On the $P_4$-components of Graphs

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Abstract

Two edges are called $P_4$-adjacent if they belong to the same $P_4$ (chordless path on 4 vertices). $P_4$-components, in our terminology, are the equivalence classes of the transitive closure of the $P_4$-adjacency relation. In this paper, new results on the structure of $P_4$-components are obtained. On the one hand, these results allow us to improve the complexity of the recognition and orientation algorithms for $P_4$-comparability and $P_4$-indifference graphs from $O(n^5)$ to $O(n^2m)$ and from $O(n^5)$ to $O(n^2m)$, respectively. On the other hand, by combining the modular decomposition with the substitution of $P_4$-components, a new unique tree representation for arbitrary graphs is derived which generalizes the homogeneous decomposition introduced by Jamison and Olariu [JO95].

1 Introduction

A $P_k$ ($C_k$) is a chordless path (cycle) on $k$ vertices. By the $P_4$ $abcd$, we denote the $P_4$ with vertices $a, b, c, d$ and edges $ab, bc$ and $cd$. An orientation $U$ of a graph $G$ is the anti-symmetric directed graph which arises from assigning a direction to each edge of $G$. A directed edge is denoted by $a \rightarrow b$ or $a \leftarrow b$.

Hoàng and Reed [HB89] suggested investigating $P_4$-comparability and $P_4$-indifference graphs which are defined as follows. An orientation is $P_4$-transitive if the orientation of every $P_4$ is transitive, ie type1 in Figure 1. Similarly, an orientation is said to be $P_4$-indifferent if every $P_4$ is indifferent, ie type2 in Figure 1. A graph which admits an acyclic $P_4$-transitive ($P_4$-indifferent) orientation is called $P_4$-comparability ($P_4$-indifference) graph.

Chvátal [Chv84] introduced perfectly orderable graphs as those graphs whose vertices can be ordered perfectly, ie the greedy algorithm proceeding along such an order computes a minimum coloring for each induced subgraph. He showed that a graph is perfectly orderable if and only if an acyclic orientation exists such that no $P_4$ $abcd$ is oriented $a \rightarrow b$ and $c \leftarrow d$ (called obstruction). Given an acyclic obstruction-free orientation, a number of problems can be solved in polynomial time which are NP-complete in general (see eg [Chv84], [Hoã94] and [AP96]). Unfortunately, it is NP-complete to decide whether a graph admits a perfect order [MP96].

On the other hand, both $P_4$-comparability and $P_4$-indifference graphs are perfectly orderable as they do not contain obstructions. In [HB89] and [HR89], Hoàng and Reed
presented an $O(n^4)$ recognition algorithm for $P_4$-comparability graphs and an $O(n^5)$ algorithm to compute the corresponding acyclic $P_4$-transitive orientation. The complexity of their recognition algorithm for $P_4$-indifference graphs is $O(n^6)$.

In this paper, we develop $O(n^2m)$ recognition and orientation algorithms for both classes of graphs. The key to our improvement lies in the detailed study of the $P_4$-adjacency relation: Two edges are $P_4$-adjacent if they belong to the same $P_4$. The equivalence classes of this $P_4$-adjacency relation are called $P_4$-components. Obviously, the orientation of an edge of a $P_4$-comparability ($P_4$-indifference) graph implies the orientation of all other edges in the same $P_4$-component.

As it turns out, the vertices incident to the edges of a $P_4$-component have a very special neighborhood relation to the other vertices. In fact, only two types of adjacency can occur. It is therefore quite natural to replace such a vertex set with two marker vertices. This substitution allows a recursive computation of the desired orientations; thus the running time is bounded by the time needed for orienting the edges in the $P_4$-components. Clearly, the latter can be done in $O(n^2m)$ with methods similar to those for the transitive orientation in [Gol77].

Moreover, the structure of the $P_4$-components allows us to refine the famous modular decomposition ([MS94], [ST94], also substitution decomposition [MR84]) in a way which maintains the uniqueness of the decomposition. Our unique decomposition tree further generalizes the homogeneous decomposition tree given by Jamison and Olariu [JO95]. Such decomposition trees can be used to solve hard problems efficiently, e.g., maximum clique, maximum stable set, minimum coloring, graph isomorphism, Hamiltonian cycle, cf [IO89], [BS89], [JO91], [JO92b], [JO92a], [HT95], [Gia96] and [BO96].

The remainder of this paper is organized as follows. The next section contains basic definitions and notations. Section 3 explores the structure of the $P_4$-components. The obtained results are used in Section 4 to design algorithms for the recognition and orientation of $P_4$-comparability and $P_4$-indifference graphs. In Section 5, we develop our new decomposition. Finally, the last section summarizes the results and poses some open problems.

2 Definitions

Let $G = (V, E)$ be an arbitrary graph and $W \subseteq V$ a subset of its vertices. By $G_W = (W, E(W))$, we denote the subgraph of $G$ induced by $W$, and $\overline{G} = (V, \overline{E})$ stands for the complement of $G$. For any vertex $v \in V$, the set of its adjacent vertices is called the neighborhood of $v$, denoted by $N(v)$. If two vertices $v$ and $w$ are adjacent in $G$, we say that $v$ sees $w$; otherwise we say that $v$ misses $w$.

A vertex $v \notin W$ is $W$-universal ($W$-null) if it sees (misses) all vertices in $W$. If $v \notin W$ is neither universal nor null, we call it $W$-partial. A $P_4$ is $W$-partial if it has at least one edge
in \( E(W) \) but not all of its vertices belong to \( W \). A vertex set \( H \subseteq V \) is called \textit{homogeneous} if \( 1 < |H| < |V| \) and no \( H \)-partial vertex exists. Graphs without homogeneous sets are called \textit{prime}.

As we are especially interested in \( P_4 \)s and \( P_4 \)-components, the following definitions and notations come in handy. Given a \( P_4 \) \( abed \), the edge \( be \) is called \textit{rib}, the edges \( ab \) and \( cd \) \textit{wings}, the vertices \( b \) and \( e \) \textit{midpoints} and the vertices \( a \) and \( d \) \textit{endpoints}. Two \( P_4 \)s are called adjacent if they have a common edge.

Throughout the whole paper, \( C^* \) stands for a \( P_4 \)-component \( C^*(vw) \) for the \( P_4 \)-component that contains the edge \( vw \). By harmless abuse of language, a \( P_4 \) with one edge (and therefore all its edges) in a \( P_4 \)-component is also said to be in \( C^* \). A vertex is covered by the \( P_4 \)-component \( C^* \) if it is incident to at least one edge in \( C^* \). The set \( V(C^*) \) of all vertices covered by \( C^* \) is called the \textit{cover} of \( C^* \). The sets of all \( V(C^*) \)-universal, -partial and -null vertices are denoted by \( P, R \) and \( Q \), respectively.

A \textit{trivial graph} has precisely one vertex and a \textit{trivial} \( P_4 \)-component consists of a single edge. A nontrivial \( P_4 \)-component \( C^* \) is called \textit{separable} if its cover \( V(C^*) \) can be partitioned into two vertex sets \( (V^1, V^2) \) such that each \( P_4 \) in \( C^* \) has its midpoints in \( V^1 \) and its endpoints in \( V^2 \).

Finally, the graph called \textit{pyramid} plays an important part in some of the theorems and proofs in this paper: A pyramid \( abedrp \) consists of a \( P_4 \) \( abed \) together with a \( \{a, b, c, d\} \)-universal vertex \( p \) and a \( \{a, b, c, d\} \)-partial vertex \( r \) which sees the midpoints of \( abed \) but misses its endpoints, see Figure 2.

![Figure 2: The pyramid abedrp.](image)

3 **Elementary properties of \( P_4 \)-components**

Most of the proofs in this section generalize an assertion \( A \) that holds for one \( P_4 \) in a \( P_4 \)-component \( C^* \) to all other \( P_4 \)s in \( C^* \). The inductive step consists of proving \( A \) for an additional \( P_4 \) based on the hypothesis that it already holds for an adjacent \( P_4 \). Given such an adjacent \( P_4 \), we have to distinguish the cases in which two ribs or two wings or a rib and a wing coincide. The first of these cases, however, can be omitted, as the next Lemma reveals. (Two \( P_4 \)s are called \textit{weak-adjacent} if two wings or a rib and a wing coincide.)

**Lemma 3.1** Two \( P_4 \)s with common ribs are connected by a sequence of weak-adjacent \( P_4 \)s.

**Proof.** Let \( abed \) and \( a'b'ed' \) denote two \( P_4 \)s with common ribs. If \( a \) and \( a' \) or \( d \) and \( d' \) coincide, the two \( P_4 \)s themselves are weak-adjacent and we are done. So assume \(|\{a, a', b, c, d, d'\}| = 6\).
If a misses \( d' \), then \( abed, abed' \) is a sequence of weak-adjacent \( P_4 \)s. The analogous argument applies if \( a' \) misses \( d \), so it remains to discuss the case \( ad', a'd \in E \).

If a misses \( a' \), we find that \( abed, aba'd, a'bed' \) is a sequence of weak-adjacent \( P_4 \)s, otherwise, if a is adjacent to \( a' \), then \( abed, ad'de, a'd'de \) denotes such a sequence. □

Consider a \( P_4 \) \( abed \) together with another vertex, say \( v \). Up to symmetry, all possible graphs induced by \( abed \) and \( v \) are enumerated in Figure 3, where bold lines indicate edges in the same \( P_4 \)-component. If we additionally assume that \( v \) is not covered by \( C^*(ab) \), the only graphs left are the \( F_1 \), the \( F_7 \) and the \( F_{10} \), i.e., \( v \) is either \{\( a, b, c, d \}\}-universal, \{\( a, b, c, d \}\}-null or it sees the midpoints but misses the endpoints of the \( P_4 \) \( abed \).

![Figure 3: All possibilities of a \( P_4 \) together with a fifth vertex \( v \).](image)

**Lemma 3.2** Let \( C^* \) be a \( P_4 \)-component and \( v \) a vertex not covered by \( C^* \). If \( v \) and a \( P_4 \) in \( C^* \) induces an \( F_7 \), then the graph induced by \( v \) and any \( P_4 \) in \( C^* \) is an \( F_7 \).

**Proof.** Our proof is by induction on the \( P_4 \)s in \( C^* \). So let \( abed \) and \( a'd'b'e'd' \) denote two weak-adjacent \( P_4 \)s in \( C^* \) and assume that the graph induced by \( abed \) and \( v \) is an \( F_7 \). According to Lemma 3.1, it suffices to distinguish the following cases:

**Case 1:** Two wings coincide. Without loss of generality, we may assume that the wing \( ab \) coincides with the wing \( a'd' \); thus either \( a' = a \) and \( b' = b \) or \( d' = b \) and \( b' = a \). The latter, however, is impossible because \( a'd' \) is a sequence of weak-adjacent \( P_4 \)s in \( C^* \) and assume that the graph induced by \( abed \) and \( v \) is an \( F_7 \). In the former case, the only possible induced graph is the \( F_7 \) as claimed.

**Case 2:** A wing coincides with a rib. A wing of \( abed \) cannot coincide with \( b'e' \) as otherwise the graph induced by \( a'd'b'e'd' \) and \( v \) would not be an \( F_1 \), \( F_7 \) or \( F_{10} \). Therefore, a wing of \( a'd'b'e'd' \) must coincide with \( be \). This implies that the graph induced by \( a'b'e'd' \) and \( v \) is an \( F_7 \); thus \( \{a, d, a', b', c, d'\} \subseteq V(G) \).

Without loss of generality (symmetry), let \( b = a' \) and \( c = b' \). Then \( d' \) sees \( a \) and \( d \), for otherwise \( abed' \) or \( dced' \) would be a \( P_4 \) in \( C^* \) that covers \( v \). So \( a'd'de \) is a \( P_4 \) in \( C^* \), a contradiction because \( a'd'de \) and \( v \) induce an \( F_7 \) as claimed. □

Let \( C^* \) be a nontrivial \( P_4 \)-component and \( r \) a vertex in \( R \). From the definition of the \( P_4 \)-components follows that a \( P_4 \) \( abed \) in \( C^* \) exists such that \( r \) is \{\( a, b, c, d \}\}-partial; hence \( abed \) and \( r \) induce an \( F_7 \). By Lemma 3.2, the vertex \( r \) sees the midpoints of every \( P_4 \) in \( C^* \) but misses its endpoints; thus \( C^* \) is separable.
Furthermore, $r$ cannot be adjacent to a vertex $q \in Q$, as otherwise any $P_4$ $abcdef$ in $C^*$ would imply a $P_4$ $qrba$ in $C^*$, a contradiction to our assumption that $r$ is not covered by $C^*$. Corollary 3.3 below restates these results.

**Corollary 3.3** Let $C^*$ be a nontrivial $P_4$-component and $R \neq \emptyset$. Then $C^*$ is separable and every vertex in $R$ is $V^1$-universal and $V^2$-null. Moreover, no edge between $R$ and $Q$ exists.

![Figure 4: Corollary 3.3 illustrated (dotted lines indicate possible edges in $V - V(C^*)$).](image)

**Lemma 3.4** Given a separable $P_4$-component $C^*$ with vertex partition $(V^1, V^2)$. Then neither a $P_3$ $abc$ with $a \in V^1$ and $b, c \in V^2$ nor a $P_3$ $abc$ with $a, b \in V^1$ and $c \in V^2$ exists.

**Proof.** In a first step, we show that no $P_3$ or $\overline{P}_3$ as described in our lemma has edges in $C^*$. Assume a $P_3$ $abc$ with $a \in V^1$ and $b, c \in V^2$. Since $C^*$ is separable, $bc$ cannot belong to $C^*$. Now suppose $ab \in C^*$. Then a $P_4$ $bade$ in $C^*$ exists with $d \in V^1$ and $e \in V^2$. If $ce \in E$, then the $P_4$ $abce$ would contradict the separability of $C^*$. Hence $ce \notin E$. But $dc \in E$ implies the $P_4$ $bedc$, and $de \notin E$ implies the $P_4$ $dabe$, in both cases a contradiction to $be \notin C^*$.

Now assume a $\overline{P}_3$ with $a, b \in V^1$, $c \in V^2$ and $ac \in C^*$. Then a $P_4$ $cabe$ in $C^*$ exists with $d \in V^1$ and $e \in V^2$. If $bd \in E$, the $P_4$ $cadb$ would violate the separability of $C^*$; hence $bd \notin E$. If $be \in E$, the $P_4$ $adeb$ would violate the separability of $C^*$; thus $be \notin E$. In fact, we have shown that given $b$ misses the vertices incident to one wing of a $P_4$ in $C^*$, the same holds for the vertices incident to the other wing. Note that Lemma 3.1 and the separability of $C^*$ imply that weak-adjacent $P_4$s in $C^*$ have a common wing. So by induction on the $P_4$s in $C^*$, no wing is incident to $b$, a contradiction to our assumption that $b$ belongs to the cover of $C^*$.

The remainder of the proof is based on the fact that a $P_3$ or a $\overline{P}_3$ as defined in our lemma has no edge in $C^*$. We call such a $P_3$ or $\overline{P}_3$ forcing because all its edges are forced out of $C^*$. Next, we show that no forcing $P_3$ $abc$ can exist. Since $C^*$ covers $b$, there is an edge $bd \in C^*$ with $d \in V^2$. If $cd \in E$, then $bd$ is a forcing $P_3$, and if $ad \notin E$, then $bd$ is a forcing $\overline{P}_3$; in both cases a contradiction to $bd \in C^*$. Therefore $cd \notin E$ and $ad \in E$; thus $cady$ is a $P_4$ in $C^*$, a contradiction to the separability of $C^*$.

It remains to prove that no forcing $P_3$ $abc$ exists. Since $C^*$ covers $c$, there is an edge $cd \in C^*$ with $d \in V^1$. Moreover $bd \in E$, for otherwise the forcing $P_3$ $decb$ would contradict $cd \in C^*$. We say that an edge $vw \in C^*$ with $v \in V^2$ and $w \in V^1$ is

- **type 1** if $b$ sees $v$ and a forcing $P_3$ $wbu$ exists, and
*type 2* if $b$ sees $w$ and a forcing $P_3$ $uvb$ exists.

Figure 5 illustrates this definition. (Solid lines indicate edges that must exist whereas dotted lines indicate edges that must not exist.)

![Diagram](image)

**Figure 5**: A type1 and type2 edge as defined in the proof of Lemma 3.4.

Obviously $cd$ is type2. We claim that any edge $vw \in C^*$ with $v \in V^2$ and $w \in V^1$ is either type1 or type2. From this follows immediately that $C^*$ cannot cover $b$, a contradiction to our assumption.

The proof of the above claim is by induction on the $P_3$s in $C^*$. Since $cd$ is type2, we have already settled the basis. For the inductive step, by Lemma 3.1 and the separability of $C^*$, it again suffices to show that given one wing in a $P_4$ in $C^*$ is type1 or type2, the same holds for the other wing in the same $P_4$. So let $vwxy$ denote an arbitrary $P_4$ in $C^*$ and assume $vw$ to be type1 or type2.

**Case 1**: $vw$ is type1. Then $v$ misses $u$, for otherwise the forcing $P_3$ $wvu$ would contradict $vw \in C^*$. We distinguish the following two subcases.

**Case 1.1**: $u = y$. If $b$ misses $x$, then $xyb$ is a forcing $P_3$, a contradiction to $xy \in C^*$. Therefore $b$ sees $x$; thus $b$ sees $y$ and $xby$ is a forcing $P_3$, i.e. $xy$ is type1.

**Case 1.2**: $u \neq y$. Then $\{|b, u, v, w, x, y\}| = 6$. Furthermore, both $bx \notin E$ and $by \notin E$ cannot hold, as otherwise the $P_4$ $buxy$ would contradict $bw \notin C^*$. If $bx \notin E$ and $by \in E$, then $xyb$ is a forcing $P_3$, a contradiction to $xy \in C^*$. If $bx \in E$ and $by \notin E$, then the $P_4$ $vbyx$ violates the separability of $C^*$. Therefore $bx \in E$ and $by \in E$ holds; thus $b$ sees $x$ and $why$ is a forcing $P_3$, i.e. $xy$ is type2.

**Case 2**: $vw$ is type2. Then $u$ sees $w$, for otherwise the forcing $P_3$ $wvu$ would contradict $vw \in C^*$. Again we distinguish two subcases.

**Case 2.1**: $x = u$. If $b$ misses $y$, the $P_4$ $vbxu$ contradicts the separability of $C^*$. Therefore $b$ sees $y$ and $xbu$ is a forcing $P_3$; thus $xy$ is type1.

**Case 2.2**: $x \neq u$. Then $\{|b, u, v, w, x, y\}| = 6$. Assume that $b$ misses $x$. Then $b$ misses $y$ as well, for otherwise the forcing $P_3$ $xyb$ would contradict $xy \in C^*$. If $u$ misses $y$, then either the $P_4$ $buxy$ contradicts $bu \notin C^*$ or the $P_4$ $uwxy$ contradicts the separability of $C^*$. So $u$ sees $y$ and both $uwxy$ and $vbyx$ are $P_4$s in $C^*$, a contradiction to $vb \notin C^*$.

Therefore our assumption was wrong; so $b$ sees $x$. Moreover $b$ sees $y$, as otherwise the $P_4$ $vbyx$ would violate the separability of $C^*$. Thus $b$ sees $x$ and $why$ is a forcing $P_3$, i.e. $xy$ is type2.

Now Theorem 3.5 follows readily from the above lemma.

**Theorem 3.5** Let $C^*$ denote an arbitrary $P_4$-component. Then no $V(C^*)$-partial $P_4$ exists.
Proof. Suppose a $V(C^*)$-partial $P_4$ $abcd$. Clearly, at least one vertex in \{a, b, c, d\} belongs to $R$; hence $C^*$ is separable with vertex partition $(V^1, V^2)$.

**Case 1:** $bc \in E(V(C^*))$. Let $a$ denote the vertex in $R$. Then $b \in V^1$ and $c \in V^2$. Since $d$ sees $c$ but misses $b$, it must belong to $V(C^*)$. Moreover $d \in V^2$ because $a$ misses $d$. So $bed$ is a $P_3$ with $b \in V^1$ and $d \in V^2$, a contradiction to Lemma 3.4.

**Case 2:** $ab \in E(V(C^*))$ or $cd \in E(V(C^*))$. Without loss of generality (symmetry), let $ab \in E(V(C^*))$. Moreover $c$ is not in the cover of $C^*$, for otherwise we are back in Case 1. Consequently $c \in R$, $a \in V^2$ and $b \in V^1$. If $d$ is not covered by $V(C^*)$, it must be $Q$-vertex. But $d$ sees $c$, a contradiction to the fact that no edge between $R$ and $Q$ can exist. Hence $d$ belongs to the cover of $C^*$; thus $d \in V^1$. So $bda$ is a $T_3$ with $d, b \in V^1$ and $a \in V^2$, a contradiction to Lemma 3.4. □

By Theorem 3.5 and Corollary 3.3, a $P_4$ with at least one but not all its vertices in $V(C^*)$ must be a $P_4$ of types (1) to (6) below.

- **type (1)** $vp_1q_2$ where $v \in V(C^*), p \in P, q_1 \in Q, q_2 \in Q$
- **type (2)** $p_1vp_2q$ where $p_1 \in P, v \in V(C^*), p_2 \in P, q \in Q$
- **type (3)** $p_1v_2p_2r$ where $p_1 \in P, v_2 \in V^2, p_2 \in P, r \in R$
- **type (4)** $v_2p_1r_2$ where $v_2 \in V^2, p_2 \in P, r_1 \in R, r_2 \in R$
- **type (5)** $rv_1pq$ where $r \in R, v_1 \in V^1, p \in P, q \in Q$
- **type (6)** $rv_1pv_2$ where $r \in R, v_1 \in V^1, p \in P, v_2 \in V^2$

Note that a $P_4$ of type (6) together with a $P_4$ $abcd$ in $C^*$ is a pyramid, see Figure 2. The graphs induced by a $P_4$ of types (3) to (5) together with a $P_4$ $abcd$ in $C^*$ are depicted in Figure 6, where bold lines indicate edges in the same $P_4$-component different from $C^*$. Obviously, the existence of a $P_4$ of types (3) to (5) implies a $P_4$ of type (6).

![Figure 6](image)

Figure 6: The subgraphs induced by a $P_4$ of types (3) to (5).

Finally, the question arises if it is possible that two $P_4$-components have the same cover. The next theorem answers this question in the negative.

**Theorem 3.6** Two different $P_4$-components have different covers.

The following lemmas prepare the proof of Theorem 3.6.
Lemma 3.7 Let $vw$ be an edge of a $P_4$ and $z$ a vertex different from $v$ and $w$.

(i) If $vw$ is a wing and $vz, wz \in E - C^*(vw)$, then $z$ sees all the vertices in the $P_4$.

(ii) If $vw$ is a wing, $z$ misses $v$ and $wz \in E - C^*(vw)$, then the $P_4$ can be labeled $vwxy$ and $z$ sees $x$ but misses $y$.

(iii) If $vw$ is a rib and $vz, wz \in E - C^*(vw)$, then the $P_4$ can be labeled $vwvx$ and either $z$ misses $v$ and $x$ or $z$ sees $v$ and $x$.

(iv) If $vw$ is a rib, $z$ misses $v$ and $wz \in E - C^*(vw)$, then $P_4$ can be labeled $vwvx$ and $wz, xz \in C^*(vw)$.

Proof. (i) Without loss of generality, let $vwxy$ be the $P_4$ in question. From Figure 3 follows that only the $F_{10}$ is possible.

(ii) The $P_4$ can be labeled $xyvw$ or $vwxy$. Again from Figure 3 follows that the former case is impossible whereas in the latter case only an $F_7$ does not contradict $wz \in E - C^*(vw)$.

(iii) A $P_4$ $xwvy$ implies an $F_1$, $F_2$ or $F_7$. But an $F_2$ cannot satisfy both $vz \notin C^*(vw)$ and $wz \notin C^*(vw)$.

(iv) In this case, only the $F_3$ does not contradict $wz \in E - C^*(vw)$, see Figure 3. $\square$

Lemma 3.8 Let $vw$ be a rib of a $P_4$ and $z$ a vertex that sees $w$ but misses $v$. If $|C^*(wz)| > 1$, then $C^*(wz) = C^*(vw)$.

Proof. Suppose the contrary, ie $C^*(wz) \neq C^*(vw)$ From Lemma 3.7(iv) follows that the $P_4$ in which $vw$ is the rib can be labeled $vwvx$ with $xz, xz \in C^*(vw)$. Moreover, as $|C^*(wz)| > 1$, the edge $wz$ belongs to $P_4$ as well.

Case 1: $wz$ is a wing. Then Lemma 3.7(iii) applies to $wz$ and $u$; hence the $P_4$ with the wing $wz$ can be labeled $wzab$. The same lemma also applies to $zw$ and $v$; therefore the same $P_4$ can be labeled $zwde$. But no $P_4$ can be labeled in both ways.

Case 2: $wz$ is a rib. Then Lemma 3.7(iv) applied to $wz$ and $u$ and $zw$ and $v$ respectively guarantees a $P_4$ $awzb$ with $uu, ab, va, vb \in C^*(wz)$. Thus either $buw$ or $bw$ is a $P_4$; in both cases a contradiction to $C^*(vw) \neq C^*(wz)$. $\square$

The next lemma deals with the pyramid, cf Figure 2.

Lemma 3.9 If $abedrp$ is a pyramid such that $C^*(rb)$ and $C^*(rc)$ are different from $C^*(ab)$, then $r$ is not covered by $C^*(ab)$. 

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Figure 7: Lemma 3.7 illustrated.
Proof. If \( \{ab, bc, cd\} = C^*(ab) \), there is nothing to prove. Therefore, assume a \( P_4 \) \( a'b'd'c' \) weak-adjacent to \( abc'd' \). Note that the \( P_4 \)'s \( rbpd \) and \( recp \) guarantee that all edges in the pyramid different from \( ab, bc \) and \( cd \) do not belong to \( C^*(ab) \).

In the following case analysis, we show that \( d'b'd'rp \) is another pyramid which satisfies \( C^*(rb') \neq C^*(ab) \) and \( C^*(rc') \neq C^*(ab) \). By induction, this holds for every \( P_4 \) in \( C^*(ab) \); thus \( r \) is incident to no edge in \( C^*(ab) \) as claimed.

Case 1: A wing of \( abc'd' \) coincides with a wing of \( d'b'd' \). Without loss of generality, let \( d'b' \) be the common edge. Then Lemma 3.7(ii) applies to \( d'b' \) and \( r \); hence \( a' = a, b' = b \) and \( r \) sees \( c' \) but misses \( d' \); thus \( C^*(rb') = C^*(ab) \). Similarly, Lemma 3.7(i) applies to \( d'b' \) and \( p \); hence \( p \) sees \( c' \) and \( d' \); thus \( a'b'd'rp \) is pyramid. Moreover \( C^*(rc') = C^*(rc) \neq C^*(ab) \) because of the \( P_4 \)s \( recp \) and \( rc'pa \).

Case 2: A wing of \( abc'd' \) coincides with the rib of \( d'b'd' \). Then Lemma 3.8 applies to \( b'd' \) and \( r \); thus \( C^*(ab) \neq C^*(rb) \) or \( C^*(ab) = C^*(rc) \), a contradiction to the premise of our lemma.

Case 3: The rib of \( abc'd' \) coincides with a wing of \( d'b'd' \). Without loss of generality, let \( a' = b \) and \( b' = c \). From Lemma 3.7(i) applied to \( d'b' \) and \( r \) follows that \( r \) sees \( c' \) and \( d' \). But the same Lemma also applies to \( a'b' \) and \( p \); so \( p \) sees \( c' \) and \( d' \). Thus \( \{a', b', c, d', r, p\} \neq 7 \). Furthermore, \( d \) sees \( d' \), as otherwise the \( P_4 \) \( dcvd \) would contradict \( C^*(ab) \neq C^*(rc) \). So \( bcvd \) and \( dd'r \) are \( P_5 \); hence \( C^*(ab) = C^*(rb) \), a contradiction to our assumption. \( \square \)

**Corollary 3.10** Let \( abcdrp \) denote a pyramid. Then \( V(C^*(rb)) = V(C^*(ab)) \) implies \( C^*(rb) = C^*(ab) \).

**Proof.** Suppose \( C^*(rb) \neq C^*(ab) \). Then \( C^*(rc) = C^*(ab) \), as otherwise a contradiction to Lemma 3.9 would arise. Therefore \( C^*(ab) \neq C^*(rc) \) is different from \( C^*(rb) \) and Lemma 3.9 applies to the pyramid \( rbpdac \); hence \( a \) cannot be covered by \( C^*(rb) \), a contradiction to our assumption. \( \square \)

**Proof of Theorem 3.6.** Suppose the contrary, ie two different \( P_4 \)-components \( C^*_1 \) and \( C^*_2 \) satisfy \( V(C^*_1) = V(C^*_2) \). Then \( C^*_1 \) (and \( C^*_2 \)) cannot be trivial and a \( P_4 \) \( abed \) in \( C^*_1 \) exists. Clearly, each vertex in \( \{a, b, c, d\} \) is incident to at least one edge in \( C^*_2 \). Therefore, the vertices \( \{a, b, c, d\} \) together with the other endpoint of such an edge, say \( v \), induce one of the graphs depicted in Figure 3. Moreover \( C^*_1 \neq C^*_2 \), which leaves the graphs \( F_1, F_2, F_3, F_4 \) and \( F_7 \). We show that each of these graphs is impossible.

**\( F_3 \):** Then \( vc \in C^*_2 \) and Lemma 3.8 applies to \( be \) and \( v \); hence \( C^*(be) = C^*(vc) \), a contradiction to \( C^*_1 \neq C^*_2 \).

**\( F_4 \):** Then \( rd \in C^*_2 \). Since the situation is symmetric relative to \( v \) and \( d \), we may assume that \( vw \) denotes another edge in a \( P_4 \) that contains \( rd \). Hence \( dvw \) is a \( P_3 \) and \( \{a, b, c, d, v, w\} \). Suppose \( w \) misses \( c \). Then \( w \) sees \( b \), as otherwise the \( P_4 \) \( bcvw \) would imply \( C^*_1 = C^*_2 \). Hence \( bwd \) is a \( P_4 \) in \( C^*_1 \); Lemma 3.8 applies to \( uw \) and \( c \); thus \( C^*(uw) = C^*(cv) \), a contradiction to \( C^*_1 \neq C^*_2 \). Therefore our supposition was wrong, so \( w \) sees \( c \).
Furthermore \( w \) misses \( a \), for otherwise the \( P_4 \)s \( awvd \) and \( awed \) would imply \( C_1^* = C_2^* \). The same contradiction arises if \( w \) sees \( b \), this time because of the \( P_4 \) \( abwv \). Hence \( abcw \) is another \( P_4 \) in \( C_1^* \).

Obviously, the same argumentation holds for the third edge of the \( P_4 \) and, by induction, for every edge in \( C_4^* \). Therefore, no edge in \( C^*(vd) \) is incident to \( a \) or \( b \), a contradiction to our assumption \( V(C_1^*) = V(C_2^*) \).

**F_2:** Without loss of generality, let \( vb \) be the edge in \( C_4^* \). Then \( vb \) cannot be the rib of a \( P_4 \), as otherwise a contradiction to Lemma 3.8 applied to \( vb \) and \( a \) would arise. Therefore \( vb \) is a wing, Lemma 3.7(ii) applies to \( vb \) and \( a \); thus our \( P_4 \) can be labeled \( vbx \) and \( a \) sees \( x \) but misses \( y \). If \( y = d \), then \( axdc \) is a \( P_4 \) which contradicts \( C_1^* \neq C_2^* \). Hence \(|\{a, b, c, d, v, x, y\}| = 7\).

**Case 1:** \( ex \notin E \). As \( xb \) is a rib, we can apply Lemma 3.8 to \( xb \) and \( c \); hence \( C_1^* = C_2^* \), the usual contradiction.

**Case 2:** \( ex \in E \). If \( d \) sees \( x \), then \( abedux \) is a pyramid which satisfies \( V(C^*(vb)) = V(C^*(ab)) \). Corollary 3.10 applies and again \( C_1^* = C_2^* \). The same contradiction arises if \( c \) sees \( y \), this time because of the pyramid \( vbxac \) and \( V(C^*(vb)) = V(C^*(ab)) \). Therefore \( dx, cy \notin E \). So \( yxev \) and \( axed \) are \( P_4 \)s; hence \( C^*(ed) = C^*(yx) \), again a contradiction to \( C_1^* \neq C_2^* \).

**F_2:** Then \( vx \in C_4^* \). Without loss of generality (symmetry), let \( vx \) be another edge in a \( P_4 \) which \( vc \) belongs to. In the following case analysis, we show that \( abvd \) together with \( x \) again induces an \( F_2 \), the structure repeats itself. Therefore, by induction, all edges in \( C_4^* \) together with \( a, b \) and \( d \) induce an \( F_2 \); thus \( a, b \) and \( d \) are not covered by \( C_2^* \), a contradiction to \( V(C_1^*) = V(C_2^*) \).

**Case 1:** \( x \) sees \( b \) and \( d \). If \( x \) sees \( a \), the \( P_4 \)s \( axdc \) and \( axve \) imply \( C_1^* = C_2^* \), a contradiction. Therefore \( x \) misses \( a \) and the \( P_4 \) \( abvd \) together with \( x \) induces an \( F_2 \) as claimed.

**Case 2:** \( x \) misses \( b \) or \( d \). If \( x \) misses \( b \), Lemma 3.8 applies to \( bv \) and \( x \), a contradiction to \( C_1^* \neq C_2^* \). Hence \( x \) sees \( b \) but misses \( d \). Then \( cv \) cannot be the wing of a \( P_4 \) that contains \( vx \), as otherwise a contradiction to Lemma 3.7(i) applied to \( vc \) and \( d \) would arise. Therefore \( cv \) is a rib, Lemma 3.7(iii) applies \( cv \) and \( d \); thus our \( P_4 \) can be labeled \( vcdx \) and, together with \( d \), induces an \( F_2 \). But we have already shown that such an \( F_2 \) leads to a contradiction. Moreover, as all other possibilities have been ruled out, \( a'b'e'd' \) and \( v \) induce another \( F_1 \). Therefore, by induction, \( v \) is \( V(C_1^*) \)-universal; thus \( v \) is not covered by \( C_1^* \), a contradiction. \( \square \)

### 4 Recognition and orientation algorithms

In order to obtain an acyclic \( P_4 \)-transitive \((P_4\)-indifferent) orientation, it suffices to compute an acyclic orientation of the edges in the \( P_4 \)s (all other edges can be oriented by topological sorting). In the following, we only discuss this part of the orientation.

If no nontrival \( P_4 \)-component covers a proper subset of the vertices of \( G \), then either \( G \) contains no \( P_4 \) or, by Theorem 3.6, precisely one nontrivial \( P_4 \)-component exists. In the former case, nothing has to be done whereas in the latter case, given \( G \) is a \( P_4 \)-comparability
(P₄-indifference) graph, a P₄-transitive (P₄-indifferent) orientation of this P₄-component is
unique (up to inversion) and therefore easy to compute. We show that all other cases can
be reduced to one of these cases.

So suppose a nontrivial P₄-component, say C*, that does not cover the whole graph. If
R = ∅, the cover of C* is a homogeneous set and we can replace it with a single marker
vertex, i.e. we choose an arbitrary vertex m ∈ V(C*) and remove all other vertices in V(C*).

(i) Replace V(C*) with a marker vertex m such that
m is P-universal and Q-null.

(ii) Recursively orient the edges of the P₄s in G_{V(C*)} and in G_{V - V(C*) + m}.

(iii) Construct an orientation of the edges of the P₄s in G by directing
vw with v, w ∈ V(C*) as in G_{V(C*)},
vw with v, w ∈ V - V(C*) as in G_{V - V(C*) + m},
vw with v ∈ V(C*) and w ∈ V - V(C*) as mw in G_{V - V(C*) + m}.

If R ≠ ∅, then C* is separable with vertex partition (V¹, V²). This time we need two
marker vertices to represent V¹ and V²: We choose an arbitrary P₄ abcd in C* and remove
all vertices in V¹ + V² except for b and d.

(i) Replace V¹ and V² with nonadjacent marker vertices b and d such that
b is P-universal, R-universal and Q-null and
d is P-universal, R-null and Q-null.

(ii) Recursively orient the edges of the P₄s in G_{V¹ + V²} and in G_{V - (V¹ + V²) + (b + d)}.

(iii) Construct an orientation of the edges of the P₄s in G by directing
vw with v, w ∈ V¹ + V² as in G_{V¹ + V²},
vw with v, w ∈ V - V¹ - V² as in G_{V - (V¹ + V²) + (b + d)},
vw with v ∈ V¹ and w ∈ V - V¹ - V² as bw in G_{V - (V¹ + V²) + (b + d)} and
vw with v ∈ V² and w ∈ V - V¹ - V² as dw in G_{V - (V¹ + V²) + (b + d)}.

Obviously, a P₄-transitive (P₄-indifferent) orientation of G induces a P₄-transitive (P₄-
indifferent) orientation of G_{V - V(C*) + m}, G_{V(C*)}, G_{V - (V¹ + V²) + (b + d)} and G_{V¹ + V²}. Lemma 4.1
and 4.2 assert that the converse holds for P₄-transitive orientations; thus the above algorithm
correctly orients a P₄-comparability graph in O(n²m), the time needed to orient the
edges in the P₄-components.

**Lemma 4.1** The orientation of the P₄s in G is P₄-transitive (P₄-indifferent) and acyclic
whenever the orientation of the P₄s in G_{V(C*)} and G_{V - V(C*) + m} is P₄-transitive (P₄-indifferent)
and acyclic.

**Lemma 4.2** The orientation of the P₄s in G is P₄-transitive and acyclic whenever the
orientation of the P₄s in G_{V¹ + V²} and G_{V - (V¹ + V²) + (b + d)} is P₄-transitive and acyclic.

Regarding P₄-indifference graphs, we have the following lemma.
Lemma 4.3 The orientation of the $P_4$s in $G$ is $P_4$-indifferent and acyclic whenever $G$ contains no pyramid and the orientation of the $P_4$s in $G_{V_1+V_2}$ and $G_{V-(V_1+V_2)+\{i+d\}}$ is $P_4$-indifferent and acyclic.

It is easy to see that no pyramid admits a $P_4$-indifferent orientation, so no $P_4$-indifference graph contains the pyramid. Thus Lemma 4.1 and 4.3 guarantee that the above algorithm correctly orients $P_4$-indifference graphs. On the other hand, as the orientation computed by the above method is always $P_4$-indifferent, it suffices to test whether our orientation is acyclic to recognize $P_4$-indifference graphs. Thus we have found an $O(n^2 m)$ recognition and orientation algorithm for $P_4$-indifference graphs.

Proof of Lemma 4.2 To begin with, we show that every $P_4$ in $G$ is oriented properly. This is obvious for $P_4$s with all vertices in $V(C^*)$ and for $P_4$s with all vertices not in $V(C^*)$. The remaining $P_4$s are of types (1) to (6), for each of which we can find a corresponding $P_4$ in $G_{V-(V_1+V_2)+\{i+d\}}$ by replacing the vertex in $V_1$ with $b$ and the vertex in $V_2$ with $d$.

Now suppose the orientation of $G$ is cyclic. As the orientation of $G_{V-(V_1+V_2)+\{i+d\}}$ and $G_{V_1+V_2}$ is acyclic, every cycle contains edges with both endpoints in $V(C^*)$ and edges with an endpoint not in $V(C^*)$. Choose a cycle with a minimal number of vertices in $V(C^*)$ and let $v \to \cdots \to w$ denote the longest part of this cycle in $V(C^*)$. Furthermore, let $u$ be the predecessor of $v$ and $x$ the successor of $w$ in this cycle; thus $u, x \not\in V(C^*)$.

Since $uv$ is directed, it must belong to a $P_4$ of types (1) to (6). Moreover $w$ cannot belong to the same partition set $V_1$ or $V_2$ as $v$ because this would imply $a \to w$, ie a cycle with fewer vertices in $V(C^*)$ would exist. Without loss of generality, let $v \in V_2$ (otherwise we invert the orientation of each directed edge). Hence $u \in P$.

But $uv$ is in no $P_4$ of types (1) or (2), as otherwise $u \to d$ and $u \to b$ in $G_{V-(V_1+V_2)+\{i+d\}}$ and therefore $u \to w$, again a contradiction because a cycle with fewer vertices in $V(C^*)$ has been found. For the same reason, $uw$ cannot belong to a $P_4$ of types (4) to (6), see Figure 6. Now assume that $uw$ is in a $P_4$ of type (3), say $p_1uvu$. Then $G_{V-(V_1+V_2)+\{i+d\}}$ contains the $P_4$s $p_1uuv$ and $rbpu$, hence $r \to b$ in $G_{V-(V_1+V_2)+\{i+d\}}$ and therefore $r \to w$ in $G$. Thus $u \to v \to \cdots \to w$ can be replaced with $u \to r \to w$, again a contradiction as we have found a cycle with fewer vertices in $V(C^*)$. □

Lemma 4.1 and Lemma 4.3 can be proven in much the same way as Lemma 4.2. Actually, these proofs are even simpler because no $P_4$ of types (3) to (6) can occur.

5 The decomposition

A vertex set $M$ is a module if no $M$-partial vertex exists. Moreover, $M$ is a proper module if additionally $M \subseteq V$. Thus, every homogeneous set is a module but not vice versa. The famous modular decomposition is based on the following theorem [MS94].

Theorem 5.1 An arbitrary graph $G = (V, E)$ satisfies at least one of the following conditions:

(i) $G$ is disconnected;
(ii) $\overline{G}$ is disconnected;
(iii) the maximal proper modules of $G$ are disjoint.
Since the connected components of $G$ (and $\overline{G}$) are disjoint modules, the above theorem guarantees the uniqueness of the modular decomposition described below.

If $G$ is trivial, then stop,
else if $G$ is disconnected, decompose the connected components of $G$,
else if $\overline{G}$ is disconnected, decompose the connected components of $\overline{G}$,
else decompose the graphs induced by the maximal proper modules of $G$.

As the decomposition operations are performed top-down, we obtain a unique decomposition tree called modular decomposition tree if we distinguish the above operations by a 0, 1 and 3-node\textsuperscript{1}. If $G$ is trivial, this is indicated by a empty node labeled $v$ where $v$ stands for the only vertex in $G$.

Procedure Build Tree($G$);
{ Input: an arbitrary graph $G = (V, E)$;
Output: the root of the decomposition tree $D(G)$ of $G$; }
begin
if $|V| = 1$ then
let $v \in V$;
return an empty node labeled $v$;
else if $G$ is disconnected then
let $G_1, G_2, \ldots, G_t$ be the connected components of $G$;
let $r_i = \text{Build\_Tree}(G_i)$ for $i = 1, \ldots, t$;
return a 0-node with children $r_1, r_2, \ldots, r_t$;
else if $\overline{G}$ is disconnected then
let $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_t$ be the connected components of $\overline{G}$;
let $r_i = \text{Build\_Tree}(\overline{G}_i)$ for $i = 1, \ldots, t$;
return a 1-node with children $r_1, r_2, \ldots, r_t$;
else (* $G$ and $\overline{G}$ are connected and $|V| > 1$ *)
let $H_1, H_2, \ldots, H_t$ be the maximal proper modules of $G$;
let $r_i = \text{Build\_Tree}(G_{H_i})$ for $i = 1, \ldots, t$;
return a 3-node with children $r_1, \ldots, r_t$;
end (* if *)
end;

Furthermore, the original graph can be reconstructed from the modular decomposition tree if we replace the maximal proper modules with marker vertices and store those prime graphs in the corresponding 3-nodes. Therefore, in some sense, any decomposition method for prime graphs can be used to refine the modular decomposition. Our decomposition of prime graphs is based on the structure of separable $P_4$-components and is defined in a way which maintains a unique decomposition tree.

To begin with, we investigate the relation between separable $P_4$-components and modules. So let $C^n$ denote a separable $P_4$-component and consider an edge $vw$ with both endpoints in $V^2$. Recall that no vertex in $V^1$ is $\{v, w\}$-partial, see Lemma 3.4; hence $v$ and $w$ have the same neighborhood relative to $V - V^2$. By induction, this holds for any two vertices in the same connected component of $G_{V^2}$, i.e. a connected component of $G_{V^2}$ is a module. The analogous argumentation applies to $V^1$ and $\overline{G}$, thus the connected components of $\overline{G}_{V^1}$ are modules.

The following lemma restates this result.

\textsuperscript{1}we use the same notation as Jamison and Olariu in [JO95]
Lemma 5.2 Let $G$ be a prime graph and $C^*$ a separable $P_4$-component with vertex partition $(V^1, V^2)$. Then $(V^1, V^2)$ is a split graph, i.e. $V^1$ is a clique and $V^2$ is a stable set.

A separable $P_4$-component is called maximal if its cover is not contained in the cover of another separable $P_4$-component. Unlike maximal proper modules, however, the covers of two maximal $P_4$-components need not be disjoint (the pyramid is a counterexample). Therefore, it is necessary to examine the relation between two maximal $P_4$-components whose covers intersect. Let us call them adjacent $P_4$-components for short.

Lemma 5.3 The $P_4$'s in adjacent $P_4$-components are of type (6) relative to one another.

Proof. We show that any $P_4$ in $C^*_2$ is of type (6) relative to $C^*_1$. Since $C^*_2$ contains a $P_4$ of types (1) to (6) relative to $C^*_1$, it suffices to prove that no $P_4$ of types (1) to (5) exists because, in this case, every $P_4$ adjacent to a $P_4$ of type (6) must be of type (6) itself.

In a $P_4 v p q_1 q_2$ of type (1), the vertex $v \in V(C^*_1)$ can be replaced with any vertex in the cover of $C^*_1$; thus the cover of $C^*_1$ is a subset of the cover of $C^*_2$, a contradiction to the fact that $C^*_2$ is a maximal $P_4$-component. Also, using similar arguments, we can show that no $P_4$ of type (2) is possible.

A $P_4$ of types (3) to (5) together with a $P_4 a b c d$ in $C^*_1$ induces one of the graphs depicted in Figure 6. But the edge $r p_2, r_1 p$ or $p q$ guarantees that, for any $P_4$ in $C^*_1$, all bold lines in the corresponding graph in Figure 6 are edges in $C^*_2$; thus all vertices in the cover of $C^*_2$ are covered by $C^*_2$, again a contradiction because $C^*_2$ is maximal. \hfill \square

By $(V^1_i, V^2_i)$, we denote the vertex partition of the cover of the separable $P_4$-component $C^*_i$. Now Corollary 5.4 follows immediately from Lemma 5.3.

Corollary 5.4 Two adjacent $P_4$-components $C^*_1$ and $C^*_2$ satisfy

(i) $V^1_1 \cap V^2_2 = \emptyset = V^2_1 \cap V^1_2$ and
(ii) $V^1_1 \cap V^2_2 \neq \emptyset \neq V^1_2 \cap V^2_1$.

A set $S = \{C^*_1, C^*_2, \ldots, C^*_k\}$ of $P_4$-components is called connected if for every pair $C^*_i$ and $C^*_j$ in $S$ a sequence $C^*_i, \ldots, C^*_j$ of $P_4$-components in $S$ exists such that two successive $P_4$-components are adjacent. Note that this definition implies that all $P_4$-components in $S$ are maximal.

Lemma 5.5 Let $G$ be a prime graph and let $\{C^*_1, C^*_2, \ldots, C^*_k\}$ be a connected set of $P_4$-components. Then $(V^1, V^2) = (\cup_{i=1}^k V^1_i, \cup_{i=1}^k V^2_i)$ induces a split graph.

Proof. This proof is by induction, and Lemma 5.2 settles the basis. For the inductive step, assume that $S = \{C^*_1, C^*_2, \ldots, C^*_j-1\}$ is connected and $C^*_j$ is adjacent to $C^*_j-1$ (after an appropriate permutation of the indices). The induction hypothesis asserts that $(V^1, V^2) = (\cup_{i=1}^{j-1} V^1_i, \cup_{i=1}^{j-1} V^2_i)$ is a split graph. By Lemma 5.2, $(V^1_j, V^2_j)$ is a split graph as well.

Clearly $V^1_j \cap V^2_j = \emptyset = V^2_j \cap V^1_j$, as otherwise a contradiction to Corollary 5.4 applied to $C^*_j$ and some $P_4$-components in $S$ would arise. Furthermore, Corollary 5.4 applied to $C^*_j-1$ and $C^*_j$ guarantees the existences of vertices $v_1 \in V^1_j \cap V^1_j$ and $v_2 \in V^2_j \cap V^2_j$.
Since each vertex \( v \) in \( V^1 \setminus V_j \) sees \( v_1 \), it must be adjacent to every vertex in \( V^1 \). Similarly, every vertex in \( V^2 \setminus V_j \) misses \( v_2 \) and, therefore, all vertices in \( V_j \). Hence \( V^1 \cup V_j \) is a clique and \( V^2 \cup V_j \) is a stable set as claimed. \( \square \)

Note that the split graphs induced by the maximal connected sets of \( P_4 \)-components are disjoint and therefore unique. Let us call them \( P_4 \)-split graphs. Moreover, it is easy to see that a vertex in \( V^1 \setminus V^2 \) which is \( P \) (\( R \) or \( Q \)) relative to \( C^*_1 \) is also \( P \) (\( R \) or \( Q \)) relative to \( C^*_2, \ldots, C^*_4 \). Therefore, we can replace \( V^1 \) and \( V^2 \) with two nonadjacent marker vertices in the same way as described in Section 4.

Next, we analyze the relation between \( P_4 \)-components and those in the complement \( \overline{G} \).

**Lemma 5.6** Let \( C^*_1 \) be a separable \( P_4 \)-component of \( G \) and \( C^*_2 \) a separable \( P_4 \)-component of \( \overline{G} \). If their covers intersect, then \( V(C^*_1) \subseteq V(C^*_2) \) or \( V(C^*_2) \supseteq V(C^*_1) \) holds.

**Proof.** Suppose the contrary. Then a \( P_4 \) \( efgh \) in \( C^*_2 \) exists with some but not all its vertices in \( V(C^*_1) \). Thus, the \( P_4 \) \( efgh \) in \( G \) is of types (1) to (6) relative to \( C^*_1 \).

But \( efgh \) cannot be of type (1), as a \( P_4 \) \( v_1v_2 \) of type (1) induces the \( P_4 \) \( v_1v_2p \) in \( \overline{G} \), which would imply \( V(C^*_1) \subseteq V(C^*_2) \). Similarly, each \( P_4 \) \( p_1v_2q \) of type (2) induces the \( P_4 \) \( vqnp_2 \) in \( \overline{G} \); thus \( efgh \) cannot be of type (2).

In all the remaining cases, see Figure 6, it is easy to verify that a \( P_4 \) \( rv_1pv_2 \) of type (6) exists that satisfies \( rp \in C^*_2 \). Consequently, for each \( P_4 \) \( abcd \) in \( C^*_2 \), we find two \( P_8 \) \( prac \) and \( prdb \) in \( C^*_2 \). Again, this implies \( V(C^*_1) \subseteq V(C^*_2) \), a contradiction to our assumption. \( \square \)

**Theorem 5.7** For every nontrivial prime graph \( G \), precisely one of the following conditions is satisfied:

(i) \( G \) is a \( P_4 \)-split graph \( (V^1, V^2) \);
(ii) \( G \) consists of a unique \( P_4 \)-split graph \( (V^1, V^2) \) together with a \( R \)-vertex \( v \);
(iii) \( G \) is no split graph, and the \( P_4 \)-split graphs in \( G \) and \( \overline{G} \) are disjoint.

**Proof.** Let \( W \) denote the set of vertices that are in no \( P_4 \). If \( W = V \), then either \( G \) or \( \overline{G} \) is disconnected, see SEINSCHEN [Sei74]. But this contradicts our assumption that \( G \) is nontrivial and prime. Therefore, \( G \) contains at least one nontrivial \( P_4 \)-component.

If \( W \neq \emptyset \), no \( P_4 \)-component covers the whole graph. Hence every \( P_4 \)-component of \( G \) is separable and a \( P_4 \)-split graph \( (V^1, V^2) \) in \( G \) exists. Furthermore, all vertices in \( R \) are adjacent to those in \( P \), as otherwise a \( P_4 \) \( rv_1pv_2 \), \( v_1 \in V^1 \), \( v_2 \in V^2 \), would exist; thus the separable \( P_4 \)-component \( C^*(rv_1) \) contradicts the fact that \( (V^1, V^2) \) is a \( P_4 \)-split graph. If \( P \cap Q \neq \emptyset \), then \( V^1 \cup V^2 \cup R \) is homogeneous. So \( P = Q = \emptyset \); hence \( R \) is a module; thus \( |R| = 1 \) and Condition (ii) is satisfied.

Now assume \( W = \emptyset \). Additionally, suppose that a \( P_4 \)-split graph \( (V^1, V^2) \) in \( G \) exists such that every vertex in \( R \) sees every vertex in \( P \). Then \( P = Q = \emptyset \), as otherwise \( V^1 \cup V^2 \cup R \) would be homogeneous. Again \( R \neq \emptyset \) implies that \( R \) is a module; thus \( R \) consists of a single vertex in no \( P_4 \), a contradiction to \( W = \emptyset \). Therefore \( R = \emptyset \) and \( V = V^1 + V^2 \), ie Condition (i) is satisfied.

Finally, suppose that for every \( P_4 \)-split graph \( (V^1, V^2) \) in \( G \) vertices \( r \in R \) and \( p \in P \) exist such that \( r \) misses \( p \), ie there is a \( P_4 \) \( rv_1pv_2 \) with \( v_1 \in V^1 \) and \( v_2 \in V^2 \). Note that
If a $P_4$-split graph of $G$ and a $P_4$-split graph of $\overline{G}$ have a common vertex, we can find a maximal $P_4$-component $C_1^*$ in $G$ and a maximal $P_4$-component $C_2^*$ in $\overline{G}$ whose covers intersect. By Lemma 5.6, one is a subset of the other. Without loss of generality, let $V(C_1^*) \subseteq V(C_2^*)$. Obviously, every $R$-vertex relative to $C_2^*$ is a $R$-vertex relative to $C_1^*$; and every $P$-vertex relative to $C_2^*$ is a $Q$-vertex relative to $C_1^*$. But our assumption that $r$ misses $p$ in $\overline{G}$ implies an edge between an $R$-vertex and a $Q$-vertex relative to $C_1^*$. But this is a contradiction to our definition of a $P_4$-split graph.

Our decomposition of prime graphs is based on Theorem 5.7. A $P_4$-split graph with clique $C$ and stable set $S$ (in $G$) is decomposed by a 4 node labeled $C$ with $|S|$ children, each of which corresponds to a vertex $v \in S$ and is labeled $N(v) \cap C$. Thus Condition (i) corresponds to a 3-node with a 4-node as its only child, Condition (ii) corresponds to a 2-node with a 4-node as its only child, and Condition (iii) is represented by a 3-node with multiple children. Only the else-part of the modular decomposition procedure is given as the rest remains the same. (Note that, below, a $P_4$-split graph of $\overline{G}$ is denoted by $(S,C)$, i.e. $S$ is a clique in $\overline{G}$ and therefore a stable set in $G$ whereas $C$ is a stable set in $\overline{G}$ and a clique in $G$.)
As in case of the modular decomposition, the original graph can be reconstructed from the unique decomposition tree if we replace the maximal homogeneous sets and the $P_4$-split graphs with marker vertices and store those graphs in the 3-nodes with multiple children.

Figure 8: An example of the module and $P_4$-split graph substitution.

Figure 8 and 9 illustrate our decomposition: Figure 8(a) shows the original graph $G$, Figure 8(b) the prime graph $G'$ after the substitution of the maximal proper modules $\{b_1, b_2\}$, $\{e_1, e_2\}$ and $\{f_1, f_2, f_3, f_4\}$, and Figure 8(c) shows the graph after the substitution of the $P_4$-split graphs $\{(e_1, e_2)\}$ in $G'$ and $\{(a_1, a_2, a_3)\}$ in $G'$. The decomposition tree is depicted in Figure 9.

Figure 9: The decomposition tree of the graph in Figure 8(a).

We should remark that Jamison and Olariu's homogeneous decomposition [JO95] performs only a part of our decomposition: A prime graph is decomposed if and only if it satisfies Condition (i) or Condition (ii) of Theorem 5.7, see also [BO96]. Thus, in a homogeneous decomposition tree, no 3-node with multiple children has a 4-node as its child. Consequently, the graph in Figure 8(b) is indecomposable relative to the homogeneous decomposition.
6 Conclusions and open problems

In this paper, we have obtained various results on the structure of the $P_4$-components. They allowed us to derive $O(n^2m)$ recognition and orientation algorithms for $P_4$-comparability and $P_4$-indifference graphs. Moreover, we proposed a unique tree representation for arbitrary graphs based on their module and $P_4$-component structure. As the modular decomposition and the orientation of comparability graphs can be computed in linear time, cf [MS94] and [MS97], we suspect that similar results are achievable for our decomposition and for the orientation of $P_4$-comparability graphs. We pose this as an open problem.

References


