From Robotics to Facility Location:  
Contraction Functions, Weber Point, Convex Core

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Abstract

We define and study the concept of Contraction Functions to solve the point formation problem in robotics. Interestingly, the discussion leads to the Weber point that plays an important role in facility location. We introduce the notion of Convex Core and show that this helps to locate the Weber point.

1 Introduction

Teams of cooperative robots are assumed to have capabilities of very high sophistication level ([Par96]). Such teams of robots have to be coordinated, and in order to avoid bottlenecks, coordination should be carried out in a distributed way.

Point formation is the problem that $n$ mobile robots in the plane should meet in one destination point. This destination point should be independent of coordinate systems, and it should not rely on an ordering of the robots, especially, there is no leader among the robots. The question is the following: Which distributed algorithm can the robots use to complete the task?

Besides practical studies, the point formation problem has been investigated in a more theoretical way [SY93, SS96, SY99, Pre01, FPSW99, FPSW00]. The solutions presented in these publications are complex, and some of them are hard to understand or verify. We present a new approach whose most distinguishing feature is its simplicity. A rough outline of our solution is as follows: Every robot takes a snapshot and detects the multi set of positions of the robots. This multi set is the input for a so called contraction function, which gives the destination point. Due to the definition of contraction functions, the destination point is independent of coordinate systems and invariant under straight line movement towards the destination point. All the robots have to do is to compute the destination point and move towards it. This solution works in an asynchronous model, where the robots take their snapshots independently and there is no global clock. Moreover, we obtain simple and easy to understand proofs.

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The results in this paper are part of the author’s PhD thesis [Sch02]
This paper has the following structure. Section 1.1 gives the exact description of the model we consider. Section 2 defines contraction functions and shows that with this concept, the point formation problem can be solved. Furthermore, this section presents contraction functions for small multi sets, derives some uniqueness results, and proves important properties. In section 3, the Weber point is introduced. The Weber point plays an important role in the field of facility location, and it turns out that this point can be used to obtain a contraction function for multi sets of arbitrary cardinality. Section 4 introduces the concept of convex core and discusses its relation to contraction functions. In section 5, we show that contraction functions can solve other formation problems as well. Section 6 concludes the paper.

1.1 The Model

We consider the following model. There are \( n \) mobile robots in the plane. The robots are anonymous in the sense that they all execute the same algorithm and they are identical copies of each other. The robots have no access to a common coordinate system, and there is no way for the robots to communicate.

A robot is able to take snapshots of the whole plane. With a snapshot, the robot is able to detect the other robots and their positions, and it can measure distances and angles. Any two robots may use different unit lengths.

The robots are assumed to be moving points, and two or more robots can occupy the same position simultaneously. The robots are able to detect multiplicity, i.e., for every point in the plane, the robots can count the number of robots on this point.

Initially, all robots are sleeping. Every robot will awake eventually, independent of the others. Then, the robot can look around (i.e., it takes a snapshot), it computes the destination point and moves towards this. The robot can do the same several times, but we do not specify this. Furthermore, it is possible that the robot has breaks, i.e., for some time, it does nothing. Each robot has its personal speed, but it is assumed that every robot will reach a computed destination point, i.e., the robot needs finite time to travel a finite distance.

1.2 Mathematical Concepts

This section presents some mathematical concepts that will be used in this paper. The first such concept is the concept of multi set.

**Definition 1.1** A mapping \( X : \mathbb{R}^2 \to \mathbb{N} \) is called a multi set of points. For a point \( p \in \mathbb{R}^2 \), we define the corresponding multi set

\[
\{p\} := \mathbb{I}_p \quad \text{where} \quad \mathbb{I}_p(q) = \begin{cases} 
0 & \text{if } q \neq p \\
1 & \text{if } q = p 
\end{cases}
\]

Throughout this paper, we only consider finite multi sets, i.e., the cardinality

\[
|X| := \sum_{p \in \mathbb{R}^2} X(p)
\]
Let \( X, X' \) be two multi sets, we define the following operations.

- **Union** \( \uplus \): \( (X \uplus X')(p) := X(p) + X'(p) \)
- **Intersection** \( \cap \): \( (X \cap X')(p) := \min\{X(p), X'(p)\} \)
- **Setminus** \( \setminus \): \( (X \setminus X')(p) := \max\{X(p) - X'(p), 0\} \)

\( X \) is called a set, iff \( X(p) \leq 1 \). Given a multi set \( X \), \( \text{uniq}(X) \) denotes the set resulting by removing multiplicities, i.e., \( p \in \text{uniq}(X) \), iff \( X(p) \geq 1 \).

For a multi set \( X \) of points, we define the convex hull \( CH(X) \) as the smallest convex set \( K \subset \mathbb{R}^2 \) with \( \text{uniq}(X) \subset K \).

Sometimes, we will use the more familiar notation \( X = \{p_1, \ldots, p_n\} \), where it is possible that \( p_i = p_j \) for \( i \neq j \). We have to mention that the enumeration in this notation is not a part of the structure of the multi set \( X \), the enumeration is arbitrary and only a help to handle multi sets.

Multi sets of collinear points will play an important role. For such a multi set, it will be crucial, whether it has a median or not. Therefore, we give a formal definition of median.

**Definition 1.2** Let \( X = \{p_1, \ldots, p_n\} \) be a multi set of collinear points, i.e., there is a straight line \( l \) such that \( p_i \in l \) for all \( i \). A point \( q \in l \) is called a median of \( X \), iff

\[
\left| \sum_{i=1}^{n} \frac{p_i - q}{|p_i - q|} \right| < X(q)
\]

A basic lemma about the existence of medians will be used frequently and is therefore shown here.

**Lemma 1.3** If a multi set \( X \) of collinear points has a median \( q \), then \( q \) is unique and \( q \in X \), i.e., \( X(q) \geq 1 \). If \( n := |X| \) is even, then \( X(q) \geq 2 \).

Furthermore, if \( n \) is odd, then the median exists.

**Proof:** Due to definition, for a median \( q \), we obtain that \( 0 < X(q) \), i.e., \( X(q) \geq 1 \). If \( n \) is even, the sum of \( n - 1 \) unit vectors \( \pm v \) cannot be zero, therefore \( X(q) \geq 2 \).

In order to show uniqueness, we choose \( q', q'' \in X \) such that \( q', q'' \) are on different sides of \( q \) and there are no points in between \( q', q'' \) other than \( q \). There are integers \( k', k'' \geq 0 \) with \( k' + k'' + X(q) = n \) and

\[
\sum_{i=1}^{n} \frac{p_i - q}{|p_i - q|} = k' \frac{q' - q}{|q' - q|} + k'' \frac{q'' - q}{|q'' - q|}
\]
Since $|k' - k''| < X(q)$ we obtain $\pm (k' - k'') < X(q)$.

$$\sum_{i=1}^{n} \frac{p_i - q'}{|p_i - q'}| = (k' - X(q')) \frac{q' - q}{|q' - q|} + (k'' + X(q)) \frac{q'' - q}{|q'' - q|}$$

$$= (X(q') + X(q) + k'' - k') \frac{q - q'}{|q - q'|}$$

This implies that $q'$ is not a median.

Let $n$ be odd. If $X$ is a set, then it has a median $q \in X$. With the following operations, every multi set of collinear points can be constructed from a set, and the median remains constant. Assume that $X$ is a multi set with median $q$. Let $p_i, p_j \in X$ be two points with the property $\frac{p_i - q}{|p_i - q|} = \frac{p_j - q}{|p_j - q|}$, then $q$ is the median of the multi set $X' := (X \setminus \{p_j\}) \cup \{p_i\}$. Furthermore, for a point $p_i \in X$ with $p_i \neq q$, $q$ is the median of the multi set $X' := (X \setminus \{p_i\}) \cup \{q\}$.

For our robot model we need to define distance preserving functions.

**Definition 1.4** A function $o : \mathbb{R}^2 \to \mathbb{R}^2$ is called isometric, if for all pair of points $p, q \in \mathbb{R}^2$, $|o(p) - o(q)| = |p - q|$ holds.

In this paper, we use several isometric functions:

- **Rotation:** For an angle $\varphi \in [0^\circ, 360^\circ]$ and for a point $p \in \mathbb{R}^2$, $R_{\varphi, p}$ denotes a rotation by $\varphi$ around $p$.
- **Mirroring or Reflection:** Given a straight line $l$, $M_l$ is the mirroring about $l$.
- **Translation:** Let $v \in \mathbb{R}^2$, the translation $T_v$ is defined as $T_v(z) := z + v$.

## 2 Contraction Functions and their properties

This section defines contraction functions and discusses their properties. Furthermore, we present examples of contraction functions for multi sets of small cardinality.

**Definition 2.1** Let $n \in \mathbb{N}$ be fixed, and let $C$ be a function that maps a multi set of $n$ points to a point $c \in \mathbb{R}^2$. The function $C$ is called a Contraction Function, if for all multi sets $X = \{p_1, \ldots, p_n\}$ the following properties are fulfilled:

1. There is no ordering among the robots, i.e., for every permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$,

$$C(p_{\pi(1)}, \ldots, p_{\pi(n)}) = C(p_1, \ldots, p_n)$$

2. The robots do not have a common coordinate system: For every isometric function $o : \mathbb{R}^2 \to \mathbb{R}^2$,

$$C(o(p_1), \ldots, o(p_n)) = o(C(p_1, \ldots, p_n))$$
3. Linear movement towards the contraction point does not change it: Let $c := C(X)$, for every vector $t = (t_1, \ldots, t_n) \in [0, 1]^n$,

$$C(\ldots, (1 - t_i)p_i + t_ic, \ldots) = c$$

If $c = C(X)$ for a contraction function $C$, then $c$ is called a Possible Contraction Point or Contraction Point of $X$.

Let $X = \{ \ldots, p_i, \ldots \}$ be a multi set. For a contraction function $C$ and a vector $t = (t_1, \ldots, t_n) \in [0, 1]^n$, $c := C(X)$, define the multi set $X' := \{ \ldots, (1 - t_i)p_i + t_ic, \ldots \}$. $X'$ is called a contraction of $X$, and we say that $X$ can be contracted to $X'$.

**Remark:** Lemma 2.3 will show that a contraction point does not exist, if the multi set consists of collinear points without median. Therefore, the “all multi sets” in definition 2.1 means that we exclude these cases.

We start with the discussion of the properties of contraction functions for small multi sets. The simplest (non empty) multi set of points consists of one point, and for this case, we get the following result.

**Lemma 2.2** Let $X = \{p\}$, then $c := p$ is the only possible contraction point.

**Proof:** It is obvious that $c$ fulfills the properties of a contraction point.

Let $o$ be a rotation by an arbitrary angle $\alpha \in [0^\circ, 360^\circ]$ around $p$. Since $o(p) = p$, for every contraction point $c$, $o(c) = c$ must hold. This can only be fulfilled with $c = p$.

The next case is $X = \{p_1, p_2\}$. Two points are trivially collinear, and if $p_1 \neq p_2$, $X$ has no median. The next lemma shows that for collinear points, the existence of a median is crucial.

**Lemma 2.3** Let $X$ be a multi set with the property that all points of $X$ are collinear, i.e., they are on a straight line $l$. Then there are two cases:

1. $X$ has a median $q$. Then $q$ is the only possible contraction point for $X$.

2. $X$ has no median. Then no contraction point exists.

*Especially in the case $n = 2$, there is a contraction point iff both points have the same position.*

**Proof:** Let $M_l$ be the mirroring about the line $l$. Since every point $p_i \in X$ lies on $l$, $M_l(p_i) = p_i$ holds. This implies that for every contraction function $C$, $C(X) = C(M_l(X)) = M_l(C(X))$. Therefore, $C(X) \in l$.

If the median $q$ exists, then a point $p_i \in X$ can move towards it, without changing the median property of $q$. This was shown in the proof of Lemma 1.3. Therefore, the median is a possible contraction point.
To show the uniqueness, we choose an arbitrary vector $v$ parallel to $l$ with $|v| \geq |p_i - q|$ for all $p_i \in X$. If $n$ is odd we define the multi set

$$X'(p) := \begin{cases} 1 & \text{if } p = q \\ \frac{n-1}{2} & \text{if } p = q - v \text{ or } p = q + v \\ 0 & \text{else} \end{cases}$$

If $n$ is even, we define

$$X'(p) := \begin{cases} 2 & \text{if } p = q \\ \frac{n-2}{2} & \text{if } p = q - v \text{ or } p = q + v \\ 0 & \text{else} \end{cases}$$

Due to its definition, $X'$ can be contracted to $X$. Since $X'$ is symmetric by an $180^\circ$ rotation around $q$, $q$ is the median of $X'$ and it is the only possible contraction point. Contraction of $X'$ to $X$ shows that $q$ is the only possible contraction point for $X$.

As to the second claim, assume that $X$ has no median, which implies that $n$ is even. Assume that the points are ordered linearly with respect to their indices. This implies that $p_{\frac{n}{2}} \neq p_{\frac{n}{2}+1}$ (otherwise, $p_{\frac{n}{2}}$ is a median). Assume for contradiction that there is a contraction function $C$ with $c := C(X)$. The point $c$ must be one of the points $p_{\frac{n}{2}}, p_{\frac{n}{2}+1}$ or must lie on the straight line segment between them (otherwise, $\frac{n}{2}$ points must pass through $p_{\frac{n}{2}}$ or $p_{\frac{n}{2}+1}$, such that this point becomes a median and the unique contraction point $c$). For the multi set

$$X'(p) := \begin{cases} \frac{n}{2} & \text{if } p = p_{\frac{n}{2}} \\ \frac{n}{2} & \text{if } p = p_{\frac{n}{2}+1} \\ 0 & \text{else} \end{cases}$$

we obtain that $C(X) = C(X') = c$. Letting $c' := \frac{1}{2}(p_{\frac{n}{2}} + p_{\frac{n}{2}+1})$, the rotation by $180^\circ$ around $c'$ transforms $X'$ into itself. Therefore, $c = c'$. But since $c'$ is not invariant under movement towards it, $X'$ has no contraction point. This contradicts the assumption. \qed

Remark: In contrast to [SY99], contraction functions do not solve the problem for an even number of collinear points without median. But the problem can be solved with a small extension and one assumption from [SY99]: Initially, the positions of the robots are different. If the number of robots is bigger than 2, then the robots inside move to the middle point defined by the two outer points, and the two robots outside do not move until the multi set of positions has a median. The only not solvable instance remains the case of two different points, but this case cannot be solved for oblivious robots in [SY99], either.

The previous lemma showed that there is no contraction function for 2 different points, and that there is no contraction function for 4 collinear points without median. However, similar to the case of one point, there is a unique contraction point for 3 points.

**Theorem 2.4** For $n \in \{3, 4\}$, there exists a unique contraction point (except in the case of 4 collinear points without median).
Proof: We discuss 4 cases. For every case, we present a solution and show uniqueness. Uniqueness is shown as follows: First, we prove uniqueness for symmetric multi sets. Second, the symmetric multi sets are contracted to arbitrary multi sets.

Case 3.a \( n = 3 \) and the three points form a triangle in which every angle is smaller than 120°. Then there is a unique point \( s \), the so called Steiner point, for which the angle between every pair of two vectors \((p_i - s), (p_j - s), i \neq j\), is equal to 120°. The Steiner point \( s \) is a possible contraction point (Proof of existence and construction of the Steiner have been presented in many publications, see for example [KM97]).

To show uniqueness, we consider an equilateral triangle. Due to the rotation symmetry, the Steiner point is the only possible contraction point. Since a contraction function must be invariant under linear movement, and since each triangle with property 3.a can be obtained from an equilateral triangle (see Figure 1) the result holds for every of these triangles.

![Figure 1: An arbitrary triangle □ derived from an equilateral triangle ⊙, \( \alpha = 120° \).](image1)

Case 3.b \( n = 3 \) and the three points form a triangle in which one angle is equal to or bigger than 120°. Then the point at the biggest angle is a possible contraction point.

It is assumed that the triangle is isosceles, we use the notation from Figure 2. Due to symmetry every contraction point must lie on the straight line which goes through \( p_1 \) and is perpendicular to the straight line through the other points. Assume for contradiction, that there is a contraction point \( c \) above \( p_1 \).

![Figure 2: Triangle with one angle bigger than 120°.](image2)

e.g., \( c = c_1 \). When \( p_2 \) moves to \( c_1 \), it has to cross the straight line through \( p_1, p_3 \). For the multi set \( \{p_1, p_2, p_3\} \), the point \( p_1 \) is the only possible contraction point.
Therefore, there is no contraction point above $p_1$. Assume for contradiction that there is a contraction point $c$ below $p_1$, e.g., $c = c_2$. For the multi set $\{p_1, c_2, p_3\}$, the Steiner point is the only possible contraction point (case 3.a). In this situation, every contraction point lies inside the triangle and cannot be equal to $c$. This is a contradiction. Therefore $c := p_1$ is the only possible contraction point for an isosceles triangle. Since every triangle can be obtained from an isosceles triangle by movement towards $p_1$, $c := p_1$ is the only solution for every case.

**Case 4.a** $n = 4$ and the four points form a convex quadrangle. In this case, the intersection of the diagonals is a possible contraction point.

If the four points form a rectangle the intersection of the diagonals is the only possible contraction point. Every arbitrary convex quadrangle can be obtained from a rectangle by movement towards the diagonals. Therefore, the intersection of the diagonals is the only possible solution.

**Case 4.b** $n = 4$ and three points form a triangle with the fourth point inside. Then the point inside is a possible contraction point.

Figure 3 shows 3 different points $c_1, c_2, c_3$ which might be possible solutions. Assume for contradiction, that there is a contraction point $c \in \{c_1, c_2, c_3\}$.

![Figure 3: 4 points in non convex position.](image)

When $c = c_1$, then in order to move to $c_1$, $p_2$ has to cross the straight line through $p_1, p_4$. When $c = c_2$, $p_3$ has to cross the line through $p_2, p_4$. When $c = c_3$, $p_3$ has to cross the line through $p_2, p_4$. In each case, there are 3 points on a line, which means that every contraction point must lie on that line. But this is a contradiction.

**Further Properties of Contraction Functions**

We presented some existence and uniqueness results for small multi sets of cardinality less than 5. For better understanding of contraction functions, we give characterizations for multi sets of arbitrary cardinality. The first result can be regarded as a majority lemma.

**Lemma 2.5 (Majority)** If there is a $q \in \mathbb{R}^2$ with $X(q) = k > \frac{n}{2}$, then $q$ is the only possible contraction point.
Proof: The property that more than half of the points have position \( q \) is invariant under linear movement towards \( q \). Therefore \( q \) is a possible contraction point. Let \( C \) be a contraction function and \( c := C(X) \), then
\[
c = C(q, \ldots, q, p_{k+1}, \ldots, p_n) = C(q, \ldots, q, c, \ldots, c) = C(c, \ldots, c, q, \ldots, q)
\]
\[
= q
\]

Due to their definition, contraction functions fulfill the contraction property, i.e., if the point move towards the destination point, the destination point does not change. Contraction functions fulfill an extension property in the following sense as well. This result is interesting for its own; moreover, it is helpful to simplify some proofs.

Lemma 2.6 (Extension) Let \( C \) be a contraction function and \( c := C(p_1, \ldots, p_n) \). For \( t = (t_1, \ldots, t_n) \in [-1, \infty)^n \), define \( p'_i := p_i + t_i(p_i - c) \). It holds
\[
C(p'_1, \ldots, p'_n) = c
\]

Proof: For \( t \in [-1, 0]^n \), the claim is the contraction property. To show the claim for \( t \in [0, \infty)^n \), we show that the lemma is true for \( t := (1, \ldots, 1) \), i.e., the distance from \( c \) is doubled. Iteration and the contraction property then show the result for arbitrary \( t \in [0, \infty)^n \).

Define \( p'_i := p_i + (p_i - c) \) and for \( c' := C(p'_1, \ldots, p'_n) \), compute the vector \( v := (c' - c) \), see Figure 4. For the points \( p''_i := \frac{1}{2}(p'_i + c') \), we obtain
\[
p''_i - p_i = \frac{1}{2}(c' - c).
\]

This leads to
\[
c' = C(p''_1, \ldots, p''_n) = C(p'_1, \ldots, p'_n) = C(p_1, \ldots, p_n + \frac{1}{2}(c' - c), \ldots)
\]
\[
= C(p_1, \ldots, p_n) + \frac{1}{2}(c' - c) = c + \frac{1}{2}(c' - c)
\]

Since \( c' = c + (c' - c) \), this implies \( c' - c = 0 \). □

Given a multi set \( X \) of points, we want to specify the possible contraction points for \( X \). For some special cases, we derived uniqueness results for the
contraction points. For other cases, we do not have uniqueness results, but the positions of contraction points can be restricted as well. The following theorem shows that the set of contraction points is a subset of the convex hull.

**Theorem 2.7** If \( c \) is a contraction point for a multi set \( X \), then \( c \) lies in the convex hull of \( X \).

**Proof:** Assume for contradiction that \( c \) is outside the convex hull of \( X \). The convex hull \( CH(X) \) is convex and bounded by a polygon. In order to move to \( c \), the points have to cross a straight line \( l \) with the property \( c \notin l \), see Figure 5. The line \( l \) can be chosen as the straight line defined by the corresponding line segment of the boundary. Therefore, it is possible that all points are on \( l \). But due to Lemma 2.3 this is either not solvable or \( c \) must lie on \( l \). This contradicts the assumption. \( \square \)

### 3 Weber Point

The previous section presented contraction functions only for small multi sets. In order to get contraction functions for multi sets of arbitrary cardinality, we look at the Weber point. This point is defined in the following way.

**Definition 3.1** Let \( X = \{p_1, \ldots, p_n\} \) be a multi set of points. The Weber point \( W(X) \) of \( X \) is defined as the point that minimizes the function

\[
z \mapsto f(z) := \sum_{p_i \in X} |p_i - z| \quad z \in \mathbb{R}^2
\]

The Weber point has a long history, and since many mathematicians worked on it, it has many different names, too, e.g., Fermat–Torricelli, Fermat–Weber. For simplicity, we avoid the discussion of this and refer to [Dör58, Wes93, Dre95, KM97, DH02]. The Weber point is named after Alfred Weber, who did some work on facility location [Web09].

If \( X \) consists of one point \( p_1 \), then \( p_1 \) is the Weber point. For a multi set of two different points, every point on the straight line segment connecting \( p_1 \) and \( p_2 \) is a Weber point. In this case, the Weber point is not unique. To characterize such cases, we cite the following lemma.
Lemma 3.2 ([KM97]) The Weber point is unique, except when $X$ consists of an even number of collinear points without median.

Our interests in Weber points are based on the fundamental result that shows a strong connection between contraction functions and Weber point.

**Theorem 3.3** If the Weber point is unique, then it is a possible contraction point.

**Proof:** We have to show that the Weber point is invariant under linear movement towards it. It is obvious that the Weber point fulfills the other properties. Let $w := W(X)$ be the Weber point of $X = \{p_1, \ldots, p_n\}$, for $t \in [0, 1]$, define $p' := (1-t)p_1 + tw$. Due to definition,

$$\sum_{i=1}^{n} |p_i - w| < \sum_{i=1}^{n} |p_i - z| \quad \forall z \in \mathbb{R}^2 \setminus \{w\}$$

holds. This and the inequality $|z - p_1| \leq |z - p'| + |p' - p_1|$ lead to

$$\sum_{i=2}^{n} |p_i - w| + |p' - w| < \sum_{i=2}^{n} |p_i - z| + |p_1 - z| + \overbrace{|p' - w| - |p_1 - w|}^{=-|p' - p_1|}$$

$$\leq \sum_{i=2}^{n} |p_i - z| + |p' - z| + |p' - p_1| - |p' - p_1|$$

$$= \sum_{i=2}^{n} |p_i - z| + |p' - z|$$

This means that $w$ is the unique Weber point of the multi set $(X \cup \{p'\}) \setminus \{p_1\}$. \qed

Theorem 3.3 directly leads to the following corollary.

**Corollary 3.4** In Theorem 2.4 it has been shown that for the cases $n \in \{3, 4\}$ there is exactly one possible contraction point. Since every Weber point is a contraction point, Theorem 2.4 characterizes the Weber point.

Furthermore, we obtain a new proof for the following well-known fact.

**Corollary 3.5 ([KM97])** $W(X) \in CH(X)$.

**Proof:** If the Weber point is not unique then the corollary is true. If the Weber point is unique, then $w := W(X)$ is a contraction point. And since every contraction point lies in the convex hull (Theorem 2.7), $w$ is in the convex hull, too. \qed

The Weber point has the following properties. A similar version for sets was given in [Wei37], we generalize this result to multi sets. The result will play an important role in our study and will be exploited extensively.
Theorem 3.6  Let $w = W(X)$ be the unique Weber point of $X$, let $X(w) \in \mathbb{N}$ be the multiplicity of $w$ in $X$. It holds:

$$\left| \sum_{i=1}^{n} \frac{p_i - w}{|p_i - w|} \right| \leq X(w)$$

Moreover, $w$ is the only point with this property. Especially, if $w \not\in X$, i.e. $X(w) = 0$, we obtain

$$\sum_{i=1}^{n} \frac{p_i - w}{|p_i - w|} = 0$$

Proof: [follows [KM97]] Define $g(z) := |p - z|$ for an arbitrary point $p \in \mathbb{R}^2$.

If $z \neq p$, we compute the gradient as

$$\nabla g(z) = \frac{p_i - z}{|p_i - z|}$$

Let $v$ be a unit vector. If $z = p$, we obtain for the directional derivative in direction $v$

$$g_v(z) := \lim_{t \to 0} \frac{g(p + tv) - g(p)}{t} = 1$$

If $w \not\in X$, the function $f$ is differentiable in $w$. Since $w$ minimizes the sum of the distances, it holds

$$0 = \nabla f(w) = \nabla \left( \sum_{i=1}^{n} |p_i - w| \right) = \sum_{i=1}^{n} \frac{p_i - w}{|p_i - w|}$$

If $w \in X$, we have to be more careful. Let $k := X(w)$. For the function

$$\tilde{f}(z) := \sum_{i=1}^{n} \frac{|p_i - z|}{|p_i - w|} = f(z) - k|w - z|$$

and the directional derivative in direction $v$ we obtain

$$f_v(w) = \left\langle \nabla \tilde{f}(w), v \right\rangle + kg_v(z) = \left\langle \sum_{i=1}^{n} \frac{p_i - w}{|p_i - w|}, v \right\rangle + k$$

Since $f$ is minimized in $w$, $f_v(w) \geq 0$ holds. The theorem is true, if $\nabla \tilde{f}(w) = 0$. Therefore, we can assume that $\nabla \tilde{f}(w) \neq 0$, and we define $v := -\frac{\nabla \tilde{f}(w)}{|\nabla \tilde{f}(w)|}$.

This leads to

$$0 \leq f_v(w) = -\left| \sum_{i=1}^{n} \frac{p_i - w}{|p_i - w|} \right| + k$$

On the other hand, assume that for $\tilde{w} \in \mathbb{R}^2$,

$$\left| \sum_{i=1}^{n} \frac{p_i - \tilde{w}}{|p_i - \tilde{w}|} \right| \leq X(\tilde{w})$$
holds. For the function
\[ \hat{f}(z) := \sum_{i=1}^{n} |p_i - z| = f(z) - X(\hat{w})|\hat{w} - z| \]
we obtain that
\[ 0 \leq -\left\langle \text{grad} \hat{f}(\hat{w}), \frac{\text{grad} \hat{f}(\hat{w})}{|\text{grad} \hat{f}(\hat{w})|} \right\rangle + X(\hat{w}) \]
Since for all \( v \in S^1 \),
\[ \left\langle \text{grad} \hat{f}(\hat{w}), \frac{\text{grad} \hat{f}(\hat{w})}{|\text{grad} \hat{f}(\hat{w})|} \right\rangle \geq \left\langle \text{grad} \hat{f}(\hat{w}), v \right\rangle \]
holds, it follows that \( f_w(\hat{w}) \geq 0 \). This means that \( f \) is minimized in \( \hat{w} \), and since \( w \) is unique, \( \hat{w} = w \).

The inequality described in theorem 3.6 shows a strong connection to the median of numbers. This is the reason why the Weber point is sometimes called spatial median.

Furthermore, the theorem shows important properties of the Weber point: For a given point \( q \in \mathbb{R}^2 \), it can be tested if \( q = W(X) \). Especially, it is easy to check whether \( W(X) \in X \) or not.

Furthermore, the theorem can be used to prove Theorem 3.3. For \( p_i \neq w \) and \( t \in [0, 1) \), \( p' := (1-t)p_i + tw \), we obtain
\[ \frac{p_i - w}{|p_i - w|} = \frac{p' - w}{|p' - w|} \]
However, we have to pay attention to the cases in which the multiplicity of the Weber point changes.

Remark: Since for \( n \in \{2, 3, 4\} \) there is at most one contraction function, the Weber point can be constructed very easily in these cases. It is remarkable that the Weber point is a solution which is minimal in the sense that it minimizes the path length the points have to move.

However, the Weber Point is not constructible by compass and ruler in general, as shown in [Baj88], [CM69]. Constructing the Weber point is equivalent to the computation of the roots of a high degree polynomial. Even for \( n = 5 \), one can find examples for which the Weber point cannot be constructed, see [CM69].

4 Convex Core

We mentioned that a contraction point must be in the convex hull. In this section we show that for an even number of points, this result can be improved. To be more precise, we define a so called convex core. The convex core is a subset of the convex hull. We show that for an even number of points, every
contraction point is inside the convex core. This is an improvement of the result, that every contraction point is in the convex hull (theorem 2.7). Since the Weber Point is a contraction point, the Weber point is in the convex core, too. To our knowledge, this has not been shown before.

**Definition 4.1** For a point \( p \in X \), the set \( K_p \subset \mathbb{R}^2 \) is defined as the convex hull of \( X \setminus \{p\} \), i.e. \( K_p := CH(X \setminus \{p\}) \). The Convex Core of \( X \) is defined as

\[
CC(X) := \bigcap_{p \in X} K_p
\]

**Example:** For \( X = \{p_1, p_2\} \) we obtain for the convex core

\[
CC(X) = \begin{cases} 
\emptyset & \text{if } p_1 \neq p_2 \\
p_1 & \text{if } p_1 = p_2
\end{cases}
\]

If \( |X| = 4 \) then \( CC(X) \) consists of a single point, or, if the points are collinear, of the line segment connecting the two middle points.

Figure 6 shows two examples. The convex cores are marked grey.

![Figure 6: Examples for convex cores](image)

The following lemma shows basic properties of the convex core. For a multi set \( X \), the convex hull \( CH(X) \) can be described by the multi set of boundary points \( \partial X \subset X \). A point \( p \in X \) belongs to the boundary \( \partial X \), if \( p \) is on the polygon that bounds \( CH(X) \). Note that, in \( \partial X \), points can occur with multiplicity bigger than 1. For instance, if \( X \) consists of collinear points, \( \partial X = X \) holds. With \( X^\circ \), we denote the multi set of inner points, i.e., \( X^\circ := X \setminus \partial X \).

**Lemma 4.2** The convex core \( CC(X) \) has the following properties

- \( CC(X) \) is a convex set.
- The convex core is monotone, i.e., for \( X \subset X' \), \( CC(X) \subset CC(X') \) holds.
- If \( p \in CC(X) \), then \( CC(X \cup \{p\}) = CC(X) \).
- For \( |X| \geq 4 \), \( CC(X) \neq \emptyset \) holds.
- The convex core \( CC(X) \) is a superset of \( CH(CC(\partial X) \cup X^\circ) \).
The convex core $CC(X)$ only depends on $\partial X, \partial X^\circ$, i.e., $CC(X) = CC(\partial X \cup \partial X^\circ)$.

**Proof:** The convex core is defined as the intersection of convex sets. Since the intersection of convex sets is convex, too, $CC(X)$ is convex.

To show monotonicity, it is enough to show it for the case $|X'| = |X| + 1$. For $X = \{p_1, \ldots, p_n\}$ and $X' = X \cup \{p\}$, we obtain

$$CC(X') = \bigcap_{i=1}^n CH((X \cup \{p\}) \setminus \{p_i\}) \cap CH(X)$$

$$\supset \bigcap_{i=1}^n CH(X \setminus \{p_i\}) = CC(X)$$

Let $p \in CC(X)$, for this we compute

$$CC(X \cup \{p\}) = \bigcap_{i=1}^n CH((X \cup \{p\}) \setminus \{p_i\}) \cap CH(X)$$

$$\subset \bigcap_{i=1}^n CH((X \cup \{p\}) \setminus \{p_i\}) = \bigcap_{i=1}^n CH(X \setminus \{p_i\}) = CC(X)$$

For every $X$ with $|X| = 4$, $CC(X) \neq \emptyset$ holds. Due to monotonicity, this implies that $CC(X) \neq \emptyset$ for $|X| \geq 4$.

For $p_i \in X^\circ$ and for $p_j \in X$, $p_i \in K_{p_j}$ holds. Therefore, $p_i \in CC(X)$ and $CH(X^\circ) \subset CC(X)$. Since $\partial X \subset X$, $CC(\partial X) \subset CC(X)$ (monotonicity). Furthermore, $CC(X)$ is convex, $CH(CC(\partial X) \cup X^\circ) \subset CC(X)$.

Let $p_i \in X \setminus \partial X \setminus \partial X^\circ$. For $p_j \neq p_i$, $CH(X \setminus \{p_j\}) = CH(X \setminus \{p_j, p_i\})$ holds. This leads to $CC(X) = CC(X \setminus \{p_i\})$. $\square$

In order to prove the main result of this section, we need the following lemma.

**Lemma 4.3** Let $n \geq 4$ be even, let $X = \{p_1, \ldots, p_n\}$ such that $p_1, \ldots, p_{n-1}$ are collinear, i.e. they lie on a straight line $l$, and $p_n \notin l$. Then the median of $p_1, \ldots, p_{n-1}$ is a contraction point and it is unique.

**Proof:** To simplify notation let the $p_1$ be the median of $p_1, \ldots, p_{n-1}$, wlog $p_1 = 0$. Let $q \in \{p_2, \ldots, p_{n-1}\}$ be a point with the greatest distance to $p_1$. Define the multi set

$$X' := \{-p_n, -q, \ldots, -q, q, \ldots, q, p_n\}$$

in which the number of points on position $-q$ is equal to the number of points on $q$, see Figure 7 for an example. Due to rotation symmetry, for $X'$ and every contraction function $C$, $p_1 = C(-p_n, -q, \ldots, -q, q, \ldots, q, p_n)$ holds. Then, the contraction property of $C$ leads to

$$p_1 = C(-p_n, -q, \ldots, -q, q, \ldots, q, p_n) = C(p_1, p_2, \ldots, p_{n-1}, p_n)$$

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This uniqueness result helps to prove the main result of this section. The following theorem improves theorem 2.7 and corollary 3.5. Moreover, the notion of contraction functions gives a short and elegant proof for this result.

**Theorem 4.4** Let $|X|$ be even. Then every contraction point for $X$ lies in the convex core $CC(X)$. Especially, the Weber point $W(X)$ is in the convex core as well.

**Proof:** The theorem is true for $n = 2$: Either there is no contraction point or the convex core consists of the contraction point only. Therefore, it can be assumed, that $n \geq 4$.

Let $c$ be a contraction point. Assume for contradiction that $c \in CH(X) \setminus CC(X)$. Then there is a $p_i \in X$ with

$$c \notin CH(X \setminus \{p_i\})$$

The set $CH(X \setminus \{p_i\})$ is compact, convex and bounded by a polygon. Therefore, there is a straight line $l$ between $c$ and $CH(X \setminus \{p_i\})$. In order to move to $c$, $n - 1$ points have to cross the line $l$. But this is the situation from Lemma 4.3, which says that $c$ is on $l$. This contradicts the assumption and proves the theorem. □

This theorem can be used to “reprove” known results.

**Lemma 4.5 (part of lemma 2.3)** If $X = \{p_1, p_2\}$ consists of two different points, then there is no contraction point.

**New Proof:** For the convex core, $CC(X) = \{p_1\} \cap \{p_2\} = \emptyset$ holds. Since every contraction point must lie in the convex core, there is no contraction point. □

**Lemma 4.6 (part of theorem 2.4)** Let $X = \{p_1, p_2, p_3, p_4\}$ such that these four points are not collinear. Then the possible contraction point mentioned in theorem 2.4 is unique.
New Proof: If the points are not collinear, the convex core $CC(X)$ consists of one single point.

It is possible that the convex core of a multi set is equal to the convex hull. If this is the case, theorem 4.4 does not help to restrict the position of possible contraction points. For a special case, theorem 4.4 can be improved.

Theorem 4.7 Let $|X|$ be even. For every $k \in \mathbb{N}$, $k \geq 1$, the property $C(kX) \in CC(X)$ holds.

Proof: Lemma 4.3 remains true for the weighted case. Therefore, the property holds.

The definition of the convex core can be used to design an algorithm that computes the convex core: Let $X$ be a multi set and let $n := |X|$. For every point $p \in X$, compute $K_p$ in $O(n \log n)$ time (see [PS85]). Then compute the intersection of these $n$ polygons. It takes at least linear time to intersect two polygons, therefore, the algorithm takes at least quadratic time. However, we can do better.

Theorem 4.8 Let $X$ be a multi set, and let $n := |X|$. The convex core of $X$ can be computed in $O(n \log n)$ time.

Proof: To show the upper bound, we present an algorithm that computes the convex core in $O(n \log n)$ time. We think of $\partial X, \partial X^\circ$ as circularly ordered lists. First, compute the convex hulls $CH(X)$ and $CH(X \setminus \partial X)$ in $O(n \log n)$ time. Let $\partial X = [p_1, p_2, \ldots, p_m]$ and let $\partial X^\circ = [q_1, \ldots, q_r]$ be the ordered lists of points. Define $K := \partial X$.

Second, for every $p_i \in \{p_1, \ldots, p_m\}$, do the following, starting with $i = 1$ and incrementing until $i = m$. Detect the points $q_i, \ldots q_r \in \partial X^\circ$ that are lying in the (possible degenerated) triangle spanned by $p_{i-1}, p_i, p_{i+1}^1$. This can be done in $O(\nu)$ time since $\partial X^\circ$ is a circularly ordered list. Detect the convex hull $H_i$ of $\{p_{i-1}, q_i, \ldots, q_r, p_{i+1}\}$ in $O(\nu)$ time in the following way, see Figure 8. For every point $q_i$, compute the angle between the two vectors $(q_i - p_{i-1}), (p_{i+1} - p_{i-1})$. Let $q_i^-$ be the first point that maximizes this angle. For every point $q_i^-$, compute the angle between the two vectors $(q_i - p_{i+1}), (p_{i-1} - p_{i+1})$. Let $q_i^+$ be the last point that maximizes this angle. The ordered list $[p_{i-1}, q_i^-, \ldots, q_i^+, p_{i+1}]$ corresponds to the boundary polygon of $H_i$. Let $D_i$ be a description of $H_i \cap K$. Then, $p_i$ is removed from $K$, $D_i$ is inserted in $K$. After the second step is done for every point $p_i$, $K = CC(X)$ holds.

To obtain the lower bound $\Omega(n \log n)$, we consider the multi set $X' := 2X$. For $X', CC(X') = CH(X') = CH(X)$ holds. Since we need $\Omega(n \log n)$ time to compute $CH(X)$, the same holds for $CC(X')$, too.  

\footnote{We omit the discussion of $p_{m+1} = p_1$ and $p_0 = p_n$}
5 Further Applications

If the Weber point is not in the given set, it can be used to get a solution for the circle formation problem, mentioned in [SS90, CFK97]. In this problem, the robots should move such that all their positions are on a circle with radius bigger than 0. The center of the circle is defined as the Weber point, the radius can be defined in different ways. If the robots all use the same unit distance, an arbitrary radius can be chosen. If this is not the case, the distances from the Weber point can be used, e.g., define the radius as the smallest distance from a point $p_i$ to $w$. For any fixed $k \in \{1, \ldots, n\}$, the $k$th smallest distance to the Weber point can be chosen as radius, too. If the Weber point is in the point set, then it can be used to solve a modified circle formation problem, in which all robots have to move to a circle or to the center of this circle. The robots at the Weber point remain there, the other robots form a circle around this center.

6 Conclusion

In order to solve the point formation problem, we defined contraction functions. This solution works in an asynchronous setting, and is quite easy to prove and to understand. For multi sets of cardinality smaller than five, we presented geometric constructions for contraction points. Furthermore, we showed that these contraction points are unique: First, we showed uniqueness for symmetric multi sets; second, the contraction property of contraction functions proved uniqueness for arbitrary multi sets. It turned out that contraction functions are interesting for their own sake, and we presented several interesting properties of contraction functions, e.g., that every contraction point for a multi set of points lies in the convex hull of these points.

The further discussion led to the Weber point. The Weber point plays an important role in facility location, and we showed that the Weber point is a possible contraction point. With this relation, we were able to ”re-derive” the result that the Weber point of a multi set of points lies in the convex hull of these points.
We proved new results as well: For an even number of points, every contraction point is in the convex core. The convex core is a subset of the convex hull. Since the Weber point is a possible contraction point, the Weber point is in the convex core, too.

It is an open question, whether the Weber point is the only possible contraction point. For multi sets with cardinality less than five, where the Weber point can be constructed by compass and ruler, we proved that the Weber point is the only possible contraction point.

If the contraction point must be on a given straight line, it is very easy to present contraction points different from the Weber point. This will be discussed in a forthcoming paper.

References


