Doctoral Thesis

Ramsey and Turán type problems in random graphs

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Publication Date:
2006

Permanent Link:
https://doi.org/10.3929/ethz-a-005271379

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Ramsey and Turán Type Problems in Random Graphs

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH

for the degree of
DOCTOR OF SCIENCES

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2006
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Abstract

The theory of random graphs was founded by Paul Erdős and Alfréd Rényi in a series of papers between 1959 and 1968. Since then this interesting and fruitful branch of combinatorics has gained attention of many experts from mathematics and theoretical computer science. We study questions from classical graph theory in the context of random graphs. In particular, we address so-called Ramsey and Turán type properties of graphs, which are central to the relatively young field of extremal graph theory.

A cornerstone of extremal graph theory is the celebrated theorem of Turán [Tur41]. It gives a tight bound on the maximal number of edges in a graph without a subgraph isomorphic to a complete graph on a fixed number of vertices and describes structural properties of such extremal examples. Kohayakawa, Łuczak, and Rödl [KLR97] studied Turán type questions in the context of random graphs. They addressed the problem of how many edges must be typically deleted from a random graph $G_{n,p}$ on $n$ vertices and edge probability $p$ in order to destroy every single copy of a given graph $F$. The authors of [KLR97] conjectured that one has to remove roughly a $(1 - (\chi(F) - 1)^{-1})$-fraction of all edges, where $\chi(F)$ denotes the chromatic number of $F$, provided the edge probability $p = p(n)$ is sufficiently large.

One approach to prove this conjecture is by utilizing Szemerédi’s regularity lemma for sparse graphs. This method requires a so-called embedding lemma for sparse graphs, which was proposed by Kohayakawa, Łuczak, and Rödl and is known as the KLR-Conjecture. It implies many more extremal results for random graphs and is considered as one of the most important open questions in this field [JLR00]. We prove a slightly weaker counting version of the KLR-Conjecture for complete graphs $K_\ell$ of size $\ell$ for $p \geq Cn^{-1/(\ell-1)}$ and a sufficiently large constant $C$.

In our second result, we transfer a classical theorem of Erdős and Gallai [EG59] from extremal graph theory into the context of random graphs. We study the minimum number of edges that must be removed from a random graph in order to destroy all long cycles. Suppose $\alpha > 0$ is a real constant and $p \gg n^{-1}$. We provide a tight bound on the number of edges that must be removed from $G_{n,p}$ in order to destroy all cycles of length at least $(1 - \alpha)n$.

The famous theorem of Ramsey [Ram30] states, in its simplest form, that for every constant $k$, there exists a number $R = R(k)$ such that every 2-coloring of the edges of a complete graph on at least $R$ vertices contains $k$ vertices that induce a monochromatic complete graph. Łuczak, Ruciński, and Voigt [LRV92] were the first to study Ramsey type questions in the context of random graphs. They proved a threshold function for the property that $G_{n,p}$ contains a monochromatic copy of some given graph $F$ in every vertex coloring. Rödl and Ruciński [RR93, RR95] proved the corresponding threshold for edge colorings. Those results were generalized by studying asymmetric Ramsey properties of random graphs in the case of vertex [Kre96] and edge colorings [KK97]. Rather than aiming for the same graph $F$ in every color, each color $k$, $1 \leq k \leq r$, is associated with a specific graph $F_k$. Kohayakawa and Kreuter [KK97] conjectured a threshold function for asymmetric Ramsey properties of random graphs and proved the case...
when each graph \( F_k \) is a cycle of fixed length. We verify this conjecture for the case when each graph \( F_k \) is a clique of fixed size, assuming the KLR-Conjecture holds in the proof of the upper bound. Suppose \( p_0 = p_0(F_1, \ldots, F_r) \) is the conjectured threshold. We show that there exists a positive constant \( B \) such that, for any \( p \geq Bp_0 \), \( G_{n,p} \) a.a.s. (asymptotically almost surely) admits no proper edge coloring with \( r \) colors, i.e., a coloring without any class \( k_0 \) that contains \( F_{k_0} \) as a subgraph. On the other hand, we prove that a.a.s. there exists a positive constant \( b \) such that, for any \( p \leq bp_0 \), there exists a proper coloring of \( G_{n,p} \). The proof of this bound is constructive through analyzing a coloring algorithm.

Friedgut et al. \[FKR^*03\] studied Ramsey properties of random graphs from an algorithmic perspective. They introduced two games for one player. In both games edges must be colored with either red or blue, where the aim is to avoid monochromatic triangles. In the online version of the game, edges of a complete graph on \( n \) vertices arrive in a random order, one at a time, and must be instantly colored. The second game is played in two rounds. Suppose \( c > 0 \) is an arbitrarily small constant. First, a random graph \( G = G_{n,p} \) with \( p = cn^{-1/2} \) is presented to the player. After the player has colored \( G \) without monochromatic triangles, new edges arrive one by one at random and must be colored in an online fashion, avoiding monochromatic triangles at all times. Utilizing the main result of \[GKRS05\], we present a new short proof for the upper bound on the duration of this semi-online triangle avoidance game with two colors. We also provide a lower bound on the duration of the second round in the case when the first round is played on a random graph \( G_{n,p} \) with \( p = cn^{-1/2-\alpha} \) for any fixed constant \( \alpha, 0 \leq \alpha \leq 1/6 \).

We give and analyze a strategy that enables the player to successfully color the first round and any \( N < n^{8\alpha} \) random edges in the second round.

Our last result provides bounds on the duration of the following balanced online edge coloring game on random graphs: the player is presented with two random edges at a time, one of which must be colored blue and the other one red immediately. The aim is to avoid monochromatic copies of some fixed graph \( F \). The study of this game was motivated by the observation that, in the games with one edge per move as described above, optimal colorings have a highly skewed distribution of edges per color. A natural question arising is: if the player is forced to keep the coloring balanced, how long can he survive without losing? As it turns out, several thresholds that we obtain in the balanced game are not the same as for the unbalanced game. For instance, the results in \[MSS05\] yield the threshold \( n^{(\ell+1)/\ell} \) for the unbalanced game avoiding monochromatic cycles \( C_\ell \) of length \( \ell \), whereas we derive the threshold \( n^{2\ell/(2\ell-1)} \) for the balanced game. Hence, the balanced online cycle avoidance game will end substantially earlier than the game without this condition.
Zusammenfassung


Ein möglicher Ansatz für den Beweis dieser Vermutung ist die Verwendung von Szemerédis Regularitätslemma für dünne Graphen. Diese Methode erfordert ein so genanntes Einbettungslemma für dünne Graphen, das von Kohayakawa, Luczak und Rödl vorgeschlagen wurde und als KLR-Vermutung in die Literatur eingegangen ist. Diese Vermutung impliziert viele weitere extreme Resultate für Zufallsgraphen und gilt als eine der wichtigsten offenen Fragen auf diesem Gebiet [JLR00]. Wir beweisen eine leicht abgeschwächte Zählervariante der KLR-Vermutung für vollständige Graphen $K_{\ell}$ der Größe $\ell$ für $p \geq Cn^{-1/(\ell-1)}$ und eine hinreichend grosse Konstante $C$.

In unserem zweiten Ergebnis stellen wir ein klassisches Theorem von Erdös und Gallai [EG59] aus der extremalen Graphentheorie in den Kontext von Zufallsgraphen. Wir untersuchen die minimale Anzahl von Kanten, die aus einem Zufallsgraphen entfernt werden müssen, um alle langen Kreise darin zu zerstören. Sei $\alpha > 0$ eine reelle Konstante und $p \gg n^{-1}$. Wir geben eine scharfe Schranke für die Anzahl von Kanten an, die aus $G_{n,p}$ entfernt werden müssen, um alle Kreise der Länge mindestens $\lceil(1 - \alpha)n\rceil$ zu vernichten.

Das berühmte Theorem von Ramsey [Ram30] besagt in seiner einfachsten Form, dass zu jeder Konstante $k$ eine Zahl $R = R(k)$ existiert, sodass jede 2-Färbung der Kanten eines vollständigen Graphen auf mindestens $R$ Knoten mindestens $k$ Knoten enthält, die einen monochromatischen vollständigen Subgraphen induzieren. Luczak, Ruciński und Voigt [LRV92] waren die Ersten, die Ramsey typische Fragestellungen im Zusammenhang mit Zufallsgraphen untersucht haben. Sie bewiesen eine Schwellenwertfunktion für die Eigenschaft, dass $G_{n,p}$ eine monochromatische Kopie eines gegebenen Graphen $F$ in jeder Knotenfärbung enthält. Rödl und Ruciński [RR93, RR95]
bewiesen einen entsprechenden Schwellenwert für Kantenfärbungen. Diese Resultate wurden verallgemeinert, indem asymmetrische Ramsey Eigenschaften von Zufallsgraphen für Knotenfärbungen [Kre96] und Kantenfärbungen [KK97] untersucht wurden. Anstatt auf denselben Graphen $F$ in jeder Farbe abzuzielen, ist jeder Farbe $k$, $1 \leq k \leq r$, ein spezifischer Graph $F_k$ zugeordnet. Kohayakawa und Kreuter [KK97] stellten eine allgemeine Vermutung über die Schwellenwertfunktion für asymmetrische Ramsey Eigenschaften von Zufallsgraphen auf und bewiesen den Fall, wenn es sich bei jedem $F_k$ um einen Kreis fester Länge handelt. Wir verifizieren diese Vermutung für den Fall, wenn es sich bei jedem $F_k$ um eine Clique fester Grösse handelt, wobei wir für den Beweis der oberen Schranke die KLR-Vermutung voraussetzen. Sei $p_0 = p_0(F_1, \ldots, F_r)$ der vermutete Schwellenwert. Wir zeigen, dass eine positive Konstante $B$ existiert, sodass für alle $p > Bp_0$ der Zufallsgraph $G_{n,p}$ a.f.s. (asymptotisch fast sicher) keine gültige Färbung mit $r$ Farben zulässt, d.h., eine Färbung ohne eine Farbklasse $k_0$, die $F_{k_0}$ als Subgraphen enthält.

Seit Friedgut et al. [FKR+03] untersuchten Ramsey Eigenschaften von Zufallsgraphen unter einem algorithmischen Blickwinkel. Sie führten zwei Spiele für einen Spieler ein. In beiden Spielen müssen Kanten entweder mit Rot oder mit Blau gefärbt werden, wobei das Ziel die Vermeidung monochromatischer Dreiecke ist. In der online Variante des Spiels erscheinen die Kanten eines vollständigen Graphen auf $n$ Knoten, zu jedem Zeitpunkt eine, in einer zufälligen Reihenfolge und müssen unmittelbar gefärbt werden. Das zweite Spiel verläuft in zwei Runden. Sei $c > 0$ eine beliebige Konstante. Zunächst wird dem Spieler ein Zufallsgraph $G = G_{n,p}$ präsentiert. Nachdem dieser den Graphen $G$ ohne monochromatische Dreiecke gefärbt hat, erscheinen neue Kanten zufällig nacheinander und müssen online gefärbt werden, wobei zu jedem Zeitpunkt monochromatische Dreiecke zu vermeiden sind. Unter Verwendung des Hauptresultats aus [GKRS05] präsentieren wir einen neuen kurzen Beweis für die obere Schranke für die Dauer dieses semi-online Spiels mit zwei Farben, in dem Dreiecke vermieden werden müssen. Des Weiteren geben wir die obere Schranke für die Dauer der zweiten Spielrunde an, wenn die erste Runde auf einem Zufallsgraphen $G_{n,p}$ mit $p = cn^{-1/2}$ präsentiert. Nachdem dieser den Graphen $G$ ohne monochromatische Dreiecke gefärbt hat, erscheinen neue Kanten zufällig nacheinander und müssen online gefärbt werden, wobei zu jedem Zeitpunkt monochromatische Dreiecke zu vermeiden sind. Unter Verwendung des Hauptresultats aus [GKRS05] präsentieren wir einen neuen kurzen Beweis für die obere Schranke für die Dauer dieses semi-online Spiels mit zwei Farben, in dem Dreiecke vermieden werden müssen. Des Weiteren geben wir eine untere Schranke für die Dauer der zweiten Spielrunde an, wenn die erste Runde auf einem Zufallsgraphen $G_{n,p}$ mit $p = cn^{-1/2 - \alpha}$ für eine gegebene Konstante $\alpha$, $0 \leq \alpha \leq 1/6$, gespielt wird. Wir präsentieren und analysieren eine Spielstrategie, die es dem Spieler ermöglicht, sowohl die Kanten der ersten als auch der zweiten Runde erfolgreich zu färben, sofern in der zweiten Runde $N \ll n^{8\alpha}$ zufällige Kanten hinzugefügt werden.

Unser letztes Resultat gibt Schranken für die Dauer des folgenden balancierten online Spiels auf Zufallsgraphen an: dem Spieler werden zu jedem Zeitpunkt zwei zufällige Kanten präsentiert, worauf die erste sofort blau und die andere rot gefärbt werden muss. Das Ziel ist die Vermeidung monochromatischer Kopien eines gegebenen Graphen $F$. Die Erforschung dieses Spiels ist auf der Beobachtung, dass optimale Färbungen in online Spielen mit einer Kante pro Zug, wie oben beschrieben, eine schiefe Verteilung der Kanten pro Farbklasse hervorrufen. Daraus ergibt sich natürlich die Frage, wie lange das Spiel andauert, wenn der Spieler die Anzahl der Kanten pro Farbklasse ausbalancieren muss. Wie sich herausstellt, unterscheiden sich einige Schwellenwerte, die für das balancierte Spiel erzielt wurden, deutlich von den Schwellenwerten für das unbalancierte Spiel, was den Einfluss dieser Bedingung zeigt. Beispielsweise ergeben die Resultate in [MSS05] den Schwellenwert $n^{(\ell+1)/\ell}$ für das unbalancierte online Spiel unter Vermeidung monochromatischer Kreise $C_\ell$ der Länge $\ell$, wohingegen wir den Schwellenwert $n^{2\ell/(2\ell-1)}$ für das balancierte Spiel ableiten. Demzufolge endet das balancierte online Spiel unter Vermeidung von Kreisen bedeutend früher als das Spiel ohne diese Bedingung.
Acknowledgments

I wish to express my deep gratitude to everyone who contributed to this work. First and foremost I am sincerely grateful to my supervisor Angelika Steger. With her unmistakable feeling for asking the right questions, she inspired and supported this work tremendously. Countless times she had the right idea when yet another proof seemed to fail. I am much obliged for what I have learned.

Many thanks go to Yoshiharu Kohayakawa, whose work had a great influence on me. I feel very honored that he is the co-examiner of this thesis. I would like to thank him and his colleagues at the Instituto de Matemática e Estatística who made my visits in São Paulo both productive and most enjoyable. In particular, I wish to thank Jozef Skokan for suggesting to work on asymmetric Ramsey properties and providing me with typical Brazilian products on every visit to Switzerland. Special thanks go to Cristina Fernandes and Blanka Homolova who introduced me to Brazilian culture and music and deserve loads of Swiss chocolate for that.

I sincerely appreciate the collaboration with my coauthors Domingos Dellamonica Jr., Stefanie Gerke, Dieter Mitsche, Reto Spöhel, Miloš Stojaković, and Andreas Weißl. I am particularly obliged to Stefanie and Reto who volunteered for endless hours of proof-reading and to Miloš who will hopefully know why. The work of Béla Bollobás and Andrzej Ruciński had a great impact on me. I deeply appreciate that they shared their knowledge on random graphs with me and pointed me to the right direction several times.

The extremely supportive and familiar atmosphere in Angelika’s research group offers unequalled opportunities. I wish to thank my colleagues for their invaluable help and the quality time we spent together. Special thanks go to my office mates Hanjo Täubig in Munich and Andreas Weis for being excellent conversational partners. Moreover, I would like to thank Florian Jug for his work on the E-Jigsaw project, Julian Lorenz for introducing me to financial mathematics, Konstantinos Panagiotou for many fruitful discussions, Jan Remy for releasing me from teaching duties during the last term, and Justus Schwartz for organizing beautiful hikes. I appreciate the advise of Alex Souza, a former member of our group, how to master the administrative issues of the Ph.D. program at ETH. I would also like to express my appreciation for our visitors Joshua Cooper, Tobias Müller, and Dirk Schlatter who brought fresh wind into our group. Last but not least, I would like to say thank you to Beate Bernhard, Barbara Heller, and Floris Tschurr for resolving all kinds of administrative problems.

Finishing this work reminds me of the people I love and who encouraged me on the way to here. I am glad you are there.
CHAPTER 1

Introduction

The theory of random graphs was founded by Paul Erdős and Alfréd Rényi in a series of papers between 1959 and 1968 (cf. [ER59, ER60, ER61]). Since then this interesting and fruitful branch of combinatorics has gained attention of many experts from mathematics and theoretical computer science (cf. [Bol01] and [JLR00]). We study questions from classical graph theory in the context of random graphs. In particular, we address so-called Ramsey and Turán type properties of graphs, which are central to the relatively young field of extremal graph theory.

A cornerstone of extremal graph theory is the celebrated theorem of Turán [Tur41]. It gives a tight bound on the maximum number of edges in a graph without a subgraph isomorphic to a complete graph on a fixed number of vertices. Later this result was generalized to arbitrary forbidden subgraphs graphs $F$ by Erdős and Stone [ES46] and Erdős and Simonovits [ES66]. They proved that there exists a constant $\delta = \delta(F)$ such that every graph on $n$ vertices with more than $(1 - \delta + o(1))\binom{n}{2}$ edges necessarily contains a copy of $F$. Surprisingly, the constant $\delta$ depends only on the chromatic number $\chi(F)$. As the presentation

The famous theorem of Ramsey [Ram30] states, in its simplest form, that for every constant $k$, there exists a number $R = R(k)$ such that every 2-coloring of the edges of a complete graph on at least $R$ vertices contains $k$ vertices that induce a monochromatic, complete graph. This is often referred to as the party problem. As is well known, among every party of at least six people, there are at least three, either all or none of whom know each other. We assume that knowing someone is a symmetric relation. The natural question is how many people have to join the party so that the same statement holds true for at least $k$ people. Ramsey’s theorem states that these numbers are finite, but the problem of proving tight bounds is unsolved yet.

In the study of random graphs, the most frequently used models are the binomial model $G_{n,p}$ and the uniform model $G_{n,m}$. In $G_{n,p}$ every edge is independently included with probability $p$, $0 \leq p \leq 1$, into an empty graph on $n$ vertices. The model $G_{n,m}$ assigns equal probability to every graph on $n$ vertices and exactly $m$ edges. We are interested in the limiting probability that a given property $P$ holds in a sequence of random graphs $(G_{n,p})_{n \in \mathbb{N}}$ or $(G_{n,m})_{n \in \mathbb{N}}$ respectively. As we are primarily interested in sparse random graphs, the edge probability $p$ typically depends on the number of vertices $n$ since, for constant $p$, $G_{n,p}$ has a.a.s. $\Omega(n^2)$ many edges. For many interesting properties, we can observe a phase transition, just like in physics between different states of aggregation. We say that $p_0$ is a threshold function for property $P$ if random graphs with edge probability substantially greater than $p_0$ satisfy $P$ in the limit, whereas the limiting probability of property $P$ tends to zero in sequences of random graphs with edge probability substantially less than $p_0$. We formalize this notion in Chapter 2.

In the remaining part of this introduction, we briefly present the main results of this work. We discuss two problems related to Turán type questions in Section 1.1 and present three results residing in Rasmey theory in Section 1.2. The presentation here is rather concise. Each result spans one chapter of this thesis with a detailed introduction.
1.1. Turán type questions for random graphs

Kohayakawa, Łuczak, and Rödl [KLR97] studied Turán type questions for random graphs. As is well known, the appearance of copies of a fixed graph $F$ in $G_{n,p}$ is determined by a certain threshold [ER59, BolSl]. Beyond that threshold, however, how many edges must be deleted from $G_{n,p}$ in order to destroy every single copy of $F$? For all graphs $G$ and $F$, let $\text{ex}(G, F)$ denote the maximum number of edges in a subgraph of $G$ without a copy of $F$, that is,

$$\text{ex}(G, F) := \max\{e(G') : F \not\subseteq G' \subseteq G\}.$$

The authors of [KLR97] conjectured the following.

**Conjecture 1.1 ([KLR97]).** Let $F$ be a non-empty graph of order at least 3, and let $0 < p = p(n) < 1$ be such that $p \gg n^{-1/m_2(F)}$, then a.a.s. we have

$$\text{ex}(G_{n,p}, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) |E(G_{n,p})|,$$

where

$$m_2(F) := \max_{F' \subseteq F, |V(F')| \geq 3} \frac{|E(F')| - 1}{|V(F')| - 2}.$$

1.1.1. A probabilistic counting lemma for complete graphs. One approach for proving Conjecture 1.1 is via the sparse version of Szemerédi’s regularity lemma, which was independently observed by Kohayakawa [Koh97] and Rödl (unpublished) (cf. [GS05]). Suppose an adversary deletes, for some constant $a > 0$,

edges from $G_{n,p}$ producing a graph $G' \subseteq G_{n,p}$. Utilizing the regularity lemma, one obtains a substructure of $G'$ in which the edges are distributed in a random like fashion. Embedding $F$ into this structure requires a proof of Conjecture 23 in [KLR97] of Kohayakawa, Łuczak, and Rödl (cf. Section 2.4.4). This so-called KLR-Conjecture formulates a probabilistic version of the classical embedding lemma for dense graphs (cf. [Die05, Lemma 7.3.2]). It implies many interesting extremal results for random graphs. In their monograph on random graphs, Janson, Łuczak, and Ruciński [JLR00] consider the verification of this conjecture as one of the most important open questions in the theory of random graphs. The conjecture is, in its full generality, still wide open [Luc00, GPS+02, GSS04, GKRS05].

In Chapter 3 we prove a slightly weaker counting version of the KLR-Conjecture for complete graphs $K_{\ell}$ of size $\ell$. That implies, amongst others, Conjecture 1.1 for $F = K_{\ell}$ and $p \gg n^{-1/(\ell-1)}$ (instead of $p \gg n^{-2/(\ell+1)}$ as conjectured).

1.1.2. An Erdős-Gallai type theorem for random graphs. Suppose $3 \leq \ell \leq n$ are fixed integers. A classical result of Erdős and Gallai [EG59] in extremal graph theory gives an upper bound on the maximum number of edges in any graph on $n$ vertices without a cycle of length greater than $\ell$. The original bound is tight if $n - 1$ is divisible by $\ell - 1$. For the remaining cases, the bound was strengthened by Woodall [Woo72].

In Chapter 4 we transfer this problem into the setting of random graphs. We study the minimum number of edges that must be removed from a random graph in order to destroy all long cycles.
Our results show that, for any real constant \( \alpha > 0 \) and \( p > n^{-1} \), any subgraph of \( G_{n,p} \) without a cycle of length at least \( [(1 - \alpha)n] \) has a.a.s. at most
\[
\left(1 - (1 - w(\alpha))(1 + w(\alpha)) + o(1)\right)|E(G_{n,p})|
\]
many edges, where
\[
w(\alpha) := (1 - \alpha)\left(\frac{1}{1 - \alpha} - \frac{1}{1 - \alpha}\right).
\]
This bound is essentially best possible since there is a way to destroy all cycles of length at least \( [(1 - \alpha)n] \) in a random graph \( G_{n,p} \) by partitioning it into \( k \) classes of size \( (1 - \alpha)n \), where \( k := \lfloor 1/(1 - \alpha) \rfloor \), and one additional class for the remaining vertices of size \( w(\alpha)n \) and deleting all edges between the partition classes.

1.2. Ramsey type questions for random graphs

Luczak, Ruciński, and Voigt [LRV92] were the first to study Ramsey type questions in the context of random graphs. Amongst other results, they proved a threshold for the property that a random graph \( G_{n,p} \) contains a monochromatic triangle in every edge-coloring with a fixed number of colors. Later Rödl and Ruciński [RR93, RR95] generalized this result to arbitrary graphs \( F \). For any fixed graph \( F \) and any integer \( k \geq 2 \), let
\[
G \rightarrow (F)^k
\]
denote the property that every edge coloring of \( G \) with \( k \) colors contains a monochromatic copy of \( F \). Then, for every graph \( F \) that is not a forest and integer \( k \), there exist constants \( b, B > 0 \) such that
\[
\lim_{n \to \infty} P[G_{n,p} \rightarrow (F)^k] = \begin{cases} 
0 & \text{if } p \leq bn^{-1/m_2(F)} \\
1 & \text{if } p \geq Bn^{-1/m_2(F)}
\end{cases}
\]
Note that the appearance of the density measure \( m_2 \) in the exponent of the threshold function here and in Conjecture 1.1 is not by coincidence. It was observed (cf. [JLR00]) that the KLR-Conjecture, which implies Conjecture 1.1, also implies the result above.

1.2.1. Asymmetric Ramsey properties. In [KK97] Kohayakawa and Kreuter studied so-called asymmetric Ramsey properties. Suppose \( F_1, F_2 \) are fixed graphs that are no forests. Let
\[
G \rightarrow (F_1, F_2)^2
\]
denote the property that in every edge-coloring of graph \( G \) with two colors, say, red and blue, there exists either a red copy of \( F_1 \) or a blue copy of \( F_2 \). Define
\[
m_2(F_1, F_2) := \max_{H \subseteq F_2, |V(H)| \geq 2} \frac{|E(H)|}{|V(H)| - 2 + 1/m_2(F_1)}.
\]
Kohayakawa and Kreuter conjectured that the sequence \( n^{-1/m_2(F_1, F_2)} \) is a sharp threshold for the property \( G_{n,p} \rightarrow (F_1, F_2)^2 \) provided we have \( m_2(F_1) \leq m_2(F_2) \). Allowing for two different constants in the 0- and the 1-statement, they settled this conjecture in [KK97] in the affirmative when \( F_1 \) and \( F_2 \) are cycles of fixed length.

In Chapter 5 we verify the case when \( F_1 \) and \( F_2 \) are cliques of fixed size. Suppose \( K_\ell \) and \( K_r \) are complete graphs on \( \ell \) and \( r \) vertices respectively with \( 3 \leq \ell \leq r \). We give an algorithm that a.a.s. finds a proper coloring of \( G_{n,p} \) in polynomial time if \( p \leq bn^{-1/m_2(K_\ell, K_r)} \) for a sufficiently small constant \( b > 0 \). On the other hand, we shall show that, assuming the KLR-Conjecture holds, \( G_{n,p} \rightarrow (K_\ell, K_r)^2 \) if \( p \geq Bn^{-1/m_2(K_\ell, K_r)} \) for a sufficiently large constant \( B \).
1.2.2. Two stage colorings. Friedgut et al. [FKR+03] studied Ramsey properties of random graphs from an algorithmic perspective. They introduced two one player games. In both games edges must be colored with either red or blue, where the aim is to avoid monochromatic triangles. In the online version of the game, edges of a complete graph on \( n \) vertices arrive in a random order, one at a time, and must be instantly colored. Friedgut et al. [FKR+03] proved that \( n^{4/3} \) is a threshold for the duration of this, that is, the number of moves until the first monochromatic triangle is created. This result was generalized in [MSS05] to a large family of graphs including cliques and cycles of fixed size. The authors of [MSS05] conjecture that, for any graph \( F \) that is no forest, the threshold on the duration of the online avoidance game with two colors is determined by the density measure

\[
\bar{m}_2(F) := \max_{H \subseteq F, |V(H)| \geq 2} \frac{|E(H)|}{|V(H)| - 2 + 1/m(F)},
\]

where

\[
m(F) := \max_{H \subseteq F} \frac{|E(H)|}{|V(H)|},
\]

in the sense that \( N_0(F, n) = n^{2-1/\bar{m}_2(F)} \). Note the similarities between the density measures \( m_2 \) as in (1.1) and \( \bar{m}_2 \). As it seems, the term \( 1/m(F) \) in the denominator of \( \bar{m}_2 \) quantifies exactly the lack of information in the online game compared to the offline game.

The second game that was studied in [FKR+03] is a combination of the offline threshold from Theorem 2.19 and the online game for triangles. This game is played in two rounds. First, a random graph \( G = G_{n,p} \) is generated with a fixed edge probability \( p = p(n) \) and is entirely presented to the player. After the player has colored \( G \) without monochromatic triangles, new edges arrive one by one and must be colored in an online fashion, always avoiding monochromatic triangles. This so-called semi-online was studied by Friedgut et al. [FKR+03] in the case when the player is given a graph \( G = G_{n,p} \) with \( p = cn^{-1/2} \) for any constant \( c > 0 \) in the first round, and there are two or three colors respectively available. If \( c \) is large, the player cannot even survive through the first round due to Theorem 2.19. Friedgut et al. [FKR+03] proved that even for arbitrarily small constants \( c > 0 \), the second round a.a.s. cannot last for any unbounded number of moves in the game with two colors.

Utilizing the main result of [GKRS05], we present a new short proof of the semi-online triangle avoidance game with two colors in Chapter 6. Our proof demonstrates a technique how to apply the regularity lemma in the context of sparse random graphs. Moreover, it employs new arguments for reasoning about \((\varepsilon)\)-regular graphs. We hope that a deeper understanding of this methods will lead the way to a proof of the KLR-Conjecture (cf. Section 2.4.4) eventually.

We also provide a lower bound on the duration of the second round in the case when the first round is played on a random graph \( G_{n,p} \) with \( p = cn^{-1/2-\alpha} \) for any fixed constant \( \alpha \), \( 0 \leq \alpha \leq 1/6 \). We give and analyze a strategy that enables the player to successfully color the first round and any \( N \ll n^{5\alpha} \) edges in the second round with two colors. Note that one cannot extend the range for \( \alpha \) in this statement. If \( \alpha \) was greater than 1/6, the player would last through any \( n^{4/3} \ll N \ll n^{8\alpha} \) moves in the second round. The analysis of the pure online game shows that this is impossible even with an empty graph in the first round. On the other hand, if \( \alpha \) is negative, one cannot even color the first round without monochromatic triangles due to Theorem 2.19.
1.2.3. Balanced online Ramsey games. Lower bounds on the duration of online graph avoidance game as described in Section 1.2.2 are proved in [FKR+03] and [MSS05] by analyzing a greedy playing strategy. This strategy produces colorings with a highly skewed distribution of edges per color. A natural question arising is: if the player is forced to keep the coloring balanced, how long can he survive without losing? In Chapter 7 we try to give an answer to this question by changing the rules of the game as follows. The player is presented with two edges at a time. One of the edges must be colored blue and the other one red.

As it turns out, several thresholds that we obtain in the balanced game are not the same as for the unbalanced game, showing that the balancedness condition makes a difference. For instance, the results in [MSS05] concerning the unbalanced game avoiding monochromatic cycles $C_\ell$ of length $\ell$ yield the threshold $n^{(\ell+1)/\ell}$, whereas we derive the threshold $n^{2\ell/(2\ell-1)}$ for balanced online colorings.

We shall prove general lower and upper bounds on the duration of the balanced online graph avoidance game, which can be applied to a variety of graphs.
CHAPTER 2

Graphs, Randomness, and Regularity

The theory on random and so-called (ε)-regular graphs forms the basis of our work. In this chapter, we give a brief introduction into these active research areas. We explain our notation and state several preliminary results from various fields, mostly without proof.

2.1. Technical preliminaries

Our notation is adopted from [JLR00], the latest monograph on random graphs. We will frequently use the Landau symbols $O, \Omega, \Theta, o,$ and $\omega$. As in the monograph, the expression $f = \Theta(g)$ means that functions $f$ and $g$ on the integers asymptotically differ at most by a multiplicative constant, i.e., $f = \Theta(g)$. The expression $f \gg g$ is equivalent to $f = \omega(g)$ and accordingly $f \ll g$ to $f = o(g)$. We write $a \sim b$ if $(1 - \varepsilon)b \leq a \leq (1 + \varepsilon)b$. The logarithm without explicit base refers to the natural logarithm.

For future reference, we collect some auxiliary, technical results, which will be used several times throughout the proofs. The first proposition gives bounds on various types of binomial coefficients.

**Proposition 2.1.** If $0 \leq x \leq 1$, then
\[
\binom{xa}{b} \leq \binom{a}{b} x^b.
\]

If $b \leq a$ and both are integral, then
\[
\binom{a}{b-c} \binom{a}{c} \leq 4^b \binom{a}{b}.
\]

For all integers $a, b, c,$ and $d$, we have
\[
\binom{a}{b} \binom{c}{d} \leq \binom{a+c}{b+d}.
\]

In the proofs we often need to estimate ratios of sums or differences. By means of the following simple observations, we can omit recurring, tedious calculations.

**Proposition 2.2.** For $a, c, C \in \mathbb{R}$ and $b, d > 0$, we have
\[
\frac{a}{b} \leq C \land \frac{c}{d} \leq C \implies \frac{a+c}{b+d} \leq C
\]

and
\[
\frac{a}{b} \geq C \land \frac{c}{d} \geq C \implies \frac{a+c}{b+d} \geq C.
\]

Similarly, if also $b > d$, we have
\[
\frac{a}{b} \leq C \land \frac{c}{d} \geq C \implies \frac{a-c}{b-d} \leq C
\]
and
\[
\frac{a}{b} \geq C \land \frac{c}{d} \leq C \implies \frac{a - c}{b - d} \geq C.
\]

In many cases we consider a random variable \( X \) that is the sum of \( n \) independent Bernoulli experiments with success probability \( p \). Thus, \( X \) is binomially distributed with parameters \( n \) and \( p \) denoted by \( X \in \text{Bin}(n, p) \). The following corollary of Chernoff’s well-known inequalities [JLR00, Theorem 2.1] yields exponentially small bounds on the tails of the binomial distribution.

**Lemma 2.3.** Let \( X \in \text{Bin}(n, p) \) and \( 0 < \varepsilon \leq 3/2 \). Then
\[
\mathbb{P} \left( |X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X] \right) \leq 2 \exp \left( -\frac{\varepsilon^2}{3} \mathbb{E}[X] \right).
\]

### 2.2. Graphs

We consider graphs \( G = (V, E) \) with edge set \( E \subseteq \binom{V}{2} \), where \( \binom{V}{2} \) denotes the set of all unordered pairs of vertices in \( V \). That is, all graphs are labeled, undirected, and simple, i.e., contain no multiple edges or loops. The vertex set of \( G \) is denoted by \( V(G) \) and the edge set by \( E(G) \). We omit the argument \( G \) in those expressions if it is clear from the context. By \( v_G \) and \( v(G) \) we denote the number of vertices of a graph \( G \) and similarly the number of edges by \( e_G \) and \( e(G) \).

We identify a clique on \( \ell \) vertices by \( K_{\ell} \), a cycle on \( \ell \) vertices by \( C_{\ell} \) and a path with \( \ell + 1 \) vertices by \( P_{\ell} \). A graph \( G \) is empty if it contains no edges.

For any vertex set \( W \subseteq V \), let \( G[W] \) denotes the graph induced by \( W \). We denote the number of edges in \( G[W] \) by \( e_G(W) = e(W) \). For any pair of not necessarily disjoint subsets \( U, W \subseteq V(G) \), let \( E_G(U, W) = E(U, W) \) denote the set of edges in \( G \) with one endpoint in \( U \) and the other endpoint in \( W \). We define \( e_G(U, W) = e(U, W) := |E_G(U, W)| \). The expression \( \Gamma(v) \) refers to the neighborhood of a vertex \( v \) in \( G \). Since we often have to consider the neighborhood into a subset \( W \subseteq V \), we abbreviate \( \Gamma(v) \cap W \) by \( \Gamma_W(v) \). Accordingly, we define \( \text{deg}_W(v) := |\Gamma_W(v)| \), which is the degree of vertex \( v \) into the set \( W \). The neighborhood spanned by a set of vertices \( Q \) is denoted by \( \Gamma(Q) \), i.e.,
\[
\Gamma(Q) := \bigcup_{v \in Q} \Gamma(v).
\]

We say that a graph \( H \) is a subgraph of \( G \), denoted by \( H \subseteq G \), if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). We say that \( G \) contains a copy of \( H \) if there exists an injection \( \phi : V(H) \to V(G) \) such that we have \( \phi(H) \subseteq G \). Abusing notation, we write, e.g., \( K_3 \subseteq G \) in order to express that \( G \) contains a triangle. We call \( G \) \( H \)-free if \( G \) contains no copy of \( H \), i.e., \( H \not\subseteq G \).

#### 2.2.1. Density measures

As we will frequently encounter throughout this work, density measures determine the growth rate of so-called threshold functions, which are described later in the context of random graphs. A density measure is a function mapping graphs to nonnegative reals.

The most well-known density measure is
\[
d(H) := \frac{e_H}{v_H}
\]
for any graph \( H \) on at least one vertex. This is exactly the average degree of \( H \) divided by 2. The density measure
\[
d_1(H) := \begin{cases} 
\frac{e_H}{v_H} & \text{if } v(H) \geq 2 \\
0 & \text{if } v(H) = 1
\end{cases}
\]

for any graph \( H \) on at least one vertex. This is exactly the average degree of \( H \) divided by 2. The density measure
\[
d_1(H) := \begin{cases} 
\frac{e_H}{v_H} & \text{if } v(H) \geq 2 \\
0 & \text{if } v(H) = 1
\end{cases}
\]
determines when copies of $H$ start to overlap on vertices in a random graph. Accordingly, the density measure
\[
d_2(H) := \begin{cases} \frac{\nu_H-1}{\nu_H-2} & \text{if } \nu(H) \geq 3 \\ \frac{1}{2} & \text{if } H \cong K_2 \\ 0 & \text{otherwise} \end{cases}
\]
determines when copies of $H$ start to overlap on edges in a random graph.

For any given density measure $d_i$, we define the maximum density
\[
m_i(H) := \max\{d_i(J) : J \subseteq H\}
\]
We say that $H$ is balanced with respect to $d_i$ if $m_i(H) = d_i(H)$. We call $H$ strictly balanced with respect to $d_i$ if $d_i(J) < d_i(H)$ for all proper subgraphs $J \subset H$. $H$ is balanced in the ordinary sense or simply balanced if it is balanced w.r.t. the density measure $d$. We call $H$ 1-balanced if it is balanced w.r.t. the density measure $d_1$ and 2-balanced if it is balanced w.r.t. the density measure $d_2$ accordingly. It is well-known that every graph $H$ satisfies $m(H) \leq m_2(H)$, and that every 2-balanced graph is also balanced.

When we study asymmetric Ramsey properties in Chapter 5, we are interested in the following generalization of $d_2$. Suppose $H_1$ is a nonempty graph. Let
\[
d_2(H_1, J) := \begin{cases} \frac{e_2 - 1}{e_2 - 2 + 1/m_2(H_1)} & \text{if } e(J) \geq 1 \\ 0 & \text{otherwise} \end{cases}
\]
\[
m_2(H_1, H_2) := \max\{d_2(H_1, J) : J \subseteq H_2\}
\]
In online graph avoidance games, which are discussed in Chapter 6, the following density measure plays an important role:
\[
\bar{d}_2(H_1, J) := \begin{cases} \frac{e_2 - 1}{e_2 - 2 + 1/m_2(H_1)} & \text{if } e(J) \geq 1 \\ 0 & \text{otherwise} \end{cases}
\]
\[
\bar{m}_2(H_1, H_2) := \max\{\bar{d}_2(H_1, J) : J \subseteq H_2\}
\]
We abbreviate $\bar{d}_2(H, H)$ by $\bar{d}_2(H)$ and $\bar{m}_2(H, H)$ by $\bar{m}_2(H)$ correspondingly. The generalization of $d_2$ as defined in (2.7) already appeared in [KK97]. In contrast to that, $\bar{d}_2$ and the corresponding maximum density $\bar{m}_2$ were apparently not studied before. Finally, we introduce the density measure
\[
d_b(H) := \begin{cases} \frac{2\nu_H - 1}{2\nu_H - 2} & \text{if } \nu(H) \geq 3 \\ \frac{1}{2} & \text{if } H \cong K_2 \\ 0 & \text{otherwise} \end{cases}
\]
which is discussed in Chapter 7.

For the sake of completeness, we conclude this section with a few interesting statements on the relations between the various density measures. As they are not used in this work, we omit the proofs. The next proposition shows that the density function $\bar{m}_2$ is sandwiched between $m$ and $m_2$.

**Proposition 2.4.** (i) Every graph $F$ satisfies
\[
m(F) \leq \bar{m}_2(F) \leq m_2(F)
\]
(ii) Every 2-balanced graph $F$ is balanced with respect to $\bar{d}_2(F, \cdot)$.
(iii) Every graph $F$ that is balanced with respect to $\bar{d}_2(F, \cdot)$ is balanced.
One can similarly show, that the extended version of the 2-density satisfies the following properties.

**Proposition 2.5.** If \( m_2(H_1) \leq m_2(H_2) \), then we have
\[
m_2(H_1) \leq m_2(H_1, H_2) \leq m_2(H_2) .
\]
In particular, if \( m_2(H_1) = m_2(H_2) \), we have \( m_2(H_1) = m_2(H_1, H_2) = m_2(H_2) \).

**Proposition 2.6.** Suppose \( m_2(H_1) \leq m_2(H_2) \). If \( H_2 \) is (strictly) balanced w.r.t. \( d_2 \), then \( H_2 \) is also (strictly) balanced w.r.t. \( d_2(H_1, \cdot) \). The converse does not hold in general.

### 2.3. Random graphs

The notion of a random graph was first introduced by Erdős [Erd47]. The theory of random graphs was founded by Erdős and Rényi in their series of papers between 1959 and 1968. The interested reader is referred to the survey about the origins of the theory of random graphs in [KR97]. The monographs of Bollobás [Bo01] and Janson, Luczak, and Rucinśki [JLR00] are the most complete collections of results in this area.

We consider the two standard models for random graphs. \( G_{n,p} \) denotes the probability space of all graphs on \( n \) labeled vertices, where each edge is present with probability \( 0 \leq p \leq 1 \) independently of all other edges. The outcome of this experiment is denoted by \( G_{n,p} \). The probability \( p \) is in many settings a function of \( n \). Observe that the number of edges in \( G_{n,p} \) is binomially distributed with parameters \( \binom{n}{2} \) and \( p \). In the model \( G_{n,m} \), the graph \( G_{n,m} \) is chosen uniformly at random among all graphs on \( n \) vertices with \( m \) edges.

We are primarily interested in structural properties of random graphs. A graph property \( P \) is formally a family of graphs. These families are typically infinite. We say that a graph \( G \) satisfies \( P \) if \( G \in P \). A graph property \( P \) is monotone increasing if \( G \in P \) implies that for all supergraphs \( G' \supseteq G \) with \( V(G') = V(G) \), we have \( G' \in P \). Correspondingly, we say that \( P \) is monotone decreasing if \( G \in P \) implies that for all subgraphs \( G' \subseteq G \) with \( V(G') = V(G) \), we have \( G' \in P \). The family of all connected graphs is an example of a monotone increasing graph property. Observe that the complement of an increasing family is monotone decreasing.

We say that \( P \) holds asymptotically almost surely (a.a.s.) in \( G_{n,p} \) if
\[
\lim_{n \to \infty} \Pr[G_{n,p} \in P] = 1 .
\]

Since \( G_{n,p} \) and \( G_{n,m} \) are equivalent in terms of the limiting probability of monotone graph properties provided \( m \asymp np^2 \) and \( p \) is sufficiently large [JLR00, Section 1.4], we often switch from one model to the other in order to simplify presentation.

Throughout Chapters 6 and 7, we consider the model of the random graph process denoted by \( (G(n,N))_{0 \leq N < \binom{n}{2}} \), where the edges appear uniformly at random one by one, i.e., in one of \( \binom{n}{2} \) possible permutations.

#### 2.3.1. Edge-distribution properties

As the next results show, the number of edges in induced subgraphs of random graphs is under tight control.

**Definition 2.7 ((\( \eta, b, p \))-bounded).** Let \( H \) be a graph, \( 0 < \eta \leq 1 \), \( 0 < p \leq 1 \), and \( b \geq 1 \). We say that \( H \) is \((\eta, b, p)\)-bounded, if for all disjoint sets \( U \) and \( W \) with \( |U|, |W| \geq \eta|V| \), we have
\[
e_H(U,W) \leq bp|U||W| .
\]
**Lemma 2.8.** For all $1/n < \varepsilon < 1$, $b > 1$, $\eta > 0$, the random graph $G_{n,p}$ is a.a.s. $(\eta, b, \varepsilon)$-bounded.

The proof of this lemma is an easy application of Chernoff’s inequality as stated in Lemma 2.3. It is more involved to prove the following strengthening [HKL95]. Chernoff’s inequality is again at the core of this proof.

**Definition 2.9 $(p, A)$-uniform.** Let $A > 0$ be fixed. A graph $G$ on $n$ vertices is $(p, A)$-uniform if, for $d = np$, we have

\[
\epsilon_G(U, W) \leq \frac{A}{d} |U||W|
\]

for all disjoint sets $U, W \subseteq V(G)$ such that $1 \leq |U| \leq |W| \leq d|U|$. We call $G$ $(p, A)$-upper-uniform if only the bound for the upper deviation in (2.10) holds, i.e.,

\[
\epsilon_G(U, W) \leq p|U||W| + \frac{A}{d} |U||W|.
\]

**Lemma 2.10.** For every $0 < p = p(n) < 1$, the random graph $G_{n,p}$ is a.a.s. $(p, e^2 \sqrt{n})$-uniform.

In $(p, A)$-uniform graphs the number of edges induced by a set $U$ of vertices is bounded. It can be observed by a double counting argument that, for any $U \subseteq V(G)$, we have

\[
\left| e_G(U) - p\left(\begin{array}{c} |U| \\ 2 \end{array}\right) \right| \leq A \sqrt{d} |U|.
\]

Note also that Lemma 2.8 follows from Lemma 2.10 since, for all $\eta > 0$, $\varepsilon > 0$, and every pair of disjoint subsets $U, W \subseteq V(H)$ with $|U|, |W| \geq \eta n$, we have

\[
A \sqrt{d |U||W|} \leq (1 + \varepsilon)p|U||W|
\]

provided that $p \gg n^{-1}$ and $n$ is sufficiently large.

**2.3.2. Thresholds for graph properties.** In the study of structural properties of random objects, we can frequently observe that even tiny modifications to the probability space produce dramatical changes in the outcome of random experiments. A little change in the expected number of edges of a random graph can effect the limiting probability of some event so that it rapidly jumps from 0 to 1. It is, for example, well-known that a.a.s. $G_{n,p}$ contains isolated vertices if $p$ grows substantially slower than $n/\log n$, but the graph is connected if $p$ is substantially greater than this function. The sequence $n/\log n$ is a threshold for the property of being connected. We formally define the notion of threshold functions in the binomial model $G_{n,p}$ as follows.

**Definition 2.11.** Let $P$ be a graph property. The sequence $p_0 = p_0(n)$ is a threshold for $P$ if

\[
\lim_{n \to \infty} P[G_{n,p} \in P] = \begin{cases} 0 & \text{if } p \ll p_0 \\ 1 & \text{if } p \gg p_0 \end{cases}.
\]

Thresholds are analogously defined in the uniform model $G_{n,m}$.

**Definition 2.12.** Let $P$ be a graph property. The sequence $m_0 = m_0(n)$ is a threshold for $P$ if

\[
\lim_{n \to \infty} P[G_{n,m} \in P] = \begin{cases} 0 & \text{if } m \ll m_0 \\ 1 & \text{if } m \gg m_0 \end{cases}.
\]
For the graph process \( (G(n, N))_{0 \leq N \leq \binom{n}{2}} \), we consider the random variable \( \hat{N} \) that measures the number of edges inserted until \( G(n, N) \) satisfies some property \( \mathcal{P} \). This is called the hitting time of \( \mathcal{P} \). There is a straightforward relation between the hitting time and the threshold function \( m_0 \) for \( \mathcal{P} \) in the uniform model \( G_{n,m} \). Since \( G(n, N) \) and \( G_{n,m} \) have the same distribution, \( m_0 \) is also a threshold for the random graph process \( (G(n, N))_{0 \leq N \leq \binom{n}{2}} \).

**Lemma 2.13.** Let \( \mathcal{P} \) be an increasing graph property with threshold \( m_0 \). Then

\[
\lim_{n \to \infty} \mathbb{P} \left[ \hat{N} \leq m \right] = \begin{cases} 0 & \text{if } m < m_0 \\ 1 & \text{if } m \geq m_0 \end{cases}
\]

As shown by Bollobás and Thomason [BT87], every monotone property follows a 0/1-law in the limiting probability. The notion of thresholds can be strengthened in the following way.

**Definition 2.14.** Let \( \mathcal{P} \) be a graph property. The sequence \( p_0 = p_0(n) \) is a sharp threshold for \( \mathcal{P} \) if, for every \( \epsilon > 0 \), we have

\[
\lim_{n \to \infty} \mathbb{P} [G_{n,p} \in \mathcal{P}] = \begin{cases} 0 & \text{if } p \leq (1 - \epsilon)p_0 \\ 1 & \text{if } p \geq (1 + \epsilon)p_0 \end{cases}
\]

One can analogously define the notion of sharp thresholds in the uniform model \( G_{n,m} \). Observe that weak threshold sequences are defined up to the binary operator \( \prec \), whereas sharp threshold sequences are defined up to \( \sim \), i.e., terms of smaller order.

2.3.3. **Small subgraphs.** A well-studied property of random graphs is the containment of subgraphs of fixed size like triangles. The following theorem of Bollobas [Bol81], which is a generalization of a result of Erdös and Rényi [ER59] from balanced to arbitrary graphs, determines the threshold for this property.

**Theorem 2.15.** Let \( H \) be a nonempty graph. Then the threshold for the property that \( G_{n,p} \) contains a copy of \( H \) is

\[
p_0(H,n) = n^{-1/m(H)},
\]

where \( m(H) \) is the maximum density corresponding to \( d(H) \) as defined in (2.4).

As was shown in [ER60], for strictly balanced graphs \( H \), the number of copies of \( H \) in \( G_{n,p} \) is concentrated around its mean (cf. [AS00, Theorem 4.4.4]). We denote the number of copies of a fixed graph \( H \) appearing in the random graph \( G_{n,p} \) by \( X(G_{n,p}, H) \).

**Theorem 2.16.** Let \( H \) be a nonempty, strictly balanced graph with \( a \geq 1 \) different automorphisms. If \( p \gg n^{-v(H)/e(H)} \), then a.a.s. we have

\[
X(G_{n,p}, H) \sim \left( \frac{1}{a} + o(1) \right) n^{v(H)} p^{e(H)}.
\]

In certain situations the following theorem of Vu [Vu01, Theorem 2.1] gives an exponentially small upper bound on the upper tail of the random variable \( X(G_{n,p}, H) \).

**Theorem 2.17.** Let \( H \) be a nonempty, balanced graph with \( a \geq 1 \) different automorphisms. If there exists a constant \( \alpha > 0 \) such that

\[
\mu := \frac{1}{a} n^{v(H)} p^{e(H)} = \Omega(n^\alpha),
\]

then, for all \( \epsilon > 0 \), we have

\[
\mathbb{P} [X(G_{n,p}, H) \geq (1 + \epsilon)\mu] \leq \exp \left\{ -\Omega \left( n^{\frac{-\alpha}{d-1}} \right) \right\}.
\]
2.3.4. Turán type properties. According to the classical monograph in the subject [Bol78], problems in extremal graph theory are often of the following form: let $\mathcal{P}$ be a graph property and $\mu$ a graph parameter; determine the least $m$ with the property that any graph $G$ with $\mu(G) > m$ has $\mathcal{P}$ and describe the so called extremal graphs, that is, those $G$ with $\mu(G) = m$ without $\mathcal{P}$. For instance, in the case of Turán's theorem, $\mathcal{P}$ is the property of containing a clique $K_t$ for some given $t$, and $\mu(G)$ is the number of edges $e(G)$ in $G$. As is well known, Turán's classical result determines the exact value of $m = m(n)$ in terms of $n = |V(G)|$ and describes all the extremal graphs.

Given a property $\mathcal{P}$ and a positive integer $n$, let $\text{ex}(n, \mathcal{P})$ be the maximal $\ell$ such that there is an $n$-vertex graph $J$ with $\ell$ edges not satisfying $\mathcal{P}$. More generally, given a graph $G$, let

$$\text{ex}(G, \mathcal{P}) := \max \{e(J) : J \subseteq G \text{ and } J \text{ does not have } \mathcal{P} \}.$$  

(2.14)

Clearly, $\text{ex}(n, \mathcal{P}) = \text{ex}(K_n, \mathcal{P})$. Since we are primarily interested in the property of containing some fixed subgraph $H$, we use the following shorthand notation. Let

$$\text{ex}(G, H) := \max \{e(J) : H \not\subseteq J \subseteq G \}.$$  

(2.15)

The celebrated result of Erdős, Stone, and Simonovits [ES46, ES66] shows that $\text{ex}(K_n, H)$ is asymptotically determined by the chromatic number $\chi(H)$ of $H$.

**Theorem 2.18.** For any nonempty graph $H$, we have

$$\text{ex}(K_n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1) \right) \binom{n}{2}.$$  

As mentioned in Chapter 1, Kohayakawa, Luczak, and Rödl [KLR97] conjectured that the result above can be transferred into a probabilistic setting (cf. Conjecture 1.1 on page 2).

2.3.5. Ramsey type properties. It follows from Ramsey's celebrated result [Ram30] that every $k$-coloring of the edges of the complete graph on $n$ vertices contains a monochromatic copy of $F$ if $n$ is sufficiently large. While that seems to rely on the fact that $K_n$ is a very dense graph, Folkman [Fol70] and, in a more general setting, Nešetřil and Rödl [NR76] showed that there also exist locally sparse graphs $G = G(F)$ with the property that every $k$-coloring of the edges of $G$ contains a monochromatic copy of $F$. By transferring the problem into a random setting, Rödl and Rucinski [RR95] showed that in fact such graphs $G$ are quite frequent. More precisely, they proved the following result. Let

$$G \rightarrow (F)_k^c$$

denote the property that every edge-coloring of $G$ with $k$ colors contains a monochromatic copy of $F$. Then their theorem reads as follows.

**Theorem 2.19 ([LRV92], [RR93], [RR95]).** Let $k \geq 2$ and $F$ be a nonempty graph that is not a forest. Then there exist constants $b, B > 0$ such that

$$\lim_{n \to \infty} \mathbb{P} [G_{n,p} \rightarrow (F)_k^c] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(F)} \\ 1 & \text{if } p \geq Bn^{-1/m_2(F)} \end{cases},$$

where $m_2(F)$ is the maximum density corresponding to $d_2(F)$ as defined in (2.6).

The value of the threshold function in Theorem 2.19 can be motivated as follows. For the sake of simplicity, suppose that $m_2(F) = (e(F) - 1)/(v(F) - 2)$. Then, for $p = c n^{-1/m_2(F)}$, the expected number of copies of $F$ containing a given edge of $G_{n,p}$ is a constant depending on $c$. If
this constant is close to zero, the copies of $F$ in $G_{n,p}$ are loosely scattered and a valid coloring should thus exist. If on the other hand this constant is large, the copies of $F$ in $G_{n,p}$ highly intersect with each other, and the existence of a valid coloring becomes unlikely.

The following theorem is a counting version of Theorem 2.19, which also appeared in [RR95]. The formulation that we chose here is slightly weaker, but it serves its purpose.

**Theorem 2.20.** Let $k \geq 1$ and $F$ be a nonempty graph. Then there exist constants $B = B(F, k)$ and $a = a(F, k)$ such that in every $k$-edge-coloring of a random graph $G_{n,p}$ with

$$p = p(n) \geq B n^{-1/m_2(F)},$$

there are a.a.s. at least $anVFp^F$ monochromatic copies of $F$.

### 2.4. The regularity lemma and $\varepsilon$-regular graphs

The notion of $\varepsilon$-regular graphs was introduced by Szemerédi in [Sze78]. It describes a property that is typically observed in random bipartite graphs. Therefore, $\varepsilon$-regular graphs are frequently referred to as pseudo-random graphs. Those graphs gained importance since it was shown in [Sze78] that every sufficiently large graph allows for a nearly complete partitioning into constantly many vertex classes of equal size such that most of them are pairwise $\varepsilon$-regular. This is, in essence, the statement of Szemerédi's regularity lemma, which we shall present in full detail later. In general, the regularity lemma imposes an additional structure on very large graphs, which can be used to identify certain small substructures like graphs of fixed size.

#### 2.4.1. $(\varepsilon, p)$-regular pairs.

Intuitively, a bipartite graph $B = (V_1 \cup V_2, E)$ is $(\varepsilon)$-regular if its edges are distributed in a random-like way. The parameter $\varepsilon$ reflects the uniformity of this distribution. As $\varepsilon$ tends to smaller values, the edge distribution in $B$ becomes more and more uniform. More formally, we measure the density $d_B(V'_1, V'_2)$ of any two nonempty subsets $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ in $B$ by

$$d_B(V'_1, V'_2) := \frac{e_B(V'_1, V'_2)}{|V'_1||V'_2|}.$$ 

We omit the subscript $B$ if it is clear from the context. The original definition of $\varepsilon$-regular pairs is as follows.

**Definition 2.21 (\varepsilon-regular pairs).** A bipartite graph $B = (V_1 \cup V_2, E)$ is $\varepsilon$-regular if for all $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $|V'_1| > \varepsilon|V_1|$ and $|V'_2| > \varepsilon|V_2|$, we have

$$|d(V'_1, V'_2) - d(V_1, V_2)| \leq \varepsilon .$$ (2.16)

If we consider sequences of bipartite graphs $(B_n)_{n \in \mathbb{N}}$ such that the overall density of $B_n$ tends to naught, this definition is meaningless in the limit. Suppose $B_n = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$ is a sequence of bipartite graphs such that $d(V_1, V_2) = o(1)$. Then (2.16) trivially holds for $n$ sufficiently large. The definition of $\varepsilon$-regular pairs was extended in the following way [Koh97] so as to allow for sequences of sparse graphs.

**Definition 2.22 ((\varepsilon, p)-regular pairs).** Let $p$ be a real number with $0 < p \leq 1$. A bipartite graph $B = (V_1 \cup V_2, E)$ is $(\varepsilon, p)$-regular if for all $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $|V'_1| \geq \varepsilon|V_1|$ and $|V'_2| \geq \varepsilon|V_2|$, we have

$$|d(V'_1, V'_2) - d(V_1, V_2)| \leq \varepsilon p .$$ (2.17)
and $B$ is $(\varepsilon)$-regular if

$$\left| d(V_1', V_2') - d(V_1, V_2) \right| \leq \varepsilon d(V_1, V_2). \quad (2.18)$$

Note the parentheses around $\varepsilon$ that distinguish the extended definition from the original one. Pairs that are $\varepsilon$-regular in the original sense are $(\varepsilon, 1)$-regular in the new terminology. A simple application of Chernoff’s inequality (cf. Lemma 2.3) shows that if we generate a random bipartite graph by choosing each of the possible edges independently with probability $p \gg n^2$, it is a.a.s. $(\varepsilon, p)$-regular for any real constant $\varepsilon > 0$. When we consider bipartite graphs that are embedded into some larger graph $F$, we say for convenience that any disjoint sets of vertices $V_1, V_2 \subseteq V(F)$ are $(\varepsilon, F, p)$-regular if and only if the induced bipartite graph $F[V_1, V_2]$ is $(\varepsilon, p)$-regular. In that case, we may also write that $(V_1, V_2)$ is an $(\varepsilon, p)$-regular pair in $F$.

In certain situations we would like to work without the bound on the upper deviation in $(\varepsilon, p)$-regular pairs.

**Definition 2.23** ($(\varepsilon, \varrho)$-lower-regular). Let $0 < \varepsilon, \varrho \leq 1$ be given. A bipartite graph $B = (V_1 \cup V_2, E)$ is called $(\varepsilon, \varrho)$-lower-regular if for all $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ with $|V_1'| \geq \varepsilon|V_1|$ and $|V_2'| \geq \varepsilon|V_2|$, we have

$$d(V_1', V_2') \geq (1 - \varepsilon) \varrho.$$

**Remark 2.24.** Every $(\varepsilon)$-regular pair $(V_1, V_2)$ is $(\varepsilon, d(V_1, V_2))$-lower-regular. Clearly, this implication also holds for $(\varepsilon, p)$-regular pairs if $p \leq d(V_1, V_2)$.

**Remark 2.25.** Suppose $(V_1, V_2)$ is an $(\varepsilon, p)$-regular pair, $\varepsilon \leq \alpha \leq 1/2$ is a real constant, and $U_1 \subseteq V_1$ and $U_2 \subseteq V_1$ satisfy $|U_1| \geq \alpha|V_1|$ and $|U_2| \geq \alpha|V_2|$ respectively. Then the subgraph of $(V_1, V_2)$ induced by $(U_1, U_2)$ forms an $(\varepsilon/\alpha, p)$-regular pair. This holds since, for every $U_1'$ and $U_2'$ with $|U_1'| \geq \varepsilon/\alpha|U_1|$ and $|U_2'| \geq \varepsilon/\alpha|U_2|$, we have

$$|U_1'| \geq \varepsilon|V_1| \quad \text{and} \quad |U_2'| \geq \varepsilon|V_2|,$$

and therefore

$$|d(U_1', U_2') - d(U_1, U_2)|$$

$$\leq |d(U_1', U_2') - d(V_1, V_2)| + |d(U_1, U_2) - d(V_1, V_2)|$$

$$\leq 2\varepsilon p \leq \frac{\varepsilon}{\alpha} p.$$

Similarly, we conclude that $(U_1, U_2)$ is $(\varepsilon/\alpha, \varrho)$-lower-regular if $(V_1, V_2)$ is $(\varepsilon, \varrho)$-lower-regular since we have

$$d(U_1', U_2') \geq (1 - \varepsilon) \varrho \geq \left(1 - \frac{\varepsilon}{\alpha}\right) \varrho.$$

That can also be stated for $(\varepsilon)$-regular graphs as follows.

**Lemma 2.26.** Let $B = (V_1 \cup V_2, E)$ be an $(\varepsilon)$-regular graph of density $d$. Then any subset $V_1'$ of $V_1$ of size at least $\alpha|V_1|$ with $\alpha > \varepsilon > 0$ induces an $(\varepsilon')$-regular graph with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon/(1 - \varepsilon)\}$ and density $d' \sim_{\varepsilon} d$.

**Proof.** Since $\alpha \geq \varepsilon$, it follows from the definition of $(\varepsilon)$-regularity that

$$d' = \frac{\varepsilon d(V_1', V_2)}{|V_1'| |V_2|} \sim_{\varepsilon} d.$$
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Let $W \subseteq V'_1$ and $Z \subseteq V_2$ be such that $|W| \geq \varepsilon |V'_1|$ and $|Z| \geq \varepsilon |V_2|$. It follows that $|W| \geq \varepsilon |V_1|$ and $|Z| \geq \varepsilon |V_2|$ and hence
\[
\frac{|E(W, Z)|}{|W||Z|} - d' \leq \frac{|E(W, Z)|}{|W||Z|} - \frac{d}{d'} + |d - d'| \leq 2\varepsilon d \leq \frac{2\varepsilon}{1 - \varepsilon} \frac{|E(V'_1, V_2)|}{|V'_1||V_2|} \leq \varepsilon d'.
\]

The following fact shows that the density of a pair is a natural choice for the parameter $p$.

**Proposition 2.27.** Let real constants $\lambda > 0$ and $\varepsilon > 0$ be given. If the pair $(V_1, V_2)$ is $(\varepsilon, p)$-regular and $e(V_1, V_2) > \lambda p|V_1||V_2|$, then $(V_1, V_2)$ is $(\varepsilon/\lambda)$-regular.

**Proof.** Observe that
\[
\lambda p = \frac{e(V_1, V_2)}{|V_1||V_2|} d(V_1, V_2) \leq \varepsilon d(V_1, V_2).
\]
Since $(V_1, V_2)$ is $(\varepsilon, p)$-regular, it satisfies
\[
d(V_1', V_2') - d(V_1, V_2) \leq \varepsilon p \leq \frac{\varepsilon}{\lambda} d(V_1, V_2),
\]
and hence (2.17) implies (2.18). \hfill \Box

The proof of the next lemma that was given in [GS05, Lemma 4.3] employs a probabilistic argument.

**Lemma 2.28.** For all $0 < \varepsilon < 1/6$, there exists a constant $C = C(\varepsilon)$ such that any $(\varepsilon)$-regular graph $B = (V_1 \cup V_2, E)$ contains a $(2\varepsilon)$-regular subgraph with $m$ edges for all $m$ satisfying $C|V(B)| \leq m \leq |E(B)|$.

The next observation asserts bounds on the degree of most vertices into any large subset of the opposite partition class. Though $\varepsilon$-regularity defines a property for vertex sets of linear size, it allows to derive bounds on the degree of most vertices. One can easily verify that the degree differs at most by an $\varepsilon$-fraction from its expectation for all but an $\varepsilon$-fraction the vertices.

**Definition 2.29 (Degree property).** Let $B = (V_1 \cup V_2, E)$ be an $(\varepsilon)$-regular graph, and $V_2'$ be a subset of $V_2$. A vertex $v \in V_1$ satisfies the lower degree property w.r.t. $V_2'$ if
\[
\deg_{V_2'}(v) = |\Gamma_{V_2'}(v)| \geq (1 - \varepsilon)d(V_1, V_2)|V_2'|, \tag{2.19}
\]
and it satisfies the upper degree property w.r.t. $V_2'$ if
\[
\deg_{V_2'}(v) = |\Gamma_{V_2'}(v)| \leq (1 + \varepsilon)d(V_1, V_2)|V_2'|. \tag{2.20}
\]

Using Definition 2.22 of $(\varepsilon)$-regular pairs, one can easily prove the following fact.

**Proposition 2.30.** Suppose $(V_1, V_2)$ is an $(\varepsilon)$-regular pair. For each set $V_2' \subseteq V_2$ of size at least $\varepsilon|V_2|$, there are at most $\varepsilon|V_1|$ vertices $v \in V_1$ that violate the lower degree property and at most as many, possibly different vertices that violate the upper degree property. Hence, we have
\[
\left| \left\{ v \in V_1 : \deg_{V_2'}(v) - d(V_1, V_2)|V_2'| \leq \varepsilon d(V_1, V_2)|V_2'| \right\} \right| \geq (1 - 2\varepsilon)|V_1|.
\]

**Remark 2.31.** For $(\varepsilon, \vartheta)$-lower-regular pairs, we easily derive
\[
\left| \left\{ v \in V_1 : \deg_{V_2'}(v) \geq (1 - \varepsilon)\vartheta|V_2'| \right\} \right| \geq (1 - \varepsilon)|V_1|,
\]
provided $|V_2'| \geq \varepsilon|V_2|$.  


This fact can be generalized from single vertices to small vertex sets $Q$, but instead of bounding the vertex degree, one considers the size of the neighborhood $\Gamma(Q)$. If $Q$ is sufficiently small, then almost each of its vertices contributes a large amount to $\Gamma(Q)$, and the lower bound is consequently again just slightly less than the expectation.

**Lemma 2.32 ([GKRS05]).** Let $\varrho > 0$ be fixed. For all positive constants $\beta$ and $\nu$, there exists a constant $\varepsilon_0 = \varepsilon_0(\beta, \nu) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and all $c$, $[2/\nu] < c \leq \nu/(3\varrho)$, the following holds: in every $(\varepsilon, \varrho)$-lower-regular graph $B = (V_1 \cup V_2, E)$, the number of sets $Q$ of size $q \geq c$ with $|\Gamma(Q)| \geq (1 - \varepsilon)cq|V_2|$ is at least

$$\left(1 - \beta^q\right)\binom{|V_1|}{q}.$$

Frequently we have to consider tiny fractions of one vertex class in regular graphs that are substantially smaller than the "admissible" size $c|V|$. The main result of [GKRS05] guarantees that most of these subsets also induce a regular pair.

**Lemma 2.33.** For all constants $0 < \beta, \varepsilon' < 1$, there exist $\varepsilon_0 = \varepsilon_0(\beta, \varepsilon') > 0$ and $C = C(\varepsilon')$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $0 < \varrho < 1$ the following holds: in every $(\varepsilon, \varrho)$-lower-regular graph $B = (V_1 \cup V_2, E)$, the number of sets $Q \subseteq V_1$ of size $q \geq C\varrho^{-1}$ that form an $(\varepsilon', \varrho)$-lower-regular graph with $V_2$ is at least

$$\left(1 - \beta^q\right)\binom{|V_1|}{q}.$$

A very similar statement holds for $(\varepsilon)$-regular graphs [GKRS05].

**Theorem 2.34.** For all constants $0 < \beta, \varepsilon' < 1$, there exist $\varepsilon_0 = \varepsilon_0(\beta, \varepsilon') > 0$ and $C = C(\varepsilon')$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the following holds: in every $(\varepsilon)$-regular graph $B = (V_1 \cup V_2, E)$ with density $d$, the number of sets $Q \subseteq V_1$ of size $q \geq Cd^{-1}$ that contain a set $\tilde{Q}$ of size at least $(1 - \varepsilon')q$ which forms an $(\varepsilon')$-regular graph of density $d' \sim \varepsilon d$ with $V_2$ is at least

$$\left(1 - \beta^q\right)\binom{|V_1|}{q}.$$

### 2.4.2. A regularity lemma for sparse graphs

Szemerédi originally introduced the regularity lemma in order to find long arithmetic progression in dense sequences [Sze78]. In the meanwhile the lemma was applied to obtain many results in graph theory. The interested reader is referred to the survey articles in [KS96, KSSS02]. However, the original statement of the lemma is meaningless for sequences of sparse graphs, i.e., for graphs on $n$ vertices with $o(n^2)$ edges. In particular, it is not useful for obtaining results about random graphs $G_{n,p}$ if $p = o(1)$. We present a generalized version of the lemma that was independently proposed by Kohayakawa [Koh97] and Rödl (unpublished).

The regularity lemma yields partitions of graphs of the following kind.

**Definition 2.35 ((\varepsilon, G, p)-regular partition).** Let $G$ be a graph on $n$ vertices. We say that a partition $\Pi = (V_0, V_1, \ldots, V_k)$ of $V$ is $(\varepsilon, G, p)$-regular if $|V_0| \leq \varepsilon|V|$ and $|V_i| = |V_j|$ for all $i, j \in \{1, 2, \ldots, k\}$, and, furthermore, at least

$$\left(1 - \varepsilon\right)\binom{k}{2}$$

pairs $(V_i, V_j)$ with $1 \leq i < j \leq k$ are $(\varepsilon, p)$-regular in $G$. 
Recall the notion of \((A,p)\)-upper-uniform graphs as in Definition 2.9. The following variant of Szemerédi's regularity lemma (see, e.g., [Koh97, KR03b, GS05]) is applicable to all graphs with edge density \(p\) that are \((A,p)\)-upper-uniform for some constant \(A \geq 1\).

**Lemma 2.36.** For all real numbers \(\varepsilon > 0\) and \(A \geq 1\) and all integers \(k_0\), there exist constants \(n_0 = n_0(\varepsilon, A, k_0) > 0\), \(d_0 = d_0(\varepsilon, A, k_0) > 0\), and \(K_0 = K_0(\varepsilon, A, k_0) > k_0\) such that the following holds: for every \((p, A)\)-upper-uniform graph \(G\) on \(n \geq n_0\) vertices with \(d = pn \geq d_0\), there exists a partition \(\Pi = (V_0, V_1, \ldots, V_k)\) of \(V\) with \(k_0 \leq k \leq K_0\) that is \((\varepsilon, G, p)\)-regular.

Lemma 2.36 also holds under weaker assumptions on \(G\), but the version above is sometimes more convenient to work with. In general, it suffices that \(G\) is \((\eta, b, p)\)-bounded as in Definition 2.7. Moreover, we can simultaneously apply the lemma to a set of graphs \(G_1, \ldots, G_a\) that share a common vertex set.

**Theorem 2.37.** [Koh97] For any \(0 < \varepsilon < 1/2\) and \(a, b, k_0 \geq 1\), there are constants \(\eta = \eta(\varepsilon, a, b, k_0) > 0\) and \(K_0 = K_0(\varepsilon, a, m_0) \geq k_0\) such that for any \(p > 0\), for all \((\eta, b, p)\)-bounded graphs \(G_1, \ldots, G_a\) with vertex set \(V\), where \(|V| \geq k_0\), there exists a partition \((V_i)^k_{i=0}\) with exceptional class \(V_0\) such that \(k_0 \leq k \leq K_0\) and \((V_i)^k_{i=0}\) is \((\varepsilon, G_j, p)\)-regular for all \(j \in [a]\).

If the regularity lemma is applied to \(G_{n,p}\), the edge probability \(p\) is the natural choice for the density since \(G_{n,p}\) a.a.s. meets the right boundedness conditions (cf. Lemmas 2.8 and 2.10). However, if one has to consider subgraphs of \(G_{n,p}\) with substantially less than \(pn^2\) edges, that choice will produce meaningless partitions. More effort is required in that case to make the proof technique work.

### 2.4.3. \((\varepsilon, p)\)-Regular Towers

In Section 2.4.1 we defined \((\varepsilon)\)-regularity for pairs of vertex sets. Here we are going to generalize this concept for \(\ell\)-partite graphs such that certain pairs of partition classes form an \((\varepsilon)\)-regular graph.

**Definition 2.38.** For a graph \(H\), let \(G(H,n,m)\) be the family of graphs on vertex set \(V = \bigcup_{x \in V(H)} V_x\), where the sets \(V_x\) are pairwise disjoint sets of vertices of size \(n\), and edge set \(E = \bigcup_{(x,y) \in E(H)} E_{xy}\), where \(E_{xy} \subseteq V_x \times V_y\) and \(|E_{xy}| = m\). Let \(G(H,n,m,\varepsilon) \subseteq G(H,n,m)\) denote the set of graphs in \(G(H,n,m)\) satisfying that each \((V_x, V_y, E_{xy})\) is an \((\varepsilon)\)-regular graph.

A graph \(G \in G(H,n,m,\varepsilon)\) looks like \(H\) with every vertex blown up to a class of \(n\) vertices and every edge to a set of \(m\) edges that form an \((\varepsilon)\)-regular graph between the corresponding vertex classes. Generalizing the notion of \((\varepsilon, p)\)-regular pairs, we call members of the family \(G(H,n,m,\varepsilon)\) \((\varepsilon, p)\)-regular towers.

We use capital, curly letters to denote families of graphs. In order to simplify notation slightly, we write \(\tilde{G}(\ell,n,m)\) instead of \(G(K_\ell,n,m)\) and \(\tilde{G}(\ell,n,m,\varepsilon)\) for \(G(K_\ell,n,m,\varepsilon)\) respectively. We define the family \(\tilde{G}(\ell,n,m,\varepsilon)\), which is a superset of \(G(\ell,n,m,\varepsilon)\), as follows. \(\tilde{G}(\ell,n,m,\varepsilon)\) is the family of \(\ell\)-partite graphs on \(\ell\) pairwise disjoint vertex sets \(V_i, \ldots, V_{\ell}\) of size \(n\) such that for all \(1 \leq i < j \leq \ell\) the bipartite graph between \(V_i\) and \(V_j\)

(i) has \(m_{i,j} \sim \varepsilon m\) edges and  
(ii) is \((\varepsilon)\)-regular.

In contrast to \(G(\ell,n,m,\varepsilon)\) the graphs in \(\tilde{G}(\ell,n,m,\varepsilon)\) do not have a fixed number of edges, but the number of edges may vary by some \(\varepsilon\)-fraction in every regular pair of vertex classes.

Theorem 2.34 yields the following consequence [GS05], which we shall inductively apply in our proofs.
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Theorem 2.39 ([GS05]). Let \( \ell \geq 3 \) be any integer. For all \( \alpha > 0, \beta > 0 \) and \( \varepsilon' > 0 \), there exist constants \( \varepsilon_0 = \varepsilon_0(\alpha, \beta, \varepsilon) > 0 \) and \( C = C(\varepsilon') > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0, \) \( n \) sufficiently large, and \( m \geq Cn^{3/2} \), the following holds: all but at most

\[
\beta^m \left( \frac{n^2}{m} \right)^\frac{1}{2}
\]

graphs in \( G(\ell, n, m, \varepsilon) \) satisfy that all but \( an^2 \) vertices \( v \in V_1 \) have neighborhoods that contain a graph in \( G(\ell - 1, x, y, \varepsilon') \) with \( x = (1 - \varepsilon')m/n \) and \( y = (1 - \varepsilon')x^2m/n^2 \).

2.4.4. The KLR-Conjecture. Graphs from the family \( G(H, n, m, \varepsilon) \) are usually obtained by applying Szemerédi's regularity lemma to some large graph \( G \) (and after removing some edges). Thus, one can find a copy of \( H \) in \( G \) if the former is a subgraph of all members of the family \( G(H, n, m, \varepsilon) \). Unfortunately, there are counterexamples to this statement if \( m = o(n^2) \). Moreover, for the family \( G(H, n, m) \), one can show that there exists a constant \( c \) such that at least a \( cm \)-fraction of all members do not contain \( H \) as a subgraph.

Definition 2.40. We denote the subfamily of \( G(H, n, m) \) consisting of all members that contain at most

\[
(1 - \delta)n^v(H)
\]

copies of \( H \) by \( F(H, n, m, \delta) \). Thus, \( F(H, n, m, 1) \) denotes the family of \( H \)-free graphs in the family \( G(H, n, m) \). We abbreviate \( F(H, n, m, 1) \) by \( F(H, n, m) \).

If the edges are randomly distributed between the vertex sets of any graph in \( G(H, n, m) \), we expect

\[
n^v(H)
\]

copies of \( H \) in any member of this family. Kohayakawa, Łuczak and Rödl [KLR97] conjectured that the fraction of graphs in \( G(H, n, m, \varepsilon) \) that does not contain \( H \) as a subgraph is subexponentially small.

KLR-Conjecture. For all \( \beta > 0 \), there exist constants \( \varepsilon_0 > 0, \) \( C > 0, \) \( n_0 > 0 \) such that for all \( m \geq Cn^{2-1/m_2(H)} \), \( n \geq n_0, \) and \( 0 < \varepsilon \leq \varepsilon_0 \), we have

\[
|F(H, n, m, \delta) \cap G(H, n, m, \varepsilon)| \leq \beta^m \left( \frac{n^2}{m} \right)^\varepsilon(H)
\]

where \( m_2(H) \) is the maximum density corresponding to \( d_2(H) \) as defined in (2.6).

The KLR-Conjecture can be easily verified for trees and it also holds when \( H \) is a cycle [GKRS05] or when \( H \) is the complete graph on 3 [Luc00], 4 [GPS+02] and 5 [GSS04] vertices. It is well-known (cf. [GS05]) that the KLR-Conjecture implies Conjecture 1.1.

We conjecture that one can replace \( F(H, n, m, \delta) \) by \( F(H, n, m, \delta) \) in the KLR-Conjecture.

Conjecture 2.41 (Counting Lemma). Let \( H \) be a fixed graph. For any \( \beta > 0 \) and \( \delta > 0 \), there exist constants \( \varepsilon_0 > 0, \) \( C > 0, \) \( n_0 > 0 \) such that for all \( m \geq Cn^{2-1/m_2(H)} \), \( n \geq n_0, \) and \( 0 < \varepsilon \leq \varepsilon_0 \), we have

\[
|F(H, n, m, \delta) \cap G(H, n, m, \varepsilon)| \leq \beta^m \left( \frac{n^2}{m} \right)^\varepsilon(H)
\]

where \( m_2(H) \) is the maximum density corresponding to \( d_2(H) \) as defined in (2.6).
CHAPTER 3

A Probabilistic Counting Lemma for Complete Graphs

In this chapter we shall prove a relaxed version of Conjecture 2.41 on the previous page for complete graphs of fixed size ℓ. We show that the statement holds for a lower bound on the edges of order \( n^{2 - 1/(\ell - 1)} \). The results presented here were published in [GMS05]. A full version of that paper is currently under review.

3.1. Introduction

The celebrated regularity lemma by Szemerédi [Sze78] states, roughly speaking, that the vertex set of every sufficiently large graph can be partitioned into a constant number of classes such that the edges between most partition classes are distributed regularly. It is particularly useful for finding subgraphs in large graphs, since one can employ the structure of the partition obtained from the lemma and show the existence of the subgraph in there. Lemmas that assert the existence of subgraphs in such a partition are called embedding or counting lemmas.

We are interested in a counting lemma for sparse graphs. As explained in Section 2.4.2, Szemerédi’s original lemma is only helpful for dense graphs (that is graphs with \( n \) vertices and \( \Theta(n^2) \) edges), but Kohayakawa [Koh97] and Rödl (unpublished) independently introduced a variant of Szemerédi’s regularity lemma for sparse graphs. Unfortunately, corresponding counting or embedding lemmas do not exist yet, and, moreover, one can show that straightforward generalizations of these are false as was noted by Luczak (cf. [KR03a, GS05]). However, Kohayakawa, Luczak, and Rödl [KLR97] formulated a probabilistic embedding lemma (cf. Section 2.4.4) that, if true, would solve some long-standing open problems in random graph theory [KLR97, Luc00, JLR00, GS05].

The KLR-Conjecture on page 18, an embedding lemma, asserts a copy of a graph \( H \) in a certain structure. Our main result of this chapter is a proof of Conjecture 2.41, a counting version of the KLR-Conjecture, in the case when \( H \) is the complete graph \( K_\ell \) of size \( \ell \) and \( m \geq Cn^{2-1/(\ell-1)} \) (instead of \( m \geq Cn^{2-2/(\ell+1)} \) as stated in Conjecture 2.41). Recall the definitions of \((\varepsilon)-regular\) pairs and towers from Section 2.4.2.

Theorem 3.1. For all \( \ell > 3, \delta > 0, \) and \( \beta > 0, \) there exist constants \( n_0 \in \mathbb{N}, C > 0, \) and \( \varepsilon_0 > 0 \) such that

\[
|\mathcal{F}(K_\ell, n, m, \delta) \cap \mathcal{G}(K_\ell, n, m, \varepsilon)| \leq \beta^m \cdot \left( \frac{n^2}{m} \right)^{\frac{\delta}{m}}.
\]

provided that \( m \geq Cn^{2-1/(\ell-1)}, \) \( n \geq n_0, \) and \( 0 < \varepsilon \leq \varepsilon_0. \)

It is well known [KLR97, Luc00, JLR00] that the KLR-Conjecture has several implications. In particular Theorem 3.1 implies the following.

Corollary 3.2. [Luc00] For all \( \ell \geq 3 \) and every \( \delta > 0, \) there exists \( c = c(\delta, \ell) \) such that the probability that a graph chosen uniformly at random from the family of all \( K_\ell \)-free labeled graphs
on $n$ vertices and $m \geq cn^{2-1/(\ell-1)}$ edges can be made $(\ell-1)$-partite by removing $\delta m$ edges tends to one as $n$ tends to infinity.

**Corollary 3.3.** [Luc00] For all $\ell \geq 3$ and for every $\varepsilon > 0$, there exist $c = c(\varepsilon, \ell)$ and $n_0 = n_0(\varepsilon, \ell)$ such that for $n \geq n_0$ and $cn^{2-1/(\ell-1)} \leq m \leq n^2/c$, a graph $G_{n,m}$ drawn uniformly at random from all labeled graphs on $n$ vertices and $m$ edges satisfies

$$\left(\frac{\ell-2}{\ell-1} - \varepsilon\right)^m \leq \mathbb{P}[G_{n,m} \text{ does not contain } K_{\ell}] \leq \left(\frac{\ell-2}{\ell-1} + \varepsilon\right)^m.$$  

Another implication of Theorem 3.1 is a version of Conjecture 1.1 under slightly stronger assumptions for cliques.

**Corollary 3.4 ([KLR97, GS05]).** Let $3 \leq \ell$ be given. For every sequence $p = p(n) \gg n^{-1/(\ell-1)}$, a.a.s. we have

$$\text{ex}(G_{n,p}, K_{\ell}) = \left(1 - \frac{1}{\ell-1} + o(1)\right) p\left(\frac{n}{2}\right).$$

This result was already shown in [SV03, KRS04]. We remark that using Theorem 3.1 together with the sparse version of the regularity lemma in fact implies a slightly stronger result. Namely, we obtain that a.a.s. every subgraph of $G_{n,p}$ with $(1 + \delta)\text{ex}(G_{n,p}, K_{\ell})$ edges contains not just one copy of a $K_{\ell}$ (which is the essential statement of the above result) but $c_{\ell}n^{2\ell/\ell(\ell-1)/2}$ copies.

We shall prove Theorem 3.1 by induction on $\ell$. Let us remark that the lower bound on $m$ in the theorem depends on the base case of the induction which is $\ell = 3$ in our proof. If we want to prove the KLR-Conjecture, we can chose $\ell = 5$ as the base case of the induction since the KLR-Conjecture has been verified for this case in [GSS04]. The induction would work analogously and yield the threshold $m > n^{2-1/(\ell-2)}$. This includes the main result of [SV03].

The organization of this chapter is as follows. Section 3.2 is devoted to the proof of our main lemma, Lemma 3.6. Roughly speaking, this lemma asserts that for $\varepsilon' \gg \varepsilon$, small subgraphs of $G \in \mathcal{G}(K_{\ell}, n, m, \varepsilon)$ are $(\varepsilon')$-regular and have further "typical" properties with very high probability. Suppose the members of a family of bad graphs $\mathcal{B}(K_{\ell}, x, y) \subseteq \mathcal{G}(K_{\ell}, x, y)$ are very rare, that is, if we choose a graph $H \in \mathcal{G}(K_{\ell}, x, y, \varepsilon)$ uniformly at random, then the probability of $H \in \mathcal{B}(K_{\ell}, x, y)$ is exponentially small for $x$ sufficiently large, $y \geq y_0(x)$, and $\varepsilon$ sufficiently small. Now consider a graph $G \in \mathcal{G}(\ell, n, m, \varepsilon)$ with $n \gg x$ vertices, and choose subsets $X_i \subseteq V_i$, $1 \leq i \leq \ell$, of size $x$ randomly. The main result of [GKRS05] essentially states that for all $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that with high probability the induced graph $G[X_1, \ldots, X_\ell]$ contains a large $(\varepsilon')$-regular subgraph $G'$, i.e., $G' \in \mathcal{G}(K_{\ell}, x', y, \varepsilon')$ with $x' \sim x$ and $y \sim x^2m/n^2$. In this chapter we shall show that $G'$ satisfies even more, namely that $G' \in \mathcal{G}(K_{\ell}, x', y, \varepsilon') \setminus \mathcal{B}(K_{\ell}, x', y)$ provided $x^2m/n^2 \geq y_0(x)$. Since the size of the neighborhood into one vertex class of a typical vertex is about $m/n$, we can apply the lemma for $x \approx m/n$ and $\ell - 1$ and deduce that most vertices in one class have nicely structured neighborhoods within the other $\ell - 1$ classes. This allows the application of the induction hypothesis in the proof of our main result Theorem 3.1, which is given in Section 3.3.

In what follows we assume that all variables denoted by small greek letters have a tiny value, say, less than $10^{-10}$. We omit the use of floors and ceilings in all expressions. As we assume that $n$ tends to infinity, this does not make any substantial difference, but greatly improves readability.
3.2. Typical Tuples of Sublinear Subsets

Before we state the main result of this section, let us first discuss some consequences of the inheritance property of $\varepsilon$-regularity stated in Theorem 2.34 on page 16. It was proved in [GS05] that given $\varepsilon' > 0$ and $\beta > 0$ we can find $\varepsilon > 0$ and $C > 0$ such that all graphs in $\mathcal{G}(\ell, n, m, \varepsilon)$ satisfy for all $x \geq Cn^2/m$ the following property: if we choose sets $X_i \subseteq V_i$ of size $(1 + \varepsilon')x$ randomly, then with probability $1 - \beta^x$ these sets contain a graph from $\mathcal{G}(\ell, x, x^2m/n^2, \varepsilon')$.

In this section we want to generalize this result as follows. Instead of just requiring that the graph induced by the tuple $(X_1, \ldots, X_\ell)$ is regular and has approximately the expected number of edges, we want to deduce that it also has further “typical” properties. Here we call a property “typical” if for suitable $n$'s and $m$'s, all but a $\beta^m$-fraction of the graphs in $\mathcal{G}(\ell, n, m, \varepsilon)$ satisfy it. We formalize this as follows.

**Definition 3.5.** We say that a family $\mathcal{B}(\ell, n, m) \subseteq \mathcal{G}(\ell, n, m)$ is small with respect to a function $m_0(n)$ if for all $\beta > 0$, there exist constants $n_\beta \in \mathbb{N}$, $C_\beta > 0$, and $\varepsilon_\beta > 0$ such that

$$|\mathcal{B}(\ell, n, m) \cap \mathcal{G}(\ell, n, m, \varepsilon_\beta)| \leq \beta^m \cdot \binom{n^2}{m}$$

for all $n > n_\beta$ and $m > C_\beta m_0(n)$.

Thus our main result Theorem 3.1 states that for $\ell > 3$ and $\delta > 0$, the family $\mathcal{S}(\ell, n, m, \delta)$ is small with respect to $n^{2-(\ell-1)/3}$ The key for proving this theorem will be our next lemma.

**Lemma 3.6.** Let $y_0(x) \geq x$ be a monotone increasing function and let $\mathcal{B}(\ell, x, y)$ be small with respect to $y_0$. For all $\beta, \varepsilon' > 0$, there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon, x$, and $y$ satisfying

\begin{align*}
0 < \varepsilon &\leq \varepsilon_0, \\
x &\geq \frac{n^2}{m} \log n, \\
\frac{C y_0(x)}{n} &\leq \frac{m}{n^2} x^2, \\
\varepsilon' \frac{m}{n^2} x^2 &\leq y \leq (1 - \varepsilon') \frac{m}{n^2} x^2,
\end{align*}

and $n$ sufficiently large, all but at most $\beta^m \binom{n^2}{m} / \mathcal{G}(\ell, n, m, \varepsilon)$ graphs $G \in \mathcal{G}(\ell, n, m, \varepsilon)$ satisfy the following property: the number of $\ell$-tuples $(X_1, \ldots, X_\ell)$ such that the induced graph $G[X_1, \ldots, X_\ell]$ contains a member of the family $\mathcal{S}(\ell, x, y, \varepsilon') \setminus \mathcal{B}(\ell, x, y)$, is at least $(1 - \beta^x) \binom{n}{(1 + \varepsilon') x}^\ell$.

The remainder of this section is devoted to the proof of Lemma 3.6.

Our proof strategy is as follows. We need to show that all but a $\beta^x$-fraction of all tuples $(X_1, \ldots, X_\ell)$ have the desired property. Therefore, we construct “good” tuples step by step. More precisely, we will introduce the notion of a good $k$-tuple $(X_1, \ldots, X_k)$ and show that if $(X_1, \ldots, X_k)$ is a good $k$-tuple, then all but a $\gamma^x$-fraction of all possible $X_{k+1}$-sets extend the given $k$-tuple to a good $(k + 1)$-tuple. This implies that we find at least

$$\left[1 - \gamma^x \left(\frac{n}{(1 + \varepsilon') x}\right)^\ell\right]$$
good $\ell$-tuples, which clearly suffices if we choose $\gamma$ appropriately and define the notion of a $k$-tuple in a suitable way.

Let us give some more intuition and postpone the precise definition of a good $k$-tuple for later. Basically, we want the properties of the lemma, i.e., that for all $1 \leq i \leq \ell$, $X_i \subseteq V_i$ and $|X_i| = (1 + e')x$. Furthermore, we want that there are a suitable subsets $X'_i \subseteq X_i$ of size $|X'_i| = x$. We will require that the subsets $X'_i$ induce a graph $G[X'_1, \ldots, X'_\ell]$ that belongs to $G(k, x, x^{2m/n^2}, e')$ and contains a graph from $G(k, x, y, e')$ that is in some still to be defined way — good. Intuitively, a $k$-partite graph $G_k$ is good, if there are many extensions of $G_k$ to an $\ell$-partite graph that does not belong to the set $B$. In particular, if $k = \ell$ then the graph is good if and only if it does not belong to the set $B$.

To simplify notation we fix some parameters for the remainder of this section. First of all, let $\beta, e' > 0$ be the constants from Lemma 3.6 and let $B(\ell, x, y) \subseteq G(\ell, x, y)$ be a family which is small with respect to $\gamma_0(x)$.

Furthermore, we assume that $\alpha, \tilde{\beta}, \tilde{\beta}$ satisfy

$$\frac{(\ell \alpha)^{e'/\ell}}{32e^2} \leq \frac{\beta}{\tilde{\beta} \cdot \varepsilon^2 + 2e + 2}, \quad \tilde{\beta} = \beta_{(1)} \quad (3.4)$$

and $C_{\tilde{\beta}}, \varepsilon_{\tilde{\beta}},$ and $n_{\tilde{\beta}}$ are defined as in (3.2). Moreover, let

$$C := \frac{C_{\tilde{\beta}}}{\varepsilon'}$$

and let $x$, $y$, $n$, and $m$ satisfy the conditions (3.3) of Lemma 3.6. Observe that this implies

$$y \geq e' \cdot \frac{m}{n^2} \geq e' \cdot \gamma_0(x) \geq C_{\tilde{\beta}} \gamma_0(x). \quad (3.5)$$

Given the set $B(\ell, x, y)$, we want to define a class of "bad" graphs denoted by $B(k, x, y, \varepsilon) \in G(k, x, y, \varepsilon)$ for $1 \leq k \leq \ell$. Intuitively, we want that $B(k, x, y, \varepsilon)$ contains those graphs from $G(k, x, y, \varepsilon)$ that can be extended "in many ways" to graphs from $B(\ell, x, y) \cap G(\ell, x, y, \varepsilon)$. For $k = \ell$ we define

$$B(\ell, x, y, \varepsilon) := B(\ell, x, y) \cap G(\ell, x, y, \varepsilon).$$

Let $G \in G(k, x, y)$. We say that a graph $G' \in G(k + 1, x, y)$ is an extension of $G$ if $G'$ restricted to the first $k$ sets is identical to $G$. With this definition at hand, we are now ready to define when we consider a graph in $G \in G(k, x, y, \varepsilon)$ for $2 \leq k < \ell$ as bad:

$$G \in B(k, x, y, \varepsilon) \iff \sum_{k' \leq k} \left[ \beta y \cdot \frac{\left(\frac{x^2}{y}\right)}{\varepsilon} \right]^{k'} \geq \frac{\beta y \cdot \varepsilon^2}{\varepsilon}.$$ 

That is, $B(k, x, y, \varepsilon)$ contains those graphs from $G(k, x, y, \varepsilon)$ that have at least $\left[ \beta y \cdot \left(\frac{x^2}{y}\right) \right]^k$ extensions belonging to $B(k + 1, x, y, \varepsilon)$. With slight abuse of notation, we set $B(1, x, y, \varepsilon) := \emptyset$. Now we easily show the following claim.

**Claim 3.7.** For all $2 \leq k \leq \ell$ and $0 < \varepsilon \leq \varepsilon_{\tilde{\beta}}$, we have

$$B(k, x, y, \varepsilon) \leq \left[ \beta y \cdot \left(\frac{x^2}{y}\right) \right]^k.$$ 

**Proof.** We prove the claim by induction on $\ell - k$. For $k = \ell$ the claim follows immediately from the fact that $B(\ell, x, y)$ is small w.r.t. $\gamma_0(x)$, Equation (3.5), the definition of $\varepsilon_{\tilde{\beta}}$, and the choice of $x$ and $y$. For the induction step consider some $2 \leq k < \ell$. By definition every graph
in $B(k, x, y, \varepsilon)$ has at least $\left[ \frac{\beta^y}{y} \cdot \left( \frac{x^2}{y} \right) \right]^k$ extensions to graphs in $B(k+1, x, y, \varepsilon)$. By the induction hypothesis we know that there exist at most $\left[ \frac{\beta^y}{y} \cdot \left( \frac{x^2}{y} \right) \right]^{\binom{k+1}{2}}$ graphs in $B(k+1, x, y, \varepsilon)$. Hence,

$$|B(k, x, y, \varepsilon)| \cdot \left[ \frac{\beta^y}{y} \cdot \left( \frac{x^2}{y} \right) \right]^k \leq \left[ \frac{\beta^y}{y} \cdot \left( \frac{x^2}{y} \right) \right]^{\binom{k+1}{2}},$$

and the bound follows immediately. \hfill \qed

**Claim 3.8.** For all $2 \leq k \leq \ell$ and $0 < \varepsilon \leq \varepsilon_0$ there exist at most $\frac{1}{k} \beta^m \binom{n^2}{m} \left( \frac{y}{x} \right)$ graphs $B \in \mathcal{G}(\ell, n, m)$ that satisfy the following properties:

- There exist $\alpha n / x$ pairwise vertex disjoint subgraphs $G_k^j \subseteq B$, $1 \leq j \leq \alpha n / x$, that belong to the family $\mathcal{G}(k, x, y, \varepsilon) \setminus B(k, x, y)$ and satisfy, for all $1 \leq i \leq k$, $|V(G_k^i) \cap V_i| = x$.
- For all $1 \leq j, s \leq \alpha n / x$, there exist pairwise disjoint sets $X_k^{j,s} \subseteq V_{k+1}$ of size $x$ such that the graph induced by $V(G_k^j) \cup X_k^{j,s}$ contains an extension of $G_k^j$ that belongs to $B(k+1, x, y, \varepsilon)$.

**Proof.** We prove the claim by constructing all graphs that satisfy the above properties. Firstly, we select bipartite graphs with $m$ edges between $V_i$ and $V_{i'}$, for all pairs $1 \leq i < i' \leq \ell$, $i' \neq k+1$. There are at most

$$\binom{n^2}{m} \left( \frac{y}{x} \right)^{-k}$$

ways to do that. Secondly, for all $1 \leq j, s \leq \alpha n / x$, we choose pairwise vertex disjoint set $X_k^j = V(G_k^j) \cap V_i$ and the extension sets $X_k^{j,s}$. There are less than

$$\frac{n^{\alpha n x + (\alpha n / x)^2}}{x} \leq 2^{(\alpha n x / x) \log n} \leq 2^{\alpha n m}$$

ways to do that. Next we choose the graphs $G_k^j$. For all $1 \leq i < i' \leq k$, one has to select $y$ edges from the edges between the pair $(X_k^j, X_k^{i'})$. Hence there are at most

$$\prod_{1 \leq i < i' \leq k} \prod_{j=1}^{\alpha n / x} \left( \frac{|E(X_k^j, X_k^{i'})|}{y} \right)^{\binom{2}{3}} \leq \prod_{1 \leq i < i' \leq k} \left( \frac{\sum_{j=1}^{\alpha n / x} |E(X_k^j, X_k^{i'})|}{\alpha n x} \right)^{\binom{2}{3}} \leq \prod_{1 \leq i < i' \leq k} \left( \frac{m}{\alpha n x} \right)^{\binom{2}{3}} \leq 2^{\alpha n m}$$

ways to choose the graphs $G_k^j$. Next we choose bad extensions of $G_k^j$ to the sets $X_k^{j,s}$. As these extensions have to belong to $B(k+1, x, y, \varepsilon)$ and $G_k^j$ is not a member of $B(k, x, y, \varepsilon)$, there at most

$$\left[ \frac{\beta^y}{y} \cdot \left( \frac{x^2}{y} \right) \right]^{\binom{\alpha n / x}{2}} \leq \beta^y \left( \frac{x^2}{y} \right) \left( \frac{\alpha^2 n^2}{(\alpha n / x)^2 y} \right)^k$$

ways to choose these extensions. Finally, we choose the remaining $m - (\alpha n / x)^2 y$ edges between $V_{k+1}$ and $V_1, \ldots, V_k$. There are at most

$$\frac{n^2}{m - (\alpha n / x)^2 y}$$

ways to do that.
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Summarizing, we have shown that there are at most

\[ \binom{n^2}{m}^{(\frac{1}{\ell}) - k} \cdot 2^{2\alpha m} \cdot 2\ell^m \cdot 3\kappa \alpha^m \cdot \binom{\alpha^2n^2}{(an/x)^2y}^k \cdot \left( \frac{n^2}{m} \right)^{n/2} \cdot \frac{\alpha^2n^2}{(an/x)^2y}^k \]

\[ \leq \left( 2^{\alpha^2\ell^2 + 3\kappa \alpha^m} \right)^m \cdot \binom{n^2}{m}^{(\frac{1}{\ell}) - k} \cdot \left( \frac{n^2}{m} \right)^{n/2} \cdot \frac{\alpha^2n^2}{(an/x)^2y}^k \]

graphs in \( G(\ell, n, m) \) that satisfy the properties of the claim. Using the technical Inequality (2.2) we obtain the upper bound

\[ (A^k \cdot 2^{\alpha^2\ell^2 + 3\kappa \alpha^m})^m \cdot \binom{n^2}{m}^{(\frac{1}{\ell}) - k} \cdot \binom{n^2}{m}^{(\frac{1}{\ell})} \]

Proof. We apply Claim 3.8 for all \( 2 < k < \ell - 1 \). This yields that all but at most

\[ \frac{\ell - 2}{\ell} \beta^m \left( \frac{n^2}{m} \right)^{\frac{1}{\ell}} \leq \beta^m \left( \frac{n^2}{m} \right)^{\frac{1}{\ell}} \]

graphs in \( G(\ell, n, m) \) satisfy the properties of Claim 3.8 for all \( 2 < k < \ell \). Now, we show that all these graphs also satisfy the properties stated in Claim 3.9.

Let \( 2 \leq k \leq \ell - 1 \) and consider a maximum set of pairwise vertex disjoint subgraph \( G^'_k \subseteq G(k, x, y, \epsilon) \setminus B(k, x, y, \epsilon) \), \( 1 \leq j \leq \alpha/m \), such that for all \( 1 \leq i \leq k \), \( |V(G^'_j) \cap V_i| = x \) and for all \( 1 \leq j \leq \alpha/m \), there exist pairwise vertex disjoint sets \( X^*_j \subseteq V_{k+1} \) of size \( \alpha/m \), such that \( G[V(G^'_k) \cup \cup X^*_k] \) contains an extension of \( G^*_k \) that belongs to \( B(k+1, x, y, \epsilon) \).

From Claim 3.8 we immediately deduce that there can exist at most \( \alpha/m \) such pairwise disjoint subgraphs \( G^*_k \). That is, if we take the union of all these graphs and subsequently take the union over all \( 2 \leq k \leq \ell - 1 \) of all their vertex sets, then we obtain sets \( W_k \) of size at most \( \ell - 2 \alpha/m \).

As we have chosen maximum sets of pairwise disjoint subgraphs \( G^*_k \), we know that whenever we choose a \( k \)-tuple \( (Z_1, \ldots, Z_k) \) with \( |Z_1 \cap (V_i \setminus W_1)| = x \) that induces a graph containing \( G_k \in G(k, x, y, \epsilon) \setminus B(k, x, y, \epsilon) \), then there exists a set \( W \subseteq V_{k+1} \) of size \( |W| = \alpha/m \) such that for any set \( Z \subseteq V_{k+1} \setminus (W_{k+1} \cup W) \) of size \( x \) the induced graph \( G[(Z_1, \ldots, Z_k, Z)] \) contains no extension of \( G_k \) that belongs to \( B(k+1, x, y, \epsilon) \).

Claim 3.10. Let \( V \) be a set of size \( n \) and let \( W \subseteq V \) be a set of size \( |W| \leq \ell \alpha |V| \). Then the number of sets \( \tilde{X} \subseteq V \) of size \( |\tilde{X}| = (1 + \epsilon')x \) that contain at least \( \frac{1}{1 + \epsilon'} x \) vertices from \( W \) is at most

\[ \left( \frac{\beta}{2\ell} \right)^x \left( \frac{n}{(1 + \epsilon')x} \right)^{\frac{1}{\ell}} \]
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PROOF. Using (2.1) and (2.2), we deduce that
\[
\left| \left\{ \bar{X} \subseteq V : |\bar{X}| = (1 + \varepsilon') x \wedge |\bar{X} \cap W| \geq \frac{1}{e} \varepsilon' x \right\} \right|
\leq \left( \frac{|W|}{\frac{1}{2} e' x} \right) \left( 1 + \frac{\varepsilon'}{e} \right) x
\leq \left( \frac{4^{1 + \varepsilon'} (\ell \alpha) \frac{1}{2} e'}{(1 + \varepsilon') x} \right)^x \left( \frac{|V|}{x} \right)
\leq \left( \frac{\beta}{2^\ell} \right)^x \left( \frac{|V|}{(1 + \varepsilon') x} \right),
\]
as claimed.

PROOF OF LEMMA 3.6. First, we specify how to choose \( \varepsilon_0 \). Let \( \varepsilon' := \min \{ \beta \varepsilon / \ell \} \). We will repeatedly apply Theorem 2.34 on page 16. So let \( e_1 := \varepsilon_0 \left( \frac{1}{2 \delta^2} \beta, \varepsilon''/4 \right) \) and let \( e_2 := \varepsilon_0 \left( \frac{1}{2 \delta^2} \beta, \varepsilon_1/2 \right) \) where \( \varepsilon_0 (\cdot, \cdot) \) is the function from Theorem 2.34. We may assume that \( \varepsilon' \gg \varepsilon'' \gg \varepsilon_1 \gg \varepsilon_0 \). In the following let \( 0 < \varepsilon \leq \varepsilon_0 \) be fixed.

Recall that Claim 3.9 states that for all but \( \beta^m \binom{n^2}{m} \delta \) of the graphs in \( G(\ell, n, m, \varepsilon) \), there exists appropriate sets \( W_i \subseteq V_i \). We will now show that the properties of the sets \( W_i \) stated in Claim 3.9 in fact suffice to show that "most" tuples \((X_1, \ldots, X_k)\) of size \((1 + \varepsilon') x\) are "good".

Let us make this more precise. For \( k \geq 2 \) we call \((X_1, \ldots, X_k)\) a good \( k \)-tuple if it satisfies the following properties:

- For \( 1 \leq i \leq k \), \( X_i \subseteq V_i \), \( |X_i| = (1 + \varepsilon') x \),
- For \( 1 \leq i \leq k \), \( 1 < j \leq \ell \), there exist subsets \( X'_i \subseteq X_i \setminus W_i \), \( |X'_i| = x \) such that \((X'_i, V_j)\) is \((\varepsilon_1)\)-regular with density \( d'_j \sim \varepsilon_1 m/n^2 \) and such that the induced graph \( G[X'_1, \ldots, X'_k] \) contains a graph \( G(X'_1, \ldots, X'_k) \in G(k, x, y, \varepsilon') \setminus B(k, x, y) \).

In addition, we call a set \( X_1 \subseteq V_1 \) of size \((1 + \varepsilon)x\) that contains a subset \( X'_1 \subseteq V_1 \setminus W_1 \) of size \(|X'_1| = x\) that is \((\varepsilon_1)\)-regular with density \( d'_j \sim \varepsilon_1 m/n^2 \) with all \( V_j \), \( 1 < j \leq \ell \), a good 1-tuple.

We will show by induction on \( k \) that for all \( 1 \leq k \leq \ell \), there exist at least
\[
\left( \binom{n}{(1 + \varepsilon') x} \right)^k
\]
good \( k \)-tuples. Observe that this will conclude the proof of the lemma as for \( k = \ell \), we have \((1 - 1/\ell \beta^2)^\ell \geq 1 - \beta^2\), and good \( \ell \)-tuples satisfy all desired properties of the lemma.

So it remains to show (3.6). Let \( k = 1 \). Observe that by Theorem 2.34 and the choice of \( \varepsilon_0 \), we know that all but
\[
(\ell - 1) \left( \frac{1}{2\delta^2} \beta \right)^{1+\varepsilon'} \left( \frac{n}{(1 + \varepsilon') x} \right)
\]
are \((\varepsilon_1/2)\)-regular with density \( d(X_1, V_j) \sim \varepsilon m/n^2 \) to \( V_j \), \( 2 \leq j \leq \ell \). Moreover, Claim 3.10 asserts that at least
\[
\left( 1 - \frac{\beta}{2\delta} \right)^x \left( \frac{n}{(1 + \varepsilon') x} \right)
\]
sets $X_i \subseteq V_i$ contain a set $\tilde{X}_1 \subseteq V_1 \setminus W_i$ of size $|\tilde{X}_1| \geq (1 + \ell^{-1}\varepsilon')x$. Let $\tilde{X}_1 := \bigcap_{j=2}^\ell \tilde{X}_j \cap \tilde{X}_1$. One easily observes that $|\tilde{X}_1| \geq x$. Note that according to Lemma 2.26, any subset $X'_i \subseteq \tilde{X}_1$ of size exactly $x$ is in fact $(\varepsilon_1)$-regular with density $d(X'_i, V_j) \sim_{\varepsilon_1} m/n^2$ to all $V_2, \ldots, V_\ell$. Here we used that for all $2 \leq j \leq \ell$,

$$
(1 - \varepsilon_1) \frac{m}{n^2} \leq \left(1 - \frac{1}{2\varepsilon_1}\right) (1 - \varepsilon) \frac{m}{n^2}
$$

$$
\leq \left(1 - \frac{1}{2\varepsilon_1}\right) d(\tilde{X}_i, V_j)
$$

$$
\leq d(\tilde{X}_1', V_j)
$$

$$
\leq \left(1 + \frac{1}{2\varepsilon_1}\right) d(\tilde{X}_1', V_j)
$$

$$
\leq \left(1 + \frac{1}{2\varepsilon_1}\right) (1 + \varepsilon) \frac{m}{n^2} \leq (1 + \varepsilon_1) \frac{m}{n^2}.
$$

Combining (3.7) and (3.8) we deduce that at least

$$
\left(1 - (\ell - 1) \cdot \left(\frac{\beta}{2\ell}\right)^{(1+\varepsilon')x} - \left(\frac{\beta}{2\ell}\right)^x\right) \cdot \left(\frac{n}{(1 + \varepsilon')x}\right)
$$

$$
\geq \left(1 - \frac{1}{\ell \beta^x}\right) \cdot \left(\frac{n}{(1 + \varepsilon')x}\right)
$$

(3.10)

sets $X_i \subseteq V_i, |X_i| = (1 + \varepsilon')x$ form good 1-tuples. This clearly completes the proof of the base case of the induction.

So assume the statement is true for some $1 \leq k < \ell$. We show that it is then also true for $k + 1$. Consider an arbitrary good $k$-tuple $(X_1, \ldots, X_k)$. We intend to show that this good $k$-tuple can be extended to a good $(k + 1)$-tuple $(X_1, \ldots, X_k, Z)$ in at least

$$
\left(1 - \frac{1}{\ell \beta^x}\right) \cdot \left(\frac{n}{(1 + \varepsilon')x}\right)
$$

ways. Clearly, this will then complete the proof of (3.6) and thus of the lemma.

We argue similarly as in the base case. According to Theorem 2.34 and the definition of $\varepsilon_0$ and $\varepsilon_1$ respectively, there exist at least

$$
\left(1 - (\ell - 1) \cdot \left(\frac{\beta}{2\ell}\right)^{(1+\varepsilon')x} - \left(\frac{\beta}{2\ell}\right)^x\right) \cdot \left(\frac{n}{(1 + \varepsilon')x}\right)
$$

sets $X_{k+1} \subseteq V_{k+1}, |X_{k+1}| = (1 + \varepsilon')x$ that contain a family of subsets $(\tilde{X}_j)_{j \in [\ell] \setminus \{k+1\}}$ such that

$$
|\tilde{X}_j| \geq \left(1 - \frac{1}{4\varepsilon''}\right) (1 + \varepsilon')x \geq \left(1 + \frac{\ell - 1}{\ell - \varepsilon'}\right) x
$$

since $\ell \varepsilon'' \leq \varepsilon'$, and the following property holds. For all $1 \leq i \leq k$, $(\tilde{X}_{k+1}', X'_i)$ is $(\varepsilon''/4)$-regular with density $d(\tilde{X}_{k+1}', X'_i) \sim_{\varepsilon_1} m/n^2$, and for all $k + 2 \leq j \leq \ell$, $(\tilde{X}_{k+1}', V_j)$ is $(\varepsilon_1/2)$-regular.
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with \( d(\tilde{X}_{k+1}^i, V_j) \sim \varepsilon \) \( m/n^2 \). The bound on \( d(\tilde{X}_{k+1}^i, X'_i) \) follows from

\[
(1 - 3\varepsilon) \frac{m}{n^2} \leq (1 - \varepsilon)^2 \frac{m}{n^2} \\
\leq (1 - \varepsilon) d(V_{k+1}, X'_i) \\
\leq d(\tilde{X}_{k+1}^i, X'_i) \\
\leq (1 + \varepsilon) d(V_{k+1}, X'_i) \\
\leq (1 + \varepsilon)^2 \frac{m}{n^2} \leq (1 + 3\varepsilon) \frac{m}{n^2}.
\]

Furthermore, since \( \tilde{G}(X'_1, \ldots, X'_k) \) is fixed, according to Claim 3.9 there exists a set \( W \subseteq V_{k+1} \) of size \( |W| \leq \alpha n \) such that all sets \( X \) of size \( x \) in \( V_{k+1} \setminus (W_{k+1} \cup W) \) satisfy that the graph induced by \( X'_1, \ldots, X'_k, X \) does not contain a bad extension of \( \tilde{G}(X'_1, \ldots, X'_k) \). Hence, according to Claim 3.10, at least

\[
\left(1 - \left(\frac{\beta}{2\ell}\right)^x\right) \left(\frac{n}{(1 + \varepsilon)x}\right)
\]

sets \( X_{k+1} \subseteq V_{k+1} \) contain a set \( \tilde{X}_{k+1} \subseteq V_{k+1} \setminus (W_{k+1} \cup W) \), each of size at least \( (1 + \frac{\ell-1}{\ell}\varepsilon') x \). As in (3.10) we obtain that at least

\[
\left(1 - \frac{1}{\ell} \beta^x\right) \left(\frac{n}{(1 + \varepsilon')x}\right)
\]

sets \( X_{k+1} \subseteq V_{k+1}, |X_{k+1}| = (1 + \varepsilon')x \) contain both a family of subsets \( (\tilde{X}_{k+1}^i)_{i \in [\ell] \setminus \{k + 1\}} \) that satisfies the regularity conditions and a subset \( \tilde{X}_{k+1} \subseteq X_{k+1} \setminus (W_{k+1} \cup W) \) of size \( |\tilde{X}_{k+1}| \geq (1 + \frac{\ell-1}{\ell}\varepsilon') x \). We deduce that the set

\[
\tilde{X}_{k+1} := \bigcap_{i=1}^k \tilde{X}_{k+1}^i \cap \tilde{X}_{k+1} \cap \bigcap_{j=k+2}^\ell \tilde{X}_{k+1}^j
\]

has size at least \( x \). Hence, we take any subset \( X'_{k+1} \subseteq \tilde{X}_{k+1} \) of size exactly \( x \). Observe that since \( |\tilde{X}_{k+1}^i| \leq 2|x'_i| \) for all \( i \in [\ell] \setminus \{k + 1\} \), regularity is inherited by \( X'_{k+1} \) due to Lemma 2.26. That is \( X'_{k+1} \) is \( (\varepsilon''/2) \)-regular with density \( d(X'_{k+1}, X'_i) \sim \varepsilon''/2 \) \( m/n^2 \) to \( X'_i \) for all \( 1 \leq i \leq k \) and \( (\varepsilon_1) \)-regular with density \( d(X'_{k+1}, V_j) \sim \varepsilon_1 \) \( m/n^2 \) to \( V_j \) for all \( k + 2 \leq j \leq \ell \). The bounds on the densities follow from

\[
\left(1 - \frac{1}{2} \varepsilon''\right) \frac{m}{n^2} \leq \left(1 - \frac{1}{4} \varepsilon''\right) (1 - 3\varepsilon_1) \frac{m}{n^2} \\
\leq \left(1 - \frac{1}{4} \varepsilon''\right) d(\tilde{X}_{k+1}^i, X'_i) \\
\leq d(X'_{k+1}, X'_i) \\
\leq \left(1 + \frac{1}{4} \varepsilon''\right) d(\tilde{X}_{k+1}^i, X'_i) \\
\leq \left(1 + \frac{1}{4} \varepsilon''\right) (1 + 3\varepsilon_1) \frac{m}{n^2} \leq \left(1 + \frac{1}{2} \varepsilon''\right) \frac{m}{n^2}
\]

and analogously to Equation 3.9.

We claim that all these sets \( X_{k+1} \) form a good \((k + 1)\)-tuple \((X_i)_{i=1}^{k+1}\). We know due to the induction hypothesis that \( G[X_1, \ldots, X'_k] \) is a member of the family \( \mathcal{G}(k, x^2m/n^2, \varepsilon'', 2) \), and by construction of \( X'_{k+1} \) that for all \( 1 \leq i \leq k \),

\[
|E(X'_{k+1}, X'_i)| \sim \varepsilon''/2 \frac{x^2 m}{n^2}
\]
since \( d(X'_{k+1}, X'_i) \sim \varepsilon'' / 2 \) \( m/n^2 \). Hence, \( G[X'_1, \ldots, X'_{k+1}] \) is a member of the family \( \tilde{G}(k + 1, 1, x, x^2 m / n^2, \varepsilon'' / 2) \). Moreover, \( G[X'_1, \ldots, X'_k] \) contains a graph \( \tilde{G}(X'_1, \ldots, X'_k) \in \mathcal{G}(k, x, y, \varepsilon'') \setminus B(k, x, y) \) since \((X_1, \ldots, X_k)\) is a good \( k \)-tuple. And since \( G[X'_1, \ldots, X'_{k+1}] \) contains no extension of \( \tilde{G}(X'_1, \ldots, X'_k) \) that belongs to \( B(k + 1, x, y, \varepsilon'') \) due to Claim 3.9 and the choice of \( X'_{k+1} \subseteq V_{k+1} \setminus (W_{k+1} \cup W) \), it remains to provide any subgraph \( \tilde{G}(X'_1, \ldots, X'_{k+1}) \) that belongs to the family \( \tilde{G}(k + 1, x, y, \varepsilon'') \). However, this follows from the application of Lemma 2.28 on page 15 to all pairs \((X'_{k+1}, X'_i)\), \( 1 \leq i \leq k \).

### 3.3. Proof of the counting lemma

Since we want to prove Theorem 3.1 by induction on \( \ell \), we rewrite Lemma 3.6 into a more convenient form. Roughly speaking the next lemma asserts that virtually all vertices in one class have neighborhoods to which we may apply the induction hypothesis.

**Lemma 3.11.** Let \( y_0(x) \geq x \) be a monotone increasing function and let \( B(\ell - 1, x, y) \) be small with respect to \( y_0 \). For all \( \beta, \varepsilon' > 0 \), there exist constants \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all \( \varepsilon, x, \) and \( y \) satisfying

\[
0 < \varepsilon \leq \varepsilon_0, \\
x = (1 - \varepsilon') m/

\[
m \gg n^{3/2} \sqrt{\log n}, \\
C_{y_0}(x) \leq \frac{m^3}{n^4}, \\
\varepsilon' \frac{m^3}{n^4} \leq y \leq (1 - \varepsilon')^3 \frac{m^3}{n^4},
\]

and \( n \) sufficiently large, all but at most \( \beta m\left(\frac{m}{n}\right)\binom{n}{\ell} \) graphs \( G \in \mathcal{G}(\ell, n, m, \varepsilon) \) satisfy the following property: there exist at least \((1 - \varepsilon') n \) vertices in \( V_1 \) that contain a member of the family

\[
\mathcal{G}(\ell - 1, x, y, \varepsilon') \setminus B(\ell - 1, x, y)
\]
in their neighborhood.

**Proof.** Let \( \hat{\beta} \) satisfy

\[
\hat{\beta}(1 - \varepsilon')^2 \frac{n^4}{2m} \leq \frac{\beta}{8}.
\]

We apply Lemma 3.6 with \( \ell \leftarrow \ell - 1, \beta \leftarrow \hat{\beta} \) and \( \varepsilon' \) to obtain constants \( \varepsilon_{\ell - 1} \leftarrow \varepsilon_0 \) and \( C_{\ell - 1} \leftarrow C \). We prove the lemma for

\[
\varepsilon_0 = \min \left\{ \varepsilon_{\ell - 1}, \frac{\varepsilon'}{4\ell}, \varepsilon'^2 \right\}
\]
and \( C = \frac{C_{\ell - 1}}{(1 - \varepsilon')^2} \).

Next, we verify the assumptions in (3.3) of Lemma 3.6. It follows from \( m \gg n^{3/2} \sqrt{\log n} \) that

\[
x = (1 - \varepsilon') \frac{m}{n} = (1 - \varepsilon') \frac{n^2}{m} \log n \frac{m^2}{n^3} \log n \geq \frac{n^2}{m} \log n
\]
for \( n \) sufficiently large and from \( C_{y_0}(x) \leq \frac{m^3}{n^4} \) that

\[
C_{\ell - 1} y_0(x) = (1 - \varepsilon')^2 \frac{m^3}{n^4} \leq (1 - \varepsilon')^2 \frac{m^3}{n^4} = \frac{m^3}{n^4} x^2.
\]

Since

\[
\varepsilon' \frac{m^3}{n^4} \leq y \leq (1 - \varepsilon')^3 \frac{m^3}{n^4},
\]
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we have

\[ \frac{m}{n^2} x^2 \leq y \leq \frac{m}{n^2} x^2. \]

Let \( G'(\ell, n, m, \varepsilon) \subseteq G(\ell, n, m, \varepsilon) \) denote the subfamily of graphs \( G \) that satisfy the following property: the number of \( \ell \)-tuples \( (X_2, \ldots, X_\ell) \) such that for each \( 2 \leq i \leq \ell, X_i \subseteq V_i \), we have \( |X_i| = (1 + \varepsilon') x \), and there exists \( X'_i \subseteq X_i \) of size \( x \) such that the induced graph \( G[X'_2, \ldots, X'_\ell] \) contains a member of the family \( G(\ell - 1, x, y, \varepsilon') \cup B(\ell - 1, x, y) \),

is at least

\[ (1 - \beta^m) \left( \frac{n}{m} \right)^{\ell - 1}. \]

Lemma 3.6 applied to \((V_2, \ldots, V_\ell)\) asserts that

\[ |G'(\ell, n, m, \varepsilon)| \geq (1 - \beta^m) \left( \frac{n^2}{m^2} \right)^{\ell - 1} \left( \frac{n^2}{m^2} \right)^{\ell - 1} \geq \left[ 1 - \left( \frac{\beta}{2} \right)^m \right] \left( \frac{n^2}{m} \right)^{\ell - 1}. \]

Observe that the factor \( \left( \frac{n^2}{m} \right)^{\ell - 1} \) accounts for the number of ways to distribute the edges incident to vertices in \( V_1 \). We shall count all graphs that belong to \( G'(\ell, n, m, \varepsilon) \), but violate the conditions of Lemma 3.11, and show that there are only few of them. Firstly, we select all edges between \( V_i, V_j \) for all \( 2 \leq i < j \leq \ell \). With a lot of room to spare, there are at most

\[ \left( \frac{n^2}{m} \right)^{\ell - 1} \]

possibilities. Secondly, we select for all the vertices \( v \in V_1 \), the degrees \( d_j(v) \) into \( V_j \) for \( j \geq 2 \). There are at most \( n^m \leq 2^m \) possibilities for sufficiently large \( n \). By Proposition 2.30 on page 15 and since we are constructing an \((\varepsilon)\)-regular graph between \( V_i \) and \( V_j \), we have to choose for at least \((1 - \varepsilon) n \) vertices \( v \in V_1 \), degrees \( d_j(v) \geq (1 - \varepsilon)m/n \) into all the sets \( V_j \). Now we choose a set of at least \( \varepsilon' n \) vertices that do not contain a graph in \( G(\ell - 1, x, y, \varepsilon') \cup B(\ell - 1, x, y) \) as a subgraph in their neighborhood. There are at most \( 2^m \leq 2^m \) possibilities to choose these vertices. We denote by \( A \) the set of all such vertices that additionally have a degree \( \sim m/n \) into each set \( V_j \) for \( 2 \leq j \leq \ell \). Note that \( |A| \geq (\varepsilon' - 2\varepsilon)n \geq (\varepsilon'/2)n \) and that each vertex in \( A \) has degree greater than

\[ (1 - \varepsilon') \frac{m}{n} \geq (1 - \varepsilon') \frac{m}{n} \geq (1 + \varepsilon') \frac{m}{n} = (1 + \varepsilon') x. \]

Now we select the neighborhoods for the vertices in \( V_1 \setminus A \). There are at most \( \left( d_j(v) \right)^n \) possibilities for each vertex \( v \) to choose its neighborhood in \( V_j \), where \( d_j(v) \) is the already fixed size of the neighborhood of \( v \) in \( V_j \). For all vertices in \( A \), we first choose a set of size \((1 + \varepsilon')x\) in each partition class \( V_2, \ldots, V_\ell \). We require that these sets do not contain subsets of size \( x \) that induce a graph which contains a member of \( G(\ell - 1, x, y, \varepsilon') \cup B(\ell - 1, x, y) \). Since we consider graphs in \( G'(\ell, n, m, \varepsilon) \), there are at most \( \beta^\ell \left( \frac{n}{1 + \varepsilon'} \right)^{\ell - 1} \) ways to choose such sets \( X_i, 2 \leq i \leq \ell \), in the neighborhoods of each vertex in \( A \). Now we have to choose the remaining neighbors for every vertex \( v \in A \). There are at most \( \prod_{j=2}^\ell \left( d_j(v) - (1 + \varepsilon') x \right)^{\ell - 1} \) ways to do this. The number of ways to
select the neighborhoods of vertices in $A$ is thus at most
\[
\prod_{v \in A} \beta^{n} \left( \frac{n}{(1 - \varepsilon')x} \right)^{\ell - 1} \prod_{j=2}^{\ell} \left( \frac{n - (1 + \varepsilon')x}{d_j(v) - (1 + \varepsilon')x} \right)
\]
\[
\leq \frac{\beta^{x'n/2}}{\beta^{x'n/4}} \left( \prod_{v \in A} \prod_{j=2}^{\ell} \frac{n}{d_j(v)} \right)
\]
\[
d_{j}(v) \leq \frac{2m}{n} \beta^{x'n/2} \left( \prod_{v \in A} \prod_{j=2}^{\ell} \frac{n}{d_j(v)} \right).
\]

We conclude from the definition of $x$ that there are at most
\[
\left( \frac{\beta^{(1 - \varepsilon')n/2}}{\beta^{n/2}} \right)^{m} \left( \prod_{v \in V_1} \prod_{j=2}^{\ell} \frac{n}{d_j(v)} \right) \leq \left( \frac{\beta}{\varepsilon'} \right)^{m} \left( \frac{n^{2}}{m} \right)^{\ell - 1}
\]
ways to select the neighborhoods of the vertices in $V_1$. Taking the graphs in $G(\ell, n, m, \varepsilon) \setminus G'(\ell, n, m, \varepsilon)$ into account, we proved that there are at most
\[
\left[ \left( \frac{\beta}{2} \right)^{m} + (2m)^{2} \left( \frac{\beta}{\varepsilon'} \right)^{m} \right] \left( \frac{n^{2}}{m} \right)^{\ell - 1} \left( \frac{n^{2}}{m} \right)^{\ell - 1} \leq \beta^{m} \left( \frac{n^{2}}{m} \right)^{\ell - 1}
\]
graphs that violate the conditions of the lemma.

Before we come to the actual proof of Theorem 3.1, we shall proof the base case $\ell = 3$ of the induction.

**Lemma 3.12.** For all $\delta > 0$, the family $\mathcal{F}(3, n, m, \delta)$ is small with respect to
\[m_3 = n^{3/2}.\]

**Proof.** Let $\delta' > 0$ satisfy
\[ (1 - \delta) \leq (1 - \delta')^4. \]
Apply Theorem 2.39 with parameters $\ell \leftarrow 3$, $\alpha \leftarrow \delta'$, $\delta' \leftarrow \delta'$, and $\beta$. Hence, we obtain constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $m \geq Cn^{3/2}$, and $n$ sufficiently large, all but at most
\[ \beta^{m} \left( \frac{n^{2}}{m} \right)^{3} \]
graphs $G \in \mathcal{G}(3, n, m, \varepsilon)$ satisfy that at least $(1 - \delta)n$ vertices $v$ in $V_1$ have neighborhoods containing a bipartite graph with at least $(1 - \delta')m^3/n^4$ edges. Since each edge forms a triangle with $v$, the number of triangles in $G$ is at least
\[ (1 - \delta')^4 \frac{m^3}{n^4} \geq (1 - \delta) \frac{m^3}{n^3}. \]

**Proof of Theorem 3.1.** We prove the theorem by induction on $\ell$. The base case is settled by Lemma 3.12. So assume that the theorem holds for some $\ell - 1 \geq 3$, i.e., for all $\delta' > 0$, the family $\mathcal{F}(\ell - 1, x, y, \delta')$ is small with respect to
\[ y_{\ell-1}(x) = x^{2-1/(\ell-2)}. \]
Choose $\delta' > 0$ such that
\[ (1 - \delta')(\ell)^{4/2} \geq 1 - \delta. \]
Apply Lemma 3.11 with the family $\mathcal{F}(\ell -1, x, y, \delta')$ and the parameters $\beta$ and $\epsilon' \leftarrow \delta'$ in order to obtain constants $\epsilon_0 > 0$ and $C > 0$. Observe that

$$m \geq C n^{2-1/(\ell-1)} \geq C n^{5/3} \gg n^{3/2} \log n$$

since $\ell \geq 4$, and that

$$C_{\ell-1}\mathcal{F}(\ell-1) \geq C_{\ell-1} \left[ (1 - \delta') \frac{m}{n} \right]^{\frac{2\ell-5}{\ell-2}} \leq C_{\ell-1} m^{-\frac{\ell-1}{2n}} m^{\frac{3}{2}} \frac{m^3}{n^4} \leq C \frac{m^3}{n^4}$$

since $m \geq C n^{2-1/(\ell-1)}$. Thus, for $y = (1 - \delta')^3 m^3/n^4$ and $n$ sufficiently large, all but at most $\beta^m \binom{n}{m} \binom{\ell}{2}$ graphs $G \in \mathcal{G}(\ell, n, m, \epsilon)$ satisfy the following property: there are at least $(1 - \delta') n$ vertices $v \in V_1$ that contain a member of the family

$$\mathcal{G}(\ell - 1, x, y, \delta') \setminus \mathcal{F}(\ell - 1, x, y, \delta')$$

in their neighborhood. But this means that each such vertex $v$ has at least

$$(1 - \delta') x^{\ell-1} \left( \frac{y}{x^2} \right)^{\binom{\ell-1}{2}}$$

cliques of size $\ell - 1$ in its neighborhood. Hence, in $G$ there are at least

$$(1 - \delta') n (1 - \delta') x^{\ell-1} \left( \frac{y}{x^2} \right)^{\binom{\ell-1}{2}}$$

$$= (1 - \delta')^2 n \left[ (1 - \delta') \frac{m}{n} \right]^{\ell-1} \left[ (1 - \delta') \frac{m}{n^2} \right]^{\binom{\ell-1}{2}}$$

$$= (1 - \delta')^{(\ell-1)/2 + \ell + 1} n^\ell \left( \frac{m}{n^2} \right)^{\binom{\ell+1}{2}}$$

$$\geq (1 - \delta) n^\ell \left( \frac{m}{n^2} \right)^{\binom{\ell}{2}}$$

complete graphs on $\ell$ vertices. \hfill \Box

As mentioned in the introduction, the threshold $m \geq n^{2-1/(\ell-1)}$ can be improved by lifting the base case of the induction. If we used $\ell = 4$, the induction would work for $m \geq n^{2-1/(\ell-3/2)}$. The case $\ell = 5$ would yield $m \geq n^{2-1/(\ell-2)}$ and so forth. Rough calculations reveal that in order to prove the conjectured threshold $m \geq n^{2-2/(\ell+1)}$ inductively, one has to use the induction assumption $\ell - 2$ instead of $\ell - 1$. 

CHAPTER 4

An Erdős-Gallai Type Theorem

In this chapter we proof a probabilistic version of the well-known theorem of Erdős and Gallai, which asserts long cycles in graphs with sufficiently many edges. We consider random graphs from which an adversary removes a fraction of all edges.

The presented results will appear in a paper joint with Domingos Dellamonica Jr., Yoshiharu Kohayakawa, and Angelika Steger.

4.1. Introduction

The circumference of a graph $G$ is the maximum length of any cycle in $G$. The following classical result of Erdős and Gallai [EG59] gives, for any graph $G$ on $n$ vertices, a sufficient condition on the number of edges in $G$ so that the circumference is greater than $\ell$, $3 \leq \ell \leq n$. For any graphs $G$ and $\{H_1, \ldots, H_k\}$, let \( \text{ex}(G, \{H_1, \ldots, H_k\}) \) denote the maximum number of edges in a subgraph of $G$ that does not contain a copy of any member of the family $\{H_1, \ldots, H_k\}$, that is,

\[
\text{ex}(G, \{H_1, \ldots, H_k\}) := \max \{e(G') : H_i \not\subseteq G' \subseteq G \text{ for all } 1 \leq i \leq k\}.
\]

This is a generalization of the notation that was introduced in Section 2.3.4. As usual, let $K_n$ denote a clique on $n$ vertices and $C_\ell$ a cycle of length $\ell$. Erdős and Gallai [EG59] proved that, for all integers $3 \leq \ell < n$, we have

\[
\text{ex}(K_n, \{C_{\ell+1}, \ldots, C_n\}) \leq \frac{\ell(n-1)}{2}.
\]

Woodall [Woo72] strengthened this bound for the case when $n-1$ is not divisible by $\ell-1$.

**Theorem 4.1.** For all integers $3 \leq \ell < n$, we have

\[
\text{ex}(K_n, \{C_{\ell+1}, \ldots, C_n\}) = \left[ \frac{n-1}{\ell-1} \right] \left( \frac{\ell}{2} \right) + \left( \frac{r+1}{2} \right),
\]

where

\[
r = n - 1 - (\ell - 1) \left[ \frac{n-1}{\ell-1} \right].
\]

For related problems and historical information, the reader is referred to the textbook of Bollobás [Bol04] and the surveys of Bondy [Bon02] and Simonovits [Sim97]. Theorem 4.1 was reproved by Caccetta and Vijayan in [CV91]. The bound in Theorem 4.1 is best possible for all integers $n$. Consider, for instance, the graph $G$ on $n$ vertices that is a collection of $\left(\frac{n-1}{\ell-1}\right)$ cliques of size $\ell - 1$ plus one additional clique of size $r$ such that all members of this collection are completely connected to another vertex $v$. Clearly, this construction does not allow for a cycle in $G$ of length greater than $\ell$.

If one is interested in copies of long cycles, i.e., cycles of length in the order of $n$, Theorem 4.1 yields the following bound on $e(G)$. 

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**Corollary 4.2.** For any real constant \( \alpha > 0 \),

\[
\text{ex}(K_n, \{ C_{\lfloor (1-\alpha)n \rfloor}, \ldots, C_n \}) = \left( 1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1) \right) \left( \frac{n}{2} \right),
\]

where

\[
w(\alpha) = (1 - \alpha) \left( \frac{1}{1 - \alpha} - \frac{1}{1 - \alpha} \right).
\]

**Proof.** We apply Theorem 4.1 with \( \ell = \lfloor (1 - \alpha)n \rfloor - 1 \). Rewrite \( r = \text{mod}(n - 1, \ell - 1) \) as follows:

\[
r = n - 1 - \lfloor (1 - \alpha)n \rfloor - 1 = n - 1 - (1 - \alpha + O(n^{-1})) \frac{n - 1}{1 - \alpha + O(n^{-1})} n
\]

\[
= \left( 1 - \alpha \right) \left( \frac{1}{1 - \alpha} - \frac{1}{1 - \alpha} \right) + O(n^{-1}) n
\]

\[
= (w(\alpha) + o(1)) n.
\]

Now, we plug \( r \) into

\[
\text{ex}(K_n, \{ C_{\lfloor (1-\alpha)n \rfloor}, \ldots, C_n \})
\]

\[
\leq \frac{n - 1}{\ell - 1} \left( \frac{\ell}{2} + \left( \frac{r + 1}{2} \right) \right)
\]

\[
= \frac{(n - 1 - r)\ell}{2} + \frac{r + 1}{2}
\]

\[
= \left( 1 - w(\alpha) + o(1) \right) \left( 1 - \alpha \right) \left( \frac{n}{2} \right) + (w(\alpha)^2 + o(1)) \left( \frac{n}{2} \right)
\]

\[
= \left( 1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1) \right) \left( \frac{n}{2} \right).
\]

Observe that \( w(\alpha) = 0 \) if \( 1 - \alpha = 1/k \) for some integer \( k \). In that case the bound on the number of edges can be simplified to \( (1 - \alpha + o(1))(\frac{n}{2}) \) in Corollary 4.2. In fact, it follows from the original theorem of Erdős and Gallai that even the \( o(1) \) error term can be omitted in this case.

We may interpret this result from a different perspective. In order to destroy all cycles of length at least \( (1 - \alpha)n \) in a complete graph on \( n \) vertices, one has to remove roughly at least an \( (1 - w(\alpha))(\alpha + w(\alpha)) \)-fraction of all edges. Our attempt is to transfer this problem into the setting of random graphs. How many edges must one typically delete in a random graph in order to destroy all long cycles? As it turns out, we obtain an analogous result to Corollary 4.2 even for very sparse random graphs.

**Theorem 4.3.** For any real constant \( \alpha > 0 \) and \( p \gg n^{-1} \), asymptotically almost surely we have

\[
\text{ex}(G_n, p, \{ C_{\lfloor (1-\alpha)n \rfloor}, \ldots, C_n \}) = \left( 1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1) \right) e(G_n, p),
\]

where

\[
w(\alpha) := (1 - \alpha) \left( \frac{1}{1 - \alpha} - \frac{1}{1 - \alpha} \right).
\]

Note that this result is essentially best possible. One way to destroy all cycles of length at least \( \lfloor (1 - \alpha)n \rfloor \) in a random graph \( G_n, p \) is to partition the graph into \( k \) classes of size \( (1 - \alpha)n \), where \( k := \lfloor 1/(1 - \alpha) \rfloor \), and one additional class for the remaining vertices of size \( w(\alpha)n \). Clearly,
if one deletes all edges between the partition classes, only cycles of length at most \((1 - \alpha)n\) remain in the graph. Since the number of edges between any pair of classes of linear size is sharply concentrated around its mean, the adversary a.a.s. deletes at most
\[
\left(\binom{k}{2}(1 - \alpha)^2 + k(1 - \alpha)w(\alpha)\right)(1 + o(1))pn^2
\]
\[
= (1 - \alpha)k\left((1 - \alpha)(k - 1) + 2w(\alpha)\right)(1 + o(1))p\left(\frac{n}{2}\right)
\]
\[
= (1 - w(\alpha))(\alpha - w(\alpha) + 2w(\alpha))(1 + o(1))p\left(\frac{n}{2}\right)
\]
\[
= ((1 - w(\alpha))(\alpha + w(\alpha)) + o(1))p
\]
edges from the random graph. The second equality follows from the identity \((1 - \alpha)k = 1 - w(\alpha)\).

In order to prove Theorem 4.3, we have to show that, for any real constant \(\beta > 0\), there exists a constant \(C > 0\) such that, for all \(p \geq Cn^{-1}\), \(G = G_{n,p}\) satisfies the following property with probability tending to 1 as \(n \to \infty\): every subgraph \(G' \subseteq G\) with at least \((1 - (1 - w(\alpha))(\alpha + w(\alpha)) + \beta)e(G)\) edges contains a copy of a cycle of length at least \((1 - \alpha)n\).

Our proof employs Szemerédi's regularity lemma for sparse graphs as presented in Section 2.4.2. Section 4.2 is primarily devoted to the proof of our main technical lemma, namely Lemma 4.4. This states that dense, regular pairs permit an almost complete covering by a long path. Finally, in Section 4.3 we use those paths as fragments of a long cycle and thereby prove Theorem 4.3.

4.2. Long paths in regular pairs

This section is devoted to the proof of the following result that guarantees long paths in \((\varepsilon, p)\)-regular pairs provided that those are \((A, \mu)\)-upper-uniform for a given constant \(A\). Lemma 4.4 is the main technical ingredient in the proof of Theorem 4.3.

Lemma 4.4. For all \(0 < \rho, \mu \leq 1\), there exists \(\varepsilon = \varepsilon(\rho, \mu) > 0\) and, for all \(0 < \nu \leq 1\) and \(A > 0\), there exists \(d_0 = d_0(\rho, \mu, \nu, A)\) such that the following holds. Let \(G\) be a \((p, A)\)-upper-uniform graph on \(n\) vertices and \(d = pn\). Suppose that \(d \geq d_0\) and \(V_1, V_2 \subseteq V(G)\) satisfy

(i) \(V_1 \cap V_2 = \emptyset\);  
(ii) \(|V_1| = |V_2| = m \geq \nu n\);  
(iii) the induced, bipartite graph \(G[V_1, V_2]\) is \((\varepsilon, p)\)-regular with density \(d_{1,2} := d_{G,p}(V_1, V_2) \geq \rho\).

Then there exist sets \(X \subseteq V_1\) and \(Y \subseteq V_2\) of size at least \(\varepsilon mn\) such that any \(x \in X\) and any \(y \in Y\) are endpoints of a path on at least \(2(1 - 2\mu)m\) vertices in \(G[V_1, V_2]\).

The reader may skip the remainder of this section on the first reading since all presented results concern exclusively the proof of Lemma 4.4. We shall prove the lemma similarly to Lemma 2.7 in [DHK05]. Understanding every detail here is not required in order to follow the proof of Theorem 4.3 in Section 4.3.

We say that a bipartite graph \(B = (U \cup W, E)\) is \((b, f)\)-expanding if for every set \(X \subseteq U\) and every set \(Y \subseteq W, |X|, |Y| \leq b\), we have
\[
|\Gamma(X)| \geq f|X| \quad \text{and} \quad |\Gamma(Y)| \geq f|Y|.
\]
Here, as usual, $\Gamma(Z)$ denotes the neighborhood of a vertex-set $Z$, that is, the set of all vertices adjacent to some $z \in Z$.

We employ the following result, which is a variant of a well known lemma due to Pósa [P6s76] (for a proof, see [Hax97]).

**Lemma 4.5.** Let $b \geq 1$ be an integer. If the bipartite graph $B$ is $(b, 2)$-expanding, then $B$ contains a path $P^{4b}$ on $4b$ vertices.

**Proof of Lemma 4.4.** Let

$$\varepsilon = \frac{\mu^2 \rho}{8}.$$  

Moreover, let

$$\delta = \varepsilon (\mu^{-1} + \varepsilon^{-1})$$

and choose $d_0$ such that

$$d_0 \left( \frac{\delta \rho \mu}{A} \right)^2 \geq 2. \quad (4.1)$$

**Claim 4.6.** For all $V_1 \subseteq V_1$ and $V_2 \subseteq V_2$ of size at least $\mu m$, there exist $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ of size at least $(\mu - \varepsilon)m$ such that for any $u_1 \in U_1$ and any $u_2 \in U_2$,

$$|\Gamma(u_1) \cap U_2| \geq (1 - \delta) d_{1,2} \mu \rho m \quad \text{and} \quad |\Gamma(u_2) \cap U_1| \geq (1 - \delta) d_{1,2} \mu \rho m. \quad (4.2)$$

**Proof.** Let us define a sequence

$$B(t) = (V_1(t), V_2(t)) \quad (t = 0, 1, 2, \ldots)$$

in the following way. Start with $B(0) = (V_1, V_2')$. Suppose now that $t \geq 0$ and that we have defined $B(t)$. If (4.2) is satisfied for $U_j = V_j(t) \ (1 \leq j \leq 2)$, then we are home. Otherwise, take

$$V_i(t + 1) = V_i(t) \setminus \{x\}$$

for some fixed $x \in V_i(t)$ and $i$ such that $|\Gamma(x) \cap V_j(t)| < (1 - \delta) d_{1,2} \mu \rho m$ for $j \neq i$ with $1 \leq j \leq 2$; moreover, take $V_j(t + 1) = V_j(t)$.

Let us suppose for a contradiction that, at some moment $T$, we have, without loss of generality, $|V_1(T)| < (\mu - \varepsilon)m$ and $|V_2(T)| \geq (\mu - \varepsilon)m$.

As $|V_2(T)| > \varepsilon m$, there exists $X \subseteq V_2(T) \setminus V_1(T)$ with cardinality greater than $\varepsilon m$ such that we have $|\Gamma(x) \cap V_2(T)| < (1 - \delta) d_{1,2} \mu \rho m$ for all $x \in X$. We conclude that

$$e(X, V_2(T)) < (1 - \delta) d_{1,2} \mu \rho m |X|,$$

which implies that the $p$-density of the pair $(X, V_2(T))$ is

$$d_{G,p}(X, V_2(T)) < (1 - \delta) d_{1,2} \mu \rho m |V_2(T)| \left( \frac{\mu m}{|V_2(T)|} \right) \leq \left( 1 - \frac{\delta \mu - \varepsilon}{\mu - \varepsilon} \right) d_{1,2} \leq d_{1,2} - \varepsilon$$

contradicting the regularity of the pair $(V_1, V_2)$. \qed

**Claim 4.7.** The bipartite graph induced by $U_1$ and $U_2$ given in Claim 4.6 is $((1 - 2\delta) d_{1,2} \mu \rho m/f, f)$-expanding for any $0 < f \leq (\delta \rho \mu / A)^2 d$. 

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Proof. Let $X \subseteq U_i$, $1 \leq i \leq 2$, be such that $|X| \leq (1 - 2\delta)d_{1,2} \mu n / f$. Let $Y = \Gamma(X) \cap U_j \subseteq U_j$, with $j \neq i$.

By the upper-uniformity condition on $G$, we have

$$e(X, Y) \leq p|X| |Y| + A \sqrt{d|X||Y|} \leq p|X|(1 - 2\delta)d_{1,2} \mu n + A \sqrt{d|X||Y|},$$

and, from (4.2), we deduce that

$$e(X, Y) = e(X, U_j) \geq (1 - \delta)d_{1,2} \mu n |X|. \quad (4.4)$$

Combining (4.3) and (4.4), we have that $(\delta d_{1,2} \mu n |X|)^2 < A^2 d|X||Y|$. Therefore,

$$|Y| > \frac{(\delta d_{1,2} \mu n |X|)^2}{A^2 d|X|} \geq \left( \frac{\delta \mu n}{A} \right)^2 d|X| \geq f|X|.$$ 

\[ \square \]

We continue the proof of Lemma 4.4 by iteratively applying Claims 4.6 and 4.7. Let

$$b = \left\lfloor \frac{1}{2} (1 - 2\delta) \mu n \right\rfloor.$$ 

Construct a sequence of disjoint paths $P(t)$, $t = 1, 2, \ldots$, on the vertices in $V_1 \cup V_2$ each of length $4b$ as follows. Suppose $P(1), \ldots, P(t - 1)$ have already been obtained. We build $P(t)$ in the following way. Let

$$V_1^t = V_1 \setminus \bigcup_{j=1}^{t-1} V(P(j)) \quad \text{and} \quad V_2^t = V_2 \setminus \bigcup_{j=1}^{t-1} V(P(j)).$$

Observe that $|V_1^t| = |V_2^t|$ since all paths have even length. If $|V_1^t| \geq \mu n$, then we can apply Claim 4.6 in order to obtain sets $U_1 \subseteq V_1^t$ and $U_2 \subseteq V_2^t$ of size at least $(\mu - \varepsilon\mu)$. It follows from Claim 4.7 and the choice of $d$ (see (4.1)) that $(U_1, U_2)$ is $(b, 2)$-expanding. Therefore, we obtain a path $P(t)$ of length $4b$ on the vertices in $U_1 \cup U_2$ by Lemma 4.5. We stop constructing new paths as soon as $|V_1^t| < \mu n$.

Suppose this procedure stopped after $T$ iterations. We concatenate the paths $P(t)$, $1 \leq t \leq T$, into a single path $P_0$ in the following way. Let $\text{head}(P(t))$ denote the first $\lfloor \varepsilon m \rfloor$ vertices of $P(t)$ in $V_1$ and analogously $\text{tail}(P(t))$ the last $\lfloor \varepsilon m \rfloor$ vertices of $P(t)$ in $V_2$ ($1 \leq t \leq T$). Since $(V_1, V_2)$ is $(\varepsilon, \mu)$-regular with density $d_{1,2}$, we have, for all $2 \leq t \leq T$,

$$e(\text{tail}(P(t - 1)), \text{head}(P(t))) \geq (d_{1,2} - \varepsilon) \varepsilon m^2 \geq 1$$

for $m$ sufficiently large. Hence, by joining $P(t - 1)$ and $P(t)$ with an arbitrary edge between $\text{tail}(P(t - 1))$ and $\text{head}(P(t))$, $P_0$ becomes at least

$$2(1 - \mu)m - 4(T - 1)\lfloor \varepsilon m \rfloor$$

vertices long. Let

$$X = \text{head}(P_0) \quad \text{and} \quad Y = \text{tail}(P_0).$$

Then any $x \in X$ and any $y \in Y$ are endpoints of a path of length at least

$$2(1 - \mu)m - 4T \lfloor \varepsilon m \rfloor \geq 2 \left( 1 - \mu - \frac{8\varepsilon}{\varepsilon \mu} \right) m \geq 2(1 - 2\mu)m$$

since

$$T \leq \frac{m}{2b} \leq \frac{m}{2(1 - 2\delta) \mu n / 4} \leq \frac{3}{2\varepsilon \mu}. $$
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In the last inequality we used that, by the choice of \( e \) and \( \delta \), we have
\[
2\delta = 2e (\mu^{-1} + \varphi^{-1}) \leq \frac{1}{4} \mu (\varphi + \mu) \leq \frac{1}{2}.
\]
\[\square\]

4.3. Proof of the Erdös-Gallai type theorem

In order to prove Theorem 4.3, we have to show that for all positive real constants \( \beta \), there exists a constant \( C = C(\beta) > 0 \) such that for all \( 0 < \alpha \leq 1 \), every subgraph \( G' \subseteq G = G_{n,p} \) with at least \( (1 - (1 - w(\alpha))(\alpha + w(\alpha)) + \beta) e(G_{n,p}) \) edges a.a.s. has circumference at least \( (1 - \alpha)n \) if \( d = pn \geq C \). Let
\[
f(\alpha) := (1 - w(\alpha))(\alpha + w(\alpha))
\]
Observe that for all \( 0 < \delta < \alpha \leq 1 \), we have
\[
\alpha \leq f(\alpha) \leq 2\alpha \quad \text{and} \quad f(\alpha) - f(\alpha - \delta) \leq 2\delta.
\]
(4.5)

Suppose the adversary creates a graph \( G' \subseteq G \) by deleting at most \( (f(\alpha) - \beta) e(G_{n,p}) \) edges from \( G \). In view of Lemma 2.10 on page 10, we may and shall assume that \( G' \) is \((p,A)\)-upper-uniform for \( A = e^2\sqrt{6} \). We shall prove the existence of a sufficiently long cycle in \( G' \) in two steps. First, we apply the regularity lemma to \( G' \) and conclude from Woodall's theorem that there exists a long cycle in the so-called reduced graph. Second, we embed a cycle in \( G' \) into this structure.

We start by defining the values of all constants involved. We abstain from simplifying the following expressions in order to facilitate readability. The particular values of the constants are of less importance provided that they are fixed. Define
\[
\beta := \beta, \quad \gamma := \beta, \quad \mu := \frac{\tau}{16}.
\]

Choose \( \epsilon_0 = \epsilon(\beta, \mu) > 0 \) according to Lemma 4.4. Suppose that \( k_1 \) is a sufficiently large integer such that Corollary 4.2 guarantees a cycle of length at least \( (1 - (\alpha - \tau/4))k \) for all graphs on \( k \geq k_1 \) vertices with at least
\[
\left(1 - f\left(\alpha - \frac{\tau}{4}\right) + \frac{\tau}{2}\right) \binom{k}{2}
\]
edges. Let
\[
\epsilon := \min\left\{\epsilon_0, \frac{1}{7}, \frac{\beta}{32}, \frac{\tau}{32}\right\}, \quad k_0 := \max\{\epsilon^{-1}, k_1\}, \quad \text{and} \quad \nu := \frac{1}{2K_0},
\]
where \( K_0 = K_0(\epsilon, A, k_0) \) comes from Lemma 2.36 on page 17. Let \( d_0 = d_0(\epsilon, A, k_0) \) be as in Lemma 2.36 and let \( d_1 = d_0(\beta, \mu, A) \) be as in Lemma 4.4. We claim that the choice of
\[
C := \max\left\{d_0, d_1, \left(\frac{8A}{\nu \beta}\right)^2\right\}
\]
satisfies the properties above.

The argument for how to produce a long cycle in the reduced graph is standard. Since \( G' \subseteq G \) is \((p,A)\)-upper-uniform, we apply Lemma 2.36 to \( G' \) with parameters \( \epsilon, A, \) and \( k_0 \). Thus, we
obtain an \((\varepsilon, G', p)\)-regular partition \(\Pi = (V_0, V_1, \ldots, V_k)\) of \(V(G')\) with \(k_0 \leq k \leq K_0\). We call a pair \((V_i, V_j)\), \(1 \leq i < j \leq k\), \textit{regular and} \(G'\)-\textit{dense} if it is \((\varepsilon, p)\)-regular and

\[
d_{G'}(V_i, V_j) \geq qp.
\]

Consider the reduced graph \(R\) in which every vertex corresponds to a class \(V_i\), \(1 \leq i \leq k\), and two vertices \(i\) and \(j\) are connected if and only if \((V_i, V_j)\) is regular and \(G'\)-dense. Our aim is to show a lower bound on the number of edges in \(R\).

Let \(n/k \geq m = |V_1| = |V_2| = \ldots = |V_k| \geq \nu n\). We need to take into account four different types of edges in \(G'\).

(i) Due to (2.11) and (2.12), the number of edges in \(G'\) that are incident to the vertices in \(V_0\) is

\[
e_{G'}(V_0) + e_{G'}(V_0, V \setminus V_0)
\[
\leq p \left( \frac{e_\mu}{2} \right) + A\sqrt{d}\varepsilon n + p\varepsilon(1 - \varepsilon)n^2 + A\sqrt{d}\varepsilon n
\leq \left( \frac{\varepsilon^2}{2} + \varepsilon(1 - \varepsilon) \right) pm^2 + 2A\sqrt{d}\varepsilon n
\leq \varepsilon pm^2 + 2A\sqrt{d}\varepsilon n
\leq 2\varepsilon \left( 1 + \frac{2A\sqrt{d}}{pm\varepsilon} \right) \frac{pn^2}{2} = 2\varepsilon \left( 1 + \frac{2A}{\sqrt{d}\varepsilon} \right) \frac{pn^2}{2}.
\]

(ii) The number of edges that belong to irregular pairs is

\[
\leq \varepsilon \left( \frac{k}{2} \right) (pm^2 + A\sqrt{d}m) \leq \varepsilon \left( 1 + \frac{Ak}{\sqrt{d}} \right) \frac{pn^2}{2}.
\]

(iii) The number of edges that belong to regular pairs that fail (4.6) is

\[
< \left( \frac{k}{2} \right) qpm^2 \leq \varrho \frac{pm^2}{2}.
\]

(iv) The number of edges whose endpoints belong to the same \(V_i\), \(1 \leq i \leq k\), is

\[
\leq k \left( p \left( \frac{m}{2} \right) + A\sqrt{d}m \right) \leq \left( \frac{1}{k} + \frac{2A}{\sqrt{d}} \right) \frac{pn^2}{2}.
\]

Summing up the estimates (4.7)-(4.10), we conclude that the number of edges in \(G'\) that are not contained in regular and \(G'\)-dense pairs is

\[
< \left( 2\varepsilon \left( 1 + \frac{2A}{\sqrt{d}\varepsilon} \right) + \varepsilon \left( 1 + \frac{Ak}{\sqrt{d}} \right) + \varrho \right) \left( 1 + \frac{2A}{\sqrt{d}} \right) \frac{pm^2}{2}
\leq \left( 4\varepsilon + \varrho + \frac{A}{\sqrt{d}} \left( 4\sqrt{\varepsilon} + \varepsilon k + 2 \right) \right) \frac{pm^2}{2}
\leq \left( 4\varepsilon + \varrho + \frac{Ak}{\sqrt{d}} \right) \frac{pm^2}{2}.
\]

We want to show that

\[
e(R) > (1 - f(\alpha) + \tau) \left( \frac{k}{2} \right).
\]

(4.12)
For the sake of contradiction, suppose \( R \) has at most this many edge. Then the number of edges in regular and \( G' \)-dense pairs in \( G' \) is
\[
\leq (1 - f(\alpha) + \tau) \left( \binom{k}{2} \left( \frac{pm^2 + A\sqrt{dm}}{2} \right) \right)
\leq \left( 1 - f(\alpha) + \tau + \frac{Ak}{\sqrt{d}} \right) \left( \frac{pm^2}{2} \right) .
\] (4.13)

Hence, adding (4.11) and (4.13), we have
\[
e(G') \leq \left( 1 - f(\alpha) + t + 4e + \theta + \beta \right) \left( p\left( \binom{n}{2} - A\sqrt{dn} \right) \right)
\leq \left( 1 - f(\alpha) + \frac{\beta}{8} + \frac{\beta}{8} + \frac{A}{\nu\sqrt{d}} \right) \left( \frac{pm^2}{2} \right)
\leq \left( 1 - f(\alpha) + \frac{\beta}{2} \right) \left( \frac{pm^2}{2} \right) .
\] (4.14)

On the other hand, since the enemy may not delete more than an \((f(\alpha) - \beta)\) fraction of the edges in \( G \), we can derive that
\[
e(G') \geq (1 - f(\alpha) + \beta) \left( p\left( \binom{n}{2} - A\sqrt{dn} \right) \right)
= (1 - f(\alpha) + \beta) \left( 1 - \frac{1}{n} - \frac{2A}{\sqrt{d}} \right) \left( \frac{pm^2}{2} \right)
\geq (1 - f(\alpha) + \beta) \left( 1 - \frac{3A}{\sqrt{d}} \right) \left( \frac{pm^2}{2} \right)
\geq \left( 1 - f(\alpha) + \beta - \frac{4A}{\sqrt{d}} \right) \left( \frac{pm^2}{2} \right) \geq \left( 1 - f(\alpha) + \frac{\beta}{2} \right) \left( \frac{pm^2}{2} \right) .
\]
This contradicts (4.14) and hence (4.12) must hold. Combining (4.12) and (4.5) we conclude
\[
e(R) > (1 - f(\alpha) + \tau) \left( \binom{k}{2} \right) \geq \left( 1 - f \left( \alpha - \frac{\tau}{4} \right) + \frac{\tau}{2} \right) \left( \frac{k}{2} \right) .
\]

Applying Corollary 4.2 to \( R \), we conclude that, by our choice of \( k_0 \), \( R \) contains a cycle of length at least \((1 - \alpha + \tau/4)k\).

Starting with the long cycle in the reduced graph, let us now embed a cycle into the original graph \( G' \). Let \( C_t \) denote the cycle in the reduced graph \( R \) of length \( t \geq (1 - \alpha + \tau/4)k \). The entire construction is illustrated in Figure 4.1 on the following page. W.l.o.g. \( C_t = (V_1, V_2, \ldots, V_t) \).

We embed a cycle of length at least \((1 - \alpha)n\) into \( G' \) as follows. For all \( 1 \leq i \leq \lfloor t/2 \rfloor \), we simultaneously apply Lemma 4.4 to each pair \((V_{2i-1}, V_{2i})\). This yields sets \( X_{2i-1} \subseteq V_{2i-1} \) and \( X_{2i} \subseteq V_{2i} \), \(|X_{2i-1}| = |X_{2i}| = \varepsilon m\), such that every pair of vertices \((x_{2i-1}, x_{2i}) \in (X_{2i-1}, X_{2i})\) is connected by a path of length at least \(2(1 - 2\varepsilon)m\) using edges from \( G'[X_{2i-1}, X_{2i}] \). We can connect these paths by putting an arbitrary edge between \( X_{2i} \) and \( X_{2i+1} \) for all \( 1 \leq i < \lfloor t/2 \rfloor \). Note that
\[
e_{G'}(X_{2i}, X_{2i+1}) \geq (d_{2i,2i+1} - \varepsilon)p(\varepsilon m)^2 \geq 1
\]

since \((V_{2i}, V_{2i+1})\) is \((\varepsilon, p)\)-regular with density \(d_{2i,2i+1} \geq \varepsilon\).

If \( t \) is even, we close the cycle by putting an arbitrary edge between \( X_1 \) and \( X_t \). Otherwise we connect \( X_1 \) and \( X_{t-1} \) through one vertex in \( V_t \). Clearly, there are at least \((1 - 2\varepsilon)m\) vertices in \( V_t \) that are adjacent to some vertex in \( X_1 \) as well as some vertex in \( X_{t-1} \).
Thus, we have constructed a cycle of length at least
\[
\left(\frac{1 - \alpha + \tau/4}{2}\right) \cdot 2(1 - 2\mu)m \\
\geq ((1 - \alpha + \tau/4)k - 2) \cdot (1 - 2\mu)m \\
\geq \left(1 - \alpha + \tau/4 - \frac{2}{k}\right) \cdot (1 - 2\mu)(1 - \varepsilon)n \\
\geq (1 - \alpha + \tau/4 - 2\varepsilon)(1 - 2\mu)(1 - \varepsilon)n \\
\geq (1 - \alpha + \tau/4 - 4\varepsilon - 2\mu)n \\
\geq (1 - \alpha)n.
\]
CHAPTER 5

Asymmetric Ramsey Properties Involving Cliques

In this Chapter we study graphs with the property that there exists an edge coloring with \( r \) colors such that, in no color \( i \), there is a monochromatic complete graph of size \( \ell_i \). We prove a threshold function that determines whether the random graph \( G_{n,p} \) satisfies this property. Our approach is constructive. We give an algorithm that computes a valid coloring of \( G_{n,p} \) a.a.s. assuming that \( p \) is sufficiently small. Our proof of the corresponding upper bound relies on the KLR-Conjecture.

The results from this chapter will appear in [MSSS06].

5.1. Introduction

The edge-chromatic number \( \chi'(G) \) is one of the classical and well studied graph parameters. It is defined as the minimum number of colors \( k \) such that \( G \) allows for an edge-coloring with no pair of adjacent edges of the same color. Viewed from a slightly different perspective, one can equivalently define \( \chi'(G) \) as the minimum number of colors \( k \) such that \( G \) admits an edge-coloring avoiding monochromatic paths of length 2. This definition has led to a fruitful and well-studied area in deterministic graph theory. For given graphs \( G \) and \( F \), is there an edge-coloring with \( k \) colors of \( G \) that avoids a monochromatic copy of \( F \)?

It follows from Ramsey's celebrated result [Ram30] that every \( k \)-coloring of the edges of the complete graph on \( n \) vertices contains a monochromatic copy of \( F \) if \( n \) is sufficiently large. While that seems to rely on the fact that \( K_n \) is a very dense graph, Folkman [Fol70] and, in a more general setting, Nešetřil and Rödl [NR76] showed that there also exist locally sparse graphs \( G = G(F) \) with the property that every \( k \)-coloring of the edges of \( G \) contains a monochromatic copy of \( F \). By transferring the problem into a random setting, Rödl and Ruciński [RR95] showed that in fact such graphs \( G \) are quite frequent (cf. Theorem 2.19 on page 12).

In Theorem 2.19 the same graph \( F \) is forbidden in every color class. We can generalize this setup by allowing for \( k \) different forbidden graphs, one per color. Within classical Ramsey theory the study of these so-called asymmetric Ramsey properties led to many interesting questions (see e.g. [CG98]) and results, most notably the celebrated paper of Kim [Kim95] where he established an asymptotically sharp bound on the Ramsey number \( R(3, \ell) \), that is, the minimum number \( n \) such that every \( 2 \)-edge-coloring of the complete graph on \( n \) vertices contains either a red triangle or a blue clique of size \( \ell \).

Within the random setting only very little is known about asymmetric Ramsey properties. Let

\[ G \rightarrow (F_1, \ldots, F_k)^e \]

denote the property that in every edge-coloring of \( G \) with \( k \) colors, there exists a color \( i \) such that \( F_i \) is contained in the subgraph of \( G \) spanned by the edges which are assigned to \( i \). In [KK97] Kohayakawa and Kreuter proved the following result for cycles \( C_\ell \) of fixed length \( \ell \).
THEOREM 5.1 ([KK97]). Let $k \geq 2$ and $3 \leq \ell_1 \leq \cdots \leq \ell_k$ be integers. Then there exist constants $b, B > 0$ such that

$$
\lim_{n \to \infty} \mathbb{P}[G_{n,p} \to (C_{\ell_1}, \ldots, C_{\ell_k})^c] = \begin{cases} 
0 & \text{if } p \leq Bn^{-1/m_2(C_{\ell_k}, C_{\ell_1})} \\
1 & \text{if } p \geq Bn^{-1/m_2(C_{\ell_k}, C_{\ell_1})}
\end{cases},
$$

where

$$
m_2(C_{\ell_k}, C_{\ell_1}) := \frac{\ell_1}{\ell_1 - 2 + 1/m_2(C_{\ell_k})}.
$$

On the basis of their results in [KK97], Kohayakawa and Kreuter formulated the following conjecture.

CONJECTURE 5.2 ([KK97]). Let $F_1, F_2$ be graphs with $1 < m_2(F_1) \leq m_2(F_2)$. Then there exists a constant $b > 0$ such that for all $\varepsilon > 0$, we have

$$
\lim_{n \to \infty} \mathbb{P}[G_{n,p} \to (F_1, F_2)^c] = \begin{cases} 
0 & \text{if } p \leq (1 - \varepsilon)bn^{-1/m_2(F_1, F_2)} \\
1 & \text{if } p \geq (1 + \varepsilon)bn^{-1/m_2(F_1, F_2)}
\end{cases},
$$

where

$$
m_2(F_1, F_2) := \max \left\{ \frac{|E(H)|}{|V(H)| - 2 + 1/m_2(F_1)} : H \subseteq F_2 \land |V(H)| \geq 2 \right\}.
$$

The threshold function from Conjecture 5.2 can be motivated similarly to the threshold function from Theorem 2.19 on page 12. The expected number of copies of $F_2$ in $G_{n,p}$ with $p = \Theta\left(n^{-1/m_2(F_1, F_2)}\right)$ is

$$
\Theta\left(n^{1/m_2(F_2)}\right) = \Omega\left(n^{2 - 1/m_2(F_1)}\right).
$$

Since every edge-coloring of $G_{n,p}$ must avoid monochromatic copies of $F_2$ in color 2, there is at least one edge of color 1 in every subgraph of $G_{n,p}$ isomorphic to $F_2$. Select one such edge from each copy of $F_2$ arbitrarily. It is plausible that these edges span a graph $G'$ with edge density $\Omega\left(n^{-1/m_2(F_1)}\right)$ that satisfies certain pseudo-random properties. As it turns out, that seems just about the right density in order to embed a copy of $F_1$ into $G'$, no matter which edges were selected from the original graph.

In this chapter, we study the limiting probability of the event $G_{n,p} \to (K_{\ell_1}, \ldots, K_{\ell_k})^c$, where $K_{\ell_i}$ denotes a clique of size $\ell_i$, $1 \leq i \leq k$. As with every monotone property, we want to establish a threshold function $p_0$. Recall that a threshold phenomenon consists of two separate statements, the so-called 0-statement and the 1-statement, which are usually proved in entirely different ways. In our setting, the two statements are as follows. For the 1-statement one has to show that if $p \geq Bp_0$, a random graph $G_{n,p}$ asymptotically almost surely (a.a.s.) satisfies $G_{n,p} \to (K_{\ell_1}, \ldots, K_{\ell_k})^c$, i.e., every $k$-edge-coloring of $G_{n,p}$ contains at least one of the forbidden monochromatic cliques. For the 0-statement we suppose that $p \leq bp_0$ for some sufficiently small constant $b > 0$ and need to provide a $k$-edge-coloring of a random graph $G_{n,p}$ that avoids every forbidden clique $K_{\ell_i}$, $1 \leq i \leq k$, in the corresponding color class $i$.

A standard way of attacking the 1-statement, which was also pursued in [KK97], is via the sparse version of Szemerédi’s regularity lemma, which was independently observed by Kohayakawa [Koh97] and Rödl (unpublished). Using properties of regularity, one can find a monochromatic copy of a forbidden subgraph in the colored graph $G_{n,p}$. Unfortunately, generalizing this argument from cycles to cliques requires a proof of Conjecture 23 in [KLR97] of Kohayakawa, Luczak, and Rödl. This so-called KLR-Conjecture (cf. Section 2.4.4) formulates
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a probabilistic version of the classical embedding lemma for dense graphs. It implies many interesting extremal results for random graphs. In their monograph on random graphs [JLR00], Janson, Łuczak, and Ruciński consider the verification of this conjecture as one of the most important open questions in the theory of random graphs. Despite recent progress [GMS05], the conjecture is, in its full generality, still wide open. However, assuming that it is true, a proof of the 1-statement is routinely obtained as discussed in Section 5.4.

From an algorithmic or constructive point of view, the proof of the 0-statement is much more interesting. The way of proving it that was pursued in [RR93] and [KK97] is by contradiction. This approach shows the existence of a coloring, but provides no efficient way of obtaining the coloring from the proof. Our approach is constructive. We provide a (polynomial-time) algorithm that computes a valid coloring for graphs that satisfy certain properties. We employ techniques similar to those in [RR93] and [KK97] in order to prove that these properties a.a.s. hold in \( G_{n,p} \) with \( p \) sufficiently small. Indeed, the results in [RR93] yield that our algorithm also computes valid colorings of \( G_{n,p} \) in the symmetric case, unless the forbidden graph is one of a few special cases, e.g., a triangle. In fact, the symmetric case of triangles was solved in [LRV92] by different methods.

We prove the threshold from Conjecture 5.2 for cliques. As in Theorems 2.19 on page 12 and 5.1, the threshold function is slightly weaker than conjectured, allowing for distinct constants in the 0- and the 1-statement.

**Theorem 5.3 (Main Result).** Let \( k \geq 2 \) and \( \ell_1 \geq \cdots \geq \ell_k \geq 3 \) be integers. Then there exist constants \( b, B > 0 \) such that

\[
\lim_{n \to \infty} P \left[ G_{n,p} \to (K_{\ell_1}, \ldots, K_{\ell_k})^c \right] = \begin{cases} 
0 & \text{if } p < bn^{-1/m_2(K_{\ell_2},K_{\ell_1})} \\
1 & \text{if } p \geq Bn^{-1/m_2(K_{\ell_2},K_{\ell_1})},
\end{cases}
\]

where

\[
m_2(K_{\ell_2}, K_{\ell_1}) := \frac{\binom{\ell_2}{2}}{\ell_1 - 2 + \frac{1}{m_2(K_{\ell_2})}},
\]

and the 1-statement holds provided Conjecture 23 in [KLR97] is true for \( K_{\ell_2} \).

The proof of the 0-statement of Theorem 5.3 is given in Section 5.2 under the additional assumption that \( \ell_2 > 3 \). For \( \ell_2 = 3 \), we face additional difficulties, which are discussed in Section 5.3.

**5.1.1. Related work.** Ramsey properties of random graphs were first studied in [LRV92]. The authors considered vertex-colorings of random graphs without any copy of a forbidden graph \( F \) in the graph induced by any color class. They presented a result analogous to Theorem 2.19 for vertex-colorings and also solved the case of edge-colorings for triangles. As it turned out, solving the general edge-coloring case required considerably more involved techniques than those employed in the vertex-coloring case.

The same situation can be observed regarding asymmetric Ramsey properties of random graphs. While the vertex case was resolved in full generality by Kreuter [Kre96], the result involving cycles [KK97] was so far the only one known for edge-colorings.

**5.2. An algorithm for computing valid edge colorings**

Suppose \( G = G_{n,p} \) with \( p \leq bn^{-1/m_2(K_{\ell_2},K_{\ell_1})} \) is given. In order to provide a valid coloring of \( G \), it suffices to compute a 2-coloring of \( E(G) \) such that there is no copy of \( K_{\ell_1} \) in color 1 and no copy of \( K_{\ell_2} \) in color 2. That implies the 0-statement of Theorem 5.3 also for \( k \)-colorings. Hence,
we focus on 2-colorings and abbreviate $\ell_1$ by $r$ and $\ell_2$ by $\ell$ in the following. For the rest of this section, $r > \ell > 3$ shall remain fixed.

We describe an algorithm that finds a valid edge-coloring of $G$ a.a.s. The basic idea of the algorithm is to remove edges from the graph successively. An edge $e$ is deleted from $G$ if there are no two cliques of size $\ell$ and $r$ respectively that intersect exactly on $e$. Assuming that all edges of $G$ can be removed in this way, it is easy to create a valid coloring by inserting them in the reverse order one by one, always assigning a valid color instantly. The actual algorithm is more complex since sometimes one has to forget about the existence of certain small cliques in order to remove really all edges from $G$. As we shall see, we can easily deal with those cliques later.

In order to simplify notation, we define, for any graph $G$, the families

$$L_G := \{L \subseteq G : L \cong K_{\ell}\} \quad \text{and} \quad R_G := \{R \subseteq G : R \cong K_r\}$$

of all $\ell$-cliques and $r$-cliques in $G$ respectively. Furthermore, we introduce the family $L_{G}^{*} \subseteq L_G$ defined as

$$L_{G}^{*} := \{L \in L_G : \forall e \in E(L) \exists R \in R_G \text{ s.t. } E(L) \cap E(R) = \{e\}\}.$$ 

The algorithm is given in Figure 5.1. Note that edges are removed from and inserted into a working copy $G' = (V, E')$ of $G$. The local variable $L$ contains the same elements as $L_G$ up to the first execution of lines 12-13. In general, we have $L \subseteq L_{G'}$.

**LEMMA 5.4.** If algorithm Asym-Edge-Col terminates without error, then it has indeed found a valid coloring of $G$.

**PROOF.** First, we argue that the algorithm never creates a blue copy of $K_{\ell}$. Observe that every copy of $K_{\ell}$ that exists in $G'$ is pushed on the stack in the first loop. Therefore, in the execution of the second loop, the algorithm must check the coloring of every such copy. Due to the order of the elements on the stack, each check is performed only after all edges of the corresponding clique were inserted and colored. For every blue copy of $K_{\ell}$, one particular edge is recolored to red. Since red edges are never flipped back to blue, no blue copy of $K_{\ell}$ can occur in the coloring found by the algorithm.

It remains to prove that the assignment of color red to some edge by the algorithm can never create an entirely red copy of $K_r$. By the condition on $f$ in line 22 of the algorithm, at the very moment there exists no copy of $K_r$ in $G'$ that intersects with $L$ exactly in $f$. So there is either no $K_r$ containing $f$ at all, or every such copy contains also another edge from $L$. In the latter case, those copies cannot become entirely red since $L$ is entirely blue.

Our last step is to show that the edge $f$ in line 22 always exists. Since the second loop inserts edges into $G'$ in the reverse order in which they were deleted during the first loop, when we select $f$ in line 22, $G'$ has the same structure as at the moment when $L$ was pushed on the stack. This could have happened either in line 7, when there exists no $r$-clique in $G'$ that intersects with $L$ on some particular edge $e \in E(L)$, or in line 12, when $L$ satisfies the condition of the if-clause in line 11. In both cases we have $L \notin L_{G'}$, and, therefore, there exists an edge $e \in E(L)$ such that all currently existing copies of $K_r$ do not intersect with $L$ exactly in $e$. 

It remains to prove the following lemma.

**LEMMA 5.5.** There exists a constant $b = b(\ell, r) > 0$ such that for $p \leq bn^{-1/\max(K_{\ell}, K_r)}$, a.a.s. algorithm Asym-Edge-Col terminates on $G_{n,p}$ without error.
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ASYM-EDGE-COL\((G = (V, E))\)

1. \(s \leftarrow \text{empty-stack}()\)
2. \(E' \leftarrow E\)
3. \(L \leftarrow L_G\)
4. \textbf{while} \(E' \neq \emptyset\)
5. \textbf{do} if \(\exists e \in E'\) s.t. \(\exists (L, R) \subseteq L \times R_{G'} = (V, E') : E(L) \cap E(R) = \{e\}\)
6. \textbf{then} for each \(L \in L : e \in E(L)\)
7. \textbf{do} \(s.\text{push}(L)\)
8. \(L.\text{remove}(L)\)
9. \(s.\text{push}(e)\)
10. \(E'.\text{remove}(e)\)
11. \textbf{else} if \(\exists L \in L \setminus L_{G'} = (V, E')\)
12. \textbf{then} \(s.\text{push}(L)\)
13. \(L.\text{remove}(L)\)
14. \textbf{else} error "stuck"
15. \textbf{while} \(s \neq \emptyset\)
16. \textbf{do} if \(s.\text{top}()\) is an edge
17. \textbf{then} \(e \leftarrow s.\text{pop}()\)
18. \(e.\text{set-color}(\text{blue})\)
19. \(E'.\text{add}(e)\)
20. \textbf{else} \(L \leftarrow s.\text{pop}()\)
21. \textbf{if} \(L\) is entirely blue
22. \textbf{then} \(f \leftarrow \text{any } e \in E(L)\) s.t.
23. \(\exists R \in R_{G'} = (V, E') : E(L) \cap E(R) = \{e\}\)
24. \(f.\text{set-color}(\text{red})\)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{The implementation of algorithm ASYM-EDGE-COL.}
\end{figure}

5.2.1. Proof of Lemma 5.5. We prove Lemma 5.5 by providing an algorithm \textsc{Grow} that, if \textsc{Asym-Edge-Col} fails on an arbitrary graph \(G\), explicitly computes a subgraph \(F \subseteq G\) which is either too large or too dense to appear in \(G_{n,p}\) with \(p\) as in the lemma. More precisely, we shall show that for any graph \(F\) that \textsc{Grow} may return, the probability that \(F\) appears in \(G_{n,p}\) is small compared to the size of \(\mathcal{F}\), the class of all graphs that \textsc{Grow} may return. It follows that \(G_{n,p}\) a.a.s. does not contain any of these graphs, which implies Lemma 5.5 by contradiction. Note that we employ algorithm \textsc{Grow} only for proving the lemma. It does not contribute to the running time of algorithm \textsc{Asym-Edge-Col}.

In order to formulate algorithm \textsc{Grow}, we need some definitions. Let

\[ \gamma = \gamma(\ell, r) := \frac{1}{m_2(K_\ell, K_r)} - 2/ (\ell + r - 3) \quad (5.1) \]

Note that for \(r > \ell > 3\), we have

\[ \gamma(\ell, r) = \frac{2((\ell^2 - 3\ell - 2)r - 2\ell(\ell - 3))}{(r - 1)(\ell + 1)(\ell + r - 3)} > 0 \quad . \]

Remark 5.6. Observe that \(\gamma(3, r)\) is negative for \(r \geq 3\). This is why we have to modify our proof for the case \(\ell = 3\), see Section 5.3. The proof we give here also covers the symmetric case of \(\ell = r \geq 5\) since then \(\gamma(\ell, \ell) > 0\).
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\text{Grow}(G' = (V, E))

1. \( i \leftarrow 0 \)
2. \( F_0 \leftarrow \text{any } R \in \mathcal{R}_G \)
3. \textbf{while } i < \log(n) \land \lambda(F_i) > -\gamma
4. \textbf{do if } \exists R \in \mathcal{R}_{G'} \setminus \mathcal{R}_F \text{ s.t. } |V(R) \cap V(F_i)| \geq 2
5. \textbf{then } F_{i+1} \leftarrow F_i \cup R
6. \text{else } e \leftarrow \text{Eligible-Edge}(F_i)
7. \quad F_{i+1} \leftarrow \text{Extend-L}(F_i, e, G')
8. \quad i \leftarrow i + 1
9. \text{return } F_i

\text{Extend-L}(F, e, G')

1. \( L \leftarrow \text{any } L \in \mathcal{L}_{G'} : e \in E(L) \)
2. \( F' \leftarrow F \cup L \)
3. \textbf{for each } e' \text{ in } E(L) \setminus E(F)
4. \textbf{do } R_{e'} \leftarrow \text{any } R \in \mathcal{R}_{G'} : E(L) \cap E(R) = \{ e' \}
5. \quad F' \leftarrow F' \cup R_{e'}
6. \text{return } F'

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{algorithm_grow}
\caption{The implementation of algorithm Grow.}
\end{figure}

For any graph \( F \), let

\[ \lambda(F) := v(F) - e(F)/m_2(K_t, K_r) \]

The definition of \( \lambda(F) \) is motivated by the fact that the number of copies of \( F \) in \( G_n,p \) with \( p = bn^{-1/m_2(K_t, K_r)} \) has order of magnitude

\[ n^{v(F)p^{e(F)}} = n^{\lambda(F)} \]

For any graph \( F \), we call an edge \( e \in E(F) \) eligible for extension if it satisfies

\[ \exists (L, R) \in \mathcal{L}_F \times \mathcal{R}_F \text{ s.t. } E(F) \cap E(R) = \{ e \} \]

The implementation of algorithm Grow is shown in Figure 5.2. The intended input is the graph \( G' \subseteq G \) after \text{Asym-Edge-Col} got stuck. It proceeds as follows: the seed \( F_0 \) is any copy of \( K_r \) in \( G' \). In every iteration \( i \), it extends \( F_i \) to \( F_{i+1} \) by adding new vertices and edges to it. As long as there are copies of \( K_r \) in \( G' \) that intersect with \( F_i \) in at least two vertices but not in all edges, it greedily adds those to \( F_i \). If there are no such copies, it calls a subroutine \text{Eligible-Edge} that takes \( F_i \) as input and returns an edge \( e \in E(F_i) \) eligible for extension that is unique up to isomorphism of \( F_i \), i.e., in such a way that for any two isomorphic graphs \( F \) and \( F' \), there exists an isomorphism \( \varphi \) with \( \varphi(F) = F' \) such that

\[ \text{Eligible-Edge}(F') = \varphi(\text{Eligible-Edge}(F)) \]

Note that this implies in particular that \( e \) depends only on the graph \( F_i \) and not on the surrounding graph \( G' \). Clearly, one way to implement this procedure would be keeping a large table of representatives for all isomorphism classes of graphs with up to \( n \) vertices and edges that maps to each entry one particular edge eligible for extension. Since we only want to show the existence of a certain structure in \( G' \) and do not care about complexity issues here, the actual implementation of that procedure is irrelevant. Procedure \text{Extend-L} then adds a graph \( L \in \mathcal{L}_{G'} \) that contains the edge \( e \) returned by \text{Eligible-Edge} to \( F_i \). It glues to each new edge \( e' \in E(L) \setminus E(F_i) \) a
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graph $R_{e'} \in \mathcal{R}_{G'}$ that intersects with $L$ only in $e'$. The algorithm stops and returns $F_i \subseteq G' \subseteq G$ as soon as $\lambda(F_i) \leq -\gamma$ or $i \geq \log(n)$.

We shall argue that $\text{GROW}$ terminates without error, i.e., that function $\text{ELIGIBLE-EDGE}$ always finds an edge eligible for extension, and that $\text{EXTEND-L}$ always finds suitable graphs $L$ and $R_{e'}$, $e' \in E(L)$. Let us consider the properties of $G'$ when $\text{ASYM-EDGE-COL}$ gets stuck. As the condition in line 5 of $\text{ASYM-EDGE-COL}$ fails, $G'$ is in the family

$$\mathcal{C}(\ell, r) := \{G = (V, E) : \forall e \in E(G) \exists (L, R) \in \mathcal{L}_G \times \mathcal{R}_G \text{ s.t. } E(L) \cap E(R) = \{e\}\}.$$ 

In fact, every edge of $G'$ is contained in a copy $L \in \mathcal{L}$, and as the condition in line 11 fails as well, $G'$ is even in the smaller family

$$\mathcal{C}^*(\ell, r) := \{G = (V, E) : \forall e \in E(G) \exists L \in \mathcal{L}_G^* \text{ s.t. } e \in E(L)\} \subseteq \mathcal{C}(\ell, r).$$

Claim 5.7. Algorithm $\text{GROW}$ terminates without error on any nonempty input graph $G' \in \mathcal{C}^*(\ell, r)$. Moreover, every iteration of the while-loop adds at least one edge to $F$.

Proof. Suppose there is no edge in $F_i$ that is eligible for extension. Then we have $F_i \in \mathcal{C}(\ell, r)$ by definition. This implies that every vertex $v \in V(F_i)$ has degree at least $(\ell - 1) + (r - 1) - 1 = \ell + r - 3$, i.e., $e(F_i)/v(F_i) \geq (\ell + r - 3)/2$. It follows that

$$\lambda(F_i) \leq e(F_i) \left(\frac{2}{\ell + r - 3} - \frac{1}{m_2(K_{\ell}, K_{r})}\right) = -e(F_i)\gamma \leq -\gamma,$$

where we used that $\gamma = \gamma(\ell, r)$ is positive. Consequently, $\text{GROW}$ terminates in line 3 without calling $\text{ELIGIBLE-EDGE}$. Hence, $\text{ELIGIBLE-EDGE}$ always returns an edge eligible for extension when called from $\text{GROW}$.

Property $\mathcal{C}^*(\ell, r)$ of $G'$ guarantees the existence of suitable graphs $L$ and $R_{e'}$, $e' \in E(L)$, when $\text{EXTEND-L}$ is called. Moreover, by definition of $\mathcal{L}_G^*$, there exists, in particular, $R_s \in \mathcal{R}_G'$ such that $e$ is the intersection of $R_s$ and $L$. When $\text{EXTEND-L}(F_i, e, G')$ is called, $R_s$ has already been added to $F$ during a previous iteration in lines 4 and 5 of $\text{GROW}$. Hence, the $L$ returned in line 1 of $\text{EXTEND-L}$ is not contained in $F$, as otherwise $e$ would not be eligible for extension. On the other hand, it is clear that an $R$ found in line 4 adds at least one new edge to $F$. Together this proves that every iteration adds at least one edge to $F$. 

Now, we will consider the evolution of $F$ in more detail. We say that iteration $i$ of the while-loop in procedure $\text{GROW}$ is non-degenerate if we have the following assertions:

- The condition in line 4 evaluates to false and, hence, $\text{EXTEND-L}$ is called.
- In line 2 of $\text{EXTEND-L}$, we have $V(F) \cap V(L) = e$.
- In every execution of line 5 of $\text{EXTEND-L}$, we have $V(F') \cap V(R_{e'}) = e'$.

Otherwise, we call iteration $i$ degenerate. In non-degenerate iterations, the graph $F_{i+1}$ is uniquely defined up to isomorphism for a given $F_i$, depending only on the implementation of subroutine $\text{ELIGIBLE-EDGE}$, which determines the position where to attach the next $K_\ell$. A graph $F_2$ that results from two non-degenerate iterations is depicted in Figure 5.3 for $r = 6$ and $\ell = 4$. The little dashed circle identifies $F_0$. The greater dotted circle circumscribes $F_1$. Observe that the structures which are added in every step are isomorphic.
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Figure 5.3. A graph $F_2$ resulting from two non-degenerate iterations for $r = 6$ and $\ell = 4$. The two central copies of $K_4$ are shaded.

Claim 5.8. If iteration $i$ of the while-loop in procedure GROW is non-degenerate, we have

$$\lambda(F_{i+1}) = \lambda(F_i).$$

Proof. In a non-degenerate iteration, the graph $L$ added in line 1 of EXTEND-L contributes $\ell-2$ new vertices and $\binom{\ell}{2} - 1$ new edges to $F$. Each of these new edges then is replaced by a copy of $K_r$. Hence, we have

$$\lambda(F_{i+1}) - \lambda(F_i) = \ell - 2 + \left( \binom{\ell}{2} - 1 \right) (r - 2) - \left( \binom{\ell}{2} - 1 \right) \frac{r}{m_2(K_\ell, K_r)}.$$

In a degenerate iteration $i$, the structure of $F_{i+1}$ does not only depend on $F_i$, but varies with the structure of $G'$. Suppose that $F_i$ is extended with an $r$-clique in line 5. This $R$ can intersect with $F_i$ in virtually every possible way. Moreover, there may be several copies of $K_r$ which satisfy the condition in line 4. The same is true for the graphs added in lines 2 and 5 of EXTEND-L. Thus, degenerate iterations cause difficulties since they enlarge the family of graphs that algorithm GROW may potentially return. However, we will show that at most a constant number of degenerate iterations can occur before the algorithm terminates. This allows us to control the
number of non-isomorphic graphs that can be the output of Grow. The key to proving this is the next claim.

**Claim 5.9.** There exists a constant $\kappa = \kappa(\ell, r) > 0$ such that if iteration $i$ of the while-loop in procedure Grow is degenerate, we have
\[
\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa.
\]

The proof of Claim 5.9 is the main technical part of our work and beyond the scope of this extended abstract. In combination with Claim 5.8, it yields the next claim, which in turn leads to a polylogarithmic bound on the number of non-isomorphic graphs that Grow can return.

**Claim 5.10.** There exists a constant $m_0 = m_0(\ell, r)$ such that algorithm Grow performs at most $m_0$ degenerate iterations before it terminates, regardless of the input instance $G'$.

**Proof.** An easy calculation yields that $\lambda(F_0) = \lambda(K_r) = 2 - 2/(\ell + 1)$. The value of the function $\lambda$ remains unchanged in every non-degenerate iteration due to Claim 5.8. However, Claim 5.9 yields a constant $\kappa$, which depends solely on $\ell$ and $r$, such that
\[
\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa
\]
for every degenerate iteration $i$. Hence, after at most
\[
m_0 := \frac{\lambda(F_0) + \gamma}{\kappa}
\]
degenerate iterations, we have $\lambda(F_i) \leq -\gamma$, and the algorithm terminates. \qed

Let $\mathcal{F}(\ell, r, n)$ denote a family of representatives for the isomorphism classes of all graphs that can be the output of Grow with parameters $n$ and $\gamma(\ell, r)$ on any input instance $G'$.

**Claim 5.11.** There exists $C = C(\ell, r)$ such that $|\mathcal{F}(\ell, r, n)| \leq \log(n)^C$.

**Proof.** For $t \geq d \geq 0$, let $\mathcal{F}(t, d)$ denote a family of representatives for the isomorphism classes of all graphs $F_t$ that algorithm Grow can generate after exactly $t$ iterations if it performs exactly $d$ degenerate iterations along the way, and let $f(t, d) := |\mathcal{F}(t, d)|$ denote its cardinality.

Observe that in every iteration, we add at most
\[
\ell - 2 + \binom{\ell}{2}(r - 2) =: K
\]
new vertices to $F$, which is exactly the number of vertices added in a non-degenerate iteration. Hence, we have $v(F_t) \leq r + Kt$. It also follows that in every iteration, the new edges $E(F_{t+1}) \setminus E(F_t)$ span a graph from $\mathcal{G}_K$, where $\mathcal{G}_K$ denotes the set of all graphs on at most $K$ vertices. $F_{t+1}$ is uniquely defined if one specifies $G \in \mathcal{G}_K$, the number $y$ of vertices in which $G$ intersects $F_t$, and two ordered lists of vertices from $G$ and $F_t$ respectively of length $y$, which specify the mapping of the intersection vertices from $G$ into $F_t$. Thus, the number of ways to extend $F_t$ is bounded from above by
\[
\sum_{G \in \mathcal{G}_K} \sum_{y=2}^{v(G)} v(G)^y(v(F_t))^y \leq C_1(r + Kt)^K \leq t^{C_2} \leq \log(n)^{C_2}
\]
for constants $C_1, C_2$ only depending on $\ell$ and $r$.

As the selection of the edge to be extended is unique up to isomorphism of $F$, the evolution of $F$ is uniquely defined if there are no degenerate iterations along the way, regardless of the input
instance $G'$. This implies in particular that $f(t,0) = 1$ for all $t$, and more generally that for $t \geq d \geq 0$

$$f(t,d) \leq \binom{t}{d} (\log(n)^{C_2})^d \leq \log(n)^{(C_2+1)d}.$$  

Here the binomial coefficient corresponds to the choice of the $d$ degenerate iterations. We conclude from Claim 5.10 that there exists a constant $C = C(\ell, r) > 0$ such that

$$|\mathcal{F}(\ell, r, n)| \leq \sum_{t=0}^{\log(n)} \sum_{d=0}^{m_0} f(t,d) \leq (\log(n) + 1)(m_0 + 1)\log(n)^{(C_2+1)m_0} \leq \log(n)^C,$$

for $n$ sufficiently large.

Claim 5.12. There exists a constant $b > 0$ such that for every sequence

$$p = p(n) \leq bn^{-1/m_2(K_t,K_r)},$$

$G_{n,p}$ does not contain any graph from $\mathcal{F}(\ell, r, n)$ a.a.s.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the families of graphs that may be output by algorithm Grow if it terminates due to the first or the second condition in line 3 respectively. Due to Claim 5.11 we have a polylogarithmic bound on the cardinality of $\mathcal{F} = \mathcal{F}(\ell, r, n) = \mathcal{F}_1 \cup \mathcal{F}_2$, and Claims 5.8 and 5.9 imply that $\lambda(F_1)$ is non-increasing. It follows that for $b := e^{-\lambda(F_0)-\gamma}$, the expected number of copies of graphs from $\mathcal{F}$ in $G_{n,p}$ with $p \leq bn^{-1/m_2(K_t,K_r)}$ is bounded from above by

$$\sum_{F \in \mathcal{F}} n^{\epsilon(F)p^{\lambda(F)}} = \sum_{F \in \mathcal{F}} b^{\lambda(F)} n^{\lambda(F)} \leq \sum_{F \in \mathcal{F}_1} e^{(-\lambda(F_0)-\gamma)\log(n)} n^{\lambda(F)} + \sum_{F \in \mathcal{F}_2} n^{-\gamma} \leq (\log(n))^C n^{-\gamma} = o(1),$$

which implies the claim due to Markov's inequality. Here we used again that $\gamma$ is positive. Note that crucially, for all $F \in \mathcal{F}_1$, we have $e(F) \geq \log(n)$ since $F$ was generated in $[\log(n)]$ iterations, each of which introduces at least one new edge.

Suppose now that algorithm Asym-Edge-Col applied to $G_{n,p}$ with $p$ as claimed gets stuck, and consider $G' \subseteq G$ at this moment. The call to Grow($G'$) returns a copy of a graph $F \in \mathcal{F}(\ell, r, n)$ that is contained in $G'$. But we just proved that a.a.s. we have $F \not\subseteq G_{n,p}$, which contradicts our assumption. This proves that Asym-Edge-Col finds a valid coloring of $G_{n,p}$ with $p \leq bn^{-1/m_2(K_t,K_r)}$ a.a.s.

5.2.2. Proof of Claim 5.9. We say that Algorithm Grow encounters a degeneracy of type 1 if the condition in line 4 evaluates to true.

Lemma 5.13. There exists a constant $\kappa_1 = \kappa_1(\ell, r) > 0$ such that if procedure Grow encounters a degeneracy of type 1 in iteration $i$ of the while-loop, we have

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa_1.$$
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Proof. The claim is trivial if $R$ overlaps with $F_i$ in $r$ vertices since we have to add at least one edge to $F_i$ but no vertex, and therefore $\kappa_1 \geq 1/m_2(K_\ell, K_r)$. Hence, assume that $R$ extends $F_i$ by $x$, $1 \leq x \leq r - 2$ vertices. Then it must add at least \( \left( \frac{x}{2} \right) + x(r - x) \) edges to $F_i$, and thus we have

$$\lambda(F_{i+1}) - \lambda(F_i) \leq x - \left( \frac{x}{2} \right) + x(r - x)/m_2(K_\ell, K_r) =: g(x).$$

We can rewrite $g(x)$ to

$$g(x) = \frac{x}{r(r-1)(\ell+1)} \left( ((\ell(r-2)+\ell)x - r^2 - (r(r-4)+2)\ell \right).$$

The function $g(x)$ is a quadratic parabola, which attains zero for $x_1 = 0$ and

$$\frac{x_2}{r_2} = \frac{r_2 + (r(r-4)+2)\ell}{(r-2)+r}.\quad \text{Since} \quad x_2 - (r-2) = \frac{2(r-\ell)}{\ell(r-2)+r} > 0.$$

$g(x)$ is negative in the entire interval $[1, r-2]$ and attains its maximum at either end of this interval. In fact, we have

$$g(r-2) - g(1) = \frac{(r-3)(\ell-1)}{(r-1)(l+1)} > 0.$$

Hence, it is safe to choose $\kappa_1$ as

$$\kappa_1 = \min(1/m_2(K_\ell, K_r), -g(r-2)).$$

We say that Algorithm Grow encounters a degeneracy of type 2 in iteration $i$ of the while-loop if, during the call to $\text{EXTEND-L}(F_i, e, G')$, the graph $L$ found in line 1 overlaps in more than two vertices with $F_i$, or if there exists an edge $e' \in E(L) \setminus E(F)$ such that the graph $R_{e'}$ found in line 4 overlaps in more than two vertices with $F'$.  

Lemma 5.14. There exists a constant $\kappa_2 = \kappa_2(\ell, r) > 0$ such that if procedure Grow encounters a degeneracy of type 2 in iteration $i$ of the while-loop, we have

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa_2.$$

Proof. Consider the graph $F := F_i$ that is passed to $\text{EXTEND-L}$ and the output of this procedure $F' := F_{i+1}$. Suppose we extend $F$ by a graph $L$ isomorphic to $K_\ell$ in iteration $i$. Let $x$ denote the number of new vertices that $L$ contributes to $F_i$, i.e., $x = |V(L) \setminus V(F)|$. Observe that $x \leq l - 2$ since $L$ must overlap with $F$ in at least one edge $e$, which was determined by subroutine ELIGIBLE-EDGE.

As all edges of $L \in L_{C_\ell}^*$ are covered by copies of $K_r$, and since the condition in line 4 of Grow evaluates to false in iteration $i$, we have

$$\left( \frac{V(F) \cap V(L)}{2} \right) \subseteq E(F). \quad (5.2)$$

Moreover, we have $x \geq 1$ due to Claim 5.7.

We need to show that there exists a constant $\kappa_2 > 0$ such that

$$\lambda(F) - \lambda(F') = v(F) - v(F') - (e(F) - e(F'))/m_2(K_\ell, K_r) \geq \kappa_2. \quad (5.3)$$

As we have argued before, the structure of $F'$ would be uniquely defined up to isomorphism just by the structure of $F$ if iteration $i$ was non-degenerate. Let $F^*$ denote the output of such
a virtual non-degenerate iteration. We transform any degenerated outcome \( F' \) into \( F^* \) in three steps

\[
F' = F^0 \overset{(i)}{\rightarrow} F^1 \overset{(ii)}{\rightarrow} F^2 \overset{(iii)}{\rightarrow} F^3 = F^*,
\]
each time carefully resolving certain kinds of degeneracies. Using a telescoping summation, we may rewrite \( \lambda(F) - \lambda(F') \) to

\[
v(F') - v(F^*) = (e(F) - e(F^*)) / m_2(K_\ell, K_r)
+ \sum_{j=1}^{3} \left( v(F^j) - v(F^{j-1}) - (e(F^j) - e(F^{j-1})) / m_2(K_\ell, K_r) \right)
= \sum_{j=1}^{3} \left( v(F^j) - v(F^{j-1}) - (e(F^j) - e(F^{j-1})) / m_2(K_\ell, K_r) \right),
\]

where the equality follows from \( \lambda(F) - \lambda(F^*) = 0 \) due to Claim 5.8. We shall show that for all \( 1 \leq j \leq 3 \), we have

\[
v(F^j) - v(F^{j-1}) - (e(F^j) - e(F^{j-1})) / m_2(K_\ell, K_r) \geq \kappa_2
\]

(5.4)

for a suitable \( \kappa_2 = \kappa_2(\ell, r) > 0 \), provided that \( F^j \) and \( F^{j-1} \) are not isomorphic. In each step \( j, 1 \leq j \leq 3 \), we consider a different structural property of \( F' \) resulting from a degeneracy of type 2. As we do not know the exact structure of \( F' \), not every step \( j \) will necessarily modify the structure of \( F^{j-1} \). However, at least for one \( j \), \( F^j \) is not isomorphic to \( F^{j-1} \) since \( F' \) is not isomorphic to \( F^* \). This suffices to conclude (5.3) from (5.4).

Let us carefully analyze the graph that procedure Extend-L appends to \( F \) when a degeneracy of type 2 occurs. First, it extends \( F \) by \( L \cong K_\ell \). Let \( L_x \subseteq L \) denote the graph where all edges in \( E(F) \cap E(L) \) were removed from \( L \). Clearly, we have \( v(L_x) = \ell \) and

\[
e(L_x) = \left( \frac{\ell}{2} \right) - \left( \frac{\ell - x}{2} \right) = \left( \frac{x}{2} \right) + x(\ell - x) .
\]

To every edge \( e' \in E(L_x) \), Extend-L glues a graph \( R_{e'} \cong K_r \) in line 5, which intersects with \( L \) only in \( e' \). (Recall that such a graph always exists since \( L \in L^*_{r, r} \).

Let \( L^R \) denote the graph

\[
L_x \cup \bigcup_{e' \in E(L_x)} R_{e'}.
\]

Due to (5.2) \( F' \) can be written as \( F \cup L^R_x \), and we have \( E(F') = E(F) \cup E(L^R_x) \). Therefore, we have

\[
e(F') - e(F) = e(L^R_x).
\]

We may also conclude that \( |V(F) \cap V(L_x)| = \ell - x \) and thus

\[
v(F') - v(F) = v(L^R_x) - |V(F) \cap V(L^R_x)|
= v(L^R_x) - (\ell - x) - |V(F) \cap (V(L^R_x) \setminus V(L_x))| .
\]

Assuming \( |V(F) \cap (V(L^R_x) \setminus V(L_x))| > 0 \), we apply transformation (i) mapping \( F^0 \) to \( F^1 \). The transformation introduces, for every vertex \( v \in V(F) \cap (V(L^R_x) \setminus V(L_x)) \), an additional vertex \( v' \). All edges incident to \( v \) belonging to \( E(F) \) remain connected to \( v \) and those belonging to \( E(L^R_x) \) are redirected to \( v' \). Thus, we disconnect the vertices in \( V(L^R_x) \setminus V(L_x) \) from the vertices in \( V(F) \). In \( L^R_x \) we replace the vertices \( V(F) \cap (V(L^R_x) \setminus V(L_x)) \) by the new vertices. Since there are no
edges in $E(L_x^R) \cap E(F)$, the output of this operation is uniquely defined, and the structure of $L_x^R$ is not affected. Hence, we have

\[
v(F^1) - v(F^0) - \left( e(F^1) - e(F^0) \right) / m_2(K_r, K_r) = |V(F) \cap (V(L_x^R) \setminus V(L_x))| \geq 1.
\]

By construction $F^1$ satisfies $V(F) \cap (V(L_x^R) \setminus V(L_x)) = \emptyset$. Next, we shall disconnect copies of $K_r$ in $L_x^R$ that mutually intersect. Since transformation (i) does not change the structure of $L_x^R$, that graph belongs to the following family. Let $K_r^x$ denote the graph isomorphic to $L_x$, i.e., a clique on $\ell$ vertices from which all edges of a clique on $\ell - x$ vertices were removed.

\[
\mathcal{H}(\ell, r, x) = \left\{ H_x = (V \cup U, E \cup D) : \right.
\]
\[
\text{H}_x \text{ is a minimal graph such that } (V, E) \cong K_r^x, \text{ and for all } e' \in E, \text{ there are sets } U(e') \subseteq U \text{ and } D(e') \subseteq D \text{ with}
\]
\[
\left( e' \cup U(e'), \{ e' \} \cup D(e') \right) \cong K_r.
\]

We refer to $V$ and $E$ as the inner vertices and edges respectively, which form the inner copy of $K_r^x$. Every edge $e' \in E$ forms together with its associated outer vertices $U(e')$ and outer edges $D(e')$ an outer copy of $K_r$. Hence, $|U(e')| = r - 2$ and $|D(e')| = \left( \binom{\ell}{2} \right) - 1$. We take $H_x$ as a minimal element with respect to subgraph inclusion, i.e., $H_x$ does not have a subgraph which satisfies the same properties. This ensures in particular that $\mathcal{H}(\ell, r, x)$ is finite. There is a uniquely defined member $H_x^* \in \mathcal{H}(\ell, r, x)$ which satisfies $U(e_1) \cap U(e_2) = \emptyset$ and $D(e_1) \cap D(e_2) = \emptyset$ for all pairs $e_1, e_2 \in E$. The following lemma, which relates the average degree of $H_x^*$ to that of all other members of the family $\mathcal{H}(\ell, r, x)$, is crucial for our proof.

**Lemma 5.15.** For all integers $r \geq \ell \geq 3$, all $1 \leq x \leq \ell - 2$, and every graph $H_x \subseteq \mathcal{H}(\ell, r, x)$, we have

\[
\frac{e(H_x)}{v(H_x)} \geq \frac{e(H_x^*)}{v(H_x^*)},
\]

where $H_x^*$ denotes the unique member of $\mathcal{H}(\ell, r, x)$ with pairwise disjoint outer cliques.

We postpone the proof of Lemma 5.15 for later since it is quite bulky. Transformation (ii) proceeds by replacing $L_x^R$ in $F^1$ by a copy $L_x^{R*}$ of $H_x^*$. Due to the assumption that $V(F) \cap (V(L_x^R) \setminus V(L_x)) = \emptyset$, $L_x^R$ intersects $F$ in exactly $\ell - x$ anchor points. We obtain the graph $F^2$ by removing $L_x^R$ and attaching $L_x^{R*}$ to the very same anchor points. $F^2$ is uniquely defined up to isomorphism due to the symmetries in $L_x^{R*}$. Observe that if $L_x^R$ is not isomorphic to $H_x^*$, the minimality condition in $\mathcal{H}(\ell, r, x)$ yields that $v(L_x^R) < v(L_x^{R*})$. That implies in combination
with Lemma 5.15 that

\[
\begin{align*}
    v(F^2) - v(F^1) - (e(F^2) - e(F^1))/m_2(K_\ell, K_r) &= v(F \cup L_{x}^{R^*}) - v(F \cup L_{x}^{R}) - \frac{e(F \cup L_{x}^{R^*}) - e(F \cup L_{x}^{R})}{m_2(K_\ell, K_r)} \\
    &= v(L_{x}^{R^*}) - v(L_{x}^{R}) - \frac{e(L_{x}^{R^*}) - e(L_{x}^{R})}{m_2(K_\ell, K_r)} \\
    \geq v(L_{x}^{R^*}) - v(L_{x}^{R}) - \frac{e(L_{x}^{R^*}) - e(L_{x}^{R})}{m_2(K_\ell, K_r)} \\
    &= \left(\frac{\ell - (x + (\ell - x))(r - 2)}{v(L_{x}^{R^*})}\right) \left(1 - \frac{e(L_{x}^{R^*})}{v(L_{x}^{R^*})m_2(K_\ell, K_r)}\right) \\
    \text{and by definition of } m_2 \\
    &\geq \left(1 - \frac{(\frac{\ell}{2})^2}{v(L_{x}^{R^*})}\right).
\end{align*}
\]

(5.6)

It remains to prove (5.4) for the last transformation (iii). We transform $F^2$ to $F^3$ by replacing the graph $L_{x}^{R^*}$ by a copy $L_{x}^{R^*2}$ of $H_{\ell-2}^{R^*}$ if $x < \ell - 2$. Note that this step does never apply if $\ell = 3$. $H_{\ell-2}^{R^*}$ consists of an inner graph $K_{\ell-2}^{x}$, a clique on $\ell$ vertices with one missing edge. On every edge of $K_{\ell-2}^{x}$ sits an outer copy of $K_{\ell}$ that is otherwise disjoint from $K_{\ell-2}^{x}$, and the outer copies do mutually not intersect. Recall that procedure ELIGIBILE-EDGE determines an edge $e \in E(F)$ which is contained in an $\ell$-clique $L \in L_{x}^{C_{x}}$ that extends $F$. In transformation (iii), we detach $L_{x}^{R^*}$ from $F^2$ by removing all but the $\ell - x$ vertices that join $L_{x}^{R^*}$ and $F$. Then, we glue a copy $L_{x}^{R^*2}$ of $H_{\ell-2}^{R^*}$ to $F$ by identifying $e$, which was returned from ELIGIBILE-EDGE($F_x$), and the edge missing in $K_{\ell-2}^{x}$ that complements it to a complete graph.

Since $0 < x < \ell - 2$, we have

\[
\begin{align*}
    v(F^3) - v(F^2) - (e(F^3) - e(F^2))/m_2(K_\ell, K_r) &= v(F \cup L_{x}^{R^*2}) - v(F \cup L_{x}^{R}) - \frac{e(F \cup L_{x}^{R^*2}) - e(F \cup L_{x}^{R})}{m_2(K_\ell, K_r)} \\
    &= (v(L_{x}^{R^*2}) - 2) - (v(L_{x}^{R}) - (\ell - x)) - \frac{e(L_{x}^{R^*2}) - e(L_{x}^{R})}{m_2(K_\ell, K_r)} \\
    \text{and since the number of edge in } L_{x} \text{ is} \\
    \left(\frac{\ell}{2}\right) - \left(\frac{\ell - x}{2}\right),
\end{align*}
\]

(5.6)
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this is equal to
\[
\left[ \binom{\ell}{2} - 1 - \left( \binom{\ell}{2} - \binom{\ell - x}{2} \right) \right] (r - 2) + c - 2 - x
- \left[ \binom{\ell}{2} - 1 - \left( \binom{\ell}{2} - \binom{\ell - x}{2} \right) \right] \frac{r}{2} / m_2(K_\ell, K_r)
= \ell - 2 - x - \left( \binom{\ell - x}{2} - 1 \right) \frac{2}{\ell + 1}
= \ell - 2 - x - \frac{\ell - 2 - x)(\ell + 1 - x)}{\ell + 1}
= (\ell - 2 - x) \frac{\ell - 3}{\ell + 1}.
\]  

Hence, setting \( \kappa_2 \) to
\[
\kappa_2 := \min \left\{ 1, \frac{2}{\binom{\ell}{2} - 1} \frac{\ell - 3}{r - 2 + \ell + 1} \right\}
\]
satisfies (5.4) due to (5.5), (5.6), and (5.7). Note that we can ignore the last term for \( \ell = 3 \).

This completes the proof of Lemma 5.14.

Claim 5.9 follows directly from Lemmas 5.13 and 5.14. It remains to prove Lemma 5.15. In fact, we shall prove a more general version of the lemma. We introduce a family \( \mathcal{H}(J, r) \) of graphs that is identical to \( \mathcal{H}(\ell, r, x) \) if \( J = K^r_{\ell} \), where \( K^r_{\ell} \) is the complete graph on \( \ell \) vertices from which the edges of a clique on \( \ell - x \) vertices are deleted.

**Definition 5.16.** For every graph \( J = (V, E) \), let
\[
\mathcal{H}(J, r) := \left\{ J^r = (V \cup U, E \cup D) : J^r \text{ is a minimal graph such that}
\right. 
\text{for all } e' \in E, \text{ there are sets } U(e') \subseteq U \text{ and } D(e') \subseteq D \text{ with}
\text{ }\left. \left( e' \cup U(e'), \{e'\} \cup D(e') \right) \cong K_r \right\}
\]

The minimality condition is understood w.r.t. subgraph inclusion. We prove Lemma 5.15 for any inner graph \( J \) that is nonempty and balanced. Observe that for any nonempty graph \( J \) and integer \( r \geq 2 \) there is a unique graph \( J^r* \in \mathcal{H}(J, r) \) in which the copies of the outer cliques are pairwise disjoint.

**Lemma 5.17.** Let \( r \geq 3 \) be a fixed integer and \( J \) be a balanced, nonempty graph. Let \( J^r* \) denote the unique member of \( \mathcal{H}(J, r) \) with pairwise disjoint outer cliques. Then every member \( J^r \in \mathcal{H}(J, r) \) satisfies
\[
\frac{e(J^r)}{v(J^r)} \geq \frac{e(J^r*)}{v(J^r*)}.
\]

**Proof.** The intuition behind our approach is the following: \( J^r* \) can be transformed into any given \( J^r \in \mathcal{H}(J, r) \) by successively merging outer copies of \( K_r \). We shall do this in \( \ell - 1 \) steps, fixing a linear ordering on the inner edges \( E \). For every edge \( f \in E \), we merge the attached outer copy \( K(f) \) to outer copies attached to edges preceding \( f \) in that ordering, keeping track of the number of edges \( \Delta_e(f) \) and vertices \( \Delta_v(f) \) vanishing in the process. One might hope that the density of \( J^r \) increases in every step of this process or, slightly stronger, that \( \Delta_e(f)/\Delta_v(f) \leq e(J^r*)/v(J^r*) \) for all \( f \in E \). Unfortunately, this does not hold, but we shall prove the existence of enough 'good' steps in the process to compensate for all 'bad' ones which
may arise. To find these good steps, we will group the edges \( f \in E \) into appropriate 'phases' such that the existence of bad steps in a phase implies the existence of some good steps earlier in that phase.

Recall that in every graph \( J^r = (V \cup U, E \cup D) \), the inner copy \( (V, E) \) is isomorphic to \( J \). By definition, for each inner edge \( f \in E \), we can identify sets of outer vertices \( U(f) \subseteq U \) and outer edges \( D(f) \subseteq D \) such that \( \tilde{K}(f) := (U(f) \cup f, D(f) \cup \{f\}) \) is isomorphic to \( K_r \). While these sets are not necessarily unique, for the rest of the proof, we fix one choice of appropriate sets \( U(f) \) and \( D(f) \). Note that by the minimality condition in Definition 5.16, every vertex and edge is included in at least one outer copy. Let \( \tilde{K}_-(f) := (U(f) \cup f, D(f)) \) denote the resulting subgraph that is isomorphic to a clique on \( r \) vertices with one edge removed. For every outer vertex \( u \in U \) and for every outer edge \( d \in D \) of \( J^r \), the sets

\[
E(u) := \{ f \in E : u \in U(f) \}
\]

and

\[
E(d) := \{ f \in E : d \in D(f) \}
\]

indicate in which outer copies \( \tilde{K}(f) \) \( u \) and \( d \) respectively participate.

Note that

\[
\sum_{d \in D} |E(d)| = \sum_{d \in D} \sum_{f \in E} 1 = \sum_{f \in E} \sum_{d \in D} 1 = \sum_{f \in E} |D(f)| = e_J \left( \binom{r}{2} - 1 \right)
\]

and analogously

\[
\sum_{u \in U} |E(u)| = e_J (r - 2)
\]

Due to

\[
e(J^r) - |D| = e_J = e(J^{r*}) - e_J \left( \binom{r}{2} - 1 \right) = e(J^{r*}) - \sum_{d \in D} |E(d)|
\]

we have

\[
e(J^r) = e(J^{r*}) - \sum_{d \in D} (|E(d)| - 1) \quad (5.8)
\]

Analogously,

\[
v(J^r) - |U| = v_J = v(J^{r*}) - e_J (r - 2) = v(J^{r*}) - \sum_{u \in U} |E(u)|
\]

yields

\[
v(J^r) = v(J^{r*}) - \sum_{u \in U} (|E(u)| - 1) \quad (5.9)
\]

Next, we impose a linear order on the vertices and edges of \( J \). For ease of notation we will use the abbreviation \( \ell := v_J \) in the remainder of the proof. Using the averaging principle, it is easy to see that for every balanced graph \( J \) there exists an ordering \( [v_1, \ldots, v_\ell] \) of its vertices such that for all \( 2 \leq i \leq \ell \), \( v_i \) has at most \( \lfloor 2d(J) \rfloor \) neighbors among \( \{v_1, \ldots, v_{i-1}\} \) in \( J \). In fact, we can compute this ordering by recursively removing the vertex of minimum degree from \( J \). W.l.o.g. we assume that the inner vertices \( V \) are ordered in this way. This ordering induces a mapping \( p : E \rightarrow \{2, \ldots, \ell\} \), which assigns every inner edge to the greater label of its two vertices. We call \( p(f) \) the phase of edge \( f \). This mapping induces a partial order on \( E \), which can be extended to a total order \( \prec \) by choosing an arbitrary order on edges of the same phase.
For $f \in E$, we define
\[
\Delta_E(f) := D(f) \cap \left( \bigcup_{f' < f} D(f') \right),
\]
\[
\Delta_V(f) := U(f) \cap \left( \bigcup_{f' < f} U(f') \right),
\]
and $\Delta_e(f) := |\Delta_E(f)|$, $\Delta_v(f) := |\Delta_V(f)|$ for the cardinalities of these sets. Intuitively, $\Delta_e(f)$ is the number of edges vanishing when $\tilde{R}_-(f)$ is merged with preceding outer copies. Analogously, $\Delta_v(f)$ is the number of vertices vanishing in this merge operation.

$\Delta_E(f)$ contains all edges $d \in D(f)$ that also belong to $D(f')$ for some edge $f' < f$. By definition, both $f$ and $f'$ are in $E(d)$. Therefore, we have
\[
\sum_{f \in E} \Delta_e(f) = \sum_{f \in E} \sum_{d \in D(f)} \sum_{d \in D(f') \text{ for } f' < f} 1 = \frac{1}{|E(d)| - 1}
\]
and by (5.8)
\[
e(J^r) = e(J^{r*}) - \sum_{f \in E} \Delta_e(f). \tag{5.10}
\]
Similarly, we have
\[
\sum_{f \in E} \Delta_v(f) = \sum_{u \in U} \sum_{u \in U(f)} \sum_{u \in U(f') \text{ for } f' < f} 1 = \frac{1}{|E(u)| - 1}
\]
and thus, by (5.9),
\[
v(J^r) = v(J^{r*}) - \sum_{f \in E} \Delta_v(f). \tag{5.11}
\]

In order to calculate the density of $J^r$, we introduce the following quantities. For every phase $i$, $2 \leq i \leq \ell$, we define
\[
\Delta_i^e := \sum_{f \in E: p(f) = i} \Delta_e(f)
\]
and
\[
\Delta_i^v := \sum_{f \in E: p(f) = i} \Delta_v(f).
\]
Due to (5.10) and (5.11), we can express the density of $J^r$ simply as
\[
e(J^r) = \frac{e(J^{r*}) - \Delta_2^e - \ldots - \Delta_{\ell}^e}{v(J^{r*}) - \Delta_2^v - \ldots - \Delta_{\ell}^v}. \tag{5.12}
\]
We call phase $i$ trivial if $\Delta_i^v = 0$, which implies $\Delta_i^e = 0$. By Proposition 2.2, in order to show
\[
\frac{e(J^r)}{v(J^r)} \geq \frac{e(J^{r*})}{v(J^{r*})}
\]
it suffices to prove that, in every non-trivial phase $i$, we have
\[
\frac{\Delta_i^e}{\Delta_i^v} \leq \frac{e(J^{r*})}{v(J^{r*})}. \]
Suppose a non-trivial phase $i \in [\ell] \setminus \{1\}$ is fixed. For every step $f \in E$ with $p(f) = i$, let $q(f) := \Delta_v(f) / \Delta_v(f)$ and

$$T(f) := \left( \Delta_V(f) \cup f, \Delta_E(f) \right) \subseteq \tilde{K}_-(f).$$

Intuitively, the graph $T(f)$ is formed by the edges and vertices that vanish in the merge step $f$. However, we have to add the two vertices of $f$ to guarantee that the graph is well-defined. We say that an edge $b \in E$, $p(b) = i$ is bad if $q(b) > d(J^{**})$. Note that, by Proposition 2.2, phase $i$ trivially satisfies the claim if it does not contain bad edges. We shall show that this holds if $J$ is a tree. In all other cases, our strategy is to show that in every phase $i$, there are sufficiently many edges that can compensate for the bad ones.

First, let us rewrite $d(J^{**}) = e(J^{**}) / v(J^{**})$. Easy algebraic transformations yield

$$d(J^{**}) = \frac{e_J(r - 2)}{2} \cdot \frac{\Delta_v(f) + 1}{\Delta_v(f)} = d_2(K_r) + \frac{d_2(K_r) - d(J)}{d(J)(r - 2) + 1}. \tag{5.13}$$

Now, suppose that $J$ is a tree. Then we have $d(J) \geq 1/2$, and we deduce from (5.13)

$$d(J^{**}) \geq d_2(K_r) - \frac{d_2(K_r) - \frac{1}{2}}{(r - 2) + 1} = \frac{r + 1}{2} - 1 = \frac{r - 1}{2}.$$

On the other hand, due to $\lfloor 2d(J) \rfloor = 1$, phase $i$ consists of a single edge $f = \{v_i, v_j\}$, $j < i$. Therefore, the vertex $v_i$ remains isolated in $T(f)$, i.e.,

$$\Delta_E(f) \subseteq \left( \Delta_V(f) \cup \{v_j\} \right).$$

Hence, we have

$$q(f) \leq \frac{(\Delta_v(f) + 1)}{\Delta_v(f)} = \frac{\Delta_v(f) + 1}{2} \leq \frac{r - 1}{2}.$$

We conclude that in the case of trees, every phase consists of just one good edge.

In the remaining proof, suppose that $J$ contains at least one cycle. As $J$ is balanced, we have $d(J) \geq 1$ and, due to (5.13),

$$d(J^{**}) \geq d_2(K_r) - \frac{d_2(K_r) - \frac{1}{2}}{r - 1} = \frac{r + 1}{2} - \frac{1}{2} = \frac{r - 1}{2}. \tag{5.14}$$

Let $b = \{v_i, v_j\} \in E$, $p(b) = i$, $j < i$, denote the worst edge in phase $i$, i.e.,

$$b = \arg\max_{f \in E, p(f) = i} q(f).$$

We analyze $T(b)$ in more detail. Clearly, there is nothing to prove if $b$ is good, i.e., $q(b) \leq d(J^{**})$. It follows from (5.14) that $\Delta_v(b) = r - 2$ since otherwise we had

$$q(b) = \frac{\Delta_v(b)}{\Delta_v(b)} \leq \frac{(\Delta_v(b) + 2)}{2} - 1 = \frac{\Delta_v(b) + 3}{2} \leq \frac{r - 2}{2} \leq d(J^{**}),$$

and thus $b$ would be good. For every edge $f = \{v_i, v_j\}$ in phase $i$, let $M(f)$ denote those edges from $\tilde{K}_-(f)[\Delta_V(f) \cup f]$ that are missing in $T(f)$, i.e.,

$$M(f) := \left( \Delta_V(f) \cup f \right) \setminus \left( \Delta_E(f) \cup \{f\} \right).$$
and let \( m(f) := |M(f)| \). Clearly, we have

\[
\Delta_v(f) = \left( \frac{\Delta_v(f) + 2}{2} \right) - 1 - m(f)
\]
\[
= \frac{\Delta_v(f) + 3}{2} \Delta_v(f) - m(f)
\]
\[
\leq d_2(K_r) \Delta_v(f) - m(f) .
\]

Observe that

\[
T(f) = \tilde{K}_-(f) \cap (G_0 \cup G_i(f)) ,
\]

where

\[
G_0 := \bigcup_{f' \in E \atop p(f') < i} \tilde{K}_-(f') \quad \text{and} \quad G_i(f) := \bigcup_{f' \in E \atop p(f') = i} \tilde{K}_-(f') .
\]

The crucial observation is that, whenever a vertex \( u \in U(f) \) is glued to some outer vertex in \( w \in V(G_0) \setminus V(G_i(f)) \), the edge \( \{v_t, u\} \) cannot vanish since \( v_t \) and \( w \) are not adjacent in \( G_0 \cup G_i(f) \). Consequently, \( \{v_t, u\} \) belongs to \( M(f) \). Analogously, we have \( \{v_y, u\} \in M(f) \) for all \( u \in U(f) \) that are glued to an outer vertex \( w \in V(G_i(f)) \setminus V(G_0) \) since \( w \) is not adjacent to \( v_y \) in \( G_0 \cup G(f) \). Denoting the set of outer vertices of \( G_0 \) and \( G_i(f) \) by

\[
U_0 := \bigcup_{f' \in E \atop p(f') < i} U(f') \quad \text{and} \quad U_i(f) := \bigcup_{f' \in E \atop p(f') = i} U(f') ,
\]

we deduce that \( M(f) \) is a superset of

\[
\{\{u, u\} : u \in (\Delta_V(f) \setminus U_i(f))\} \cup \{\{v_y, u\} : u \in (\Delta_V(f) \setminus U_0)\} .
\]

This yields for \( f = b \),

\[
m(b) \geq |\Delta_V(b) \setminus U_i(b)| + |\Delta_V(b) \setminus U_0|
\]
\[
\geq |\Delta_V(b) \setminus (U_i(b) \cap U_0)|
\]
\[
\geq \Delta_v(b) - |U_0 \cap U_i(b)|
\]
\[
= r - 2 - |\Phi| ,
\]

where

\[
\Phi := U_0 \cap U_i(b) .
\]

We can express the set \( \Phi \) as the disjoint union of contributions from edges preceding \( b \) in phase \( i \).

\[
\Phi = U_0 \cap \bigcup_{f < b \atop p(f) = i} U(f) = U_0 \cap \bigcup_{f < b \atop p(f) = i} (U(f) \setminus U_i(f))
\]
\[
= \bigcup_{f < b \atop p(f) = i} [U_0 \cap (U(f) \setminus U_i(f))] = \bigcup_{f < b \atop p(f) = i} [(U_0 \cap U(f)) \setminus U_i(f)]
\]
\[
= \bigcup_{f < b \atop p(f) = i} [((U_0 \cup U_i(f)) \cap U(f)) \setminus U_i(f)] = \bigcup_{f < b \atop p(f) = i} [\Delta_V(f) \setminus U_i(f)] .
\]

For all edges \( f < b \) in phase \( i \), we shall use the weaker implication of (5.16)

\[
m(f) \geq |\Delta_V(f) \setminus U_i(f)| =: |\Psi(f)| .
\]
Observe that
\[ |\Phi| = \sum_{f,\sigma,\tau} |\Psi(f)|. \tag{5.19} \]

In order to simplify notation, we define \( \Psi(f) := \emptyset \) for all \( f \geq b \). In particular, we have \( \Psi(b) = \emptyset \).

Let
\[ G := \{ f \in E : p(f) = i \land \Psi(f) \neq \emptyset \} \]
denote the edges in phase \( i \) that contribute to \( \Phi \). The remaining edges of phase \( i \) form the set \( \overline{G} \), i.e.,
\[ \overline{G} := \{ f \in E \setminus G : p(f) = i \} . \]

From (5.15) and (5.18), we derive the following upper bound on the number of edges vanishing in every merge step \( g \in \mathcal{G} \)
\[ \Delta_v(g) \leq d_2(K_r) \Delta_v(g) - |\Psi(g)| . \tag{5.20} \]

For all remaining edges \( f \in \overline{G} \), we use a trivial upper bound on the number of vanishing edges due to the choice of \( b \). Clearly,
\[ \Delta_v(f) \leq q(b) \Delta_v(f) = \left( \frac{m(b)}{r-2} \right) \Delta_v(f) \tag{5.21} \]

Due to (5.19), (5.20), and (5.21), we obtain
\[ \frac{\Delta^i_v}{\Delta^i_v} = \frac{\sum_{g \in G} \Delta_v(g) + \sum_{f \in \overline{G}} \Delta_v(f)}{\Delta^i_v} \leq \frac{\sum_{g \in \overline{G}} (d_2(K_r) \Delta_v(g) - |\Psi(g)|) + \sum_{f \in \overline{G}} \left( d_2(K_r) - \frac{m(b)}{r-2} \right) \Delta_v(f)}{\Delta^i_v} \tag{5.22} \]

Clearly, this expression attains its maximum for \( |\Phi| \) and \( \sum_{f \in \overline{G}} \Delta_v(f) \) being minimal and \( \Delta^i_v \) maximal. We derive from (5.17) that \( |\Phi| \geq r - 2 - m(b) \). Since, by definition, \( b \in \overline{G} \), we have \( \sum_{f \in \overline{G}} \Delta_v(f) \geq \Delta_v(b) = r - 2 \). Recall that each phase consists of at most \( \lfloor 2d(J) \rfloor \) merge operations. Hence, we have \( \Delta^i_v \leq \lfloor 2d(J) \rfloor \) \((r-2)\). By plugging those bounds into (5.22), we obtain
\[ \frac{\Delta^i_v}{\Delta^i_v} \leq d_2(K_r) - \frac{r - 2 - m(b) + \frac{m(b)}{r-2}(r-2)}{\lfloor 2d(J) \rfloor (r-2)} = d_2(K_r) - \frac{1}{\lfloor 2d(J) \rfloor} . \]

Comparing with (5.13), the claim follows if
\[ \frac{d_2(K_r) - d(J)}{d(J)(r-2) + 1} \leq \frac{1}{\lfloor 2d(J) \rfloor} . \]

Dropping the floor function and expanding yields the quadratic inequality
\[ (d(J) - 1)(2d(J) - 1) \geq 0 , \]
which is obviously satisfied due to \( d(J) \geq 1 \).

This completes the proof of Lemma 5.17.
5.3. An algorithm for triangles

As stated in Remark 5.6, the proof presented in Section 5.2 does not cover the case $\ell = 3$ since $\gamma(3, r) = -1/(r^2 - r) < 0$. This implies, in particular, that, for any $b > 0$, $G_{n,p}$ with $p = bn^{-1/m2(K_3,K_r)}$ may contain copies of $K_{r+1}$. Since $K_{r+1}$ is a member of the family $C^*(3,r)$, the coloring algorithm ASYM-EDGE-COL will terminate with an error. We formulate a refined version of this algorithm, namely ASYM-EDGE-COL-K3, which handles those situations correctly. The proof that ASYM-EDGE-COL-K3 colors $G_{n,p}$ successfully is similar to the proof for the original algorithm, but requires some extra work. We shall state and prove counterparts to several lemmas from Section 5.2, which will be referenced along with the statement of the specialized versions. For the rest of this section, let $\ell = 3$ and $r \geq 4$ be fixed.

The refinement of ASYM-EDGE-COL is based on the observation that, for $r \geq 6$, $K_{r+1}$ is essentially the only member of $C^*(3,r)$ that appears in $G_{n,p}$ and causes errors. Hence, if algorithm ASYM-EDGE-COL gets stuck, $G'$ is a.a.s. the union of edge-disjoint copies of $K_{r+1}$ and can be easily colored. For $r \in \{4,5\}$, the argument is similar, but becomes more tedious to formalize since $r + 1$-cliques can overlap to some extent.

In order to state ASYM-EDGE-COL-K3, we need to introduce some notation. We define the following families $A = A(r)$ of graphs.

$$A(r) = \{K_{r+1}\} \quad \text{for } r \geq 6,$$

$$A(5) = \{K_6, K_6 \cap K_6\},$$

$$A(4) = \{K_5, K_5 \cap K_5, K_5 \cap K_5\},$$

where $K_k \cap_x K_k$ denotes the graph consisting of two $k$-cliques that overlap in exactly $x$ vertices. For any given graph $G'$ and fixed $r$, the family $S_{G'}$ identifies all maximal subgraphs of $G'$ isomorphic to a member of $A(r)$, i.e.,

$$S_{G'} := \{S \subseteq G' : S \cong A \in A(r) \land S \text{ is maximal}\}.$$ 

The maximality condition is understood w.r.t. subgraph inclusion, that is, there are no two members $S, T \in S_{G'}$ such that $S \subseteq T$. Note that this matters only for $r \in \{4,5\}$. For any edge $e \in E(G')$, let

$$S_{G'}(e) := \{S \in S_{G'} : e \in E(S)\}.$$ 

Note that for $r \geq 6$, $S_{G'}(e)$ denotes just the set of all copies of $K_{r+1}$ in $G'$ that contain $e$. We call $G'$ an $A$-graph if, for all $e \in E(G')$, we have

$$|S_{G'}(e)| = 1,$$

i.e., if $G'$ is an edge-disjoint union of graphs from $A(r)$.

It is easily verified that, for all $r \geq 4$, there exists a valid coloring of all graphs $A \in A(r)$. We employ a simple subroutine A-COLOR($G'$) that computes, for any $A$-graph $G'$, a coloring as follows: assign a valid coloring to every subgraph $S \in S_{G'}$ locally, that is, regardless of the remaining structure of $G'$. In that way all edges of $G'$ will be assigned a color since $G'$ is an $A$-graph. Assuming that there is no triangle in $G'$ that contains edges from at least two different members of $S_{G'}$, procedure A-COLOR($G'$) will thus compute a valid coloring of $G'$ since it cannot produce monochromatic triangles, let alone monochromatic $r$-cliques. Hence, we call an $A$-graph $G'$ trivially colorable if, for all triangles $T \subseteq G'$, there exists a (unique) member $S \in S_{G'}$ such that $T \subseteq S$. 


Chapter 5. Asymmetric Ramsey Properties

Asym-Edge-Col-K3(G = (V, E))

1. \( s \leftarrow \text{empty-stack}() \)
2. \( E' \leftarrow E \)
3. \( \mathcal{L} \leftarrow \mathcal{L}_G \)
4. while \( G' = (V, E') \) is no trivially colorable \( A \)-graph
5. do if \( \exists e \in E' \) s.t. \( \#(L, R) \in \mathcal{L} \times \mathcal{R}_{G'}(V, E') : E(L) \cap E(R) = \{e\} \)
6. then for each \( L \in \mathcal{L} : e \in E(L) \)
7. do \( s\text{-push}(L) \)
8. \( \mathcal{L} \text{-remove}(L) \)
9. \( s\text{-push}(e) \)
10. \( E'\text{-remove}(e) \)
11. else if \( \exists L \in \mathcal{L} \setminus \mathcal{L}_{G'}(V, E') \)
12. then \( s\text{-push}(L) \)
13. \( \mathcal{L} \text{-remove}(L) \)
14. else error "stuck"
15. A-Color(\( G' = (V, E') \))
16. while \( s \neq \emptyset \)
17. do if \( s\text{-top}() \) is an edge
18. then \( e \leftarrow s\text{-pop}() \)
19. \( e\text{-set-color}(\text{blue}) \)
20. \( E'\text{-add}(e) \)
21. else \( L \leftarrow s\text{-pop}() \)
22. if \( L \) is entirely blue
23. then \( f \leftarrow \text{any } e \in E(L) \text{ s.t.} \)
24. \( \#R \in \mathcal{R}_{G'}(V, E') : E(L) \cap E(R) = \{e\} \)
25. \( f\text{-set-color}(\text{red}) \)

Figure 5.4. The refined coloring algorithm that avoids blue triangles. (cf. Figure 5.1 on page 45). \( \mathcal{L}_G \) denotes the family of all triangles in \( G \).

With those definitions at hand, we are ready to formulate the refined version of the coloring algorithm Asym-Edge-Col-K3 in Figure 5.4. Note the modifications in lines 4 and 15. The correctness of the algorithm follows directly from the specification of procedure A-Color and the proof of Lemma 5.4.

Lemma 5.18 (Lemma 5.4'). If algorithm Asym-Edge-Col-K3 terminates without error, then it has indeed found a valid coloring of \( G \).

Hence, it remains to prove that no error occurs.

Lemma 5.19 (Lemma 5.5'). There exists a positive constant \( b = b(r) \) such that the algorithm Asym-Edge-Col-K3 a.a.s. terminates on \( G_{n,p} \) with \( p \leq bn^{-1/m_2(K_3,S_r)} \) without error.

5.3.1. Proof of Lemma 5.19. We shall prove Lemma 5.19 in a similar way as Lemma 5.5. We formulate a refined version of algorithm Grow, namely Grow-K3, that produces a very dense or very large subgraph of \( G \) if algorithm Asym-Edge-Col-K3(\( G \)) gets stuck with \( G' \).
Chapter 5. Asymmetric Ramsey Properties

Grow-K$_3$($G' = (V, E)$)

1. if $\forall e \in E : |S_{G'}(e)| = 1$
2. then $T \leftarrow$ any member of $T_{G'}$
3. return $\bigcup_{T \in S_{G'}}(T) S$
4. if $\exists e \in E : |S_{G'}(e)| \geq 2$
5. then $S_1, S_2 \leftarrow$ any two distinct members of $S_{G'}(e)$
6. return $S_1 \cup S_2$
7. $e \leftarrow$ any $e \in E : |S_{G'}(e)| = 0$
8. $F_0 \leftarrow$ any $R \in R_{G'} : e \in E(R)$
9. $i \leftarrow 0$
10. while $i < \log(n) \wedge \forall \tilde{F} \subseteq F_i : \lambda(\tilde{F}) > \gamma$
11. do if $\exists R \in R_{G'} \setminus R_{F_i}$ s.t. $|V(R) \cap V(F_i)| \geq 2$
12. then $F_{i+1} \leftarrow F_i \cup R$
13. else $e \leftarrow$ Eligible-Edge($F_i$)
14. $F_{i+1} \leftarrow$ Extend-L($F_i, e, G'$)
15. $i \leftarrow i + 1$
16. return Densest-Subgraph($F_i$)

Figure 5.5. The refined algorithm Grow-K$_3$ (cf. Figure 5.2 on page 46).

Suppose $G'$ is fixed. For any subgraph $F \subseteq G'$, let

$$S_{G'}(F) := \bigcup_{e \in E(F)} S_{G'}(e).$$

We define the family $T_{G'}$ of all dangerous triangles in $G'$ as follows:

$$T_{G'} := \{T \subseteq G' : T \cong K_3 \wedge |S_{G'}(T)| \geq 2\}.$$ 

Algorithm Grow-K$_3$ utilizes a procedure Densest-Subgraph($F$) that returns a graph $\tilde{F} \subseteq F$ with maximal density, i.e., satisfying $d(\tilde{F}) = m(F)$. The output of this procedure is unique up to isomorphism. Similarly to procedure Eligible-Edge, Densest-Subgraph could be implemented using a huge lookup-table.

The algorithm Grow-K$_3$ is shown in Figure 5.5. Before Grow-K$_3$ enters the while-loop, it tests for two special cases. The first case holds if and only if $G'$ is an $A$-graph. Since the algorithm Asym-Edge-Col-K$_3$ terminated with an error, $G'$ is not trivially colorable, that is, the family $T_{G'}$ is not empty. Hence, Grow-K$_3$ returns any member $T \in T_{G'}$ together with the surrounding graphs from $S_{G'}$ that contain $T$. In the second case, Grow-K$_3$ produces members of $S_{G'}$ that cluster locally. If neither of those special cases occurs, Grow-K$_3$ proceeds essentially as before, but chooses the seed of the growing procedure more carefully. As we excluded the two special cases, there must be an edge $e \in E(G')$ that is not contained in any member of $S_{G'}$. On the other hand, there must be a member of $R_{G'}$ that contains $e$ since $G'$ is a member of $C^* (3, r)$. We choose any such copy of $K_r$ for the seed. Observe that thus $F_i$ cannot become an $A$-graph in any iteration $i$ since we always have $F_i \subseteq G'$ and $e \in F_i$, but $S_{G'}(e)$ is empty. The terminating condition of the while-loop is strengthened in the sense that Grow-K$_3$ stops not only if $\lambda(F_i)$ falls below $\gamma$, but also if $F_i$ contains some subgraph that satisfies this condition. Accordingly, algorithm Grow-K$_3$ returns always a densest subgraph of $F_i$. 
We choose the value of $\gamma$ in GROW-K3 slightly different than before in order to guarantee that the invocation of subroutine ELIGIBLE-EDGE($F_i$) is successful in every iteration $i$. Recall the corresponding argument for the case $\ell > 3$. By a simple degree-counting argument, we showed that for all graphs $F \in C(\ell, r)$ satisfy $d(F) < m_2(K_3, K_r) + \epsilon$. By the choice of $\gamma := \frac{1}{m_2(K_3, K_r)} - \frac{1}{m_2(K_3, K_r) + \epsilon}$ (cf. (5.1)), ELIGIBLE-EDGE was never invoked on graphs from the family $C(\ell, r)$ and thus always produced an edge eligible for extension. Here, we use a similar statement, whose proof is more involved than the simple degree-counting argument from before. For any given $\ell > 0$, let denote the family of all graphs in $C(\ell, r)$ that contain no subgraph denser than $m_2(K_3, K_r) + \epsilon$.

**Lemma 5.20.** There exists a constant $\epsilon = \epsilon(r) > 0$ such that every graph in $C_\ell(3, r)$ is an $A$-graph.

The proof of Lemma 5.20 is deferred to Section 5.3.2. Now, choosing $\epsilon = \epsilon(r)$ according to Lemma 5.20 and letting $\gamma = \gamma(r) := \frac{1}{m_2(K_3, K_r)} - \frac{1}{m_2(K_3, K_r) + \epsilon}$, we can prove that the invocation of ELIGIBLE-EDGE always produces an edge eligible for extension.

**Claim 5.21 (Claim 5.7').** Algorithm GROW-K3 terminates without error on any nonempty input graph $G' \in C^*(3, r)$ that is no trivially colorable $A$-graph.

**Proof of Claim 5.21.** Clearly, nothing can go wrong if one of the two special cases in lines 1 to 6 occurs. As mentioned before, the choice of $F_0$ in line 8 ensures that $F_i$ never becomes an $A$-graph. Suppose there is no edge in $F_i$ that is eligible for extension. Then we have $F_i \in C(3, r)$ by definition. As $F_i$ is no $A$-graph and by the choice of $\epsilon$, Lemma 5.20 implies $m(F_i) \geq m_2(K_3, K_r) + \epsilon$.

Consequently, for $\overline{F} \subseteq F_i$ with $d(\overline{F}) = m(F_i)$, we have

$$
\lambda(\overline{F}) \leq \epsilon(\overline{F}) \left( \frac{1}{m(F_i)} - \frac{1}{m_2(K_3, K_r)} \right) \\
\leq \epsilon(\overline{F}) \left( \frac{1}{m_2(K_3, K_r) + \epsilon} - \frac{1}{m_2(K_3, K_r)} \right) \\
= -\epsilon(\overline{F}) \gamma \leq -\gamma.
$$

Therefore, algorithm GROW-K3 exits from the while-loop before calling ELIGIBLE-EDGE. Hence, every invocation of ELIGIBLE-EDGE returns an edge eligible for extension.

Now, we are ready to prove Lemma 5.19. Suppose that $G'$ is an $A$-graph and, therefore, algorithm GROW-K3 returns a graph $F := \bigcup_{S \in S_{\text{col}}(T)} S$. 


for some triangle $T \in T_{Gr}$. For all $r \geq 6$, the graph $F$ is uniquely defined up to isomorphism. It consists of a triangle, each edge of which is embedded into a separate copy of $K_{r+1}$. Hence, we have

$$m(F) \geq d(F) = \frac{\binom{r+1}{2}}{3(r+1)} = \frac{r+1}{2} = m_2(K_r),$$

and this structure does a.a.s. not appear in $G = G_{n,p}$ if we have $p = bn^{-1/m_2(K_3,K_r)}$ for any constant $b > 0$. This contradicts that algorithm ASYM-EDGE-COL-K3 terminated with error on $G$. If $r = 5$, two edges of the triangle $T$ could be embedded into a member of the family $S_{G'}$ that is isomorphic to $K_6 \cap \delta K_6$ and the third edge into a copy of $K_6$. Note that $K_6 \cap \delta K_6$ is isomorphic to $K_7$ with one missing edge. In that case we have

$$m(F) = \frac{\binom{5}{2} + \binom{6}{2} - 1}{6+7-2} = \frac{35}{11} > \frac{20}{7} = m_2(K_3,K_6).$$

In the case $r = 4$, this translates to a structure of two cliques of size 5 and 6 respectively that overlap exactly on one edge. Consequently, we have

$$m(F) = \frac{\binom{4}{2} + \binom{5}{2} - 1}{5+6-2} = \frac{8}{3} > \frac{12}{5} = m_2(K_3,K_4).$$

Alternatively, two edges of the triangle $T$ could be embedded into a member of the family $S_{G'}$ that is isomorphic to $K_5 \cap \delta K_5$ and the third edge into a copy of $K_5$. Then we have

$$m(F) = \frac{\binom{5}{2} - 1}{3 \cdot 5 - 4} = \frac{29}{11} > \frac{12}{5} = m_2(K_3,K_4).$$

Hence, all possible graphs returned in line 3 of algorithm GROW-K3 are too dense to appear in $G_{n,p}$.

Suppose there exists an edge $e \in E(G')$ with $|S_{G'}(e)| \geq 2$ and GROW-K3 returns

$$F := S_1 \cup S_2.$$

Then, for all $r \geq 6$, the graph $F$ is isomorphic to $K_{r+1} \cap K_{r+1}$ for $2 \leq x \leq r$. This, however, implies that we have

$$m(F) \geq d(F) = \frac{2\binom{r}{2} - \binom{x}{2}}{2(r+1) - x} \geq \frac{r(r-1)}{2(r+2)} = m_2(K_3,K_r),$$

where the minimum is attained for $x = r$. If $r = 5$, (5.23) holds for all $2 \leq x \leq 4$, and if $r = 4$, it is only true for $x = 3$. Similar calculations show that there cannot be a third copy of $K_{r+1}$ sharing edges with two edge-intersecting copies of $K_{r+1}$ for all $r \geq 4$. As GROW can construct at most constantly many non-isomorphic graphs $F$ in that way, $F$ is a.a.s. not contained in $G_{n,p}$ due to Markov's inequality.

Now, suppose that GROW-K3 enters the while-loop. As in Section 5.2.1, let $F(3,r,n)$ denote a family of representatives for the isomorphism classes of all graphs that can be the output of GROW-K3 with parameters $n$ and $\gamma(r)$ on any input instance $G'$. Since this part of algorithm GROW-K3 differs from the original algorithm only by returning the densest subgraph $\bar{F} \subseteq F_1$ rather than $F_1$ itself, and since, by specification of procedure DENSEST-SUBGRAPH, there is a mapping between $\bar{F}$ and $F_1$ up to isomorphism, Claim 5.11 applies to $F(3,r,n)$, that is, there exists $C = C(3,r)$ such that $|F(3,r,n)| \leq \log(n)^C$. Note that Claims 5.8, 5.9, and 5.10 also hold for $\ell = 3$. Therefore, we may prove Claim 5.12 similarly as before and obtain a constant $b > 0$ such that $G_{n,p}$ with $p \leq bn^{-1/m_2(K_3,K_r)}$ contains no member of $F(3,r,n)$ a.a.s.

This concludes the proof of Lemma 5.19.
5.3.2. Proof of Lemma 5.20. In this section we shall prove the main technical lemma of Section 5.3. Since the actual value of $\varepsilon$ is irrelevant, we do not specify it, but only prove its existence. Suppose $r \geq 4$ is fixed and $F \in \mathcal{C}_r(3, r)$ is given. We shall show that $F$ is an $A$-graph provided that $\varepsilon$ is sufficiently small.

Let $\widehat{F} \subseteq F$ denote the graph spanned by all copies of $K_{r+1}$ in $G$, i.e.,

$$\widehat{F} := \bigcup_{S \in \mathcal{S}_F} S .$$

We call an $A$-graph $F$ an $A$-forest if the family

$$\{ C \subseteq F : C \text{ is a cycle } \land |\mathcal{S}_F(C)| \geq 2 \}$$

is empty, i.e., there is no cycle in $\widehat{F}$ whose edges are embedded into at least two different members of $\mathcal{S}_F$.

**Claim 5.22.** There exists a constant $\varepsilon = \varepsilon(r) > 0$ such that, for all graphs $F \in \mathcal{C}_r(3, r)$, $F$ is an $A$-forest.

**Proof.** As argued in the proof of Lemma 5.19, if there is an edge $e \in E(\widehat{F})$ with $|e(S)| > 2$ (cf. (5.23)), then there exists a subgraph $\overline{F}$ in $\overline{F}$ satisfying

$$d(\overline{F}) \geq m_2(K_3, K_r) + \varepsilon$$

for $\varepsilon > 0$ sufficiently small. Hence, all members of $\mathcal{S}_F$ are pairwise edge-disjoint and, therefore, $\widehat{F}$ is an $A$-graph.

It remains to show that $\widehat{F}$ is actually an $A$-forest. Suppose that $\widehat{F}$ contains a cycle $C$ formed by edges from $k := |\mathcal{S}_F(C)| \geq 2$ distinct members of $\mathcal{S}_F$. Let

$$\widehat{C} := \bigcup_{S \in \mathcal{S}_F(C)} S$$

denote this so-called $A$-cycle. Then we have

$$d(\widehat{C}) = \frac{\sum_{S \in \mathcal{S}_F(C)} e(S)}{\sum_{S \in \mathcal{S}_F(C)} v(S) - k}$$

$$= \frac{\sum_{S \in \mathcal{S}_F(C)} e(S)}{\sum_{S \in \mathcal{S}_F(C)} (v(S) - 1)} \geq m_2(K_3, K_r) + \varepsilon$$

for $r \geq 4$ and all graphs $A \in \mathcal{A}(r)$, we have

$$\frac{e(A)}{v(A) - 1} \geq m_2(K_3, K_r) + \varepsilon \quad (5.24)$$

for $\varepsilon > 0$ sufficiently small. Hence, $\widehat{F}$ cannot contain an $A$-cycle.

In order to prove that $\widehat{F} = F$, we inspect the vertices of $\widehat{F}$ that are adjacent to vertices in $F \setminus \widehat{F}$ more closely.

**Claim 5.23.** There exists a constant $\varepsilon > 0$ such that all graphs $F \in \mathcal{C}_r(3, r)$ satisfy the following property. For every vertex $v_0 \in V(\widehat{F})$ that is contained in exactly one member of $\mathcal{S}_F$, $\deg_F(v_0) > \deg_{\mathcal{S}_F}$ implies that

$$\deg_F(v_0) \geq \begin{cases} \deg_{\mathcal{S}_F}(v_0) + 3 & \text{if } r = 4 , \\ \deg_{\mathcal{S}_F}(v_0) + r & \text{if } r \geq 5 . \end{cases}$$
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Proof. Let \( S \in \mathcal{S}_F \) be the graph containing \( v_0 \). Note that \( \deg_F(v_0) = \deg_S(v_0) \). Set

\[
\Delta := \deg_F(v_0) - \deg_S(v_0)
\]

and let \( T := F[\Gamma(v_0) \setminus V(S)] \) denote the graph induced by the \( \Delta \) neighbors of \( v_0 \) that are not in \( S \). Note that \( |V(T)| = \Delta \).

Consider the graph

\[
J := F[V(S) \cup \Gamma(v_0)].
\]

We shall prove that \( d(J) > m_2(K_r+1, r) + \varepsilon \) for \( \varepsilon > 0 \) sufficiently small if \( v_0 \) violates the claim.

As \( F \) is a member of the family \( \mathcal{C}(3, r) \), each one of the \( \Delta \) edges connecting \( v_0 \) and \( T \) is covered by at least \( r - 1 \) triangles. Therefore, each vertex of \( T \) has at least \( r - 1 \) common neighbors with \( v_0 \). Since no more than \( \Delta - 1 \) of these are in \( T \), every vertex of \( T \) has at least \( (r - 1) - (\Delta - 1) = r - \Delta \) neighbors in \( V(S) \setminus \{v_0\} \). In fact, \( J \) achieves the lowest average degree if \( T \) is a complete graph. In that case, \( J \) is composed of \( S \), a complete graph \( T \), \( \Delta \) edges connecting \( S \) and \( T \) via \( v_0 \), and another \( \Delta(r - \Delta) \) edges running between \( T \) and \( S \setminus \{v_0\} \). This yields that

\[
e(J) \geq e(S) + \Delta + \Delta(r - \Delta)
\]

and, consequently, that

\[
d(J) \geq \frac{e(S) + \Delta(r - \Delta)}{\nu(S) + \Delta}.
\]

Recall that \( S \in \mathcal{S}_F \) is isomorphic to a graph from \( \mathcal{A}(r) \) by definition. If \( S \) is a copy of \( K_{r+1} \), the right hand side of (5.25) is at least \( m_2(K_3, K_r) + \varepsilon \) for \( \varepsilon > 0 \) sufficiently small if \( v_0 \) violates the claim. As \( F \) is a member of the family \( \mathcal{C}(3, r) \), each one of the \( \Delta \) edges connecting \( v_0 \) and \( T \) is covered by at least \( r - 1 \) triangles. Therefore, each vertex of \( T \) has at least \( r - 1 \) common neighbors with \( v_0 \). Since no more than \( \Delta - 1 \) of these are in \( T \), every vertex of \( T \) has at least \( (r - 1) - (\Delta - 1) = r - \Delta \) neighbors in \( V(S) \setminus \{v_0\} \). In fact, \( J \) achieves the lowest average degree if \( T \) is a complete graph.

Hence, if \( \Delta < 3 \) for \( r = 4 \) or \( \Delta < r \) for \( r \geq 5 \), then \( F \) contains a subgraph \( J \) with \( d(J) \geq m_2(K_3, K_r) + \varepsilon \), contradicting that \( F \) is a member of \( \mathcal{C}_r(3, r) \).

Continuing the proof of Lemma 5.20, suppose that \( \widetilde{F} \subseteq F \). We shall show that this implies \( m(F) \geq m_2(K_3, K_r) + \varepsilon \) for some sufficiently small constant \( \varepsilon > 0 \), contradicting the assumption that \( F \in \mathcal{C}_r(3, r) \). As usual, we call a maximal connected subgraph of \( F \) without a cut-vertex a block. W.l.o.g. we may assume that \( F \) contains no isolated vertices. Moreover, as a member of \( \mathcal{C}(3, r) \), \( F \) cannot contain cut edges. Hence, every block \( B \subseteq F \) is a maximal 2-connected subgraph.

Observe that \( B \) is also a member of \( \mathcal{C}(3, r) \) since otherwise it could be extended as we have \( F \in \mathcal{C}(3, r) \) and \( B \subseteq F \). Since decomposing \( F \) into blocks partitions the edge set of \( F \), we can identify a block \( B_0 \subseteq F \) such that \( \widetilde{F}[V(B_0)] \subseteq B_0 \). Moreover, since we have \( m(F) \geq m(B_0) \), it suffices to prove the statement for \( F = B_0 \), that is, assuming \( F \) is 2-connected.

We shall argue that \( F \) contains at least one copy of \( K_{r+1} \). As we have \( F \in \mathcal{C}(3, r) \), every vertex \( v \in V(F) \) has degree at least \( (3 - 1) + (r - 1) - 1 = r \). Moreover, any two adjacent vertices have at least \( (3 - 2) + (r - 2) = r - 1 \) common neighbors. It follows that any vertex of degree exactly \( r \) and its neighborhood induce a clique of size \( r + 1 \) in \( F \). This implies that the degree of
all vertices in $V(F)$ that are not contained in a copy of $K_{r+1}$ is at least $r+1$. Hence, $F$ contains at least one copy of $K_{r+1}$ since otherwise we had

$$m(F) \geq d(F) = \frac{\sum_{v \in V(F)} \text{deg}(v)}{2v(F)} \geq \frac{r+1}{2} = m_2(K_r) \geq m_2(K_3, K_r) + \varepsilon$$

for $\varepsilon > 0$ sufficiently small.

Consider a fixed connected component $\hat{C} \subseteq \hat{F}$. Since $F$ is 2-connected and $\hat{C}$ is a connected $A$-forest, there are at least two vertices $v_1, v_2 \in V(\hat{C})$ with $\text{deg}_F(v) > \text{deg}_\hat{C}(v)$ that are not cut-vertices of $\hat{C}$, i.e., that are contained in exactly one member of $S_F$. Let $k := |S_C|$ denote the number of (edge-disjoint) members of $S_F$ that are subgraphs of $\hat{C}$. Then we have

$$d(\hat{C}) = \frac{\sum_{S \in S_C} e(S)}{\sum_{S \in S_C} v(S) - (k-1)} = \frac{\sum_{S \in S_C} e(S)}{1 + \sum_{S \in S_C} (v(S) - 1)} + \sum_{S \in S_C} e(S).$$

We deduce from Claim 5.23 that, for $r \geq 5$, we have

$$\sum_{v \in V(\hat{C})} \text{deg}_F(v) v(\hat{C}) \geq 2r + \sum_{v \in V(\hat{C})} \text{deg}_\hat{C}(v) v(\hat{C}) = 2 \frac{r + \sum_{S \in S_C} e(S)}{1 + \sum_{S \in S_C} (v(S) - 1)}$$

due to (5.24) and $r \geq m_2(K_3, K_r) + \varepsilon$. For $r = 4$, we arrive at the same conclusion replacing $r$ by 3 in (5.27). Summing up over all connected components of $\hat{F}$ and applying Proposition 2.2, we conclude that

$$\sum_{v \in V(\hat{F})} \text{deg}_F(v) v(\hat{F}) \geq 2(m_2(K_3, K_r) + \varepsilon).$$

As we have shown that

$$\sum_{v \in F \setminus \hat{F}} \text{deg}_F(v) v(F \setminus \hat{F}) \geq 2(m_2(K_3, K_r) + \varepsilon)$$

in (5.26), it follows that $d(F) \geq m_2(K_3, K_r) + \varepsilon$, contradicting that $F$ is a member of $C_\varepsilon(3, r)$.

This concludes the proof of Lemma 5.20.

### 5.4. Proof of the 1-statement

Before addressing the 1-statement of Theorem 5.3 in full generality, we shall explain how to deduce it for 2-colorings. Suppose that $r > \ell \geq 3$ are integers. We need to show that there exists a constant $B = B(\ell, r)$ such that we have $G_{n,p} \rightarrow (K_\ell, K_r)$ a.a.s. for $p \geq B n^{-1/m_2(K_\ell, K_r)}$.

For $\ell = 3$, this was already proved by Kohayakawa and Kreuter.

**Theorem 5.24 ([KK97], Theorem 2).** Let $\ell \geq 3$ be a fixed integer and $H$ be a 2-balanced graph with $m_2(H) > m_2(C_\ell) = (\ell - 1)/(\ell - 2)$. Then there is a constant $B$ such that, setting $p = p(n) = B n^{-1/m_2(C_\ell, H)}$, we have

$$\mathbb{P}[G_{n,p} \rightarrow (H, C_\ell)^\varepsilon] = 1 - o(1).$$

Clearly, a triangle is a cycle of length 3 and every clique is 2-balanced. Assuming the KLR-Conjecture holds, we can generalize this statement to arbitrary graphs $J$ and $H$ satisfying certain restrictions.
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Theorem 5.25. Let \( J \) be a graph for which Conjecture 5.27 holds and \( H \) be a 2-balanced graph with \( m_2(H) > m_2(J) > 1 \). Then there is a constant \( B \) such that, setting \( p = p(n) = Bn^{-1/m_2(J,H)} \), we have

\[
P[G_{n,p} \to (H,J)^e] = 1 - o(1)
\]

Our proof of Theorem 5.25 proceeds along the same lines as the proof of Theorem 5.24 in [KK97]. Before presenting the full details in Section 5.4.1, we shall briefly discuss the crucial differences between both proofs. Inspection of the proof of Theorem 5.24 given in [KK97] reveals that only Lemma 17 from there actually depends on the structure of \( C_\ell \). All other statements and arguments also hold true when \( C_\ell \) is replaced by an arbitrary graph \( J \) with \( 1 < m_2(J) < m_2(H) \). Lemma 17 yields an upper bound on the number of graphs satisfying certain pseudo-random properties. Those are based on the notion of \((\varepsilon,p)\)-regular pairs, as introduced in Section 2.4.1.

Let \( J \) be graph of order \( v(J) = \ell \) on the vertex set \( \{v_1, \ldots, v_\ell\} \). Let also \( V = (V_i)_{i=1}^\ell \) be a family of \( \ell \) pairwise disjoint sets, each of cardinality \( m \). Let reals \( 0 < \gamma \leq 1 \), \( D \geq 1 \) and an integer \( T \) be given. We shall be interested in counting the number of \( \ell \)-partite graphs \( F \) in the family denoted by

\[
F(J,\varepsilon,\bar{p},\gamma,D;V,T)
\]

with partition classes \( V_i, 1 \leq i \leq \ell \), and exactly \( T \) edges that satisfy the following properties:

1. \((V_i,V_j)\) is \((\varepsilon,F,\bar{p})\)-regular and has \( \bar{p} \)-density \( \gamma \leq d_{F,\bar{p}}(V_i,V_j) \leq D \) whenever \( v_iv_j \in E(J) \).
2. \( F \) is \( J \)-free.

Then Lemma 17 from [KK97] reads as follows.

Lemma 5.26 ([KK97], Lemma 17). Let an integer \( \ell \geq 3 \) be fixed and let constants \( 0 < \alpha \leq 1 \), \( 0 < \gamma \leq 1 \), \( C \geq 1 \) and \( D \geq 1 \) be given. Then there are constants \( 0 < \varepsilon \leq 1 \), \( \bar{B} \geq 0 \), and \( m_0 \) that depend only on \( \ell \), \( \alpha \), \( \gamma \), \( C \), and \( D \) such that, if \( \bar{p} = \bar{p}(m) \geq \bar{B}m^{-1/m_2(C_\ell)} \), for all integers \( m \geq m_0 \) and \( T \geq 1 \), we have

\[
|F(C_\ell,\varepsilon,\bar{p},\gamma,D;V,T)| \leq \alpha^T \left( \frac{(\ell + 2)m^2}{T} \right),
\]

provided each member \( F \) additionally satisfies the following property: for all \( U \subseteq V_{i-1} \) and \( W \subseteq V_i, 2 \leq i \leq \ell \), with

\[
|U| \leq |W| \leq \bar{d}|U| \leq \bar{d}^{\ell-2},
\]

where \( \bar{d} = \bar{p}m \), we have

\[
e_{F}(U,W) \leq C|W|.
\]

We state a version of the KLR-Conjecture that is equivalent to the formulation in Section 2.4.4 and immediately implies Lemma 5.26.

Conjecture 5.27 ([KLR97], Conjecture 23). Let \( J \) be a graph on \( \ell \geq 3 \) vertices and and let constants \( 0 < \alpha \leq 1 \), \( 0 < \gamma \leq 1 \), and \( D \geq 1 \) be given. Then there are constants \( 0 < \varepsilon \leq 1 \), \( \bar{B} \geq 0 \), and \( m_0 \) that depend only on \( J \), \( \alpha \), \( \gamma \), and \( D \) such that, if \( \bar{p} = \bar{p}(m) \geq \bar{B}m^{-1/m_2(J)} \), for all integers \( m \geq m_0 \) and \( T \geq 1 \), we have

\[
|F(J,\varepsilon,\bar{p},\gamma,D;V,T)| \leq \alpha^T \left( \frac{(\ell)m^2}{T} \right).
\]
Note that the factors \((\ell + 2)\) and \((\ell^2)\) respectively in the binomial coefficients are negligible since they contribute only a factor of \(O(1)^T\) to the total expression, which may be suppressed by choosing \(\alpha\) sufficiently small. Moreover, the constant \(D\) was arbitrarily set to 2 in the original formulation of Conjecture 5.27. The additional assumption on each member \(F \in F(C_\ell, \varepsilon, \delta, \gamma, D; V, T)\) in Lemma 5.26 turned out to be dispensable as Conjecture 5.27 has been verified for cycles of fixed length in [GKRS05].

Theorem 5.25 is basically proved by substituting Lemma 5.26 by Conjecture 5.27 and every occurrence of \(C_\ell\) by \(J\) in the proof of Theorem 5.24. Similarly, we can replace the assertion \(m_2(J) < m_2(H)\) in the theorem by \(m_2(J) = m_2(H)\) assuming that \(H\) is strictly 2-balanced. Moreover, as discussed in Section 5.4.4, we can deduce the following upper bound on the threshold for asymmetric Ramsey properties with multiple colors similarly to Theorem 4 in [KK97].

**Theorem 5.28.** Let \(k\) be an integer and \(H_1, \ldots, H_k\) be graphs such that \(H_1\) is 2-balanced with
\[
m_2(H_1) > m_2(H_2) \geq \cdots \geq m_2(H_k) > 1 .
\]
Suppose Conjecture 5.27 holds for all \(H_2, \ldots, H_k\). Then there is a constant \(B\) such that, setting \(p = p(n) = B n^{-1/m_2(H_2,H_1)}\), we have
\[
P[G_n,p \rightarrow (H_1, \ldots, H_k)^e] = 1 - o(1) .
\]
If \(H_1, \ldots, H_k\) are cliques, the conclusion of Theorem 5.28 follows even if Conjecture 5.27 is only known for the second greatest clique among those. All we need to observe is that, for given integers \(\ell_1 > \ell_2 \geq 3\),
\[
G \rightarrow (K_{\ell_1}, K_{\ell_2}, K_{\ell_2}, \ldots, K_{\ell_2})^e
\]
implies
\[
G \rightarrow (K_{\ell_1}, K_{\ell_2}, K_{\ell_3}, \ldots, K_{\ell_k})^e
\]
provided that \(\ell_i \leq \ell_2\) for all \(3 \leq i \leq k\) since we have \(K_{\ell_i} \subseteq K_{\ell_2}\). And the 1-statement of Theorem 5.3 follows because the threshold depends only on \(\ell_1\) and \(\ell_2\).

**5.4.1. Proof of Theorem 5.25.** We give a brief outline of the proof first. It heavily relies on the following notion: Let \(G\) and \(H\) be two graphs. A copy of \(H\) in \(G\) is isolated if it does not intersect any other copy of \(H\) in \(G\) on an edge. Suppose \(G_{\text{blue}} \cup G_{\text{red}}\) is any 2-coloring of \(G = G_{n,p}\) such that \(G_{\text{red}}\) contains no copy of \(H\). Thus, every copy of \(H\) in \(G\) has to contain at least one blue edge. Hence, the number of isolated copies of \(H\) in \(G\) provides a lower bound on the number of edges in \(G_{\text{blue}}\). In our case, that number is just given by the expected number of copies of \(H\) in \(G\), that is,
\[
\Theta(p^{e(H)} n^{\nu(H)}) = \Omega(n^{2-1/m_2(J)}) .
\]
Let \(\overline{G^H}\) be a graph constructed by taking exactly one blue edge from each isolated copy of \(H\) in \(G\). Since \(e(\overline{G^H}) = \Omega(n^{2-1/m_2(J)})\), we would expect a copy of \(J\) in \(\overline{G^H}\) provided it was a subgraph of a truly random graph \(G_{n,p_0}\), where \(p_0 = n^{-1/m_2(J)}\) (cf. Conjecture 1.1 on page 2). Although this is not quite so, we can still prove (cf. Lemmas 5.34 and 5.37) that \(\overline{G^H}\) has certain pseudo-random properties.

We denote the spanning subgraph of \(G\) that consists of all isolated copies of \(H\) by \(G^H\). Note that \(\nu(G^H) = v(G)\). We write \(E \subseteq E(G^H)\) if \(E \subseteq E(G^H)\) and every copy of \(H\) in \(G\) contains at most one element of \(E\). By \(\mathcal{I}(G, H)\) we denote the family of graphs \(\overline{G^H} \subseteq G^H\) in which all but one arbitrary edge are removed from every copy of \(H\) in \(G^H\). We now define two graph properties
and prove that \( G^H \in \mathcal{I}(G, H) \) satisfies both of them provided \( G = G_{n, p} \) with \( p \) sufficiently large. The first property is as follows:

**Property 5.29 (Regularization).** Let \( 0 < \gamma_0 \leq K_0 \) be fixed integers and \( \gamma_0, p_0, \varepsilon > 0 \) be real constants. Suppose \( G \) is a graph on the vertex set \([n]\) and \( J \) a graph on \([\ell]\). \( G \) satisfies Property 5.29 if there exists \( V_0 \subseteq V(G) \) of the form \( V_0 = (V_i)_{i=1}^\ell \), where the sets \( V_i \) are pairwise disjoint and have equal size \( \bar{n} \) with \( n/2K_0 \leq \bar{n} \leq n/K_0 \), such that for all \( \{i, j\} \in E(J) \), \( (V_i, V_j) \) is an \((\varepsilon, p_0)\)-regular pair in \( G[V_0] \) with density at least \( \gamma_0 \).

**Lemma 5.30.** Suppose \( H \) and \( J \) are graphs as in Theorem 5.25. There exist real constants \( \gamma_0 = \gamma_0(H, J) \) and \( \varepsilon_0 = \varepsilon_0(H, J) > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \), there exists integers \( 0 < k_0 = k_0(H, J, \varepsilon) \leq K_0 = K_0(H, J, \varepsilon) \) such that for all \( B > 0 \),

\[
p = Bn^{-(\varepsilon(H)-2+1/m_2(J))/e(H)},
\]

and

\[
p_0 = \frac{(\gamma_0 e(H))^2}{|\Aut(H)|^2} = (2 + o(1)) \frac{e(H)P\varepsilon(H)}{|\Aut(H)|} n^{-1/m_2(J)},
\]

the graph \( G = G_{n, p} \) a.a.s. satisfies that Property 5.29 holds for every member \( G^H \) of the family \( \mathcal{I}(G, H) \).

We continue with the second property.

**Property 5.31 (Embedding).** Let \( 0 < k_0 \leq K_0 \) be fixed integers and \( \gamma_0, p_0, \varepsilon > 0 \) be real constants. Suppose \( G \) is a graph on the vertex set \([n]\) and \( J \) a graph on \([\ell]\). \( G \) satisfies Property 5.31 if for all \( V \subseteq V(G) \) of the form \( V = (V_i)_{i=1}^\ell \), where the sets \( V_i \) are pairwise disjoint and have equal size \( \bar{n} \) with \( n/2K_0 \leq \bar{n} \leq n/K_0 \), the induced subgraph \( G[V] \) contains a copy of \( J \) provided the following condition holds: for all \( \{i, j\} \in E(J) \), \( (V_i, V_j) \) is an \((\varepsilon, p_0)\)-regular pair in \( G[V] \) with density at least \( \gamma_0 \).

**Lemma 5.32.** Suppose \( H \) and \( J \) are graphs as in Theorem 5.25. If the KLR-Conjecture holds, then for all \( 0 < \gamma_0 \leq 1 \), there exist real constants \( \varepsilon_0 = \varepsilon_0(H, J, \gamma_0) > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \), all integers \( 0 < k_0 \leq K_0 \), there is \( B = B(H, J, \gamma_0, k_0, K_0) > 0 \) such that for \( p \) as in (5.28) and \( p_0 \) as in (5.29), \( G = G_{n, p} \) a.a.s. satisfies that Property 5.31 holds for every member \( G^H \) of the family \( \mathcal{I}(G, H) \).

The proof of Theorem 5.25 follows from the combination of Lemmas 5.30 and 5.32.

**Proof of Theorem 5.25.** Let \( H \) and \( J \) be as in Theorem 5.25. We define the constants as follows. Invoke Lemma 5.30 to obtain constants \( \gamma_0 \) and \( \varepsilon_0(H, J) \). By plugging \( \gamma_0 \) into Lemma 5.32, we obtain the constant \( \varepsilon_0(H, J, \gamma_0) \). Let

\[
\varepsilon := \min\{\varepsilon_0(H, J), \varepsilon_0(H, J, \gamma_0)\}
\]

in Lemma 5.30, which yields integers \( k_0 \) and \( K_0 \).

Suppose the edges of \( G = G_{n, p} \) are colored with two colors, say red and blue, without a red copy of \( H \). We shall show that this implies a blue copy of \( J \).

Observe that if the coloring of \( G \) contains no red copy of \( H \), then in each isolated copy of \( H \) in \( G^H \), there must be at least one blue edge. Consider the graph \( G^H_0 \subseteq G^H \) where we pick exactly one arbitrary blue edge from each copy of \( H \) in \( G^H \). Clearly, \( G^H_0 \) is member of the family \( \mathcal{I}(G, H) \). We know from Lemma 5.30 that \( G^H_0 \) satisfies Property 5.29. That is, there exists a vector \( V_0 \subseteq V(G^H_0) \) of the form \( V_0 = (V_i)_{i=1}^\ell \), where \( \ell := v(J) \), the sets \( V_i \) are pairwise
disjoint and have equal size \( \bar{n} \) with \( n/2K_0 \leq \bar{n} \leq n/k_0 \), such that for all \( \{i, j\} \in E(J) \), \((V_i, V_j)\) is an \((\epsilon, p_0)\)-regular pair in \( G^n_0[V_0] \) with density at least \( \gamma_0 \). By the choice of the constants, \( G \) a.a.s. also satisfies Property 5.31, and hence there must be a copy of \( J \) in \( G^n_0[V_0] \). Since all edges of this copy of \( J \) are colored blue, the proof of Theorem 5.25 is complete. \( \square \)

What remains to prove are Lemmas 5.30 and 5.32.

5.4.2. Proof of Lemma 5.30. Throughout this section we shall assume that \( p \) and \( p_0 \) are of the form as in (5.28) and (5.29) respectively, \( H \) is a 2-balanced graph, and \( J \) satisfies \( m_2(J) < m_2(H) \). We start with some technical lemmas.

**Lemma 5.33 ([KK97], Lemma 12).** For any graphs \( H_1, \ldots, H_r \), there exist positive constants \( c = c(H_1, \ldots, H_r) \) and \( k_0 = k_0(H_1, \ldots, H_r) \) for which the following property holds. For all \( k > k_0 \); in every \( r \)-edge-coloring of the complete graph on \( k \) vertices, there exists a color \( i, 1 \leq i \leq r \), that contains at least \( ck^v(H_i) \) copies of \( H_i \).

The proof of Lemma 5.33 is a rather simple application of Ramsey’s theorem and the averaging principle.

**Lemma 5.34 ([KK97], Lemma 13).** Let \( G = G_{n,p} \) and \( p_0 \) be as in (5.29). For any fixed set \( E \subseteq \binom{[n]}{2} \),

\[ \mathbb{P} \left[ E \subseteq E(G^H) \right] \leq p_0^{\left| E \right|} . \]

**Lemma 5.35 ([KK97], Lemma 14).** Let \( \omega = \omega(n) \to \infty \) as \( n \to \infty \) and \( G = G_{n,p} \). Then the graph \( G\) a.a.s. has the following property. For all \( U, W \subseteq V(G^H) \) with \( U \cap W = \emptyset \) that satisfy \( |U||W| \geq \omega n^2-1/m^2(J) \), we have

\[ e_{G^n}(U, W) \leq 3e(H)p_0|U||W| . \]

The proof of Lemma 5.35 makes use of Lemma 5.34.

**Definition 5.36.** Let \( G \) and \( H \) be a graphs on the vertex sets \([n]\) and \([h]\) respectively. Suppose \( W = (W_1, \ldots, W_h) \) is a vector of disjoint subsets of \( V(G) \). We denote the number \( Z_W = Z_W(G) \) of embeddings of \( H \) into \( G \) such that for all \( 1 \leq i \leq h \), vertex \( i \) of \( H \) is mapped into \( W_i \). Let \( Y_W = Y_W(G) \) denote the number of such embeddings into \( G^H \).

**Lemma 5.37 ([KK97], Lemma 16).** For any \( B > 0 \), \( G_{n,p} \) a.a.s. satisfies the following property. For any vector \( W = (W_i)_{i=1}^h \) of pairwise disjoint sets \( W_i \subseteq [n], 1 \leq i \leq h \), each of cardinality at least \( \log(n)/n \), we have \( Y_W = Y_W(G) \geq E[Z_W]/2 \).

The proof of Lemma 5.37 relies on the fact that \( m_2(J) < m_2(H) \) and that \( H \) is 2-balanced. It employs Janson’s inequality. Assuming that \( H \) is strictly 2-balanced, Lemma 5.37 and hence Theorem 5.25 hold even for graphs \( J \) and \( H \) with \( m_2(J) = m_2(H) \).

Now we are ready to prove Lemma 5.30.

**Proof of Lemma 5.30.** Let \( c = c(J, H, K_2) \) and \( k_0 = k_0(J, H, K_2) \) be guaranteed by Lemma 5.33. Set

\[ b := 3e(H), \]

\[ \gamma_0 := \min \left\{ 1, \frac{c}{2^W(H)+3e(H)} \right\}, \]

\[ \epsilon_0 := \min\{1/2, 2c\} . \]
For any given $0 < \varepsilon < \varepsilon_0$, let $\eta = \eta(\varepsilon, 1, b, k_0)$ and $K_0 = M_0(\varepsilon, 1, k_0)$ be given by Theorem 2.37 on page 17.

Consider $G = G_{n,b}$ and $\overline{G^H} \subseteq G^H$. By Lemma 5.35, $\overline{G^H}$ is $(\eta, b, p_0)$-bounded. By applying Theorem 2.37 on page 17 to $\overline{G^H}$, we obtain an $(\varepsilon, \overline{G^H}, p_0)$-regular partition $(C_i)_{i=0}^k$ with exceptional class $C_0$ such that $k_0 \leq k \leq K_0$. Set $n := |C_1| = \ldots = |C_k|$ and note that

$$n/\log n < n/2K_0 < n < n/k_0.$$ 

We 3-color the edges of the complete graph $K_k$ as follows:

- $ij$ has color 1 if $(C_i, C_j)$ is $(\varepsilon, p_0)$-regular and we have $|E_{\overline{G^H}}(C_i, C_j)| \geq \gamma_0 p_0 n^2$;
- $ij$ has color 2 if $(C_i, C_j)$ is $(\varepsilon, p_0)$-regular and we have $|E_{\overline{G^H}}(C_i, C_j)| < \gamma_0 p_0 n^2$;
- $ij$ has color 3 if $(C_i, C_j)$ is not $(\varepsilon, p_0)$-regular.

If there is copy of $J$ in color 1, then we are done. Hence assume this is not the case. Furthermore, since the partition $(C_i)_{i=0}^k$ is $(\varepsilon, \overline{G^H}, p_0)$-regular, there are only $\varepsilon(k^2) < ck^2$ edges in color 3. By Lemma 5.33, there must be at least $ck^2$ copies of $H$ in $K_k$ colored by color 2.

For every such copy of $H$ (on vertices $i_1,\ldots,i_h \in [k]$), sets $(C_{i_j})_{j=1}^h$ satisfy the assumptions of Lemma 5.37 and determine $p(H)\cdot n^2$ isolated copies of $H$ in $G$. Since distinct copies of $H$ in $K_k$ determine distinct isolated copies of $H$ in $G$, we obtain that the number of edges in pairs $(C_i, C_j)$ such that $ij$ is colored by 2, is at least

$$ck^2 \cdot p(H)\cdot n^2/2 \geq ck^2 \cdot p(H)\cdot n/2k^2 \geq c2^{-v(H)-1}B(H)n^{2-1/m_2(J)}.$$ 

On the other hand, by the definition of our 3-coloring, the number of edges in pairs $(C_i, C_j)$, where $ij$ is colored by 2, is at most

$$\sum_{ij \text{ has color 2}} |E_{\overline{G^H}}(C_i, C_j)| < \left(\frac{k^2}{2}\right) \cdot \gamma_0 p_0 n^2 \leq 4\gamma_0 e(H)B(H)\cdot n^{2-1/m_2(J)}.$$ 

The last two inequalities together with (5.30b) yield a contradiction. \hfill \square

5.4.3. **Proof of Lemma 5.32.** Suppose that $\gamma_0 > 0$ and graphs $H$ and $J$ are given. We apply the KLR-Conjecture for

$$\beta := \left(\frac{2e(J)}{\alpha^2}\right)^{e(J)}$$

and obtain constants $n_\beta = n_0(\beta, J)$, $\varepsilon_\beta = \varepsilon_0(\beta, J)$ and $B_\beta = B(\beta, J)$. Set $\varepsilon_0 := \min\{1/6, \varepsilon_\beta/2\}$ and let $\varepsilon \leq \varepsilon_0$ and $k_0 \leq K_0$ be given. We define

$$B := B_\beta.$$ 

Let $p$ and $p_0$ be as in (5.28) and (5.29), i.e.,

$$p := Bn^{-(\nu(H)+2+1/m_2(J))/e(H)}$$

and

$$p_0 := \frac{(n)_{\varepsilon(H)}e(H)}{|\text{Aut}(H)|\binom{n}{2}p^{e(H)}} = (2 + o(1))\frac{e(H)B^{e(H)}}{|\text{Aut}(H)|}n^{-1/m_2(J)}.$$
Suppose that Lemma 5.32 is not true and there exists a graph $\overline{G^H} \subseteq G^H$ in the family $I(G, H)$ for which there exists a vector $V = (V_i)_{i=1}^\ell$, where $\ell := v(J)$, such that

- all sets $V_i$ are pairwise disjoint and have equal size $\bar{n}$ with $n/2K_0 \leq \bar{n} \leq n/k_0$,
- for all $\{i, j\} \in E(J)$, $(V_i, V_j)$ is an $(\varepsilon, p_0)$-regular pair in $\overline{G^H}[V]$ with density at least $\gamma_0p_0$,
- $G^H[V]$ does not contain a copy of $J$.

For $G = G_{n,p}$, we call a vector $V = (V_i)_{i=1}^\ell$, where $V_i \subseteq V(G)$ for all $i \in [\ell]$, bad if

- all sets $V_i$ are pairwise disjoint and have equal size $\bar{n}$ with $n/2K_0 \leq \bar{n} \leq n/k_0$,
- there exists a copy of $F \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)$ such that $m := \gamma_0\bar{n}^2p_0$, the vertex set of $F$ is $\bigcup_{i=1}^\ell V_i$, and $E(F) \subseteq E(G^H)$. Consequently, $V$ is bad.

We claim that there exists a copy of $F \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)$ such that $F \subseteq \overline{G^H}[V]$, and, therefore, $E(F) \subseteq E(G^H)$. Indeed, for $n$ sufficiently large we have that $\bar{n} \geq n/2K_0 > n_\beta$. Furthermore, let $C(\varepsilon_\beta)$ be the constant guaranteed by Lemma 2.28 on page 15. Observe that we have that

$$m = \gamma_0\bar{n}^2p_0 \geq \frac{\gamma_0 B}{\|\text{Aut}(H)\|} \bar{n}^{2-1/m_2(J)} > C(\varepsilon_\beta)\bar{n}.$$ (The last inequality holds because $m_2(J) > 1$ and $n$ is large.) Then Lemma 2.28 and $2\varepsilon \leq \varepsilon_\beta$ imply that $\overline{G^H}[V]$ contains a member of $F(J, \bar{n}, \bar{m}, \varepsilon_\beta)$.

We will prove Lemma by computing the expected number $E[X]$ of bad vectors $V$. By Markov’s inequality it suffices to show that this expectation is $o(1)$. For given $n/2K_0 \leq \bar{n} \leq n/k_0$, $V = (V_i)_{i=1}^\ell$, and $F \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)$ denote by $X(\bar{n}, V, F)$ the indicator random variable for the event that

- all sets $V_i$ are pairwise disjoint and have equal size $\bar{n}$ with $n/2K_0 \leq \bar{n} \leq n/k_0$,
- $F$ has vertex set $V$ and $E(F) \subseteq E(G^H)$.

By Lemma 5.34, the probability of this event is at most

$$p_0^{e(F)} = p_0^{e(J)\bar{m}}$$

Clearly,

$$X = \sum_{n/2K_0 \leq \bar{n} \leq n/k_0} \sum_{V \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)} \sum_{F \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)} X(\bar{n}, V, F)$$

It follows from the KLR-Conjecture that

$$\sum_{F \in F(J, \bar{n}, \bar{m}, \varepsilon_\beta)} E[X(\bar{n}, V, F)] \leq |F(J, \bar{n}, \bar{m}, \varepsilon_\beta)| \left( \frac{\beta^m}{\bar{m}} \right) = e^{e(J)\bar{m}}$$

For any given $\bar{n}$ there are at most $(\ell + 1)^n$ vectors $V$. Furthermore,

$$e^{-e(J)\bar{m}} = e^{-e(J)\gamma_0\bar{n}^2p_0}$$
is maximized for \( \bar{n} = n/2K_0 \), therefore, we obtain
\[ \mathbb{E}[X] \leq n \times (\ell + 1)^n \times e^{-\varepsilon(J)n^2p_0/4K_0^2} \overset{(5.29)}{=} o(1), \]
as desired. (The last equality holds because \( m_2(J) > 1 \).)

5.4.4. Proof of Theorem 5.28. In this section we show how to adjust the proof of Theorem 5.25 to the case when we have more than two colors.

Suppose there exists a valid \( r \)-coloring \( G_1 \cup G_2 \cup \cdots \cup G_r \) of \( G = G_{n,p} \). Then every copy of \( H_1 \) in \( G \) has to contain at least one edge of some other color than 1. Hence, the number of isolated copies of \( H_1 \) in \( G \) provides a lower bound on the number of edges in \( G_2 \cup \cdots \cup G_r \). Again, the number of the isolated copies of \( H_1 \) in \( G \) is just given by the expected number of copies of \( H_1 \) in \( G \), that is,
\[ \Theta\left(p^{e(H_1)}n^{v(H_1)}\right) = \Omega\left(n^{2-1/m_2(H_2)}\right). \]
Hence, in some color \( c \in \{2, \ldots, r\} \), the graph \( G_c^{H_1} \) constructed by taking one edge in color \( c \) from each isolated copy of \( H_1 \) in \( G \) satisfies
\[ e\left(G_c^{H_1}\right) = \Omega\left(n^{2-1/m_2(H_2)}\right) \geq \Omega\left(n^{2-1/m_2(H_c)}\right). \]
Similarly to the 2-color case, \( G_c^{H_c} \) satisfies Properties 5.29 and 5.31 that guarantee a copy of \( H_c \) in \( G_c^{H_c} \).
CHAPTER 6

Two Stage Ramsey Games

In this chapter, we present a short proof of a result by Friedgut et al. [FKR+03], utilizing the main result of [GKRS05]. We consider the following two stage edge coloring game with two colors: the first round consists of a random graph with edge probability $p = cn^{-1/2}$ for any arbitrarily small constant $c > 0$; the second round is a sequence of $N = N(n)$ many random edges, which must be colored one by one in an online fashion. We show that the player a.a.s. creates a monochromatic triangle for any sequence $N \gg 1$ in the second round. We also provide lower bounds on the duration of the second round, which indicate a more general threshold phenomenon.

6.1. Introduction

We study an edge coloring problem for random graphs in which the aim is to avoid monochromatic triangles. We follow the taxonomy of Friedgut et al. [FKR+03], who distinguish three different variants of the game.

6.1.1. Offline game. In the offline game, the entire random graph $G_n = G_{n,p}$ is presented to the player before the coloring starts. Luczak, Ruciński, and Voigt [LRV92] proved that $n^{-1/2}$ is a threshold the property that there exists a coloring of $G$ without any monochromatic triangle. More precisely, they showed that if the player is allowed to use two colors, there are constants $b > 0$ and $B > 0$ such that a.a.s. there exists an edge coloring of $G$ without a monochromatic triangle provided $p < bn^{-1/2}$, but every two-coloring contains at least one such triangle if $p \geq Bn^{-1/2}$. Rödl and Ruciński [RR93, RR95] generalized this result from triangles to arbitrary graphs (cf. Theorem 2.19 on page 12).

The natural question arising in view of Theorem 2.19 is whether this threshold is sharp, i.e., we have $b = B$. This conjecture is a major open problem in random graph theory. Recently, these questions motivated a great deal of research activities (see [FK96], [Fri99], [FK00], [Fri05]) and culminated in a paper by Friedgut, Rödl, Ruciński and Tetali [FRR06] who established the existence of a sharp threshold for triangles and two colors, but its value is yet unknown.

6.1.2. Online game. Friedgut et al. [FKR+03] transferred these problems into an algorithmic or 'online' setting. They studied the following one player game. The board is a graph with $n$ vertices, which initially contains no edges. The edges are presented to the player, henceforth called Painter, one by one in an order chosen uniformly at random among all permutations of the underlying complete graph. One of two colors, red or blue, has to be assigned to each edge immediately. Painter's objective is to color as many edges as possible without creating a monochromatic copy of some fixed graph $F$. As soon as the first monochromatic copy of $F$ is closed, the game ends. We call this the online $F$-avoidance game and refer to the number of properly colored edges as its duration. Friedgut et al. [FKR+03] showed that for triangles,
there is a threshold that differs dramatically from the one in the offline case. The online game a.a.s. stops whenever Painter has seen substantially more than \( n^{4/3} \) edges.

This result was generalized in [MSS05] from triangles to a large class of graphs, which contains cliques and cycles of fixed size. It was shown that, for all graphs \( F \) in this class, there exists a function \( N_0 = N_0(F, n) \) and a strategy such that Painter a.a.s. 'survives' with this strategy as long as the number of edges is substantially less than \( N_0 \). It was also shown that this strategy is best possible, that is, there is no strategy enabling Painter to survive substantially more than \( N_0 \) edges.

Recall the density measures for graphs introduced in Section 2.2.1. With these definitions at hand, the main result of [MSS05] reads as follows.

**Theorem 6.1.** Let \( F \) be a 2-balanced graph that is not a tree. If there exists a subgraph \( F_- \subseteq F \) with \( e_F - 1 \) edges such that

\[
m_2(F_-) \leq \frac{e_F^2}{v_F + e_F(v_F - 2)},
\]

then the threshold for the online \( F \)-avoidance game with two colors is

\[
N_0(F, n) = n \left( 2 - \frac{1}{m(F)} \right) \left( 1 + \frac{1}{e_F} \right).
\]

More precisely, Painter a.a.s. succeeds with the greedy strategy (always use one color, say red, if this does not close a monochromatic copy of \( F \), and blue otherwise) if the number of edges \( N < N_0 \), and any strategy fails if \( N > N_0 \).

In particular, this theorem yields the threshold formula

\[
N_0(\ell, n) = n^{(2-1/m(F)) \left( 1 + 1/e_F \right)}
\]

for cliques of fixed size \( \ell \) and the formula

\[
N_0(\ell, n) = n^{1+1/\ell}
\]

for cycles of length \( \ell \).

The methods from [MSS05] yield more general lower and upper bounds than stated in Theorem 6.1. These bounds indicate that \( N_0(F, n) = n^{2 - 1/m_2(F)} \) as defined in (2.8) could be the threshold for all non-forests \( F \). This remains an open problem.

### 6.1.3. Semi-online game

The so-called semi-online version is a combination of both the offline and the online game. It is played in two rounds. First, a random graph \( G = G_{n,p} \) is generated with a fixed edge probability \( p = p(n) \) and is entirely presented to Painter. After Painter assigned a valid coloring to \( G \), i.e., a coloring without monochromatic triangles, new edges appear one by one and must be colored in an online fashion, always maintaining a valid coloring. The semi-online game was the main focus of Friedgut et al. in [FKR+03]. They studied the case when the edge probability is \( p = cn^{-1/2} \) for any (arbitrarily small) constant \( c > 0 \), and two or three colors respectively are available to Painter. Note that, up to the constant factor \( c \), this probability corresponds to the threshold of the offline version of the game. We give a short proof of the main result in [FKR+03] for the game with two colors. We shall show that no matter how small \( c > 0 \) is, the second round of the game a.a.s. ends after any unbounded number of moves.

**Theorem 6.2.** Let \( c > 0 \) be fixed. If the first round of the semi-online triangle avoidance game with two colors is played on a random graph \( G_{n,p} \) with \( p = cn^{-1/2} \), then the second round a.a.s. ends after any \( N = N(n) \gg 1 \) moves.
The lower bound on the duration of the second round is trivial as long as the graph presented in the first round is relatively dense. Starting off with a random graph with edge probability substantially less than \( cn^{-1/2} \) in the first round enables Painter to survive more moves in the second round with a suitable strategy. The following theorem quantifies the tradeoff between the density of the graph from the first round and the duration of the second round.

**Theorem 6.3.** Let \( c > 0 \) and \( 0 < \alpha \leq 1/6 \) be fixed. If the first round of the semi-online triangle avoidance game with two colors is played on a random graph \( G_{n,p} \) with \( p = cn^{-1/2-\alpha} \), then Painter can a.a.s. survive any \( N = N(n) \ll n^{8\alpha} \) moves in the second round without creating a monochromatic triangle.

We conjecture that this theorem can be matched from above. This would include both the online and the offline triangle avoidance game with two colors as extremal cases for \( \alpha = 1/6 \) and \( \alpha = 0 \) respectively. Unfortunately, our methods do not seem powerful enough to prove this.

**Conjecture 6.4.** Let \( c > 0 \) and \( 0 < \alpha \leq 1/6 \) be fixed. If the first round of the semi-online triangle avoidance game with two colors is played on a random graph \( G_{n,p} \) with \( p = cn^{-1/2-\alpha} \), then the threshold for the duration of the second round is

\[
N_0(n) = n^{8\alpha}.
\]

Before we present our proofs of Theorems 6.2 and 6.3, we shall compare the online and the semi-online game in more detail as they are closely related. The proof of the upper bound in Theorem 6.1 utilizes a two round exposure. Suppose the online triangle avoidance game with two colors lasts for \( N \gg n^{4/3} \) edges. The game is reduced to an offline two-round game, where Painter is granted a mercy period of \( N_1 \) edges. She may wait until the end of this phase with the coloring of those edges. Then another \( N_2 = N - N_1 \) random edges are simultaneously added, and Painter must color them. The argument shows that, regardless of her strategy, Painter will a.a.s. create many 'threats' in the first round, which force her to create a monochromatic triangle in the second round.

Let \( R \) and \( B \) denote the subgraphs of \( G(n, N_1) \) spanned by the red and blue edges respectively, and let the base graph of \( R \), denoted by Base(\( R \)), be the set of all vertex-pairs that, joined by an edge, would complete a triangle in \( R \). Base(\( B \)) and Base(\( G(n, N_1) \)) are defined analogously. Clearly, if an edge from Base(\( R \)) (or Base(\( B \))) is added to the graph, it has to be colored blue (resp. red). The 'threats' are copies of triangles in Base(\( R \)) or Base(\( B \)). Showing that either Base(\( R \)) or Base(\( B \)) contains \( \Omega(n^2) \) triangles after the first round suffices to prove that Painter a.a.s. creates a monochromatic triangle in the second round, thereby ending the game.

One can try to argue along these lines in order to prove the upper bound of the semi-online game. This, however, yields a weaker result. Suppose \( G = G_{n,p} \) with \( p = cn^{-1/2} \) was colored offline with red and blue, where w.l.o.g. red was used more frequently than blue. Then a.a.s. we have

\[
e(R) \geq \frac{1}{4} p \binom{n}{2} = \Omega \left( n^{3/2} \right).
\]

By counting triples in the neighborhood in \( R \) of every vertex \( v \in V(R) \), we can establish approximately

\[
n \cdot \left( \frac{e(R)}{n} \right)^3 = \Omega \left( n^{5/2} \right)
\]

many triangles in Base(\( R \)) after the first round. Now \( N \) many new edges appear. Clearly, if \( n^{5/2} \cdot (N/n^2)^3 \to \infty \), then, in expectation, at least one of the threats will be hit in the
second round. Consequently, the second round ends after any \( N \gg \sqrt{n} \) edges. Hence, proving Theorem 6.2 requires a more sophisticated approach.

The main idea in the proof of Theorem 6.2 is to establish a lower bound on the number of edges in \( \text{Base}(R) \cap \text{Base}(B) \). We consider the critical pairs of vertices \((u, v)\) in \( G \) that are connected by a red as well as a blue path of length two in \( G \). Clearly, the game stops in the online phase as soon as any of those pairs becomes an edge. We shall show that the colored graph contains \( \Omega(n^2) \) many critical pairs after the first round, ending the game in the second round very quickly. The main difficulty is to establish this lower bound on the number of those bichromatic threats.

### 6.2. Upper bound on the duration of the game

In this section, we present the proof of Theorem 6.2. We show that every 2-coloring of a random graph \( G = G_{n,p} \) a.a.s. contains sufficiently many critical pairs, that is, pairs of vertices that end the game immediately once they become an edge. Let \( G = R \cup B \) be a 2-coloring of \( G \), i.e., we have \( E(G) = E(R) \cup E(B) \). Critical pairs of vertices are connected by a path of length two in \( R \) as well as in \( B \).

**Lemma 6.5.** For all \( c > 0 \), there exists a constant \( \tau > 0 \) such that the random graph \( G = G_{n,p} \) with \( p = cn^{-1/2} \) a.a.s. contains at least \( \tau n^2 \) critical pairs in every valid 2-coloring.

This lemma enables us to prove Theorem 6.2.

**Proof of Theorem 6.2.** Assume that the coloring of the graph \( G \) from the first stage is fixed and \( N = N(n) \) tends to infinity arbitrarily slowly. We shall show that the game is over after any \( N > 1 \) with probability tending to 1 as \( n \to \infty \). According to Lemma 6.5 there are a.a.s. \( \tau n^2 \) critical pairs in \( G \). Let \( X_i \) denote the indicator random variable for the event that the \( i \)-th move of the second round closes any of those pairs. Since there are less than \( n^2 \) pairs of vertices in \( G \) in total, for all \( i \leq N \), we have

\[
P[X_i = 0 | X_1 = 0 \land \ldots \land X_{i-1} = 0] \leq 1 - \frac{\tau n^2}{n^2} = 1 - \tau.
\]

Let \( X = \sum_{i=1}^{N} X_i \). The game ends within the second round if \( X \geq 1 \). Hence,

\[
P[X = 0] = 1 - P[X = 1] = 1 - \prod_{i=2}^{N} P[X_i = 0 | X_1 = 0 \land \ldots \land X_{i-1} = 0]
\]

\[
\geq 1 - (1 - \tau)^N
\]

\[
\geq 1 - \exp[-\tau N] = 1 - o(1).
\]

This concludes the proof of Theorem 6.2.

Note that with a slightly more complicated proof one arrives at the same conclusion no matter if the second round is played online or offline. We employ the regularity lemma (cf. Section 2.4.2) in order to prove Lemma 6.5. Our proof strategy is to find four classes \( V_1, V_2, V_R, \) and \( V_B \) of vertices in the regularized graph such that there are sufficiently many critical pairs \((v_1, v_2)\) in \( V_1 \times V_2 \), which are connected by a red path of length two via some vertex in \( V_R \) and a blue path of length two via some vertex in \( V_B \). More precisely, the graph induced by \( V_1, V_2, V_R, \) and \( V_B \) has the following structure.
PROPERTY 6.6 (Regularization). Let $0 < k_0 \leq K_0$ be fixed integers and $\lambda$, $p$, and $\varepsilon > 0$ be real constants. Suppose $G$ is a graph on the vertex set $[n]$, and $R \cup B$ is an edge coloring of $G$. $G$ satisfies Property 6.6 if there exist sets $V_1$, $V_2$, $V_R$, and $V_B \subseteq V(G)$, $V_1 \cap V_2 = \emptyset$, of equal size $n$ with $n/K_0 \leq n \leq n/k_0$ that satisfy the following property: for all $i \in \{1, 2\}$ and $C \in \{R, B\}$, the pair $(V_i, V_C)$ is $(\varepsilon, p)$-regular in $C$ with density at least $\lambda p$.

The substructure contained in graphs with Property 6.6 is depicted in Figure 6.1. The first one of our more technical lemmas states that there are suitable constants such that this structure exists in any 2-coloring of $G_{n,p}$ with $p = cn^{-1/2}$.

**LEMMA 6.7.** Let $0 < c < 1/5$ be fixed. There exist real constants $\lambda_c$, $k_c$, and $\varepsilon_c > 0$ such that for all $0 < \varepsilon \leq \varepsilon_c$, there exists an integer $K_c = K_c(\varepsilon)$ such that, for every valid coloring $R \cup B = G$, the graph $G = G_{n,p}$ with $p = cn^{-1/2}$ a.a.s. satisfies Property 6.6 with $k_0 \leftarrow k_c$, $K_0 \leftarrow K_c$, and $\lambda \leftarrow \lambda_c$.

The proof of Lemma 6.7 is postponed to Section 6.2.1. It is a fairly standard application of the regularity lemma. The second technical lemma is a consequence of the first one, Lemma 6.7, and certain properties of $(\varepsilon, p)$-regular pairs. It states that there are a.a.s. sufficiently many critical pairs in partition classes as described in Property 6.6 provided those are embedded into $G_{n,p}$.

**LEMMA 6.8.** Let $0 < c < 1/5$ be fixed. There exist constants $K_c > 0$ and $\tau_c > 0$ such that, for every valid coloring $R \cup B = G$, the graph $G = G_{n,p}$ with $p = cn^{-1/2}$ a.a.s. contains disjoint
sets $V_1, V_2 \subseteq V(G)$ of size $\bar{n} \geq n/K_c$ that satisfy the following property: there are at least $\tau_c \bar{n}^2$ pairs $\{u, w\} \in V_1 \times V_2$ that are connected by a red and a blue path of length two.

By choosing appropriate constants, we can easily deduce Lemma 6.5 from Lemma 6.8.

**Proof of Lemma 6.5.** W.l.o.g. we may assume that $c \leq 1/5$ since the number of critical pairs is an increasing graph property. We apply Lemma 6.8 with argument $c$ and obtain constants $K_c$ and $\tau_c$. Let

$$\tau := \frac{\tau_c}{K_c^2}. $$

Thus, for every valid coloring $R \cup B = G$, Lemma 6.8 guarantees

$$\tau_c \bar{n}^2 \geq \tau_c \left( \frac{n}{K_c} \right)^2 = \tau n^2$$

critical pairs. \qed

What remains to prove are the two more technical lemmas, namely Lemma 6.7 and 6.8. Those proofs are presented in the two following sections.

6.2.1. **Proof of Lemma 6.7.** Suppose that $G = G_{n,p}$ was colored with red and blue such that there is no monochromatic triangle. We call this a valid coloring denoted by $G = R \cup B$. We begin with proving lower bounds on the number of edges in $R$ and $B$. A triangle is isolated in $G$ if it does not intersect on an edge with another triangle in $G$. As observed in [FKR+03], the number of isolated triangles in $G$ is a.a.s. at least

$$\frac{e(G)^3}{1} n^3/2,$$

provided that $c \leq 1/5$. This is an immediate consequence of the expected number of triangles in $G$,

$$\left( \frac{1}{6} + o(1) \right) c^3 n^3/2,$$

and the expected number of triangle pairs sharing an edge,

$$\left( \frac{1}{4} + o(1) \right) c^3 n^3/2,$$

both of which are concentrated around the mean. Since every isolated triangle in $G$ contains at least one edge in every color, we have

$$e(R), e(B) \geq \frac{c^2 n^3}{1} \frac{3}{2} =: \lambda_0 cn^{3/2}.$$ 

As stated before we use the sparse version of the regularity lemma in order to prove Lemma 6.7. Once the graph has been partitioned by the lemma, we must find regular pairs that are sufficiently dense. The next claim was proved in [FKR+03, Claim 22]. It is a consequence of the concentration of the number of edges in large subgraphs of $G_{n,p}$.

**Claim 6.9.** Suppose $C \subseteq G$ is a subgraph of $G = G_{n,p}$ with $\kappa(C) \geq \lambda_0 cn^{3/2}$. There exist $\delta_0 > 0$ and $k_0 \geq 1$ such that, for all $0 < \delta \leq \delta_0$ and $K_0 \geq k_0$, any $(\delta, C, p)$-regular partition $(V_i)_{i=0}^k$, $k_0 \leq k \leq K_0$, with exceptional class $V_0$ a.a.s. satisfies the following property. Let $\Phi$ be the set of pairs $\{i, j\}$ $(1 \leq i < j \leq k)$ for which $(V_i, V_j)$ is $(\delta, p)$-regular and

$$\epsilon_C(V_i, V_j) \geq \frac{1}{2} \lambda_0 p |V_i||V_j|.$$  \,(6.1)
Then we have
\[ |\Phi| \geq \frac{5}{8} \lambda_0 \binom{k}{2}. \] (6.2)

The following claim asserts that dense, regular pairs of \( R \) and \( B \) intersect in at least one partition class. Consider the complete graph \( K_k \) on the vertex set \([k]\). We say that graphs \( K^R_k, K^B_k, \) and \( K^S_k \) on the same set of vertices are a decomposition of \( K_k \) if we have
\[ K_k = K^R_k \cup K^B_k \cup K^S_k. \]

**Claim 6.10.** Let \( 0 < \gamma \leq 1 \) be fixed. Suppose \( K^R_k \cup K^B_k \cup K^S_k \) is a decomposition of \( K_k \) satisfying
\[ e(K^R_k), e(K^B_k) \geq \gamma \binom{k}{2} \quad \text{and} \quad e(K^S_k) \leq 2\gamma \binom{k}{2}. \]

Then there exists a vertex \( v_0 \in [k] \) with positive degree in both \( K^R_k \) and \( K^B_k \).

**Proof.** For \( C \in \{R, B\} \), let \( U_C \subseteq [k] \) denote the set of vertices with positive degree in \( K^C_k \), i.e.,
\[ U_C := \left\{ v \in [k] : \deg_{K^C_k}(v) > 0 \right\}. \]

Clearly, we are done if \( U_R \cap U_B \neq \emptyset \). Hence, suppose \( U_R \) and \( U_B \) have no common element. The lower bound on the number of edges in \( K^R_k \) and \( K^B_k \) implies that
\[ |U_C|^2 \geq 2\gamma \binom{k}{2}, \]

since necessarily we have
\[ e(K^C_k) \leq \frac{|U_C|}{2} \leq \frac{|U_C|^2}{2} \]
for all \( C \in \{R, B\} \). As \( K^R_k \cup K^B_k \cup K^S_k \) decompose \( K_k \), however, we have
\[ e(K^S_k) \geq |U_R| \cdot |U_B| \geq 2\gamma \binom{k}{2}, \]
a contradiction. \( \square \)

Now we are ready to prove Lemma 6.7. Let
\[ \lambda_c := \frac{\lambda_0}{4} \]
and let \( k_0 \) and \( \delta_0 \) be the constants from Claim 6.9. Moreover, let
\[ \varepsilon_c := \min \left\{ \delta_0, \frac{\lambda_0}{2}, \frac{1}{2} \right\} \quad \text{and} \quad k_c := k_0. \] (6.3)

Suppose \( \varepsilon \) was chosen in the range \( 0 < \varepsilon < \varepsilon_c \). Let
\[ \delta := \frac{\varepsilon}{2}. \]

We apply the regularity lemma, Lemma 2.37, with parameters \( \varepsilon \rightarrow \delta, a \rightarrow 2, b \rightarrow 2, \) and \( k_0 \rightarrow k_c \) and obtain constants \( \eta \) and \( K_0 \). As \( G_{n,p} \) is a.a.s. \((\eta, b, p)\)-bounded due to Lemma 2.8, so are its subgraphs \( R \) and \( B \), and Lemma 2.37 guarantees a partition \((V_i)_{i=0}^k, k_0 \leq k \leq K_0\), with exceptional class \( V_0 \) that is \((\delta, C, p)\)-regular for all \( C \in \{R, B\} \).
Suppose the partition is fixed. Consider the following decomposition of the complete graph $K_k$.

For all $C \in \{R, B\}$, let $K^C_k$ be the subgraph of $K_k$ on the edge set

$$E(K^C_k) := \left\{ \{i, j\} \in \binom{[k]}{2} : (V_i, V_j) \text{ is } (\varepsilon, p)\text{-regular in } C \text{ and } e_C(V_i, V_j) \geq \frac{1}{2} \lambda_0 p|V_i||V_j| \right\}.$$

Let $K^S_k \subseteq K_k$ be the subgraph with

$$E(K^S_k) := \left\{ \{i, j\} \in \binom{[k]}{2} : (V_i, V_j) \text{ is not } (\varepsilon, p)\text{-regular in both } R \text{ and } B \right\}.$$

Clearly, the graphs $K^R_k \cup K^B_k \cup K^S_k$ decompose $K_k$ since, every edge $e \in E(K_k)$ that is not in $E(K^S_k)$ must be either in $E(K^R_k)$ or $E(K^B_k)$. This follows from the concentration of the number of edges between any pairs of vertex classes due to Chernoff’s inequality (cf. Lemma 2.3), that is, for any $1 \leq i < j \leq k$, we have a.a.s.

$$e_{G_n,p}(V_i, V_j) \geq \frac{1}{2} p|V_i||V_j|.$$

Therefore, at least one color $C \in \{R, B\}$ satisfies

$$e_C(V_i, V_j) \geq \frac{1}{2} \lambda_0 p|V_i||V_j|,$$

since $\lambda_0$ was chosen less than $1/2$. Claim 6.9 guarantees that, for all $C \in \{R, B\}$, we have

$$e(K^C_k) \geq \frac{5}{8} \lambda_0 \binom{k}{2} =: \gamma_0 \binom{k}{2}.$$

On the other hand, we have

$$e(K^S_k) \leq 2\delta \binom{k}{2} \leq \varepsilon \binom{k}{2} \leq \lambda_0 \binom{k}{2} < 2\gamma_0 \binom{k}{2}.$$

Applying Claim 6.10 with $\gamma \leftarrow \gamma_0$, we obtain a vertex $v_0 \in [k]$ that has positive degree in both $K^R_k$ and $K^B_k$, that is, vertex $v_0$ is incident to some vertex $v_R$ in $K^R_k$ and some vertex $v_B$ in $K^B_k$. Note that $v_R$ and $v_B$ are not necessarily distinct.

By construction, vertices $v_0$, $v_R$, and $v_B$ correspond to partition classes $V_0$, $V'_R$, and $V'_B$ respectively with the following property: for all $C \in \{R, B\}$, we have $(V_0, V_C)$ is a $(\delta, p)$-regular pair in $C$ with density $\lambda_0 p/2$. We split $V_0$ into two sets $V_1$ and $V_2$ of equal size $\bar{n}$ arbitrarily. For convenience, we pick arbitrary sets $V_R \subseteq V'_R$ and $V_B \subseteq V'_B$ of the same size. Due to Remark 2.25 for all $i \in \{1, 2\}$ and $C \in \{R, B\}$, the pair $(V_i, V_C)$ is $(\delta/2, p) = (\varepsilon, p)$-regular in $C$ with density

$$d(V_i, V_C) \geq d(V_0, V'_R) - \delta p \geq \left(\frac{\lambda_0}{2} - \delta\right) p = (\lambda_0 - \varepsilon) \frac{p}{2} \geq \lambda_0 \frac{p}{4} = \lambda_c p.$$

Setting

$$K_c := \frac{K_0}{4},$$

and observing that the size $\bar{n}$ of $V_1$, $V_2$, $V_R$, and $V_B$ satisfies

$$\frac{n}{K_c} = \frac{n}{k_0} \geq \bar{n} \geq (1 - \delta) \frac{n}{2K_0} \geq \frac{n}{K_c},$$

concludes the proof of Lemma 6.7.
6.2.2. Proof of Lemma 6.8. The proof of Lemma 6.8 requires some preparatory work. Our aim is to embed pairs connected by red and blue paths of length two into structures as obtained from Lemma 6.7. In the first part of the proof, we shall study properties of this family of graphs and show that, except for a very tiny fraction of the family, each member contains sufficiently many critical pairs. This is formally stated in Lemma 6.16, a consequence of Lemma 6.14, which in turn is deduced from Lemma 6.12. We conclude in the second part of the proof (cf. Corollary 6.17) from Lemma 6.16 and Markov's inequality that those exceptional members of the family are very unlikely to appear as a subgraph of $G_{n,p}$, from which the proof of Lemma 6.8 easily follows.

Suppose $G = G_{n,p} = R \cup B$ was 2-colored with no monochromatic triangle. Recall that Lemma 6.8 returns constants $\lambda_c$, $k_c$, and $K_c = K_c(\varepsilon)$ for any sufficiently small $\varepsilon > 0$ and sets $V_1, V_2, V_R$, and $V_B \subseteq V(G)$, $V_1 \cap V_2 = \emptyset$, of equal size $\bar{n}$ with $n/K_c \leq \bar{n} \leq n/k_c$ that satisfy the following property: for all $i \in \{1, 2\}$ and $C \in \{R, B\}$, the pair $(V_i, V_C)$ is $(\varepsilon, p)$-regular in $C$ with density at least $\lambda_p$. Owing to Lemma 2.28, this structure contains an $(\bar{\varepsilon})$-regular subgraph with exactly $\bar{m} := \lambda_p p\bar{n}^2$ edges between each of the four pairs of vertex classes, where $\bar{\varepsilon}$ is just slightly greater than $\varepsilon$. This graph is a member of the family $G(C_4, \bar{n}, \bar{m}, \bar{\varepsilon})$ as defined in Section 2.4.3. However, since we may have $V_R = V_B$, the cycle $C_4$ of length 4 can degenerate to a graph on 3 vertices with multiple edges, where two non-adjacent vertices were contracted into one. As we are merely interested in how many different ways the edges of the four edge-disjoint bipartite graphs of the "blown-up" $C_4$ can be arranged, this degenerate configuration is also admissible. For the sake of simplicity, one can think of pairwise disjoint sets $V_1, V_2, V_R,$ and $V_B$. The proof covers the degenerate case without any modifications as well.

In the following Lemmas 6.12, 6.14, and 6.16, we consider members of the family $G(C_4, \bar{n}, \bar{m}, \varepsilon)$ with uncolored edges. In order to simplify notation, we refer to the vertex sets of graphs from this family in the standard way, namely as $V_1, V_2, V_3,$ and $V_4$. When we apply Lemma 6.16 in the proof of Lemma 6.8, the correspondence will be

$$V_1 \leftrightarrow V_1, \ V_2 \leftrightarrow V_R, \ V_3 \leftrightarrow V_2, \ V_4 \leftrightarrow V_B.$$ 

We deal with different kinds of "black sheep" $B(C_4, \bar{n}, \bar{m}, \varepsilon)$ from the family $G(C_4, \bar{n}, \bar{m}, \varepsilon)$ in each of the Lemmas 6.12, 6.14, and 6.16. The first unpopular members are defined as follows.

**Definition 6.11.** $B_1(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon') \subseteq G(C_4, \bar{n}, \bar{m}, \varepsilon)$ is the subfamily of graphs that satisfy the following property: there exists a set $P_2 \subseteq V_2$ of size greater than $\bar{n}/100$ which admits a partitioning into $\ell > 0$ classes $Q_1, \ldots, Q_\ell$ such that, for each $i \in [\ell]$, we have

$$|Q_i| - \frac{\bar{m}_i}{\bar{n}} \leq \frac{1}{2} \frac{\bar{m}}{\bar{n}},$$

$$|\Gamma_V(Q_i)| \geq \frac{1}{4} \frac{\bar{m}_i^2}{\bar{n}^2},$$

and

$$|\Gamma_V(Q_i), V_4| \text{ is not } (\varepsilon', \frac{\bar{m}_i}{\bar{n}^2})\text{-lower-regular}.$$

Definition 6.11 is illustrated in Figure 6.2 on the following page.

**Lemma 6.12.** Let $\lambda > 0$ be fixed. For all $\beta > 0$ and $\epsilon' > 0$, there exist $\epsilon = \epsilon(\beta, \epsilon') > 0$ and $n_0 = n_0(\epsilon', \lambda)$ such that for all $n \geq n_0$, $\bar{n} \geq n/\log n$, and $\bar{m} \geq \lambda p\bar{n}^2$, we have

$$|B_1(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')| \leq \beta \bar{m} \left(\frac{\bar{n}^2}{\bar{m}}\right)^4.$$
Proof. Suppose $\lambda, \beta, \epsilon', n, \text{ and } m$ are given as in the lemma. Let

$$\nu := \frac{1}{100} \quad \text{and} \quad \beta := \left(\frac{\beta}{2n^3}\right)^{\beta/\nu}.\quad (6.4)$$

Applying Lemma 2.33 on page 16 with arguments $\beta \leftarrow \beta$ and $\epsilon'$ yields a constant $\epsilon_0 > 0$. We choose $\epsilon := \epsilon_0$. With those constants at hand, we prove the upper bound on the number of ways to construct members of the family $B_1(C_4, \bar{n}, m, \epsilon, \epsilon')$ as stated in the lemma. Start by choosing the degree $d$ of each vertex $v \in V_2$ into $V_3$, which yields at most $nn$ possibilities. Next, choose the set of bad vertices $F_2 \subseteq V_2$ (at most $2^n$ possibilities) and partition it into $\ell$ classes $Q_1, \ldots, Q_\ell$. Since we can assign each vertex in $F_2$ to one of $\ell \leq \bar{n}$ classes, there are at most $\bar{n}^\ell$ ways for partitioning $F_2$. Clearly, this determines the size of $Q_i$ for all $i \in [\ell]$. Now, for each $Q_i$, $1 \leq i \leq \ell$, fix the cardinality $t_i := |\Gamma_{V_3}(Q_i)|$ of the neighborhood set. Since we have at most $\bar{n}^\ell$ ways for partitioning $F_2$, and each neighborhood has size at most $\bar{n}$, $\bar{n}^\ell$ is an upper bound on the number of choices for all $t_i$. Summing up over the number of choices so far yields at most

$$2^{\bar{n}^3 n} \leq 2^{4n^3 \log n} \leq 2^{\lambda n^3 / \log n} \leq 2^{\bar{n}}$$

possibilities provided $n \geq n_0$ is sufficiently large. The key observation is that the number of ways for choosing the neighborhood $\Gamma_{V_3}(Q_i)$ of each set $Q_i \in F_2$ can be controlled using Lemma 2.33. Due to Remark 2.24, the pair $(V_3, V_4)$ is $(\epsilon, m/n^2)$-lower-regular. Moreover, the lower bounds on $t_i$, $1 \leq i \leq \ell$, and $m$ assert that, for all constants $C = C(\epsilon')$ as in Lemma 2.24, there exists $n_0$
such that for all $n \geq n_0$, we have

$$t_i = |\Gamma_{V_3}(Q_i)| \geq \frac{1}{4} \frac{m^2}{n^2} = \frac{1}{4} \frac{m^2}{n^4} \cdot \frac{n^2}{m} \geq \frac{1}{4} \frac{(\lambda p n^2)^3}{n^4} \cdot \frac{n^2}{m} \geq \frac{\lambda^3 c_3}{(\log n)^2} \cdot \frac{n^2}{m} \geq C \frac{n^2}{m} .$$

Under these assumptions Lemma 2.33 guarantees that there are at most

$$\beta n^3 \binom{n}{t_i}$$

choices for the neighborhood $\Gamma_{V_3}(Q_i) \subseteq V_3$ such that $(\Gamma_{V_3}(Q_i), V_4)$ is not $(\varepsilon, \bar{m}/n^2)$-lower-regular. Therefore, the number of ways to determine these neighborhoods and to distribute $\bar{m}_i := \sum_{v \in Q_i} d_v$ edges for each set $Q_i$ is at most

$$\prod_{i=1}^{t} \beta n^3 \binom{n}{t_i} \left( \frac{t_i|Q_i|}{\bar{m}_i} \right) \leq \beta n^3 \frac{m^2}{n^2} \prod_{i=1}^{t} \left( \frac{e t_i|Q_i|}{\bar{m}_i} \right)^{\bar{m}_i}$$

$$\leq \beta n^3 \frac{m^2}{n^2} \left( \frac{e^{2m}}{\beta^{2m/8}} \exp \left( 2 \sum_{i=1}^{t} \bar{m}_i \right) \prod_{i=1}^{t} \left( \frac{\bar{m}_i|Q_i|}{\bar{m}_i} \right) \right)$$

$$\leq \left( \beta^{2m/8} e^{2m} \right)^{\bar{m}} \left( \frac{\bar{m}_i|Q_i|}{\bar{m}_i} \right) \leq \left( \beta^{2m/8} e^{2m} \right)^{\bar{m}} \left( \frac{|E|}{\bar{m}_i} \right) \right) .$$

(6.5)

It remains to choose the edges between $V_2 \setminus F_2$ and $V_3$, which yields at most

$$\left( \frac{m^2}{n^2} \right)^3$$

(6.6)

possibilities, and the edges between the pairs $(V_1, V_2)$, $(V_3, V_4)$, and $(V_4, V_1)$, which is bounded from above by

$$\left( \frac{m^2}{n^2} \right)^3$$

(6.7)

Combining (6.4), (6.5), (6.6), and (6.7) yields that there are at most

$$\left( \beta^{2m/8} e^{2m} \right)^{\bar{m}} \left( \frac{|E|}{\bar{m}_i} \right) \leq \beta \left( \frac{n^2}{m} \right)^4$$

elements in the family $B_1(C_4, n, \bar{m}, \varepsilon, \varepsilon')$ due to the choice of $\beta$.

Another type of members of family $G(C_4, n, \bar{m}, \varepsilon)$ that we would rather avoid is the following.

**Definition 6.13.** $B_2(C_4, n, \bar{m}, \varepsilon, \varepsilon') \subseteq G(C_4, n, \bar{m}, \varepsilon)$ is the subfamily of graphs that satisfy the following property: there exists a set $F_1 \subseteq V_1$ of size greater than $n/100$ such that, for each $v \in F_1$,

$$N_v := \Gamma_{V_3}(\Gamma_{V_3}(v))$$

has size less than $\frac{1}{4} \frac{m^2}{n^2}$ ,

(6.8a)

or

$$\forall N_v \subseteq N_v, |N_v| \geq \frac{1}{4} \frac{m^2}{n^2} : (N_v, V_4) \text{ is not } (\varepsilon, \bar{m}/n^2) \text{-lower-regular} .$$

(6.8b)

Definition 6.13 is illustrated in Figure 6.3 on the following page.
**Figure 6.3.** The family $B_2(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon')$.

**Lemma 6.14.** Let $\lambda > 0$ be fixed. For all $\beta > 0$ and $\varepsilon' > 0$, there exist $\varepsilon = \varepsilon(\beta, \varepsilon') > 0$ and $n_0 = n_0(\varepsilon', \lambda)$ such that for all $n \geq n_0$, $\tilde{n} \geq n / \log n$, and $\lambda p \tilde{n}^2 \leq \tilde{m} \leq 2p \tilde{n}^2$, we have

$$|B_2(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon')| \leq \beta^m \left( \frac{n^2}{\tilde{m}} \right)^4.$$ 

**Proof.** We prove this lemma similarly to Lemma 6.12 by counting the number of ways to construct graphs from the family $B_2(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon')$. Let constants $\lambda, \beta, \varepsilon'$ be given. Invoking Lemma 6.12 with parameters $\lambda, \beta \leftarrow \beta/2$, and $\varepsilon'$ yields constants $\varepsilon(\beta, \varepsilon') =: \varepsilon_1$ and $n_0(\varepsilon', \lambda) =: n_1$. Let

$$\nu := \frac{1}{100} \quad \text{and} \quad \overline{\beta} := \left( \frac{\beta}{4} \right)^{4/\nu}.$$ 

Applying Lemma 2.32 on page 16 with parameters $\beta \leftarrow \overline{\beta}$ and $\nu \leftarrow 1/2$ yields $\varepsilon_0(\beta, \nu) =: \varepsilon_2$. We choose $\varepsilon := \min\{\varepsilon_1, (1 - \nu)\varepsilon_2, \nu/4\}$. Then Lemma 6.12 states that we have

$$|B_1(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon')| \leq |B_1(C_4, \tilde{n}, \tilde{m}, \varepsilon_1, \varepsilon')| \leq \left( \frac{\beta}{2} \right)^{\tilde{m}} \left( \frac{n^2}{\tilde{m}} \right)^4 \leq \frac{\beta^m}{2} \left( \frac{n^2}{\tilde{m}} \right)^4,$$

provided that $n \geq n_1$. Hence, it suffices to show that the family

$$B'_2(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon') := B_2(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon') \setminus B_1(C_4, \tilde{n}, \tilde{m}, \varepsilon, \varepsilon')$$

contains at most

$$\frac{\beta^m}{2} \left( \frac{n^2}{\tilde{m}} \right)^4$$

graphs for $n$ being sufficiently large.
Graphs from the family $B_2'(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')$ can be constructed as follows. Choosing the edges between the pairs $(V_2, V_3)$, $(V_3, V_4)$, and $(V_4, V_1)$ can be done in at most
\[ \binom{n^2}{m^3} \] ways. Since we construct a graph that is not member of the family $B_1(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')$, every set $F_2 \subseteq V_2$ that admits a partitioning as in Definition 6.11 has size at most $\nu \bar{n}$. Fix any such set $F_2^{\text{max}} \subseteq V_2$ of maximal size. Let $\widehat{V}_2 := V_2 \setminus F_2^{\text{max}}$. For each vertex $v \in V_1$, choose appropriate degrees $\text{deg}_{F_2'}(v)$ into $F_2'$ and $\text{deg}_{F_2}(v)$ into $\widehat{V}_2$ (at most $n^2$ possibilities). Note that this determines the set $F_\varepsilon \subseteq V_1$ of vertices that violate the degree property with respect to $\widehat{V}_2$ as defined in 2.29, i.e., for each $v \in V_1 \setminus F_\varepsilon$, we have
\[ \left| \text{deg}_{F_2'}(v) - (1 - \nu) \frac{\bar{m}}{n} \right| \leq \varepsilon (1 - \nu) \frac{\bar{m}}{n}, \] which implies
\[ \left| \text{deg}_{F_2'}(v) - \frac{\bar{m}}{n} \right| \leq \frac{1}{2} \frac{\bar{m}}{n} \] by our choice of $\varepsilon$ and $\nu$. Proposition 2.30 on page 15 states that we have
\[ |F_\varepsilon| \leq 2 \varepsilon \bar{n} \leq \frac{\nu}{2} \bar{n} \leq \frac{1}{2} |F_1|. \] Now select the set of bad vertices $F_1 \subseteq V_1$ (at most $2^5$ possibilities). Since we have $\bar{m} \geq \lambda \nu n^2$, altogether this results in at most
\[ 2^n \bar{n}^2 \bar{m} \leq 2^{3n} n \bar{m} \log n \leq 2^{\lambda \nu n^2 / \log n} \leq 2^{\bar{m}} \] possibilities for $n$ being sufficiently large.

It remains to select the neighborhood of each vertex $v \in V_1 \setminus V_2$. Let $F'_v := F_1 \setminus F_\varepsilon$. Due to (6.11), we have $|F'_v| \geq \nu \bar{n}/2$, and each vertex in $F'_v$ satisfies the degree property with respect to $\widehat{V}_2$ as in (6.10). Moreover, let
\[ \bar{m}_1 := e(F'_v, F_2^{\text{max}}) = \sum_{v \in F'_v} \text{deg}_{F_2^{\text{max}}}(v), \]
\[ \bar{m}_2 := e(F'_v, \widehat{V}_2) = \sum_{v \in F'_v} \text{deg}_{\widehat{V}_2}(v), \]
and
\[ \bar{m}_\nu := e(F'_v, V_2) = \bar{m}_1 + \bar{m}_2. \] Note that due to (6.10) we have
\[ \bar{m}_2 \geq |F'_v| \frac{1}{2} \frac{\bar{m}}{n} \geq \frac{1}{4} \nu \bar{m}. \] Next, we distribute the edges between $V_1 \setminus F'_v$ and $V_2$ and the edges between $F'_v$ and $F_2^{\text{max}}$, which are no more than
\[ \left( \frac{|V_1 \setminus F'_v| \bar{n}}{\bar{m}_1} \right) \left( \frac{|F'_v| \bar{m}_2}{\bar{m}_1} \right) \] possibilities. And finally, we select the neighborhood $Q_v := \Gamma_{\widehat{V}_2}(v)$ for each vertex $v \in F'_v$. Observe that any vertex $v \in F'_v$ satisfies
\[ |N'_v| < \frac{1}{4} \frac{\bar{m}_1^2}{n^2}, \]
where $N'_v := \Gamma_{V_3}(Q_v)$. For the sake of contradiction, suppose some vertex $v_0 \in F_\nu$ violates this inequality. Then we have

$$|N_{v_0}| = |\Gamma_{V_3}(\Gamma_{V_3}(v_0))| \geq |N'_v| \geq \frac{1}{4} \frac{m^2}{n^2},$$

that is, (6.8a) is false for $v_0$ and, consequently, $v_0$ must satisfy (6.8b) because we have $F_\nu \subseteq F_1$. This implies that the pair $(N'_v, V_4)$ is not $(\varepsilon', \frac{m}{n^2})$-lower-regular. Under this assumption we can add the set $Q_{v_0}$ to $F_2^\max$ maintaining the properties of $F_2^\max$ (cf. Definition 6.11) as we have

$$|Q_{v_0}| - \frac{m}{n} \leq \frac{1}{2} \frac{m}{n}$$

due to (6.10) and

$$|\Gamma_{V_3}(Q_{v_0})| = |N'_v| \geq \frac{1}{4} \frac{m^2}{n^2}.$$

This, however, contradicts the maximality of set $F_2^\max$.

Utilizing Lemma 2.32 limits the number of choices for the neighborhood sets $Q_v, v \in F_\nu$. Since $(V_1, V_2)$ is $(\varepsilon)$-regular, the pair $(V_1, \tilde{V}_2)$ is $(\varepsilon_2, \frac{m}{n^2})$-lower-regular due to the choice of $\varepsilon$ and Remarks 2.24 and 2.25 on page 14. For any vertex $v \in F_\nu$, let $q_v := |Q_v|$. Then (6.10) and $m \leq 2pm^2$ yield that

$$1 < \frac{1}{2} \frac{n}{m} \leq q_v < \frac{3}{2} \frac{n}{m} = \frac{1}{6} \frac{9m^2 n^2}{m^2} \leq \frac{1}{6} \frac{18c^2 m^2 n^2}{m^2} \leq \frac{1}{6} \frac{n^2}{m^2} \leq \frac{1}{6} \frac{n^2}{m^2}.$$  \hspace{1cm}  (6.15)

Thus, for each $v \in F_\nu$, we can apply Lemma 2.32 with parameter $c \leftarrow q_v$ to conclude that there are at most

$$\tilde{\beta}'_{q_v}(\tilde{V}_2)$$

sets of size $q_v$ in $\tilde{V}_2$ that cover less than

$$\frac{1}{2} q_v \frac{m}{n^2} \frac{n}{m} \geq \frac{1}{4} \frac{m^2}{n^2}$$

vertices of $V_3$, thus satisfying (6.14). Hence, the $m_2$ edges between $F_\nu$ and $\tilde{V}_2$ can be distributed in at most

$$\sum_{v \in F_\nu} \tilde{\beta}^{m_2}_{q_v}(\tilde{V}_2) \leq \tilde{\beta}^{m_2}_{m_2}(\Gamma_{V_3}|\tilde{V}_2)$$
\hspace{1cm}  (6.16)

many ways. Putting (6.9), (6.12), (6.13), and (6.16) together, we conclude that the number of graphs in $B'_2(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')$ can be bounded from above by

$$2^{m} \tilde{\beta}^{m_2}_{m_2}\left(\frac{|F_{\nu}|}{m_1}\right)\left(\frac{|\Gamma_{V_3}|}{m_2}\right)\left(\frac{|V_1 \setminus V_1|}{m_2 - m_\nu}\right)\left(\frac{\bar{n}^2}{m_2}\right)^3 \leq \left(2^{\tilde{\beta}'/4}\right) \left(\frac{m}{m_1}\right)^4 \leq \left(\frac{\beta}{2}\right)^m \left(\frac{\bar{n}^2}{m_2}\right)^4 \leq \frac{\beta^m}{2} \left(\frac{\bar{n}^2}{m_2}\right)^4.$$  \hspace{1cm}  \square

The last members of family $\mathcal{G}(C_4, \bar{n}, \bar{m}, \varepsilon)$ with bad characteristics is defined as follows.

**Definition 6.15.** $B_3(C_4, \bar{n}, \bar{m}, \varepsilon) \subseteq \mathcal{G}(C_4, \bar{n}, \bar{m}, \varepsilon)$ is the subfamily of graphs that satisfy the following property: there exists a set $F_1 \subseteq V_3$ of size greater than $\bar{n}/50$ such that, for each $v \in F_1$,

$$\tilde{N}_v := \Gamma_{V_3}(\Gamma_{V_3}(v)) \cap \Gamma_{V_3}(\Gamma_{V_3}(v))$$

has size less than $\frac{1}{16}\frac{\bar{m}^4}{\bar{n}^6}$.  \hspace{1cm}  (6.17)

Definition 6.15 is illustrated in Figure 6.4 on the following page.
Lemma 6.16. Let $\lambda > 0$ be fixed. For all $\beta > 0$, there exist $\varepsilon = \varepsilon(\beta) > 0$ and $n_0 = n_0(\lambda)$ such that for all $n \geq n_0$, $\bar{n} \geq n/\log n$, and $\lambda \bar{n}^2 \leq \bar{m} \leq 2\bar{n}^2$, we have

$$|B_3(C_4, \bar{n}, \bar{m}, \varepsilon)| \leq \beta^m \left(\frac{\bar{n}^2}{\bar{m}}\right)^4.$$ 

Proof. We prove Lemma 6.16 analogously to Lemma 6.14 by counting the number of ways to construct graphs from the family $B_3(C_4, \bar{n}, \bar{m}, \varepsilon)$. Let constants $\lambda$ and $\beta$ be given and set

$$\nu := \frac{1}{100} \quad \text{and} \quad \bar{\beta} := \left(\frac{\beta}{\nu}\right)^{4/\nu}.$$ 

Applying Lemma 2.32 on page 16 with parameters $\beta \leftarrow \bar{\beta}$ and $\nu \leftarrow 1/2$ yields $\varepsilon_0(\beta, \nu) =: \varepsilon'$. Invoking Lemma 6.14 with parameters $\lambda$, $\beta \leftarrow \beta/2$, and $\varepsilon'$ yields constants $\varepsilon(\beta, \varepsilon') =: \varepsilon_1$ and $n_0(\varepsilon', \lambda) =: n_1$. We choose $\varepsilon := \min\{\varepsilon_1, \nu/4\}$. Then due to Lemma 6.14 we have

$$|B_2(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')| \leq \left(\frac{\beta}{2}\right)^m \left(\frac{\bar{n}^2}{\bar{m}}\right)^4 \leq \frac{\beta^m}{2} \left(\frac{\bar{n}^2}{\bar{m}}\right)^4$$

provided $n \geq n_1$. Hence, it suffices to show that the family

$$B'_3(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon') := B_3(C_4, \bar{n}, \bar{m}, \varepsilon) \setminus B_2(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')$$

contains at most

$$\frac{\beta^m}{2} \left(\frac{\bar{n}^2}{\bar{m}}\right)^4$$

graphs if $n$ is sufficiently large.
Graphs from the family \( B'_3(C_4, \bar{n}, \bar{m}, \epsilon, \epsilon') \) can be constructed as follows. First, choose the edges between the pairs \((V_1, V_2), (V_2, V_3), \) and \((V_3, V_4), \) where we have at most
\[
\left( \frac{\bar{n}^2}{\bar{m}} \right)^3
\] (6.18)
possibilities. Select the set of bad vertices \( F_1 \subseteq V_1 \) (at most \( 2^\bar{n} \) possibilities) and, for each vertex \( v \in V_1, \) choose appropriate degrees \( \deg_{V_4}(v) \) into \( V_4 \) (at most \( \bar{m}^\bar{n} \) possibilities). Since we have \( \bar{m} \geq \lambda \bar{n}^2, \) altogether this results in at most
\[
2^\bar{n} \bar{n}^\bar{m} \leq 2^{\bar{n} \log \bar{n}} \leq 2^{\lambda \bar{n}^2/\log n} \leq 2^\bar{m}
\] (6.19)
possibilities for \( n \) being sufficiently large. Note that the degree of the vertices in \( V_1 \) determines the set \( F_\epsilon \subseteq V_1 \) of vertices that violate the degree property with respect to \( V_4 \) as defined in 2.29, i.e., for each \( v \in V_1 \setminus F_\epsilon, \) we have
\[
\left| \deg_{V_4}(v) \right| - \frac{\bar{m}}{\bar{n}} \leq \epsilon \frac{\bar{m}}{\bar{n}} \leq \frac{\bar{m}}{2 \bar{n}}
\] (6.20)
by our choice of \( \epsilon. \) Proposition 2.30 on page 15 states that we have
\[
|F_\epsilon| \leq 2 \bar{n} \bar{n} \bar{m} \leq \frac{\nu \bar{n}}{2} \leq \frac{1}{4} |F_1|
\] (6.21)
Moreover, since we construct a graph that is not member of the family \( B_2(C_4, \bar{n}, \bar{m}, \epsilon, \epsilon'), \) every subset of \( F'_1 \subseteq V_1 \) such that every vertex \( v \in F'_1 \) satisfies (6.8a) or (6.8b) from Definition 6.13 has size at most \( \nu \bar{n} \leq |F_1|/2. \) Fix any such set \( F'_1 \subseteq V_1 \) of maximal size. Let \( F_v := F_1 \setminus (F_\epsilon \cup F'_1). \) Clearly, we have
\[
|F_v| \geq \frac{1}{4} |F_1| \geq \frac{\nu \bar{n}}{2},
\]
and each vertex in \( F_v \) satisfies the degree property with respect to \( V_4 \) as in (6.20) and the following property: \( N_v = \Gamma_{V_4}(\Gamma_{V_3}(v)) \) contains a subset \( N'_v \) of size at least
\[
\frac{1}{4} \frac{\bar{m}^2}{\bar{n}^2}
\]
that forms an \((\epsilon', \bar{m}/\bar{n}^2)-\)lower-regular pair with \( V_4. \) Let
\[
\bar{m}_v := \epsilon(F_v, V_4) = \sum_{v \in F_v} \deg_{V_4}(v).
\]
Note that due to (6.20) we have
\[
\bar{m}_v \geq \left| F_v \right| \frac{\bar{m}}{2 \bar{n}} \geq \frac{1}{4} \nu \bar{m}_v.
\]
Next, we distribute the edges between \( V_1 \setminus F_v \) and \( V_4, \) which are no more than
\[
\left( \frac{|V_1 \setminus F_v|}{\bar{m} - \bar{m}_v} \right)
\] (6.22)
possibilities. And finally, we select the neighborhood \( Q_v := \Gamma_{V_4}(v) \) for each vertex \( v \in F_v. \) Utilizing Lemma 2.32 again limits the number of choices for the neighborhood sets \( Q_v, v \in F_v. \) For any vertex \( v \in F_v, \) let \( q_v := |Q_v|. \) Then (6.20) and \( \bar{m} \leq 2\bar{n}^2 \) yield that
\[
1 < q_v \leq \frac{1}{6} \frac{\bar{n}^2}{\bar{m}}
\]
as in (6.15). Moreover, the pair \((N'_v, V_4)\) is \((\epsilon', \bar{m}/\bar{n}^2)-\)lower-regular. Thus, for each \( v \in F_v, \) we can apply Lemma 2.32 with parameter \( c \leftarrow q_v \) to conclude that there are at most
\[
\bar{m}_v \left( \frac{\bar{n}}{q_v} \right)
\]
sets of size $q_v$ in $V_4$ that cover less than

$$\frac{1}{2} q_v \frac{\bar{m}}{\bar{n}^2} |N'_v| \geq \frac{1}{16} \frac{\bar{m}}{\bar{n}^6}$$

vertices of $N'_v$, thus satisfying (6.17) in Definition 6.15. Hence, the $\bar{m}_v$ many edges between $F_v$ and $V_4$ can be distributed in at most

$$\sum_{v \in F_v} \tilde{\beta}^{2m} \left( \frac{\bar{n}}{q_v} \right) \leq \tilde{\beta}^{2m_v} \left( \frac{|F_v|^{\bar{n}}}{\bar{m}_v} \right)$$

(6.23)

many ways. Putting (6.18), (6.19), (6.22), and (6.23) together, we conclude that the number of graphs in $B_3'(C_4, \bar{n}, \bar{m}, \varepsilon, \varepsilon')$ can be bounded from above by

$$2^n \tilde{\beta}^{m_v} \left( \frac{|F_v|^{\bar{n}}}{\bar{m}_v} \right) \left( \frac{|V_1 \setminus B_v|^{\bar{n}}}{\bar{m}} \right) \left( \frac{\bar{n}^2}{\bar{m}} \right)^3$$

$$\leq \left( \frac{2^n}{4} \right) \tilde{\beta}^{n^2} \left( \frac{\bar{n}^2}{\bar{m}} \right)^4 \leq \left( \frac{\beta}{2} \right)^n \left( \frac{\bar{n}^2}{\bar{m}} \right)^4 \leq \frac{\beta^n}{2} \left( \frac{\bar{n}^2}{\bar{m}} \right)$$

The following corollary is an immediate consequence of Lemma 6.16 and Markov's inequality for the random graph $G_{n,p}$.

**Corollary 6.17.** Let $\lambda > 0$ be fixed. There exist $\varepsilon_\lambda > 0$ such that for all constants $0 < \varepsilon < \varepsilon_0$ and all sequences $\bar{n} = \bar{n}(n)$ with $n \geq \bar{n} \geq n / \log n$ and $\bar{m} = \bar{m}(n) := \lambda \bar{n}^2$, $G_{n,p}$ a.a.s. contains no member of the family $B_3(C_4, \bar{n}, \bar{m}, \varepsilon)$.

**Proof.** Let

$$\beta := \frac{\lambda^4}{e^6}.$$

We apply Lemma 6.16 with parameters $\lambda$ and $\beta$ and obtain $\varepsilon =: \varepsilon_\lambda$ such that, for all $0 < \varepsilon \leq \varepsilon_\lambda$, we have

$$|B_3(C_4, \bar{n}, \bar{m}, \varepsilon)| \leq |B_3(C_4, \bar{n}, \bar{m}, \varepsilon_\lambda)| \leq \beta^n \left( \frac{\bar{n}^2}{\bar{m}} \right)^4$$

provided $n$ is sufficiently large. Therefore, the expected number of graphs from the family $B_3(C_4, \bar{n}, \bar{m}, \varepsilon)$ in $G_{n,p}$ is at most

$$n^{4n} |B_3(C_4, \bar{n}, \bar{m}, \varepsilon)| p^{\bar{m}} \leq e^{4n} \log n \beta^n \left( \frac{\bar{n}^2}{\bar{m}} \right)^4 p^{\bar{m}}$$

$$\leq e^{\bar{m}} \beta^n \left( \frac{\bar{n}^2}{\bar{m}} \right)^4 p^{\bar{m}}$$

$$= \left( \frac{\beta^e}{\lambda^4} \right)^{\bar{m}}$$

$$= e^\varepsilon \bar{m} = o(1).$$

Corollary 6.17 enables us to prove Lemma 6.8. Suppose $G = G_{n,p} = R \cup B$ was colored with red and blue without a monochromatic triangle. Applying Lemma 6.7 yields constants $\lambda_c$, $k_c$ and $\varepsilon_c$. Invoking Corollary 6.17 with parameter $\lambda \mapsto \lambda_c$ yields constant $\varepsilon_\lambda$. Let

$$\varepsilon := \min \left\{ \varepsilon_c, \frac{\lambda_c \varepsilon_\lambda}{2} \right\}.$$
Plugging $\varepsilon$ into Lemma 6.8 yields a constant $K_c = K_c(\varepsilon)$ such that $G$ a.a.s. satisfies Property 6.6, that is, there exist sets $V_1, V_2, V_R, \text{ and } V_B \subseteq V(G)$, $V_1 \cap V_2 = \emptyset$, of equal size $\bar{n}$ with $n/K_c \leq \bar{n} \leq n/k_c$ that satisfy the following property: for all $i \in \{1, 2\}$ and $C \in \{R, B\}$, the pair $(V_i, V_C)$ is $(\varepsilon, p)$-regular in $C$ with density at least $\lambda_{C}$. Owing to Proposition 2.27 on page 15, these four pairs are $(\varepsilon, \lambda_C)$-regular, and each of them contains a $(2\varepsilon/\lambda_C)$-regular subgraph on exactly $\bar{m} := \lambda_{C} \bar{n}^2$ edges due to Lemma 2.28 on page 15. Indeed, the constant $C = C(\varepsilon)$ in Lemma 2.28 is negligible since we have $\bar{m} \gg \bar{n}$.

Let $J \subseteq G$ be the subgraph formed by the four vertex sets $V_1, V_2, V_R, \text{ and } V_B$ and $4\bar{n}$ many edges such that, for all $i \in \{1, 2\}$ and $C \in \{R, B\}$, the pair $(V_i, V_C)$ is $(2\varepsilon/\lambda_C)$-regular in $C$ with exactly $\bar{m}$ edges. We shall argue that $J$ is a.a.s. not a member of the family $B_3(C_4, \bar{n}, \bar{m}, 2\varepsilon/\lambda_C)$. The constant $\varepsilon$ was chosen in such a way that we have $2\varepsilon/\lambda_C \leq \varepsilon$. Moreover, we have

$$\bar{n} \geq \frac{n}{K_c} \geq \frac{n}{\log n}$$

for $n$ sufficiently large. Hence, if $J$ was a member of $B_3(C_4, \bar{n}, \bar{m}, 2\varepsilon/\lambda_C)$, that would contradict Corollary 6.17.

Embedding sufficiently many critical pairs into $J$ concludes the proof of Lemma 6.8. Since $J$ is not a member of $B_3(C_4, \bar{n}, \bar{m}, 2\varepsilon/\lambda_C)$, there are at least

$$\frac{49}{50} \bar{n} \geq \frac{49}{50K_c} n$$

vertices in $v \in V_1$, each of which satisfies that

$$\bar{N}_v = \Gamma_{V_2}(\Gamma_{V_R}(v)) \cap \Gamma_{V_2}(\Gamma_{V_B}(v))$$

has size at least

$$\frac{1}{16} \frac{\lambda_C^4 p^4 \bar{n}^8}{\bar{n}^5} \geq \frac{c^4 \lambda_C^4 n^{-2} n^3}{16 \frac{K^3_c}{K^3_c}} = \frac{c^4 \lambda_C^4}{16 K_c^3 n}$$

as stated in Definition 6.15. This yields at least

$$\frac{49}{50K_c} n \cdot \frac{c^4 \lambda_C^4}{16 K_c^3 n} = \frac{49 c^4 \lambda_C^4}{800 K_c^3 n^2} =: \tau_c n^2$$

many critical pairs in $J$, and Lemma 6.8 is proved.

### 6.3. Lower bound on the duration of the game

In this section, we prove Theorem 6.3. We have to argue that Painter a.a.s. succeed in both, finding an edge coloring without monochromatic triangles of $G = G_{n,p}$ with $p = cn^{-1/2}$ in the offline phase and subsequently extending this coloring for $N \ll n^{3a}$ random edges in the online phase.

The statement is trivial for $\alpha = 0$ since then the success in the first round depends only on the constant $c$ due to Theorem 2.19 on page 12, and there are no edges to come in the second round. Hence, suppose we have $\alpha > 0$. Since this implies $p \ll n^{-1/2}$, Theorem 2.19 guarantees that a.a.s. there exists a valid coloring of $G = G_{n,p}$ in the first round with two colors, say, red and blue. Fix that coloring $G = R_0 \cup B_0$, decomposing $G$ into a red graph $R_0$ and a blue graph $B_0$. We alter the coloring as follows: consider all edges $e \in E(R_0)$ in an arbitrary order, each time moving $e$ into $E(B_0)$ if this does not create a triangle in $B_0$. After all edges in $E(R_0)$ have been processed in that way, we obtain a new valid coloring $G = R \cup B$. It remains to show that this coloring may be properly extended in the online phase.
Claim 6.18. Painter can a.a.s. color any $N \ll n^{8a}$ edges in the online phase without a monochromatic triangle by playing greedily, i.e., assigning blue to every new edge unless this would close a blue triangle.

Proof. Suppose Painter has just lost the game. According to her strategy, she never creates a blue triangle. That is, after the final move the graph contains a red triangle $T$. By construction, the endpoints of each edge $e \in E(T)$ must be connected by a blue path of length two as otherwise $e$ would have been colored blue in the first as well as in the second round. We distinguish two cases. If all three blue paths run through pairwise distinct middle vertices, the final edge ending the game must be an inner edge of a so-called pyramid graph, which consists of an inner triangle each of whose edges is contained in another outer triangle. Note that a pyramid graph consists of six vertices and nine edges. If at least two of the blue paths have their middle vertex in common, the final edge necessarily belongs to a $K_4$.

The probability of losing the game in one particular step of the online phase is therefore dominated by the number of vertex pairs $(u, v)$ in the graph at this moment that close either the inner triangle of a pyramid graph or a $K_4$. We refer to such pairs as threats. An upper bound on the number of threats in the graph can be computed by counting the number of subgraphs isomorphic to a pyramid graph without one inner edge (henceforth denoted by $P^-$) and the number of subgraphs isomorphic to a $K_4$ without one edge (henceforth denoted by $K_4^-$). Note that not every such threat is actually dangerous for Painter since the coloring of the surrounding structure is not taken into account. Thus, we potentially overestimate the risk of losing the game.

Let $G_i$ denote the random graph after the insertion of the $i$-th edge in the online phase and $G_0 = G$. Let the random variable $X(G_i, H)$ count the number of subgraphs isomorphic to $H$ in $G_i$. Similarly, let $X(G_{n,p}, H)$ count the number of subgraphs isomorphic to $H$ in the random graph $G_{n,p}$. Let $Q(H, x)$ denote the property that a given graph contains at least $x$ subgraphs isomorphic to $H$. Clearly, $Q(H, x)$ is a monotone increasing property. We want to give a bound on the probability that $G_i \in Q(H, x)$ for $H = P^-$ and $H = K_4^-$ and appropriate values $x$.

Every graph $G_i$, $0 \leq i \leq N$, appearing in the random process is distributed like $G_{n,m+i}$, where $m := e(G_0)$. Hence,

$$P[G_i \in Q(H, x)] = \sum_{m=0}^{(2)} P[G_{n,m+i} \in Q(H, x)] \cdot P[e(G_{n,p}) = m]$$

$$\leq P[G_{n,m_0+i} \in Q(H, x)] + P[e(G_{n,p}) \geq m_0] ,$$

as $Q(H, x)$ is monotone increasing. Setting $m_0 := n^2 p$ we obtain that

$$P[e(G_{n,p}) \geq m_0] = e^{-\Theta(n^2 p)}$$

by application of Chernoff bounds, and since we have

$$m_0 + i \leq m_0 + N = m_0 + o(n^{8a}) = m_0 + o(m_0) \leq 2m_0$$

for $\alpha \leq 1/6$. Hence, we get

$$P[G_i \in Q(H, x)] \leq P[G_{n,2m_0} \in Q(H, x)] + e^{-\Theta(n^2 p)} .$$
Again using the monotonicity of \(Q(H,x)\), for all \(1 \leq \tilde{p} \leq 1\), we have
\[
P[G_{n,\tilde{p}} \in Q(H,x)] = \sum_{m=0}^{\binom{n}{2}} P[G_{n,m} \in Q(H,x)] \cdot P[e(G_{n,\tilde{p}}) = m] \\
\geq P[G_{n,2m_0} \in Q(H,x)] \cdot P[e(G_{n,\tilde{p}}) \geq 2m_0].
\]

If we set \(\tilde{p} := 8p\), Chernoff bounds again imply that \(P[e(G_{n,\tilde{p}}) \geq 2m_0] \geq 1 - e^{-\Theta(n^2p)}\). Hence, we have
\[
P[G_i \in Q(H,x)] \leq \frac{P[G_{n,8p} \in Q(H,x)]}{1 - e^{-\Theta(n^2p)}} + e^{-\Theta(n^2p)} \\
\leq 2 \cdot P[G_{n,8p} \in Q(H,x)] + e^{-\Theta(n^2p)}.
\]

Now we consider the cases \(P^-\) and \(K_4^-\). Let
\[
\mu := \mathbb{E}[X(G_{n,8p}, P^-)] = \Theta(n^6p^6) = \Theta(n^{2-8\alpha}) = \Omega\left(n^{2/3}\right)
\]
and
\[
\lambda := \mathbb{E}[X(G_{n,8p}, K_4^-)] = \Theta(n^4p^5) = \Theta(n^{3/2-5\alpha}) = \Omega\left(n^{4/6}\right).
\]

One can easily verify that \(P^-\) and \(K_4^-\) are balanced graphs. Therefore, we may apply Theorem 2.17 on page 11 with parameters \(a \leftarrow |\text{Aut}(P^-)|\) and \(\alpha \leftarrow (2 - 8\alpha)\) and conclude that
\[
P[G_{n,8p} \in Q(P^-, 2\mu)] \leq \exp\left\{-\Omega\left(n^{(2-8\alpha)/5}\right)\right\}.
\]

Analogously, applying Theorem 2.17 with parameters \(a \leftarrow |\text{Aut}(K_4^-)|\) and \(\alpha \leftarrow (3/2-5\alpha)\) yields
\[
P[G_{n,8p} \in Q(K_4^-, 2\lambda)] \leq \exp\left\{-\Omega\left(n^{(3/2-5\alpha)/3}\right)\right\}.
\]

Let \(Z_i\) be the random variable indicating that a red triangle was created in step \(i\), and let
\[
Z := \sum_{i=1}^{N} Z_i
\]
denote the total number of red triangles. We conclude from the previous calculations that
\[
P[Z > 0] \leq \sum_{i=1}^{N} P[Z_i > 0] \\
\leq \sum_{i=1}^{N} \left[P[Z_i > 0 | G_i \not\in Q(P^-, 2\mu) \land G_i \not\in Q(K_4^-, 2\lambda)] \\
+ 2P[G_{n,8p} \in Q(P^-, 2\mu)] \\
+ 2P[G_{n,8p} \in Q(K_4^-, 2\lambda)] + e^{-\Theta(n^2p)}\right] \\
\leq N \left[\frac{2(\mu + \lambda)}{\binom{n}{2} - 2m_0} + \exp\left\{-\Omega\left(n^{(2-8\alpha)/5}\right)\right\}
\right. \left. + \exp\left\{-\Omega\left(n^{(3/2-5\alpha)/3}\right)\right\} + e^{-\Theta(n^2p)}\right] \\
= N \left[O\left(n^{-8\alpha}\right) + \exp\left\{-\Omega\left(n^{(2-8\alpha)/5}\right)\right\}\right] = o(1)
\]
since \(N = o(n^{6\alpha})\) and \(\alpha \leq 1/6\).

This completes the proof of Theorem 6.3.
CHAPTER 7

Balanced Ramsey Games

The games studied in this chapter are similar to the online games as studied in [FKR+03] and [MSS05]. This time, however, the player receives the edges of a random graph $G_{n,p}$ two at a time in a random order. He has to color one of the edges red and the other one blue avoiding a monochromatic copy of some fixed graph $F$. Thus, the player is forced to keep the number of edges in every color class even. As we shall show, this additional assumption has a dramatical influence on the threshold for the game in the case of cycles of fixed length.

Some of the results presented here were published in [MMS05]. A journal version was accepted for publication.

7.1. Introduction

The games we study are played by a single player, whom we call Painter. He maintains a balanced 2-coloring in the random graph process, coloring two edges at a time in an online fashion. His goal is to avoid creating a monochromatic copy of a fixed graph $F$ for as long as possible.

The precise description of the game’s setup, its rules, and Painter’s objective are as follows. Let $e_1, e_2, \ldots, e_M$ be the edges of $K_n$ where $M = \binom{n}{2}$, and let $\pi \in S_M$ be a permutation of the set $[M]$, chosen uniformly at random. By $G(n, i)$, $i = 1, \ldots, M$, we denote the graph on $n$ vertices with the edge-set $E(G(n, i)) = \{e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(i)}\}$. In the $i$th move of the game, Painter is presented with edges $e_{\pi(2i-1)}$ and $e_{\pi(2i)}$. He then immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. Therefore, after playing the first $i$ moves, Painter has created a balanced 2-coloring of the graph $G(n, 2i)$. Note that at move $i$, he has no knowledge of the order in which the remaining edges will be presented to him in the future.

Let $F$ be a fixed graph. Painter loses the game as soon as he creates a monochromatic copy of $F$, i.e., Painter loses in move $\text{min}\{i : G(n, 2i) \text{ contains a monochromatic copy of } F\}$. His goal is to play as long as possible without losing. It is well-known that for $n$ large enough, every 2-coloring of edges of $K_n$ contains a monochromatic copy of $F$. Therefore, Painter cannot survive to the end of the game. Assuming that his strategy is fixed, for every graph process, there is an integer $i$ such that Painter loses in his $i$th move playing on that particular graph process. Since the graph process on which the game is played is chosen uniformly at random, for fixed $n$ and $i$, we can reason about the probability that Painter loses before his $i$th move. Note that, generally speaking, Painter can lose the game in two ways. If one of the two edges to be colored closes both a red and a blue copy of $F$, then he obviously cannot properly color it. We call this a bichromatic threat. Also, if both edges to be colored close a monochromatic copy of $F$ of the same color, the game is over. We refer to this as monochromatic threat. It is easy to see that these are the only two possibilities for losing.
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7.1.1. Our Results. In this paper, we attempt to determine the maximal number of moves that Painter can a.a.s. play without losing. More precisely, we would like to find a threshold function \( N_0 = N_0(F, n) \) for which

- there exists a strategy for Painter, such that for \( N \ll N_0 \), we have
  \[ P[\text{Painter loses in the first } N \text{ moves}] \to 0, \]
- no matter how Painter plays, for \( N \gg N_0 \), we have
  \[ P[\text{Painter loses in the first } N \text{ moves}] \to 1, \]

as \( n \to \infty \). Our interests lie in determining this threshold for a number of graph-theoretic structures. Observe that the existence of this threshold is not guaranteed—there may exist a graph \( F \) for which there is no such threshold.

In Section 7.2 we study constraints that imply non-trivial lower and upper bounds on the duration of the balanced graph avoidance game for certain families of graphs. Namely, Proposition 7.2 gives a lower bound and Proposition 7.5 an upper bound on the length of the game under certain conditions. As we will show, these bounds are tight in some but not in all cases. However, there are families of graphs for which they give rise to a threshold. This holds for cycles of fixed length.

**Theorem 7.1.** For any fixed integer \( \ell \geq 3 \), the threshold of the online balanced avoidance game for cycles of length \( \ell \) exists and is given by

\[ N_0(C_{\ell}, n) := n^{2^{\ell-1}}. \]

7.1.2. Motivation and Related Work. In [FKR+03] Friedgut et al. introduce the concept of an online game played on the random graph process. In this game the player colors edges with two colors, one at a time in an online fashion. His goal is to avoid a monochromatic copy of a triangle for as long as possible. Note that one color may be used more frequently than the other by the player.

The results in [MSS05] determine the threshold for this game for a large family of graphs. In that setting the optimal strategy is essentially to play greedily—using the first color whenever possible, and the second one only to prevent from losing immediately. The colorings obtained by following this strategy are typically unbalanced. A natural question arising is: if the player is forced to keep his coloring balanced, how long can he survive without losing? We try to give an answer to this question by looking at the analogous game in which the coloring of the graph is balanced. As it turns out, several thresholds that we obtain in the balanced game are not the same as for the unbalanced game, showing that the condition of being balanced makes a difference. For instance, in the cycle avoidance game, the authors of [MSS05] obtain the threshold \( n^{(\ell+1)/\ell} \) in the unbalanced case, whereas we derive the threshold \( n^{2\ell/(2^\ell-1)} \) from our results for balanced online colorings. Hence, the balanced online cycle avoidance game will end substantially earlier than the unbalanced game.

In both [FKR+03] and [MSS05], generalizations of the online games with more than two colors are mentioned and analyzed in some cases. One obvious generalization is the game in which \( s \) edges at a time are introduced in the random graph process, where \( s \geq 2 \) is a fixed integer. Then Painter immediately colors them with \( s \) distinct colors. Painter's objective remains to avoid creating a monochromatic copy of \( F \) for as long as possible.
Another motivation comes from Beck's Chooser-Picker games on graphs \[\text{[Bec02a, Bec02b]}\]. During the game, the balanced coloring of the subset of edges of \(K_n\) is maintained. In the "misère" version of the game, Chooser wins if at the end of the game (when all the edges are colored) there is no red copy of a fixed graph \(F\). Otherwise, Picker wins.

When Picker is playing randomly, this game is quite similar to the balanced avoidance games that we introduce here. A balanced two coloring of the edges of the random graph process is maintained by Chooser, and he colors them two at a time. The only difference is that in our game we investigate a Ramsey-type \[\text{[Ram30]}\] property, where \(F\) has to be avoided in both colors.

Generally speaking, studying games on graphs and random graphs in parallel often uncovers surprising connections between thresholds for winning a game on one side, and thresholds for certain properties of random graphs on the other – see \[\text{[Bec97]}\] and \[\text{[SS05]}\], where this phenomenon is pointed out for Maker-Breaker games on graphs. As the reader will see, we also encounter some relationships of this kind while dealing with balanced avoidance games.

### 7.2. Bounds on the duration of the game

In this section we prove a lower and an upper bound on the duration of the balanced \(F\)-avoidance game that hold under certain conditions. In some cases, e.g., for cycles of fixed length, the bounds match and yield a threshold, but there are classes of graphs for which we do not obtain such sharp results.

#### 7.2.1. Lower Bound

The following proposition gives a lower bound on the duration of the balanced \(F\)-avoidance game under certain conditions. The statement holds if all subgraphs of \(F\) with one edge less are balanced in the ordinary sense. We define \(C(F)\) as the following family of subgraphs of \(F\):

\[
C(F) := \{ C \subseteq F : \exists e \in E(F) \text{ s.t. } C \text{ is a connected component of } F \setminus \{ e \} \}
\]

Recall that the density measure \(d_b(F)\) was defined in (2.9) as

\[
d_b(F) := \frac{2e_F - 1}{2v_F - 2}
\]

**Proposition 7.2.** Let \(F\) be a nonempty graph. If every graph \(C \in C(F)\) is balanced, then Painter can a.a.s. play any

\[
N(n) \ll n^{2-1/d_b(F)}
\]

moves in the balanced online game avoiding a monochromatic copy of \(F\), where \(d_b(F)\) is defined as in (2.9).

Before we present the proof of Proposition 7.2, we introduce some notation. Let the random variable \(X(G_i, H)\) count the number of subgraphs isomorphic to \(H\) in \(G_i\), where \(G_i\) is the graph consisting of the first \(i\) edges in the random graph process. Similarly, let \(X(G_{n,p}, H)\) count the number of subgraphs isomorphic to \(H\) in the random graph \(G_{n,p}\). Let \(Q(H, x)\) denote the family of graphs that contain at least \(x\) copies of \(H\). Clearly, \(Q(H, x)\) is a monotone increasing family. We need the following technical lemma, which, generally speaking, expresses the asymptotic equivalence of the models \(G_{n,p}\) and \(G_i\) with respect to \(Q\).

**Lemma 7.3.** For \(p = 8N/n^2\) and all \(0 \leq i \leq N \leq \binom{n}{2}/2\), we have

\[
P\left[G_{2i} \in Q(H, x)\right] \leq P\left[G_{n,p} \in Q(H, x)\right] + e^{-\Theta(N)}
\]
PROOF. Observe that each graph $G_{2i}$, $0 \leq i \leq N$, appearing in the random process is distributed like $G_{n,2i}$, the uniform random graph with exactly $2i$ edges. Since $Q$ is a monotone increasing property, we have

$$P[G_{2i} \in Q(H,x)] \leq P[G_{n,2N} \in Q(H,x)]. \quad (7.1)$$

Applying the law of total probability and the monotonicity of $Q$, we obtain

$$P[G_{n,p} \in Q(H,x)] = \sum_{m=0}^{\binom{n}{2}} P[G_{n,m} \in Q(H,x)] \cdot P[e(G_{n,p}) = m]$$

$$\geq P[G_{n,2N} \in Q(H,x)] \cdot P[e(G_{n,p}) \geq 2N].$$

It follows that

$$P[G_{n,2N} \in Q(H,x)] \leq \frac{P[G_{n,p} \in Q(H,x)]}{P[e(G_{n,p}) \geq 2N]}.$$ 

For $p = 8N/n^2$, Chernoff bounds (cf. Lemma 2.3) imply that

$$P[e(G_{n,p}) \geq 2N] \geq 1 - e^{-\Theta(N)}.$$

Hence, together with (7.1), we have

$$P[G_{2i} \in Q(H,x)] \leq \frac{P[G_{n,p} \in Q(H,x)]}{1 - e^{-\Theta(N)}}$$

$$= P[G_{n,p} \in Q(H,x)] + e^{-\Theta(N)}.$$

Now we can prove Proposition 7.2.

PROOF. Proof of Proposition 7.2 We have to argue that there exists a strategy for Painter that a.a.s. enables him to avoid monochromatic copies of $F$ in every step of the random process, up to $G_{2N}$. He plays greedily: if one of the two possibilities to complete a move would create a monochromatic copy of $F$, then he choose the other one. Otherwise, he plays arbitrarily.

Let $\mathcal{F}_-$ denote the family of pairwise non-isomorphic subgraphs of $F$ with $e_F - 1$ edges. For $F_- \in \mathcal{F}_-$, we have that $v(F_-) = v(F)$ and $e(F_-) = e(F) - 1$. Since all edges in the random graph process appear independently uniformly at random, the probability of losing the game in one particular step is determined by the number of edges $uv$ that close a monochromatic copy of $F_-$ to $F$. There are two different configurations that force Painter to create a monochromatic copy of $F$.

In the first case, a new edge may appear as a vertex pair $uv$ that is covered by both a red copy $F^r$ and a blue copy $F^b$ with $F^r, F^b \in \mathcal{F}_-$. But this implies the existence of a graph $F^{(2)}$ in $G_{n,2N}$ consisting of two subgraphs isomorphic to $F$, which share exactly one edge and possibly more vertices. Suppose

$$N = \frac{n^{2-1/d_F}}{\omega},$$

where $\omega$ tends to infinity as $n \to \infty$ arbitrarily slowly. Then the expected number of copies of $F^{(2)}$ in $G_{n,2N}$ is, up to a multiplicative constant,

$$\eta^{v(F^{(2)})} \left( \frac{N}{n^2} \right)^{e(F^{(2)})} \leq \omega^{-2(2\omega-1)} = o(1).$$

It follows from Markov's inequality that Painter is unlikely to create a copy of $F^{(2)}$ in the first $N$ moves.
However, Painter can create a monochromatic copy of $F$ differently. In this event, two edges $v_1v_2$ and $v_3v_4$ that are covered by a monochromatic members of $\mathcal{F}_-$ which have the same color show up in the same move. We refer to the pair of edges $\{v_1v_2, v_3v_4\}$ as a threat. An upper bound on the number of threats in the graph can be derived by counting the number of subgraphs isomorphic to some member of $\mathcal{F}_-$ and taking its square. Note that not every such threat is actually dangerous to Painter since we disregard the coloring of the surrounding structure. Thus, we overestimate the risk of losing the game.

Let $p = 8N/n^2$, and $F^- \in \mathcal{F}_-$ such that $F^-$ consists of $k$ balanced components, $F_1^-, \ldots, F_k^- \in \mathcal{C}(F)$. W.l.o.g. for each $F_i^-$, $1 \leq i \leq k$, we have

$$d(F_i^-) < d_b(F^-)$$

(7.2)

since otherwise we have $m(F) \geq d(F_i^-) \geq d_b(F^-)$, and thus $F$ does a.a.s. not appear in $G_{2N}$ due to Theorem 2.15 on page 11. In that case Painter will a.a.s. survive $N$ moves. We denote the expected number of copies of $F^-$ in $G_{n,p}$ by $\mu(F^-)$, i.e.,

$$\mu(F^-) := \mathbb{E}[X(G_{n,p}, F^-)] \simeq n^{v_p}p^{\epsilon_p-1}$$

(7.3)

Since otherwise we have $m(F) \geq d(F_i^-) \geq d_b(F^-)$, and thus $F$ does a.a.s. not appear in $G_{2N}$ due to Theorem 2.15 on page 11. In that case Painter will a.a.s. survive $N$ moves. We denote the expected number of copies of $F^-$ in $G_{n,p}$ by $\mu(F^-)$, i.e.,

$$\mu(F^-) := \mathbb{E}[X(G_{n,p}, F^-)] \simeq n^{v_p}p^{\epsilon_p-1}$$

W.l.o.g. we may assume that $\omega = o(\log(n))$. Then for all $1 \leq i \leq k$, we have

$$\mu(F_i^-) = \Omega \left( \frac{n^{v_p}}{\log(n)} \left( 4n^{-1/d_b(F^-)} / \log(n) \right)^{\epsilon_p(F^-)} \right)$$

for a suitable $\epsilon_i = \epsilon_i(F_i) > 0$, where the last step follows from (7.2). According to (7.3) there exists a constant $A = A(F^-) \geq 1$ such that

$$A \mu(F^-) \geq \prod_{i=1}^{k} \mu(F_i^-)$$

We conclude that

$$\mathbb{P}[X(G_{n,p}, F^-) \geq 2A\mu(F^-)]$$

$$\leq \mathbb{P} \left[ \prod_{i=1}^{k} X(G_{n,p}, F_i^-) \geq \prod_{i=1}^{k} 2^{1/k} \mu(F_i^-) \right]$$

$$\leq \mathbb{P} \left[ \bigvee_{i=1}^{k} X(G_{n,p}, F_i^-) \geq 2^{1/k} \mu(F_i^-) \right]$$

$$\leq \mathbb{P} \left[ X(G_{n,p}, F_i^-) \geq 2^{1/k} \mu(F_i^-) \right]$$

$$\leq \exp \left\{ -\Omega \left( n^{\epsilon_i/(v(F_i^-) - 1)} \right) \right\} \leq \exp \left\{ -\Omega (n^{\alpha}) \right\}$$

for a suitable constant $\alpha = \alpha(F^-) > 0$. The last line was obtained by application of Theorem 2.17 on page 11 with parameters $H \leftarrow F_i^-, (1 + \varepsilon) \leftarrow 2^{1/k}$, and $\alpha \leftarrow \epsilon_i$. Note that the expression $\exp \left\{ -\Omega \left( n^{\epsilon_i/(v(F_i^-) - 1)} \right) \right\}$ can be replaced by $\exp \left\{ -\Omega (n^{\epsilon_i}) \right\}$ for any $\epsilon_i > 0$, if $v(F_i^-) = 1$. 


Let $Z_i$ be the indicator random variable for the event that both new edges close a monochromatic threat in step $i$, and let $Z$ denote the sum over all steps. From the previous calculations and Lemma 7.3, we conclude that

$$
P[Z > 0] \leq \sum_{i=1}^{N} P[Z_i > 0]
$$

$$
\leq \sum_{i=1}^{N} \left\{ P\left[Z_i > 0 \mid \bigwedge_{F- \in \mathcal{F}-} G_{2i-2} \notin \mathcal{Q}(F-, 2A\mu(F_-))\right]

+ P\left[ \bigvee_{F- \in \mathcal{F}-} G_{2i-2} \in \mathcal{Q}(F-, 2A\mu(F_-))\right] \right\}.
$$

And this is at most

$$
N \left\{ \left( \frac{\sum_{F- \in \mathcal{F}-} 2A\mu(F_-)}{2} \right)^2

+ \sum_{F- \in \mathcal{F}-} \left( P[G_{n,p} \in \mathcal{Q}(F-, 2A\mu(F_-))] + e^{-\Theta(N)} \right) \right\}
$$

$$
\leq N \left\{ O\left( n^{2v_F - 2 - 2(e_F - 1)/d_b(F)} \right) + \sum_{F- \in \mathcal{F}-} \left( e^{-\Omega(n^{\alpha_F})} + e^{-\Theta(N)} \right) \right\}
$$

$$
\leq o\left( n^{2v_F - 2 - (2e_F - 1)/d_b(F)} \right) + o(1)
$$

since $N = o\left( n^{2 - 1/d_b(F)} \right)$ and $2v_F - 2 - (2e_F - 1)/d_b(F) = 0$.

As Painter will not create a monochromatic copy of $F$ in a different way, the statement follows.

If $F$ is a tree on $t > 1$ vertices, then the removal of an edge yields a forest consisting of two smaller trees $T_1$ and $T_2$, which are clearly balanced. Thus, we can rewrite Proposition 7.2 in the following way.

**Corollary 7.4.** Let $F$ be a tree on $t > 1$ vertices. Then Painter can a.a.s. play any

$$
N(n) = o\left( n^{1/(2t - 3)} \right)
$$

moves in the balanced online game avoiding a monochromatic copy of $F$.

As shown in [MMS05], this bound is tight for stars, but not for paths with four edges.

### 7.2.2. Upper bound.

**Proposition 7.5.** Let $F$ be a non-empty graph that is strictly balanced w.r.t. $d_b$ and contains a subgraph $F_-$ of $F$ with $e_F - 1$ edges satisfying $d_b(F) \geq \mathfrak{m}_2(F_-)$. Then, no matter how he plays, Painter will a.a.s. lose the balanced $F$ avoidance game in any

$$
N(n) \gg n^{2 - 1/d_b(F)}
$$

moves.
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Proof. Let \( F_- \) be fixed such that \( m_2(F_-) \leq d_b(F) \). We switch between the binomial model \( G_{n,p} \) and the uniform random graph model \( G_{n,N} \), exploiting their asymptotic equivalence via \( p = \Theta(N/n^2) \) \([\text{Bol01, Theorem 2.2}]\). We split the games into two rounds \( N_1 = N_2 := N/2 \gg n^{2-1/d_b(F)} \) of equal length and assume w.l.o.g. that \( N \leq n^{2-1/d_b(F)} \log n \). Observe that after the first \( N_1 \) moves, \( N \) edges have been revealed to Painter. Let \( X(G_{n,N}, F_-) \) denote the number of subgraphs isomorphic to \( F_- \) in \( G_{n,N} \).

Claim 7.6. In every 2-edge-coloring of the random graph \( G_{n,N} \), there are a.a.s.
\[
\Omega(\mathbb{E}[X(G_{n,N}, F_-)])
\]
pairs \( uv \in \binom{[n]}{2}\setminus E(G_{n,N}) \) that complete a monochromatic copy of \( F_- \) in the same color, say red, to \( F \).

Proof. We call an edge critical, if it completes an entirely red copy of \( F_- \) to \( F \). Theorem 2.20 on page 13 yields that a.a.s. the number of (w.l.o.g.) monochromatically red subgraphs of \( G_{n,N} \) isomorphic to \( F_- \) is
\[
\Omega(\mathbb{E}[X(G_{n,N}, F_-)])
\]
since
\[
N \gg n^{2-1/m_2(F_-)}
\]
holds due to the assumptions in Proposition 7.5. Every such copy induces one critical edge in \( G_{n,N} \), but we may over-count if there are many pairs of monochromatic copies of \( F_- \) that cover the same vertex pair.

If one critical edge \( e = uv \) is induced by multiple copies of \( F_- \), then \( G_{n,N} \) contains a subgraph \( (F_-)_H \) of the following structure: \( (F_-)_H \) is the union of two graphs isomorphic to \( F_- \) such that their intersection complemented with \( e \) is a copy of a proper subgraph \( H \subsetneq F \). For any graph \( (F_-)_H \), we have
\[
e((F_-)_H) = e(F_-) + e(F_-) - (e_H - 1) = e(F_-) + e_F - e_H,
\]
and
\[
v((F_-)_H) = v(F_-) + v_F - v_H.
\]
We denote the number of subgraphs isomorphic to \( (F_-)_H \) in \( G_{n,N} \) by \( X(G_{n,N}, (F_-)_H) \). It follows that
\[
\mathbb{E}[X(G_{n,N}, (F_-)_H)] \leq n^{v(F_-) + v_F - v_H + e(F_-) + e_F - e_H}
\]
\[
\ll \mathbb{E}[X(G_{n,N}, F_-)] \left[ \frac{n^{v_F - e_F - v_H}d_b(F)}{n^{v_H - e_H}d_b(F)} \right]^{e_F - e_H}
\]
\[
\ll \mathbb{E}[X(G_{n,N}, F_-)].
\]
The last step holds since for every subgraph \( H \subsetneq F \), we can write
\[
v_F - e_F/d_b(F) < v_H - e_H/d_b(F)
\]
equivalently as
\[
\frac{2e_F - 1}{2v_F - 2} = d_b(F) < \frac{e_F - e_H}{v_F - v_H} = \frac{2e_F - 1}{2v_F - 2} - (2v_H - 2).
\]
Due to Proposition 2.2, this inequality holds if
\[
\frac{2e_F - 1}{2v_F - 2} > \frac{2e_H - 1}{2v_H - 2},
\]
i.e., the graph \( F \) is strictly balanced w.r.t. \( d_b \).
Observe that the number of copies \((F_\alpha)_H\) is a.a.s. bounded from above due to Markov's inequality, which yields that, for every \(\varepsilon > 0\), we have

\[
P\left[ X(G_{n,N}, (F_\alpha)_H) \geq \varepsilon \mathbb{E}[X(G_{n,N}, F_\alpha)] \right] = o(1) .
\]

Since the number of critical edges induced by a fixed occurrence of a graph \((F_\alpha)_H\) is bounded by a constant only depending on \(F\), the multiply counted copies of \(F_\alpha\) are a.a.s. of lower order of magnitude than \(\mathbb{E}[X(G_{n,N}, F_\alpha)]\) and thus negligible. Moreover, only a negligible fraction of the critical pairs was actually revealed in \(G_{n,N}\) since \(\mathbb{E}[X(G_{n,N}, F_\alpha)] \ll \mathbb{E}[X(G_{n,N}, F_\alpha)]\). This concludes the proof of Claim 7.6.

Continuing the proof of Proposition 7.5, we apply the claim to show that the game does a.a.s. not last for more than \(N_2\) moves. Suppose that Painter played his first \(N_1\) moves, thus fixing a coloring of the first \(2N_1\) edges. By Claim 7.6, there are \(M = \Omega(X(G_{n,N}, F_\alpha))\) critical pairs of vertices in \(\binom{[n]}{2} \setminus E(G_{n,2N_1})\). If two of these pairs are simultaneously presented to Painter, he loses the game. In every step \(i\), the probability of this event is determined by the number of remaining critical pairs. However, Painter will not lose if only one of the edges presented to him is critical. Moreover, that edge will neutralize a critical pair – it will not be critical any more. On the other hand, Painter can neutralize only a tiny fraction of all critical pairs since we have

\[
N_2 \ll M .
\] (7.4)

In order to justify (7.4), observe that, for \(e_F \geq 3\), (7.4) is equivalent to

\[
N_2 \gg n \left( \frac{2(e_F - 1) - v_F}{e_F - 2} \right) .
\]

As we assumed that \(N_2 \gg n^{2 - 1/d_\alpha(F)}\), (7.4) holds provided that

\[
2 - \frac{1}{d_\alpha(F)} \geq \frac{2(e_F - 1) - v_F}{e_F - 2} ,
\]

which can be rewritten to

\[
3v_F \geq 2e_F + 2 .
\] (7.5)

This, however, easily follows from \(d_\alpha(F) \geq m_2(F_\alpha)\) since that implies that we have

\[
\frac{2e_F - 1}{2v_F - 2} \geq \frac{e_F - 2}{v_F - 2} ,
\]

which is equivalent to (7.5) for \(v_F \geq 3\). If \(e_F = 2\), the path on two edges \(P_2\) is the only graph that is strictly balanced w.r.t. \(d_\alpha\). In that case we have

\[
M = \Omega \left( \frac{n^2 N}{n^2} \right) = \Omega(nN) ,
\]

which is clearly substantially greater than \(N_2\). The remaining cases for \(e_F \leq 1\) are trivial.

Let \(X_i\) be the random variable indicating that the game was lost in step \(i\) of the second round. Since, in every move, the probability of being presented with a pair of critical edges that ends the game is at least

\[
\Omega \left( \frac{M}{2^i} \right) = \Omega \left( \frac{M^2}{n^4} \right) ,
\]
the probability of surviving the whole second round is bounded from above by

\[
\left(1 - \Omega \left( \frac{M^2}{n^2} \right) \right)^N \leq \exp \left\{ -\Omega \left( \frac{M^2 N_2}{n^4} \right) \right\} \\
\leq \exp \left\{ -\Omega \left( \frac{\left( \frac{n^{\epsilon_F} \left( \frac{N}{n^2} \right)^{(\epsilon_F - 1)}}{n^{4\epsilon_F - 4\epsilon_F N^{2\epsilon_F - 1}}} \right) N}{n^4} \right) \right\} \\
= e^{-\omega(1)} = o(1) .
\]

This concludes the proof of Proposition 7.5.

7.2.3. The cycle game. We remark that Proposition 7.5 only applies to families of rather sparse graphs. According to (7.5), the number of edges in the forbidden graph \( F \) must be linear in the number of vertices. However, there exist families of graphs that satisfy the conditions of both Proposition 7.2 and Proposition 7.5. In that case the lower bound obtained in Proposition 7.2 matches the upper bound from Proposition 7.5 up to a multiplicative constant, giving the exact threshold for several games. In particular, we obtain the threshold for the cycle game.

\textbf{Proof.} Proof of Theorem 7.1 Observe that for a cycle \( C_\ell \), the only member in the family \( \mathcal{F}_- \) is a path \( P_{\ell - 1} \) with \( \ell - 1 \) edges. Clearly, this graph is balanced in the ordinary sense, and we have

\[
d_b(C_\ell) = \frac{2\ell - 1}{2\ell} > 1 = \frac{2}{2} = m_2(P_{\ell - 1}).
\]

Moreover, \( C_\ell \) is strictly balanced w.r.t. \( d_b \). Hence, the statement follows from Proposition 7.2 and Proposition 7.5.
Bibliography


Chapter 7. Bibliography


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