Report

Faulhaber's triangle

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Publication Date:
2011

Permanent Link:
https://doi.org/10.3929/ethz-a-006804427

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Faulhaber’s Triangle

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December 22, 2011

Like Pascal’s triangle, Faulhaber’s triangle is easy to draw: all you need is pen, paper and a little recursion. The rows of Faulhaber’s triangle are the coefficients of polynomials that represent sums of integer powers. Such polynomials are often called \textit{Faulhaber formulae} [2], after Johann Faulhaber (1580–1635); hence we dub the triangle Faulhaber’s triangle.

\textbf{Constructing Faulhaber’s triangle}

Draw a right triangle, similar to the one shown in Figure 1. Number the rows, starting with row 0; number the columns from left to right, starting with column 1. The numbers on row \( i \) are found using the following recursive rules:

\begin{itemize}
  \item The leftmost element of each row is chosen such that the row sums to 1. In particular, the only number on row 0 is 1.
  \item The element at row \( i \) and column \( j \) (\( 1 < j \leq i + 1 \)) is found by multiplying the number directly above and to the left by \( \frac{i}{j} \).
\end{itemize}

\textbf{Sums of integer powers}

The sum of integer powers \( 1^p + 2^p + \cdots + n^p \), with integers \( n, p \geq 0 \), is a polynomial in \( n \) of degree \( p + 1 \). That is \( f_p(n) = a_{p+1}n^{p+1} + a_p n^p + \cdots + a_1 n + a_0 \). Taking \( n = 0 \), it follows immediately that \( a_0 = 0 \). In order to find the coefficients of the polynomial, we draw Faulhaber’s triangle. Row \( p \) of the triangle gives the coefficients \( a_1, \cdots, a_{p+1} \).

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Figure 1: Faulhaber’s triangle

| row 0 | 1 |
| row 1 | 1/2 | 1/2 |
| row 2 | 1/6 | 1/2 | 1/3 |
| row 3 | 0 | 1/4 | 1/2 | 1/4 |
| row 4 | -1/30 | 0 | 1/3 | 1/2 | 1/5 |
| row 5 | 0 | -1/72 | 0 | 5/72 | 1/2 | 1/6 |
| row 6 | 1/12 | 0 | -1/6 | 0 | 1/2 | 1/7 |

For instance, to find \( f_4(n) \) we use row 4 of Figure 1:

\[
\begin{align*}
  a_1 &= -\frac{1}{30}, \\
  a_2 &= 0, \\
  a_3 &= \frac{1}{3}, \\
  a_4 &= \frac{1}{2}, \\
  a_5 &= \frac{1}{5}.
\end{align*}
\]

That is

\[
f_4(n) = \sum_{i=1}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.
\]

We now observe that \( f_p(n) \) is always of the shape

\[
\frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + a_{p-1}n^{p-1} + a_{p-3}n^{p-3} + \ldots,
\]

with all coefficients \( a_{p-2k} = 0 \) for \( k > 0 \). We also note that the numbers appearing on the vertical leg (leftmost column) of Faulhaber’s triangle are the Bernoulli numbers, namely \( B_0 = 1, \ B_1 = \frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_3 = 0, \ B_4 = -\frac{1}{30}, \) etc. This is due to the well-know Bernoulli formula stating \( f_p(n) = \frac{1}{p+1} \sum_{i=0}^{p} \binom{p+1}{i} B_i n^{p+1-i}. \)

Why it works

Suppose the coefficient of \( n^a \) is \( \alpha \) in \( f_b(n) \), for some \( 1 < a \leq b+1 \), and the coefficient of \( n^{a-1} \) in \( f_{b-1}(n) \) is \( \beta \). It can be shown that \( \alpha = \frac{b}{a} \beta \), cf. [1, 3]. In Faulhaber’s triangle, this corresponds to row \( b-1 \) containing \( \beta \) at column \( a-1 \), and row \( b \) containing \( \alpha \) at column \( a \). Note that our construction of Faulhaber’s triangle ensures \( \alpha = \frac{b}{a} \beta \).

Next, observe that \( f_p(1) = a_{p+1} + \cdots + a_1 = 1 \), for all \( p \), so that \( a_1 = 1 - (a_{p+1} + \cdots + a_2) \). This is the reason the leftmost element of each row is chosen such that the values on the row sum up to 1.

Now, by a straightforward induction, if the numbers on row \( p \) are the coefficients of \( f_p(n) \), then the numbers on row \( p + 1 \) are the coefficients
of $f_{p+1}(n)$. The base case is immediate, as $f_0(n) = n$.

References

