Report

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On the $P_4$-components of Graphs

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Abstract

Two edges are called $P_4$-adjacent if they belong to the same $P_4$ (chordless path on 4 vertices). $P_4$-components, in our terminology, are the equivalence classes of the transitive closure of the $P_4$-adjacency relation. In this paper, new results on the structure of $P_4$-components are obtained. On the one hand, these results allow us to improve the complexity of the recognition and orientation algorithms for $P_4$-comparability and $P_4$-indifference graphs from $O(n^5)$ to $O(n^3m)$ and from $O(n^5)$ to $O(n^3m)$, respectively. On the other hand, by combining the modular decomposition with the substitution of $P_4$-components, a new unique tree representation for arbitrary graphs is derived which generalizes the homogeneous decomposition introduced by Jamison and Olariu [JO95].

1 Introduction

A $P_k$ ($C_k$) is a chordless path (cycle) on $k$ vertices. By the $P_4$ $abcd$, we denote the $P_4$ with vertices $a, b, c, d$ and edges $ab, bc$ and $cd$. An orientation $U$ of a graph $G$ is the antisymmetric directed graph which arises from assigning a direction to each edge of $G$. A directed edge is denoted by $a \to b$ or $a \leftarrow b$.

Hoàng and Reed [HB89] suggested investigating $P_4$-comparability and $P_4$-indifference graphs which are defined as follows. An orientation is $P_4$-transitive if the orientation of every $P_4$ is transitive, ie type 1 in Figure 1. Similarly, an orientation is said to be $P_4$-indifferent if every $P_4$ is indifferent, ie type 2 in Figure 1. A graph which admits an acyclic $P_4$-transitive ($P_4$-indifferent) orientation is called $P_4$-comparability ($P_4$-indifference) graph.

Chvátal [Chv84] introduced perfectly orderable graphs as those graphs whose vertices can be ordered perfectly, ie the greedy algorithm proceeding along such an order computes a minimum coloring for each induced subgraph. He showed that a graph is perfectly orderable if and only if an acyclic orientation exists such that no $P_4$ $abcd$ is oriented $a \to b$ and $c \leftarrow d$ (called obstruction). Given an acyclic obstruction-free orientation, a number of problems can be solved in polynomial time which are NP-complete in general (see eg [Chv84], [Hoâ94] and [AP96]). Unfortunately, it is NP-complete to decide whether a graph admits a perfect order [MP96].

On the other hand, both $P_4$-comparability and $P_4$-indifference graphs are perfectly orderable as they do not contain obstructions. In [HB89] and [HR89], Hoàng and Reed...
presented an $O(n^4)$ recognition algorithm for $P_4$-comparability graphs and an $O(n^5)$ algorithm to compute the corresponding acyclic $P_4$-transitive orientation. The complexity of their recognition algorithm for $P_4$-indifference graphs is $O(n^6)$.

In this paper, we develop $O(n^2m)$ recognition and orientation algorithms for both classes of graphs. The key to our improvement lies in the detailed study of the $P_4$-adjacency relation: Two edges are $P_4$-adjacent if they belong to the same $P_4$. The equivalence classes of this $P_4$-adjacency relation are called $P_4$-components. Obviously, the orientation of an edge of a $P_4$-comparability ($P_4$-indifference) graph implies the orientation of all other edges in the same $P_4$-component.

As it turns out, the vertices incident to the edges of a $P_4$-component have a very special neighborhood-relation to the other vertices. In fact, only two types of adjacency can occur. It is therefore quite natural to replace such a vertex set with two marker vertices. This substitution allows a recursive computation of the desired orientations; thus the running time is bounded by the time needed for orienting the edges in the $P_4$-components. Clearly, the latter can be done in $O(n^2m)$ with methods similar to those for the transitive orientation in [Gol77].

Moreover, the structure of the $P_4$-components allows us to refine the famous modular decomposition ([MS94], [ST94], also substitution decomposition [MR84]) in a way which maintains the uniqueness of the decomposition. Our unique decomposition tree further generalizes the homogeneous decomposition tree given by Jamison and Olariu [JO95]. Such decomposition trees can be used to solve hard problems efficiently, eg maximum clique, maximum stable set, minimum coloring, graph isomorphism, Hamiltonian cycle, cf [IO89], [BS89], [JO91], [JO92b], [JO92a], [HT95], [Gia96] and [BO96].

The remainder of this paper is organized as follows. The next section contains basic definitions and notations. Section 3 explores the structure of the $P_4$-components. The obtained results are used in Section 4 to design algorithms for the recognition and orientation of $P_4$-comparability and $P_4$-indifference graphs. In Section 5, we develop our new decomposition. Finally, the last section summarizes the results and poses some open problems.

2 Definitions

Let $G = (V, E)$ be an arbitrary graph and $W \subseteq V$ a subset of its vertices. By $G_W = (W, E(W))$, we denote the subgraph of $G$ induced by $W$, and $\overline{G} = (V, E)$ stands for the complement of $G$. For any vertex $v \in V$, the set of its adjacent vertices is called the neighborhood of $v$, denoted by $N(v)$. If two vertices $v$ and $w$ are adjacent in $G$, we say that $v$ sees $w$; otherwise we say that $v$ misses $w$.

A vertex $v \notin W$ is $W$-universal ($W$-null) if it sees (misses) all vertices in $W$. If $v \notin W$ is neither universal nor null, we call it $W$-partial. A $P_4$ is $W$-partial if it has at least one edge
in $E(W)$ but not all of its vertices belong to $W$. A vertex set $H \subseteq V$ is called \textit{homogeneous} if $1 < |H| < |V|$ and no $H$-partial vertex exists. Graphs without homogeneous sets are called \textit{prime}.

As we are especially interested in $P_4$s and $P_4$-components, the following definitions and notations come in handy. Given a $P_4$ $abcd$, the edge $bc$ is called \textit{rib}, the edges $ab$ and $cd$ \textit{wings}, the vertices $b$ and $c$ \textit{midpoints} and the vertices $a$ and $d$ \textit{endpoints}. Two $P_4$s are called adjacent if they have a common edge.

Throughout the whole paper, $C^*$ stands for a $P_4$-component and $C^*(vw)$ for the $P_4$-component that contains the edge $vw$. By harmless abuse of language, a $P_4$ with one edge (and therefore all its edges) in a $P_4$-component is also said to be in $C^*$. A vertex is covered by the $P_4$-component $C^*$ if it is incident to at least one edge in $C^*$. The set $V(C^*)$ of all vertices covered by $C^*$ is called the \textit{cover} of $C^*$. The sets of all $V(C^*)$-universal, -partial and -null vertices are denoted by $P$, $R$ and $Q$, respectively.

A \textit{trivial graph} has precisely one vertex and a \textit{trivial $P_4$-component} consists of a single edge. A nontrivial $P_4$-component $C^*$ is called \textit{separable} if its cover $V(C^*)$ can be partitioned into two vertex sets ($V^1$, $V^2$) such that each $P_4$ in $C^*$ has its midpoints in $V^1$ and its endpoints in $V^2$.

Finally, the graph called \textit{pyramid} plays an important part in some of the theorems and proofs in this paper: A pyramid $abcdrp$ consists of a $P_4$ $abcd$ together with a \{$a, b, c, d\}$-universal vertex $p$ and a \{$a, b, c, d\}$-partial vertex $r$ which sees the midpoints of $abcd$ but misses its endpoints, see Figure 2.

![Figure 2: The pyramid $abcdrp$.](image)

### 3 Elementary properties of $P_4$-components

Most of the proofs in this section generalize an assertion $A$ that holds for one $P_4$ in a $P_4$-component $C^*$ to all other $P_4$s in $C^*$. The inductive step consists of proving $A$ for an additional $P_4$ based on the hypothesis that it already holds for an adjacent $P_4$. Given such an adjacent $P_4$, we have to distinguish the cases in which two ribs or two wings or a rib and a wing coincide. The first of these cases, however, can be omitted, as the next Lemma reveals. (Two $P_4$s are called \textit{weak-adjacent} if two wings or a rib and a wing coincide.)

**Lemma 3.1** Two $P_4$s with common ribs are connected by a sequence of weak-adjacent $P_4$s.

**Proof.** Let $abcd$ and $a'd'b'c'd'$ denote two $P_4$s with common ribs. If $a$ and $a'$ or $d$ and $d'$ coincide, the two $P_4$s themselves are weak-adjacent and we are done. So assume $|\{a, a', b, c, d, d'\}| = 6$. 

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If \( a \) misses \( d' \), then \( abed, abed' \), \( a' b' c' d' \) is a sequence of weak-adjacent \( P_4 \)s. The analogous argument applies if \( a' \) misses \( d \), so it remains to discuss the case \( ad', a' d' \in E \).

If \( a \) misses \( d' \), we find that \( abed, abd', a' b' c' d' \) is a sequence of weak-adjacent \( P_4 \)s, otherwise, if \( a \) is adjacent to \( a' \), then \( abed, ad'ce, d'ad'c \) denotes such a sequence. \( \square \)

Consider a \( P_4 \) \( abed \) together with another vertex, say \( v \). Up to symmetry, all possible graphs induced by \( abed \) and \( v \) are enumerated in Figure 3, where bold lines indicate edges in the same \( P_4 \)-component. If we additionally assume that \( v \) is not covered by \( C^*(ab) \), the only graphs left are the \( F_1, \) the \( F_7 \) and the \( F_{10} \), i.e. \( v \) is either \{a, b, c, d\}-universal, \{a, b, c, d\}-null or it sees the midpoints but misses the endpoints of the \( P_4 \) \( abed \).

![Figure 3: All possibilities of a \( P_4 \) together with a fifth vertex \( v \).](image)

**Lemma 3.2** Let \( C^* \) be a \( P_4 \)-component and \( v \) a vertex not covered by \( C^* \). If \( v \) and a \( P_4 \) in \( C^* \) induces an \( F_7 \), then the graph induced by \( v \) and any \( P_4 \) in \( C^* \) is an \( F_7 \).

**Proof.** Our proof is by induction on the \( P_4 \)s in \( C^* \). So let \( abed \) and \( a'b'c'd' \) denote two weak-adjacent \( P_4 \)s in \( C^* \) and assume that the graph induced by \( abed \) and \( v \) is an \( F_7 \). According to Lemma 3.1, it suffices to distinguish the following cases:

**Case 1:** Two wings coincide. Without loss of generality, we may assume that the wing \( ab \) coincides with the wing \( a'b' \); thus either \( a' = a \) and \( b' = b \) or \( a' = b \) and \( b' = a \). The latter, however, is impossible because \( a'b'c'd' \) and \( v \) would not induce an \( F_1, F_7 \) or \( F_{10} \). In the former case, the only possible induced graph is the \( F_7 \) as claimed.

**Case 2:** A wing coincides with a rib. A wing of \( abed \) cannot coincide with \( b'c' \) as otherwise the graph induced by \( a'b'c'd' \) and \( v \) would not be an \( F_1, F_7 \) or \( F_{10} \). Therefore, a wing of \( a'b'c'd' \) must coincide with \( bc \). This implies that the graph induced by \( a'b'c'd' \) and \( v \) is an \( F_1 \); thus \(|\{a, d, a', b', c, d'\}| = 6\).

Without loss of generality (symmetry), let \( b = a' \) and \( c = b' \). Then \( d' \) sees \( a \) and \( d \), for otherwise \( abed' \) or \( dcvd' \) would be a \( P_4 \) in \( C^* \) that covers \( v \). So \( ad'ce \) is a \( P_4 \) in \( C^* \), a contradiction because \( ad'ce \) and \( v \) induce an \( F_5 \). \( \square \)

Let \( C^* \) be a nontrivial \( P_4 \)-component and \( r \) a vertex in \( R \). From the definition of the \( P_4 \)-components follows that a \( P_4 \) \( abed \) in \( C^* \) exists such that \( r \) is \{a, b, c, d\}-partial; hence \( abed \) and \( r \) induce an \( F_7 \). By Lemma 3.2, the vertex \( r \) sees the midpoints of every \( P_4 \) in \( C^* \) but misses its endpoints; thus \( C^* \) is separable.
Furthermore, $r$ cannot be adjacent to a vertex $q \in Q$, as otherwise any $P_4 \ abed$ in $C^*$ would imply a $P_4 \ qrba$ in $C^*$, a contradiction to our assumption that $r$ is not covered by $C^*$. Corollary 3.3 below restates these results.

**Corollary 3.3** Let $C^*$ be a nontrivial $P_4$-component and $R \neq \emptyset$. Then $C^*$ is separable and every vertex in $R$ is $V^1$-universal and $V^2$-null. Moreover, no edge between $R$ and $Q$ exists.

![Diagram](image)

Figure 4: Corollary 3.3 illustrated (dotted lines indicate possible edges in $V - V(C^*)$).

**Lemma 3.4** Given a separable $P_4$-component $C^*$ with vertex partition $(V^1, V^2)$. Then neither a $P_3 \ abc$ with $a \in V^1$ and $b, c \in V^2$ nor a $\overline{P}_3 \ abc$ with $a, b \in V^1$ and $c \in V^2$ exists.

**Proof.** In a first step, we show that no $P_3$ or $\overline{P}_3$ as described in our lemma has edges in $C^*$. Assume a $P_3 \ abc$ with $a \in V^1$ and $b, c \in V^2$. Since $C^*$ is separable, $bc$ cannot belong to $C^*$. Now suppose $ab \in C^*$. Then a $P_4 \ bade$ in $C^*$ exists with $d \in V^1$ and $e \in V^2$. If $ce \in E$, then the $P_4 \ abce$ would contradict the separability of $C^*$. Hence $ce \notin E$. But $dc \in E$ implies the $P_4 \ bedc$, and $de \notin E$ implies the $P_4 \ dabc$, in both cases a contradiction to $be \notin C^*$.

Now assume a $\overline{P}_3$ with $a, b \in V^1$, $c \in V^2$ and $ac \in C^*$. Then a $P_4 \ cad e$ in $C^*$ exists with $d \in V^1$ and $e \in V^2$. If $bd \in E$, the $P_4 \ cadb$ would violate the separability of $C^*$; hence $bd \notin E$. If $be \in E$, the $P_4 \ adec$ would violate the separability of $C^*$; thus $be \notin E$. In fact, we have shown that given $b$ misses the vertices incident to one wing of a $P_4$ in $C^*$, the same holds for the vertices incident to the other wing. Note that Lemma 3.1 and the separability of $C^*$ imply that weak-adjacent $P_4$s in $C^*$ have a common wing. So by induction on the $P_4$s in $C^*$, no wing is incident to $b$, a contradiction to our assumption that $b$ belongs to the cover of $C^*$.

The remainder of the proof is based on the fact that a $P_3$ or a $\overline{P}_3$ as defined in our lemma has no edge in $C^*$. We call such a $P_3$ or $\overline{P}_3$ forcing because all its edges are forced out of $C^*$. Next, we show that no forcing $\overline{P}_3 \ abc$ can exist. Since $C^*$ covers $b$, there is an edge $bd \in C^*$ with $d \in V^2$. If $cd \in E$, then $bd$ is a forcing $P_3$, and if $ad \notin E$, then $bad$ is a forcing $\overline{P}_3$; in both cases a contradiction to $bd \in C^*$. Therefore $cd \notin E$ and $ad \in E$; thus $cadb$ is a $P_4$ in $C^*$, a contradiction to the separability of $C^*$.

It remains to prove that no forcing $P_3 \ abc$ exists. Since $C^*$ covers $c$, there is an edge $cd \in C^*$ with $d \in V^1$. Moreover $bd \in E$, for otherwise the forcing $P_3 \ deb$ would contradict $cd \in C^*$. We say that an edge $uv \in C^*$ with $v \in V^2$ and $w \in V^1$ is

**type 1** if $b$ sees $v$ and a forcing $P_3 \ wbu$ exists, and
type 2 if \(b\) sees \(w\) and a forcing \(P_3\ wbu\) exists.

Figure 5 illustrates this definition. (Solid lines indicate edges that must exist whereas dotted lines indicate edges that must not exist.)

![Figure 5: A type1 and type2 edge as defined in the proof of Lemma 3.4.](image)

Obviously \(cd\) is type2. We claim that any edge \(vw \in C^*\) with \(v \in V^2\) and \(w \in V^1\) is either type1 or type2. From this follows immediately that \(C^*\) cannot cover \(b\), a contradiction to our assumption.

The proof of the above claim is by induction on the \(P_4\)s in \(C^*\). Since \(cd\) is type2, we have already settled the basis. For the inductive step, by Lemma 3.1 and the separability of \(C^*\), it again suffices to show that given one wing in a \(P_4\) in \(C^*\) is type1 or type2, the same holds for the other wing in the same \(P_4\). So let \(vwxy\) denote an arbitrary \(P_4\) in \(C^*\) and assume \(vw\) to be type1 or type2.

Case 1: \(vw\) is type1. Then \(v\) misses \(u\), for otherwise the forcing \(P_3\ wvu\) would contradict \(vw \in C^*\). We distinguish the following two subcases.

Case 1.1: \(u = y\). If \(b\) misses \(x\), then \(xyb\) is a forcing \(P_3\), a contradiction to \(xy \in C^*\). Therefore \(b\) sees \(x\); thus \(b\) sees \(y\) and \(xbu\) is a forcing \(P_3\), ie \(xy\) is type1.

Case 1.2: \(u \neq y\). Then \(|\{b, u, v, w, x, y\}| = 6\). Furthermore, both \(bx \notin E\) and \(by \notin E\) cannot hold, as otherwise the \(P_4\ bwxu\) would contradict \(bw \notin C^*\). If \(bx \notin E\) and \(by \in E\), then \(xyb\) is a forcing \(P_3\), a contradiction to \(xy \in C^*\). If \(bx \in E\) and \(by \notin E\), then the \(P_4\ vbxy\) violates the separability of \(C^*\). Therefore \(bx \in E\) and \(by \notin E\) holds; thus \(b\) sees \(x\) and \(why\) is a forcing \(P_3\), ie \(xy\) is type2.

Case 2: \(vw\) is type2. Then \(u\) sees \(w\), for otherwise the forcing \(\overline{P_3}\ wuv\) would contradict \(vw \in C^*\). Again we distinguish two subcases.

Case 2.1: \(x = u\). If \(b\) misses \(y\), the \(P_4\ vbxy\) contradicts the separability of \(C^*\). Therefore \(b\) sees \(y\) and \(xbu\) is a forcing \(P_3\); thus \(xy\) is type1.

Case 2.2: \(x \neq u\). Then \(|\{b, u, v, w, x, y\}| = 6\). Assume that \(b\) misses \(x\). Then \(b\) misses \(y\) as well, for otherwise the forcing \(P_3\ xyb\) would contradict \(xy \in C^*\). If \(u\) misses \(y\), then either the \(P_4\ bxy\) contradicts \(bu \notin C^*\) or the \(P_4\ uwy\) contradicts the separability of \(C^*\). So \(u\) sees \(y\) and both \(uvy\) and \(ebu\) are \(P_4\)s in \(C^*\), a contradiction to \(eb \notin C^*\).

Therefore our assumption was wrong; so \(b\) sees \(x\). Moreover \(b\) sees \(y\), as otherwise the \(P_4\ vbxy\) would violate the separability of \(C^*\). Thus \(b\) sees \(x\) and \(why\) is a forcing \(P_3\), ie \(xy\) is type2.

Now Theorem 3.5 follows readily from the above lemma.

**Theorem 3.5** Let \(C^*\) denote an arbitrary \(P_4\)-component. Then no \(V(C^*)\)-partial \(P_4\) exists.
Proof. Suppose a $V(C^*)$-partial $P_4$ $abcd$. Clearly, at least one vertex in $\{a, b, c, d\}$ belongs to $R$; hence $C^*$ is separable with vertex partition $(V^1, V^2)$.

Case 1: $bc \in E(V(C^*))$. Let $a$ denote the vertex in $R$. Then $b \in V^1$ and $c \in V^2$. Since $d$ sees $c$ but misses $b$, it must belong to $V(C^*)$. Moreover $d \in V^2$ because $a$ misses $d$. So $bed$ is a $P_3$ with $b \in V^1$ and $c, d \in V^2$, a contradiction to Lemma 3.4.

Case 2: $ab \in E(V(C^*))$ or $cd \in E(V(C^*))$. Without loss of generality (symmetry), let $ab \in E(V(C^*))$. Moreover $c$ is not in the cover of $C^*$, for otherwise we are back in Case 1. Consequently $c \in R$, $a \in V^2$ and $b \in V^1$. If $d$ is not covered by $V(C^*)$, it must be $Q$-vertex. But $d$ sees $c$, a contradiction to the fact that no edge between $R$ and $Q$ can exist. Hence $d$ belongs to the cover of $C^*$; thus $d \in V^1$. So $bda$ is a $P_3$ with $d, b \in V^1$ and $a \in V^2$, a contradiction to Lemma 3.4.

By Theorem 3.5 and Corollary 3.3, a $P_4$ with at least one but not all its vertices in $V(C^*)$ must be a $P_4$ of types (1) to (6) below.

- type (1) $vp_1q_2$ where $v \in V(C^*)$, $p \in P$, $q_1 \in Q$, $q_2 \in Q$
- type (2) $p_1vp_2q$ where $p_1 \in P$, $v \in V(C^*)$, $p_2 \in P$, $q \in Q$
- type (3) $p_1v_2p_2r$ where $p_1 \in P$, $v_2 \in V^2$, $p_2 \in P$, $r \in R$
- type (4) $v_2pr_1r_2$ where $v_2 \in V^2$, $p \in P$, $r_1 \in R$, $r_2 \in R$
- type (5) $rv_1pq$ where $r \in R$, $v_1 \in V^1$, $p \in P$, $q \in Q$
- type (6) $rv_1pv_2$ where $r \in R$, $v_1 \in V^1$, $p \in P$, $v_2 \in V^2$

Note that a $P_4$ of type (6) together with a $P_4$ $abcd$ in $C^*$ is a pyramid, see Figure 2. The graphs induced by a $P_4$ of types (3) to (5) together with a $P_4$ $abcd$ in $C^*$ are depicted in Figure 6, where bold lines indicate edges in the same $P_4$-component different from $C^*$. Obviously, the existence of a $P_4$ of types (3) to (5) implies a $P_4$ of type (6).

![Figure 6: The subgraphs induced by a $P_4$ of types (3) to (5).](image)

Finally, the question arises if it is possible that two $P_4$-components have the same cover. The next theorem answers this question in the negative.

**Theorem 3.6** Two different $P_4$-components have different covers.

The following lemmas prepare the proof of Theorem 3.6.
**Lemma 3.7** Let $vw$ be an edge of a $P_4$ and $z$ a vertex different from $v$ and $w$.

(i) If $vw$ is a wing and $vz, wz \in E - C^*(vw)$, then $z$ sees all the vertices in the $P_4$.

(ii) If $vw$ is a wing, $z$ misses $v$ and $wz \in E - C^*(vw)$, then the $P_4$ can be labeled $vwxy$ and $z$ sees $x$ but misses $y$.

(iii) If $vw$ is a rib and $vz, wz \in E - C^*(vw)$, then the $P_4$ can be labeled $vwzx$ and either $z$ misses $u$ and $x$ or $z$ sees $u$ and $x$.

(iv) If $vw$ is a rib, $z$ misses $v$ and $wz \in E - C^*(vw)$, then $P_4$ can be labeled $vwzx$ and $wz, xz \in C^*(vw)$.

**Proof.** (i) Without loss of generality, let $vwxy$ be the $P_4$ in question. From Figure 3 follows that only the $F_{10}$ is possible.

(ii) The $P_4$ can be labeled $xyuw$ or $vwxy$. Again from Figure 3 follows that the former case is impossible whereas in the latter case only an $F_7$ does not contradict $wz \in E - C^*(vw)$.

(iii) A $P_4$ $xwvy$ implies an $F_1, F_2$ or $F_7$. But an $F_2$ cannot satisfy both $vz \notin C^*(vw)$ and $wz \notin C^*(vw)$.

(iv) In this case, only the $F_3$ does not contradict $wz \in E - C^*(vw)$, see Figure 3. \qed

**Lemma 3.8** Let $vw$ be a rib of a $P_4$ and $z$ a vertex that sees $w$ but misses $v$. If $|C^*(wz)| > 1$, then $C^*(wz) = C^*(vw)$.

**Proof.** Suppose the contrary, ie $C^*(wz) \neq C^*(vw)$ From Lemma 3.7(iv) follows that the $P_4$ in which $vw$ is the rib can be labeled $vwzx$ with $uz, xz \in C^*(vw)$. Moreover, as $|C^*(wz)| > 1$, the edge $wz$ belongs to $P_4$ as well.

**Case 1:** $wz$ is a wing. Then Lemma 3.7(ii) applies to $wz$ and $u$; hence the $P_4$ with the wing $wz$ can be labeled $wzab$. The same lemma also applies to $zw$ and $v$; therefore the same $P_4$ can be labeled $zwde$. But no $P_4$ can be labeled in both ways.

**Case 2:** $wz$ is a rib. Then Lemma 3.7(iv) applied to $wz$ and $u$ and $zw$ and $v$ respectively guarantees a $P_4$ $awzb$ with $aa, ab, va, vb \in C^*(wz)$. Thus either $bwzx$ or $abzw$ is a $P_4$; in both cases a contradiction to $C^*(vw) \neq C^*(wz)$. \qed

The next lemma deals with the pyramid, cf Figure 2.

**Lemma 3.9** If $abedrp$ is a pyramid such that $C^*(rh)$ and $C^*(re)$ are different from $C^*(ab)$, then $r$ is not covered by $C^*(ab)$.  

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[Figure 7: Lemma 3.7 illustrated.]
Proof. If \( \{ab, bc, cd\} = C^*(ab) \), there is nothing to prove. Therefore, assume a \( P_4 \) \( a'b'd'd' \) weak-adjacent to \( abcd \). Note that the \( P_4 \)'s rbpd and rcpa guarantee that all edges in the pyramid different from \( ab, bc \) and \( cd \) do not belong to \( C^*(ab) \).

In the following case analysis, we show that \( a'b'd'd'pr \) is another pyramid which satisfies \( C^*(r'b') \neq C^*(ab) \) and \( C^*(r'd') \neq C^*(ab) \). By induction, this holds for every \( P_4 \) in \( C^*(ab) \); thus \( r \) is incident to no edge in \( C^*(ab) \) as claimed.

**Case 1:** A wing of \( abcd \) coincides with a wing of \( a'b'd'd' \). Without loss of generality, let \( a'b' \) be the common edge. Then Lemma 3.7(ii) applies to \( a'b' \) and \( r \); hence \( a' = a \), \( b' = b \) and \( r \) sees \( c' \) but misses \( d' \); thus \( C^*(r'b') = C^*(rb) \neq C^*(ab) \). Similarly, Lemma 3.7(i) applies to \( d'b' \) and \( p \); hence \( p \) sees \( c' \) and \( d' \); thus \( a'b'd'd'pr \) is pyramid. Moreover \( C^*(r'd') = C^*(rc) \neq C^*(ab) \) because of the \( P_4 \)'s rcpa and repa.

**Case 2:** A wing of \( abcd \) coincides with the rib of \( a'b'd'd' \). Then Lemma 3.8 applies to \( b'd' \) and \( r \); thus \( C^*(ab) = C^*(rb) \) or \( C^*(ab) = C^*(rc) \), a contradiction to the premise of our lemma.

**Case 3:** The rib of \( abcd \) coincides with a wing of \( a'b'd'd' \). Without loss of generality, let \( a' = b \) and \( b' = c \). From Lemma 3.7(i) applied to \( a'b' \) and \( r \) follows that \( r \) sees \( c' \) and \( d' \). But the same Lemma also applies to \( a'b' \) and \( p \); so \( p \) sees \( c' \) and \( d' \). Thus \( \{a', b', c', d', d, r, p\} \) = 7. Furthermore \( d \) sees \( d' \), as otherwise the \( P_4 \) dcrd would contradict \( C^*(ab) \neq C^*(rc) \). So \( bcd'd'd'd'rb \) are \( P_4 \); hence \( C^*(ab) = C^*(rb) \), a contradiction to our assumption. 

\[ \square \]

**Corollary 3.10** Let \( abedrp \) denote a pyramid. Then \( V(C^*(rb)) = V(C^*(ab)) \) implies \( C^*(rb) = C^*(ab) \).

**Proof.** Suppose \( C^*(rb) \neq C^*(ab) \). Then \( C^*(rc) = C^*(ab) \), as otherwise a contradiction to Lemma 3.9 would arise. Therefore \( C^*(ab) = C^*(rc) \) is different from \( C^*(rb) \) and Lemma 3.9 applies to the pyramid rbpdace; hence \( a \) cannot be covered by \( C^*(rb) \), a contradiction to our assumption. \[ \square \]

**Proof of Theorem 3.6.** Suppose the contrary, i.e. two different \( P_4 \)-components \( C_1^* \) and \( C_2^* \) satisfy \( V(C_1^*) = V(C_2^*) \). Then \( C_1^* \) and \( C_2^* \) cannot be trivial and a \( P_4 \) abed in \( C_1^* \) exists. Clearly, each vertex in \( \{a, b, c, d\} \) is incident to at least one edge in \( C_2^* \). Therefore, the vertices \( \{a, b, c, d\} \) together with the endpoint of such an edge, say \( v \), induce one of the graphs depicted in Figure 3. Moreover \( C_1^* \neq C_2^* \), which leaves the graphs \( F_1, F_2, F_3, F_4 \) and \( F_7 \). We show that each of these graphs is impossible.

**F_3:** Then \( ve \in C^*_1 \) and Lemma 3.8 applies to \( bc \) and \( v \); hence \( C^*(bc) = C^*(vc) \), a contradiction to \( C_1^* \neq C_2^* \).

**F_4:** Then \( rd \in C^*_2 \). Since the situation is symmetric relative to \( v \) and \( d \), we may assume that \( vw \) denotes another edge in a \( P_4 \) that contains \( rd \). Hence \( dvw \) is a \( P_3 \) and \( \{a, b, c, d, v, w\} \) = 6.

Suppose \( w \) misses \( c \). Then \( w \) sees \( b \), as otherwise the \( P_4 \) bcwv would imply \( C_1^* = C_2^* \). Hence \( bwv \) is a \( P_4 \) in \( C_2^* \). Lemma 3.8 applies to \( uw \) and \( c \); thus \( C^*(uv) = C^*(cv) \), a contradiction to \( C_1^* \neq C_2^* \). Therefore our supposition was wrong, so \( w \) sees \( c \).
Furthermore \( w \) misses \( a \), for otherwise the \( P_4 \)'s \( awvd \) and \( awed \) would imply \( C^*_1 = C^*_2 \). The same contradiction arises if \( w \) sees \( b \), this time because of the \( P_4 \) \( abwr \). Hence \( abcw \) is another \( P_4 \) in \( C^*_1 \).

Obviously, the same argumentation holds for the third edge of the \( P_4 \) and, by induction, for every edge in \( C^*_2 \). Therefore, no edge in \( C^*(vd) \) is incident to \( a \) or \( b \), a contradiction to our assumption \( V(C^*_1) = V(C^*_2) \).

**F_2:** Without loss of generality, let \( vb \) be the edge in \( C^*_2 \). Then \( vb \) cannot be the rib of a \( P_4 \), as otherwise a contradiction to Lemma 3.8 applied to \( vb \) and \( a \) would arise. Therefore \( vb \) is a wing, Lemma 3.7(ii) applies to \( vb \) and \( a \); thus our \( P_4 \) can be labeled \( vbxy \) and \( a \) sees \( x \) but misses \( y \). If \( y = d \), then \( axdc \) is a \( P_4 \) which contradicts \( C^*_1 \neq C^*_2 \). Hence \( \{|a, b, c, d, v, x, y\}| = 7 \).

**Case 1:** \( ex \notin E \). As \( xb \) is a rib, we can apply Lemma 3.8 to \( xb \) and \( c \); hence \( C^*_1 = C^*_2 \), the usual contradiction.

**Case 2:** \( ex \in E \). If \( d \) sees \( x \), then \( abcdvx \) is a pyramid which satisfies \( V(C^*(vb)) = V(C^*(ab)) \), Corollary 3.10 applies and again \( C^*_1 = C^*_2 \). The same contradiction arises if \( c \) sees \( y \), this time because of the pyramid \( vbxyac \) and \( V(C^*(vb)) = V(C^*(ab)) \). Therefore \( dx, ey \notin E \). So \( yxve \) and \( axcd \) are \( P_4 \)'s; hence \( C^*(ed) = C^*(yx) \), again a contradiction to \( C^*_1 \neq C^*_2 \).

**F_2:** Then \( vx \in C^*_2 \). Without loss of generality (symmetry), let \( vx \) be another edge in a \( P_4 \) which \( vc \) belongs to. In the following case analysis, we show that \( abed \) together with \( x \) again induces an \( F_2 \), i.e. the structure repeats itself. Therefore, by induction, all edges in \( C^*_2 \) together with \( a, b \) and \( d \) induce an \( F_2 \); thus \( a, b \) and \( d \) are not covered by \( C^*_2 \), a contradiction to \( V(C^*_1) = V(C^*_2) \).

**Case 1:** \( x \) sees \( b \) and \( d \). If \( x \) sees \( a \), the \( P_4 \)'s \( axdc \) and \( axve \) imply \( C^*_1 = C^*_2 \), a contradiction. Therefore \( x \) misses \( a \) and the \( P_4 \) \( abvd \) together with \( x \) induces an \( F_2 \) as claimed.

**Case 2:** \( x \) misses \( b \) or \( d \). If \( x \) misses \( b \), Lemma 3.8 applies to \( bv \) and \( x \), a contradiction to \( C^*_1 \neq C^*_2 \). Hence \( x \) sees \( b \) but misses \( d \). Then \( vx \) cannot be the wing of a \( P_4 \) that contains \( vx \), as otherwise a contradiction to Lemma 3.7(i) applied to \( vc \) and \( d \) would arise. Therefore \( cv \) is a rib, Lemma 3.7(iii) applies \( cv \) and \( d \); thus our \( P_4 \) can be labeled \( axcvz \) and, together with \( d \), induces an \( F_2 \). But we have already shown that such an \( F_2 \) leads to a contradiction.

**F_1:** Let \( d'bl'c'd' \) be a \( P_4 \) weak-adjacent to \( abed \). Obviously, \( v \notin \{d', b', c', d'\} \). Moreover, as all other possibilities have been ruled out, \( d'bl'c'd' \) and \( v \) induce another \( F_1 \). Therefore, by induction, \( v \) is \( V(C^*_1) \)-universal; thus \( v \) is not covered by \( C^*_1 \), a contradiction. \( \square \)

## 4 Recognition and orientation algorithms

In order to obtain an acyclic \( P_4 \)-transitive (\( P_4 \)-indifferent) orientation, it suffices to compute an acyclic orientation of the edges in the \( P_4 \)'s (all other edges can be oriented by topological sorting). In the following, we only discuss this part of the orientation.

If no nontrivial \( P_4 \)-component covers a proper subset of the vertices of \( G \), then either \( G \) contains no \( P_4 \) or, by Theorem 3.6, precisely one nontrivial \( P_4 \)-component exists. In the former case, nothing has to be done whereas in the latter case, given \( G \) is a \( P_4 \)-comparability
(P₄-indifference) graph, a P₄-transitive (P₄-indifferent) orientation of this P₄-component is unique (up to inversion) and therefore easy to compute. We show that all other cases can be reduced to one of these cases.

So suppose a nontrivial P₄-component, say C*, that does not cover the whole graph. If R = ∅, the cover of C* is a homogeneous set and we can replace it with a single marker vertex, i.e., we choose an arbitrary vertex m ∈ V(C*) and remove all other vertices in V(C*).

(i) Replace V(C*) with a marker vertex m such that m is P-universal and Q-null.

(ii) Recursively orient the edges of the P₄-s in Gᵥ(C*) and in Gᵥ⁻ᵥ(C*⁺m).

(iii) Construct an orientation of the edges of the P₄-s in G by directing

vw with v, w ∈ V(C*) as in Gᵥ(C*),
vw with v, w ∈ V - V(C*) as in Gᵥ⁻ᵥ(C*⁺m),
wv with v ∈ V(C*) and w ∈ V - V(C*) as mw in Gᵥ⁻ᵥ(C*⁺m).

If R ≠ ∅, then C* is separable with vertex partition (V¹, V²). This time we need two marker vertices to represent V¹ and V²: We choose an arbitrary P₄ abcd in C* and remove all vertices in V¹ + V² except for b and d.

(i) Replace V¹ and V² with nonadjacent marker vertices b and d such that b is P-universal, R-universal and Q-null and d is P-universal, R-null and Q-null.

(ii) Recursively orient the edges of the P₄-s in Gᵥ⁺ᵥ² and in Gᵥ⁻(V¹⁺V²⁺(b+d)).

(iii) Construct an orientation of the edges of the P₄-s in G by directing

vw with v, w ∈ V¹ + V² as in Gᵥ⁺ᵥ²,
wv with v, w ∈ V - V¹ - V² as in Gᵥ⁻(V¹⁺V²⁺(b+d)),
wv with v ∈ V¹ and w ∈ V - V¹ - V² as bw in Gᵥ⁻(V¹⁺V²⁺(b+d)) and
vw with v ∈ V² and w ∈ V - V¹ - V² as dw in Gᵥ⁻(V¹⁺V²⁺(b+d)).

Obviously, a P₄-transitive (P₄-indifferent) orientation of G induces a P₄-transitive (P₄-indifferent) orientation of Gᵥ⁻ᵥ(C*⁺m), Gᵥ(C*), Gᵥ⁻(V¹⁺V²⁺(b+d)) and Gᵥ⁺ᵥ². Lemma 4.1 and 4.2 assert that the converse holds for P₄-transitive orientations; thus the above algorithm correctly orients a P₄-comparability graph in O(n²m), the time needed to orient the edges in the P₄-components.

**Lemma 4.1.** The orientation of the P₄-s in G is P₄-transitive (P₄-indifferent) and acyclic whenever the orientation of the P₄-s in Gᵥ(C*) and Gᵥ⁻ᵥ(C*⁺m) is P₄-transitive (P₄-indifferent) and acyclic.

**Lemma 4.2.** The orientation of the P₄-s in G is P₄-transitive and acyclic whenever the orientation of the P₄-s in Gᵥ⁺ᵥ² and Gᵥ⁻(V¹⁺V²⁺(b+d)) is P₄-transitive and acyclic.

Regarding P₄-indifference graphs, we have the following lemma.
**Lemma 4.3** The orientation of the $P_4$s in $G$ is $P_4$-indifferent and acyclic whenever $G$ contains no pyramid and the orientation of the $P_4$s in $G_{V_1+V_2}$ and $G_{V-(V_1+V_2)+\{b+d\}}$ is $P_4$-indifferent and acyclic.

It is easy to see that no pyramid admits a $P_4$-indifferent orientation, so no $P_4$-indifference graph contains the pyramid. Thus Lemma 4.1 and 4.3 guarantee that the above algorithm correctly orients $P_4$-indifference graphs. On the other hand, as the orientation computed by the above method is always $P_4$-indifferent, it suffices to test whether our orientation is acyclic to recognize $P_4$-indifference graphs. Thus we have found an $O(n^2m)$ recognition and orientation algorithm for $P_4$-indifference graphs.

**Proof of Lemma 4.2** To begin with, we show that every $P_4$ in $G$ is oriented properly. This is obvious for $P_4$s with all vertices in $V(C^*)$ and for $P_4$s with all vertices not in $V(C^*)$.

The remaining $P_4$s are of types (1) to (6), for each of which we can find a corresponding $P_4$ in $G_{V-(V_1+V_2)+\{b+d\}}$ by replacing the vertex in $V_1$ with $b$ and the vertex in $V_2$ with $d$.

Now suppose the orientation of $G$ is cyclic. As the orientation of $G_{V-(V_1+V_2)+\{b+d\}}$ and $G_{V_1+V_2}$ is acyclic, every cycle contains edges with both endpoints in $V(C^*)$ and edges with an endpoint not in $V(C^*)$. Choose a cycle with a minimal number of vertices in $V(C^*)$ and let $v \rightarrow \cdots \rightarrow w$ denote the longest part of this cycle in $V(C^*)$. Furthermore, let $u$ be the predecessor of $v$ and $x$ the successor of $w$ in this cycle; thus $u, x \not\in V(C^*)$.

Since $uv$ is directed, it must belong to a $P_4$ of types (1) to (6). Moreover $w$ cannot belong to the same partition set $V_1$ or $V_2$ as $v$ because this would imply $u \rightarrow w$, ie a cycle with fewer vertices in $V(C^*)$ would exist. Without loss of generality, let $v \in V_2$ (otherwise we invert the orientation of each directed edge). Hence $u \in P$.

But $uv$ is in no $P_4$ of types (1) or (2), as otherwise $u \rightarrow d$ and $u \rightarrow b$ in $G_{V-(V_1+V_2)+\{b+d\}}$ and therefore $u \rightarrow w$, again a contradiction because a cycle with fewer vertices in $V(C^*)$ has been found. For the same reason, $uv$ cannot belong to a $P_4$ of types (4) to (6), see Figure 6. Now assume that $uv$ is in a $P_4$ of type (3), say $p_1uvr$. Then $G_{V-(V_1+V_2)+\{b+d\}}$ contains the $P_4$s $p_1dvr$ and $rbp_1d$; hence $r \rightarrow b$ in $G_{V-(V_1+V_2)+\{b+d\}}$ and therefore $r \rightarrow w$ in $G$. Thus $u \rightarrow v \rightarrow \cdots \rightarrow w$ can be replaced with $u \rightarrow r \rightarrow w$, again a contradiction as we have found a cycle with fewer vertices in $V(C^*)$.

Lemma 4.1 and Lemma 4.3 can be proven in much the same way as Lemma 4.2. Actually, these proofs are even simpler because no $P_4$ of types (3) to (6) can occur.

## 5 The decomposition

A vertex set $M$ is a module if no $M$-partial vertex exists. Moreover, $M$ is a proper module if additionally $M \subset V$. Thus, every homogeneous set is a module but not vice versa. The famous modular decomposition is based on the following theorem [MS94].

**Theorem 5.1** An arbitrary graph $G = (V, E)$ satisfies at least one of the following conditions:

(i) $G$ is disconnected;

(ii) $\overline{G}$ is disconnected;

(iii) the maximal proper modules of $G$ are disjoint.
Since the connected components of $G$ (and $\overline{G}$) are disjoint modules, the above theorem guarantees the uniqueness of the modular decomposition described below.

If $G$ is trivial, then stop,
else if $G$ is disconnected, decompose the connected components of $G$,
else if $\overline{G}$ is disconnected, decompose the connected components of $\overline{G}$,
else decompose the graphs induced by the maximal proper modules of $G$.

As the decomposition operations are performed top-down, we obtain a unique decomposition tree called modular decomposition tree if we distinguish the above operations by a 0, 1 and 3-node\(^1\). If $G$ is trivial, this is indicated by a empty node labeled $v$ where $v$ stands for the only vertex in $G$.

**Procedure** Build\_Tree($G$);
{ Input: an arbitrary graph $G = (V,E)$;
 Output: the root of the decomposition tree $D(G)$ of $G$; }
begin
  if $|V| = 1$ then
    let $v \in V$;
    return an empty node labeled $v$;
  else if $G$ is disconnected then
    let $G_1, G_2, \ldots, G_t$ be the connected components of $G$;
    let $r_i = \text{Build\_Tree}(G_i)$ for $i = 1, \ldots, t$;
    return a 0-node with children $r_1, r_2, \ldots, r_t$;
  else if $\overline{G}$ is disconnected then
    let $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_t$ be the connected components of $\overline{G}$;
    let $r_i = \text{Build\_Tree}(\overline{G}_i)$ for $i = 1, \ldots, t$;
    return a 1-node with children $r_1, r_2, \ldots, r_t$;
  else (* $G$ and $\overline{G}$ are connected and $|V| > 1$ *)
    let $H_1, H_2, \ldots, H_t$ be the maximal proper modules of $G$;
    let $r_i = \text{Build\_Tree}(G_{H_i})$ for $i = 1, \ldots, t$;
    return a 3-node with children $r_1, \ldots, r_t$;
  end (* if *)
end;

Furthermore, the original graph can be reconstructed from the modular decomposition tree if we replace the maximal proper modules with marker vertices and store those prime graphs in the corresponding 3-nodes. Therefore, in some sense, any decomposition method for prime graphs can be used to refine the modular decomposition. Our decomposition of prime graphs is based on the structure of separable $P_4$-components and is defined in a way which maintains a unique decomposition tree.

To begin with, we investigate the relation between separable $P_4$-components and modules. So let $C^*$ denote a separable $P_4$-component and consider an edge $vw$ with both endpoints in $V^2$. Recall that no vertex in $V^1$ is $\{v, w\}$-partial, see Lemma 3.4; hence $v$ and $w$ have the same neighborhood relative to $V - V^2$. By induction, this holds for any two vertices in the same connected component of $G_{V^2}$, i.e. a connected component of $G_{V^2}$ is a module. The analogous argumentation applies to $V^1$ and $\overline{G}$, thus the connected components of $\overline{G}_{V^1}$ are modules.

The following lemma restates this result.

\(^1\) we use the same notation as Jamison and Olariu in \cite{JO95}
Lemma 5.2 Let \( G \) be a prime graph and \( C^* \) a separable \( P_4 \)-component with vertex partition \((V^1, V^2)\). Then \((V^1, V^2)\) is a split graph, ie \( V^1 \) is a clique and \( V^2 \) is a stable set.

A separable \( P_4 \)-component is called *maximal* if its cover is not contained in the cover of another separable \( P_4 \)-component. Unlike maximal proper modules, however, the covers of two maximal \( P_4 \)-components need not be disjoint (the pyramid is a counterexample). Therefore, it is necessary to examine the relation between two maximal \( P_4 \)-components whose covers intersect. Let us call them adjacent \( P_4 \)-components for short.

Lemma 5.3 The \( P_4 \)’s in adjacent \( P_4 \)-components are of type (6) relative to one another.

Proof. We show that any \( P_4 \) in \( C^*_2 \) is of type (6) relative to \( C^*_1 \). Since \( C^*_2 \) contains a \( P_4 \) of types (1) to (6) relative to \( C^*_1 \), it suffices to prove that no \( P_4 \) of types (1) to (5) exists because, in this case, every \( P_4 \) adjacent to a \( P_4 \) of type (6) must be of type (6) itself.

In a \( P_4 \) \((v, p, q, w)\) of type (1), the vertex \( v \in V(C^*_1) \) can be replaced with any vertex in the cover of \( C^*_1 \); thus the cover of \( C^*_1 \) is a subset of the cover of \( C^*_2 \), a contradiction to the fact that \( C^*_1 \) is a maximal \( P_4 \)-component. Also, using similar arguments, we can show that no \( P_4 \) of type (2) is possible.

A \( P_4 \) of types (3) to (5) together with a \( P_4 \) \(abcd\) in \( C^*_1 \) induces one of the graphs depicted in Figure 6. But the edge \( r_p, r_1p \) or \( pq \) guarantees that, for any \( P_4 \) in \( C^*_1 \), all bold lines in the corresponding graph in Figure 6 are edges in \( C^*_2 \); thus all vertices in the cover of \( C^*_1 \) are covered by \( C^*_2 \), again a contradiction because \( C^*_1 \) is maximal.

By \((V^1_i, V^2_i)\), we denote the vertex partition of the cover of the separable \( P_4 \)-component \( C^*_i \). Now Corollary 5.4 follows immediately from Lemma 5.3.

Corollary 5.4 Two adjacent \( P_4 \)-components \( C^*_1 \) and \( C^*_2 \) satisfy

(i) \( V^1_1 \cap V^2_2 = \emptyset = V^2_1 \cap V^1_2 \) and \( V^1_1 \cap V^2_1 \neq \emptyset \neq V^1_2 \cap V^2_2 \).

A set \( S = \{C^*_1, C^*_2, \ldots, C^*_k\} \) of \( P_4 \)-components is called *connected* if for every pair \( C^*_i \) and \( C^*_j \) in \( S \) a sequence \( C^*_i, \ldots, C^*_j \) of \( P_4 \)-components in \( S \) exists such that two successive \( P_4 \)-components are adjacent. Note that this definition implies that all \( P_4 \)-components in \( S \) are maximal.

Lemma 5.5 Let \( G \) be a prime graph and let \( \{C^*_1, C^*_2, \ldots, C^*_k\} \) be a connected set of \( P_4 \)-components. Then \((V^1, V^2) = (\cup_{i=1}^k V^1_i, \cup_{i=1}^k V^2_i) \) induces a split graph.

Proof. This proof is by induction, and Lemma 5.2 settles the basis. For the inductive step, assume that \( S = \{C^*_1, C^*_2, \ldots, C^*_j-1\} \) is connected and \( C^*_j \) is adjacent to \( C^*_j-1 \) (after an appropriate permutation of the indices). The induction hypothesis asserts that \((V^1, V^2) = (\cup_{i=1}^{j-1} V^1_i, \cup_{i=1}^{j-1} V^2_i) \) is a split graph. By Lemma 5.2, \((V^1_j, V^2_j) \) is a split graph as well.

Clearly \( V^1_j \cap V^2_j = \emptyset = V^2_j \cap V^1_j \), as otherwise a contradiction to Corollary 5.4 applied to \( C^*_j \) and some \( P_4 \)-components in \( S \) would arise. Furthermore, Corollary 5.4 applied to \( C^*_j-1 \) and \( C^*_j \) guarantees the existences of vertices \( v_1 \in V^1_j \cap V^1_j \) and \( v_2 \in V^2_j \cap V^2_j \).
Since each vertex $v$ in $V^1 - V_j$ sees $v_1$, it must be adjacent to every vertex in $V^1_j$. Similarly, every vertex in $V^2 - V^1_j$ misses $v_2$ and, therefore, all vertices in $V^2_j$. Hence $V^1 \cup V^1_j$ is a clique and $V^2 \cup V^2_j$ is a stable set as claimed. \hfill \Box

Note that the split graphs induced by the maximal connected sets of $P_k$-components are disjoint and therefore unique. Let us call them $P_k$-split graphs. Moreover, it is easy to see that a vertex in $V - V^1 - V^2$ which is $P_k$-graph relative to $C^+_1$ is also $P_k$-graph relative to $C^+_2, \ldots, C^+_6$. Therefore, we can replace $V^1$ and $V^2$ with two nonadjacent marker vertices in the same way as described in Section 4.

Next, we analyze the relation between $P_4$-components and those in the complement $\overline{G}$.

**Lemma 5.6** Let $C^+_1$ be a separable $P_4$-component of $G$ and $C^+_2$ a separable $P_4$-component of $\overline{G}$. If their covers intersect, then $V(C^+_1) \subseteq V(C^+_2)$ or $V(C^+_1) \supseteq V(C^+_2)$ holds.

**Proof.** Suppose the contrary. Then a $P_4$ $fgh$ in $C^+_2$ exists with some but not all its vertices in $V(C^+_1)$. Thus, the $P_4$ $efgh$ in $G$ is of types (1) to (6) relative to $C^+_1$.

But $efgh$ cannot be of type (1), as a $P_4 vpq_1q_2$ of type (1) induces the $P_4 q_1vq_2p$ in $\overline{G}$, which would imply $V(C^+_1) \subseteq V(C^+_2)$. Similarly, each $P_4 p_1vpq_2$ of type (2) induces the $P_4 vqpq_2p$ in $\overline{G}$; thus $efgh$ cannot be of type (2).

In all the remaining cases, see Figure 6, it is easy to verify that a $P_4 rvp_1pv_2$ of type (6) exists which satisfies $rp \in C^+_1$. Consequently, for each $P_4$ $abed$ in $C^+_2$, we find two $P_4$ $prac$ and $prdb$ in $C^+_2$. Again, this implies $V(C^+_1) \subseteq V(C^+_2)$, a contradiction to our assumption. \hfill \Box

**Theorem 5.7** For every nontrivial prime graph $G$, precisely one of the following conditions is satisfied:

(i) $G$ is a $P_4$-split graph $(V^1, V^2)$;

(ii) $G$ consists of a unique $P_4$-split graph $(V^1, V^2)$ together with a $R$-vertex $v$;

(iii) $G$ is no split graph, and the $P_4$-split graphs in $G$ and $\overline{G}$ are disjoint.

**Proof.** Let $W$ denote the set of vertices that are in no $P_4$. If $W = V$, then either $G$ or $\overline{G}$ is disconnected, see SEINSCH [Sei74]. But this contradicts our assumption that $G$ is nontrivial and prime. Therefore, $G$ contains at least one nontrivial $P_4$-component.

If $W \neq \emptyset$, no $P_4$-component covers the whole graph. Hence every $P_2$-component of $G$ is separable and a $P_4$-split graph $(V^1, V^2)$ in $G$ exists. Furthermore, all vertices in $R$ are adjacent to those in $P$, as otherwise a $P_4 rvp_1pv_2, v_1 \in V^1, v_2 \in V^2$, would exist; thus the separable $P_4$-component $C^+(rvv_1)$ contradicts the fact that $(V^1, V^2)$ is a $P_4$-split graph. If $P \cup Q \neq \emptyset$, then $V^1 \cup V^2 \cup R$ is homogeneous. So $P = Q = \emptyset$; hence $R$ is a module; thus $|R| = 1$ and Condition (ii) is satisfied.

Now assume $W = \emptyset$. Additionally, suppose that a $P_4$-split graph $(V^1, V^2)$ in $G$ exists such that every vertex in $R$ sees every vertex in $P$. Then $P = Q = \emptyset$, as otherwise $V^1 \cup V^2 \cup R$ would be homogeneous. Again $R \neq \emptyset$ implies that $R$ is a module; thus $R$ consists of a single vertex in no $P_4$, a contradiction to $W = \emptyset$. Therefore $R = \emptyset$ and $V = V^1 + V^2$, i.e., Condition (i) is satisfied.

Finally, suppose that for every $P_4$-split graph $(V^1, V^2)$ in $G$ vertices $r \in R$ and $p \in P$ exist such that $r$ misses $p$, i.e., there is a $P_4$ $rvpv_2$ with $v_1 \in V^1$ and $v_2 \in V^2$. Note that
$C^*(rv_1)$ cannot be separable, as otherwise a contradiction to our definition of a $P_4$-split graph would arise. But every $P_4$-component of a split graph is separable; hence $G$ cannot be a split graph and the first part of Condition (iii) is satisfied. Furthermore, we may assume that for every $P_4$-split graph in $G$ vertices $r \in R$ and $p \in P$ exist such that $r$ misses $p$, as otherwise the argumentation of the previous paragraph applied to $G$ would imply that $G$ is a split graph. By Lemma 5.5, it remains to show that a $P_4$-split graph in $G$ and a $P_4$-split graph in $\overline{G}$ are disjoint.

If a $P_4$-split graph of $G$ and a $P_4$-split graph of $\overline{G}$ have a common vertex, we can find a maximal $P_4$-component $C^*_1$ in $G$ and a maximal $P_4$-component $C^*_2$ in $\overline{G}$ whose covers intersect. By Lemma 5.6, one is a subset of the other. Without loss of generality, let $V(C^*_1) \subseteq V(C^*_2)$. Obviously, every $R$-vertex relative to $C^*_2$ is a $R$-vertex relative to $C^*_1$; and every $P$-vertex relative to $C^*_2$ is a $Q$-vertex relative to $C^*_1$. But our assumption that $r$ misses $p$ in $\overline{G}$ implies an edge between an $R$-vertex and a $Q$-vertex relative to $C^*_1$. But this is a contradiction to Corollary 3.3.

Our decomposition of prime graphs is based on Theorem 5.7. A $P_4$-split graph with clique $C$ and stable set $S$ (in $G$) is decomposed by a 4 node labeled $C$ with $|S|$ children, each of which corresponds to a vertex $v \in S$ and is labeled $N(v) \cap C$. Thus Condition (i) corresponds to a 3-node with a 4-node as its only child, Condition (ii) corresponds to a 2-node with a 4-node as its only child, and Condition (iii) is represented by a 3-node with multiple children.

Only the else-part of the modular decomposition procedure is given as the rest remains the same. (Note that, below, a $P_4$-split graph of $G$ is denoted by $(S,C)$, i.e., $S$ is a clique in $G$ and therefore a stable set in $G$ whereas $C$ is a stable set $G$ and a clique in $G$.)

\begin{verbatim}
else (* G and \overline{G} are connected and |V| > 1 *)
    let $H_1, H_2, \ldots, H_t$ be the maximal proper modules of $G$;
    let $r_i = \text{Build Tree}(G_{H_i})$ for $i = 1, \ldots, t$;
    let $G' = (V', E')$ be the graph with $h_1, \ldots, h_t$ substituted for $H_1, \ldots, H_t$;
    let $(C'_1, S'_1), (C'_2, S'_2), \ldots, (C'_k, S'_k)$ be the $P_4$-split graphs in $G'$;
    let $(S_{k+1}, C_{k+1}), (S_{k+2}, C_{k+2}), \ldots, (S_{k'}, C_{k'})$ be the $P_4$-split graphs in $\overline{G'}$;
    let $C_1, C_2, \ldots, C_h$ and $S_1, S_2, \ldots, S_h$ denote the corresponding vertex sets in $G$;
    for $i = 1, \ldots, h$ do
        create a 4-node $c_i$ labeled $C_i$;
        for all $h_j \in C'_i$ with $|H_j| > 1$ do
            make $r_j$ a child of $c_i$;
        for all $h_j \in S_i$ do
            make $r_j$ a child of $c_i$;
            give $r_j$ the label $N(H_j) \cap C_i$;
        end (* for all *)
    end (* for *)
    if $k = 1$ and $V' \not= C'_1 \cup S'_1$ then
        let $h_j$ be the vertex in $V' - C'_1 - S'_1$;
        return a 2-node with children $c_1$ and $r_j$;
    else
        return a 3-node with children $c_1, \ldots, c_h$ and $r_j$ such that $h_j \in V' - C'_1 - S'_1 - \cdots - C'_k - S'_k$;
    end (* if *)
end (* if *)
\end{verbatim}
As in case of the modular decomposition, the original graph can be reconstructed from the unique decomposition tree if we replace the maximal homogeneous sets and the $P_4$-split graphs with marker vertices and store those graphs in the 3-nodes with multiple children.

![Diagram](image)

**Figure 8:** An example of the module and $P_4$-split graph substitution.

Figure 8 and 9 illustrate our decomposition: Figure 8(a) shows the original graph $G$, Figure 8(b) the prime graph $G'$ after the substitution of the maximal proper modules $\{b_1, b_2\}$, $\{e_1, e_2\}$ and $\{f_1, f_2, f_3, f_4\}$, and Figure 8(c) shows the graph after the substitution of the $P_4$-split graphs ($\{e_1, e_2\}$, $\{d_1, d_2\}$) in $G'$ and ($\{a_1, a_2, a_3\}$, $\{b_{12}, b_3, b_4\}$) in $G'$. The decomposition tree is depicted in Figure 9.

![Diagram](image)

**Figure 9:** The decomposition tree of the graph in Figure 8(a).

We should remark that Jamison and Olariu’s homogeneous decomposition [JO95] performs only a part of our decomposition: A prime graph is decomposed if and only if it satisfies Condition (i) or Condition (ii) of Theorem 5.7, see also [BO96]. Thus, in a homogeneous decomposition tree, no 3-node with multiple children has a 4-node as its child. Consequently, the graph in Figure 8(b) isindecomposable relative to the homogeneous decomposition.
6 Conclusions and open problems

In this paper, we have obtained various results on the structure of the $P_4$-components. They allowed us to derive $O(n^2m)$ recognition and orientation algorithms for $P_4$-comparability and $P_4$-indifference graphs. Moreover, we proposed a unique tree representation for arbitrary graphs based on their module and $P_4$-component structure. As the modular decomposition and the orientation of comparability graphs can be computed in linear time, cf [MS94] and [MS97], we suspect that similar results are achievable for our decomposition and for the orientation of $P_4$-comparability graphs. We pose this as an open problem.

References


