

Markov processes involving q -Stirling numbers

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Markov Processes Involving q -Stirling Numbers

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Abstract

In this paper we consider the Markov process defined by

$$P_{1,1} = 1, \quad P_{n,\ell} = (1 - \lambda_{n,\ell}) \cdot P_{n-1,\ell} + \lambda_{n,\ell-1} \cdot P_{n-1,\ell-1}$$

for transition probabilities $\lambda_{n,\ell} = q^\ell$ and $\lambda_{n,\ell} = q^{n-1}$. We give closed forms for the distributions and the moments of the underlying random variables. Thereby we observe that the distributions can be easily described in terms of q -Stirling numbers of the second kind. Their occurrence in a purely time dependent Markov process allows a natural approximation for these numbers through the normal distribution. We also show that these Markov processes describe some parameters related to the study of random graphs as well as to the analysis of algorithms.

Keywords: q -Stirling numbers, Markov processes, random graphs, approximate counting.

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In this paper we will study the Markov process defined by

$$\begin{aligned}
P_{1,1} &= 1 \\
P_{n,\ell} &= 0 \text{ for } \ell \notin \{1, \dots, n\} \text{ and} \\
P_{n,\ell} &= (1 - \lambda_{n,\ell}) P_{n-1,\ell} + \lambda_{n,\ell-1} P_{n-1,\ell-1} \text{ otherwise,}
\end{aligned} \tag{1}$$

where $P_{n,\ell}$ is the probability distribution of the underlying random variable at discrete time n and $\lambda_{n,\ell}$ is the transition probability from state ℓ to state $\ell + 1$ at time n . In particular we will consider some special choices of transition probabilities for which the random variable described by the process assumes a concrete interpretation.

The motivation behind this work arises from the average case analysis of combinatorial algorithms, in particular the Simon algorithm [Sim88, CS94] for computing the transitive closure on acyclic digraphs and the so-called ‘‘Approximate Counting Algorithm’’ (see [Fla85, Mor78]), a method for ‘‘keeping approximate counts of large numbers in small registers’’. In [Sim88] two variants of the above process have been analyzed, namely:

$$\lambda_{n,\ell} = 1 - q^\ell \quad \text{and} \quad \lambda_{n,\ell} = q^\ell.$$

For the first transition probability, the underlying random variable models the size of the transitive closure of a node in the random acyclic digraph $G_{n,p}$. Closed forms for its distribution and moments involving q -hypergeometric series are given in [SCC93]. The second one describes the width of a greedy constructed decomposition of $G_{n,p}$ into node disjoint paths [Sim88] as well as the value of the counting register in the ‘‘Approximate Counting Algorithm’’. Flajolet [Fla85] showed that the distribution of this random variable is given by

$$P_{n,\ell} = \sum_{j=0}^{\ell-1} (-1)^j \frac{q^{\binom{j}{2}} (1 - q^{\ell-j})^{n-1}}{(q)_j (q)_{\ell-j-1}}.$$

After having introduced some definitions in Section 2, in Section 3.1 we will simplify this expression to

$$P_{n,\ell} = (1 - q)^{n-\ell} q^{\binom{\ell}{2}} S_2(\ell, n - \ell),$$

where $S_2(n, h)$ is the q -Stirling number of the second kind [Gou61], and we will present closed forms for the first two moments, results which are missing in [Sim88] and [Fla85]. As a nice byproduct we will observe that the same process also models the size of a greedy clique and the size of a greedy stable set.

In Section 3.2 then we will deal with another distribution describing the random variable counting number of sources (or sinks) in a random acyclic digraph. In this case the transition probability will be given by

$$\lambda_{n,\ell} = q^{n-1},$$

which will lead to

$$P_{n,\ell} = (1 - q)^{n-\ell} q^{\binom{n}{2}} S_2(-n, n - \ell). \tag{2}$$

n . This allows a simple approximation for the discrete time process, leading to the well-known backward equation of Kolmogorov (see [Fel71, Volume II, Diffusion Processes], [Gne80]):

$$\frac{\partial f(n, \ell)}{\partial n} = -a(n, \ell) \frac{\partial f(n, \ell)}{\partial \ell} + \frac{1}{2} b(n, \ell) \frac{\partial^2 f(n, \ell)}{\partial \ell^2} \quad (3)$$

In our case we have

$$a(n, \ell) = a(n) \quad \text{and} \quad b(n, \ell) = b(n),$$

conditions which allow to reduce (3) to

$$\frac{\partial f(n, \ell)}{\partial n} = \frac{1}{2} \frac{\partial^2 f(n, \ell)}{\partial \ell^2},$$

whose solution is given by the normal distribution. This approximation together with (2) results in an approximation for the q -Stirling numbers of the second kind.

2 Some Definitions

In the following let $G = (V, E)$ be a graph with a set of vertices V and a set of edges $E \subseteq V \times V$. Without loss of generality we will assume that the graph does not contain loops (i.e. edges with two equal endpoints). In later sections we will present some interpretations of the Markov process (1) related to random graphs. Let us therefore introduce the following model:¹

Definition 1 *The probability space of graphs $G_{n,p}$ is defined as the set of graphs with vertex set $V = \{1, \dots, n\}$ and edge set*

$$E = \{ \{i, j\} \mid i, j \in V, i \neq j \} \text{ for an } \mathbf{undirected},$$

$$E = \{ (i, j) \mid i, j \in V, i \neq j \} \text{ for a } \mathbf{directed} \text{ and}$$

$$E = \{ (i, j) \mid i, j \in V, i < j \} \text{ for a } \mathbf{directed, acyclic graph},$$

where every edge occurs independently with probability $p \in (0, 1)$. In the following we will use the convention of writing q instead of $1 - p$.

Throughout this paper we will deal with the so-called q -notation, a pure numbertheoretical notation which seems to be the appropriate tool to deal with our kind of problems. Let us therefore introduce the following definitions.²

¹For more details on random graphs see [Pal85].

²For more details on these topics see [GR90], [HW89] and [Gou61].

$$(A; q)_0 \equiv (A)_0 = 1, \quad (4)$$

$$(A; q)_n \equiv (A)_n = \prod_{j=0}^{n-1} (1 - Aq^j). \quad (5)$$

Whenever no misunderstanding can arise, by the q -shifted factorials as well as by the following q -formulae we will drop the q .

- The q -natural numbers are defined for any integer n by

$$[n]_q \equiv [n] = \frac{1 - q^n}{1 - q}. \quad (6)$$

- The q -Stirling numbers of the first kind are defined for integers n, k as the sum of the $\binom{n}{k}$ products with k different factors which may be formed from the first n q -natural numbers $[1], [2], \dots, [n]$ (for $k = 0$ we will define them as 1). We will follow the notation of Gould [Gou61] and denote them by $S_1(n, k; q) \equiv S_1(n, k)$. As an illustration let us compute a few examples:

$$\begin{aligned} S_1(3, 0) &= [1] = 1 \quad (\text{definition}) \\ S_1(3, 1) &= [1] + [2] + [3] = 3 + 2q + q^2 \\ S_1(3, 2) &= [1][2] + [1][3] + [2][3] = 3 + 4q + 3q^2 + q^3 \end{aligned}$$

- Similarly, the q -Stirling numbers of the second kind $S_2(n, k; q) \equiv S_2(n, k)$, are defined for integers n, k as the sum of the $\binom{n+k-1}{k}$ products, each with k factors (repeated factors allowed), which may be formed from the first n q -natural numbers $[1], [2], \dots, [n]$ (for $k = 0$ they will be also defined as 1). Again let us compute a few examples:

$$\begin{aligned} S_2(2, 0) &= [1] = 1 \quad (\text{definition}) \\ S_2(2, 1) &= [1] + [2] = 2 + q \\ S_2(2, 2) &= [1][1] + [1][2] + [2][2] = 3 + 3q + q^2 \\ S_2(2, 3) &= [1][1][1] + [1][1][2] + [1][2][2] + [2][2][2] \\ &= 4 + 6q + 4q^2 + q^4 \end{aligned}$$

While these definitions are quite simple, they soon become useless because of the large number of terms involved. Let us therefore present recurrences which are more practical in many circumstances:

$$S_1(n, k) = S_1(n-1, k) + [n]S_1(n-1, k-1) \quad (7)$$

$$S_2(n, k) = S_2(n-1, k) + [n]S_2(n, k-1) \quad (8)$$

bers of the second kind is given by

$$S_1(-n-1, k; 1/q) = q^k S_2(n, k; q) \quad (9)$$

$$S_2(-n-1, k; 1/q) = q^k S_1(n, k; q) \quad (10)$$

which can be regarded as definition of q -Stirling numbers for negative coefficient n .

- Finally we will define the *broken Lambert series* as

$$L_n(q) \stackrel{\text{def}}{=} \sum_{\ell=1}^n \frac{q^\ell}{1-q^\ell}; \quad (11)$$

notice that the usual *Lambert Series*, $L_\infty(q)$, corresponds to the generating function of the divisor-function.

3 The Markov Process

In this section we will consider the Markov process (1) for two particular transition probabilities, namely

$$\lambda_{n,\ell} = q^\ell \quad \text{and} \quad \lambda_{n,\ell} = q^{n-1}.$$

The process with the former one defines a random variable whose distribution was provided in closed form by Flajolet [Fla85]. In Section 3.1 we will simplify this result to an expression involving q -Stirling numbers and we will also provide closed forms for the first two moments. In Section 3.2 then we will consider the Markov process with the second transition probability. A graphtheoretical interpretation for the underlying random variable will be given together with closed forms for its distribution and moments.

3.1 Transition Probability $\lambda_{n,\ell} = q^\ell$

A possible graphtheoretical interpretation for the random variable defined by the process (1) with transition probability $\lambda_{n,\ell} = q^\ell$ is given by the so-called *chain decomposition* of a graph.

Definition 2 For an acyclic digraph G we inductively define a chain decomposition as a partition $\mathcal{Z} = Z_1 + \dots + Z_k$ of V satisfying:

1. For $n = 1$ the partition \mathcal{Z} contains only the path $Z_1 = 1$.
2. Let $n \geq 2$ and $\mathcal{Z} = Z_1 + \dots + Z_\ell$ be the partition of G not containing the node n . Then either one of the paths $Z_i, 1 \leq i \leq \ell$, can be extended to n or there is no Z_i for which there is an edge between the last vertex of Z_i and n . In the first case we extend that Z_i to n and in the second one a new path $Z_{\ell+1}$ containing only the node n is added to \mathcal{Z} .

of the chain decomposition of a graph belonging to the $G_{n,p}$ class; further we will write $K_{n,\ell}$ for the probability of having $\mathcal{K}_n = \ell$.

It is now easy to show that $K_{n,\ell}$ satisfies the following recursion:

$$K_{1,1} = 1 \quad \text{and} \quad K_{n,\ell} = (1 - q^\ell) K_{n-1,\ell} + q^{\ell-1} K_{n-1,\ell-1}. \quad (12)$$

Consider the process in which we build a $G_{n,p}$ graph G_n from a graph G_{n-1} through addition of the node n ; then in G_n we will have ℓ chains if in G_{n-1} either there are already ℓ chains and the node n is connected to at least one of the chain ends, or there are only $\ell - 1$ chains and n is not connected to any of their ends. The corresponding probabilities are

$$K_{n-1,\ell} (1 - q^\ell) \quad \text{and} \quad K_{n-1,\ell-1} q^{\ell-1},$$

and thus $K_{n,\ell}$ satisfies the recursion (12).

In [Fla85] the following closed form for the distribution $K_{n,\ell}$ was found using other, mainly analytical, techniques:

$$K_{n,\ell} = \sum_{j=0}^{\ell-1} (-1)^j \frac{q^{\binom{j}{2}} (1 - q^{\ell-j})^{n-1}}{(q)_j (q)^{\ell-j-1}}. \quad (13)$$

We will simplify this to

Theorem 3 *The probability $K_{n,\ell}$ that in $G_{n,p}$ the chain decomposition has width ℓ is given by*

$$K_{n,\ell} = (1 - q)^{n-\ell} q^{\binom{\ell}{2}} S_2(\ell, n - \ell) \quad (14)$$

Proof. We will prove (14) by induction on n . Considering first that $K_{1,1} = 1$, the case $n = 1$ is established. With (12) we find further for $n \geq 2$:

$$\begin{aligned} K_{n,\ell} &= (1 - q^\ell) K_{n-1,\ell} + q^{\ell-1} K_{n-1,\ell-1} \\ &\stackrel{\text{I.H.}}{=} (1 - q^\ell) (1 - q)^{n-\ell-1} q^{\binom{\ell}{2}} S_2(\ell, n - \ell - 1) \\ &\quad + q^{\ell-1} (1 - q)^{n-\ell} q^{\binom{\ell-1}{2}} S_2(\ell - 1, n - \ell) \\ &= (1 - q)^{n-\ell} q^{\binom{\ell}{2}} \underbrace{([\ell] S_2(\ell, n - \ell - 1) + S_2(\ell - 1, n - \ell))}_{\stackrel{(8)}{=} S_2(\ell, n - \ell)}. \square \end{aligned}$$

In Figure 1 we have plotted the evolving distribution for different values of q . We clearly see that expectation and variance both rise with increasing values of q .

Next we will find closed forms for the first two moments of \mathcal{K}_n . Let us start by considering the probability generating function of \mathcal{K}_n .

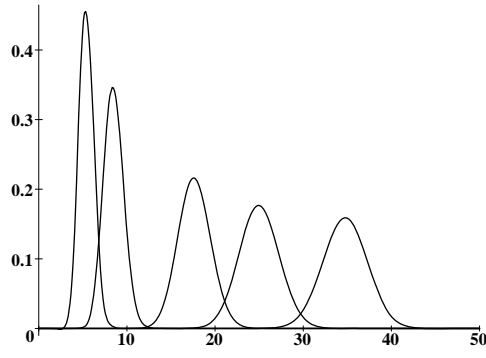


Figure 1: The evolving distribution of the number of chains in a random acyclic digraph with 50 vertices.

Lemma 4 *The probability generating function*

$$g_n(z) \stackrel{\text{def}}{=} \sum_{\ell=1}^n z^\ell K_{n,\ell}$$

of the random variable K_n has the following closed form:

$$g_n(z) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-q)^\ell z (z)_\ell. \quad (15)$$

Proof. Let us define

$$G(t, y) = \sum_{\substack{n \geq 1 \\ \ell \geq 0}} K_{n,\ell+1} t^n y^\ell. \quad (16)$$

Then we have

$$\begin{aligned} G(t, y) &= t + \sum_{\substack{n \geq 2 \\ \ell \geq 0}} K_{n,\ell+1} t^n y^\ell \\ &= t + \sum_{\substack{n \geq 1 \\ \ell \geq 0}} K_{n+1,\ell+1} t^{n+1} y^\ell \\ &\stackrel{(12)}{=} t + \sum_{\substack{n \geq 1 \\ \ell \geq 0}} ((1 - q^{\ell+1}) K_{n,\ell+1} + q^\ell K_{n,\ell}) t^{n+1} y^\ell \\ &= t + t G(t, y) - t q G(t, y q) + y q t G(t, y q) \end{aligned}$$

so that

$$G(t, y) = \frac{t}{1-t} - \frac{t q (1-q)}{1-t} G(t, y q).$$

Iteration finally yields

$$G(t, y) = \sum_{n \geq 0} \frac{(-1)^n t^{n+1} q^n (y)_n}{(1-t)^{n+1}}. \quad (17)$$

$$\begin{aligned}
g_n(z) &= \sum_{\ell \geq 1} z^\ell K_{n,\ell} \\
&= z \sum_{\ell \geq 1} z^{\ell-1} K_{n,\ell} \\
&= z G(t, z) \Big|_{t^n}^\ddagger \\
&= z \sum_{\ell \geq 0} \frac{(-1)^\ell t^{\ell+1} q^\ell(z)_\ell}{(1-t)^{\ell+1}} \Big|_{t^n} \\
&= z \sum_{\ell \geq 0} \sum_{h \geq 0} (-1)^\ell t^{\ell+h+1} \binom{\ell+h}{h} q^\ell(z)_\ell \Big|_{t^n} \\
&= z \sum_{\ell \geq 0} (-1)^\ell \binom{n-1}{\ell} q^\ell(z)_\ell. \quad \square
\end{aligned}$$

From (15) we can easily derive the expectation and the second moment of \mathcal{K}_n .

Theorem 5 *The expected value of \mathcal{K}_n is given by*

$$\mathbf{E}[\mathcal{K}_n] = 1 + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} (-1)^{\ell-1} q^\ell(q)_{\ell-1} \quad (18)$$

and the second moment by

$$\mathbf{E}[(\mathcal{K}_n)^2] = \mathbf{E}[\mathcal{K}_n] + 2 \sum_{\ell=1}^{n-1} (-1)^{\ell+1} q^\ell(q)_{\ell-1} \binom{n-1}{\ell} (1 - L_{\ell-1}(q)). \quad (19)$$

Proof. To prove this theorem we notice that

$$\left[\frac{d}{dz} g_n(z) \right]_{z=1} = \left[\sum_{\ell=1}^n \ell z^{\ell-1} K_{n,\ell} \right]_{z=1} = \mathbf{E}[\mathcal{K}_n]$$

and

$$\left[\frac{d^2}{dz^2} g_n(z) \right]_{z=1} = \left[\sum_{\ell=1}^n \ell(\ell-1) z^{\ell-2} K_{n,\ell} \right]_{z=1} = \mathbf{E}[(\mathcal{K}_n)^2] - \mathbf{E}[\mathcal{K}_n].$$

In $g_n(z)$ the only term containing z is $z(z)_\ell$ so that we only need to determine the first two derivatives of this term. For $\ell = 0$ we have

$$\left[\frac{d}{dz} z(z)_0 \right]_{z=1} = \left[\frac{d}{dz} z \right]_{z=1} = 1$$

and for $\ell \geq 1$ we find

$$\left[\frac{d}{dz} z(z)_\ell \right]_{z=1} = \left[\frac{d}{dz} z(1-z)(zq)_{\ell-1} \right]_{z=1} = (q)_{\ell-1}.$$

[‡]Thereby we denote the coefficient of t^n .

$$\begin{aligned}
& \left[\frac{d^2}{dz^2} z(z)_\ell \right]_{z=1} \\
&= \left[\frac{d^2}{dz^2} z(1-z)(zq)_{\ell-1} \right]_{z=1} \\
&= \left[\frac{d}{dz} \left((1-2z)(zq)_{\ell-1} + z(1-z) \frac{d}{dz} (zq)_{\ell-1} \right) \right]_{z=1} \\
&= -2(q)_{\ell-1} - 2 \underbrace{\left[z \frac{d}{dz} (zq)_{\ell-1} \right]_{z=1}}_{-(q)_{\ell-1} \sum_{j=1}^{\ell-1} \frac{q^j}{1-q^j}} \\
&= -(q)_{\ell-1} (2 - 2L_{\ell-1}(q)).
\end{aligned}$$

Expressions (18) and (19) follow directly therefrom. \square

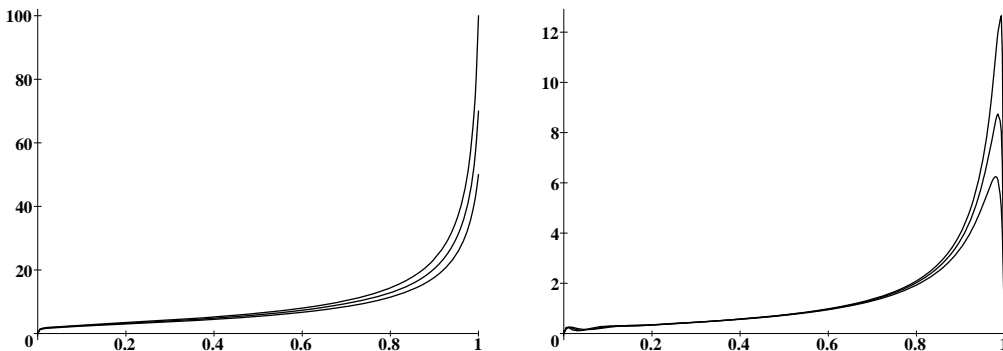


Figure 2: The evolving expectation (left) and variance for the number of chains.

In Figure 2 we have plotted the evolution of the expectation and the variance of \mathcal{K}_n for a random graph with 50 (lowest curves), 70 and 100 nodes.

Note that we can model the number of vertices in a greedy constructed stable set using the same probability process. Instead of dealing with the end vertices of a chain we look at the vertices in the stable set. Further, if we substitute q with p we can also model the number of vertices in a greedy constructed maximal clique using a dual argument.

3.2 Transition Probability $\lambda_{n,\ell} = q^{n-1}$

In this section we will consider the Markov process (1) with the transition probability $\lambda_{n,\ell} = q^{n-1}$. It turns out that the random variable modeled by this process has

acyclic digraph.

Definition 6 A node v in a digraph G is called a source if it does not have any predecessor. We will denote by \mathcal{S}_n the number of sources in a random acyclic digraph drawn from the $G_{n,p}$ class. Furthermore we denote by $S_{n,\ell}$ the probability of having exactly ℓ sources in such a graph.

Again it is not difficult to show that $S_{n,\ell}$ satisfies the Markov process (1) with transition probability

$$\lambda_{n,\ell} = q^{n-1}. \quad (20)$$

Consider in fact the process in which we build a graph G_n from G_{n-1} through addition of the node n . The graph G_n can have exactly ℓ sources if in G_{n-1} either there exist only $\ell - 1$ sources and no node of G_{n-1} is connected to n , or there are already ℓ sources and there is at least a node in G_{n-1} connected to n . The corresponding probabilities are

$$S_{n-1,\ell-1} q^{n-1} \quad \text{and} \quad S_{n-1,\ell} (1 - q^{n-1}),$$

proving therefore what we have asserted.

At this point we can show:

Theorem 7 Let G be a random acyclic digraph drawn from the $G_{n,p}$ model. Then the random variable \mathcal{S}_n counting the number of sources in G has the following distribution, expectation and variance:

$$S_{n,\ell} = (1 - q)^{n-\ell} q^{\binom{n}{2}} S_2(-n, n - \ell) \quad (21)$$

$$\mathbf{E}[\mathcal{S}_n] = [n] \quad (22)$$

and

$$\mathbf{Var}[\mathcal{S}_n] = [n]_q - [n]_{q^2} \quad (23)$$

Proof. We first prove (21) by induction on n . For $n = 1$ the equation is satisfied, so let us assume $n \geq 2$. Using (1) and (20) we get

$$\begin{aligned} S_{n,\ell} &= (1 - q^{n-1}) S_{n-1,\ell} + q^{n-1} S_{n-1,\ell-1} \\ &\stackrel{\text{I.H.}}{=} (1 - q^{n-1}) (1 - q)^{n-\ell-1} q^{\binom{n-1}{2}} S_2(-n+1, n-\ell-1) \\ &\quad + q^{n-1} (1 - q)^{n-\ell} q^{\binom{n-1}{2}} S_2(-n+1, n-\ell) \\ &= (1 - q)^{n-\ell} q^{\binom{n}{2}} \\ &\quad \times \left(\frac{[n-1]}{q^{n-1}} S_2(-n+1, n-\ell-1) + S_2(-n+1, n-\ell) \right) \\ &\stackrel{\text{(8)}}{=} (1 - q)^{n-\ell} q^{\binom{n}{2}} \left(\frac{[n-1]}{q^{n-1}} S_2(-n+1, n-\ell-1) \right) \end{aligned}$$

$$\begin{aligned}
& + S_2(-n, n - \ell) + [-n + 1] S_2(-n + 1, n - \ell - 1) \\
= & (1 - q)^{n - \ell} q^{\binom{n}{2}} \left(S_2(-n, n - \ell) \right. \\
& \left. + \underbrace{\left(\frac{[n - 1]}{q^{n-1}} + [-n + 1] \right)}_0 S_2(-n + 1, n - \ell - 1) \right).
\end{aligned}$$

Let us now define the random variables $\mathcal{V}_i, 1 \leq i \leq n$, taking value 1 if node i is a source and 0 otherwise. \mathcal{V}_i has therefore a Bernoulli distribution with probability q^{i-1} , namely the probability that node i is a source in G . Accordingly we get

$$\mathbf{E}[\mathcal{V}_i] = q^{i-1} \quad \text{and} \quad \mathbf{Var}[\mathcal{V}_i] = q^{i-1}(1 - q^{i-1}).$$

This leads immediately to

$$\mathbf{E}[\mathcal{S}_n] = \mathbf{E}\left[\sum_{i=1}^n \mathcal{V}_i\right] = \sum_{i=1}^n \mathbf{E}[\mathcal{V}_i] = \sum_{i=1}^n q^{i-1} = \frac{1 - q^n}{1 - q} = [n].$$

On the other hand, the \mathcal{V}_i 's are pairwise independent random variables and hence we get

$$\begin{aligned}
\mathbf{Var}[\mathcal{S}_n] &= \sum_{i=1}^n \mathbf{Var}[\mathcal{V}_i] = \sum_{i=1}^n q^{i-1}(1 - q^{i-1}) \\
&= \frac{1 - q^n}{1 - q} - \frac{1 - (q^2)^n}{1 - q^2} = [n]_q - [n]_{q^2}. \quad \square
\end{aligned}$$

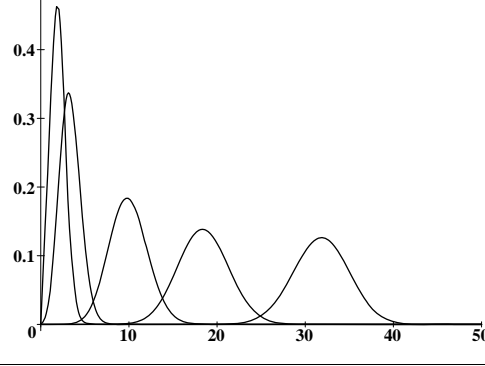


Figure 3: The evolving distribution of the number of sources in a random acyclic digraph with 50 vertices.

We have plotted in Figure 3 the evolution of the distribution representing the number of sources in a random acyclic digraph with 50 vertices and q going from 0.5 to 0.98.

In Figure 4 we have plotted the evolution of the expectation and the variance of \mathcal{S}_n for a random graph with 50 (lowest curves), 70 and 100 nodes.

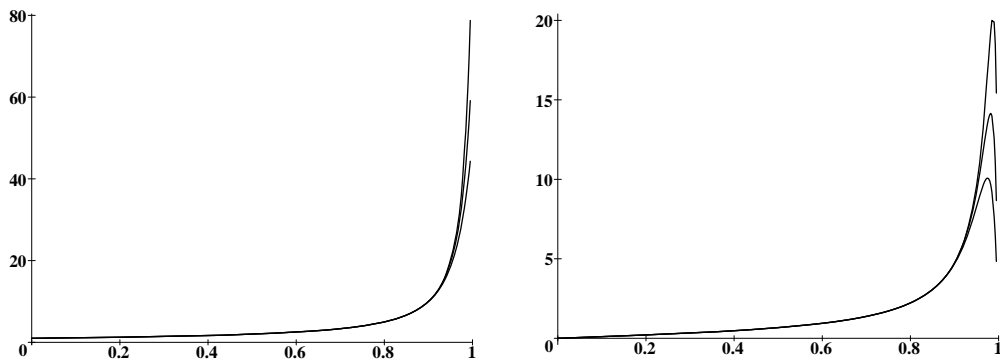


Figure 4: The evolving expectation (left) and variance for the number of sources.

In this section we have presented a distribution which describes the number of sources in a random directed acyclic graph. Note that for symmetry reasons the same process can also be used to describe the number of sinks in a random acyclic digraph. This holds even though the process which describes the number sinks is completely different from the process describing the number sources.

4 Approximations

In this section we will consider the approximation of the discrete distribution $S_{n,\ell}$ by a function $f(n, \ell)$, continuous in both variables. The classical example for this kind of approach is the binomial distribution $BIN(n, p)$

$$B_{n,\ell} = \binom{n}{\ell} p^\ell (1-p)^{n-\ell} \quad (24)$$

which can be approximated by the normal distribution $N(\mu, \sigma^2)$

$$N_{n,\ell} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ell-\mu)^2}{2\sigma^2}\right) \quad (25)$$

with expectation $\mu = np$ and variance $\sigma^2 = p(1-p)n$. A simple approach leading to these results comes from the interpretation of the normal distribution as a diffusion process. One dimensional diffusion processes can be described by the partial differential equation

$$\frac{\partial f(x, t)}{\partial t} = -a(x, t) \frac{\partial f(x, t)}{\partial x} + \frac{1}{2} b(x, t) \frac{\partial^2 f(x, t)}{\partial x^2}, \quad (26)$$

known as Kolmogorov backward equation (see [Fel71]).

Let $\mathcal{Y}(t)$ be a random variable describing the position of a particle at time t and let $f(x, t)$ be the probability function of $\mathcal{Y}(t)$. Further let the random variable

$\mathcal{Y}(t) = x$. Then $f(x, t)$ satisfies equation (26) and the coefficients $a(x, t)$ and $b(x, t)$ are given by

$$a(x, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \mathbf{E} [\mathcal{Y}_{x,t}(t + \Delta t) - \mathcal{Y}_{x,t}(t)] \quad (27)$$

$$b(x, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \mathbf{E} [(\mathcal{Y}_{x,t}(t + \Delta t) - \mathcal{Y}_{x,t}(t))^2]. \quad (28)$$

From a physical point of view $a(x, t)$ represents the mean speed value of $\mathcal{Y}(t)$ at position x and $b(x, t)$ is proportional to the mean kinetic energy of the system.

The simplest nontrivial case in which we can solve equation (26) is given by

$$a(x, t) = 0 \quad \text{and} \quad b(x, t) = 1,$$

reducing the Kolmogorov equation to

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 f(x, t)}{\partial x^2}. \quad (29)$$

This process describes the Brownian motion and in this case the solution of (29) is the normal distribution $N(0, 1)$.

The connection between a Markov process of type (1) and a diffusion process arises from the interpretation of n as time and ℓ as position. Then the Markov process corresponds to a diffusion process (see [Ris84]) with

$$a(n, \ell) \approx \lambda_{n,\ell} \quad \text{and} \quad b(n, \ell) \approx \lambda_{n,\ell}.$$

Unfortunately equation (26) is not easy to solve in the general case. However, if a and b are dependent only on the time and not on the position, i.e.

$$a(t, x) = a(t) \quad \text{and} \quad b(t, x) = b(t),$$

then we can apply a variable substitution to (26) (see [Gne80, §53, §54]) and reduce it to (29), whose solution will be given by $N(\mu, \sigma^2)$.

This is exactly the case for the random variable \mathcal{S}_n . As we already have closed forms for its expectation and variance it seems natural to use them in the normal distribution. Accordingly we get

$$\begin{aligned} S_{n,\ell} &\approx \frac{1}{\sqrt{2\pi \mathbf{Var}[\mathcal{S}_n]}} \exp\left(-\frac{1}{2} \frac{(\ell - \mathbf{E}[\mathcal{S}_n])^2}{\mathbf{Var}[\mathcal{S}_n]}\right) \\ &= \frac{1}{\sqrt{2\pi ([n]_q - [n]_{q^2})}} \exp\left(-\frac{1}{2} \frac{(\ell - [n])^2}{[n]_q - [n]_{q^2}}\right). \end{aligned}$$

This together with the exact formula (21) for the distribution results in the following approximation for the q -Stirling numbers of the second kind:

$$S_2(-n, \ell) \approx \frac{(1-q)^{-\ell} q^{-\binom{n}{2}}}{\sqrt{2\pi ([n]_q - [n]_{q^2})}} \exp\left(-\frac{1}{2} \frac{(\ell - [n])^2}{[n]_q - [n]_{q^2}}\right). \quad (30)$$

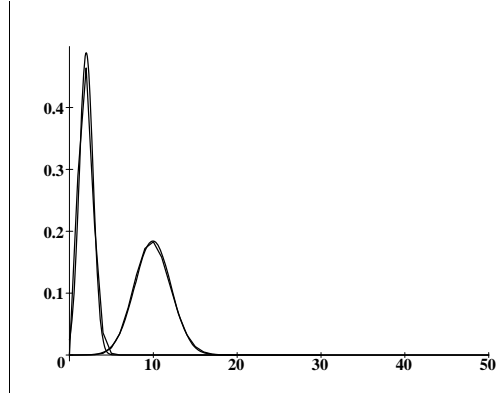


Figure 5: Source distribution and its approximation in a graph with 50 nodes for $q = 0.5$ (left curves) and $q = 0.9$.

This approximation for $S_{n,\ell}$ seems to be pretty good as we can see in Figure 5. However, there are some remarks to be done. While the distribution is very well approximated for values of ℓ such that $S_{n,\ell}$ is significantly above 0, the approximation becomes too small for values of ℓ such that $S_{n,\ell}$ is close to 0. In the latter case the approximation may deviate from the correct value by several orders of magnitude. This is still a very small difference in absolute terms, but if we let $\tilde{S}_{n,\ell}$ be the value computed with the approximation formula, then the relative error, computed as

$$\frac{S_{n,\ell} - \tilde{S}_{n,\ell}}{\tilde{S}_{n,\ell}},$$

may be a very large factor. In Figure 6 we have plotted the absolute and relative error between the distribution and the approximation for $n = 50$ and $q = 0.9$ (see Figure 5).

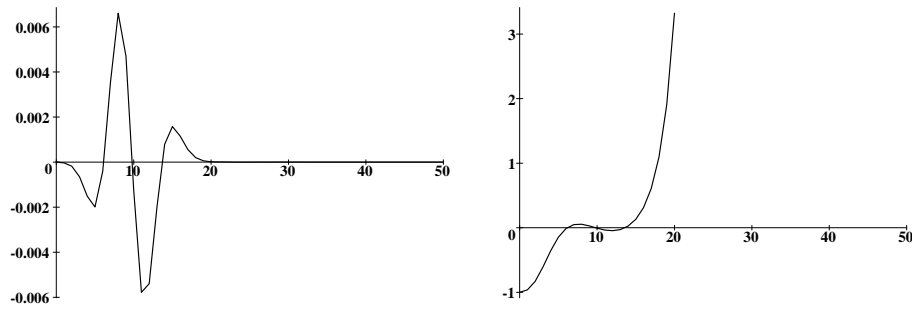


Figure 6: Absolute (left) and relative (right) error between the source distribution and its approximation for $n = 50$, $q = 0.9$.

of the second kind and their approximation.

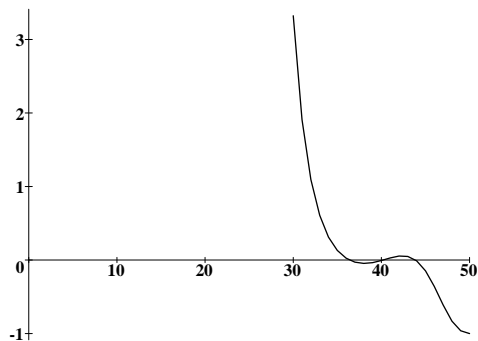


Figure 7: Relative difference between q -Stirling numbers of the second kind and their approximation for $n = 50$ and $q = 0.9$.

As we can see, the situation for the q -Stirling numbers is similar. The approximation is pretty accurate around the expected value of the source distribution, but the accuracy is rather poor in the regions where the source distribution has values close to 0.

If we look at Figure 4 it becomes clear that expectation and variance become larger for values of q near 1. As a large variance implies a flat curve, for values of q close to 1 there will be a good number of values ℓ for which $S_{n,\ell}$ is sufficiently large. On the other hand, for values of q significantly deviating from 1 the approximation becomes inaccurate for most of the ℓ -values. Nonetheless, for graph theory purposes the values of q close to 1 are the most interesting, since we usually deal with sparse graphs.

5 Concluding Remarks

In this paper we have studied the random variable described by a general Markov process for two particular transition probabilities. It turns out that the distributions of these two random variables can be easily described in terms of q -Stirling numbers of the second kind. As far as we know this is the first time that these numbertheoretical functions arise in conjunction with a probability distribution.

One of the two random variables, the number of sources in a random acyclic digraph, has a transition probability which is only time dependent. This has allowed us to determine an approximation for its distribution through the normal distribution, leading finally to an approximation for some q -Stirling numbers of the second kind. By the transform (9) we get then an approximation for some q -Stirling numbers of the first kind.

they definitely represent a first approach to the approximation of q -Stirling numbers, a result that would be very meaningful to the theory of q -hypergeometric series.

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