{\text{flat}}^{n-1}(E_r)^{\frac{n-2}{n-1}}, \tag{A.8}$$
which by Equation (A.7) leads to
$$\begin{aligned}\tilde{f}(r)\omega(r)^{n-2}\mathscr{H}_{\text{flat}}^{n-2}(\partial E_r) &\geq c_n \tilde{f}(r)^{\frac{1}{n-1}}\Big(\omega(r)^{n-1}\tilde{f}(r)\mathscr{H}_{\text{flat}}^{n-1}(E_r)\Big)^{\frac{n-2}{n-1}} \\ &\geq c_n \tilde{f}(0)^{\frac{1}{n-1}}\mathscr{H}_f^{n-1}(E_r)^{\frac{n-2}{n-1}} \\ &> c_n \tilde{f}(0)^{\frac{1}{n-1}} P_f(r)^{\frac{-1}{n-1}}\mathscr{H}_f^{n-1}(E_r),\end{aligned}$$
where we used that $\tilde{f}(r) = \exp(h(2\operatorname{artanh}(r)))$ is non-decreasing. Since
$$-P_f'(r) = \tilde{f}(r)\omega(r)^{n-1}\mathscr{H}_{\text{flat}}^{n-2}(\partial E_r), \qquad -V_f'(r) = \omega(r)\mathscr{H}_f^{n-1}(E_r),$$
we obtain that
$$-(P_f(r)^{\frac{n}{n-1}})' > -c_n \tilde{f}(0)^{\frac{1}{n-1}} V_f'(r),$$
which integrated from r to 1 leads to
$$P_f(r)^{\frac{n}{n-1}} > 2cV_f(r), \tag{A.9}$$
with $2c = c_n \tilde{f}(0)^{\frac{1}{n-1}}$. The last thing we need to do is to take advantage of the optimality of the set E by operating a small perturbation of its boundary. Let $K \subset H_{\mathbb{R}}^n$ be a compact subset of $H_{\mathbb{R}}^n$ and $\Gamma : (-\varepsilon_0, \varepsilon_0) \times H_{\mathbb{R}}^n \to H_{\mathbb{R}}^n$ be any variation inside K, that is $f(0,\cdot) = \text{id}$ and for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the map $\Gamma(\varepsilon, \cdot)$ is a smooth diffeomorphism equal to the identity outside

K. Then the first order expansion of the volume and perimeter operators can be computed as

$$V_f(E_\varepsilon) = V_f(E) + \varepsilon \int_{\partial E} g_H(\nu, X)\, d\mathscr{H}_f + o(\varepsilon),$$
$$P_f(E_\varepsilon) = P_f(E) + \varepsilon \int_{\partial E} \mathbf{H}_f g_H(\nu, X)\, d\mathscr{H}_f + o(\varepsilon),$$

where $X(x) := \frac{\partial \Gamma}{\partial \varepsilon}(0, x)$, $E_\varepsilon := \Gamma(\varepsilon, E)$, and $\mathbf{H}_f = H + \partial_\nu \ln(f)$ stands for the weighted mean curvature of E, which is constant at every regular point if E is isoperimetric. We refer to [76, Chapter 17.3] and [101, Section 3] for the careful proof of this fact. This allows us to perturb the set E inside $K = B(r_0)$ for some r_0 close enough to 1 according to a small parameter $\varepsilon \in (0, \varepsilon_0)$, such that the resulting perturbed sets $(E_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ satisfy

$$V_f(E_\varepsilon) = V_f(E) + \varepsilon,$$

and

$$P_f(E_\varepsilon) \le P_f(E) + \varepsilon(\mathbf{H}_f + 1),$$

Choose now $\varepsilon^{\frac{1}{n}} < \frac{c}{\mathbf{H}_f + 1}$ and $R_0 > r_0$ so that $\varepsilon = V_f(r) < \varepsilon_0$. Then, $F = E_\varepsilon \cap B(R_0)$ has weighted volume equal to E, and from

$$\begin{aligned} P_f(F) = P_f(E_\varepsilon) - P_f(r) + \mathscr{H}_f^{n-1}(E_r) &\le P_f(E) + \varepsilon(\mathbf{H}_f + 1) - 2c\varepsilon^{\frac{n-1}{n}} + \mathscr{H}_f^{n-1}(E_r) \\ &< P_f(E) - c\varepsilon^{\frac{n-1}{n}} + \mathscr{H}_f^{n-1}(E_r), \end{aligned}$$

and the optimality of E, we infer that

$$\mathscr{H}_f^{n-1}(E_r) > c\varepsilon^{\frac{n-1}{n}}.$$

Hence

$$-V_f'(r) = \omega(r)\mathscr{H}_f^{n-1}(E_r) > c\omega(r)\varepsilon^{\frac{n-1}{n}} = c\omega(r)V_f(r)^{\frac{n-1}{n}},$$

implies that

$$-(V_f(r)^{\frac{1}{n}})' > c\omega(r) = c(2\,\mathrm{artanh}(r))'.$$

Integrating both sides, since $\mathrm{artanh}(r)$ tends to $+\infty$ as r tends to 1, we obtain a contradiction with the assumption that $V_f(r) > 0$ for every $r \in (0, 1)$. □

We complete the section by proving the Riemannian mean-convexity of the isoperimetric sets.

Theorem A.4 (Mean-convexity). *Let $h : \mathbb{R} \to \mathbb{R}$ be a convex and even function, and let $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$ be defined through $f(x) = \exp(h(d_H(x, o)))$ for some base point $o \in H^n_{\mathbb{R}}$. Then, every set attaining the isoperimetric infimum in Equation (2.1) is mean-convex.*

Proof. Let E be an isoperimetric set. Thanks to Theorem A.3, E is bounded, and therefore there exists $z \in \partial E$ maximizing the distance from the base point o. By the regularity properties summarized in Theorem 1.23, z is a regular point, and

$$H(z) \geq (n-1),$$

where $H(z)$ denotes the unweighted mean curvature of ∂E at z. Let now $x \in \partial E$ be another regular point. Since the weighted mean curvature $\mathbf{H}_f = H + \partial_\nu \ln(f)$ is constant, we have in particular that

$$\begin{aligned} H(x) &= H(z) + \partial_\nu \ln(f)(z) - \partial_\nu \ln(f)(x) \\ &\geq (n-1) + h'(d_H(o,z)) - h'(d_H(o,x)) \geq (n-1), \end{aligned}$$

where in the last inequality we used the convexity of the exponent h. □

Bibliography

[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*. Vol. 75. Springer Science & Business Media, 2012.

[2] R. A. Adams and J. J. Fournier. *Sobolev spaces*. Elsevier, 2003.

[3] A. D. Alexandrov. "A characteristic property of spheres". In: *Annali di Matematica Pura ed Applicata* 58.1 (1962), pp. 303–315.

[4] L. Ambrosio. *Lectures on elliptic partial differential equations*. Appunti Lecture Notes 18. Pisa: Edizioni della Normale, 2018.

[5] L. Ambrosio et al. "Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case". In: *Comm. in Partial Differential Equations* 37.12 (2012), pp. 2209–2227.

[6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Courier Corporation, 2000.

[7] G. Antonelli and F. Glaudo. "Nonexistence of isoperimetric sets in spaces of positive curvature". In: *arXiv preprint arXiv:2306.13052* (2023).

[8] G. Antonelli et al. *Sharp isoperimetric comparison on non collapsed spaces with lower Ricci bounds*. 2022. URL: https://arxiv.org/abs/2201.04916.

[9] J.-P. Aubin. "Un théoreme de compacité". In: *CR Acad. Sci. Paris* 256.24 (1963), pp. 5042–5044.

[10] J. Barbosa, M. do Carmo, and J. Eschenburg. "Stability of Hypersurfaces of Constant Mean Curvature in Riemannian Manifolds". In: *Mathematische Zeitschrift* 197 (Mar. 1988), pp. 123–138.

[11] E. Battaglia, F. Montefalcone, and R. Monti. "Stability of the CMC hypersurfaces in the complex hyperbolic space." In: *cvgmt.sns.it* (2010). URL: https://cvgmt.sns.it/media/doc/paper/4468/CHn_Annales.pdf.

[12] J.-D. Benamou, C. Cotter, and H. Malamut. "Entropic optimal transport solutions of the semigeostrophic equations". In: *Journal of Computational Physics* 500 (2024), p. 112745.

[13] J.-D. Benamou and Y. Brenier. "Weak Existence for the Semigeostrophic Equations Formulated as a Coupled Monge-Ampère/Transport Problem". In: *SIAM J. Appl. Math.* 58.5 (1998), pp. 1450–1461.

[14] I. Benjamini and J. Cao. "A new isoperimetric comparison theorem for surfaces of variable curvature". In: *Duke Math. J.* 85.2 (1996), pp. 359–396.

[15] L. B. Bergery and J.-P. Bourguignon. "Laplacians and Riemannian submersions with totally geodesic fibres". In: *Illinois Journal of Mathematics* 26.2 (1982), pp. 181–200.

[16] J. Berndt. "Real hypersurfaces with constant principal curvatures in complex hyperbolic space." In: *Journal für die reine und angewandte Mathematik* 395 (1989), pp. 132–141.

[17] R. G. Bettiol, E. A. Lauret, and P. Piccione. "Full Laplace spectrum of distance spheres in symmetric spaces of rank one". In: *Bulletin of the London Mathematical Society* 54.5 (2022), pp. 1683–1704.

[18] V. Blåsjö. "The isoperimetric problem". In: *The American Mathematical Monthly* 112.6 (2005), pp. 526–566.

[19] V. Bögelein, F. Duzaar, and C. Scheven. "A sharp quantitative isoperimetric inequality in hyperbolic n-space". In: *Calculus of Variations and Partial Differential Equations* 54 (Dec. 2015).

[20] E. Bongiovanni et al. "Isoperimetry in Surfaces of Revolution with Density". In: *Missouri journal of Mathematical Sciences* 30.2 (2018), pp. 150–165.

[21] A. A. Borisenko. "On the global structure of Hopf hypersurfaces in a complex space form". In: *Illinois Journal of Mathematics* 45.1 (2001), pp. 265–277.

[22] U. Bottazzini. *Il flauto di Hilbert: storia della matematica moderna e contemporanea.* Utet, 1990.

[23] D. P. Bourne et al. "Semi-discrete optimal transport methods for the semi-geostrophic equations". In: *Calculus of Variations and Partial Differential Equations* 61 (2022), pp. 1–34.

[24] W. Boyer et al. "Isoperimetric regions in $\mathbb{R}^n$ with density r^p". In: *Analysis and Geometry in Metric Spaces* 4.1 (2016).

[25] Y. Brenier. "Polar factorization and monotone rearrangement of vector-valued functions". In: *Communications on Pure and Applied Mathematics* 44.4 (1991), pp. 375–417.

[26] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Springer Science & Business Media, 2013.

[27] L. Capogna et al. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Vol. 259. Springer Science & Business Media, 2007.

[28] C. Carathéodory and E. Study. "Zwei Beweise des Satzes, daß der Kreis unter allen Figuren gleichen Umfanges den größten Inhalt hat". In: *Mathematische Annalen* 68.1 (1909), pp. 133–140.

[29] G. Carron. "Stabilité isopérimétrique". In: *Mathematische Annalen* 306.1 (1996), pp. 323–340.

[30] G. R. Chambers. "Proof of the log-convex density conjecture". In: *Journal of the European Mathematical Society* 21.8 (2019), pp. 2301–2332.

[31] I. Chavel. *Eigenvalues in Riemannian geometry*. Academic press, 1984.

[32] I. Chavel. *Riemannian geometry: a modern introduction*. Vol. 98. Cambridge university press, 2006.

[33] J. Cheng, M. Cullen, and M. Feldman. "Classical Solutions to Semi-geostrophic System with Variable Coriolis Parameter". In: *Archive for Rational Mechanics and Analysis* 227.1 (2018), pp. 215–272.

[34] M. Cicalese and G. P. Leonardi. "A selection principle for the sharp quantitative isoperimetric inequality". In: *Archive for Rational Mechanics and Analysis* 206.2 (2012), pp. 617–643.

[35] D. Cordero-Erausquin. "Sur le transport de mesures périodiques". In: *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics* 329.3 (1999), pp. 199–202.

[36] M. J. Cullen. *A mathematical theory of large-scale atmosphere/ocean flow*. London: Imperial College Press, 2006.

[37] E. De Giorgi. "Sulla proprieta isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita". In: *Ennio De Giorgi* (1958), p. 198.

[38] G. De Philippis and A. Figalli. "Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps". In: *Anal. PDE* 6 (Feb. 2012).

[39] G. De Philippis and A. Figalli. "$W^{2,1}$ regularity for solutions of the Monge-Ampère equation". In: *Inventiones Mathematicae* 192.1 (2013), pp. 55–69.

[40] P. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago lectures in mathematics series. Chicago: University of Chicago Press, 1996.

[41] C. P. Egan et al. "A new implementation of the geometric method for solving the Eady slice equations". In: *Journal of Computational Physics* 469 (2022), p. 111542.

[42] J. Eichhorn. *Global analysis on open manifolds*. New York: Nova Science, 2007.

[43] A. Eliassen. *The quasi-static equations of motion with pressure as independent variable*. Grøndahl & sons boktr., I kommisjon hos Cammermeyers boghandel, 1949.

[44] L. Euler. "Principes généraux du mouvement des fluides". In: *Mémoires de l'académie des sciences de Berlin* (1757), pp. 274–315.

[45] L. C. Evans. *Partial differential equations*. 2nd ed., repr. with corr. Graduate Studies in Mathematics vol. 19, ed. 2, repr. Providence, R.I: American Mathematical Society, 2015.

[46] J. C. Faria, M. C. Lopes Filho, and H. J. N. Lopes. "Weak stability of Lagrangian solutions to the semigeostrophic equations". In: *Nonlinearity* 22.10 (2009), p. 2521.

[47] M. Feldman and A. Tudorascu. "On Lagrangian solutions for the semi-geostrophic system with singular initial data". In: *SIAM Journal on Mathematical Analysis* 45.3 (2013), pp. 1616–1640.

[48] A. Figalli. "Regularity of codimension-1 minimizing currents under minimal assumptions on the integrand". In: *Journal of Differential Geometry* 106.3 (2017), pp. 371–391.

[49] A. Figalli and F. Glaudo. *An invitation to optimal transport, Wasserstein distances, and gradient flows*. EMS Textbooks in Mathematics, 2021.

[50] A. Figalli, F. Maggi, and A. Pratelli. "A mass transportation approach to quantitative isoperimetric inequalities". In: *Inventiones Mathematicae* 182 (Oct. 2010), pp. 167–211.

[51] A. Figalli and F. Maggi. "On the isoperimetric problem for radial log-convex densities". In: *Calculus of Variations and Partial Differential Equations* 48.3 (2013), pp. 447–489.

[52] B. Fuglede. "Stability in the Isoperimetric Problem for Convex or Nearly Spherical Domains in $\mathbb{R}^n$". In: *Transactions of the American Mathematical Society* 314.2 (1989), pp. 619–638.

[53] N. Fusco, F. Maggi, and A. Pratelli. "The sharp quantitative isoperimetric inequality". In: *Annals of Mathematics* (2008), pp. 941–980.

[54] M. Galli and M. Ritoré. "Existence of isoperimetric regions in contact sub-Riemannian manifolds". In: *Journal of Mathematical Analysis and Applications* 397.2 (2013), pp. 697–714.

[55] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Vol. 2. Springer, 1990.

[56] E. Giusti and G. H. Williams. *Minimal surfaces and functions of bounded variation*. Vol. 80. Springer, 1984.

[57] M. Gromov. "Paul Levy's isoperimetric inequality". In: *IHÉS* (1980).

[58] M. Gromov et al. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Boston, MA : Birkhäuser Boston, 2014.

[59] E. Hebey. *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*. Courant Lecture Notes in Mathematics ; Volume 5. New York, New York ; Courant Institute of Mathematical Sciences, 2000 - 1999.

[60] E. Hebey. *Sobolev spaces on Riemannian manifolds*. Vol. 1635. Springer Science & Business Media, 1996.

[61] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Graduate studies in mathematics, volume 34. Providence, Rhode Island: American Mathematical Society, 2001 - 1978.

[62] B. J. Hoskins. "The geostrophic momentum approximation and the semi-geostrophic equations". In: *JAS* 32.2 (1975), pp. 233–242.

[63] H. Howards, M. Hutchings, and F. Morgan. “The isoperimetric problem on surfaces”. In: *The American Mathematical Monthly* 106.5 (1999), pp. 430–439.

[64] S. Howe. “The Log-Convex Density Conjecture and vertical surface area in warped products”. In: *Advances in Geometry* 15.4 (2015), pp. 455–468.

[65] W.-t. Hsiang and W.-Y. Hsiang. “On the uniqueness of isoperimetric solutions and imbedded soap bubbles in non-compact symmetric spaces, I”. In: *Inventiones Mathematicae* 98.1 (1989), pp. 39–58.

[66] W.-Y. Hsiang. “On soap bubbles and isoperimetric regions in noncompact symmetric spaces, I”. In: *Tohoku Mathematical Journal* 44.2 (1992), pp. 151–175.

[67] J. Jost and J. Jost. *Riemannian geometry and geometric analysis*. Vol. 42005. Springer, 2008.

[68] T. Kato and G. Ponce. “Commutator estimates and the Euler and Navier-Stokes equations”. In: *Communications on Pure and Applied Mathematics* 41.7 (1988), pp. 891–907.

[69] A. Kolesnikov and R. Zhdanov. “On isoperimetric sets of radially symmetric measures”. In: *Concentration, Functional inequalities and Isoperimetry, Contemp. Math.* 545 (Feb. 2010).

[70] P. Kreyssig. “An introduction to harmonic manifolds and the Lichnerowicz conjecture”. In: *arXiv preprint arXiv:1007.0477* (2010).

[71] P. Lévy and F. Pellegrino. “Problèmes concrets d’analyse fonctionnelle”. In: *(No Title)* (1951).

[72] H. Li and B. Xu. “A class of weighted isoperimetric inequalities in hyperbolic space”. In: *Proceedings of the American Mathematical Society* 151.05 (2023), pp. 2155–2168.

[73] J.-L. Lions. “Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires”. In: *Dunod* (1969).

[74] G. Loeper. “A Fully Nonlinear Version of the Incompressible Euler Equations: The Semigeostrophic System”. In: *SIAM J. Math. Anal.* 38 (Jan. 2006), pp. 795–823.

[75] E. N. Lorenz. “Deterministic nonperiodic flow”. In: *Journal of atmospheric sciences* 20.2 (1963), pp. 130–141.

[76] F. Maggi. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012.

[77] A. J. Majda and A. L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2001.

[78] V. Maz’ya and T. O. Shaposhnikova. *Sobolev Spaces : With Applications to Elliptic Partial Differential Equations*. Berlin, Heidelberg : Springer, 2011.

[79] I. E. McGillivray. “A weighted isoperimetric inequality on the hyperbolic plane”. In: (2017). URL: https://arxiv.org/abs/1712.07690.

[80] E. Milman. “Sharp isoperimetric inequalities and model spaces for the curvature-dimension-diameter condition”. In: *Journal of the European Mathematical Society* 17.5 (2015), pp. 1041–1078.

[81] J. A. Montero. “Hodge decomposition with degenerate weights and the Gross–Pitaevskii energy”. In: *Journal of Functional Analysis* 254.7 (2008), pp. 1926–1973.

[82] S. Montiel. “Real hypersurfaces of a complex hyperbolic space”. In: *Journal of the Mathematical Society of Japan* 37.3 (1985), pp. 515–535.

[83] F. Morgan. “Clusters minimizing area plus length of singular curves”. In: *Mathematische Annalen* 299.1 (1994), pp. 697–714.

[84] F. Morgan. *Geometric measure theory: a beginner's guide*. Academic press, 2016.

[85] F. Morgan, M. Hutchings, and H. Howards. “The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature”. In: *Transactions of the American Mathematical Society* 352.11 (2000), pp. 4889–4909.

[86] F. Morgan and A. Pratelli. “Existence of isoperimetric regions in $\mathbb{R}^n$ with density”. In: *Annals of Global Analysis and Geometry* 43.4 (2013), pp. 331–365.

[87] G. D. Mostow. *Strong rigidity of locally symmetric spaces*. 78. Princeton University Press, 1973.

[88] S. Nardulli. “The isoperimetric profile of a smooth Riemannian manifold for small volumes”. In: *Annals of Global Analysis and Geometry* 36 (Sept. 2009), pp. 111–131.

[89] S. Nardulli and L. E. Osorio Acevedo. “Sharp Isoperimetric Inequalities for Small Volumes in Complete Noncompact Riemannian Manifolds of Bounded Geometry Involving the Scalar Curvature”. In: *International Mathematics Research Notices* 2020.15 (June 2018), pp. 4667–4720.

[90] C. Navier. “Mémoire sur les lois du mouvement des fluides. Memoir on the laws of fluid motion”. In: *Mémoires de l'Académie Royale des Sciences de l'Institut de France, VI* (1823), pp. 389–440.

[91] B. O'Neill. “The fundamental equations of a submersion.” In: *Michigan Mathematical Journal* 13.4 (1966), pp. 459–469.

[92] P. Pansu. “Sur la régularité du profil isopérimétrique des surfaces riemanniennes compactes”. In: *Annales de l'institut Fourier*. Vol. 48. 1. 1998, pp. 247–264.

[93] R. H. Pedrosa. “The isoperimetric problem in spherical cylinders”. In: *Annals of Global Analysis and Geometry* 26.4 (2004), pp. 333–354.

[94] R. H. Pedrosa and M. Ritoré. “Isoperimetric domains in the Riemannian product of a circle with a simply connected space form and applications to free boundary problems”. In: *Indiana University Mathematics Journal* (1999), pp. 1357–1394.

[95] L. F. Richardson. *Weather prediction by numerical process*. University Press, 1922.

[96] M. Ritoré. “Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces”. In: *Communications in Analysis and Geometry* 9.5 (2001), pp. 1093–1138.

[97] M. Ritoré. “Stable periodic projective planes”. In: *Proceedings of the American Mathematical Society* 124.12 (1996), pp. 3851–3856.

[98] M. Ritoré and A. Ros. “Stable constant mean curvature tori and the isoperimetric problem in three space forms”. In: *Comment. Math. Helv* 67.2 (1992), pp. 293–305.

[99] H. J. Rivertz and P. Tomter. “Stability of Geodesic Spheres”. In: *Geometry and topology of submanifolds* (1994), pp. 320–324.

[100] A. Ros. “The isoperimetric problem”. In: *Global theory of minimal surfaces* 2 (2001), pp. 175–209.

[101] C. Rosales et al. “On the isoperimetric problem in Euclidean space with density”. In: *Calculus of Variations and Partial Differential Equations* 31.1 (2008), pp. 27–46.

[102] S. Rosenberg. *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*. 31. Cambridge University Press, 1997.

[103] L. A. Santaló. *Integral geometry and geometric probability*. Cambridge university press, 2004.

[104] J. Scheuer and C. Xia. “Locally constrained inverse curvature flows”. In: *Transactions of the American Mathematical Society* 372.10 (2019), pp. 6771–6803.

[105] E. Schmidt. “Der Brunn-Minkowskische Satz und sein Spiegeltheorem sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und hyperbolischen Geometrie”. In: *Mathematische Annalen* 120.1 (1947), pp. 307–422.

[106] E. Schmidt. “Über das isoperimetrische Problem im Raum von n Dimensionen”. In: *Mathematische Zeitschrift* 44 (1939), pp. 689–788.

[107] E. Schmidt. “Über die isoperimetrische Aufgabe im n-dimensionalen Raum konstanter negativer Krümmung: I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n-dimensionalen hyperbolischen Raum”. In: *Mathematische Zeitschrift* 46 (1940), pp. 204–230.

[108] H. A. Schwarz. “Beweis des Satzes, dass die Kugel kleinere Oberfläche besitzt, als jeder andere Körper gleichen Volumens”. In: *Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen* 1884 (1884), pp. 1–13.

[109] L. Silini. “Approaching the isoperimetric problem in $H^m_{\mathbb{C}}$ via the hyperbolic log-convex density conjecture”. In: *Calculus of Variations and Partial Differential Equations* 63.1 (2024), p. 11.

[110] L. Silini. “Local existence of smooth solutions for the semigeostrophic equations on curved domains”. In: *SIAM Journal on Mathematical Analysis* 55.6 (2023), pp. 6554–6579.

[111] L. Silini. “Quantitative C^1-stability of spheres in rank one symmetric spaces of non-compact type”. In: *arXiv preprint arXiv:2304.02412* (2023).

[112] J. Simon. “Compact sets in the space $L^p(O, T; B)$”. In: *Annali di Matematica pura ed applicata* 146 (1986), pp. 65–96.

[113] J. Steiner. “Einfache Beweise der isoperimetrischen Hauptsätze.:” in: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 1838.18 (1838), pp. 281–296.

[114] G. G. Stokes. *Transactions of the Cambridge Philosophical Society*. Vol. 8. University Press, 1845.

[115] M. Taylor. “Euler equation on a rotating surface”. In: *Journal of Functional Analysis* 270.10 (2016), pp. 3884–3945.

[116] M. E. Taylor. *Pseudodifferential operators and nonlinear PDE*. Progress in Mathematics Vol. 100. Boston, Mass: Birkhäuser, 1991.

[117] P. Tomter. “Constant mean curvature surfaces in the Heisenberg group.” In: *Proc. Sympos. Pure Math.* (1993), pp. 485–495.

[118] P. Topping. “The isoperimetric inequality on a surface”. In: *Manuscripta Mathematica* 100.1 (1999), pp. 23–34.

[119] C. Viana. “The isoperimetric problem for lens spaces”. In: *Mathematische Annalen* 374.1 (2019), pp. 475–497.

[120] K. Weierstrass. *Mathematische Werke von Karl Weierstrass: bd. Abhandlungen I. 1894*. Vol. 1. Mayer & Müller, 1894.