Diss. ETH No. 20341

Tensor Norms
And Non-Locality

A dissertation submitted to

ETH ZURICH

for the degree of
Doctor of Sciences

presented by

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2012

Acknowledgments

This thesis would not have been possible without the support of many people.

First of all I would like to thank my two advisors Stefan Wolf and Renato Renner. They have been great advisors giving me the opportunity to work in an interdisciplinary research environment. They gave me a lot of freedom in my research, but still, their doors were always open for discussions. I am grateful to Ueli Maurer and Stephanie Wehner for being co-examiners and investing their time in reviewing this thesis.

My studies would not have been the same without the present and past members of the different Quantum Information Research Groups at ETH. During my first two years I was employed at the computer science department where I had the pleasure to meet Daniel Burgarth, Roger Colbeck, Matthias Fitzi, Manuel Forster, Viktor Galliard, Esther Hänngi, Melanie Raemy, Severin Winkler, Stefan Wolf, and Jürg Wullschleger. Special thanks go to Severin Winkler, who was my office mate during this time. After the first two years, I was employed at the physics department where I had the pleasure to meet Normand Beaudry, Mario Berta, Matthias Christandl, Oscar Dahlsten, Frédéric Dupuis, Philippe Faist, David Gross, Stefan Hengl, Lea Krämer, Anthony Leverrier, Christopher Portmann, Joseph Renes, Renato Renner, Lídia del Rio, Christian Schilling, Volkher Scholz, Cyril Stark, Marco Tomamichel, Michael Walter, Mário Ziman, and Johan Åberg. I thank them all for countless research discussions, as well as enjoyable lunch and coffee breaks.

Special thanks go to Johan Åberg who was not only a great office mate during my time at the physics department, but also had always an answer to my research questions and helped me solve several problems I got stuck on. I also want to thank Earl Campbell, Christian Gogolin, Thomas Holenstein, Miguel Navascués, and Carlos Palazuelos for interesting and stimulating discussions about tensor norms.

This thesis has improved substantially by the valuable comments of those who read preliminary versions of it. In particular, I would like to thank Frédéric Dupuis, Matthias Fitzi, Manuel Forster, Cyril Stark, Marco Tomamichel, and Johan Åberg for their valuable suggestions and proof-reading.

I would also like to thank my family for their continuous support. Finally and most of all, I want to thank Andrea for everything.

Abstract

Non-locality is arguably one of the most striking aspects of the difference between quantum and classical physics. In this thesis, we investigate non-locality in an abstract framework where classical and quantum theories are special cases. In this framework, we investigate devices with classical inputs and outputs that are shared between two separate parties and where the internal workings of the devices are unknown to us. We refer to such devices as bipartite systems, and we assume that they can be modelled by conditional probability distributions. If the behaviour of a bipartite system cannot be explained by local processing on shared randomness, the bipartite system is called non-local. Such non-local behaviour can be observed in nature when two separate parties share entangled quantum states on which they can perform local but non-commuting measurements. Bipartite non-local systems are powerful resources for various communication and information theoretic tasks. For example, they can be used to establish a secret key between two parties, to reduce the communication complexity of distributed computing, or to increase the winning probability of two-prover games.

The contribution of this thesis is two-fold. First, we develop a framework, based on the theory of tensor norms, that allows us to study properties of bipartite systems, two-prover games, and Bell inequalities. Second, in order to demonstrate the power and usefulness of the framework we present four applications in quantum information theory.

The framework contains three main parts. The first part treats the embedding of bipartite systems and two-prover games into tensor product spaces. This embedding allows us to evaluate different tensor norms on bipartite systems (yielding convex sets of bipartite systems) and two-prover games (yielding winning probabilities of two-prover games). The second part introduces the composition of bipartite systems and two-prover games. This allows us to combine bipartite systems in order to ob-

tain larger systems and to study parallel repetition of two-prover games. Finally, the third part exposes how wirings can be represented as linear maps on tensor product spaces. We prove that the values of tensor norms for bipartite systems do not increase under wirings. This result enables us to study sets of bipartite systems that are closed under wirings.

We prove four main results with the help of the framework. First, we derive an upper bound on the maximal winning probability of two-prover games, where the provers have entanglement as resources. In order to prove this, we first derive a generalized version of Grothendieck's inequality that includes settings of arbitrary output alphabet sizes. We furthermore establish close connections between quantum systems and the Hilbertian tensor norm and between local systems and the projective tensor norm.

Second, we provide an alternative proof of the perfect parallel repetition theorem for entangled XOR games. We prove this by showing that the dual Hilbertian tensor norm obeys a direct-product theorem and that the winning probability of entangled XOR games can be computed by the dual Hilbertian tensor norm.

Third, we show that there exist quantum systems that cannot be obtained by wirings of isotropic quantum systems. In order to prove this result, we show that the dual Hilbertian tensor norm induces a convex set of bipartite systems that is closed under wirings and that this convex set is closely related to the set of binary quantum systems.

Fourth, we prove sufficient conditions for tensor norms that imply the impossibility of non-locality distillation for isotropic systems. We also construct a continuous hierarchy of cross norms and prove, based on two conjectures and the sufficient conditions, that non-locality distillation is impossible for isotropic quantum systems.

Zusammenfassung

Nicht-Lokalität ist wohl einer der auffälligsten Unterschiede zwischen Quantenphysik und klassischer Physik. In dieser Arbeit untersuchen wir Nicht-Lokalität in einem abstrakten Framework, in dem die klassische- und die Quantentheorie Spezialfälle sind. Wir betrachten Geräte mit klassischen Ein-und Ausgängen, die von zwei getrennten Parteien geteilt werden und deren interne Funktionsweise unbekannt für uns ist. Wir nennen solche Geräte bipartite Systeme, und wir nehmen an, dass sie durch bedingte Wahrscheinlichkeitsverteilungen modelliert werden können. Wenn das Verhalten eines bipartiten Systems nicht durch lokale Verarbeitung auf gemeinsamer zufälliger Information erklärt werden kann, dann wird das bipartite System nicht-lokal genannt. Solches nicht-lokales Verhalten kann in der Natur beobachtet werden, wenn zwei getrennte Parteien verschränkte Quantenzustände teilen, auf denen sie lokale Messungen durchführen können. Nicht-lokale Systeme sind leistungsstarke Ressourcen für verschiedene kommunikations-und informationstheoretische Aufgaben. Zum Beispiel können sie verwendet werden, um einen geheimen Schlüssel zwischen zwei Parteien zu erzeugen, um die Kommunikations-Komplexität von verteilten Berechnungen zu verringern oder um die Gewinnwahrscheinlichkeit von Two-Prover Spielen zu erhöhen.

Der Beitrag dieser Arbeit ist zweigeteilt. Zunächst entwickeln wir ein Framework basierend auf der Theorie der Tensor Normen, welches uns erlaubt Eigenschaften von bipartiten Systemen, Two-Prover Spielen, und Bell Ungleichungen zu untersuchen. Um den Nutzen des Frameworks zu demonstrieren, präsentieren wir zweitens vier Anwendungen in der Quanten-Informationstheorie.

Das Framework besteht aus drei Hauptteilen. Der erste Teil behandelt die Einbettung von bipartiten Systemen und Two-Prover Spielen in Tensorprodukt-Räume. Diese Einbettung ermöglicht es uns, unterschiedliche Tensor Normen auf bipartiten Systemen und Two-Prover Spielen

auszuwerten. Der zweite Teil führt die Komposition bipartiter Systemen und Two-Prover Spielen ein. Dies erlaubt uns bipartite Systeme zu kombinieren, um grössere Systeme zu erhalten und die parallele Wiederholung von Two-Prover Spielen zu studieren. Schliesslich zeigt der dritte Teil wie lokale Verdrahtungen von Systemen als lineare Abbildungen auf Tensorprodukt-Räume dargestellt werden können. Wir beweisen, dass die Werte der Tensor Normen für bipartite Systeme nicht erhöht werden können unter der Anwendung von lokalen Verdrahtungen. Dieses Ergebnis ermöglicht es uns Mengen von bipartiten Systemen zu studieren, welche unter der Anwendung von lokalen Verdrahtungen abgeschlossen sind.

Wir beweisen vier Anwendungen in der Quanten-Informationstheorie. Zunächst leiten wir Obergrenzen für die maximale Gewinnwahrscheinlichkeit von Two-Prover Spielen her, in denen die Prüfer Verschränkung als Ressource haben. Um dies zu beweisen, leiten wir eine verallgemeinerte Version der Grothendieck Ungleichung her, die es erlaubt beliebigen Alphabet Grössen zu studieren. Darüber hinaus stellen wir eine enge Verbindung zwischen Quanten-Systemen und der Hilbertschen Tensor Norm und zwischen lokalen Systemen und der projektiven Tensor Norm her.

Zweitens zeigen wir wie sich die Gewinnwahrscheinlichkeit von parallel wiederholten und verschränkten XOR-Spielen verhält. Wir beweisen dies, indem wir zeigen, dass die duale Hilbertsche Tensor Norm einer Produktregel gehorcht und dass die Gewinnwahrscheinlichkeit von verschränkten XOR-Spielen durch die duale Hilbertsche Tensor Norm berechnet werden kann.

Drittens zeigen wir, dass es Quanten-Systeme gibt, die nicht durch lokalen Verdrahtungen von isotropen Quanten-Systemen erhalten werden können. Um dieses Ergebnis zu beweisen zeigen wir, dass die duale Hilbertsche Tensor Norm eine konvexe Menge von bipartiten Systemen induziert, die abgeschlossen ist unter der Anwendung von lokalen Verdrahtungen und dass diese konvexe Menge eng mit der Menge aller binären Quanten-Systeme verbunden ist.

Viertens beweisen wir hinreichende Bedingungen für Tensor Normen, welche die Unmöglichkeit der Nicht-Lokalitäts Destillation für isotrope Systeme impliziert. Ausserdem konstruieren wir eine kontinuierliche Hierarchie von Kreuz-Normen und beweisen, basierend auf zwei Vermutungen und den hinreichenden Bedingungen, dass Nicht-Lokalitäts Destillation unmöglich ist für isotrope Quanten-Systeme.

Contents

Chapter 1

Introduction

1.1 Non-Locality and Non-Signalling Systems

Entanglement is one of the central and most fascinating properties of quantum mechanics. The strange consequences of entangled quantum states already puzzled Einstein, Podolsky, and Rosen [EPR35], in their seminal paper of 1935 where they raise the issue whether quantum mechanics is complete. This leads to the question whether it is possible to augment quantum mechanics with additional (yet) unknown parameters, so called *local hidden variables* (LHV), in order to obtain a local, realistic, and complete theory. It took almost 30 years until John Bell gave a negative answer to this question [Bel64].

Let us first introduce some notation and terminology before we state the precise content of Bell's answer. Assume two separate parties, called Alice and Bob, share a device that on inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ for Alice and Bob, respectively, produces outputs $a \in \mathcal{A}$ and $b \in \mathcal{B}$, respectively (see Figure 1.1 and note that the calligraphic letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$, and $\mathcal{Y}$ will always denote finite sets in this thesis). We call such a device a *bipartite system*. In this thesis we model the behaviour of a bipartite system by a conditional probability distribution $P(a, b|x, y)$. Furthermore, we imagine that each bipartite system is used only once and that all used bipartite systems are independent of each other. We use the letter P to denote the given bipartite system and we say that $P(a, b|x, y)$ is the corresponding conditional probability distribution that describes the behaviour of P. Note that a bipartite system can be represented as an element of the real vector space $\mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$.

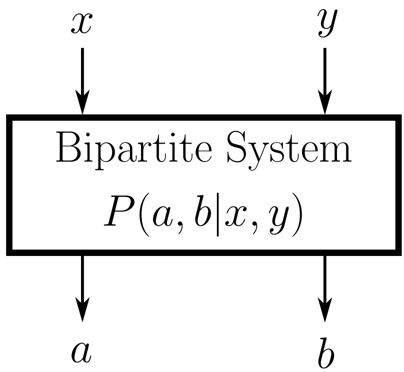

Figure 1.1: The behaviour of a bipartite system P is described by the conditional probability distribution $P(a,b|x,y)$, with input $x \in \mathcal{X}$ and output $a \in \mathcal{A}$ for Alice and input $y \in \mathcal{Y}$ and output $b \in \mathcal{B}$ for Bob.

In this thesis we consider only bipartite systems that cannot transmit information between the parties. Consequently, such systems are called *non-signalling*. Since Alice and Bob could be far apart from each other, this assumption is necessary if we want to be in agreement with the special theory of relativity. Two subsets of the set all non-signalling systems are of special interest to us. First, we call a bipartite system *quantum* if its corresponding conditional probability distribution $P(a,b|x,y)$ can be obtained by local measurements on a bipartite quantum state. In this setup, x and y denote the chosen measurement settings of Alice and Bob, respectively, and a and b are the corresponding measurement outcomes. Second, a bipartite system is called *local* if its corresponding conditional probability distribution $P(a,b|x,y)$ can be obtained by local computations by Alice and Bob on their respective inputs x and y and some shared random bits. Note that the shared randomness corresponds to the abovementioned LHV. We then also say that $P(a,b|x,y)$ admits a LHV theory description.

Note that our use of the word "system" is slightly different from how a physicist intuitively might interpret this word. Given an underlying quantum state, a "system" in our sense not only contains the underlying quantum object, but does also include the measurements. For example, if Alice and Bob share a collection of bipartite quantum systems they can only manipulate the classical measurement outputs and the chosen measurement settings. However, Alice cannot perform an arbitrary joint measurement on her collection of quantum states.

Let us come back to Bell's answer about the incompleteness of quantum theory. The original paper of EPR gives the impression that the authors hoped that not only classical physics but also quantum mechanics admits (in a modern terminology) a LHV theory description. However, Bell showed [Bel64] that there exist quantum systems that cannot be described by a LHV theory. Consequently, such bipartite systems are called *non-local*.

In the last two decades, non-locality has become an extensively studied subject within quantum information theory. It has applications in subjects ranging from device-independent quantum key distribution [BHK05, ABG$^+$07, HRW10], over questions about the foundations of quantum mechanics [BBL$^+$06, ABL$^+$09, NW10], to multi-prover games [BOGKW88, CHTW04, CSUU07, KRT08, KKM$^+$08, KR10].

1.2 Two-Prover Games and Bell Inequalities

In a *two-prover game*, Alice and Bob, the provers, are separated from each other and receive each a classical question, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively, from a verifier. The probability distribution $\pi(x, y)$ of these inputs is known to all parties. Alice and Bob send back answers $a \in \mathcal{A}$ and $b \in \mathcal{B}$, respectively, to the verifier. However, Alice and Bob are not allowed to communicate with each other. The goal of the provers is to maximize the winning probability for the two-prover game that is defined by the probability distribution $\pi(x, y)$ and a predicate $V(a, b, x, y) \in \{0, 1\}$. We say that the game is won for inputs x and y if Alice and Bob return answers a and b such that the predicate evaluates to 1. Note that a two-prover game is completely described by the non-negative real numbers $\pi(x, y) \cdot V(a, b, x, y)$, for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$, and therefore, can be represented as an element of the real vector space $\mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$. Since the strategy of Alice and Bob is described by some non-signalling system with behaviour $P \in \mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$, a two-prover game can also be interpreted as a linear functional on the space $\mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$ that assigns a real number, the winning probability, to P.

The maximal winning probability of a two-prover game depends on the kind of non-signalling systems Alice and Bob share. Typically, it is higher if they share non-local systems as resources instead of only local systems. In order to gain a better understanding of the power and limitations of quantum non-locality it is, therefore, of interest to compare the winning probabilities of two-prover games for the case where Alice and Bob share quantum systems, to the case where they share local systems.

Bell inequalities [Bel64] are a generalization of two-prover games in the sense that they are described by $|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|$ arbitrary real numbers, and not only non-negative ones as in the case of two-prover games. Bell inequalities have the property that they are satisfied by all local systems. This yields an alternative view on non-locality, namely, a bipartite system is called non-local if and only if it violates a Bell inequality.

Similarly as in the case of two-prover games the question arises of how strongly a given Bell inequality can be violated by quantum systems. It was this problem that in 1987 lead Tsirelson to establish the first connection between tensor norms and quantum information theory [Tsi87].

1.3 Local Processing of Non-Signalling Systems

Since we consider non-signalling systems as resources in two-party computations (e.g., to compute a distributed function or to win a two-prover game), the involved parties typically share several copies of such non-signalling systems. Alice and Bob can locally process the inputs and outputs of their non-signalling systems to produce a new non-signalling system. For example, the input to one non-signalling system can depend on the output of another one. The most general such protocol Alice and Bob can follow is called a *wiring*. A wiring consists of a classical circuit for Alice and Bob, respectively, which acts *locally*, i.e., no communication is involved between Alice and Bob, on the shared non-signalling systems. Furthermore, they share a common random string.

Let $\mathcal{C}$ be some subset of all non-signalling systems. Assume that we want to analyse the maximal winning probability of a two-prover game given that Alice and Bob can use (as part of some wiring) an unlimited number of elements from the set $\mathcal{C}$. Since Alice and Bob have shared randomness, they can compute convex combinations of non-signalling systems in $\mathcal{C}$. It is, therefore, reasonable to require that all non-signalling systems that can be obtained by convex combinations from elements in $\mathcal{C}$ to also be members of $\mathcal{C}$. But this is not yet the end of the story, since Alice and Bob can apply wirings on the elements of $\mathcal{C}$ to obtain new non-signalling systems that are not part of $\mathcal{C}$. Adding all non-signalling systems to $\mathcal{C}$ that can be obtained by wirings of elements in $\mathcal{C}$ can be regarded as computing the *closure* of $\mathcal{C}$ under wirings [ABL$^+$09]. The reason for considering the closure of $\mathcal{C}$ is that we want the set $\mathcal{C}$ to contain all elements that are allowed as resources in the two-party computations. Note that the set of local, and the set of quantum systems, are both closed under wirings. Hence, obtaining a deeper understanding of the power and limitations of wirings is of fundamental importance to the analysis of two party protocols with non-signalling systems as resources.

1.4 Tensor Norms

We have seen in the previous sections that bipartite systems and two-prover games can be represented as elements of the real vector space $\mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$. The insight that $\mathbb{R}^{|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|}$ is isomorphic to the tensor product space $\mathbb{R}^{|\mathcal{X}||\mathcal{A}|} \otimes \mathbb{R}^{|\mathcal{Y}||\mathcal{B}|}$ turns out to be crucial. The embedding of bipartite systems and two-prover games into tensor product spaces allows us to use tools and techniques from functional analysis, in particular from the theory of tensor norms. We show in this thesis that computing these norms for bipartite systems and two-prover games yields new insights in quantum information theory.

Tensor norms have already been studied for quite a long time. In the late thirties, the additional structure of tensor product spaces lead Murray and von Neumann to think about new classes of norms with extra properties, later known as *tensor norms*. However, it was Schatten who put the theory of tensor norms onto firm grounds [Sch50]. Independently of Schatten, Grothendieck developed a theory of tensor norms on his own. The results were published in his seminal paper of 1953, which nowadays is just called the “Résumé” [Gro53]. The “Résumé” has deeply influenced functional analysis and its results will be used several times in this thesis[1].

1.5 Related Work

In this section, we only list related work that directly inspired this thesis. An in-depth comparison with related research results is provided at the beginning of each chapter and/or at the beginning of selected sections.

The first to observe that there is a connection between tensor norms and quantum information theory was Tsirelson. He showed that the maximal violation of correlation Bell inequalities by quantum mechanics is proportional to Grothendieck's constant $1.68 \lesssim K_G \lesssim 1.78$ [Tsi87].

[1] For example, Grothendieck's inequality.

Thirteen years after Tsirelson's seminal work, Rudolph investigated properties of the *projective tensor norm* for bipartite quantum states [Rud00, Rud01a, Rud01b, DHR02, Rud03, Rud05]. In particular, he proved that the projective tensor norm induces the set of separable quantum states. This work inspired us to investigate other tensor norms and the set of bipartite systems they induce. However, since we analyse bipartite systems and not quantum states in this thesis we need to consider different normed vector spaces than Rudolph (he used spaces of trace class operators on Hilbert spaces equipped with the trace norm).

A priori, there exist different possibilities for the normed vector spaces into which the bipartite systems and two-prover games can be embedded. However, Junge, Pérez-García, and co-workers have recently introduced normed vector spaces that seem to be the right choice to study bipartite systems, Bell inequalities and two-prover games. In particular, they generalized the work of Tsirelson to the case of Bell inequalities that correspond to multiple outputs and to more than two parties [PGWP+08, JPPG+10a, JPPG+10b, JP11].

1.6 Contributions and Thesis Outline

The contribution of this thesis is two-fold. First, we extend and clarify the connection between tensor norms and quantum information theory. Second, we show that embedding bipartite systems and two-prover games into the context of tensor norms allows us to prove new results in quantum information theory.

The first contribution is exposed in Chapter 3. There, we develop a framework which is based on the theory of tensor norms, that allows us to study the properties of bipartite systems, two-prover games, and Bell inequalities[2]. The second contribution, the application of the framework, is presented in Chapter 6.

Let us now expose in detail the main components of our framework and their connection to quantum information theory. The framework consists of three main components (see Figure 1.2 and Chapter 3). The first component deals with the representation of bipartite systems and two-prover games as elements of tensor product spaces and the definition of tensor norms on these spaces (see Section 2.3, Section 2.5, Section 3.1.1,

[2]From now on, we will only use the notion of two-prover games, although most of our results hold also for Bell inequalities. See also Section 2.5.2 which explains the relation between two-prover games and Bell inequalities.

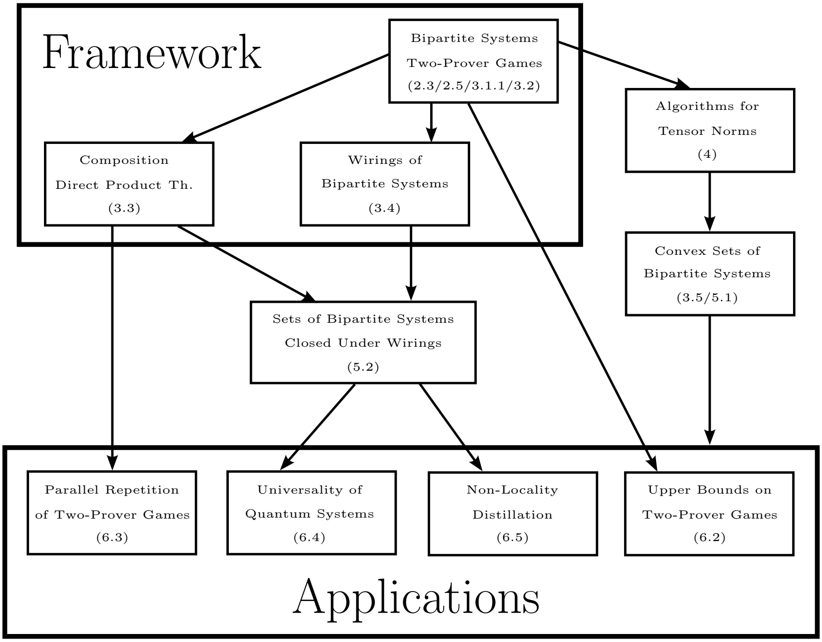

Figure 1.2: The main components of this thesis are shown. The numbers indicate in which section the corresponding subject is covered.

and Section 3.2). The second component treats the composition of bipartite systems and direct-product theorems for two-prover games (see Section 3.3) . Finally, the representation of wirings as linear maps on tensor product spaces is covered by the third component (see Section 3.4).

Let us elaborate on the first component of the framework. In Section 2.3 and Section 2.5, we show that bipartite systems and two-prover games can be represented as elements of tensor product spaces. We then define tensor norms on these spaces in Section 3.2 and we will see that the values of tensor norms are related to the winning probability of two-prover games (see Section 3.1.1). This first component is really the foundation of the framework and its ideas and results are used throughout this thesis.

The second component of the framework is worked out in Section 3.3 and can be split into two parts. The first part deals with the composition of bipartite systems. We will use this in order to assemble large systems from small ones. The second part treats the composition of two-prover games in order to obtain parallel-repetition results for them. We will show in Section 3.3.3 that composite systems/games for the projective and the dual Hilbertian tensor norms obey direct-product rules (see Theorem 3.1 and Theorem 3.2).

The direct-product result for the dual Hilbertian tensor norm, together with the results of Section 5.1, will enable us to provide an alternative proof of the parallel-repetition theorem for entangled XOR games (see Theorem 6.5 in Section 6.3).

In Section 3.4, we show how wirings of non-signalling systems can be represented as linear maps on tensor product spaces. The main result of this section is that values of tensor norms for bipartite systems do not increase under wirings (see Theorem 3.3). Combining this result with the composition of bipartite systems enables us to study convex sets of bipartite systems that are closed under wirings (see Section 5.2). In particular, we prove sufficient conditions for (tensor) norms to induce sets of bipartite systems[3] that are closed under wirings (see Theorem 5.3 and Corollary 5.1). By using these sufficient conditions we show that the projective and the dual Hilbertian tensor norms induce sets of bipartite systems that are closed under wirings (see Section 5.2.3).

Using these closed sets allows us to prove in Section 6.4 that isotropic quantum systems cannot be used as resources to obtain arbitrary bipartite quantum systems by means of wirings (see Theorem 6.6).

Based on a conjecture, we construct in Section 6.5 a hierarchy of convex sets such that each set in the hierarchy is closed under wirings. This result can then be used to show that non-locality distillation is impossible for isotropic quantum systems (see Theorem 6.8).

It is important to note that the framework is formulated in a very general language which is not restricted to quantum information theory. Actually, it is not until Section 5.1, where a relation between quantum mechanics and tensor norms is established. There, we show how the Hilbertian and dual Hilbertian tensor norms are related to the set of bipartite quantum systems (see Theorem 5.1 and Theorem 5.2). In order to prove these results we derive semidefinite programs that compute these tensor norms (see Theorem 4.2 and Theorem 4.3). The results of Section 5.1 will be used in all applications of Chapter 6.

Finally, by first proving a generalized Grothendieck inequality (see Theorem 6.1 in Section 6.2.3) and using the results of Section 5.1 we can

[3]i.e., this set contains all bipartite systems for which the norm under consideration evaluates to a value smaller or equal to one.

find upper bounds on the entangled value of two-prover games (see Theorem 6.4 in Section 6.2).

Chapter 2

Preliminaries

2.1 Normed Vector Spaces

2.1.1 Definition of Normed Vector Spaces

We call the function $\|\cdot\|_X : \mathbb{R}^n \to \mathbb{R}$, for $1 \leq n < \infty$, a *norm* over the vector space $\mathbb{R}^n$ if it fulfils the following three conditions:

1. $\|P\|_X = 0$ if and only if $P = 0$.
2. $\|c \cdot P\|_X = |c| \cdot \|P\|_X$, for all $c \in \mathbb{R}$ and $P \in \mathbb{R}^n$.
3. $\|P + Q\|_X \leq \|P\|_X + \|Q\|_X$, for all $P, Q \in \mathbb{R}^n$.

Given a vector space $\mathbb{R}^n$ and a norm $\|\cdot\|_X$ on it, the tuple $X := (\mathbb{R}^n, \|\cdot\|_X)$ is called a *normed vector space*.

The *algebraic dual space* of $\mathbb{R}^n$, denoted by $(\mathbb{R}^n)^\#$, is the vector space of all linear functionals from the vector space $\mathbb{R}^n$ to the real numbers. We write $\langle G, P \rangle \in \mathbb{R}$ for the application of the linear functional $G : \mathbb{R}^n \to \mathbb{R}$ on the element $P \in \mathbb{R}^n$. Since $(\mathbb{R}^n)^\#$ is isomorphic to $\mathbb{R}^n$ the bracket $\langle \cdot, \cdot \rangle$ can be interpreted as the usual inner product of real vectors. The corresponding *dual norm* is defined by

$$\|G\|_{X^*} := \sup_{P \in \mathbb{R}^n} \left\{ |\langle G, P \rangle| \;:\; \|P\|_X \leq 1 \right\}, \tag{2.1}$$

for $G \in (\mathbb{R}^n)^\# \cong \mathbb{R}^n$. The *dual normed vector space* is then given by $X^* := ((\mathbb{R}^n)^\#, \|\cdot\|_{X^*})$. The normed vector space X^* is then called the *topological dual* of X.

We write $\langle f_i, P\rangle$, where $f_i \in (\mathbb{R}^n)^\# \cong \mathbb{R}^n$ is the all-zero vector with a one at position i, to access the i'th entry of the vector $P \in \mathbb{R}^n$. And similarly, if $G \in (\mathbb{R}^n)^\# \cong \mathbb{R}^n$ we use $\langle G, e_i\rangle$, where $e_i \in \mathbb{R}^n$ is the all-zero vector with a one at position i, to access the i'th entry of G. The inner product $\langle G, P\rangle$ can, therefore, also be written as

$$\langle G, P\rangle = \sum_{i=1}^{n} \langle G, e_i\rangle \cdot \langle f_i, P\rangle \ ,$$

for $P \in \mathbb{R}^n$ and $G \in (\mathbb{R}^n)^\# \cong \mathbb{R}^n$. Since $\mathbb{R}^n$ and $(\mathbb{R}^n)^\#$ are isomorphic it will be sufficient to just use the space $\mathbb{R}^n$ from now on.

2.1.2 Norm Inequalities

The Cauchy-Schwarz inequality says that for all $P, G \in \mathbb{R}^n$ it holds that

$$|\langle G, P\rangle|^2 \leq \langle G, G\rangle \cdot \langle P, P\rangle \ .$$

In particular, let $\mathbb{I} \in \mathbb{R}^n$ with $\langle f_i, \mathbb{I}\rangle = 1$ for all $1 \leq i \leq n$. Then

$$\left(\sum_{i=1}^{n} \langle G, e_i\rangle\right)^2 = |\langle G, \mathbb{I}\rangle|^2 \leq n \cdot \sum_{i=1}^{n} |\langle G, e_i\rangle|^2 \ ,$$

for all $G \in \mathbb{R}^n$.

Given two *finite*-dimensional normed vector spaces $X := (\mathbb{R}^n, \|\cdot\|_X)$ and $Y := (\mathbb{R}^n, \|\cdot\|_Y)$, there exist positive constants $C, D \in \mathbb{R}$ such that

$$C \cdot \|P\|_X \leq \|P\|_Y \leq D \cdot \|P\|_X \ ,$$

for all $P \in \mathbb{R}^n$. For a proof see for example [Wer95].

Furthermore, we have the following relation between norms and their duals.

Lemma 2.1. *Let $X := (\mathbb{R}^n, \|\cdot\|_X)$ and $Y := (\mathbb{R}^n, \|\cdot\|_Y)$ be normed vector spaces. Then, for any constant $c > 0$,*

$$\|P\|_X \leq c \cdot \|P\|_Y \ , \forall P \in \mathbb{R}^n \ \Leftrightarrow \|G\|_{Y^*} \leq c \cdot \|G\|_{X^*} \ , \forall G \in \mathbb{R}^n \ .$$

Proof. By the definition of the dual norm given in (2.1), we obtain

$$\begin{aligned}
\|G\|_{Y^*} &= \sup\{|\langle G, P\rangle| \ : \ \|P\|_Y \leq 1\} \\
&= c \cdot \sup\{|\langle G, P\rangle| \ : \ c \cdot \|P\|_Y \leq 1\} \\
&\leq c \cdot \sup\{|\langle G, P\rangle| \ : \ \|P\|_X \leq 1\} \\
&= c \cdot \|G\|_{X^*} \ ,
\end{aligned}$$

and similarly for the other direction. □

2.1.3 Tensor Product

Let $X := (\mathbb{R}^n, \|\cdot\|_X)$ and $Y := (\mathbb{R}^m, \|\cdot\|_Y)$ be arbitrary finite-dimensional normed vector spaces. The algebraic tensor product of X and Y is denoted by $X \otimes Y$. Although this is a finite-dimensional vector space again, it is not yet a *normed* vector space since we have not yet defined a norm on it. We will construct and investigate such norms, called *tensor norms*, on tensor product spaces in Chapter 3.

Given that $P \in X$ and $Q \in Y$ the *product tensor* $P \otimes Q$ is an element of $X \otimes Y$. In general, elements in $X \otimes Y$ are linear combinations of product tensors. In particular, we say that $P \in X \otimes Y$ if and only if

$$P = \sum_{i=1}^{k} P_A^i \otimes P_B^i ,$$

with $1 \le k < \infty$ and $P_A^i \in X$ and $P_B^i \in Y$, respectively. It is important to note that this decomposition is not unique since there are infinitely many such representations.

The tensor product has the following properties:

1. $(P_1 + P_2) \otimes Q = P_1 \otimes Q + P_2 \otimes Q$,
2. $P \otimes (Q_1 + Q_2) = P \otimes Q_1 + P \otimes Q_2$,
3. $c \cdot (P \otimes Q) = (c \cdot P) \otimes Q = P \otimes (c \cdot Q)$,
4. $0 \otimes Q = P \otimes 0 = 0$,

for all $P, P_1, P_2 \in X$, $Q, Q_1, Q_2 \in Y$, and $c \in \mathbb{R}$. Furthermore, let X^* and Y^* be the dual normed vector spaces of X and Y, respectively. Then, the inner product on the tensor product space is defined as

$$\langle G_A \otimes G_B, P_A \otimes P_B \rangle = \langle G_A, P_A \rangle \cdot \langle G_B, P_B \rangle ,$$

with $G_A \in X^*$, $G_B \in Y^*$, $P_A \in X$, $P_B \in Y$, $G_A \otimes G_B \in X^* \otimes Y^*$, and $P_A \otimes P_B \in X \otimes Y$. In particular, we have

$$\langle G, P \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle G, e_i \otimes e_j \rangle \cdot \langle f_i \otimes f_j, P \rangle ,$$

for $P \in X \otimes Y$ and $G \in X^* \otimes Y^*$. We also use the notation $e_{i,j} := e_i \otimes e_j$ and $f_{i,j} := f_i \otimes f_j$.

If $\{e_i\}_{1 \leq i \leq n}$ and $\{e_j\}_{1 \leq j \leq m}$ are orthonormal bases for X and Y, respectively, then $\{e_i \otimes e_j\}_{i,j}$ is an orthonormal basis for $X \otimes Y$. Therefore, if X and Y have dimensions n and m, respectively, the tensor product space $X \otimes Y$ has dimension $n \cdot m$.

Let us now provide a vector representation of the tensor product known as the *Kronecker product*. Let $P \in X$ and $Q \in Y$. The Kronecker product of P and Q is then given by

$$P \otimes Q = \left(\begin{array}{c} \langle f_1 \otimes f_1, P \otimes Q \rangle \\ \vdots \\ \langle f_1 \otimes f_m, P \otimes Q \rangle \\ \hline \langle f_2 \otimes f_1, P \otimes Q \rangle \\ \vdots \\ \langle f_2 \otimes f_m, P \otimes Q \rangle \\ \hline \vdots \\ \hline \langle f_n \otimes f_1, P \otimes Q \rangle \\ \vdots \\ \langle f_n \otimes f_m, P \otimes Q \rangle \end{array} \right) = \left(\begin{array}{c} \langle f_1, P \rangle \cdot \langle f_1, Q \rangle \\ \vdots \\ \langle f_1, P \rangle \cdot \langle f_m, Q \rangle \\ \hline \langle f_2, P \rangle \cdot \langle f_1, Q \rangle \\ \vdots \\ \langle f_2, P \rangle \cdot \langle f_m, Q \rangle \\ \hline \vdots \\ \hline \langle f_n, P \rangle \cdot \langle f_1, Q \rangle \\ \vdots \\ \langle f_n, P \rangle \cdot \langle f_m, Q \rangle \end{array} \right) . \tag{2.2}$$

Note that since $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{n \cdot m}$ the product tensor $P \otimes Q$ can also be seen as an element of $\mathbb{R}^{n \cdot m}$.

2.1.4 Examples of Normed Vector Spaces

The normed vector space that is most important to us is denoted by

$$\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) := (\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}, \| \cdot \|_{\infty(1)}) ,$$

with $\mathcal{X} := \{1, 2, ..., |\mathcal{X}|\}$ and $\mathcal{A} := \{1, 2, ..., |\mathcal{A}|\}$ sets of finite cardinality. The $\infty(1)$-norm is defined as

$$\|P\|_{\infty(1)} := \max_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} |\langle f_x \otimes f_a, P \rangle| ,$$

for $P \in \mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}$.

The dual normed vector space is given by $\left(\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})\right)^* \equiv \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) := (\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}, \|\cdot\|_{1(\infty)})$ with

$$\|G\|_{1(\infty)} \equiv \|G\|_{\infty(1)^*} := \sum_{x\in\mathcal{X}} \max_{a\in\mathcal{A}} |\langle G, e_x \otimes e_a\rangle| \, ,$$

for $G \in (\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|})^\# \cong \mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}$. It is easy to verify that

$$\|G\|_{1(\infty)} = \sup_P \{|\langle G, P\rangle| \ : \ \|P\|_{\infty(1)} \leq 1\} \, ,$$

and, therefore, the $1(\infty)$-norm is indeed the dual of the $\infty(1)$-norm.

Note that for $|\mathcal{A}| = 1$ we obtain the normed vector space $\ell_\infty^{|\mathcal{X}|} := (\mathbb{R}^{|\mathcal{X}|}, \|\cdot\|_\infty)$, where $\|P\|_\infty := \max_{x\in\mathcal{X}} |\langle f_x, P\rangle|$, and for $|\mathcal{X}| = 1$ the normed vector space $\ell_1^{|\mathcal{A}|} := (\mathbb{R}^{|\mathcal{A}|}, \|\cdot\|_1)$, where $\|P\|_1 := \sum_{a\in\mathcal{A}} |\langle f_a, P\rangle|$.

Another normed vector space we will regularly use is $\ell_2^n := (\mathbb{R}^n, \|\cdot\|_2)$ with the 2-norm defined as

$$\|P\|_2 := \left(\sum_{i=1}^n |\langle f_i, P\rangle|^2\right)^{1/2} \, .$$

The special properties the space ℓ_2^n has make it to a *Hilbert space*. A finite-dimensional Hilbert space $\mathcal{H}$ is a normed vector space where the norm fulfils the parallelogram identity, i.e.,

$$\|P+Q\|_{\mathcal{H}}^2 + \|P-Q\|_{\mathcal{H}}^2 = 2\|P\|_{\mathcal{H}}^2 + 2\|Q\|_{\mathcal{H}}^2 \, ,$$

for all $P, Q \in \mathcal{H}$. Furthermore, in that case the norm $\|\cdot\|_{\mathcal{H}}$ uniquely induces an *inner product* on the Hilbert space. The normed vector space ℓ_2^n has the nice property that it is *self dual*, i.e., $\ell_2^n \cong (\ell_2^n)^*$ which means that

$$\|P\|_2 = \sup_{G\in\mathbb{R}^n} \{|\langle G, P\rangle| \ : \ \|G\|_{2^*} \leq 1\} = \|P\|_{2^*} \, ,$$

for all $P \in \mathbb{R}^n$. Note that in particular $\|P\|_2^2 = \langle P, P\rangle$.

The 2-norm behaves 'nicely' on product tensors, i.e., it is easy to see that

$$\|P \otimes Q\|_2 = \|P\|_2 \cdot \|Q\|_2 \, ,$$

for all $P \in \ell_2^n$ and $Q \in \ell_2^m$.

2.1.5 Operator Norms

Let $X := (\mathbb{R}^n, \|\cdot\|_X)$ and $Y := (\mathbb{R}^m, \|\cdot\|_Y)$ be finite-dimensional normed vector spaces and $\mathcal{T} : X \to Y$ be a linear map. The *operator norm* of $\mathcal{T}$ is defined as

$$\|\mathcal{T}\|_{X\to Y} := \sup_{P\in X} \{\|\mathcal{T}(P)\|_Y \ : \ \|P\|_X \leq 1\} \ .$$

The *transposed operator* of $\mathcal{T}$ is denoted by $\mathcal{T}^T : Y^* \to X^*$ and defined as

$$\langle f_i, \mathcal{T}^T(e_j)\rangle := \langle f_j, \mathcal{T}(e_i)\rangle \ ,$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. In particular, we have that

$$\langle G, \mathcal{T}(P)\rangle = \langle \mathcal{T}^T(G), P\rangle \ ,$$

for all $P \in X$ and all $G \in Y^*$.

Lemma 2.2. *Let $X := (\mathbb{R}^n, \|\cdot\|_X)$ be a finite-dimensional normed vector space and $\mathcal{T} : \ell_2^m \to X$ a linear operator. Then*

$$\|\mathcal{T}\|_{2\to X} = \|\mathcal{T}^T\|_{X^*\to 2} \ .$$

A proof of this fact is provided in Appendix A.2.

2.2 Quantum Physics

Let $\mathcal{H}_A$ be a (complex) Hilbert space of dimension n. We use Dirac's bracket notation[1] to denote elements of $\mathcal{H}_A$ and write the inner product associated to the Hilbert space as $\langle\phi|\psi\rangle \in \mathbb{C}$, for $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A$. Then, a *pure quantum state* $|\phi\rangle$ is an element of $\mathcal{H}_A$ such that $\langle\phi|\phi\rangle = 1$.

Let M and O be linear operators on $\mathcal{H}_A$. Then, M is called a *projector* if $M^2 = M$ and $M \succeq 0$ (i.e., $\langle\phi|M|\phi\rangle \geq 0$ for all $|\phi\rangle \in \mathcal{H}_A$) and O is called *Hermitian* if $O = O^\dagger$ (i.e., $\langle f_i, O^\dagger(e_j)\rangle = \langle f_j, O(e_i)\rangle^*$ for all $1 \leq i, j \leq n$ and where * computes the complex conjugate). Note that every projector is also Hermitian.

Every Hermitian operator has a spectral decomposition, i.e., there exist projectors $\{M^a\}_{a\in\mathcal{A}}$ and real numbers $a \in \mathbb{R}$ such that

$$O = \sum_a a \cdot M^a \ ,$$

[1] See also [NC00] for a good introduction to quantum information theory.

where the projectors satisfy the *completeness relation* $\sum_a M^a = id_{\mathcal{H}_A}$ (where $id_{\mathcal{H}_A}$ is the identity operator on the space $\mathcal{H}_A$) and the *orthogonality relation* $M^{a_1} \cdot M^{a_2} = 0$ for all $a_1 \neq a_2$ (see also [NC00]). In particular, if we consider only one-dimensional projectors $M^a := |a\rangle\langle a|$, where $\{|a\rangle\}_{1 \leq a \leq n}$ is an orthonormal basis of $\mathcal{H}_A$, the completeness relation becomes

$$\sum_{a=1}^{n} |a\rangle\langle a| = id_{\mathcal{H}_A} \ .$$

A *projective measurement* is described by a Hermitian operator O, also called *observable*, on the Hilbert space $\mathcal{H}_A$. The probability of measuring $a \in \mathcal{A}$, given that we apply the measurement on the pure quantum state $|\phi\rangle \in \mathcal{H}_A$, is postulated to be $\langle\phi|M^a|\phi\rangle$, with M^a a projector of the spectral decomposition $O = \sum_a a \cdot M^a$. Using linearity and the fact that the projectors sum to the identity, yields

$$\sum_{a \in \mathcal{A}} \langle\phi|M^a|\phi\rangle = \langle\phi| \sum_{a \in \mathcal{A}} M^a|\phi\rangle = \langle\phi|id_{\mathcal{H}_A}|\phi\rangle = 1 \ .$$

Furthermore, since $M^a \succeq 0$ we have $\langle\phi|M^a|\phi\rangle \geq 0$ and, therefore, the values $\{\langle\phi|M^a|\phi\rangle\}_{a \in \mathcal{A}}$ indeed correspond to a valid probability distribution.

A *bipartite* quantum state is an element of the tensor product of Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Note that $\mathcal{H}_A \otimes \mathcal{H}_B$ is again a Hilbert space. Assume we are given another projective measurement $\{N^b\}_{b \in \mathcal{B}}$ on $\mathcal{H}_B$. The probability of measuring $a \in \mathcal{A}$ and $b \in \mathcal{B}$ on the pure quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is then computed by

$$\langle\Psi|M^a \otimes N^b|\Psi\rangle \ . \tag{2.3}$$

Remark 1. It is no restriction to assume pure quantum states and projective measurements since the most general setting with mixed quantum states and POVM measurements can always be transformed to the former one by enlarging the underlying Hilbert spaces (see for example [NC00]).

2.3 Bipartite Systems

A *system*[2] P is a device which on an input $x \in \mathcal{X}$, immediately provides an output $a \in \mathcal{A}$. We model the behaviour of a system P by a conditional

[2]The quantum information community also uses the notion of a *box*, whereas classical information theorists would call it a *memoryless channel*.

probability distribution $P(a|x) := P_{A|X}(a,x)$. In this thesis, we consider systems with *finite* alphabets $\mathcal{X}$ and $\mathcal{A}$, respectively. By definition, every system P fulfils the following conditions:

$$\begin{aligned} P(a|x) &\geq 0\ (positivity)\ , \\ \sum_{a\in\mathcal{A}} P(a|x) &= 1\ (normalization)\ , \end{aligned}$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$. Furthermore, if $P(a|x) \in \{0,1\}$, for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$, the system P is called *deterministic*.

Every system P can be represented as an element of the tensor product space $\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}$ by the following identification:

$$\langle f_x \otimes f_a, P\rangle := P(a|x)\ , \tag{2.4}$$

for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ and with $P \in \mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}$. As we have discussed in Section 2.1.4, the vector space $\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|} \cong \mathbb{R}^{|\mathcal{X}||\mathcal{A}|}$ can be turned into a normed vector space by defining a norm on it. If we choose the $\infty(1)$-norm, every system P has the property that $\|P\|_{\infty(1)} = 1$. We will see in the next chapter that it is, therefore, convenient to consider systems as elements of the normed vector space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$.

Example 1. Let $\mathcal{X} = \mathcal{A} = \{1,2\}$. A system P written as an element in the vector space $\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|}$ is given by (see (2.2) and (2.4)):

$$P = \left(\begin{array}{c} P(1|1) \\ P(2|1) \\ \hline P(1|2) \\ P(2|2) \end{array}\right) = \left(\begin{array}{c} \langle f_1 \otimes f_1, P\rangle \\ \langle f_1 \otimes f_2, P\rangle \\ \hline \langle f_2 \otimes f_1, P\rangle \\ \langle f_2 \otimes f_2, P\rangle \end{array}\right)\ .$$

We are particularly interested in *bipartite systems* shared between two parties, called Alice and Bob. A bipartite system P has inputs $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and outputs $a \in \mathcal{A}$, $b \in \mathcal{B}$ for Alice and Bob, respectively (see Figure 1.1), with finite input/output alphabets. We model the behaviour of a bipartite system P by a conditional probability distribution $P(a,b|x,y) := P_{AB|XY}(a,b,x,y)$. A bipartite system P can be represented as an element of the tensor product space $\mathbb{R}^{|\mathcal{X}|} \otimes \mathbb{R}^{|\mathcal{A}|} \otimes \mathbb{R}^{|\mathcal{Y}|} \otimes \mathbb{R}^{|\mathcal{B}|}$ by the following identification:

$$\langle f_{x,a} \otimes f_{y,b}, P\rangle := P(a,b|x,y)\ ,$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$. Therefore, we also write $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

In this thesis we consider bipartite systems as *resources* in information processing tasks that are executed by two separate parties. We make two crucial assumptions about the system which are used in the two-party computations: first, each (bipartite) system can only be used once. And second, Alice gets her output immediately after she has provided her input to the bipartite system, independent of whether Bob has already provided his input (and similarly for the other direction).

Let us denote the set of all bipartite systems with alphabet sizes at most $m \in \mathbb{N}$ by $\mathcal{S}_m$. In other words, $P \in \mathcal{S}_m$ if and only if $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, with $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{X}|, |\mathcal{Y}| \leq m$, and P is a bipartite system. In this thesis we will be only interested in bipartite systems which do not allow for message transmission between Alice and Bob. We, therefore, have to restrict the space of bipartite systems $\mathcal{S}_m$ by adding constraints.

2.3.1 Non-Signalling Systems

A bipartite system $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is called *non-signalling* if it satisfies the following conditions:

$$\begin{aligned} P(b|y) \equiv \sum_{a \in \mathcal{A}} P(a,b|x,y) &= \sum_{a \in \mathcal{A}} P(a,b|x',y) \text{ , for all } b,x,x',y \text{ ,} \\ P(a|x) \equiv \sum_{b \in \mathcal{B}} P(a,b|x,y) &= \sum_{b \in \mathcal{B}} P(a,b|x,y') \text{ , for all } a,x,y,y' \text{ ,} \end{aligned}$$

i.e., the output b of Bob is independent of the input x of Alice and Alice's output a is independent of Bob's input y. We denote by $\mathcal{NS}_m$ the set of all non-signalling systems with alphabet sizes upper bounded by $m \in \mathbb{N}$. Clearly, we have $\mathcal{NS}_m \subseteq \mathcal{S}_m$. In the special case where Alice and Bob have only binary inputs and binary outputs, respectively, the according set is denoted by $\mathcal{NS}_{\mathrm{CHSH}}$. We also say that the elements of $\mathcal{NS}_{\mathrm{CHSH}}$ are *binary non-signalling systems*.

Example 2. Let $\mathcal{A} = \mathcal{B} = \mathcal{X} = \mathcal{Y} = \{1,2\}$. We define the system $P_{\mathrm{PR}} \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ by $P_{\mathrm{PR}}(a,b|1,1) = P_{\mathrm{PR}}(a,b|1,2) = P_{\mathrm{PR}}(a,b|2,1) := 1/2$ if $a = b$ and $P_{\mathrm{PR}}(a,b|2,2) := 1/2$ if $a \neq b$. All other entries are, therefore, zero. Computing the marginals then yields

$$P_{\mathrm{PR}}(b|x,y) = P_{\mathrm{PR}}(a|x,y) = 1/2 \text{ ,}$$

for all $a,b,x,y \in \{1,2\}$, and, hence, P_{PR} is non-signalling. As we have seen in Section 2.1.3, any element of $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ can be decomposed into a sum of product tensors. One possible decomposition for the bipartite system P_{PR} is given by

$$\begin{aligned} P_{\mathrm{PR}} &= \frac{1}{2}\cdot\begin{pmatrix}0\\1\\\hline 0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\\\hline 1\\0\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}1\\0\\\hline 0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\\\hline 1\\0\end{pmatrix}\\ &+ \frac{1}{2}\cdot\begin{pmatrix}0\\1\\\hline 1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\\hline 0\\1\end{pmatrix}-\frac{1}{2}\cdot\begin{pmatrix}0\\1\\\hline 0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\\\hline 1\\0\end{pmatrix}. \end{aligned}$$

Note that the local vector elements of Alice and Bob given in the decomposition of Example 2 are all *systems*. This is a generic feature of non-signalling systems:

Lemma 2.3. *$P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is a non-signalling system if and only if there exists a decomposition*

$$P = \sum_i p_i \cdot P_A^i \otimes P_B^i \ ,$$

such that $\sum_i p_i = 1$ and $P_A^i \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B^i \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ are systems.

A proof can be found in [DKLR09].

2.3.2 Quantum Systems

A bipartite system $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is called *quantum* if it can be simulated by appropriate product measurements on a bipartite quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, i.e.,

$$P(a,b|x,y) = \langle\Psi|M_x^a \otimes N_y^b|\Psi\rangle \ , \text{ for all } a,b,x,y \ ,$$

for projective measurements $\{M_x^a\}_{a\in\mathcal{A}}$ and $\{N_y^b\}_{b\in\mathcal{B}}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively (see also (2.3)). Since we do not restrict the dimension of the underlying Hilbert spaces, we can without loss of generality assume projective measurements and pure quantum states (see also Remark 1).

The set of all quantum systems with alphabet sizes upper bounded by m is denoted by $\mathcal{Q}_m$. Again, $\mathcal{Q}_{\mathrm{CHSH}}$ denotes the restriction to binary systems. It is easy to see that any bipartite quantum system is non-signalling and, hence, we have $\mathcal{Q}_m \subseteq \mathcal{NS}_m$.

2.3.3 Local Systems

A bipartite system $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is called *local* if it can be expressed as

$$P(a,b|x,y) = \sum_i P(\mu_i) \cdot P(a|x,\mu_i) \cdot P(b|y,\mu_i) \ , \text{ for all } a,b,x,y \ ,$$

with $\sum_i P(\mu_i) = 1$, $P(\mu_i) \geq 0$, and $P(a|x,\mu_i)$ and $P(b|y,\mu_i)$ arbitrary systems which depend on the *local hidden variable* μ_i. We denote the set of all local systems with alphabet sizes at most m by $\mathcal{L}_m$ and its restriction to binary systems by $\mathcal{L}_{\mathrm{CHSH}}$.

The following relations hold between the sets that have just been introduced:

$$\mathcal{L}_m \subsetneq \mathcal{Q}_m \subsetneq \mathcal{NS}_m \subsetneq \mathcal{S}_m \ ,$$

for all $m \geq 2$. The first *strict* subset symbol can be read as "quantum mechanics is non-local" and the second one as "but quantum mechanics is not maximally non-local" (see also Figure 2.1 in Section 2.6.1). Furthermore, we use the notation $\partial\mathcal{S}$ to denote the *boundary* of a set $\mathcal{S}$ and $\emptyset$ to denote the empty set. By $\mathcal{NS}_m \backslash \mathcal{L}_m$ we denote the set of all non-signalling systems which are non-local.

2.3.4 Correlation Systems

Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be a bipartite system with *binary* outputs on Alice's and Bob's side, respectively. The *correlation system* $C \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ associated with the system P is then defined as

$$\langle f_x \otimes f_y, C\rangle := P(a=b|x,y) - P(a \neq b|x,y) \ . \tag{2.5}$$

Hence, in the case where $\mathcal{A} = \mathcal{B} = \{-1,+1\}$, the expression $\langle f_x \otimes f_y, C\rangle$ denotes the *expectation value* of the system P for the inputs x and y.

The set $\mathcal{NS}_{\mathrm{co}}$ contains all correlation systems which can be obtained by computing (2.5) for non-signalling systems with binary outputs. Similarly, we define the sets $\mathcal{L}_{\mathrm{co}}$ and $\mathcal{Q}_{\mathrm{co}}$ which contain all local and all quantum correlation systems, respectively.

Let us provide an alternative definition of the set of all quantum correlation systems: $C \in \mathcal{Q}_{\mathrm{co}}$ if and only if there exists a pure quantum state $|\Psi\rangle$ and Hermitian operators A_x and B_y with the two eigenvalues $+1$ and -1, respectively, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, such that

$$\langle f_x \otimes f_y, C\rangle = \langle\Psi|A_x \otimes B_y|\Psi\rangle \ ,$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Note that $\ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ is equivalent to $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ given that $|\mathcal{A}| = |\mathcal{B}| = 1$ (see also Section 2.1.4). In this thesis, we will prove results for bipartite systems in $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ that hold for any alphabet sizes, in particular for the case $|\mathcal{A}| = |\mathcal{B}| = 1$. Then, if we set $|\mathcal{A}| = |\mathcal{B}| = 1$ we indicate that we consider correlation systems and *not* bipartite systems that have output alphabets of cardinality 1.

2.4 Wirings of Non-Signalling Systems

Assume Alice is given several systems $P_{A_1}, ..., P_{A_n}$. These systems can be considered as resources which can be used to build a new system $P_A := \mathcal{W}_A(P_{A_1}, ..., P_{A_n})$. We call $\mathcal{W}_A$ a *local strategy* for Alice. A local strategy can be seen as a circuit which describes how the systems $P_{A_1}, ..., P_{A_n}$ are *wired* together (see Figure 3.2 in Section 3.4.2 for an example). Assume that Bob has also a local strategy $\mathcal{W}_B$. If Alice and Bob now share several non-signalling systems $P_{A_1B_1}, ..., P_{A_nB_n}$ they can apply their local strategies on their respective parts of the non-signalling systems in order to obtain a new non-signalling system. Therefore, roughly speaking, a wiring is a non-interactive protocol between Alice and Bob which takes as inputs several non-signalling systems, wires them together using classical local circuits on Alice's and Bob's side, respectively, and outputs a non-signalling system.

Formally, we use the following definition of a wiring [BP05]. Let Alice and Bob have inputs x and y, respectively, and assume they share n non-signalling systems $P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n} \in \mathcal{NS}_m$ and have shared randomness with distribution $P(\mu)$. Then, assume that Alice applies the following protocol [BP05]:

1. She inputs $x_1 := f_{\mathcal{W}}^1(x, \mu)$ into her part of the non-signalling system $P_{A_{i_1}B_{i_1}}$, with $i_1 := g_{\mathcal{W}}^1(x, \mu)$. She obtains output a_1.

2. She inputs $x_2 := f_{\mathcal{W}}^2(x, a_1, \mu)$ into her part of the non-signalling system $P_{A_{i_2}B_{i_2}}$, with $i_2 := g_{\mathcal{W}}^2(x, a_1, \mu)$. She obtains output a_2.

3. She continues this protocol until all n systems have been used. Her final output is $a := f_{\mathcal{W}}^{n+1}(x, a_1, ..., a_n, \mu)$.

Hence, the function $f_{\mathcal{W}}^i$ computes the input for the next system which is selected by the function $g_{\mathcal{W}}^i$. Bob proceeds along similar lines applying the functions $h_{\mathcal{W}}^1, ..., h_{\mathcal{W}}^{n+1}$ and $k_{\mathcal{W}}^1, ..., k_{\mathcal{W}}^n$ on his input y, the shared randomness μ, and his part of the shared non-signalling systems. Assume he obtains as his final output b. We call the map $\mathcal{W} : \mathcal{NS}_m^{\times n} \to \mathcal{NS}_{m'}$, with $Q := \mathcal{W}(P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n})$, a *wiring* if and only if there exist functions $f_{\mathcal{W}}^1, ..., f_{\mathcal{W}}^{n+1}$, $g_{\mathcal{W}}^1, ..., g_{\mathcal{W}}^n$ and $h_{\mathcal{W}}^1, ..., h_{\mathcal{W}}^{n+1}$, $k_{\mathcal{W}}^1, ..., k_{\mathcal{W}}^n$ for Alice and Bob, respectively, applied on $P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n}$ by the above described protocol and a distribution on the shared randomness $P(\mu)$ such that the resulting conditional probability distribution $\sum_\mu P(\mu) \cdot P(a, b|x, y, \mu)$ is equal to the non-signalling system Q.

Remark 2. It is crucial to assume that the input systems to a wiring are non-signalling since there exist wirings on *signalling* systems that have an undefined behaviour. In order to see this, consider the following example. Alice and Bob share two signalling systems $P_{A_1B_1}$ and $P_{A_2B_2}$, respectively, with the same behaviour. In particular, Bob's input is Alice's output and the output of Bob is Alice's input. The wiring is then defined as follows: Alice gets input x which she feeds into her part of the system $P_{A_1B_1}$ and Bob gets input y and uses his part of the system $P_{A_2B_2}$. The local strategy of Alice is such that she should then use the output of $P_{A_1B_1}$ as the input for the system $P_{A_2B_2}$. Similarly, Bob's local strategy is such that he uses the output of $P_{A_2B_2}$ as the input for $P_{A_1B_1}$. However, since Bob has not yet used his part of the system $P_{A_1B_1}$, Alice does not get an output. Similarly, Bob does not get an output for the system $P_{A_2B_2}$ since Alice has not yet provided an input to her part of $P_{A_2B_2}$. Hence, there is a deadlock and no final output is obtained.

2.5 Bell Inequalities and Two-Prover Games

2.5.1 Bell Inequalities

Let $c \in \mathbb{R}$ be a positive constant and $G : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}) \to \mathbb{R}$ a linear functional from the space of systems to the real numbers. Then, if there exists a *quantum* system $Q \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ such that $|\langle G, Q\rangle| > c$ and

$$|\langle G, P\rangle| \leq c\,, \tag{2.6}$$

for all *local* systems $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, the expression in (2.6) is called a *Bell inequality* [Bel64]. Note that since G is a linear functional from the space of systems to the real numbers it is an element of the dual space $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$.

The constant c in (2.6) is usually the smallest c which fulfils this inequality for all local systems. Hence, a typical Bell inequality is given by $|\langle G, P\rangle| \leq \omega_{\mathcal{L}}(G)$ with

$$\omega_{\mathcal{L}}(G) := \sup_P \{|\langle G, P\rangle| \ : \ P \textit{ is local}\} \ . \tag{2.7}$$

Consequently, if the constant c is equal to the expression $\omega_{\mathcal{L}}(G)$ we call $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ a Bell inequality.

The largest quantum violation of a Bell inequality G is defined as

$$\omega_{\mathcal{Q}}(G) := \sup_P \{|\langle G, P\rangle| \ : \ P \textit{ is quantum } \} \ .$$

Hence, any Bell inequality G must fulfil $\omega_{\mathcal{L}}(G) < \omega_{\mathcal{Q}}(G)$.

We say that a Bell inequality $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ is violated by a bipartite system P if

$$|\langle G, P\rangle| \equiv \left| \sum_{x,y,a,b} \langle G, e_{x,a} \otimes e_{y,b}\rangle \cdot \langle f_{x,a} \otimes f_{y,b}, P\rangle \right| > \omega_{\mathcal{L}}(G) \ .$$

Example 3. The most prominent example of a Bell inequality is the so-called Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [CHSH69]. Let $\mathcal{A} = \mathcal{B} = \mathcal{X} = \mathcal{Y} = \{1, 2\}$. The CHSH Bell inequality G_{CHSH} is then defined as

$$\langle G_{\mathrm{CHSH}}, e_{x,a} \otimes e_{y,b}\rangle := \begin{cases} +1 \ , & \textit{if } (a-1) \oplus (b-1) = (x-1) \wedge (y-1) \\ -1 \ , & \textit{otherwise} \end{cases} \ ,$$

for all $x, y, a, b \in \{1, 2\}$. It is not difficult to see that $\omega_{\mathcal{L}}(G_{\mathrm{CHSH}}) = 2$ and, therefore, $|\langle G_{\mathrm{CHSH}}, P\rangle| \leq 2$ for all $P \in \mathcal{L}_{\mathrm{CHSH}}$. Furthermore, there exists a quantum system $Q \in \mathcal{Q}_{\mathrm{CHSH}}$ such that $|\langle G_{\mathrm{CHSH}}, Q\rangle| = 2\sqrt{2}$. This was shown to be maximal for quantum theory in [Tsi80]. By allowing all non-signalling system we obtain $|\langle G_{\mathrm{CHSH}}, P_{\mathrm{PR}}\rangle| = 4$ with P_{PR} the non-signalling system given in Example 2 in Section 2.3.1 [PR94].

2.5.2 Two-Prover Games

In a *classical one-round two-prover cooperative game of incomplete information* [BOGKW88] two classical and spatially separated provers, usually called Alice and Bob, try to win a game by interacting with a verifier. The two provers can agree on a strategy before the game. During the game the two provers are not allowed to communicate. The messages which are exchanged by the verifier and the two provers are classical bit strings. Let $\pi : \mathcal{X} \times \mathcal{Y} \to [0,1]$ be a probability distribution known by the verifier and the two provers. The verifier selects $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ according to the probability distribution π and sends the value x to Alice and y to Bob. Alice and Bob send to the verifier the values $s_A(x) = a \in \mathcal{A}$ and $s_B(y) = b \in \mathcal{B}$ where we call the pair (s_A, s_B) a strategy for the game. Note that it is sufficient to consider deterministic strategies only as the optimal (shared) randomness can be selected in advance. The provers win the game $G := (\pi, V)$ if the publicly known predicate $V : \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ evaluates to 1 for the four-tuple (a, b, x, y). We will consider two classes of games:

Definition 2.1. Let $G = (\pi, V)$ be a game. Then

- G is called a *unique* game if there exist permutations $\sigma_{x,y}$ for all inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $V(a, b, x, y) = 1$ if and only if $b = \sigma_{x,y}(a)$.
- G is called an *XOR* game if $\mathcal{A} = \mathcal{B} = \{1, 2\}$ with $V(1, 1, x, y) = V(2, 2, x, y)$ and $V(1, 2, x, y) = V(2, 1, x, y)$ for all inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Note that a strategy is just a deterministic system and, therefore, any strategy $s_A : \mathcal{X} \to \mathcal{A}$ can be represented as an element of $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$. Let us denote the systems corresponding to s_A and s_B by P_A and P_B, respectively. Furthermore, let us define the two-prover game $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ as

$$\langle G, e_{x,a} \otimes e_{y,b} \rangle := \pi(x, y) \cdot V(a, b, x, y) \ . \tag{2.8}$$

Then, the *classical value* of a game, denoted by $\omega_{\mathcal{L}}(G)$, is defined as the maximal value that can be achieved by any two strategies s_A and s_B for a given game $G = (\pi, V)$, i.e.,

$$\begin{aligned}
\omega_{\mathcal{L}}(G) \quad &:= \quad \max_{s_A, s_B} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \pi(x, y) \cdot V(s_A(x), s_B(y), x, y) \ , \\
&= \quad \sup_{P_A, P_B} \sum_{x,y} \sum_{a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \langle f_{x,a} \otimes f_{y,b}, P_A \otimes P_B \rangle \\
&= \quad \sup_P \{ |\langle G, P \rangle| \ : \ P \textit{ is local } \} \ , \qquad (2.9)
\end{aligned}$$

where we used in the third line that if P is local then we can conclude that the optimal P in (2.9) is given by $P_A \otimes P_B$ for some systems P_A and

P_B. Note that this definition of $\omega_{\mathcal{L}}$ is really the same as in (2.7). It will be clear from the context whether G represents a pair (π, V) or an element of $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$.

We can give the two provers more power by allowing them to share entangled quantum states. Alice and Bob can then select measurements depending on their inputs x and y, respectively, and measure the entangled state $|\Psi\rangle$, obtaining measurement results a and b, respectively. The *entangled value* of a game $G = (\pi, V)$, denoted by $\omega_{\mathcal{Q}}(G)$, is defined as

$$\begin{aligned} \omega_{\mathcal{Q}}(G) &:= \sup_{|\Psi\rangle} \sup_{M_x^a, N_y^b} \sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b,x,y) \cdot \langle \Psi | M_x^a \otimes N_y^b | \Psi \rangle \,, \\ &= \sup_P \{ |\langle G, P \rangle| \ : \ P \textit{ is quantum } \} \,, \end{aligned}$$

with $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and projective measurements $\{M_x^a\}_{a\in\mathcal{A}}$ and $\{N_y^b\}_{b\in\mathcal{B}}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. It is clear that $\omega_{\mathcal{Q}}(G) \geq \omega_{\mathcal{L}}(G)$ for all games G.

Note that two-prover games, interpreted as elements of $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, are closely related to Bell inequalities. However, there are some crucial differences:

- Two-prover games have only non-negative entries, i.e., $\langle G, e_{x,a} \otimes e_{y,b} \rangle \geq 0$ for all x, y, a, b. This is in contrast to Bell inequalities which a priori can have arbitrary entries.

- Two-prover games fulfil a normalization condition (see definition given in (2.8)).

- Bell inequalities must fulfil the the inequality $\omega_{\mathcal{L}}(G) < \omega_{\mathcal{Q}}(G)$ which is not required for games.

2.6 Non-Locality and Isotropic Systems

In the following we consider binary systems with input/output alphabets given by $\mathcal{A} = \mathcal{B} = \mathcal{X} = \mathcal{Y} = \{0,1\}$. The set $\mathcal{L}_{\text{CHSH}}$, which is a subset of $\mathbb{R}^{16}$, is a convex polytope (see for example [BLM+05]). Its non-trivial facets (i.e., not given by normalization, positivity or non-signalling constraints) — there are eight of them — are called *CHSH facets*. We denote the according hyperplanes by $F^{\eta\nu\sigma}$, with $\eta, \nu, \sigma \in \{0,1\}$. These hyperplanes are given by $F^{\eta\nu\sigma}(P) = 0$ with

$$\begin{aligned} F^{\eta\nu\sigma}(P) \quad &:= \quad (-1)^{\sigma} E_{00}(P) + (-1)^{\nu\oplus\sigma} E_{01}(P) \\ &+ \quad (-1)^{\eta\oplus\sigma} E_{10}(P) - (-1)^{\eta\oplus\nu\oplus\sigma} E_{11}(P) - 2 \,, \end{aligned}$$

and correlation functions

$$E_{xy}(P) := P(0,0|x,y) - P(0,1|x,y) - P(1,0|x,y) + P(1,1|x,y) \,,$$

for all $x, y \in \{0,1\}$. $P \in \mathcal{NS}_{\text{CHSH}}$ is a local system if and only if $F^{\eta\nu\sigma}(P) \leq 0$ for all $\eta, \nu, \sigma \in \{0,1\}$. The eight inequalities $F^{\eta\nu\sigma}(P) \leq 0$, for $\eta, \nu, \sigma \in \{0,1\}$, are called the *CHSH Bell inequalities*. By setting $\eta = \nu = \sigma = 0$ we obtain the canonical CHSH Bell inequality given by

$$F^{000}(P) = E_{00}(P) + E_{01}(P) + E_{10}(P) - E_{11}(P) - 2 \leq 0 \,.$$

Note that this is exactly the Bell inequality given in Example 3 of Section 2.5.1.

2.6.1 Measure of Non-Locality

The CHSH measure of non-locality [CHSH69] for a non-signalling system $P \in \mathcal{NS}_{\text{CHSH}}$ is proportional to the distance of P to the local polytope $\mathcal{L}_{\text{CHSH}}$, i.e., the distance to one of the eight hyperplanes $F^{\eta\nu\sigma}$, with $\eta, \nu, \sigma \in \{0,1\}$, which are associated to the CHSH facets (see Figure 2.1). It follows from elementary geometry that the Euclidean distance of an arbitrary $P \in \mathcal{NS}_{\text{CHSH}}$ to the hyperplane $F^{\eta\nu\sigma}$ is given by

$$d(P, F^{\eta\nu\sigma}) = \frac{|F^{\eta\nu\sigma}(P)|}{4} \,.$$

Note that only for $F^{\eta\nu\sigma}(P) > 0$ we can conclude that P lies *outside* the local polytope and is, therefore, non-local. Then, since the CHSH non-locality of $P \in \mathcal{NS}_{\text{CHSH}}$, denoted by $NL(P)$, is proportional to the distance of P to the local polytope, we have the following definition.

Definition 2.2 (CHSH Non-Locality)**.** Let $P \in \mathcal{NS}_{\text{CHSH}}$. Then, the CHSH non-locality of P is defined as

$$NL(P) := \max_{\eta,\nu,\sigma\in\{0,1\}} F^{\eta\nu\sigma}(P) + 2 \,.$$

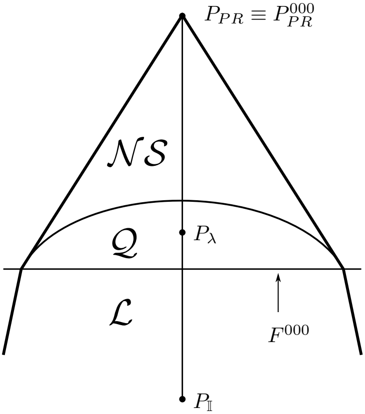

Figure 2.1: A two-dimensional slice of the eight-dimensional non-signalling polytope $\mathcal{NS}_{\text{CHSH}}$ is shown. The set of all local binary systems is denoted by $\mathcal{L}_{\text{CHSH}}$ and the set of all binary quantum systems by $\mathcal{Q}_{\text{CHSH}}$. Note that $\mathcal{L}_{\text{CHSH}} \subsetneq \mathcal{Q}_{\text{CHSH}} \subsetneq \mathcal{NS}_{\text{CHSH}}$. The isotropic system P_λ is a convex combination of the maximally mixed system $P_{\mathbb{I}}$ and the PR-box P_{PR}. The systems which lie above the hyperplane F^{000} are non-local and violate the CHSH Bell inequality $F^{000}(P) \leq 0$.

Note that there is a rescaling of the distance from P to the local polytope in order to get the "standard" values for the amount of non-locality.

For each $P \in \mathcal{NS}_{\text{CHSH}} \setminus \mathcal{L}_{\text{CHSH}}$ there exists exactly one choice of $\eta, \nu, \sigma \in \{0, 1\}$ such that $F^{\eta\nu\sigma}(P) > 0$, and we then say that P lies above the facet associated with $F^{\eta\nu\sigma}$ (see also Figure 2.1). The non-signalling system lying above $F^{\eta\nu\sigma}$ which is farthest away from the facet, and is, therefore, maximally non-local, is called a PR-box [PR94]. They are denoted by $P_{\text{PR}}^{\eta\nu\sigma}$ and are explicitly given by

$$P_{\text{PR}}^{\eta\nu\sigma}(a, b|x, y) := \begin{cases} \frac{1}{2}\ , \ \textit{if}\ a \oplus b = (x \wedge y) \oplus (\eta \wedge x) \oplus (\nu \wedge y) \oplus \sigma \\ 0\ , \ \textit{otherwise} \end{cases} . \tag{2.10}$$

Note that these eight PR-boxes can be reversibly transformed from one to the other by using only local operations of Alice and Bob. Due to this symmetry, we will be mainly interested in the canonical CHSH Bell inequality

$$F^{000}(P) = E_{00}(P) + E_{01}(P) + E_{10}(P) - E_{11}(P) - 2 \leq 0\ .$$

and the corresponding PR-box $P_{\mathrm{PR}} := P_{\mathrm{PR}}^{000}$.

2.6.2 Isotropic Systems

Binary non-signalling systems that have special symmetry properties are *isotropic* systems. They are convex combinations of the PR-box P_{PR} and the maximally mixed system $P_{\mathbb{I}}$ which is defined by $P_{\mathbb{I}}(a,b|x,y) = 1/4$ for all $a,b,x,y \in \{0,1\}$. The definition of isotropic systems reads then as follows (see also Figure 2.1):

Definition 2.3 (Isotropic System)**.** A non-signalling system $P \in \mathcal{NS}_{\mathrm{CHSH}}$ is called isotropic if it can be written as

$$P = \lambda \cdot P_{\mathrm{PR}} + (1-\lambda) \cdot P_{\mathbb{I}} =: P_\lambda \ ,$$

for some $0 \le \lambda \le 1$. The set of all isotropic non-signalling systems is denoted by $\mathcal{NS}_{\mathrm{iso}}$.

We denote the corresponding local and quantum sets associated with isotropic systems by $\mathcal{L}_{\mathrm{iso}} := \mathcal{L}_{\mathrm{CHSH}} \cap \mathcal{NS}_{\mathrm{iso}}$ and $\mathcal{Q}_{\mathrm{iso}} := \mathcal{Q}_{\mathrm{CHSH}} \cap \mathcal{NS}_{\mathrm{iso}}$, respectively. Computing the CHSH non-locality of an isotropic system as well as deciding whether it is local or quantum is straightforward.

Lemma 2.4. *Let P_λ be isotropic. Then*

1. *P_λ is a local system if and only if $0 \le \lambda \le 1/2$.*
2. *P_λ is a quantum system if and only if $0 \le \lambda \le 1/\sqrt{2}$.*
3. *$NL(P_\lambda) = 4 \cdot \lambda$ for $0 \le \lambda \le 1$.*

Proof. That $NL(P_\lambda) = 4 \cdot \lambda$ follows immediately from Definition 2.2 with $\eta = \nu = \sigma = 0$. Hence, $0 \le NL(P_\lambda) \le 2$ for $0 \le \lambda \le 1/2$ and, therefore, P_λ is local [Bel64, CHSH69]. For the quantum case we know by [Tsi80] that $NL(P_\lambda) \le 2\sqrt{2}$ for all isotropic quantum systems P_λ (which can be achieved) and, therefore, $\lambda \le 1/\sqrt{2}$ by using $NL(P_\lambda) = 4 \cdot \lambda$. $\square$

2.7 Semidefinite Programming

We denote the vector space of all real and symmetric $n \times n$ - matrices by $\mathbb{S}^n$. Then, $M \in \mathbb{S}^n$ is called positive-semidefinite, denoted by $M \succeq 0$, if

$$\langle v, M(v) \rangle \ge 0 \ ,$$

for all $v \in \mathbb{R}^n$. There is a useful alternative characterization of positive-semidefinite matrices (see [HJ85] for example):

Lemma 2.5. *$M \in \mathbb{S}^n$ is positive-semidefinite if and only if there exists a $k \in \mathbb{N}$ and vectors $v_1, ..., v_n \in \mathbb{R}^k$ such that*

$$\langle f_i, M(e_j)\rangle = \langle v_i, v_j\rangle \ ,$$

for all $1 \leq i, j \leq n$.

A *semidefinite program* (SDP) is an optimization problem which extends linear programming. See [VB96] for a nice review article about SDPs. Given $C_1, ..., C_m, A \in \mathbb{S}^n$ and real numbers $b_1, .., b_m \in \mathbb{R}$, the following optimization problem is called an SDP:

$$\begin{array}{rll} \min_{M \in \mathbb{S}^n} & \mathrm{tr}(A \cdot M) & \\ s.t. & \mathrm{tr}(C_i \cdot M) = b_i \ , & 1 \leq i \leq m \\ & M \succeq 0 \ , & \end{array} \tag{2.11}$$

where we define $\mathrm{tr}(A \cdot M) := \sum_i \langle f_i, (A \cdot M)(e_i)\rangle \equiv \sum_{i,j} \langle f_i, A(e_j)\rangle \cdot \langle f_i, M(e_j)\rangle$. We call n the *dimension of the SDP* and m the *number of constraints of the SDP*.

If the matrix M in (2.11) is enforced to be *diagonal* the resulting optimization problem is called a *linear program* (which is, therefore, a special case of an SDP).

The SDP of (2.11) is *equivalent* to the following maximization problem:

$$\begin{array}{rll} \max_{\tilde{M} \in \mathbb{S}^{\tilde{n}}} & \mathrm{tr}(\tilde{A} \cdot \tilde{M}) & \\ s.t. & \mathrm{tr}(\tilde{C}_i \cdot \tilde{M}) \leq b_i \ , & 1 \leq i \leq m \\ & \tilde{M} \succeq 0 \ , & \end{array} \tag{2.12}$$

for some $\tilde{C}_1, ..., \tilde{C}_m, \tilde{A} \in \mathbb{S}^{\tilde{n}}$.

Let us show that these two optimization problems really yield the same result. We only show one direction, namely that any problem in the form of (2.12) can be transformed into (2.11). This implies that if we have an algorithm which computes the optimum of (2.11) we also have an algorithm which computes the optimum of (2.12). First, note that $\max_{\tilde{M} \in \mathbb{S}^{\tilde{n}}} \mathrm{tr}(\tilde{A} \cdot \tilde{M}) = \min_{\tilde{M} \in \mathbb{S}^{\tilde{n}}} \mathrm{tr}((-\tilde{A}) \cdot \tilde{M})$ given the constraints in (2.12). We then introduce m additional variables $y_1, ..., y_m \geq 0$ such that

$$\begin{array}{rll} \min_{\tilde{M} \in \mathbb{S}^{\tilde{n}}} & \mathrm{tr}((-\tilde{A}) \cdot \tilde{M}) & \\ s.t. & \mathrm{tr}(\tilde{C}_i \cdot \tilde{M}) + y_i = b_i \ , & 1 \leq i \leq m \\ & y_i \geq 0 \ , & 1 \leq i \leq m \\ & \tilde{M} \succeq 0 \ , & \end{array}$$

which is still equivalent to (2.12). Let $Y := diag(y_1, ..., y_m)$ be an $m \times m$ diagonal matrix and $D_i := diag(0, .., 0, 1, 0, .., 0)$ be an $m \times m$ matrix which is all zeros but has a single one at the i'th diagonal position. Furthermore, let $n := \tilde{n} + m$. We define the $n \times n$ block matrices

$$M := \begin{pmatrix} \tilde{M} & 0 \\ 0 & Y \end{pmatrix} , \; A := \begin{pmatrix} -\tilde{A} & 0 \\ 0 & 0 \end{pmatrix} , \; C_i := \begin{pmatrix} \tilde{C}_i & 0 \\ 0 & D_i \end{pmatrix} .$$

Hence, we have $\mathrm{tr}((-\tilde{A}) \cdot \tilde{M}) = \mathrm{tr}(A \cdot M)$ and $\mathrm{tr}(C_i \cdot M) = \mathrm{tr}(\tilde{C}_i \cdot \tilde{M}) + y_i$. Finally, note that $\tilde{M} \succeq 0$ and $Y \succeq 0$ if and only if $M \succeq 0$ (just diagonalize the matrix $\tilde{M}$ in the block matrix M, see also [HJ85]). Note that with this transformation we enlarged the space over which we optimize from $\mathbb{S}^{\tilde{n}}$ to $\mathbb{S}^{\tilde{n}+m}$ but we did not increase the number of constraints m.

Due to the convex nature of an SDP it "can be solved very efficiently, both in theory and practice"[VB96].

Lemma 2.6 (Alizadeh [Ali93])**.** *An SDP as given in (2.11) or (2.12) can be solved efficiently in polynomial time. More precisely, in time* $\mathrm{poly}(n + m + \log(1/\epsilon))$*, where* $\epsilon > 0$ *is the desired accuracy of the SDP solution.*

2.7.1 SDP for Computing $\mathcal{Q}_m^1$

Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be a bipartite system with $|\mathcal{A}|, |\mathcal{B}|, |\mathcal{X}|, |\mathcal{Y}| \leq m$. Then, we say that $P \in \mathcal{Q}_m^1$ if and only if there exist vectors $m_{x,a}, n_{y,b} \in \ell_2^n$ such that

$$\begin{aligned}
\langle m_{x,a_1}, m_{x,a_2} \rangle &= 0 \, , \textit{ for all } \; x, a_1 \neq a_2 \, , \\
\langle n_{y,b_1}, n_{y,b_2} \rangle &= 0 \, , \textit{ for all } \; y, b_1 \neq b_2 \, , \\
\sum_a \langle m_{x,a}, m_{x,a} \rangle &= 1 \, , \textit{ for all } \; x, \\
\sum_b \langle n_{y,b}, n_{y,b} \rangle &= 1 \, , \textit{ for all } \; y, \\
\langle m_{x,a}, n_{y,b} \rangle &= \langle f_{x,a} \otimes f_{y,b}, P \rangle \, , \textit{ for all } \; x, y, a, b \, .
\end{aligned}$$

It has been shown that $\mathcal{Q}_m^1$ is an approximation to the quantum set $\mathcal{Q}_m$ in the sense that $\mathcal{Q}_m \subsetneq \mathcal{Q}_m^1$ [NPA07, NPA08, DLTW08]. The convex set $\mathcal{Q}_m^1$ is the first set in a hierarchy of sets $\{\mathcal{Q}_m^i\}_{i \geq 1}$ which in the limit converges to the quantum set, i.e., $\lim_{i \to \infty} \mathcal{Q}_m^i = \mathcal{Q}_m$ and where $\mathcal{Q}_m \subseteq \mathcal{Q}_m^{i+1} \subseteq \mathcal{Q}_m^i$ for all i [NPA07, NPA08, DLTW08].

It can be shown that deciding whether a bipartite system P is an element of $\mathcal{Q}_m^1$ or not is equivalent to solving a certain *semidefinite feasibility* *problem* [NPA07, NPA08, DLTW08]. In other words, there exists an SDP which solves the above *constraint satisfaction problem*. The main insight one needs in order to prove this fact is provided by Lemma 2.5. See also Section 4.5 where this transition from conditions on vectors to an SDP is made explicitly.

Chapter 3

Framework for Bipartite Systems and Two-Prover Games

3.1 Introduction and Motivation

In this chapter we introduce a framework that allows us to study bipartite systems, two-prover games, and Bell inequalities. This framework is based on the theory of tensor norms. Since tensor norms are well established and well studied objects in functional analysis many powerful tools and techniques to study them are already available. For example, Grothendieck's inequality and the duality relations between tensor norms will turn out to be fruitful in our studies.

We have seen in Section 2.3 and Section 2.5 that bipartite systems and two-prover games can be interpreted as elements of tensor product spaces. Defining a special class of *norms* on these tensor product spaces, called *tensor norms*, will allow us to assign non-negative values to these tensor elements. We will see that these values have an interpretation in the context of quantum information theory. For example, we can compute (or find an upper bound to) the value of entangled two-prover games, study wirings of non-signalling systems (like non-locality distillation protocols), or analyse parallel repetition of two-prover games.

The main goal of this chapter is to make the connection between quantum information theory and tensor norms explicit. We prove various results about the connection of tensor norms to bipartite systems and two-prover games. These results are also of independent interest from a purely mathematical point of view and lead to new insights in quantum information theory.

3.1.1 Relation of Tensor Norms to Systems and Games

In this section, we provide arguments why *tensor norms* are convenient tools to study bipartite systems and two-prover games. Note, however, that the full power of this framework will be unfolded only in the application chapter.

Tensor norms are a special class of norms defined over a tensor product space. We have seen in Section 2.3 that bipartite systems can be represented as elements of the tensor product space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. One can define a *tensor norm* on this tensor product space and consider all bipartite systems that have norm at most 1. This set of systems is convex since every norm fulfils the triangle inequality. Furthermore, depending on the used tensor norm, this convex set has additional interesting properties.

More formally, we have the following definition: let $\mathcal{S}_m$ be the set of all bipartite systems of alphabet sizes at most m (see Section 2.3) and α : $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}) \to \mathbb{R}$, for $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{A}|, |\mathcal{B}| \leq m$, be a tensor norm. Then, we define the set

$$\mathcal{R}_m^\alpha := \{P \,:\, \alpha(P) \leq 1 \wedge P \in \mathcal{S}_m\} \,. \tag{3.1}$$

This definition immediately implies that $\mathcal{R}_m^\alpha \subseteq \mathcal{S}_m$ for all tensor norms α. However, it could be the case that $\mathcal{R}_m^\alpha = \mathcal{S}_m$ for all tensor norms α and, therefore, tensor norms would not further restrict the set of all bipartite systems. Fortunately, as we will see later, this is not the case since different tensor norms typically yield different convex sets. To summarize, we can say that a better understanding of the sets $\mathcal{R}_m^\alpha$ requires insights concerning the underlying tensor norms. Why this is desirable will become clear in Chapter 6, where we focus on the applications of this framework.

Let us now show how tensor norms are related to two-prover games. Assume the set $\mathcal{R}_m^\alpha$, for some tensor norm α, denotes an "interesting" subset of all bipartite systems. Suppose that we are interested in the value of a certain two-prover game G, given that Alice and Bob can use systems from the set $\mathcal{R}_m^\alpha$ as resources. This value, denoted by $\omega_{\mathcal{R}_m^\alpha}(G)$ (see also

Section 2.5), can then be upper bounded by

$$\begin{aligned}\omega_{\mathcal{R}_m^\alpha}(G) &:= \sup_P \{|\langle G, P\rangle| \,:\, P \in \mathcal{R}_m^\alpha\} \\ &\leq \sup_P \{|\langle G, P\rangle| \,:\, \alpha(P) \leq 1\} \\ &= \alpha^*(G)\,,\end{aligned}$$

where we used (3.1) in the second line and the duality of norms in the third line. Hence, the value of the dual tensor norm α^* immediately implies an upper bound on the value of a two-prover game executed over bipartite systems coming from a set induced by the tensor norm α.

3.1.2 Related Work

That there exists a connection between *Banach space theory*[1], in particular the tensor product in Banach space theory, and quantum information theory was first observed by Tsirelson [Tsi87]. He used Grothendieck's inequality (which establishes a connection between the projective and the Hilbertian tensor norm) in order to upper bound the value of correlation Bell inequalities (see also Section 6.2 in the application chapter for more details). More recently, this line of research was continued in [PGWP$^+$08, JPPG$^+$10a, JPPG$^+$10b, JP11] which generalize the work of Tsirelson to arbitrary Bell inequalities and multipartite settings. Furthermore, [JPPG$^+$10a, JPPG$^+$10b] show that operator space theory, which can be seen as a *non-commutative* generalization of Banach space theory (see for example [JP10]), is a natural framework to study arbitrary Bell inequalities.

The work of Rudolph [Rud00, Rud01a, Rud01b, DHR02, Rud03, Rud05] shows that the projective tensor norm defines an entanglement measure and can be used as a criterion for separability. Note, however, that he defines the tensor norm over different local normed vector spaces than we do. In particular, he uses the the space of trace class operators on Hilbert spaces equipped with the trace norm.

The theory of tensor norms has not only applications in quantum information theory but is also used to compute lower bounds in communication complexity [LS07, LMSS07, LSS08] and to derive approximation algorithms [AN04, CW04].

[1] Since we consider only *finite*-dimensional spaces in this thesis we restrict our attention to *normed vector space* since in finite dimensions every normed vector space is also a Banach space.

3.1.3 Chapter Outline

In Section 3.2, we first provide a definition of *tensor norms* and then introduce four specific tensor norms that we investigate in this thesis: the projective, the injective, the Hilbertian, and the dual Hilbertian tensor norm. The composition of bipartite systems and two-prover games is investigated in Section 3.3. After working out the right definitions for studying composite systems and games, we show that the projective and dual Hilbertian tensor norms obey direct-product theorems. In Section 3.4, we show how wirings can be represented as linear maps on tensor product spaces. As a main result of this section, we prove that no tensor norm increases under wirings. Finally, in Section 3.5, we study the relation between convex sets of bipartite systems and cross norms[2]. In particular, we show that for any convex set of bipartite systems (which is larger than the set of local systems) there exists a cross norm which induces this set in the sense of (3.1).

3.2 Introducing Tensor Norms

3.2.1 Definitions

In what follows, X and Y will be finite-dimensional normed vector spaces and $X \otimes Y$ denotes the algebraic tensor product of these two spaces. We will call X and Y *local spaces*. A *tensor norm* $\alpha : X \otimes Y \to \mathbb{R}$ is a norm which depends on the local normed vector spaces X and Y and has some additional properties, as introduced below[3]. Since α depends on X and Y, we will write $\alpha(S; X, Y)$ to denote the norm of $S \in X \otimes Y$ given the local normed vector spaces X and Y. The *dual tensor norm* of α is denoted by α^* and defined as

$$\alpha^*(R; X^*, Y^*) := \sup_S \{ |\langle R, S \rangle| \ : \ \alpha(S; X, Y) \leq 1 \} .$$

Let us first define a broader class of norms than tensor norms, called *cross norms* [Rya02]:

Definition 3.1 (Cross Norm)**.** Let X and Y be finite-dimensional normed vector spaces. A *cross norm* $\alpha : X \otimes Y \to \mathbb{R}$ is a norm on $X \otimes Y$ which satisfies the following two conditions:

[2]which is a broader class of norms than tensor norms. See Definition 3.1 in Section 3.2.1.
[3]See also [DF93, Rya02], which give a good introduction to the subject of tensor norms.

1. $\alpha(S_A \otimes S_B; X, Y) = \|S_A\|_X \cdot \|S_B\|_Y$,$\forall\, S_A \in X, S_B \in Y$.
2. $\alpha^*(R_A \otimes R_B; X^*, Y^*) = \|R_A\|_{X^*} \cdot \|R_B\|_{Y^*}$,$\forall\, R_A \in X^*, R_B \in Y^*$.

The definition of tensor norms reads then as follows [DF93]:

Definition 3.2 (Tensor Norm)**.** A *tensor norm* $\alpha : X \otimes Y \to \mathbb{R}$ is a cross norm for all pairs X, Y of finite-dimensional normed vector spaces such that

$$\|\mathcal{T}_A \otimes \mathcal{T}_B\|_{X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2} \leq \|\mathcal{T}_A\|_{X_1 \to X_2} \cdot \|\mathcal{T}_B\|_{Y_1 \to Y_2}$$

for all linear maps $\mathcal{T}_A : X_1 \to X_2$ and $\mathcal{T}_B : Y_1 \to Y_2$.

The additional property of tensor norms compared to cross norms is called the *metric mapping property*. The operator norm $\|\mathcal{T}\|_{X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2}$ is defined as

$$\|\mathcal{T}\|_{X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2} := \sup_S \{\alpha(\mathcal{T}(S); X_2, Y_2) \ : \ \alpha(S; X_1, Y_1) \leq 1\} \ . \quad (3.2)$$

Remark 3. Let us make a remark about the terminology in the literature. In [Rya02], a norm which fulfils Definition 3.1 is called a *reasonable cross norm*. Furthermore, norms which fulfil Definition 3.2 are called *uniform cross norms* in [Sch50, Rya02] and *tensor norms* in [DF93].

Lemma 3.1. *Let $\alpha : X \otimes Y \to \mathbb{R}$ be a tensor norm. Then*

$$\alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(S)) \leq \alpha(S) \ ,$$

for all $S \in X_1 \otimes Y_1$ and all linear maps $\mathcal{T}_A : X_1 \to X_2$ and $\mathcal{T}_B : Y_1 \to Y_2$ with $\|\mathcal{T}_A\|_{X_1 \to X_2} \leq 1$ and $\|\mathcal{T}_B\|_{Y_1 \to Y_2} \leq 1$.

Note that $\alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(S)) \leq \alpha(S)$ is just a shorter notation for $\alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(S); X_2, Y_2) \leq \alpha(S; X_1, Y_1)$.

Proof. Since α is a tensor norm it holds by definition that

$$\|\mathcal{T}_A \otimes \mathcal{T}_B\|_{X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2} \leq \|\mathcal{T}_A\|_{X_1 \to X_2} \cdot \|\mathcal{T}_B\|_{Y_1 \to Y_2} \leq 1 \ ,$$

for all linear maps $\mathcal{T}_A : X_1 \to X_2$ and $\mathcal{T}_B : Y_1 \to Y_2$ with $\|\mathcal{T}_A\|_{X_1 \to X_2} \leq 1$ and $\|\mathcal{T}_B\|_{Y_1 \to Y_2} \leq 1$. Hence, by the definition of the operator norm given in (3.2), we obtain

$$\sup_S \{\alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(S); X_2, Y_2) \ : \ \alpha(S; X_1, Y_1) \leq 1\} \leq 1 \ ,$$

and, therefore,

$$\alpha\left((\mathcal{T}_A \otimes \mathcal{T}_B)(S); X_2, Y_2\right) \leq 1 \ ,$$

for all $S \in X_1 \otimes Y_1$ with $\alpha(S; X_1, Y_1) = 1$. $\square$

3.2.2 Four Different Tensor Norms

We consider *four* tensor norms in this thesis. They are all well-studied objects in functional analysis, and their properties and relations among each other have already been investigated by Grothendieck [Gro53].

Projective and Injective Tensor Norm

The *projective tensor norm* of $S \in X \otimes Y$ is defined by

$$\pi(S; X, Y) := \inf \left\{ \sum_{i=1}^{k} \|S_A^i\|_X \cdot \|S_B^i\|_Y \ : \ S = \sum_{i=1}^{k} S_A^i \otimes S_B^i \right\} ,$$

where the infimum is over all decompositions (or representations) of S. The *injective tensor norm* of $S \in X \otimes Y$ is defined by

$$\varepsilon(S; X, Y) := \sup \left\{ |\langle R_A \otimes R_B, S\rangle| \ : \ \|R_A\|_{X^*} \leq 1, \|R_B\|_{Y^*} \leq 1 \right\} ,$$

where the supremum is over $R_A \in X^*$ and $R_B \in Y^*$.

One can show [Rya02] that these two norms are the duals of each other, i.e.,

$$\begin{aligned} \pi(S; X, Y) &= \sup\{|\langle R, S\rangle| \ : \ \varepsilon(R; X^*, Y^*) \leq 1\} , \\ \varepsilon(S; X, Y) &= \sup\{|\langle R, S\rangle| \ : \ \pi(R; X^*, Y^*) \leq 1\} , \end{aligned}$$

for $S \in X \otimes Y$ and $R \in X^* \otimes Y^*$. Hence, we have $\pi = \varepsilon^*$ and $\varepsilon = \pi^*$.

Hilbertian Tensor Norm and its Dual

The *Hilbertian tensor norm* of $S \in X \otimes Y$ can be defined as [DF93]:

$$\gamma_2(S; X, Y) := \inf \left\{ \sup \left(\sum_i |\langle R_A, S_A^i\rangle|^2 \right)^{1/2} \sup \left(\sum_i |\langle R_B, S_B^i\rangle|^2 \right)^{1/2} \right\}$$

where the infimum is over all decompositions $S = \sum_i S_A^i \otimes S_B^i \in X \otimes Y$ and the supremums are over $R_A \in X^*$ and $R_B \in Y^*$, respectively, such that $\|R_A\|_{X^*} \leq 1$ and $\|R_B\|_{Y^*} \leq 1$.

The *dual Hilbertian tensor norm* can be written as [DF93]:

$$\gamma_2^*(S; X, Y) := \inf \left\{ \|(\mu_{ij})\|_{2\to 2} \cdot \left(\sum_{i=1}^{n} \|S_A^i\|_X^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{n} \|S_B^j\|_Y^2 \right)^{1/2} \right\} ,$$

where the infimum is over all decompositions $S = \sum_{i,j}^{n} \mu_{ij} \cdot S_A^i \otimes S_B^j \in X \otimes Y$ and (μ_{ij}) is a real $n \times n$-matrix.

There is a useful alternative representation of the Hilbertian tensor norm whereof it got its name. A tensor element $S \in X \otimes Y$ can be interpreted as a linear operator $\hat{S} : X^* \to Y$ by the following identification:

$$\hat{S}(R_A) := \sum_i \langle R_A, S_A^i \rangle \cdot S_B^i \ ,$$

with $S = \sum_i S_A^i \otimes S_B^i$ and $R_A \in X^*$. Note that $\hat{S}(R_A)$ does not depend on the actual decomposition of S. We are now ready to state the alternative representation of the γ_2-norm [Rya02]:

$$\gamma_2(S; X, Y) = \inf_{\hat{S} = W \cdot V} \|W\|_{2 \to Y} \cdot \|V\|_{X^* \to 2} \ , \tag{3.3}$$

where the infimum is over all factorizations of $\hat{S}$ into linear operators $W : \ell_2 \to Y$ and $V : X^* \to \ell_2$. In other words, $\hat{S}$ is factored through the Hilbert space ℓ_2. See Appendix A.1 for a proof of this equivalence.

3.2.3 Relation Between Tensor Norms

The projective and injective tensor norms are the "extremal" tensor norms, i.e., all tensor norms are larger than the injective and smaller than the projective tensor norm for some fixed local normed vector spaces X and Y. Furthermore, this fact can also be used to obtain an alternative definition of cross norms. Formally, we have [Rya02]:

Lemma 3.2. *Let X and Y be normed vector spaces and α a norm on $X \otimes Y$. Then, α is a cross norm if and only if*

$$\varepsilon(S; X, Y) \le \alpha(S; X, Y) \le \pi(S; X, Y) \ ,$$

for every $S \in X \otimes Y$.

Furthermore, it can be shown that $\gamma_2(S; X, Y) \le \gamma_2^*(S; X, Y)$, for all $S \in X \otimes Y$ [Rya02], and, therefore,

$$\varepsilon(S; X, Y) \le \gamma_2(S; X, Y) \le \gamma_2^*(S; X, Y) \le \pi(S; X, Y) \ , \tag{3.4}$$

for all $S \in X \otimes Y$.

3.2.4 Notational Conventions

Since we mainly work over the tensor product spaces $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ and $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, it is convenient to simplify the notation for these cases. In particular, given that α is a tensor norm we define

$$\begin{aligned}\alpha(P) &:= \alpha(P; \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}), \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}))\ , \\ \alpha(G) &:= \alpha(G; \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}), \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}))\ .\end{aligned}$$

Hence, like in Section 2.3, we use the letter P (sometimes also with subscripts and/or superscripts) to indicate that we work over the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Similarly, as in Section 2.5, we use the letter G (sometimes also with subscripts and/or superscripts) to indicate that we work over the space $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$.

Furthermore, we will sometimes call elements of $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$, $\ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ and $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ *systems* although they may *not* represent a conditional probability distribution. Similarly, we sometimes call elements of $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$, $\ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ *games* even though they may not be. It will be clear from the context whether we mean a "real" system (or game) or just an element of the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ (or $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$).

3.3 Composition of Systems and Games

3.3.1 Introduction

Assume we are given several bipartite systems $P_{A_kB_k}$, for $1 \leq k \leq n$, and we know the values $\alpha(P_{A_kB_k})$, for all $1 \leq k \leq n$, where α is some tensor norm. Furthermore, we denote by $P_{A_1B_1} \odot ... \odot P_{A_nB_n}$ the *composition* of the given bipartite systems (see Figure 3.1 and the next section for a formal definition). A *direct-product theorem* for the tensor norm α would allow us to derive an upper bound on the value of α for the composite system $P_{A_1B_1} \odot ... \odot P_{A_nB_n}$ by computing the product of the the α-norm of the individual systems, i.e.,

$$\alpha(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \leq \alpha(P_{A_1B_1}) \cdot ... \cdot \alpha(P_{A_1B_1})\ .$$

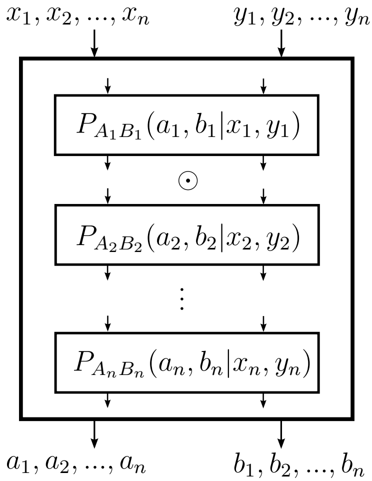

Figure 3.1: The composition of bipartite systems $P_{A_1B_1}, ..., P_{A_nB_n}$ is denoted by $P_{A_1B_1} \odot ... \odot P_{A_nB_n}$. The composition $P_{A_1B_1} \odot ... \odot P_{A_nB_n}$ can be considered again as a bipartite system with enlarged input/output alphabets.

Contribution

We prove direct-product theorems for the dual Hilbertian and the projective tensor norm. This allows us to derive upper bounds on the γ_2^*-norm of composite systems (and games) by computing the product of the γ_2^*-norms of the individual systems (and games). These individual systems (and games) can have arbitrary input and output alphabets and, therefore, live on the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ (and $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$). The results of this section have been published in [Duk11].

Related Work

Our direct-product theorem for the γ_2^* tensor norm generalizes the results of [LSS08, CSUU07]. In particular, the authors of [LSS08, CSUU07] prove their result for the case where the vector space of a single game is given by $\ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|}$. In other words, they consider the special case of XOR games (i.e., where the output alphabet sizes are set to $|\mathcal{A}| = |\mathcal{B}| = 1$, see also the comment at the end of Section 2.3.4) whereas we consider arbitrary output alphabet sizes.

Applications

There are two fields of application. First, if we consider the composition of games the, direct-product theorems can be used to compute upper bounds on the value of parallel repeated two-prover games. In particular, the direct-product theorem for the γ_2^* tensor norm, where a single games is defined on $\ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|}$, is used in [CSUU07] to prove a perfect parallel-repetition theorem for entangled XOR games. In Section 6.3, we drive an alternative proof of this parallel-repetition theorem by using our generalized direct-product theorem for γ_2^*, where in our case a single two-prover game will be defined on $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, with $|\mathcal{A}| = |\mathcal{B}| = 2$.

Second, if we consider the composition of systems, direct-product theorems can be used derive sets of bipartite systems which are closed under wirings[4] (together with the results of the next section which is about wirings). In particular, in Section 5.2 we use the direct-product theorem in order to prove that γ_2^* induces a set of bipartite systems that is closed under wirings.

3.3.2 Multipartite Projective and Injective Tensor Norms

Since we want to evaluate tensor norms for composite systems/games we have to define new local norms since the $\infty(1)$ and $1(\infty)$-norms have been defined for *single* systems/games. A priori, there exist several possible choices for this new local norm. However, it turns out that *multipartite* versions of the projective and injective tensor norms allow us to treat composite systems and games in a very convenient and powerful way. Hence, we also use tensor norms for the *local* spaces.

See also [FH01] where multipartite tensor norms are defined and [PG04, PGWP$^+$08] for applications of multipartite versions of the projective and injective tensor norms[5].

The normed vector space associated with the multipartite generalization of the projective tensor norm is defined as

$$\Pi_{\infty(1)}^{A^n} := \left(\ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes \ldots \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|}), \pi_{\infty(1)}^{A^n}\right) ,$$

[4]see Section 5.2 for a definition.

[5]Note that they are working over different local normed vector spaces than we do in this thesis.

where the $\pi^{A^n}_{\infty(1)}$-norm is given by

$$\pi^{A^n}_{\infty(1)}(P_A) := \inf\left\{\sum_i \|P^i_{A_1}\|_{\infty(1)} \cdot \ldots \cdot \|P^i_{A_n}\|_{\infty(1)}\right\} , \tag{3.5}$$

with $P_A \in \ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes \ldots \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|})$ and where the infimum is over all decompositions $P_A = \sum_i P^i_{A_1} \otimes \ldots \otimes P^i_{A_n}$. Similarly, we define the normed vector space

$$\Pi^{A^n}_{1(\infty)} := \left(\ell_1^{|\mathcal{X}_1|}(\ell_\infty^{|\mathcal{A}_1|}) \otimes \ldots \otimes \ell_1^{|\mathcal{X}_n|}(\ell_\infty^{|\mathcal{A}_n|}), \pi^{A^n}_{1(\infty)}\right) ,$$

where the $\pi^{A^n}_{1(\infty)}$-norm is defined as

$$\pi^{A^n}_{1(\infty)}(G_A) := \inf\left\{\sum_i \|G^i_{A_1}\|_{1(\infty)} \cdot \ldots \cdot \|G^i_{A_n}\|_{1(\infty)}\right\} , \tag{3.6}$$

with $G_A \in \ell_1^{|\mathcal{X}_1|}(\ell_\infty^{|\mathcal{A}_1|}) \otimes \ldots \otimes \ell_1^{|\mathcal{X}_n|}(\ell_\infty^{|\mathcal{A}_n|})$ and where the infimum is over all decompositions $G_A = \sum_i G^i_{A_1} \otimes \ldots \otimes G^i_{A_n}$.

It follows immediately from (3.5) and (3.6) that

$$\begin{aligned}
\pi^{A^n}_{\infty(1)}(P_{A_1} \otimes \ldots \otimes P_{A_n}) &= \|P_{A_1}\|_{\infty(1)} \cdot \ldots \cdot \|P_{A_n}\|_{\infty(1)} , && (3.7)\\
\pi^{A^n}_{1(\infty)}(G_{A_1} \otimes \ldots \otimes G_{A_n}) &= \|G_{A_1}\|_{1(\infty)} \cdot \ldots \cdot \|G_{A_n}\|_{1(\infty)} , && (3.8)
\end{aligned}$$

for all $P_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$ and $G_{A_k} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|})$. Therefore, if we restrict to the case of a single system/game we obtain as a special case the $\infty(1)/1(\infty)$-norms:

$$\begin{aligned}
\pi^{A^1}_{\infty(1)}(P_A) &= \|P_A\|_{\infty(1)} ,\\
\pi^{A^1}_{1(\infty)}(G_A) &= \|G_A\|_{1(\infty)} ,
\end{aligned}$$

for all $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $G_A \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$.

The normed vector space associated with the multipartite generalization of the injective tensor norm is defined as

$$E^{A^n}_{\infty(1)} := \left(\ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes \ldots \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|}), \varepsilon^{A^n}_{\infty(1)}\right) ,$$

where the $\varepsilon^{A^n}_{\infty(1)}$-norm is given by

$$\varepsilon^{A^n}_{\infty(1)}(P_A) := \sup_{G_{A_1},\ldots,G_{A_n}} |\langle G_{A_1} \otimes \ldots \otimes G_{A_n}, P_A\rangle| ,$$

with $P_A \in \ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|})$ and where the supremum is over $G_{A_k} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|})$ such that $\|G_{A_k}\|_{1(\infty)} \leq 1$, for all $1 \leq k \leq n$.

Similarly, we define the normed vector space

$$E_{1(\infty)}^{A^n} := \left(\ell_1^{|\mathcal{X}_1|}(\ell_\infty^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_1^{|\mathcal{X}_n|}(\ell_\infty^{|\mathcal{A}_n|}), \varepsilon_{1(\infty)}^{A^n}\right) ,$$

where the $\varepsilon_{1(\infty)}^{A^n}$-norm is defined as

$$\varepsilon_{1(\infty)}^{A^n}(G_A) := \sup_{P_{A_1},...,P_{A_n}} |\langle G_A, P_{A_1} \otimes ... \otimes P_{A_n}\rangle| ,$$

with $G_A \in \ell_1^{|\mathcal{X}_1|}(\ell_\infty^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_1^{|\mathcal{X}_n|}(\ell_\infty^{|\mathcal{A}_n|})$ and where the supremum is over $P_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$ such that $\|P_{A_k}\|_{\infty(1)} \leq 1$, for all $1 \leq k \leq n$.

Again, it follows immediately from the definitions of $\varepsilon_{\infty(1)}^{A^n}$ and $\varepsilon_{1(\infty)}^{A^n}$ that

$$\begin{aligned} \varepsilon_{\infty(1)}^{A^n}(P_{A_1} \otimes ... \otimes P_{A_n}) &= \|P_{A_1}\|_{\infty(1)} \cdot ... \cdot \|P_{A_n}\|_{\infty(1)} , \\ \varepsilon_{1(\infty)}^{A^n}(G_{A_1} \otimes ... \otimes G_{A_n}) &= \|G_{A_1}\|_{1(\infty)} \cdot ... \cdot \|G_{A_n}\|_{1(\infty)} , \end{aligned}$$

for all $P_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$ and $G_{A_k} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|})$. Hence, if we restrict to the case of a single system/game we also obtain as a special case the $\infty(1)/1(\infty)$-norms:

$$\begin{aligned} \varepsilon_{\infty(1)}^{A^1}(P_A) &= \|P_A\|_{\infty(1)} , \\ \varepsilon_{1(\infty)}^{A^1}(G_A) &= \|G_A\|_{1(\infty)} , \end{aligned}$$

for all $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $G_A \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$.

The same duality relations as in the bipartite case hold also for the multipartite generalizations of the projective and injective tensor norms (see for example [PGWP+08]):

$$\Pi_{\infty(1)}^{A^n} = (E_{1(\infty)}^{A^n})^* , \tag{3.9}$$

$$\Pi_{1(\infty)}^{A^n} = (E_{\infty(1)}^{A^n})^* . \tag{3.10}$$

3.3.3 Direct-Product Theorems

The *composition* of bipartite systems and two-prover games is indicated by the symbol $\odot$. Formally, we define the composition of bipartite systems

$P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$, for $1 \leq k \leq n$, as

$$P_{A_1B_1} \odot ... \odot P_{A_nB_n} := \sum_{i_1,..,i_n} (P_{A_1}^{i_1} \otimes ... \otimes P_{A_n}^{i_n}) \otimes (P_{B_1}^{i_1} \otimes ... \otimes P_{B_n}^{i_n}) \,,$$

with $P_{A_kB_k} = \sum_{i_k} P_{A_k}^{i_k} \otimes P_{B_k}^{i_k}$, for $1 \leq k \leq n$, arbitrary decompositions. This definition is independent of the choice of decomposition since we just perform a permutation of the local systems in $P_{A_1B_1} \otimes ... \otimes P_{A_nB_n}$, such that all local systems of Alice are followed by all local systems of Bob. Analogously, we define the composition of games $G_{A_1B_1} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|}) \otimes \ell_1^{|\mathcal{Y}_k|}(\ell_\infty^{|\mathcal{B}_k|})$, for $1 \leq k \leq n$, as

$$G_{A_1B_1} \odot ... \odot G_{A_nB_n} := \sum_{i_1,..,i_n} (G_{A_1}^{i_1} \otimes ... \otimes G_{A_n}^{i_n}) \otimes (G_{B_1}^{i_1} \otimes ... \otimes G_{B_n}^{i_n}) \,,$$

with $G_{A_kB_k} = \sum_{i_k} G_{A_k}^{i_k} \otimes G_{B_k}^{i_k}$, for $1 \leq k \leq n$, arbitrary decompositions.

By doing this permutation of systems/games allows us to write the composite system $P_{A_1B_1} \odot ... \odot P_{A_nB_n}$ as an element of the space $\Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$ and the composite game $G_{A_1B_1} \odot ... \odot G_{A_nB_n}$ as an element of $\Pi_{1(\infty)}^{A^n} \otimes \Pi_{1(\infty)}^{B^n}$. As already introduced in Section 3.2.4, the letters P and G denote the local normed vector spaces over which a tensor norm α is defined. We now overload the notation even more in order to be able to handle the composition of systems and games. In particular, we define

$$\begin{aligned} \alpha(P_{A_1B_1} \odot .. \odot P_{A_nB_n}) &:= \alpha\left(P_{A_1B_1} \odot .. \odot P_{A_nB_n}; \Pi_{\infty(1)}^{A^n}, \Pi_{\infty(1)}^{B^n}\right) \,, \\ \alpha(G_{A_1B_1} \odot .. \odot G_{A_nB_n}) &:= \alpha\left(G_{A_1B_1} \odot .. \odot G_{A_nB_n}; \Pi_{1(\infty)}^{A^n}, \Pi_{1(\infty)}^{B^n}\right) \,. \end{aligned}$$

Projective Tensor Norm

The next result shows that the projective tensor norm of the composition of systems is upper bounded by the product of the projective tensor norm of the individual systems.

Theorem 3.1. *Let $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$, for $1 \leq k \leq n$. Then*

$$\pi(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \leq \pi(P_{A_1B_1}) \cdot ... \cdot \pi(P_{A_nB_n}) \,,$$

with $P_{A_1B_1} \odot ... \odot P_{A_nB_n} \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$.

Proof. For every $\epsilon > 0$ there exist decompositions $P_{A_kB_k} = \sum_{i_k} P^{i_k}_{A_k} \otimes P^{i_k}_{B_k}$, for all $1 \le k \le n$, such that $\pi(P_{A_kB_k}) + \epsilon \ge \sum_{i_k} \|P^{i_k}_{A_k}\|_{\infty(1)} \cdot \|P^{i_k}_{B_k}\|_{\infty(1)}$. We therefore obtain

$$\begin{aligned}
& \pi(P_{A_1B_1} \odot \ldots \odot P_{A_nB_n}) \\
\le & \sum_{i_1,\ldots,i_n} \pi^{A^n}_{\infty(1)}(P^{i_1}_{A_1} \otimes \ldots \otimes P^{i_n}_{A_n}) \cdot \pi^{B^n}_{\infty(1)}(P^{i_1}_{B_1} \otimes \ldots \otimes P^{i_n}_{B_n}) \\
= & \sum_{i_1,\ldots,i_n} \|P^{i_1}_{A_1}\|_{\infty(1)} \cdot \ldots \cdot \|P^{i_n}_{A_n}\|_{\infty(1)} \cdot \|P^{i_1}_{B_1}\|_{\infty(1)} \cdot \ldots \cdot \|P^{i_n}_{B_n}\|_{\infty(1)} \\
= & \prod_k \left(\sum_{i_k} \|P^{i_k}_{A_k}\|_{\infty(1)} \cdot \|P^{i_k}_{B_k}\|_{\infty(1)} \right) \\
\le & \prod_k \left(\pi(P_{A_kB_k}) + \epsilon \right) \ ,
\end{aligned}$$

where we used (3.7) in the second line. Since ϵ was arbitrary, the lemma follows. □

Dual Hilbertian Tensor Norm

Let us now prove a direct-product theorem for the γ_2^* tensor norm. For this proof we need the following lemma.

Lemma 3.3 (Bennett [Ben77]). *Let A and B be $n \times n$ and $m \times m$ matrices over $\mathbb{R}$, respectively. Then*

$$\|A \otimes B\|_{2\to 2} = \|A\|_{2\to 2} \cdot \|B\|_{2\to 2} \ .$$

Theorem 3.2. *Let $G_{A_kB_k} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|}) \otimes \ell_1^{|\mathcal{Y}_k|}(\ell_\infty^{|\mathcal{B}_k|})$, $1 \le k \le n$, and $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$, for $1 \le k \le n$. Then*

$$\begin{aligned}
\gamma_2^*(G_{A_1B_1} \odot \ldots \odot G_{A_nB_n}) &\le \gamma_2^*(G_{A_1B_1}) \cdot \ldots \cdot \gamma_2^*(G_{A_nB_n}) \ , \\
\gamma_2^*(P_{A_1B_1} \odot \ldots \odot P_{A_nB_n}) &\le \gamma_2^*(P_{A_1B_1}) \cdot \ldots \cdot \gamma_2^*(P_{A_nB_n}) \ ,
\end{aligned}$$

for $G_{A_1B_1} \odot \ldots \odot G_{A_nB_n} \in \Pi^{A^n}_{1(\infty)} \otimes \Pi^{B^n}_{1(\infty)}$ and $P_{A_1B_1} \odot \ldots \odot P_{A_nB_n} \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.

Proof. For every $\epsilon > 0$ there exist decompositions

$$G_{A_kB_k} = \sum_{i_k,j_k} \mu_{i_kj_k} \cdot G^{i_k}_{A_k} \otimes G^{j_k}_{B_k} \ ,$$

for all $1 \leq k \leq n$, such that

$$\gamma_2^*(G_{A_kB_k})+\epsilon \geq \|(\mu_{i_kj_k})\|_{2\to 2}\cdot\left(\sum_{i_k}\|G_{A_k}^{i_k}\|_{1(\infty)}^2\right)^{1/2}\cdot\left(\sum_{j_k}\|G_{B_k}^{j_k}\|_{1(\infty)}^2\right)^{1/2}.$$

In order to shorten the formulas, we define $c_A := \sum_{i_1,\ldots,i_n}\pi_{1(\infty)}^{A^n}(G_{A_1}^{i_1}\otimes \ldots \otimes G_{A_n}^{i_n})^2$ and $c_B := \sum_{j_1,\ldots,j_n}\pi_{1(\infty)}^{B^n}(G_{B_1}^{j_1}\otimes \ldots \otimes G_{B_n}^{j_n})^2$. We therefore obtain

$$\begin{aligned}\gamma_2^*(G_{A_1B_1}\odot \ldots \odot G_{A_nB_n}) &\leq \|(\mu_{i_1j_1})\otimes \ldots \otimes(\mu_{i_nj_n})\|_{2\to 2}\cdot\sqrt{c_A\cdot c_B}\\ &= \|(\mu_{i_1j_1})\|_{2\to 2}\cdot \ldots \cdot\|(\mu_{i_nj_n})\|_{2\to 2}\cdot\sqrt{c_A\cdot c_B}\,,\end{aligned}$$

where we used Lemma 3.3 in the second line. Furthermore, since

$$c_A \equiv \sum_{i_1,\ldots,i_n}\pi_{1(\infty)}^{A^n}(G_{A_1}^{i_1}\otimes \ldots \otimes G_{A_n}^{i_n})^2 = \prod_k\left(\sum_{i_k}\|G_{A_k}^{i_k}\|_{1(\infty)}^2\right),$$

by (3.8), we obtain

$$\begin{aligned}&\gamma_2^*(G_{A_1B_1}\odot \ldots \odot G_{A_nB_n})\\ \leq\ &\prod_k\left(\|(\mu_{i_kj_k})\|_{2\to 2}\left(\sum_{i_k}\|G_{A_k}^{i_k}\|_{1(\infty)}^2\right)^{1/2}\cdot\left(\sum_{j_k}\|G_{B_k}^{j_k}\|_{1(\infty)}^2\right)^{1/2}\right)\\ \leq\ &\prod_k(\gamma_2^*(G_{A_kB_k})+\epsilon)\ .\end{aligned}$$

Since ϵ was arbitrary, the first part of the theorem follows. The proof of the second part of the theorem is analogous. □

3.4 Introducing Dynamics into the Framework

3.4.1 Introduction

In the previous section we have seen how one can combine systems to obtain systems with larger input/output alphabet sizes. However, the composition of systems just puts them "next to each other" and, hence, there is no interaction between the individual systems. In this section we will show how systems can interact with each other and how a description of this interaction can be incorporated into the setting of tensor products of normed vector spaces. More precisely, we investigate wirings, as introduced in Section 2.4, applied on composite systems and analyse how the values of tensor norms are affected by this local processing.

Contribution

First, we show how wirings can be represented as linear maps on tensor product spaces. This representation allows us to use tools from functional analysis and linear algebra. Hence, we do not have a *combinatorial problem* any more (i.e., which systems should be selected next and which input should we use) but a *continuous* one.

The representation of wirings as linear maps allows us to study wiring protocols as maps which act on tensor product spaces. This enables us to prove our main result of this section: tensor norms do not increase under wirings (see Theorem 3.3).

Related Work

To the best of our knowledge, the connection between wirings and tensor norms has not been studied before. However, in [Bar07], Barrett analysed linear maps that correspond to wirings (actually even more general maps) on single and bipartite systems.

Applications

The results will be used in Section 5.2 to prove that the γ_2^* tensor norm induces a set of bipartite systems that is closed under wirings. This result will then further be used in the application chapter, namely in Section 6.4 (universality of quantum systems) and Section 6.5 (non-locality distillation).

3.4.2 Wirings Represented as Linear Maps

Let us now put the notion of a wiring as introduced in Section 2.4 into the context of tensor products of normed vector spaces. In particular, we show that one can associate a linear map to each wiring.

Let us first assume that the wiring does not involve shared randomness. Let us denote by $\mathcal{W}_A$ Alice's *local strategy* which is associated with the functions $f^1_{\mathcal{W}},...,f^{n+1}_{\mathcal{W}}$ and $g^1_{\mathcal{W}},...,g^n_{\mathcal{W}}$ (see Section 2.4). The local strategy $\mathcal{W}_A$ has the property that if $P_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$, for $1 \leq k \leq n$, are systems then also

$$P_A := \mathcal{W}_A(P_{A_1}, P_{A_2}, ..., P_{A_n}) \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) , \tag{3.11}$$

is a system. In the following we show that a local strategy $\mathcal{W}_A$ is *multilinear*, i.e., that

$$P_A = \sum_{i_k} \mathcal{W}_A(P_{A_1}, ..., P_{A_{k-1}}, P^{i_k}_{A_k}, P_{A_{k+1}}, ..., P_{A_n}) , \tag{3.12}$$

with $P_{A_k} = \sum_{i_k} P^{i_k}_{A_k}$, for all $1 \leq k \leq n$, and

$$c_k \cdot P_A = \mathcal{W}_A(P_{A_1}, ..., P_{A_{k-1}}, c_k \cdot P_{A_k}, P_{A_{k+1}}, ..., P_{A_n}) , \tag{3.13}$$

for all $1 \leq k \leq n$ and $c_k \in \mathbb{R}$. The intuition behind $\mathcal{W}_A$ being multilinear is the following: let $P_{A_k} = \sum_{i_k} p_{i_k} \cdot P^{i_k}_{A_k}$, with $\sum_{i_k} p_{i_k} = 1$ and $p_{i_k} \geq 0$ and $P^{i_k}_{A_k}$ systems. It does not matter if we directly provide the system P_{A_k} to the local strategy or if we provide the system $P^{i_k}_{A_k}$ with probability p_{i_k} and then forget the value i_k (see also [Bar07]).

We will now represent the output system P_A in terms of the functions $f^1_{\mathcal{W}},...,f^{n+1}_{\mathcal{W}}$, $g^1_{\mathcal{W}},...,g^n_{\mathcal{W}}$ and the systems $P_{A_1},...,P_{A_n}$ which completely determine the local strategy $\mathcal{W}_A$. In what follows, we also use the shorter notation $P_i := P_{A_i}$ to denote Alice's systems. We denote by $P(a_1, a_2, ..., a_n, a|x)$ the probability that the first used system outputs a_1, the second one a_2, and so on, and the final output is a, conditioned on the input being x. The output system P_A can then be written as

$$P_A(a|x) = \sum_{a_1,a_2,...,a_n} P(a_1, a_2, ..., a_n, a|x) . \tag{3.14}$$

By using the definition of the conditional probability, $P(a_1, a_2, ..., a_n, a|x)$ can be represented as

$$\begin{aligned} & P(a_1, ..., a_n, a|x) \\ = \;& P(a_1|x) \cdot P(a_2|a_1, x) \cdot ... \cdot P(a_n|a_1, ..., a_{n-1}, x) \cdot P(a|a_1, ..., a_n, x) , \\ = \;& P(a_1|x_1, x) \cdot ... \cdot P(a_n|x_n, a_1, ..., a_{n-1}, x) \cdot P(a|a_1, ..., a_n, x) , \\ = \;& P_{i_1}(a_1|x_1) \cdot P_{i_2}(a_2|x_2) \cdot ... \cdot P_{i_n}(a_n|x_n) \cdot P(a|a_1, ..., a_n, x) , \end{aligned} \tag{3.15}$$

where we used in the second equality that $x_k := f^k_{\mathcal{W}}(x, a_1, ..., a_{k-1})$ and, therefore, $P(a_k|a_1, ..., a_{k-1}, x) = P(a_k|x_k, a_1, ..., a_{k-1}, x)$, and in the third equality that $a_1, ..., a_{k-1}, x$ uniquely determine which system should by used next, i.e., $i_k := g^k_{\mathcal{W}}(x, a_1, a_2, ..., a_{k-1})$. Furthermore, note that $P(a|a_1, ..., a_n, x) \in \{0, 1\}$ since $a := f^{n+1}_{\mathcal{W}}(x, a_1, ..., a_n)$. Putting (3.14) and (3.15) together yields

$$P_A(a|x) = \sum_{a_1,...,a_n} P(a|a_1, ..., a_n, x) \cdot P_{i_1}(a_1|x_1) \cdot ... \cdot P_{i_n}(a_n|x_n) \; . \quad (3.16)$$

Note that $(i_1, i_2, ..., i_n)$ is a permutation of $(1, 2, ..., n)$ which is determined by the functions $g^1_{\mathcal{W}}, ..., g^n_{\mathcal{W}}$, the input x and the outputs $a^n := (a_1, ..., a_n)$. Let us denote this permutation by π_{x,a^n}. Then, we can write (3.16) as

$$P_A(a|x) = \sum_{a_1,...,a_n} P(a|a_1, ..., a_n, x) \prod_{i=1}^{n} P_i(a_{\pi^{-1}_{x,a^n}(i)}|x_{\pi^{-1}_{x,a^n}(i)}) \; . \quad (3.17)$$

Let us now show that (3.12) holds. First, let $P_k = P^1_k + P^2_k$. Plugging this into (3.17) yields

$$\begin{aligned} &\langle f_{x,a}, P_A \rangle \\ = \;& \sum P(a|a_1, .., a_n, x) P^1_k(a_{\pi^{-1}_{x,a^n}(k)}|x_{\pi^{-1}_{x,a^n}(k)}) \prod_{i \neq k} P_i(a_{\pi^{-1}_{x,a^n}(i)}|x_{\pi^{-1}_{x,a^n}(i)}) \\ + \;& \sum P(a|a_1, .., a_n, x) P^2_k(a_{\pi^{-1}_{x,a^n}(k)}|x_{\pi^{-1}_{x,a^n}(k)}) \prod_{i \neq k} P_i(a_{\pi^{-1}_{x,a^n}(i)}|x_{\pi^{-1}_{x,a^n}(i)}) \; . \end{aligned}$$

Since $\langle f_{x,a}, P_A \rangle = \langle f_{x,a}, \mathcal{W}_A(P_1, .., P^1_k + P^2_k, .., P_n) \rangle$ by (3.11), it follows that

$$\langle f_{x,a}, P_A \rangle = \langle f_{x,a}, \mathcal{W}_A(P_1, .., P^1_k, .., P_n) \rangle + \langle f_{x,a}, \mathcal{W}_A(P_1, .., P^2_k, .., P_n) \rangle \; ,$$

and, hence, (3.12) indeed holds. Similarly, one can show that (3.13) holds, and, therefore, $\mathcal{W}_A$ is multilinear.

It is well known (see for example [DF93]) that if $\mathcal{W}_A$ is multilinear then there exists a *linear* map $\mathcal{T}_A$ on a tensor product space such that

$$\mathcal{W}_A(P_1, P_2, ..., P_n) = \mathcal{T}_A(P_1 \otimes P_2 \otimes ... \otimes P_n) \; ,$$

for all $P_1, ..., P_n$. Hence, the output of the map $\mathcal{T}_A$ can be computed by

$$\begin{aligned} \langle f_{x,a}, P_A \rangle &= \langle f_{x,a}, \mathcal{T}_A(P_1 \otimes ... \otimes P_n) \rangle \\ &= \sum_{a_1,..,a_n} P(a|a_1, ..., a_n, x) \prod_{i=1}^{n} P_i(a_{\pi^{-1}_{x,a^n}(i)}|x_{\pi^{-1}_{x,a^n}(i)}) \; . \end{aligned} \quad (3.18)$$

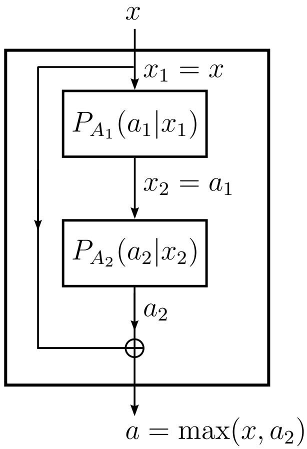

Figure 3.2: This figure shows the local strategy for Alice from Example 4. The behaviour of the resulting system $P_A(a|x)$ is determined by the circuit and the two systems P_{A_1} and P_{A_2}.

Example 4. Let us provide an example for a local strategy $\mathcal{T}_A$ of Alice. First, let all input/output alphabets be equal to $\{1, 2\}$, i.e., all involved systems have binary inputs and binary outputs. Then, assume Alice is in possession of two systems, denoted by $P_{A_1}(a_1|x_1)$ and $P_{A_2}(a_2|x_2)$. The behaviour of the resulting system $P_A(a|x)$, which is computed by

$$P_A = \mathcal{T}_A(P_{A_1} \otimes P_{A_2}) \ ,$$

is described as follows (see also Figure 3.2): first, Alice uses x as the input of the system P_{A_1}. The resulting output is then used as the input for the next system P_{A_2}. The final output a is then just the maximum of the input x and the output a_2 of the system P_{A_2}.

How does the matrix $\mathcal{T}_A$ look like for this local strategy? First, it has dimension $(2 \cdot 2) \times (2 \cdot 2 \cdot 2 \cdot 2)$ and, hence, 64 entries. We only list entries that are equal to one, all other entries will be zero. For $x = 1$ and $a = 1$ only two entries out of 16 are not equal to zero, namely

$$\langle f_{1,1}, \mathcal{T}_A(e_{1,1} \otimes e_{1,1})\rangle = 1 \ , \ \langle f_{1,1}, \mathcal{T}_A(e_{1,2} \otimes e_{2,1})\rangle = 1 \ .$$

See also (3.18). The reason is as follows. We know that $x_1 = x$ and, hence, $x_1 = 1$. Furthermore, $a = \max(x, a_2)$, $a = 1$ and $x = 1$ imply that $a_2 = 1$.

Hence, we can only choose a_1 (or x_2, which must be equal to a_1). By setting $a_1 = x_2 = 1$ we obtain $\langle f_{1,1}, \mathcal{T}_A(e_{1,1} \otimes e_{1,1})\rangle = 1$ and by setting $a_1 = x_2 = 2$ we obtain $\langle f_{1,1}, \mathcal{T}_A(e_{1,2} \otimes e_{2,1})\rangle = 1$. By similar arguments one can derive the following non-zero entries of $\mathcal{T}_A$:

$$\begin{aligned}
\langle f_{1,2}, \mathcal{T}_A(e_{1,1} \otimes e_{1,2})\rangle &= 1\ ,\ \langle f_{1,2}, \mathcal{T}_A(e_{1,2} \otimes e_{2,2})\rangle = 1\ ,\\
\langle f_{2,2}, \mathcal{T}_A(e_{2,1} \otimes e_{1,1})\rangle &= 1\ ,\ \langle f_{2,2}, \mathcal{T}_A(e_{2,2} \otimes e_{2,1})\rangle = 1\ ,\\
\langle f_{2,2}, \mathcal{T}_A(e_{2,1} \otimes e_{1,2})\rangle &= 1\ ,\ \langle f_{2,2}, \mathcal{T}_A(e_{2,2} \otimes e_{2,2})\rangle = 1\ .
\end{aligned}$$

Note that there is no entry for $x = 2$ and $a = 1$ since this is incompatible with the circuit. Using (3.18), this yields the following expression for the system P_A:

$$P_A = \left(\begin{array}{c} P_A(1|1) \\ P_A(2|1) \\ \hline P_A(1|2) \\ P_A(2|2) \end{array}\right) = \left(\begin{array}{c} P_{A_1}(1|1) \cdot P_{A_2}(1|1) + P_{A_1}(2|1) \cdot P_{A_2}(1|2) \\ P_{A_1}(1|1) \cdot P_{A_2}(2|1) + P_{A_1}(2|1) \cdot P_{A_2}(2|2) \\ \hline 0 \\ 1 \end{array}\right) .$$

Note that the first two rows sum to 1 and the last two rows sum to 1, for *any* two systems P_{A_1} and P_{A_2}. And, hence, P_A is again a system. This finishes this example.

By doing the same reasoning on Bob's side one can conclude that there exists a linear map $\mathcal{T}_B$ for every local strategy $\mathcal{W}_B$ of Bob such that

$$\mathcal{W}_B(P_{B_1}, P_{B_2}, ..., P_{B_2}) = \mathcal{T}_B(P_{B_1} \otimes P_{B_2} \otimes ... \otimes P_{B_n})\ .$$

Hence, by using linearity and the definition of non-signalling systems given in Section 2.3.1, we obtain that any wiring $\mathcal{W} : \mathcal{NS}_m^{\times n} \to \mathcal{NS}_{m'}$ of Alice and Bob (without shared randomness) can be written as

$$\begin{aligned}
&\mathcal{W}(P_{A_1B_1}, ..., P_{A_nB_n})\\
=\ &\sum_{i_1,\dots,i_n} p_{i_1} \cdot ... \cdot p_{i_n} \cdot \mathcal{T}_A(P_{A_1}^{i_1} \otimes ... \otimes P_{A_n}^{i_n}) \otimes \mathcal{T}_B(P_{B_1}^{i_1} \otimes ... \otimes P_{B_n}^{i_n})\ ,
\end{aligned}$$

with $P_{A_kB_k} = \sum_{i_k} p_{i_k} \cdot P_{A_k}^{i_k} \otimes P_{B_k}^{i_k}$. Note that since $P_{A_k}^{i_k}$ and $P_{B_k}^{i_k}$ are systems (see Lemma 2.3) we can conclude that $\mathcal{T}_A(P_{A_1}^{i_1} \otimes ... \otimes P_{A_n}^{i_n})$ and $\mathcal{T}_B(P_{B_1}^{i_1} \otimes ... \otimes P_{B_n}^{i_n})$ are systems as well and, therefore, $\mathcal{W}(P_{A_1B_1}, ..., P_{A_nB_n})$ is again a non-signalling system by Lemma 2.3.

Recall that we denote the composition of systems by $P_{A_1B_1} \odot P_{A_2B_2} \odot ... \odot P_{A_nB_n} \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$ (see Section 3.3.3). Due to linearity, an arbitrary wiring that also includes shared randomness can be written as

$$\begin{aligned}
&\mathcal{W}(P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n}) \\
= &\sum_s \mu_s \sum_{i_1,..,i_n} p_{i_1} \cdot .. \cdot p_{i_n} \cdot \mathcal{T}^s_A(P^{i_1}_{A_1} \otimes .. \otimes P^{i_n}_{A_n}) \otimes \mathcal{T}^s_B(P^{i_1}_{B_1} \otimes .. \otimes P^{i_n}_{B_n}) \;, \\
= &\sum_s \mu_s \cdot (\mathcal{T}^s_A \otimes \mathcal{T}^s_B)(P_{A_1B_1} \odot P_{A_2B_2} \odot ... \odot P_{A_nB_n}) \;,
\end{aligned}$$

with $P_{A_kB_k} = \sum_{i_k} p_{i_k} \cdot P^{i_k}_{A_k} \otimes P^{i_k}_{B_k}$ and μ_s representing the shared randomness with $\sum_s \mu_s = 1$ and $\mu_s \geq 0$. Hence, we have shown that any wiring $\mathcal{W}$ can be represented as a linear map

$$\mathcal{T}_{\mathcal{W}} := \sum_s \mu_s \cdot (\mathcal{T}^s_A \otimes \mathcal{T}^s_B) : \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)} \to \ell^{|\mathcal{X}|}_\infty(\ell^{|\mathcal{A}|}_1) \otimes \ell^{|\mathcal{Y}|}_\infty(\ell^{|\mathcal{B}|}_1) \;. \quad (3.19)$$

A special wiring which acts on single systems is the *depolarization wiring* [MAG06] (also called *twirling* [Sho09]), denoted by $\mathcal{W}_{\text{iso}} : \mathcal{NS}_{\text{CHSH}} \to \mathcal{NS}_{\text{iso}}$. It maps binary non-signalling systems to isotropic systems while preserving the non-locality.

Lemma 3.4 (Masanes et al. [MAG06]). *The depolarization wiring* $\mathcal{W}_{\text{iso}} : \ell^{|\mathcal{X}|}_\infty(\ell^{|\mathcal{A}|}_1) \otimes \ell^{|\mathcal{Y}|}_\infty(\ell^{|\mathcal{B}|}_1) \to \ell^{|\mathcal{X}|}_\infty(\ell^{|\mathcal{A}|}_1) \otimes \ell^{|\mathcal{Y}|}_\infty(\ell^{|\mathcal{B}|}_1)$*, for* $|\mathcal{X}| = |\mathcal{Y}| = |\mathcal{A}| = |\mathcal{B}| = 2$*, maps every* $P \in \mathcal{NS}_{\text{CHSH}}$ *to an isotropic system such that* $NL(\mathcal{W}_{\text{iso}}(P)) = NL(P)$*. It can be written as*

$$\mathcal{W}_{\text{iso}}(P) := \sum_{s=1}^{8} \frac{1}{8} \cdot (\mathcal{T}^s_A \otimes \mathcal{T}^s_B)(P) \;,$$

where $\mathcal{T}^s_A : \ell^{|\mathcal{X}|}_\infty(\ell^{|\mathcal{A}|}_1) \to \ell^{|\mathcal{X}|}_\infty(\ell^{|\mathcal{A}|}_1)$ *and* $\mathcal{T}^s_B : \ell^{|\mathcal{Y}|}_\infty(\ell^{|\mathcal{B}|}_1) \to \ell^{|\mathcal{Y}|}_\infty(\ell^{|\mathcal{B}|}_1)$ *are local strategies.*

3.4.3 Tensor Norms and Wirings

In the following, we need three crucial properties of the map $\mathcal{T}_A$ introduced in the previous section. First, $\mathcal{T}_A$ is linear. Second, $\mathcal{T}_A(P_{A_1} \otimes P_{A_2} \otimes ... \otimes P_{A_n})$ is a system for all systems $P_{A_1}, ..., P_{A_n}$ since the local strategy $\mathcal{W}_A$ maps systems to systems. Third, the matrix $\mathcal{T}_A$ has only non-negative entries. This is the case since $\langle f_{x,a}, \mathcal{T}_A(e_{x_1,a_1} \otimes ... \otimes e_{x_n,a_n}) \rangle \geq 0$ by (3.18) and $P(a|a_1, ..., a_n, x) \in \{0, 1\}$.

We now have all the necessary tools in order to prove the following result which shows how the $\infty(1)$-norm behaves under local strategies.

Lemma 3.5. *Let* $\mathcal{T}_A : \ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes \ell_\infty^{|\mathcal{X}_2|}(\ell_1^{|\mathcal{A}_2|}) \otimes \ldots \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|}) \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ *be a local strategy of Alice. It then holds that*

$$\|\mathcal{T}_A(P_{A_1} \otimes P_{A_2} \otimes \ldots \otimes P_{A_n})\|_{\infty(1)} \leq \prod_{k=1}^{n} \|P_{A_k}\|_{\infty(1)} \,,$$

for all $P_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$.

Proof. Let $\|P_{A_k}\|_{\infty(1)} = 1$ for all $k \in \{1, 2, \ldots, n\}$, and, hence

$$\sum_{a_k} |\langle f_{x_k,a_k}, P_{A_k}\rangle| \leq 1 \,, \tag{3.20}$$

for all x_k. Showing that $\|\mathcal{T}_A(P_{A_1} \otimes P_{A_2} \otimes \ldots \otimes P_{A_n})\|_{\infty(1)} \leq 1$ finishes the proof. By using the triangle inequality we obtain

$$\begin{aligned}
& \sum_a \left| \sum_{\substack{a_1,\ldots,a_n \\ x_1,\ldots,x_n}} \langle f_{x,a}, \mathcal{T}_A(e_{x_1,a_1} \otimes \ldots \otimes e_{x_n,a_n})\rangle \prod_k \langle f_{x_k,a_k}, P_{A_k}\rangle \right| \\
\leq \;& \sum_a \sum_{\substack{a_1,\ldots,a_n \\ x_1,\ldots,x_n}} |\langle f_{x,a}, \mathcal{T}_A(e_{x_1,a_1} \otimes \ldots \otimes e_{x_n,a_n})\rangle| \prod_k |\langle f_{x_k,a_k}, P_{A_k}\rangle| \\
\leq \;& \sum_a \sum_{x_1,\ldots,x_n} \left|\langle f_{x,a}, \mathcal{T}_A(e_{x_1,\hat{a}_1(x_1)} \otimes \ldots \otimes e_{x_n,\hat{a}_n(x_n)})\rangle\right| \,, \\
= \;& \sum_a \sum_{x_1,\ldots,x_n} \langle f_{x,a}, \mathcal{T}_A(e_{x_1,\hat{a}_1(x_1)} \otimes \ldots \otimes e_{x_n,\hat{a}_n(x_n)})\rangle \,,
\end{aligned} \tag{3.21}$$

for all $x \in \mathcal{X}$ and with functions $\hat{a}_k : \mathcal{X}_k \to \mathcal{A}_k$, and where we used (3.20) in the second inequality and that $\mathcal{T}_A$ has only non-negative entries in the last equality.

Let us define the systems $\tilde{P}_{A_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|})$ as follows:

$$\langle f_{x_k,a_k}, \tilde{P}_{A_k}\rangle := \begin{cases} 1 \,, & \text{if } a_k = \hat{a}_k(x_k) \,, \\ 0 \,, & \text{otherwise} \,. \end{cases}$$

Using the definition of the systems $\tilde{P}_{A_k}$ leads to

$$\begin{aligned}
& \sum_a \langle f_{x,a}, \mathcal{T}_A(\tilde{P}_{A_1} \otimes \tilde{P}_{A_2} \otimes \ldots \otimes \tilde{P}_{A_n})\rangle \\
= \;& \sum_a \sum_{x_1,\ldots,x_n} \langle f_{x,a}, \mathcal{T}_A(e_{x_1,\hat{a}_1(x_1)} \otimes \ldots \otimes e_{x_n,\hat{a}_n(x_n)})\rangle \,.
\end{aligned} \tag{3.22}$$

Since $\mathcal{T}_A$ maps systems to systems, the output $\tilde{P}_A := \mathcal{T}_A(\tilde{P}_{A_1} \otimes \tilde{P}_{A_2} \otimes ... \otimes \tilde{P}_{A_n})$ is a system as well. Therefore, $\sum_a \langle f_{x,a}, \tilde{P}_A \rangle = 1$ for all x, and, hence, by combining (3.21) and (3.22) we obtain

$$\begin{aligned}
& \|\mathcal{T}_A(P_{A_1} \otimes P_{A_2} \otimes ... \otimes P_{A_n})\|_{\infty(1)} \\
= \quad & \max_x \sum_a \left| \sum_{\substack{a_1,\ldots,a_n \\ x_1,\ldots,x_n}} \langle f_{x,a}, \mathcal{T}_A(e_{x_1,a_1} \otimes ... \otimes e_{x_n,a_n}) \rangle \prod_k \langle f_{x_k,a_k}, P_{A_k} \rangle \right| \\
\leq \quad & 1 \, ,
\end{aligned}$$

which finishes the proof. $\square$

The previous result only treated the case where the inputs to the map $\mathcal{T}_A$ are product systems. Let us now prove a more general result which handles arbitrary input systems. Since we work on a space that allows us to represent several systems, we need the $\pi^{A^n}_{\infty(1)}$-norm that was introduced in Section 3.3.2.

Lemma 3.6. *Let $\mathcal{T}_A : \Pi^{A^n}_{\infty(1)} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ be a local strategy of Alice. It then holds that*

$$\|\mathcal{T}_A(P_A)\|_{\infty(1)} \equiv \pi^{A^1}_{\infty(1)}(\mathcal{T}_A(P_A)) \leq \pi^{A^n}_{\infty(1)}(P_A) \, ,$$

for all $P_A \in \Pi^{A^n}_{\infty(1)}$.

Proof. For every $\epsilon > 0$ there exists a decomposition $P_A = \sum_i P^i_{A_1} \otimes ... \otimes P^i_{A_n}$ such that $\pi^{A^n}_{\infty(1)}(P_A) + \epsilon \geq \sum_i \|P^i_{A_1}\|_{\infty(1)} \cdot ... \cdot \|P^i_{A_n}\|_{\infty(1)}$. We then obtain

$$\begin{aligned}
\|\mathcal{T}_A(P_A)\|_{\infty(1)} \quad & \leq \quad \sum_i \|\mathcal{T}_A(P^i_{A_1} \otimes ... \otimes P^i_{A_n})\|_{\infty(1)} \\
& \leq \quad \sum_i \|P^i_{A_1}\|_{\infty(1)} \cdot ... \cdot \|P^i_{A_n}\|_{\infty(1)} \\
& \leq \quad \pi^{A^n}_{\infty(1)}(P_A) + \epsilon \, ,
\end{aligned}$$

where we used the triangle inequality and the linearity of $\mathcal{T}_A$ in the first line and Lemma 3.5 in the second line. Since ϵ was arbitrary the result follows. $\square$

The next lemma shows that tensor norms cannot increase under local and transposed local strategies.

Lemma 3.7. *Let $\mathcal{T}_A : \Pi^{A^n}_{\infty(1)} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $\mathcal{T}_B : \Pi^{B^n}_{\infty(1)} \to \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be local strategies and α be an arbitrary tensor norm. Then*

$$\begin{aligned} \alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(P)) &\leq \alpha(P) , \\ \alpha^*((\mathcal{T}_A^T \otimes \mathcal{T}_B^T)(G)) &\leq \alpha^*(G) , \end{aligned}$$

for all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$ and all $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ with $\mathcal{T}_A^T : \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \to E^{A^n}_{1(\infty)}$ and $\mathcal{T}_B^T : \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}) \to E^{B^n}_{1(\infty)}$ being the transposed matrices of $\mathcal{T}_A$ and $\mathcal{T}_B$, respectively.

Proof. If $\mathcal{T}_A$ is a local strategy it follows from Lemma 3.6 that

$$\|\mathcal{T}_A\|_{\Pi^{A^n}_{\infty(1)} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})} = \sup_{P_A} \left\{ \|\mathcal{T}_A(P_A)\|_{\infty(1)} \ : \ \pi^{A^n}_{\infty(1)}(P_A) \leq 1 \right\} \leq 1 \, .$$

Hence, since α is a tensor norm it follows from Lemma 3.1 that $\alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(P)) \leq \alpha(P)$ for all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.

Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. The second part of the lemma can be proved as follows:

$$\begin{aligned} & \alpha^*((\mathcal{T}_A^T \otimes \mathcal{T}_B^T)(G)) \\ = \ & \sup_P \left\{ |\langle (\mathcal{T}_A^T \otimes \mathcal{T}_B^T)(G), P \rangle| \ : \ \alpha(P) \leq 1 \right\} \\ = \ & \sup_P \left\{ |\langle G, (\mathcal{T}_A \otimes \mathcal{T}_B)(P) \rangle| \ : \ \alpha(P) \leq 1 \right\} \\ \leq \ & \sup_P \left\{ |\langle G, (\mathcal{T}_A \otimes \mathcal{T}_B)(P) \rangle| \ : \ \alpha((\mathcal{T}_A \otimes \mathcal{T}_B)(P)) \leq 1 \right\} \\ \leq \ & \sup_{\tilde{P}} \left\{ |\langle G, \tilde{P} \rangle| \ : \ \alpha(\tilde{P}) \leq 1 \right\} \\ = \ & \alpha^*(G) , \end{aligned}$$

with $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$ and $\tilde{P} \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, and where we used the first part of the lemma in the first inequality. $\square$

Recall that every wiring of bipartite systems can be represented as a linear map $\mathcal{T}_\mathcal{W} : \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ such that $\mathcal{T}_\mathcal{W} = \sum_s \mu_s \cdot (\mathcal{T}_A^s \otimes \mathcal{T}_B^s)$, with $\mu_s \geq 0$, $\sum_s \mu_s = 1$ and $\mathcal{T}_A^s$ and $\mathcal{T}_B^s$ local strategies (see (3.19)). Hence, due to convexity and linearity, Lemma 3.7 implies the following theorem.

Theorem 3.3. *Let* $\mathcal{T}_{\mathcal{W}} : \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)} \to \ell^{|\mathcal{X}|}_{\infty}(\ell^{|\mathcal{A}|}_{1}) \otimes \ell^{|\mathcal{Y}|}_{\infty}(\ell^{|\mathcal{B}|}_{1})$ *represent a wiring and* α *be an arbitrary tensor norm. Then*

$$\alpha\left(\mathcal{T}_{\mathcal{W}}(P)\right) \leq \alpha\left(P\right) \ ,$$
$$\alpha^*\left(\mathcal{T}^T_{\mathcal{W}}(G)\right) \leq \alpha^*\left(G\right) \ ,$$

for all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$ *and all* $G \in \ell^{|\mathcal{X}|}_{1}(\ell^{|\mathcal{A}|}_{\infty}) \otimes \ell^{|\mathcal{Y}|}_{1}(\ell^{|\mathcal{B}|}_{\infty})$, *and where* $\mathcal{T}^T_{\mathcal{W}} : \ell^{|\mathcal{X}|}_{1}(\ell^{|\mathcal{A}|}_{\infty}) \otimes \ell^{|\mathcal{Y}|}_{1}(\ell^{|\mathcal{B}|}_{\infty}) \to E^{A^n}_{1(\infty)} \otimes E^{B^n}_{1(\infty)}$ *is the transposed matrix of* $\mathcal{T}_{\mathcal{W}}$.

3.5 Convex Sets of Bipartite Systems

3.5.1 Introduction

Equation (3.1) in Section 3.1.1 shows how a tensor norm α can be used in order to define a convex set of bipartite systems. In this section, we investigate the converse problem: for every convex set of systems, does there exist a tensor norm α that induces this set?

Contribution

We prove that for every convex set $\mathcal{C}_m$ of bipartite systems, with $\mathcal{L}_m \subseteq \mathcal{C}_m \subseteq \mathcal{S}_m$, there exists a *cross norm* α such that $\mathcal{R}^{\alpha}_m = \mathcal{C}_m$. Hence, cross norms seem to be the right mathematical object for the study of convex sets of non-local systems.

Open Problems

Is it possible to find a definition of a norm α (similarly as the one given in Lemma 3.8) such that every convex set of bipartite systems is obtainable and α is a *tensor norm*, and not only a cross norm? If we allow all convex sets it is not possible. The convex set has at least to be closed under local operations on single systems because of the metric mapping property of tensor norms. In order to see this, assume that we have a tensor norm α which has the property that there exist $P \in \mathcal{NS}\backslash\mathcal{R}^{\alpha}_m$, $\tilde{P} \in \mathcal{R}^{\alpha}_m$, and local strategies $\mathcal{T}_A$ and $\mathcal{T}_B$ such that $(\mathcal{T}_A \otimes \mathcal{T}_B)(\tilde{P}) = P$ (i.e., the set $\mathcal{R}^{\alpha}_m$ is not closed under local operations on a single system). Since $P \in \mathcal{NS}\backslash\mathcal{R}^{\alpha}_m$ and $\tilde{P} \in \mathcal{R}^{\alpha}_m$ imply that $\alpha(P) > 1$ and $\alpha(\tilde{P}) \leq 1$, respectively, we obtain a contradiction by using Lemma 3.7. Therefore, the convex sets induced by tensor norms are all closed under local operations on single bipartite systems (see also Section 5.2).

3.5.2 Cross Norms From Convex Sets

Let us define

$$\bar{\mathcal{L}}_m := \{P_A \otimes P_B \ : \ \|P_A\|_{\infty(1)} \leq 1 \ , \ \|P_B\|_{\infty(1)} \leq 1\} \ ,$$

for $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, with $|\mathcal{X}|, |\mathcal{Y}|, |\mathcal{A}|, |\mathcal{B}| \leq m$. For some set $\mathcal{C}_m$ of elements from the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, we define the corresponding set $\widehat{\mathcal{C}}_m$ as follows:

$$\widehat{\mathcal{C}}_m := \mathcal{C}_m \cup \bar{\mathcal{L}}_m \ .$$

Based on the set $\widehat{\mathcal{C}}_m$, we can define a norm.

Lemma 3.8. *Let $\mathcal{C}_m$ be a subset of all elements of the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, the function $\delta_{\mathcal{C}_m} : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}) \to \mathbb{R}$, defined as*

$$\delta_{\mathcal{C}_m}(P) := \inf \left\{ \sum_{i=1}^{n} |\alpha_i| \ : \ P = \sum_{i=1}^{n} \alpha_i \cdot P_i \ , \ \alpha_i \in \mathbb{R} \ , \ P_i \in \widehat{\mathcal{C}}_m \right\} \ ,$$

is a norm.

Proof. First, note that *every* $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ can be written as $P = \sum_i \alpha_i \cdot P_i$ with $P_i \in \widehat{\mathcal{C}}_m$ and $\alpha_i \in \mathbb{R}$. This follows from the fact that $\|e_{x,a}\|_{\infty(1)} = 1$ and $\|e_{y,b}\|_{\infty(1)} = 1$ for all $a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{X}, y \in \mathcal{Y}$ and, therefore, the basis vectors $e_{x,a} \otimes e_{y,b}$ are elements of $\bar{\mathcal{L}}_m \subseteq \widehat{\mathcal{C}}_m$.

Let us now show that $\delta_{\mathcal{C}_m}$ is indeed a norm for any set $\mathcal{C}_m$. We first prove the triangle inequality. For every $\epsilon > 0$ there exist decompositions $P^1 = \sum_i \alpha_i^1 \cdot P_i^1$ and $P^2 = \sum_j \alpha_j^2 \cdot P_j^2$, with $P_i^1, P_j^2 \in \widehat{\mathcal{C}}_m$, such that $\delta_{\mathcal{C}_m}(P^1) + \epsilon \geq \sum_i |\alpha_i^1|$ and $\delta_{\mathcal{C}_m}(P^2) + \epsilon \geq \sum_j |\alpha_j^2|$. By considering the decomposition $P^1 + P^2 = \sum_i \alpha_i^1 \cdot P_i^1 + \sum_j \alpha_j^2 \cdot P_j^2$ we obtain

$$\begin{aligned} \delta_{\mathcal{C}_m}(P^1 + P^2) &\leq \sum_i |\alpha_i^1| + \sum_j |\alpha_j^2| \\ &\leq \delta_{\mathcal{C}_m}(P^1) + \epsilon + \delta_{\mathcal{C}_m}(P^2) + \epsilon \ . \end{aligned}$$

Since ϵ was arbitrary we obtain $\delta_{\mathcal{C}_m}(P^1 + P^2) \leq \delta_{\mathcal{C}_m}(P^1) + \delta_{\mathcal{C}_m}(P^2)$.

Next, we show that $\delta_{\mathcal{C}_m}(c \cdot P) = |c| \cdot \delta_{\mathcal{C}_m}(P)$ for all $c \in \mathbb{R}\backslash\{0\}$ and all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. For every $\epsilon > 0$ there exists a decomposition

$c \cdot P = \sum_i \alpha_i \cdot P_i$ such that $\delta_{\mathcal{C}_m}(c \cdot P) + \epsilon \geq \sum_i |\alpha_i|$. Considering the decomposition $P = \sum_i (\alpha_i/c) \cdot P_i$ then yields

$$\begin{aligned} |c| \cdot \delta_{\mathcal{C}_m}(P) &\leq |c| \cdot \sum_i \left| \frac{\alpha_i}{c} \right| \\ &\leq \delta_{\mathcal{C}_m}(c \cdot P) + \epsilon \, . \end{aligned}$$

Since ϵ was arbitrary we obtain $|c| \cdot \delta_{\mathcal{C}_m}(P) \leq \delta_{\mathcal{C}_m}(c \cdot P)$. Furthermore, for every $\epsilon > 0$ there exists a decomposition $P = \sum_i \alpha_i \cdot P_i$ such that $\delta_{\mathcal{C}_m}(P) + \epsilon \geq \sum_i |\alpha_i|$. Then, by considering the decomposition $c \cdot P = \sum_i c \cdot \alpha_i \cdot P_i$ we obtain

$$\begin{aligned} \delta_{\mathcal{C}_m}(c \cdot P) &\leq \sum_i |c \cdot \alpha_i| \\ &\leq |c| \cdot (\delta_{\mathcal{C}_m}(P) + \epsilon) \, . \end{aligned}$$

Since ϵ was arbitrary we obtain $|c| \cdot \delta_{\mathcal{C}_m}(P) \geq \delta_{\mathcal{C}_m}(c \cdot P)$ and, therefore, $|c| \cdot \delta_{\mathcal{C}_m}(P) = \delta_{\mathcal{C}_m}(c \cdot P)$.

That $\delta_{\mathcal{C}_m}(P) = 0$ for $P = 0$ is obvious. For the converse, assume that $\delta_{\mathcal{C}_m}(P) = 0$. Then, for every $\epsilon > 0$ there exists a decomposition $P = \sum_i \alpha_i \cdot P_i$ such that $\sum_i |\alpha_i| \leq \epsilon$. Since ϵ was arbitrary we can conclude that $\sum_i |\alpha_i| = 0$ and, therefore, $\alpha_i = 0$ for all i which implies that $P = 0$. □

Lemma 3.9. *Let $\mathcal{C}_m$ be a set such that $\mathcal{L}_m \subseteq \mathcal{C}_m \subseteq \mathcal{S}_m$. Then, $\delta_{\mathcal{C}_m}$ is a* cross norm, *i.e.,*

$$\varepsilon(P) \leq \delta_{\mathcal{C}_m}(P) \leq \pi(P) \, ,$$

for all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

Proof. Let us first prove that $\delta_{\mathcal{C}_m}(P) \leq \pi(P)$. First, we need to show that

$$\delta_{\mathcal{C}_m}(P_A \otimes P_B) \leq \|P_A\|_{\infty(1)} \cdot \|P_B\|_{\infty(1)} \, , \tag{3.23}$$

for all $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Let $\|P_A\|_{\infty(1)} = 1$ and $\|P_B\|_{\infty(1)} = 1$. Since $P_A \otimes P_B \in \bar{\mathcal{L}}_m$ we obtain that $P_A \otimes P_B \in \widehat{\mathcal{C}}_m$ and, hence, $\delta_{\mathcal{C}_m}(P_A \otimes P_B) \leq 1$, which implies (3.23). Then, for every $\epsilon > 0$ there exists a decomposition $P = \sum_i P_A^i \otimes P_B^i$ such that $\sum_i \|P_A^i\|_{\infty(1)} \cdot \|P_B^i\|_{\infty(1)} \leq \pi(P) + \epsilon$. Using (3.23) and the fact that $\delta_{\mathcal{C}_m}$ obeys the triangle inequality implies

$$\begin{aligned}\delta_{\mathcal{C}_m}(P) &\leq \sum_i \delta_{\mathcal{C}_m}(P_A^i \otimes P_B^i) \\ &\leq \sum_i \|P_A^i\|_{\infty(1)} \cdot \|P_B^i\|_{\infty(1)} \\ &\leq \pi(P) + \epsilon \, .\end{aligned}$$

Since ϵ was arbitrary we obtain $\delta_{\mathcal{C}_m}(P) \leq \pi(P)$ for all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

Let us now show that $\varepsilon(P) \leq \delta_{\mathcal{C}_m}(P)$. Let $\delta_{\mathcal{C}_m}(P) = 1$. Then, for every $\epsilon > 0$ there exists a decomposition $P = \sum_i \alpha_i \cdot P_i$, with $P_i \in \widehat{\mathcal{C}}_m$, such that $\sum_i |\alpha_i| \leq 1+\epsilon$. The injective tensor norm of P can then be upper-bounded by

$$\begin{aligned}\varepsilon(P) &= \sup_{G_A, G_B} |\langle G_A \otimes G_B, P\rangle| \\ &\leq \sup_{G_A, G_B} \sum_i |\alpha_i| |\langle G_A \otimes G_B, P_i\rangle| \\ &\leq (1+\epsilon) \max_i \sup_{G_A, G_B} |\langle G_A \otimes G_B, P_i\rangle| \\ &= (1+\epsilon)\varepsilon(P_i) \, ,\end{aligned}$$

where the supremum is over $G_A \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$ and $G_B \in \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ such that $\|G_A\|_{1(\infty)} \leq 1$ and $\|G_B\|_{1(\infty)} \leq 1$, respectively, and where we used the triangle inequality in the second line. Then, since ϵ was arbitrary, showing that $\varepsilon(P_i) \leq 1$ for all $P_i \in \widehat{\mathcal{C}}_m$ implies $\varepsilon(P) \leq 1$. First, if $P_i \in \mathcal{C}_m \subseteq \mathcal{S}_m$ we obtain by Lemma 5.3 in Section 5.1.4 that $\varepsilon(P_i) = 1$. Second, if $P_i \in \bar{\mathcal{L}}_m$ we can conclude that $P_i = P_A \otimes P_B$ for some $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ with $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$. Then, by using that ε is a cross norm we obtain $\varepsilon(P_i) \equiv \varepsilon(P_A \otimes P_B) = \|P_A\|_{\infty(1)} \cdot \|P_B\|_{\infty(1)} \leq 1$. □

The next result states that for every convex set of non-local systems, there exists a cross norm which induces this set.

Theorem 3.4. *Let $\mathcal{C}_m$ be a convex set such that $\mathcal{L}_m \subseteq \mathcal{C}_m \subseteq \mathcal{S}_m$. Then, the cross norm $\delta_{\mathcal{C}_m}$ has the property that*

$$\mathcal{C}_m = \mathcal{R}_m^{\delta_{\mathcal{C}_m}} \, .$$

Proof. We first prove that $\mathcal{C}_m \subseteq \mathcal{R}_m^{\delta_{\mathcal{C}_m}}$. Let $P \in \mathcal{C}_m$. Then, by the definition of the $\delta_{\mathcal{C}_m}$-norm we obtain that $\delta_{\mathcal{C}_m}(P) \leq 1$. Hence, since $\mathcal{C}_m \subseteq \mathcal{S}_m$ it follows that $P \in \mathcal{S}_m$ and, therefore, $P \in \mathcal{R}_m^{\delta_{\mathcal{C}_m}}$.

Let us now prove that $\mathcal{R}_m^{\delta_{\mathcal{C}_m}} \subseteq \mathcal{C}_m$. Let $P \in \mathcal{R}_m^{\delta_{\mathcal{C}_m}}$. The definition of the set $\mathcal{R}_m^{\delta_{\mathcal{C}_m}}$ implies that $P \in \mathcal{S}_m$ and $\delta_{\mathcal{C}_m}(P) \leq 1$. Hence, for every $\epsilon > 0$ there exists a decomposition $P = \sum_i \alpha_i \cdot P_i$ such that $\sum_i |\alpha_i| \leq 1 + \epsilon$ and $P_i \in \widehat{\mathcal{C}}_m$. We now show that $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$ and $P_i \in \mathcal{C}_m$ since these facts imply that $P \in \mathcal{C}_m$ (because $\mathcal{C}_m$ is a convex set). First, since P is a bipartite system we have

$$\begin{aligned}
1 &= \left| \sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P \rangle \right| \\
&= \left| \sum_i \alpha_i \cdot \left(\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle \right) \right| \\
&\leq \sum_i |\alpha_i| \cdot \left| \sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle \right| \\
&\leq \sum_i |\alpha_i| \leq 1 + \epsilon\,, \qquad (3.24)
\end{aligned}$$

where we applied the triangle inequality in the third line and we used that $|\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle| \leq 1$ for all $P_i \in \widehat{\mathcal{C}}_m$ in the fourth line. Since ϵ was arbitrary we can conclude that all inequalities in (3.24) are actually equalities. Hence, the fourth line implies that $\sum_i |\alpha_i| = 1$ and $|\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle| = 1$. The third line implies that α_i and $\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle$ have the same sign. Hence, we can choose the signs such that $\alpha_i \geq 0$ and $\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle = 1 \geq 0$ for all i.

What remains to be shown is that $P_i \in \mathcal{C}_m$ for all i. Since we have that $P_i \in \widehat{\mathcal{C}}_m$ we only have to prove that for $P_i \in \bar{\mathcal{L}}_m$ it holds that P_i is an element of $\mathcal{C}_m$. By using that $P_i \in \bar{\mathcal{L}}_m$, and, hence $P_i = P_A \otimes P_B$, and $\sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_i \rangle = 1$, we obtain

$$\left(\sum_a \langle f_{x,a}, P_A \rangle \right) \left(\sum_b \langle f_{y,b}, P_B \rangle \right) = 1\,, \qquad (3.25)$$

with $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$. Since $\|P_A\|_{\infty(1)} \leq 1$ implies that $|\sum_a \langle f_{x,a}, P_A \rangle| \leq 1$ we can conclude that $|\sum_a \langle f_{x,a}, P_A \rangle| = 1$ and

$|\sum_b \langle f_{y,b}, P_B \rangle| = 1$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and that $\sum_a \langle f_{x,a}, P_B \rangle$ and $\sum_b \langle f_{y,b}, P_B \rangle$ have the same sign. Since (3.25) holds for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we can assume without loss of generality that $\sum_a \langle f_{x,a}, P_A \rangle = 1$ and $\sum_b \langle f_{y,b}, P_B \rangle = 1$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Together with $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$ this implies that $\langle f_{x,a}, P_A \rangle \geq 0$ and $\langle f_{y,b}, P_B \rangle \geq 0$ for all a, b, x, y, and, hence, P_A and P_B are systems. Therefore, $P_A \otimes P_B$ is a local system, which by the assumption $\mathcal{L}_m \subseteq \mathcal{C}_m$ implies that $P_A \otimes P_B \in \mathcal{C}_m$. $\square$

Chapter 4

Algorithms for Computing Tensor Norms

4.1 Introduction

In the previous chapter we have introduced the notion of tensor norms. Among other things, we have provided definitions for the Hilbertian and dual Hilbertian tensor norms. By just having a short look at these definitions, it is not obvious at all whether these tensor norms can be "easily" computed or not. Having algorithms at our disposal which can compute these tensor norms will be crucial in the analysis of their properties. Hence, the main goal of this chapter is to derive algorithms that compute the injective, the projective, the Hilbertian, and the dual Hilbertian tensor norms.

There is one subtlety one has to keep in mind. Since the calculations of these algorithms are over *real* vector spaces, the outputs usually are an *approximation* to the analytical solution. We then have to guess the right analytical solutions from the outputs of the algorithms. Fortunately, we can actually test whether our guessing was successful or not. Let us justify this claim with the help of an example.

Assume we are given a two-prover game G and we want to compute the Hilbertian tensor norm $\gamma_2(G)$. Then, by just cleverly guessing[1] a decomposition of G (see the definition given in Section 3.2.2) we obtain an *upper bound* on $\gamma_2(G)$ since γ_2 computes an *infimum*. On the other hand,

[1]e.g., by using the algorithm derived in this chapter.

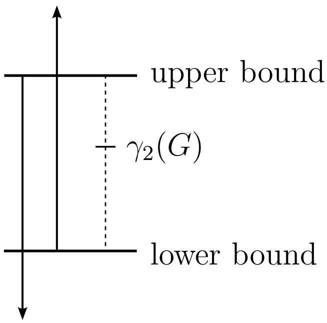

Figure 4.1: An upper bound on $\gamma_2(G)$ is given by some guessed, but fixed, decomposition $\hat{G} = W \cdot V$ and has value $\|W\|_{2\to\infty(1)} \cdot \|V\|_{\infty(1)\to 2}$ (see the definition of γ_2 given in (3.3)). A lower bound on $\gamma_2(G)$ is given by some guessed, but fixed, P with $\gamma_2^*(P) \leq 1$ and has value $|\langle G, P\rangle|$ (since γ_2^* is the dual of γ_2). The value $\gamma_2(G)$ is then located somewhere on the dashed line between these two bounds. If the lower bound matches the upper bound, we know that $\gamma_2(G)$ must actually be equal to this value.

if we consider γ_2 as the *dual tensor norm* of γ_2^*, by cleverly guessing a system[2] P such that $\gamma_2^*(P) \leq 1$, we obtain a *lower bound* on $\gamma_2(G)$ since this definition computes a *supremum*. This guessing strategy could still leave open a gap between the upper and lower bounds, depending on how well we guessed. However, if the lower and upper bound actually coincide[3] we can be sure that we have computed $\gamma_2(P)$ and not only an approximation (see Figure 4.1).

Contribution

We derive semidefinite programs for the Hilbertian and dual Hilbertian tensor norm, which compute the values of these norms up to any desired accuracy. We also compute the running times of these SDP algorithms

[2]Here, P does not necessarily represent a conditional probability distribution but just an element of the space $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Nevertheless, we call it a system. See also Section 3.2.4 where we introduced this notational convention.

[3]Note that this relation between the lower and upper bounds is of the same kind as in the case of the strong duality theorem for SDPs.

which are exponential in the alphabet sizes. Furthermore, we show that computing the injective tensor norm corresponds to solving a certain discrete optimization problem and the projective tensor norm can be computed by solving a linear program.

The results of this this chapter are summarized in Table 4.1. The entries show the number of constraints over which the linear programs (LP) and semidefinite programs (SDP) optimize, respectively. In the case of the injective tensor norm ε, the two entries denote the number of possible solutions which have to be tested in order to obtain the optimum. The running times to evaluate the tensor norms is then a polynomial in the number of constraints (see Section 2.7).

For example, the Hilbertian tensor norm $\gamma_2(P)$ can be computed by an SDP which has $O(2^{|\mathcal{A}|}|\mathcal{X}| + 2^{|\mathcal{B}|}|\mathcal{Y}|)$ constraints. Hence, if the output alphabets $|\mathcal{A}|$ and $|\mathcal{B}|$ are constant, $\gamma_2(P)$ can be computed in polynomial time (in the input alphabet sizes) by Lemma 2.6 in Section 2.7.

	$P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$	$G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$	Algo
π	$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|})$	$O(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}|)$	LP
ε	$O(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}|)$	$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|})$	
γ_2	$O(2^{|\mathcal{A}|}|\mathcal{X}| + 2^{|\mathcal{B}|}|\mathcal{Y}|)$	$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|} + 2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|})$	SDP
γ_2^*	$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|} + 2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|})$	$O(2^{|\mathcal{A}|}|\mathcal{X}| + 2^{|\mathcal{B}|}|\mathcal{Y}|)$	SDP

Table 4.1: The number of constraints which have to be satisfied and the used algorithms that compute the different tensor norms are shown.

Related Work

The special case where the Hilbertian tensor norm is defined over $\ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ has been solved in [LMSS07]. They prove that $\gamma_2(P; \ell_\infty^{|\mathcal{X}|}, \ell_\infty^{|\mathcal{Y}|})$ can be computed to any desired accuracy by an SDP.

A constant factor approximation algorithm for the dual Hilbertian tensor norm $\gamma_2^*(G; \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}), \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}))$ has been proposed in [JP11]. Their algorithm is an SDP as well. The advantage of their SDP is that it has only *polynomially* many constraints, and not exponentially many as ours. However, the price for this is that they can only compute an approximation.

We cannot hope to significantly improve the running time for computing the injective tensor norm on games, $\varepsilon(G)$, since (1) $\omega_{\mathcal{L}}(G) = \varepsilon(G)$ (i.e., the classical value of a two-prover game G is equal to $\varepsilon(G)$, see Lemma 6.1 in the application chapter) and (2) it has been shown [Raz95, AS98, ALM$^+$98] that computing (even approximating) $\omega_{\mathcal{L}}(G)$ is NP-hard, even for *constant* output alphabet sizes (in particular, it is even true for XOR games [Hås01]).

The computation of the projective tensor norm π has also been investigated over *different* local normed vector spaces. It has been shown in [PG04] that $\pi(P; \ell_2^n, \ell_2^n)$ (i.e., the local normed vector spaces are Hilbert spaces) can be approximated by a linear program with exponentially many constraints in the dimension n of the local Hilbert spaces.

Applications

The results of this chapter will be regularly used throughout the remaining parts of this thesis. Sometimes explicitly, but most of the time implicitly to obtain the right decompositions in the tensor norm computations.

Section Outline

In Section 4.2 and Section 4.3, we show algorithms that compute the injective and projective tensor norm, respectively. In Section 4.4.1, we show that the matrices in the definition of the Hilbertian tensor norm (see (3.3)) can be replaced by a collection of vectors. This interpretation will then enable us in Section 4.4.2 to derive alternative representations for the $1(\infty) \to 2$,$2 \to \infty(1)$, $\infty(1) \to 2$ and $2 \to 1(\infty)$ operator norms. Putting the results of Section 4.4.1 and Section 4.4.2 together allows us then to show in Section 4.5 and Section 4.6 that the Hilbertian and dual Hilbertian tensor norms, respectively, can be computed by SDPs.

4.2 Algorithm for Injective Tensor Norm

Let us first show that calculating the injective tensor norm is equivalent to a combinatorial optimization problem which has to iterate over exponentially many possibilities.

Lemma 4.1. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, the injective tensor norm $\varepsilon(G)$ can be computed by optimizing over*

$$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|}) ,$$

possible solutions and where each possibility takes time $O(|\mathcal{X}||\mathcal{Y}|)$ to evaluate.

Proof. The injective tensor norm for two-prover games is computed by (see Section 3.2.2):

$$\varepsilon(G) = \sup_{P_A, P_B} \left\{ |\langle G, P_A \otimes P_B \rangle| \ : \ \|P_A\|_{\infty(1)} \leq 1 \ , \ \|P_B\|_{\infty(1)} \leq 1 \right\} \ ,$$

with $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$, and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. The term $\langle G, P_A \otimes P_B \rangle$ can be expanded as

$$\langle G, P_A \otimes P_B \rangle = \sum_{x,y} \left(\sum_{a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \langle f_{x,a}, P_A \rangle \langle f_{y,b}, P_B \rangle \right) \ .$$

Then, $\max_x \sum_a |\langle f_{x,a}, P_A \rangle| \leq 1$ and $\max_y \sum_b |\langle f_{y,b}, P_B \rangle| \leq 1$ imply that the injective tensor norm can be computed by

$$\varepsilon(G) = \sup_{f,g,s,t} \left| \sum_{x,y} \langle G, e_{x,f(x)} \otimes e_{y,g(y)} \rangle \cdot s_{x,f(x)} \cdot t_{y,g(y)} \right| \ , \qquad (4.1)$$

for functions $f : \mathcal{X} \to \mathcal{A}$ and $g : \mathcal{Y} \to \mathcal{B}$, and where $s_{x,f(x)}, t_{y,g(y)} \in \{-1, +1\}$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Hence, $\varepsilon(G)$ can be computed by iterating over all functions $f : \mathcal{X} \to \mathcal{A}$ and $g : \mathcal{Y} \to \mathcal{B}$ (which are $|\mathcal{A}|^{|\mathcal{X}|}|\mathcal{B}|^{|\mathcal{Y}|}$ possibilities) and all possible signs for $s_{x,f(x)}$ and $t_{y,g(y)}$ (which are $2^{|\mathcal{X}|}2^{|\mathcal{Y}|}$ possibilities). Furthermore, computing the sum in (4.1) takes $O(|\mathcal{X}||\mathcal{Y}|)$ steps. □

Going along similar lines as in the previous proof one can show the following result.

Lemma 4.2. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, the injective tensor norm $\varepsilon(P)$ can be computed by optimizing over*

$$O(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}|) \ ,$$

possible solutions, and where each possibility takes time $O(|\mathcal{A}||\mathcal{B}|)$ to evaluate.

Proof. By using that $\|G_A\|_{1(\infty)} \leq 1$ and $\|G_B\|_{1(\infty)} \leq 1$, we obtain that the injective tensor norm can be computed by

$$\begin{aligned} \varepsilon(P) &= \sup_{G_A, G_B} \left\{ |\langle G_A \otimes G_B, P \rangle| \ : \ \|G_A\|_{1(\infty)} \leq 1 \ , \ \|G_B\|_{1(\infty)} \leq 1 \right\} \\ &= \max_{x,y,s_a,t_b} \left| \sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P \rangle \cdot s_a \cdot t_b \right| \ , \qquad (4.2) \end{aligned}$$

with $s_a, t_b \in \{-1, +1\}$, for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence, $\varepsilon(P)$ can be computed by iterating over all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ (which are $|\mathcal{X}||\mathcal{Y}|$ possibilities) and all possible signs for s_a and t_b (which are $2^{|\mathcal{A}|}2^{|\mathcal{B}|}$ possibilities). Furthermore, computing the sum in (4.2) takes $O(|\mathcal{A}||\mathcal{B}|)$ steps. □

4.3 LP for Projective Tensor Norm

In order to compute the projective tensor norm we will use that it is the dual norm of the injective tensor norm.

Lemma 4.3. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, the projective tensor norm $\pi(P)$ can be computed by an LP of dimension $O(|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|)$ which has*

$$O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|}) .$$

constraints. The running time of the LP is then given by

$$poly(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|} + \log(1/\epsilon)) ,$$

with $\epsilon > 0$ the accuracy of the LP solution.

Proof. Since π is the dual tensor norm of ε (see Section 3.2.2) we have that

$$\begin{aligned} \pi(P) &= \sup_G \{|\langle G, P\rangle| \ : \ \varepsilon(G) \leq 1\} \\ &= \sup_G \left\{ \left| \sum_{x,y} \sum_{a,b} \langle G, e_{x,a} \otimes e_{y,b}\rangle \langle f_{x,a} \otimes f_{y,b}, P\rangle \right| \ : \ \varepsilon(G) \leq 1 \right\} . \end{aligned}$$

Then, by using the proof of Lemma 4.1, we can rewrite the constraint $\varepsilon(G) \leq 1$ as

$$-1 \leq \sum_{x,y} \langle G, e_{x,f(x)} \otimes e_{y,g(y)}\rangle \cdot s_{x,f(x)} \cdot s_{y,g(y)} \leq 1$$

for all functions $f : \mathcal{X} \to \mathcal{A}$ and $g : \mathcal{Y} \to \mathcal{B}$ and all $s_{x,f(x)}, t_{y,g(y)} \in \{-1, +1\}$. Note that these are $O(2^{|\mathcal{X}|}|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{Y}|}|\mathcal{B}|^{|\mathcal{Y}|})$ *linear* constraints in G. Furthermore, the objective function $\langle G, P\rangle$ is linear as well. Then, the running time follows from Lemma 2.6. □

Lemma 4.4. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, the projective tensor norm $\pi(G)$ can be computed by an LP of dimension $O(|\mathcal{X}||\mathcal{A}||\mathcal{Y}||\mathcal{B}|)$ which has*

$$O(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}|) .$$

constraints. The running time of the LP is then given by

$$poly(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}| + \log(1/\epsilon)) ,$$

with $\epsilon > 0$ the accuracy of the LP solution.

Proof. By using the proof of Lemma 4.2 and that π is the dual tensor norm of ε (see Section 3.2.2) implies that computing $\pi(G)$ is a linear program with objective function $|\langle G, P\rangle|$ and constraints

$$-1 \leq \sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P\rangle \cdot s_a \cdot s_b \leq 1$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ and all $s_a, t_b \in \{-1, +1\}$. Note that these are $O(2^{|\mathcal{A}|}|\mathcal{X}|2^{|\mathcal{B}|}|\mathcal{Y}|)$ *linear* constraints in P. Furthermore, the objective function $\langle G, P\rangle$ is linear as well. Then, the running time follows from Lemma 2.6. $\square$

4.4 Technical Preliminaries for (Dual) Hilbertian Tensor Norm

4.4.1 Hilbertian Tensor Norm for Systems and Games

In Section 3.2.2, we have provided a definition for the Hilbertian tensor norm over arbitrary finite-dimensional normed vector spaces X and Y. Let us now analyse this tensor norm for the case where the local normed vector spaces are $X := \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$ and $Y := \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, respectively. By using (3.3) we obtain

$$\gamma_2(G) = \inf_{\hat{G} = W \cdot V} \|W\|_{2 \to 1(\infty)} \cdot \|V\|_{\infty(1) \to 2} ,$$

where the infimum is over all factorizations of $\hat{G}$ into linear operators $W : \ell_2 \to \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and $V : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \to \ell_2$.

Let us introduce some new notation. Given column vectors $v_{i,j} \in \mathbb{R}^n$, with $1 \leq i \leq N$ and $1 \leq j \leq M$, we write $(v_{i,j})$ for the $n \times N \cdot M$-matrix which has the vectors $v_{i,j}$ as columns. The order is such that the first M columns of $(v_{i,j})$ are given by the vectors $v_{1,1}, v_{1,2}, ..., v_{1,M}$. The second M columns are $v_{2,1}, v_{2,2}, ..., v_{2,M}$, and so forth. We write $(v_{i,j})^T$ for the transposed matrix, i.e., the matrix $(v_{i,j})^T$ has the vectors $v_{i,j}^T$ as rows.

Now, we interpret W and V as matrices of dimension $|\mathcal{Y}||\mathcal{B}| \times n$ and $n \times |\mathcal{X}||\mathcal{A}|$, respectively, such that their matrix product yields $\hat{G}$. Representing W as a row matrix $W := (w_{y,b})^T$ and V as a column matrix $V := (v_{x,a})$ yields as entries of $\hat{G} = W \cdot V$ the values $\langle G, e_{x,a} \otimes e_{y,b} \rangle = \langle w_{y,b}, v_{x,a} \rangle = \langle v_{x,a}, w_{y,b} \rangle$. An immediate consequence is the following result:

Lemma 4.5. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. The Hilbertian tensor norm is then computed by*

$$\gamma_2(G) = \inf \|(w_{y,b})^T\|_{2\to 1(\infty)} \cdot \|(v_{x,a})\|_{\infty(1)\to 2} \ ,$$

where the infimum is over vectors $v_{x,a}, w_{y,b} \in \ell_2$ such that $\langle G, e_{x,a} \otimes e_{y,b} \rangle = \langle v_{x,a}, w_{y,b} \rangle$.

A similar argument for the Hilbertian tensor norm defined over the local normed vector spaces $X := \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $Y := \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ yields:

Lemma 4.6. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. The Hilbertian tensor norm is then computed by*

$$\gamma_2(P) = \inf \|(n_{y,b})^T\|_{2\to\infty(1)} \cdot \|(m_{x,a})\|_{1(\infty)\to 2} \ ,$$

where the infimum is over vectors $m_{x,a}, n_{y,b} \in \ell_2$ such that $\langle f_{x,a} \otimes f_{y,b}, P \rangle = \langle m_{x,a}, n_{y,b} \rangle$.

4.4.2 Alternative Expressions for Operator Norms

In the following two sections we prove alternative expressions for the $1(\infty) \to 2$,$2 \to \infty(1)$, $\infty(1) \to 2$ and $2 \to 1(\infty)$ operator norms. These new expressions will then allow us in Section 4.5 and Section 4.6 to show how the γ_2 and γ_2^* tensor norms can be written as semidefinite programs (SDP).

The $1(\infty) \to 2$ and $2 \to \infty(1)$ Operator Norms

In this section we prove new equivalent expressions for the operator norms $\|\cdot\|_{1(\infty)\to 2}$ and $\|\cdot\|_{2\to\infty(1)}$ which we have introduced in the previous section.

Lemma 4.7. *Let $n_{y,b} \in \ell_2$ and $m_{x,a} \in \ell_2$ for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, with $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, and $\mathcal{B}$ finite sets. Then*

$$\|(m_{x,a})\|_{1(\infty)\to 2} = \max_{x\in\mathcal{X}} \max_{s:\mathcal{A}\to\{-1,+1\}} \left\|\sum_{a\in\mathcal{A}} s(a)\cdot m_{x,a}\right\|_2 ,$$

and

$$\|(n_{y,b})^T\|_{2\to\infty(1)} = \max_{y\in\mathcal{Y}} \max_{t:\mathcal{B}\to\{-1,+1\}} \left\|\sum_{b\in\mathcal{B}} t(b)\cdot n_{y,b}\right\|_2 .$$

In particular, $\|(m_{x,a})\|^2_{1(\infty)\to 2} \geq \sum_{a\in\mathcal{A}}\|m_{x,a}\|_2^2$ and $\|(n_{y,b})^T\|^2_{2\to\infty(1)} \geq \sum_{b\in\mathcal{B}}\|n_{y,b}\|_2^2$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively.

Proof. By using the definition of the $\|\cdot\|_{1(\infty)\to 2}$-norm, we obtain

$$\|(m_{x,a})\|_{1(\infty)\to 2} = \sup_{\|G\|_{1(\infty)}\leq 1} \left\|\sum_{x\in\mathcal{X}}\sum_{a\in\mathcal{A}} \langle G, e_x\otimes e_a\rangle \cdot m_{x,a}\right\|_2 .$$

Since every G with $\|G\|_{1(\infty)} \leq 1$ can be written as $\langle G, e_x \otimes e_a\rangle = \kappa_x \cdot \mu_{x,a}$, with $\sum_{x\in\mathcal{X}}|\kappa_x| \leq 1$ and $\mu_{x,a} \in [-1,1]$, we get

$$\begin{aligned}
\|(m_{x,a})\|_{1(\infty)\to 2} \quad &\leq \quad \sup_{\|G\|_{1(\infty)}\leq 1} \sum_{x\in\mathcal{X}} |\kappa_x| \cdot \left\|\sum_{a\in\mathcal{A}} \mu_{x,a}\cdot m_{x,a}\right\|_2 \\
&\leq \quad \sup_{\|G\|_{1(\infty)}\leq 1} \max_{x\in\mathcal{X}} \left\|\sum_{a\in\mathcal{A}} \mu_{x,a}\cdot m_{x,a}\right\|_2 \\
&= \quad \sup_{\|\mu\|_\infty\leq 1} \max_{x\in\mathcal{X}} \left\|\sum_{a\in\mathcal{A}} \langle \mu, e_a\rangle\cdot m_{x,a}\right\|_2 , \qquad (4.3)
\end{aligned}$$

where we used the triangle inequality in the first line. That the expression $\|(m_{x,a})\|_{1(\infty)\to 2}$ is greater or equal than the upper bound of (4.3) is obvious, by setting $\kappa_x = 1$ for the optimal x and, hence, we have equality. That the optimal vector $\mu \in \ell_\infty^{|\mathcal{A}|}$ in (4.3) can be chosen to consist only of entries $+1$ and -1 follows from the convexity of norms. That $\|(n_{y,b})^T\|_{2\to\infty(1)} = \max_{s,y}\|\sum_{b\in\mathcal{B}} t(b)\cdot n_{y,b}\|_2$ holds as well follows from Lemma 2.2.

Let us now show that $\|(m_{x,a})\|^2_{1(\infty)\to 2} \geq \sum_{a\in\mathcal{A}}\|m_{x,a}\|_2^2$. By using the above result we obtain

$$\begin{aligned}\|(m_{x,a})\|^2_{1(\infty)\to 2} &\geq \left\langle \sum_{a_1\in\mathcal{A}} s(a_1)\cdot m_{x,a_1}, \sum_{a_2\in\mathcal{A}} s(a_2)\cdot m_{x,a_2}\right\rangle \\ &= \sum_{a\in\mathcal{A}}\langle m_{x,a}, m_{x,a}\rangle + \sum_{a_1\neq a_2} s(a_1)s(a_2)\langle m_{x,a_1}, m_{x,a_2}\rangle\ ,\end{aligned}$$

for all $x \in \mathcal{X}$ and all $s : \mathcal{A} \to \{-1,+1\}$. If we show that there exists a function $s : \mathcal{A} \to \{-1,+1\}$ such that $\sum_{a_1\neq a_2} s(a_1)\cdot s(a_2)\cdot\langle m_{x,a_1}, m_{x,a_2}\rangle \geq 0$, we conclude that

$$\|(m_{x,a})\|^2_{1(\infty)\to 2} \geq \sum_{a\in\mathcal{A}}\langle m_{x,a}, m_{x,a}\rangle = \sum_{a\in\mathcal{A}}\|m_{x,a}\|_2^2\ .$$

We will now construct a function with this property. First, we write

$$\begin{aligned}&\sum_{a_1\neq a_2} s(a_1)\cdot s(a_2)\cdot\langle m_{x,a_1}, m_{x,a_2}\rangle \\ =\ & 2\cdot\sum_{a_1=2}^{|\mathcal{A}|} s(a_1)\cdot\left(\sum_{a_2=1}^{a_1-1} s(a_2)\cdot\langle m_{x,a_1}, m_{x,a_2}\rangle\right)\ . \qquad (4.4)\end{aligned}$$

For $a = 1$ we set $s(1) := 1$. We then set the value for $s(2)$ which will depend on $s(1)$. Then we set $s(3)$ which will depend on $s(1)$ and $s(2)$. Hence, the value for $s(a_1)$ will depend on all $s(1), s(2), ..., s(a_1-1)$. In particular, we define $s(a_1)$ to be

$$s(a_1) := sign\left(\sum_{a_2=1}^{a_1-1} s(a_2)\cdot\langle m_{x,a_1}, m_{x,a_2}\rangle\right)\ .$$

By defining the function s in this way, the right-hand side of (4.4) is always non-negative which is what we wanted to prove. By the same reasoning, one can show that $\|(n_{y,b})^T\|^2_{2\to\infty(1)} \geq \sum_{b\in\mathcal{B}}\|n_{y,b}\|_2^2$ holds as well. □

Note that, for $|\mathcal{A}| = |\mathcal{B}| = 1$, we have that $\|U\|_{2\to\infty}$ is the largest 2-norm of a row of U and $\|V\|_{1\to 2}$ is the largest 2-norm of a column of V.

The $\infty(1)\to 2$ and $2\to 1(\infty)$ Operator Norms

Let us now provide alternative expressions for the $\infty(1)\to 2$ and $2\to 1(\infty)$ operator norms.

Lemma 4.8. *Let $v_{x,a} \in \ell_2$ and $w_{y,b} \in \ell_2$ for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$, with $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, and $\mathcal{B}$ finite sets. Then*

$$\|(v_{x,a})\|_{\infty(1)\to 2} = \max_{f:\mathcal{X}\to\mathcal{A}} \max_{s:\mathcal{X}\to\{-1,+1\}} \left\| \sum_{x\in\mathcal{X}} s(x)\cdot v_{x,f(x)} \right\|_2 ,$$

and

$$\|(w_{y,b})^T\|_{2\to 1(\infty)} = \max_{g:\mathcal{Y}\to\mathcal{B}} \max_{t:\mathcal{Y}\to\{-1,+1\}} \left\| \sum_{y\in\mathcal{Y}} t(y)\cdot w_{y,g(y)} \right\|_2 .$$

In particular, $\|(v_{x,a})\|^2_{\infty(1)\to 2} \geq \sum_{x\in\mathcal{X}} \|v_{x,f(x)}\|_2^2$ and $\|(w_{y,b})^T\|^2_{2\to 1(\infty)} \geq \sum_{y\in\mathcal{Y}} \|w_{y,g(y)}\|_2^2$ for all $f : \mathcal{X} \to \mathcal{A}$ and all $g : \mathcal{Y} \to \mathcal{B}$, respectively.

Proof. By the definition of the $\infty(1) \to 2$ operator norm, we get

$$\begin{aligned}
\|(v_{x,a})\|_{\infty(1)\to 2} &= \sup_{\|P\|_{\infty(1)}\leq 1} \|(v_{x,a})(P)\|_2 \\
&= \sup_{\|P\|_{\infty(1)}\leq 1} \left\| \sum_{x\in\mathcal{X}} \sum_{a\in\mathcal{A}} \langle f_x \otimes f_a, P\rangle \cdot v_{x,a} \right\|_2 .
\end{aligned}$$

Note that $\|P\|_{\infty(1)} \leq 1$ implies $\sum_{a\in\mathcal{A}} |\langle f_x \otimes f_a, P\rangle| \leq 1$ for all $x \in \mathcal{X}$. Showing that there exist functions $s : \mathcal{X} \to \{-1,+1\}$ and $f : \mathcal{X} \to \mathcal{A}$ such that

$$\left\| z + \sum_a \langle f_x \otimes f_a, P\rangle \cdot v_{x,a} \right\|_2 \leq \left\| z + s(x)\cdot v_{x,f(x)} \right\|_2 ,$$

for any $z \in \ell_2^n$ implies the first part of the lemma. Let $P_{x,a} := \langle f_x \otimes f_a, P\rangle$. Using that ℓ_2^n is self-dual yields

$$\begin{aligned}
\left\| z + \sum_a P_{x,a}\cdot v_{x,a} \right\|_2 &= \sup_{\|\lambda\|_2\leq 1} \left| \langle \lambda, z\rangle + \sum_a P_{x,a}\cdot \langle \lambda, v_{x,a}\rangle \right| \\
&\leq \sup_{\|\lambda\|_2\leq 1} |\langle \lambda, z\rangle| + \sum_a |P_{x,a}|\cdot |\langle \lambda, v_{x,a}\rangle| \\
&\leq \sup_{\|\lambda\|_2\leq 1} |\langle \lambda, z\rangle| + |\langle \lambda, v_{x,f(x)}\rangle| \\
&= \sup_{\|\lambda\|_2\leq 1} |\langle \lambda, z\rangle + s(x)\cdot \langle \lambda, v_{x,f(x)}\rangle| \\
&= \sup_{\|\lambda\|_2\leq 1} |\langle \lambda, z + s(x)\cdot v_{x,f(x)}\rangle| \\
&= \|z + s(x)\cdot v_{x,f(x)}\|_2 ,
\end{aligned}$$

where we used the triangle inequality in the second line, the fact that $\sum_{a\in\mathcal{A}} |P_{x,a}| \le 1$ in the third line and that $s(x)$ is such that $s(x)\cdot v_{x,f(x)}$ and $\langle\lambda, z\rangle$ have the same sign. Hence, we get that $\|(v_{x,a})\|_{\infty(1)\to 2} = \max_{s,f} \|\sum_{x\in\mathcal{X}} s(x)\cdot v_{x,f(x)}\|_2$. By Lemma 2.2 we get $\|(w_{y,b})^T\|_{2\to 1(\infty)} = \max_{t,g} \|\sum_{y\in\mathcal{Y}} t(y)\cdot w_{y,g(y)}\|_2$ as well.

Let us now show that $\|(v_{x,a})\|^2_{\infty(1)\to 2} \ge \sum_{x\in\mathcal{X}} \|v_{x,f(x)}\|_2^2$. Employing the fact that $\|(v_{x,a})\|_{\infty(1)\to 2} = \max_{s,f} \|\sum_{x\in\mathcal{X}} s(x)\cdot v_{x,f(x)}\|_2$, yields

$$\begin{aligned}
\|(v_{x,a})\|^2_{\infty(1)\to 2} &\ge \left\langle \sum_{x_1\in\mathcal{X}} s(x_1)\cdot v_{x_1,f(x_1)}, \sum_{x_2\in\mathcal{X}} s(x_2)\cdot v_{x_2,f(x_2)} \right\rangle \\
&= \sum_{x\in\mathcal{X}} \langle v_{x,f(x)}, v_{x,f(x)}\rangle \\
&+ \sum_{x_1\neq x_2} s(x_1)\cdot s(x_2)\cdot \langle v_{x_1,f(x_1)}, v_{x_2,f(x_2)}\rangle ,
\end{aligned}$$

for all $s : \mathcal{X} \to \{-1,+1\}$ and $f : \mathcal{X} \to \mathcal{A}$. If there exists a function $s : \mathcal{X} \to \{-1,+1\}$ such that

$$\sum_{x_1\neq x_2} s(x_1)\cdot s(x_2)\cdot \langle v_{x_1,f(x_1)}, v_{x_2,f(x_2)}\rangle \ge 0 ,$$

we conclude that

$$\|(v_{x,a})\|^2_{\infty(1)\to 2} \ge \sum_{x\in\mathcal{X}} \langle v_{x,f(x)}, v_{x,f(x)}\rangle = \sum_{x\in\mathcal{X}} \|v_{x,f(x)}\|_2^2 .$$

So let us construct a function with this property. First, we write

$$\begin{aligned}
&\sum_{x_1\neq x_2} s(x_1)\cdot s(x_2)\cdot \langle v_{x_1,f(x_1)}, v_{x_2,f(x_2)}\rangle \\
= \quad & 2\cdot \sum_{x_1=2}^{|\mathcal{X}|} s(x_1)\cdot \left(\sum_{x_2=1}^{x_1-1} s(x_2)\cdot \langle v_{x_1,f(x_1)}, v_{x_2,f(x_2)}\rangle \right) . \qquad (4.5)
\end{aligned}$$

For $x_1 = 1$ we set $s(1) := 1$. We then set the value for $s(2)$ which will depend on $s(1)$. Then we set $s(3)$ which will depend on $s(1)$ and $s(2)$. Hence, the value for $s(x_1)$ will depend on all $s(1), s(2), ..., s(x_1-1)$. In particular, we define $s(x_1)$ to be

$$s(x_1) := sign\left(\sum_{x_2=1}^{x_1-1} s(x_2)\cdot \langle v_{x_1,f(x_1)}, v_{x_2,f(x_2)}\rangle \right) .$$

By defining the function s in this way, the right hand side of equation (4.5) is always non-negative which is what we wanted to prove. That $\|(w_{y,b})^T\|^2_{2\to 1(\infty)} \geq \sum_{y\in\mathcal{Y}} \|w_{y,g(y)}\|_2^2$ follows by an analogous argument. ☐

4.5 SDP for Hilbertian Tensor Norm

We will first analyse the Hilbertian tensor norm for the case of two-prover games.

Theorem 4.1. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, computing the Hilbertian tensor norm $\gamma_2(G)$ is equivalent to solving the following SDP:*

$$\gamma_2(G) = \inf z \ ,$$

such that

$$\begin{aligned} v_{x,a}, w_{y,b} &\in \ell_2 \ , \\ \langle G, e_{x,a} \otimes e_{y,b}\rangle &= \langle v_{x,a}, w_{y,b}\rangle \ , \\ \left\| \sum_{x\in\mathcal{X}} s(x)\cdot v_{x,f(x)} \right\|_2^2 &\leq z \ , \textit{for all } s : \mathcal{X} \to \{-1,+1\} \textit{ and } f : \mathcal{X} \to \mathcal{A} \ , \\ \left\| \sum_{y\in\mathcal{Y}} t(y)\cdot w_{y,g(y)} \right\|_2^2 &\leq z \ , \textit{for all } t : \mathcal{Y} \to \{-1,+1\} \textit{ and } g : \mathcal{Y} \to \mathcal{B} \ . \end{aligned}$$

Proof. By Lemma 4.5 the Hilbertian tensor norm γ_2 on $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})\otimes\ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ can be written as

$$\gamma_2(G) = \inf \|(w_{y,b})^T\|_{2\to 1(\infty)} \cdot \|(v_{x,a})\|_{\infty(1)\to 2} \ ,$$

with $\langle G, e_{x,a}\otimes e_{y,b}\rangle = \langle v_{x,a}, w_{y,b}\rangle$ and $v_{x,a}, w_{y,b} \in \ell_2$. Hence, the constraints of the above optimization problem together with Lemma 4.8 imply $\|(v_{x,a})\|^2_{\infty(1)\to 2} \leq z$ and $\|(w_{y,b})^T\|^2_{2\to 1(\infty)} \leq z$ for all $v_{x,a}, w_{y,b} \in \ell_2$ with $\langle G, e_{x,a}\otimes e_{y,b}\rangle = \langle v_{x,a}, w_{y,b}\rangle$. Since we minimize over z and we can assume without loss of generality that $\|(v_{x,a})\|^2_{\infty(1)\to 2} = \|(w_{y,b})^T\|^2_{2\to 1(\infty)}$ we can infer that $\inf z = \inf_{\hat{G}=(w_{y,b})^T\cdot(v_{x,a})} \|(v_{x,a})\|_{\infty(1)\to 2}\|(w_{y,b})^T\|_{2\to 1(\infty)}$, and hence, $\gamma_2(G) = \inf z$.

What remains to be shown is that the above optimization problem is indeed an SDP. In order to achieve this we show that the above program is equivalent to the standard SDP given in (2.11). First, let $A :=$

$diag(0, 0, ..., 0, 1)$ be an $(|\mathcal{X}||\mathcal{A}|+|\mathcal{Y}||\mathcal{B}|+1) \times (|\mathcal{X}||\mathcal{A}|+|\mathcal{Y}||\mathcal{B}|+1)$-diagonal matrix. And second, let us define the $n \times m$-matrix $M^{(\bar{v},\bar{w})}$ as

$$\langle f_i, M^{(\bar{v},\bar{w})}(e_j)\rangle := \langle v_i, w_j\rangle \ ,$$

with $\bar{v} := (v_1, ..., v_n)$ and $\bar{w} := (w_1, ..., w_m)$. In particular, we define $\bar{v} := (v_{1,1}, v_{1,2}, ..., v_{x,a}, ..., v_{|\mathcal{X}|,|\mathcal{A}|})$ and $\bar{w} := (w_{1,1}, w_{1,2}, ..., w_{y,b}, ..., w_{|\mathcal{Y}|,|\mathcal{B}|})$, with $v_{x,a}, w_{y,b} \in \ell_2$, and, hence,

$$\langle f_x \otimes f_a, M^{(\bar{v},\bar{w})}(e_y \otimes e_b)\rangle = \langle v_{x,a}, w_{y,b}\rangle \ .$$

Furthermore, we define $\bar{u} := (u)$ for some $u \in \ell_2$. Now, let us define the following $(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}| + 1) \times (|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}| + 1)$ block matrix:

$$M := \begin{pmatrix} M^{(\bar{v},\bar{v})} & M^{(\bar{v},\bar{w})} & M^{(\bar{v},\bar{u})} \\ M^{(\bar{w},\bar{v})} & M^{(\bar{w},\bar{w})} & M^{(\bar{w},\bar{u})} \\ M^{(\bar{u},\bar{v})} & M^{(\bar{u},\bar{w})} & M^{(\bar{u},\bar{u})} \end{pmatrix} \ .$$

Note that M is symmetric since $(M^{(\bar{w},\bar{v})})^T = M^{(\bar{v},\bar{w})}$ and that $M^{(\bar{u},\bar{u})} = \langle u, u\rangle$. Then, since $\langle f_i, M(e_j)\rangle = \langle v_i, v_j\rangle$ for some vectors $v_i, v_j \in \ell_2$, we can conclude according to Lemma 2.5 that M is positive-semidefinite.

We will interpret the parameter z as $\langle u, u\rangle$ since we have $\mathrm{tr}(A \cdot M) = \langle u, u\rangle$. This is our objective function (see also (2.11)) where we minimize over $M \succeq 0$. Hence, we have brought the objective function into standard form. Let us now do the same for the constraints. The first constraint in the above program can be written as

$$\langle f_x \otimes f_a, M^{(\bar{v},\bar{w})}(e_y \otimes e_b)\rangle = \langle G, e_{x,a} \otimes e_{y,b}\rangle \ .$$

These are of course linear constraints and, hence, there exist matrices $C_{x,a,y,b} \in \mathbb{S}^{|\mathcal{X}||\mathcal{A}|+|\mathcal{Y}||\mathcal{B}|+1}$ such that $\mathrm{tr}(C_{x,a,y,b} \cdot M) = \langle G, e_{x,a} \otimes e_{y,b}\rangle$ for all $x \in \mathcal{X}, y \in \mathcal{Y}, a \in \mathcal{A}, b \in \mathcal{B}$. Furthermore, since $\left\|\sum_{x\in\mathcal{X}} s(x) \cdot v_{x,f(x)}\right\|_2^2 = \sum_{x,\tilde{x}} s(x)s(\tilde{x})\langle v_{x,f(x)}, v_{\tilde{x},f(\tilde{x})}\rangle$ we can write the next constraints as

$$\sum_{x,\tilde{x}} s(x)s(\tilde{x})\langle f_x \otimes f_{f(x)}, M^{(\bar{v},\bar{v})}(e_{\tilde{x}} \otimes e_{f(\tilde{x})})\rangle - \langle u, u\rangle \leq 0 \ ,$$

for all $s : \mathcal{X} \to \{-1, +1\}$ and $f : \mathcal{X} \to \mathcal{A}$. Again, these are linear constraints and, hence, there exist matrices $C_{s,f}$ such that $\mathrm{tr}(C_{s,f} \cdot M) \leq 0$ for all $s : \mathcal{X} \to \{-1, +1\}$ and $f : \mathcal{X} \to \mathcal{A}$. For the last constraints, we similarly get matrices $C_{t,g}$ such that $\mathrm{tr}(C_{t,g} \cdot M) \leq 0$ for all $t : \mathcal{Y} \to \{-1, +1\}$ and $g : \mathcal{Y} \to \mathcal{B}$. Finally, we can transform the inequality constraints into equality constraints by adding additional variables (see the argument in Section 2.7). □

Based on the proof of Theorem 4.1 we obtain by a straightforward counting argument and the use of Lemma 2.6 the following result:

Lemma 4.9. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, the dimension of the SDP in Theorem 4.1 is*

$$O(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$$

and the number of constraints is

$$O(|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{X}|} + |\mathcal{B}|^{|\mathcal{Y}|}2^{|\mathcal{Y}|}) \; .$$

The running time of the SDP in Theorem 4.1 is then given by

$$poly(|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{X}|} + |\mathcal{B}|^{|\mathcal{Y}|}2^{|\mathcal{Y}|} + \log(1/\epsilon)) \; ,$$

with $\epsilon > 0$ the accuracy of the SDP solution.

Let us now analyse the case where the underlying vector spaces represent systems. We do not provide a proof since it would be almost identical to the proof of Theorem 4.1, the only difference is that we use Lemma 4.6 and Lemma 4.7 instead of Lemma 4.5 and Lemma 4.8.

Theorem 4.2. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, computing the Hilbertian tensor norm $\gamma_2(P)$ is equivalent to solving the following SDP:*

$$\gamma_2(P) = \inf z \; ,$$

such that

$$\begin{array}{rcl} m_{x,a}, n_{y,b} & \in & \ell_2 \; , \\ \langle f_{x,a} \otimes f_{y,b}, P \rangle & = & \langle m_{x,a}, n_{y,b} \rangle \; , \\ \left\| \sum_{a \in \mathcal{A}} s(a) \cdot m_{x,a} \right\|_2^2 & \leq & z \; , \textit{for all } \; s : \mathcal{A} \to \{-1, +1\} \textit{ and } x \in \mathcal{X} \; , \\ \left\| \sum_{b \in \mathcal{B}} t(b) \cdot n_{y,b} \right\|_2^2 & \leq & z \; , \textit{for all } \; t : \mathcal{B} \to \{-1, +1\} \textit{ and } y \in \mathcal{Y} \; . \end{array}$$

Lemma 4.10. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, the dimension of the SDP in Theorem 4.2 is*

$$O(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$$

and the number of constraints is

$$O(|\mathcal{X}|2^{|\mathcal{A}|} + |\mathcal{Y}|2^{|\mathcal{B}|}) \ .$$

The running time of the SDP in Theorem 4.2 is then given by

$$poly(|\mathcal{X}|2^{|\mathcal{A}|} + |\mathcal{Y}|2^{|\mathcal{B}|} + \log(1/\epsilon)) \ ,$$

with $\epsilon > 0$ *the accuracy of the SDP solution.*

Note that the running time for constant *output alphabet sizes* $|\mathcal{A}|$ and $|\mathcal{B}|$ of the SDP given in Theorem 4.2 is polynomial in the input alphabet sizes. This is in contrast to the running time of the SDP given in Theorem 4.1 which is exponential, even for fixed output alphabet sizes. However, if we fix the *input alphabet sizes* $|\mathcal{X}|$ and $|\mathcal{Y}|$ the situation is opposite, namely the SDP in Theorem 4.1 is polynomial and the SDP in Theorem 4.2 is exponential.

4.6 SDP for Dual Hilbertian Tensor Norm

Theorem 4.3. *Let* $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. *Then, computing the dual Hilbertian tensor norm* $\gamma_2^*(P)$ *is equivalent to solving the following SDP:*

$$\gamma_2^*(P) = \sup_{\{v_{x,a}\},\{w_{y,b}\}} \left| \sum_{x,y} \sum_{a,b} \langle v_{x,a}, w_{y,b} \rangle \cdot \langle f_{x,a} \otimes f_{y,b}, P \rangle \right| \ ,$$

such that

$$\begin{aligned} v_{x,a}, w_{y,b} \quad &\in \quad \ell_2^n \ , \\ \left\| \sum_{x \in \mathcal{X}} s(x) \cdot v_{x,f(x)} \right\|_2^2 \quad &\leq \quad 1 \ , \textit{ for all } \ s : \mathcal{X} \to \{-1,+1\} \textit{ and } f : \mathcal{X} \to \mathcal{A} \ , \\ \left\| \sum_{y \in \mathcal{Y}} t(y) \cdot w_{y,g(y)} \right\|_2^2 \quad &\leq \quad 1 \ , \textit{ for all } \ t : \mathcal{Y} \to \{-1,+1\} \textit{ and } g : \mathcal{Y} \to \mathcal{B} \ . \end{aligned}$$

Proof. By norm duality (see also (2.1)) we have

$$\gamma_2^*(P) = \sup_G \{ |\langle G, P \rangle| \ : \ \gamma_2(G) \leq 1 \} \ , \tag{4.6}$$

with $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. By Lemma 4.5 the Hilbertian tensor norm γ_2 can be written as

$$\gamma_2(G) = \inf \|(w_{y,b})^T\|_{2\to 1(\infty)} \cdot \|(v_{x,a})\|_{\infty(1)\to 2} ,$$

with $\langle G, e_{x,a} \otimes e_{y,b}\rangle = \langle v_{x,a}, w_{y,b}\rangle$. Lemma 4.8 provides an alternative representation of the operator norms $\|\cdot\|_{2\to 1(\infty)}$ and $\|\cdot\|_{\infty(1)\to 2}$. Therefore, since $\langle G, P\rangle = \sum_{x,y,a,b}\langle G, e_{x,a} \otimes e_{y,b}\rangle\langle f_{x,a} \otimes f_{y,b}, P\rangle$ Lemma 4.8 together with (4.6) imply the result.

What remains to be shown is that the above optimization problem is indeed an SDP. In order to achieve this we show that the above program is equivalent to the standard SDP given in (2.12). By using the same notation as in the proof of Theorem 4.1, we define the following $(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|) \times (|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$ block matrix

$$M := \begin{pmatrix} M^{(\bar{v},\bar{v})} & M^{(\bar{v},\bar{w})} \\ M^{(\bar{w},\bar{v})} & M^{(\bar{w},\bar{w})} \end{pmatrix} ,$$

with $\bar{v} := (v_{1,1}, ..., v_{x,a}, ..., v_{|\mathcal{X}|,|\mathcal{A}|})$ and $\bar{w} := (w_{1,1}, ..., w_{y,b}, ..., w_{|\mathcal{Y}|,|\mathcal{B}|})$, with vectors $v_{x,a}, w_{y,b} \in \ell_2$. Note that M is symmetric since $(M^{(\bar{w},\bar{v})})^T = M^{(\bar{v},\bar{w})}$. Then, since $\langle f_i, M(e_j)\rangle = \langle v_i, v_j\rangle$ for some vectors $v_i, v_j \in \ell_2$, we can conclude according to Lemma 2.5 that M is positive-semidefinite. Furthermore, let us define the symmetric $(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|) \times (|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$-block matrix A as

$$A := \frac{1}{2}\begin{pmatrix} 0 & \hat{P} \\ \hat{P}^T & 0 \end{pmatrix} ,$$

where $\hat{P}$ is a $|\mathcal{X}||\mathcal{A}| \times |\mathcal{Y}||\mathcal{B}|$-matrix such that $\langle f_{x,a}, \hat{P}(e_{y,b})\rangle := \langle f_{x,a} \otimes f_{y,b}, P\rangle$. Hence, we have that

$$\mathrm{tr}(A \cdot M) = \mathrm{tr}(\hat{P} \cdot M^{(\bar{v},\bar{w})}) = \sum_{x,y,a,b} \langle v_{x,a}, w_{y,b}\rangle \cdot \langle f_{x,a} \otimes f_{y,b}, P\rangle ,$$

which corresponds to the objective function of the above program.

Since the constraints are linear (see also the proof of Theorem 4.1) there exists matrices $C_{s,f}$ and $C_{t,g}$ such that $\mathrm{tr}(C_{s,f} \cdot M) \leq 1$ and $\mathrm{tr}(C_{t,g} \cdot M) \leq 1$ for all $s : \mathcal{X} \to \{-1, +1\}$, $f : \mathcal{X} \to \mathcal{A}$, $t : \mathcal{Y} \to \{-1, +1\}$ and $g : \mathcal{Y} \to \mathcal{B}$. ☐

Based on the proof of Theorem 4.3 we obtain by a straightforward counting argument and the use of Lemma 2.6 the following result:

Lemma 4.11. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, the dimension of the SDP in Theorem 4.3 is*

$$O(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$$

and the number of constraints is

$$O(|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{X}|} + |\mathcal{B}|^{|\mathcal{Y}|}2^{|\mathcal{Y}|}) \ .$$

The running time of the SDP in Theorem 4.3 is then given by

$$poly(|\mathcal{A}|^{|\mathcal{X}|}2^{|\mathcal{X}|} + |\mathcal{B}|^{|\mathcal{Y}|}2^{|\mathcal{Y}|} + \log(1/\epsilon)) \ ,$$

with $\epsilon > 0$ the accuracy of the SDP solution.

Let us now analyse the case where the underlying vector spaces represent games. We do not provide a proof since it would be almost identical to the proof of Theorem 4.3: the only difference is that we use Lemma 4.6 and Lemma 4.7 instead of Lemma 4.5 and Lemma 4.8.

Theorem 4.4. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, computing the dual Hilbertian tensor norm $\gamma_2^*(G)$ is equivalent to solving the following SDP:*

$$\gamma_2^*(G) = \sup_{\{m_{x,a}\},\{n_{y,b}\}} \left| \sum_{x,y} \sum_{a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \cdot \langle m_{x,a}, n_{y,b} \rangle \right| \ ,$$

such that

$$\begin{aligned} m_{x,a}, n_{y,b} \quad &\in \quad \ell_2 \ , \\ \left\| \sum_{a \in \mathcal{A}} s(a) \cdot m_{x,a} \right\|_2^2 \quad &\leq \quad 1 \ , \textit{ for all } \ s : \mathcal{A} \to \{-1, +1\} \textit{ and } x \in \mathcal{X} \ , \\ \left\| \sum_{b \in \mathcal{B}} t(b) \cdot n_{y,b} \right\|_2^2 \quad &\leq \quad 1 \ , \textit{ for all } \ t : \mathcal{B} \to \{-1, +1\} \textit{ and } y \in \mathcal{Y} \ . \end{aligned}$$

Lemma 4.12. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. Then, the dimension of the SDP in Theorem 4.4 is*

$$O(|\mathcal{X}||\mathcal{A}| + |\mathcal{Y}||\mathcal{B}|)$$

and the number of constraints is

$$O(|\mathcal{X}|2^{|\mathcal{A}|} + |\mathcal{Y}|2^{|\mathcal{B}|}) \ .$$

The running time of the SDP in Theorem 4.4 is then given by

$$poly(|\mathcal{X}|2^{|\mathcal{A}|} + |\mathcal{Y}|2^{|\mathcal{B}|} + \log(1/\epsilon)) \ ,$$

with $\epsilon > 0$ *the accuracy of the SDP solution.*

Similarly as in the case of the SDPs for the Hilbertian tensor norm we have that for fixed *input alphabet sizes* the SDP in Theorem 4.3 can be solved in polynomial time and the SDP in Theorem 4.11 only in exponential time. Conversely, for fixed *output alphabet sizes* the SDP in Theorem 4.11 can be solve in polynomial time and the SDP in Theorem 4.3 only in exponential time.

Chapter 5

Sets of Bipartite Systems Induced by Tensor Norms

In this chapter, we establish a first explicit connection between tensor norms and quantum theory. However, the results presented in this chapter are mainly preparatory work for Chapter 6. The goal of this chapter is two-fold. First, by evaluating different tensor norms on bipartite systems we want to obtain a deeper understanding of them, from a purely mathematical point of view. Second, we want to derive alternative characterizations of the set of local and quantum systems by means of tensor norms.

Recall that a tensor norm α induces the following convex set of bipartite systems (see Section 3.1.1):

$$\mathcal{R}_m^\alpha := \{P \,:\, \alpha(P) \leq 1 \wedge P \in \mathcal{S}_m\} \,.$$

Then, we have shown in Section 3.5 that any convex set of bipartite systems, $\mathcal{C}_m$, which obeys $\mathcal{L}_m \subseteq \mathcal{C}_m \subseteq \mathcal{S}_m$, is induced by some cross norm. In the first part of this chapter (i.e., Section 5.1) we investigate the converse problem. In particular, we compute the convex sets of bipartite systems which are induced by the projective tensor norm π, the injective tensor norm ε, the Hilbertian tensor norm γ_2, and the dual Hilbertian tensor norm γ_2^*.

In the second part (i.e., Section 5.2), we investigate convex sets of bipartite systems induced by tensor norms that satisfy an additional property: we want the sets $\mathcal{R}_m^\alpha$ to be *closed under wirings*. In other words, we demand that any local processing of bipartite systems from the set $\mathcal{R}_m^\alpha$ yields again a bipartite system from the set $\mathcal{R}_m^\alpha$.

5.1 Convex Sets of Bipartite Systems

5.1.1 Introduction

In this section we will analyse the tensor norms π, ε, γ_2, and γ_2^* and calculate for which sets of bipartite systems they have norm smaller or equal to one and, hence, compute the sets $\mathcal{R}_m^\pi$, $\mathcal{R}_m^\varepsilon$, $\mathcal{R}_m^{\gamma_2}$, and $\mathcal{R}_m^{\gamma_2^*}$.

Contribution

We prove relations between tensor norms and the set of local systems $\mathcal{L}_m$, the set of quantum systems $\mathcal{Q}_m$, and the set of all bipartite systems $\mathcal{S}_m$. The results of this first part of this chapter are summarized in Table 5.1.

	$\ell_\infty^{\lvert\mathcal{X}\rvert} \otimes \ell_\infty^{\lvert\mathcal{Y}\rvert}$	$\ell_\infty^{\lvert\mathcal{X}\rvert}(\ell_1^{\lvert\mathcal{A}\rvert}) \otimes \ell_\infty^{\lvert\mathcal{Y}\rvert}(\ell_1^{\lvert\mathcal{B}\rvert})$
π	$= \mathcal{L}_{\text{co}}$ (5.1.3)	$= \mathcal{L}_m$ (5.1.3)
ε	$= \mathcal{S}_{\text{co}}$ (5.1.4)	$= \mathcal{S}_m$ (5.1.4)
γ_2	$= \mathcal{Q}_{\text{co}}$ [Tsi87, Tsi93] (5.1.5)	$\supseteq \mathcal{Q}_m$ (5.1.5)
γ_2^*	$\subseteq \mathcal{Q}_{\text{co}}$ and $\supseteq K_G \cdot \mathcal{Q}_{\text{co}}$ (5.1.6)	$= \mathcal{Q}_{\text{iso}}$ (5.1.6)

Table 5.1: Relations of tensor norms to convex sets of bipartite systems.

For example, by the entry "$= \mathcal{L}_m$", we mean that the projective tensor norm π induces the set of local systems, i.e., $\mathcal{R}_m^\pi = \mathcal{L}_m$. The entry "$\supseteq K_G \cdot \mathcal{Q}_{\text{co}}$" denotes the relation $\mathcal{Q}_{\text{co}} \subseteq K_G^{-1} \cdot \mathcal{R}_{\text{co}}^{\gamma_2^*}$, where $P \in K_G^{-1} \cdot \mathcal{R}_{\text{co}}^{\gamma_2^*}$ means that P is a bipartite correlation system with $K_G^{-1} \cdot \gamma_2^*(P) \leq 1$, where $1.68 \lesssim K_G \lesssim 1.78$ is the Grothendieck constant (see also Section 6.2).

Furthermore, we also prove that the dual Hilbertian tensor norm γ_2^* is not comparable to the set of quantum systems, i.e., neither is $\mathcal{R}_m^{\gamma_2^*} \subseteq \mathcal{Q}_m$ nor $\mathcal{Q}_m \subseteq \mathcal{R}_m^{\gamma_2^*}$ for $m \geq 3$.

Related Work

Rudolph has shown [Rud00] that the projective tensor norm defined over *Hilbert spaces* induces the set of all separable density operators. Tsirelson has proved [Tsi87, Tsi93] that the Hilbertian tensor norm γ_2 induces the set of all quantum correlation systems, i.e., $\mathcal{R}_{\text{co}}^{\gamma_2} = \mathcal{Q}_{\text{co}}$.

Open Problems

In Section 5.1.6, we conjecture that for every *binary quantum system* the dual Hilbertian tensor norm evaluates to one, i.e., $\mathcal{Q}_{\text{CHSH}} \subseteq \mathcal{R}_{\text{CHSH}}^{\gamma_2^*}$. Although we have numerical evidence that support this conjecture we do not have a proof. More generally, a better understanding of the connection between the Hilbertian and dual Hilbertian tensor norms and the set of quantum systems is desirable.

5.1.2 Relation Between Tensor Norms and Systems

Let us first prove a property every tensor norm over the space of bipartite systems has.

Lemma 5.1. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be a bipartite system. Then*

$$\alpha(P) \geq 1 \ ,$$

for all tensor norms α over $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

Proof. By the definition of the injective tensor norm given in Section 3.2.2 we have

$$\begin{aligned} \varepsilon(P) &= \sup\left\{|\langle G_A \otimes G_B, P\rangle| \ : \ \|G_A\|_{1(\infty)} \leq 1, \|G_B\|_{1(\infty)} \leq 1\right\} \\ &\geq |\langle \mathbb{I}_A \otimes \mathbb{I}_B, P\rangle| \ , \end{aligned}$$

where $\mathbb{I}_A$ is the all-1 vector multiplied by $1/|\mathcal{X}|$ and $\mathbb{I}_B$ is the all-1 vector multiplied by $1/|\mathcal{Y}|$. Hence, it holds that $\|\mathbb{I}_A\|_{1(\infty)} = 1$ and $\|\mathbb{I}_B\|_{1(\infty)} = 1$, respectively. Taking the tensor product of $\mathbb{I}_A$ and $\mathbb{I}_B$ yields the all-1 vector multiplied by $1/(|\mathcal{X}||\mathcal{Y}|)$. Then, by using the fact that P is a bipartite system and, therefore, $\sum_{a,b}\langle f_{x,a} \otimes f_{y,b}, P\rangle = 1$, we obtain

$$|\langle \mathbb{I}_A \otimes \mathbb{I}_B, P\rangle| = \frac{1}{|\mathcal{X}||\mathcal{Y}|} \cdot |\mathcal{X}||\mathcal{Y}| = 1 \ .$$

Applying Lemma 3.2 results in

$$\alpha(P) \geq \varepsilon(P) \geq 1 \, ,$$

for all tensor norms α. □

5.1.3 Projective Tensor Norm and Local Systems

Let us now prove a tight connection between the projective tensor norm π and the set of local systems.

Lemma 5.2. *Let $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be a bipartite system. Then, P is a local system if and only if $\pi(P) = 1$.*

Proof. We first prove the "only if" part. Since P is a local system it can be written as

$$P = \sum_{i=1}^{n} \mu_i \cdot P_A^i \otimes P_B^i \, ,$$

with P_A^i and P_B^i systems and $\sum_{i=1}^n \mu_i = 1$ and $\mu_i \geq 0$ for all $1 \leq i \leq n$. Hence, by using the definition of the projective tensor norm π given in Section 3.2.2, we obtain

$$\begin{aligned}
\pi(P) &\leq \sum_{i=1}^{n} \|\mu_i \cdot P_A^i\|_{\infty(1)} \cdot \|P_B^i\|_{\infty(1)} \\
&= \sum_{i=1}^{n} |\mu_i| \cdot \|P_A^i\|_{\infty(1)} \cdot \|P_B^i\|_{\infty(1)} \\
&= 1 \, ,
\end{aligned}$$

where we used the fact that if P_A is a system then $\|P_A\|_{\infty(1)} = 1$ in the third line. That $\pi(P) \geq 1$ follows from Lemma 5.1.

Let us now prove the "if" part. For every $\epsilon > 0$ there exists a decomposition $P = \sum_i P_A^i \otimes P_B^i$ such that

$$\pi(P) + \epsilon \geq \sum_i \|P_A^i\|_{\infty(1)} \cdot \|P_B^i\|_{\infty(1)} \, .$$

Since P is a bipartite system we obtain for any x and y that

$$\begin{aligned} 1 &= \left|\sum_{a,b}\langle f_{x,a}\otimes f_{y,b},P\rangle\right| \\ &= \left|\sum_i \left(\sum_a \langle f_{x,a},P_A^i\rangle\right)\left(\sum_b \langle f_{y,b},P_B^i\rangle\right)\right| \\ &\leq \sum_i \left|\sum_a \langle f_{x,a},P_A^i\rangle\right|\left|\sum_b \langle f_{y,b},P_B^i\rangle\right| \\ &\leq \sum_i \|P_A^i\|_{\infty(1)}\cdot\|P_B^i\|_{\infty(1)} \\ &\leq \pi(P)+\epsilon \\ &= 1+\epsilon\,, \end{aligned} \tag{5.1}$$

where we used the triangle inequality in the third line. Since ϵ was arbitrary we can conclude that the first two inequalities in (5.1) are actually equalities. The first inequality gives us the condition

$$\left(\sum_a \langle f_{x,a},P_A^i\rangle\right)\cdot\left(\sum_b \langle f_{y,b},P_B^i\rangle\right)\geq 0\,, \tag{5.2}$$

for all x, y, i, and the second inequality yields $\|P_A^i\|_{\infty(1)} = |\sum_a \langle f_{x,a},P_A^i\rangle|$ and $\|P_B^i\|_{\infty(1)} = |\sum_b \langle f_{y,b},P_B^i\rangle|$ for all x, y, i.

Let us define new vectors $\tilde{P}_A^i \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $\tilde{P}_B^i \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ by

$$\tilde{P}_A^i := \frac{1}{\|P_A^i\|_{\infty(1)}}\cdot P_A^i\,,\ \tilde{P}_B^i := \frac{1}{\|P_B^i\|_{\infty(1)}}\cdot P_B^i\,,$$

for all i, and

$$\mu_i := \|P_A^i\|_{\infty(1)}\cdot\|P_B^i\|_{\infty(1)}\,.$$

We can then rewrite $P = \sum_i P_A^i \otimes P_B^i$ as

$$P = \sum_i \mu_i \cdot \tilde{P}_A^i \otimes \tilde{P}_B^i\,,$$

with $\|\tilde{P}_A^i\|_{\infty(1)} = 1$, $\|\tilde{P}_B^i\|_{\infty(1)} = 1$, and $\sum_i \mu_i = 1$ with $\mu_i \geq 0$, by using (5.1). What remains to be shown is that $\tilde{P}_A^i$ and $\tilde{P}_B^i$ can be chosen in such a way that $\tilde{P}_A^i \geq 0$ and $\tilde{P}_B^i \geq 0$ for all i. By (5.2) we obtain

$$\left(\sum_a \langle f_{x,a}, \tilde{P}_A^i \rangle\right) \cdot \left(\sum_b \langle f_{y,b}, \tilde{P}_B^i \rangle\right) \geq 0 \,. \tag{5.3}$$

Let us fix i and let us assume there is x such that $\sum_a \langle f_{x,a}, \tilde{P}_A^i \rangle < 0$. But this implies by equation (5.3) that $\sum_b \langle f_{y,b}, \tilde{P}_B^i \rangle \leq 0$ for all $y \in \mathcal{Y}$. And this again implies that $\sum_a \langle f_{x,a}, \tilde{P}_A^i \rangle \leq 0$ for all $x \in \mathcal{X}$. Hence, either all elements of the sets $\{\sum_a \langle f_{x,a}, \tilde{P}_A^i \rangle\}_x$ and $\{\sum_b \langle f_{y,b}, \tilde{P}_B^i \rangle\}_y$ are non-positive or all of them are non-negative, for a fixed i. Hence, we can define new vectors by

$$\hat{P}_A^i := \begin{cases} (-1) \cdot \tilde{P}_A^i \;, & \textit{if there exists } x \in \mathcal{X} \textit{ s.t. } \sum_a \langle f_{x,a}, \tilde{P}_A^i \rangle < 0 \\ (+1) \cdot \tilde{P}_A^i \;, & \textit{otherwise} \end{cases}$$

and

$$\hat{P}_B^i := \begin{cases} (-1) \cdot \tilde{P}_B^i \;, & \textit{if there exists } y \in \mathcal{Y} \textit{ s.t. } \sum_b \langle f_{y,b}, \tilde{P}_B^i \rangle < 0 \\ (+1) \cdot \tilde{P}_B^i \;, & \textit{otherwise} \end{cases}$$

which have the property that

$$\hat{P}_A^i \otimes \hat{P}_B^i = \tilde{P}_A^i \otimes \tilde{P}_B^i \;,$$

for all i, and $\sum_a \langle f_{x,a}, \hat{P}_A^i \rangle \geq 0$, and $\sum_b \langle f_{y,b}, \hat{P}_B^i \rangle \geq 0$, for all x, y and i. Finally, $\sum_a \langle f_{x,a}, \hat{P}_A^i \rangle \geq 0$ together with $\|\hat{P}_A^i\|_{\infty(1)} = |\sum_a \langle f_{x,a}, \hat{P}_A^i \rangle| = 1$ imply that $\sum_a |\langle f_{x,a}, \hat{P}_A^i \rangle| = \sum_a \langle f_{x,a}, \hat{P}_A^i \rangle = 1$ for all $x \in \mathcal{X}$ and, therefore, all entries of $\hat{P}_A^i$ must be non-negative, i.e., $\hat{P}_A^i \geq 0$. Hence, we can represent P by

$$P = \sum_i \mu_i \cdot \hat{P}_A^i \otimes \hat{P}_B^i \;,$$

with $\hat{P}_A^i \geq 0$, $\sum_a \langle f_{x,a}, \hat{P}_A^i \rangle = 1$, $\hat{P}_B^i \geq 0$, $\sum_b \langle f_{y,b}, \hat{P}_B^i \rangle = 1$, $\sum_i \mu_i = 1$, and $\mu_i \geq 0$, which shows that P is a local system. □

By using the definition of the set $\mathcal{R}_m^\alpha$, we immediately obtain that $\mathcal{R}_m^\pi = \mathcal{L}_m$. Hence, the projective tensor norms yields an alternative characterization of the set of local systems. The projective tensor norm, therefore, opens a new route to a better understanding of the properties of local systems.

5.1.4 Injective Tensor Norm and Signalling Systems

Since the injective tensor norm is smaller than the projective tensor norm it will induce a larger set of systems. In particular, it corresponds to the set of all bipartite systems, even the signalling ones. Hence, we have that $\mathcal{R}^{\varepsilon}_m = \mathcal{S}_m$.

Lemma 5.3. *If $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is a bipartite system, then $\varepsilon(P) = 1$.*

Proof. That $\varepsilon(P) \geq 1$ follows from Lemma 5.1. On the other hand, by the definition of the injective tensor norm we have

$$\varepsilon(P) = \sup\{|\langle G_A \otimes G_B, P\rangle| \ : \ \|G_A\|_{1(\infty)} \leq 1 \ , \ \|G_B\|_{1(\infty)} \leq 1\} \ .$$

We then obtain for any $\|G_A\|_{1(\infty)} \leq 1$ and $\|G_B\|_{1(\infty)} \leq 1$ that

$$\begin{aligned} |\langle G_A \otimes G_B, P\rangle| &= \left| \sum_{x,y,a,b} \langle G_A, e_{x,a}\rangle \langle G_B, e_{y,b}\rangle \langle f_{x,a} \otimes f_{y,b}, P\rangle \right| \\ &\leq \sum_{x,y,a,b} |\langle G_A, e_{x,a}\rangle| |\langle G_B, e_{y,b}\rangle| |\langle f_{x,a} \otimes f_{y,b}, P\rangle| \\ &\leq \max_{x,y} \sum_{a,b} |\langle f_{x,a} \otimes f_{y,b}, P\rangle| \\ &= 1 \ , \end{aligned}$$

where we used the triangle inequality in the second line, that $\|G_A\|_{1(\infty)} \leq 1$ and $\|G_B\|_{1(\infty)} \leq 1$ in the third line and that P is a bipartite system in the last line. Hence, we have shown that $\varepsilon(P) \leq 1$ as well. $\square$

5.1.5 Hilbertian Tensor Norm and Quantum Systems

We will establish a connection, although not a tight one as in the case of local systems, between quantum systems and the Hilbertian tensor norm. Let us first investigate the special case of *correlation systems* (see Section 2.3.4). Recall that $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ is a *quantum correlations system* if there exists a pure quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, and observables $A_1, ..., A_{|\mathcal{X}|}$ and $B_1, ..., B_{|\mathcal{Y}|}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, with eigenvalues ± 1, such that

$$\langle f_x \otimes f_y, P\rangle = \langle \Psi | A_x \otimes B_y | \Psi\rangle \ .$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

In order to establish a connection to the Hilbertian tensor norm we need a theorem by Tsirelson, which says that the correlations which can be obtained by measurements on a quantum state can be represented by inner products of real unit vectors, and vice versa.

Lemma 5.4 (Tsirelson's Theorem [Tsi80]). *Let $A_1, ..., A_{|\mathcal{X}|}$ and $B_1, ..., B_{|\mathcal{Y}|}$ be observables with eigenvalues in $[-1,+1]$. Then, for any state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exist real unit vectors $m_1, ..., m_{|\mathcal{X}|} \in \mathbb{R}^{2\cdot\max\{|\mathcal{X}|,|\mathcal{Y}|\}}$ and $n_1, ..., n_{|\mathcal{Y}|} \in \mathbb{R}^{2\cdot\max\{|\mathcal{X}|,|\mathcal{Y}|\}}$ such that*

$$\langle m_x, n_y \rangle = \langle \Psi | A_x \otimes B_y | \Psi \rangle \ ,$$

for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$.

Conversely, let $m_1, .., m_{|\mathcal{X}|}, n_1, ..., n_{|\mathcal{Y}|} \in \mathbb{R}^N$ be real vectors with $\|m_x\|_2 \leq 1$ and $\|n_y\|_2 \leq 1$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively, and $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be any maximally entangled state where $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2^{\lceil N/2+1 \rceil}$. Then, there exist observables $A_1, ..., A_{|\mathcal{X}|}$ on $\mathcal{H}_A$ and $B_1, ..., B_{|\mathcal{Y}|}$ on $\mathcal{H}_B$ with eigenvalues ± 1 such that

$$\langle m_x, n_y \rangle = \langle \Psi | A_x \otimes B_y | \Psi \rangle \ ,$$

for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$.

Note that this is a slightly generalized version of Tsirelson's theorem where the vectors $m_x \in \mathbb{R}^N$ and $n_y \in \mathbb{R}^N$ do not need to be unit vectors. So let us show that Lemma 5.4 indeed holds. In order to be allowed to apply the standard Tsirelson theorem, we need unit vectors. So let us construct them. Define $\tilde{m}_x \in \mathbb{R}^{N+2}$ to be $\langle \tilde{m}_x, e_i \rangle := \langle m_x, e_i \rangle$ for all $1 \leq i \leq N$, $\langle \tilde{m}_x, e_{N+1} \rangle := \sqrt{1 - \|m_x\|_2^2}$ and $\langle \tilde{m}_x, e_{N+2} \rangle := 0$. And similarly, for $\tilde{n}_y \in \mathbb{R}^{N+2}$ we set $\langle \tilde{n}_y, e_i \rangle := \langle n_y, e_i \rangle$ for all $1 \leq i \leq N$, $\langle \tilde{n}_y, e_{N+1} \rangle := 0$ and $\langle \tilde{n}_y, e_{N+2} \rangle := \sqrt{1 - \|n_y\|_2^2}$. We then have $\langle m_x, n_y \rangle = \langle \tilde{m}_x, \tilde{n}_y \rangle$ and $\|\tilde{m}_x\|_2 = 1$ and $\|\tilde{n}_y\|_2 = 1$ for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$. Hence, we can apply the standard Tsirelson theorem and get $\langle m_x, n_y \rangle = \langle \tilde{m}_x, \tilde{n}_y \rangle = \langle \Psi | A_x \otimes B_y | \Psi \rangle$.

Using Lemma 5.4, we are now ready to provide a proof of the results in [Tsi87, Tsi93], which show a tight connection between the Hilbertian tensor norm and quantum correlation systems.

Lemma 5.5 ([Tsi87, Tsi93]). *$P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ is a quantum correlation system if and only if $\gamma_2(P) \leq 1$.*

Proof. Since P is a quantum correlation system we can write it, according to Lemma 5.4, as

$$\langle f_x \otimes f_y, P \rangle = \langle m_x, n_y \rangle \ ,$$

with $\|m_x\|_2 = 1$ and $\|n_x\|_2 = 1$ for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$, respectively. Furthermore, the matrices (m_x) and $(n_y)^T$ give a factorization of $\hat{P}$ (see Section 4.4.1 for the notation), i.e., we have $\hat{P} = (n_y)^T \cdot (m_x)$. Using Lemma 4.6 (with $|\mathcal{A}| = |\mathcal{B}| = 1$) yields

$$\gamma_2(P) \leq \|(n_y)^T\|_{2\to\infty} \cdot \|(m_x)\|_{1\to 2} \ .$$

By applying Lemma 4.7 and using that $\|m_x\|_2 = 1$ and $\|n_x\|_2 = 1$, we get $\|(n_y)^T\|_{2\to\infty} = \|(m_x)\|_{1\to 2} = 1$ and, hence, $\gamma_2(P) \leq 1$.

For the converse, assume that $\gamma_2(P) \leq 1$. Then, by using Lemma 4.6, we can conclude that there exist real vectors $\{m_x\}$ and $\{n_y\}$ such that $\|(n_y)^T\|_{2\to\infty} \leq 1$ and $\|(m_x)\|_{1\to 2} \leq 1$ with $\langle f_x \otimes f_y, P \rangle = \langle m_x, n_y \rangle$. Then, the second part of Lemma 4.7 implies that $\|m_x\|_2 \leq 1$ and $\|n_y\|_2 \leq 1$ for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$. Applying the second part of Lemma 5.4 on the vectors $\{m_x\}$ and $\{n_y\}$ implies that P is indeed a quantum correlation system. ☐

We would also like to prove a similar result as given by Lemma 5.5 for quantum systems $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Unfortunately, no generalization of Tsirelson's theorem to many outputs is known to exist[1]. In particular, the second part of Tsirelson's theorem is the problem, as the first part can be generalized as will be seen in the proof of Theorem 5.1. We therefore get a weaker result.

Theorem 5.1. *If $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ is a quantum system, then $\gamma_2(P) = 1$.*

Proof. Since P is a quantum system there exists a pure quantum state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and projective measurements $\{M_x^a\}_{a\in\mathcal{A}}$ and $\{N_y^b\}_{b\in\mathcal{B}}$ with $\sum_{a\in\mathcal{A}} M_x^a = id_{\mathcal{H}_A}$ and $\sum_{b\in\mathcal{B}} N_y^b = id_{\mathcal{H}_B}$, respectively, such that

$$\langle f_{x,a} \otimes f_{y,b}, P \rangle = \langle \Psi | M_x^a \otimes N_y^b | \Psi \rangle = \langle \Psi | (M_x^a \otimes id_{\mathcal{H}_B}) \cdot (id_{\mathcal{H}_A} \otimes N_y^b) | \Psi \rangle \ .$$

So far we have only dealt we the inner product of real vectors. We define the inner product of complex vectors $v, w \in \mathbb{C}^n$, with $v := (v_1, ..., v_n)$

[1]although, one might consider the *quantum rounding* method in [KRT08] as some kind of approximated version of Tsirelson's theorem for unique games.

and $w := (w_1, ..., w_n)$, to be

$$\langle v, w \rangle := \sum_i v_i^* \cdot w_i \ ,$$

where v_i^* is the complex conjugate of v_i. Let $\{|i\rangle\}_i$ be an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$. We define the complex vectors $\tilde{m}_{x,a}$ and $\tilde{n}_{y,b}$ by

$$\begin{aligned} \langle f_i, \tilde{m}_{x,a} \rangle &:= \langle i|(M_x^a \otimes id_{\mathcal{H}_B})|\Psi\rangle \ , \\ \langle f_i, \tilde{n}_{y,b} \rangle &:= \langle i|(id_{\mathcal{H}_A} \otimes N_y^b)|\Psi\rangle \ . \end{aligned}$$

We therefore obtain

$$\begin{aligned} \langle \tilde{m}_{x,a}, \tilde{n}_{y,b} \rangle &= \sum_i \langle f_i, \tilde{m}_{x,a} \rangle^* \cdot \langle f_i, \tilde{n}_{y,b} \rangle \\ &= \sum_i \langle \Psi|(M_x^a \otimes id_{\mathcal{H}_B})|i\rangle\langle i|(id_{\mathcal{H}_A} \otimes N_y^b)|\Psi\rangle \\ &= \langle \Psi|M_x^a \otimes N_y^b|\Psi\rangle \ , \end{aligned} \tag{5.4}$$

by using the fact that $\sum_i |i\rangle\langle i| = id_{\mathcal{H}_A} \otimes id_{\mathcal{H}_B}$. Note that

$$\|\tilde{m}_{x,a}\|_2^2 = \sum_i |\langle f_i, \tilde{m}_{x,a} \rangle|^2 = \langle \Psi|(M_x^a \otimes id_{\mathcal{H}_B})|\Psi\rangle \ ,$$

since M_x^a is a projector and, therefore,

$$\sum_{a=1}^{|\mathcal{A}|} \|\tilde{m}_{x,a}\|_2^2 = \sum_{a=1}^{|\mathcal{A}|} \langle \Psi|(M_x^a \otimes id_{\mathcal{H}_B})|\Psi\rangle = \langle \Psi|\Psi\rangle = 1 \ , \tag{5.5}$$

for all $x \in \{1, ..., |\mathcal{X}|\}$. Similarly $\sum_{b=1}^{|\mathcal{B}|} \|\tilde{n}_{y,b}\|_2^2 = 1$ for all $y \in \{1, ..., |\mathcal{Y}|\}$. Furthermore, the vectors $\{\tilde{m}_{x,a}\}_a$ are mutually orthogonal for a given x, i.e., $\langle \tilde{m}_{x,a_1}, \tilde{m}_{x,a_2} \rangle = \delta_{a_1,a_2} \cdot \|\tilde{m}_{x,a_1}\|_2^2$, with $a_1, a_2 \in \{1, 2, ..., |\mathcal{A}|\}$, and for all $x \in \{1, 2, ..., |\mathcal{X}|\}$. This is the case since

$$\begin{aligned} \langle \tilde{m}_{x,a_1}, \tilde{m}_{x,a_2} \rangle &= \langle \Psi|(M_x^{a_1} \cdot M_x^{a_2} \otimes id_{\mathcal{H}_B})|\Psi\rangle \\ &= \langle \Psi|(M_x^{a_1} \otimes id_{\mathcal{H}_B})|\Psi\rangle \cdot \delta_{a_1,a_2} \\ &= \|\tilde{m}_{x,a_1}\|_2^2 \cdot \delta_{a_1,a_2} \ , \end{aligned} \tag{5.6}$$

where we used that $M_x^{a_1}$ and $M_x^{a_2}$ are projectors with the property $M_x^{a_1} \cdot M_x^{a_2} = M_x^{a_1} \cdot \delta_{a_1,a_2}$. By an analogous argument, one can show that

$\langle \tilde{n}_{y,b_1}, \tilde{n}_{y,b_2} \rangle = \delta_{b_1,b_2} \cdot \|\tilde{n}_{y,b_1}\|_2^2$, with $b_1, b_2 \in \{1, 2, ..., |\mathcal{B}|\}$, and for all $y \in \{1, 2, ..., |\mathcal{Y}|\}$.

Let us define the real vectors $m_{x,a}$ and $n_{y,b}$ which are computed from $\tilde{m}_{x,a}$ and $\tilde{n}_{y,b}$, respectively, by the following rule:

$$\begin{aligned} \langle f_{2i-1}, m_{x,a} \rangle &:= Re(\langle f_i, \tilde{m}_{x,a} \rangle) \, , \\ \langle f_{2i}, m_{x,a} \rangle &:= (-1) \cdot Im(\langle f_i, \tilde{m}_{x,a} \rangle) \, , \\ \langle f_{2i-1}, n_{y,b} \rangle &:= Re(\langle f_i, \tilde{n}_{y,b} \rangle) \, , \\ \langle f_{2i}, n_{y,b} \rangle &:= Im(\langle f_i, \tilde{n}_{y,b} \rangle) \, . \end{aligned}$$

Since $\langle \Psi | M_x^a \otimes N_y^b | \Psi \rangle$ is a real number, we can rewrite (5.4) as

$$\langle m_{x,a}, n_{y,b} \rangle = \sum_i \langle f_i, m_{x,a} \rangle \cdot \langle f_i, n_{y,b} \rangle = \langle \Psi | M_x^a \otimes N_y^b | \Psi \rangle \, .$$

Furthermore, since $\|m_{x,a}\|_2 = \|\tilde{m}_{x,a}\|_2$ and $\|n_{y,b}\|_2 = \|\tilde{n}_{y,b}\|_2$, (5.5) implies that

$$\sum_{a=1}^{|\mathcal{A}|} \|m_{x,a}\|_2^2 = 1 \, , \; \sum_{b=1}^{|\mathcal{B}|} \|n_{y,b}\|_2^2 = 1 \, . \tag{5.7}$$

And since $\langle \tilde{m}_{x,a_1}, \tilde{m}_{x,a_2} \rangle$ is a real number as well, (5.6) implies that

$$\langle m_{x,a_1}, m_{x,a_2} \rangle = \delta_{a_1,a_2} \cdot \|m_{x,a_1}\|_2^2 \, , \; \langle n_{y,b_1}, n_{y,b_2} \rangle = \delta_{b_1,b_2} \cdot \|n_{y,b_1}\|_2^2 \, . \tag{5.8}$$

In order to compute γ_2 we will use Lemma 4.6. Let us now show that $\|(n_{y,b})^T\|_{2\to\infty(1)} = 1$ and $\|(m_{x,a})\|_{1(\infty)\to 2} = 1$ and, therefore, prove that $\gamma_2(P) \leq 1$. The fact that $\gamma_2(P) \geq 1$ follows from Lemma 5.1. By applying Lemma 4.7 we get

$$\begin{aligned} \|(m_{x,a})\|_{1(\infty)\to 2}^2 &= \max_{s,x} \left\| \sum_{a\in\mathcal{A}} s(a) \cdot m_{x,a} \right\|_2^2 \\ &= \max_{s,x} \sum_a \langle m_{x,a}, m_{x,a} \rangle \\ &= \max_{s,x} \sum_a \|m_{x,a}\|_2^2 = 1 \, , \end{aligned}$$

where we used the orthogonality relations of (5.8) in the second line and (5.7) in the third line. By the same argument, one can also show that $\|(n_{y,b})^T\|_{2\to\infty(1)} = 1$ holds. $\square$

If we translate Theorem 5.1 into the language of convex sets of systems we immediately obtain $\mathcal{Q}_m \subseteq \mathcal{R}_m^{\gamma_2}$.

5.1.6 Dual Hilbertian Tensor Norm and Quantum Systems

Here, we establish a connection between the dual Hilbertian tensor norm γ_2^* and quantum systems. Note first that by (3.4), we know that

$$\gamma_2(P) \leq \gamma_2^*(P) ,$$

for all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Hence, we can conclude that γ_2^* will induce a convex set of systems which is not larger than the one induced by γ_2, i.e., $\mathcal{R}_m^{\gamma_2^*} \subseteq \mathcal{R}_m^{\gamma_2}$. Since we know from the previous section that $\mathcal{Q}_m \subseteq \mathcal{R}_m^{\gamma_2}$, it is not clear yet what the relation between the quantum set $\mathcal{Q}_m$ and $\mathcal{R}_m^{\gamma_2^*}$ will be.

An immediate observation about the connection between quantum correlation system and the dual Hilbertian tensor norms is the following result.

Lemma 5.6. *Let $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$. If $\gamma_2^*(P) \leq 1$ then P is a quantum correlation system.*

Proof. By (3.4), we get that $\gamma_2(P) \leq \gamma_2^*(P)$ for all $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$. Furthermore, Lemma 5.5 implies that if $\gamma_2(P) \leq 1$, then P is a quantum correlation system. These two facts imply the result. □

The converse is not true since there exists a quantum correlation system with ternary inputs which has dual Hilbertian tensor norm larger than one.

Lemma 5.7. *There exists a quantum correlation system $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$, with $|\mathcal{X}| = |\mathcal{Y}| = 3$, such that $\gamma_2^*(P) \geq 1.125$.*

Proof. We consider the quantum correlation system P, defined as $\langle f_x \otimes f_y, P\rangle := \langle m_x, n_y\rangle$, with

$$m_1 = n_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; m_2 = n_2 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} , \; m_3 = n_3 = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} .$$

By Lemma 5.4 this corresponds to a quantum correlation system since all vectors are of unit length. By using Theorem 4.3, with $|\mathcal{A}| = |\mathcal{B}| = 1$, and the fact that $\langle m_x, n_y\rangle\langle v_x, w_y\rangle = \langle m_x \otimes v_x, n_y \otimes w_y\rangle$, we can write the dual Hilbertian tensor norm $\gamma_2^*(P)$ as the following SDP:

$$\gamma_2^*(P) = \sup_{\{v_x\},\{w_y\}} \left| \sum_{x,y} \langle m_x \otimes v_x, n_y \otimes w_y\rangle \right| , \tag{5.9}$$

such that

$$\left\| \sum_{x=1}^{|\mathcal{X}|} s(x) \cdot v_x \right\|_2 \leq 1 \,,\ \forall\, s : \mathcal{X} \to \{-1, +1\} \,, \tag{5.10}$$

$$\left\| \sum_{y=1}^{|\mathcal{Y}|} t(y) \cdot w_y \right\|_2 \leq 1 \,,\ \forall\, t : \mathcal{Y} \to \{-1, +1\} \,, \tag{5.11}$$

and $v_x, w_y \in \ell_2^n$. Numerically solving this SDP suggests the following assignment for the solution vectors: $v_1 = w_1 := 1/2 \cdot m_1$, $v_2 = w_2 := 1/2 \cdot m_2$ and $v_3 = w_3 := 1/2 \cdot m_3$. Putting these vector into (5.9) yields

$$\begin{aligned}
\gamma_2(P) &\geq \left| \sum_{x,y} \langle m_x \otimes v_x, n_y \otimes w_y \rangle \right| \\
&= \left| \left\langle \sum_x m_x \otimes v_x, \sum_y m_y \otimes v_y \right\rangle \right| \\
&= \left\| \sum_x m_x \otimes v_x \right\|_2^2 \\
&= \left\| 1/2 \cdot \sum_x m_x \otimes m_x \right\|_2^2 \\
&= 9/8 \,,
\end{aligned}$$

where we used in the first line that the vectors v_1, v_2, v_3, w_1, w_2, and w_3 fulfil the constraints given in (5.10) and (5.11). □

However, the next result shows that the dual Hilbertian tensor norm cannot become too large for quantum correlation systems.

Lemma 5.8. *If $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ is a quantum correlation system, then $\gamma_2^*(P) \leq K_G$, where K_G is the Grothendieck constant.*

Proof. By Grothendieck's inequality (see (6.3)) and Lemma 3.2 we have $\gamma_2^*(P) \leq \pi(P) \leq K_G \cdot \gamma_2(P)$, where K_G is the Grothendieck constant. Furthermore, using Lemma 5.5, which says that if $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ is a quantum correlation system then $\gamma_2(P) \leq 1$, results in $\gamma_2^*(P) \leq K_G$. □

If we restrict to correlation systems which have only binary inputs we even obtain a tight connection (together with Lemma 5.6) between the dual Hilbertian tensor norm and quantum correlation systems.

Lemma 5.9. *If $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$, with $|\mathcal{X}| = |\mathcal{Y}| = 2$, is a quantum correlation system, then $\gamma_2^*(P) \leq 1$.*

Proof. By using Theorem 4.3, with $|\mathcal{A}| = |\mathcal{B}| = 1$, we can write the dual Hilbertian tensor norm $\gamma_2^*(P)$ as the following SDP.

$$\gamma_2^*(P) = \sup_{\{v_x\},\{w_y\}} \left| \sum_{x,y} \langle f_x \otimes f_y, P \rangle \cdot \langle v_x, w_y \rangle \right| , \tag{5.12}$$

such that

$$\left\| \sum_{x=1}^{|\mathcal{X}|} s(x) \cdot v_x \right\|_2 \leq 1 \, , \; \forall \, s : \mathcal{X} \to \{-1, +1\} \, , \tag{5.13}$$

$$\left\| \sum_{y=1}^{|\mathcal{Y}|} t(y) \cdot w_y \right\|_2 \leq 1 \, , \; \forall \, t : \mathcal{Y} \to \{-1, +1\} \, , \tag{5.14}$$

and $v_x, w_y \in \ell_2^n$.

Since P is a quantum correlation system, it can be written as $\langle f_x \otimes f_y, P \rangle = \langle m_x, n_y \rangle$, with $\|m_x\|_2 = 1$ and $\|n_y\|_2 = 1$, by Lemma 5.4. Hence, (5.12) becomes

$$\begin{aligned} \gamma_2^*(P) &= \sup_{\{v_x\},\{w_y\}} \left| \sum_{x,y} \langle m_x, n_y \rangle \cdot \langle v_x, w_y \rangle \right| \\ &= \sup_{\{v_x\},\{w_y\}} \left| \sum_{x,y} \langle m_x \otimes v_x, n_y \otimes w_y \rangle \right| \\ &= \sup_{\{v_x\},\{w_y\}} \left| \left\langle \sum_x m_x \otimes v_x, \sum_y n_y \otimes w_y \right\rangle \right| , \\ &\leq \sup_{\{v_x\},\{w_y\}} \left\| \sum_x m_x \otimes v_x \right\|_2 \cdot \left\| \sum_y n_y \otimes w_y \right\|_2 , \end{aligned}$$

where the last line is implied by the Cauchy-Schwarz inequality. Showing that $\| \sum_x m_x \otimes v_x \|_2 \leq 1$ and $\| \sum_y n_y \otimes w_y \|_2 \leq 1$ finishes the proof. This can be done as follows:

$$
\begin{aligned}
& \left\| \sum_x m_x \otimes v_x \right\|_2^2 \\
= \; & \|m_1 \otimes v_1 + m_2 \otimes v_2\|_2^2 \\
= \; & \langle m_1, m_1 \rangle \langle v_1, v_1 \rangle + 2 \cdot \langle m_1, m_2 \rangle \langle v_1, v_2 \rangle + \langle m_2, m_2 \rangle \langle v_2, v_2 \rangle \\
= \; & \langle v_1, v_1 \rangle + 2 \cdot \langle m_1, m_2 \rangle \langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle \\
\leq \; & \langle v_1, v_1 \rangle + 2 \cdot \|m_1\|_2 \cdot \|m_2\|_2 \cdot \langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle \\
\leq \; & \langle v_1, v_1 \rangle + 2 \cdot sign\langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle \\
= \; & \langle v_1 + sign\langle v_1, v_2 \rangle \cdot v_2, v_1 + sign\langle v_1, v_2 \rangle \cdot v_2 \rangle \\
= \; & \|v_1 + sign\langle v_1, v_2 \rangle \cdot v_2\|_2^2 \\
\leq \; & 1 \,,
\end{aligned}
$$

where we used the fact that $\langle m_1, m_1 \rangle = \|m_1\|_2^2 = 1$ and $\langle m_2, m_2 \rangle = \|m_2\|_2^2 = 1$ in the third equality and in the second inequality, the Cauchy-Schwarz inequality in the first inequality and equation (5.13) in the last line. By an analogous argument one can show that $\| \sum_y n_y \otimes w_y \|_2 \leq 1$ holds as well. ☐

So far, we have only analysed the connection between the dual Hilbertian tensor norm and *correlation systems*. Let us now show a tight connection between γ_2^* and binary quantum systems that are isotropic. Recall that P_λ is an isotropic quantum system if and only if $0 \leq \lambda \leq 1/\sqrt{2}$ (see Lemma 2.4).

Theorem 5.2. *Let P_λ be an isotropic system. Then, $\gamma_2^*(P_\lambda) = 1$ if and only if $0 \leq \lambda \leq 1/\sqrt{2}$.*

Proof. Due to convexity, proving that $\gamma_2^*(P_{1/\sqrt{2}}) \leq 1$ implies $\gamma_2^*(P_\lambda) \leq 1$ for all $0 \leq \lambda \leq 1/\sqrt{2}$. The isotropic quantum system $P_{1/\sqrt{2}}$ can be obtained by projective measurements on a Bell state, i.e., there exist real projectors $\{M_x^a\}_{a \in \mathcal{A}}$ and $\{N_y^b\}_{b \in \mathcal{B}}$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that

$$\langle f_{x,a} \otimes f_{y,b}, P_{1/\sqrt{2}} \rangle = \langle \Phi^+ | M_x^a \otimes N_y^b | \Phi^+ \rangle \,,$$

with $|\Phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ for all $x, y, a, b \in \{1, 2\}$. Let $\{|i\rangle\}_i$ be an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$, and let us define real vectors $m_{x,a}, n_{y,b} \in \mathbb{R}^4$ by

$$
\begin{aligned}
\langle f_i, m_{x,a} \rangle \; &:= \; \langle i | (M_x^a \otimes id_B) | \Phi^+ \rangle \\
\langle f_i, n_{y,b} \rangle \; &:= \; \langle i | (id_A \otimes N_y^b) | \Phi^+ \rangle \,.
\end{aligned}
$$

Since $(M_x^a)^\dagger = M_x^a$ and $(N_y^b)^\dagger = N_y^b$ are real we obtain $\langle f_{x,a} \otimes f_{y,b}, P_{1/\sqrt{2}} \rangle = \langle m_{x,a}, n_{y,b} \rangle$. Furthermore, it holds that

$$\langle m_{x_1,a_1}, m_{x_2,a_2} \rangle = \frac{1}{2} \operatorname{tr}(M_{x_1}^{a_1} \cdot M_{x_2}^{a_2}) \, ,$$

which can be used in order to show the following conditions on the vectors $\{m_{x,a}\}$:

$$\begin{aligned} \langle m_{x,a_1}, m_{x,a_2} \rangle &= 0 \, , \, \forall \, a_1 \neq a_2 \in \{1,2\} \, , && (5.15) \\ \langle m_{x_1,a_1}, m_{x_2,a_2} \rangle &\geq 0 \, , \, \forall \, x_1, x_2, a_1, a_2 \in \{1,2\} \, , && (5.16) \\ \langle m_{1,a}, m_{1,a} \rangle &= \langle m_{1,a}, m_{2,1} \rangle + \langle m_{1,a}, m_{2,2} \rangle \, , \forall \, a \in \{1,2\}, && (5.17) \\ \langle m_{2,a}, m_{2,a} \rangle &= \langle m_{2,a}, m_{1,1} \rangle + \langle m_{2,a}, m_{1,2} \rangle \, , \forall \, a \in \{1,2\}, && (5.18) \end{aligned}$$

and

$$1 = \langle m_{1,1}, m_{2,1} \rangle + \langle m_{1,1}, m_{2,2} \rangle + \langle m_{1,2}, m_{2,1} \rangle + \langle m_{1,2}, m_{2,2} \rangle \, . \qquad (5.19)$$

The same conditions can be derived for the vectors $\{n_{y,b}\}$.

By using Theorem 4.3, we can conclude that for every $\epsilon > 0$ there exist vectors $\{v_{x,a}\}$ and $\{w_{y,b}\}$ such that

$$\gamma_2^*(P_{1/\sqrt{2}}) - \epsilon \leq \left| \sum_{x,y} \sum_{a,b} \langle f_{x,a} \otimes f_{y,b}, P_{1/\sqrt{2}} \rangle \langle v_{x,a}, w_{y,b} \rangle \right| \, ,$$

with $\|v_{1,a_1} \pm v_{2,a_2}\|_2 \leq 1$, for all $a_1, a_2 \in \{1,2\}$, and $\|w_{1,b_1} \pm w_{2,b_2}\|_2 \leq 1$, for all $b_1, b_2 \in \{1,2\}$. Furthermore, by using the Cauchy-Schwarz inequality and that $\langle f_{x,a} \otimes f_{y,b}, P_{1/\sqrt{2}} \rangle = \langle m_{x,a}, n_{y,b} \rangle$, we obtain

$$\begin{aligned} \gamma_2^*(P_{1/\sqrt{2}}) - \epsilon &\leq \left| \left\langle \sum_{x,a} m_{x,a} \otimes v_{x,a}, \sum_{y,b} n_{y,b} \otimes w_{y,b} \right\rangle \right| \\ &\leq \left\| \sum_{x,a} m_{x,a} \otimes v_{x,a} \right\|_2 \cdot \left\| \sum_{y,b} n_{y,b} \otimes w_{y,b} \right\|_2 \, , \end{aligned}$$

where we used that $\langle m_{x,a} \otimes v_{x,a}, n_{y,b} \otimes w_{y,b} \rangle = \langle m_{x,a}, n_{y,b} \rangle \langle v_{x,a}, w_{y,b} \rangle$ and the linearity of the inner product in the first line. By using the conditions given in (5.15), (5.17) and (5.18) one can derive the following:

$$\begin{aligned}
& \left\| \sum_{x,a} m_{x,a} \otimes v_{x,a} \right\|_2^2 \\
= \quad & \langle m_{1,1}, m_{2,1} \rangle \cdot \|v_{1,1} + v_{2,1}\|_2^2 + \langle m_{1,1}, m_{2,2} \rangle \cdot \|v_{1,1} + v_{2,2}\|_2^2 \\
+ \quad & \langle m_{1,2}, m_{2,1} \rangle \cdot \|v_{1,2} + v_{2,1}\|_2^2 + \langle m_{1,2}, m_{2,2} \rangle \cdot \|v_{1,2} + v_{2,2}\|_2^2 \, .
\end{aligned}$$

Furthermore, by using conditions (5.16) and (5.19) together with $\|v_{1,a_1} + v_{2,a_2}\|_2 \leq 1$, for all $a_1, a_2 \in \{1, 2\}$, we obtain $\| \sum_{x,a} m_{x,a} \otimes v_{x,a}\|_2^2 \leq 1$. Similarly, one can show that $\| \sum_{y,b} n_{y,b} \otimes w_{y,b}\|_2^2 \leq 1$ holds as well, and we therefore obtain $\gamma_2^*(P_{1/\sqrt{2}}) \leq 1$ since ϵ was arbitrary. That $\gamma_2^*(P_{1/\sqrt{2}}) \geq 1$ follows from Lemma 5.1.

Let us now prove the converse by showing that if $\lambda > 1/\sqrt{2}$ then $\gamma_2^*(P_\lambda) > 1$. Since $\gamma_2^*(P_\lambda)$ corresponds to an SDP by Theorem 4.3 we can efficiently solve it and guess the solution vectors $\{v_{x,a}\}$ and $\{w_{y,b}\}$. In particular, we define them as follows:

$$\begin{aligned}
v_{1,1} \quad &:= \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \; v_{1,2} := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \, , \\
v_{2,1} \quad &:= \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, , \; v_{2,2} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \, , \\
w_{1,1} \quad &:= \quad \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \, , \; w_{1,2} := \frac{1}{2} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \, , \\
w_{2,1} \quad &:= \quad \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \, , \; w_{2,2} := \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \, .
\end{aligned}$$

Note that $\|v_{1,a_1} \pm v_{2,a_2}\|_2 = 1$, for all $a_1, a_2 \in \{1, 2\}$, and $\|w_{1,b_1} \pm w_{2,b_2}\|_2 = 1$, for all $b_1, b_2 \in \{1, 2\}$, and that $2\sqrt{2} \cdot |\langle v_{x,a}, w_{y,b} \rangle| = 1$ for all a, b, x, y. Using Theorem 4.3 and the vectors $\{v_{x,a}\}$ and $\{w_{y,b}\}$, followed by some straightforward calculations, yields

$$\gamma_2^*(P_\lambda) \geq \sqrt{2} \cdot \lambda \, ,$$

and, therefore, for $\lambda > 1/\sqrt{2}$, we have $\gamma_2^*(P_\lambda) > 1$. □

Putting this result into the context of convex sets of bipartite systems yields the relation $\mathcal{R}_{\text{iso}}^{\gamma_2^*} = \mathcal{Q}_{\text{iso}}$.

Conjectured Relation Between Dual Hilbertian Tensor Norm and Quantum Systems

By having a close look at the proof of Theorem 5.2 we notice that the only assumption which does not hold in general for all quantum systems is condition (5.16) for the cases where $x_1 \neq x_2$. That generalized versions of the conditions (5.15), (5.17), (5.18), and (5.19) are true for arbitrary quantum systems can be seen by having a look at the proof of Theorem 5.1. However, we conjecture that for the special case where $P \in \mathcal{Q}^1_{\text{CHSH}}$ (see Section 2.7.1 for a definition of this set) there exist vectors $m_{x,a}$ and $n_{y,b}$ such that *all* the conditions (5.15), (5.16), (5.17), (5.18), and (5.19) are satisfied[2]. Hence, we have the following conjecture.

Conjecture 1. If $P \in \mathcal{Q}^1_{\text{CHSH}}$ then $\gamma_2^*(P) = 1$.

This conjecture can also be written as $\mathcal{Q}^1_{\text{CHSH}} \subseteq \mathcal{R}^{\gamma_2^*}_{\text{CHSH}}$. Note that this would also imply that for all binary quantum systems P we have $\gamma_2^*(P) = 1$ since $\mathcal{Q}_{\text{CHSH}} \subseteq \mathcal{Q}^1_{\text{CHSH}}$ (see Section 2.7.1).

Let us provide numerical evidence that Conjecture 1 is indeed true. By using the semidefinite programming solver SeDuMi [Stu99], together with the MATLAB toolbox YALMIP [Löf04], we compute the boundaries of the sets $\mathcal{R}^{\gamma_2^*}_{\text{CHSH}}$, $\mathcal{Q}^1_{\text{CHSH}}$ and $\mathcal{Q}^2_{\text{CHSH}}$ for two different two-dimensional slices of the eight-dimensional non-signalling polytope $\mathcal{NS}_{\text{CHSH}}$. That computing the set $\mathcal{R}^{\gamma_2^*}_{\text{CHSH}}$ corresponds to solving an SDP is shown in Theorem 4.3. Furthermore, determining the sets $\mathcal{Q}^1_{\text{CHSH}}$ and $\mathcal{Q}^2_{\text{CHSH}}$ corresponds also to solving some SDP as is shown in [NPA07, NPA08, DLTW08]. See also Section 2.7.1.

We use the parametrization given in [ABPS09] for the two-dimensional slices. The first slice we consider consists of binary non-signalling systems $P^1_{\alpha,\beta} \in \mathcal{NS}_{\text{CHSH}}$ which are defined as

$$P^1_{\alpha,\beta} := \alpha \cdot P^{000}_{\text{PR}} + \beta \cdot P^{010}_{\text{PR}} + (1 - \alpha - \beta) \cdot P_{\mathbb{I}} \,, \tag{5.20}$$

with $\alpha \in [1/2, 1]$, $\beta \in [0, 1/2]$ and $P^{\eta\nu\sigma}_{\text{PR}}$, with $\eta, \nu, \sigma \in \{0, 1\}$, defined as in (2.10). The resulting boundaries are shown in Figure 5.1. One can see that all three boundaries $\partial\mathcal{R}^{\gamma_2^*}_{\text{CHSH}}$, $\partial\mathcal{Q}^1_{\text{CHSH}}$, and $\partial\mathcal{Q}^2_{\text{CHSH}}$ are identical. Furthermore, it is shown in [ABPS09] that $\mathcal{Q}_{\text{CHSH}} = \mathcal{Q}^1_{\text{CHSH}} = \mathcal{Q}^2_{\text{CHSH}}$ for this particular slice. Therefore, we have strong evidence that $\partial\mathcal{R}^{\gamma_2^*}_{\text{CHSH}} = \partial\mathcal{Q}_{\text{CHSH}}$ holds for this slice.

[2] Note that the vectors cannot be constructed as in the proof of Theorem 5.1 since there exist counterexamples where $\langle m_{x_1,a_1}, m_{x_2,a_2} \rangle < 0$, for $x_1 \neq x_2$, by this construction.

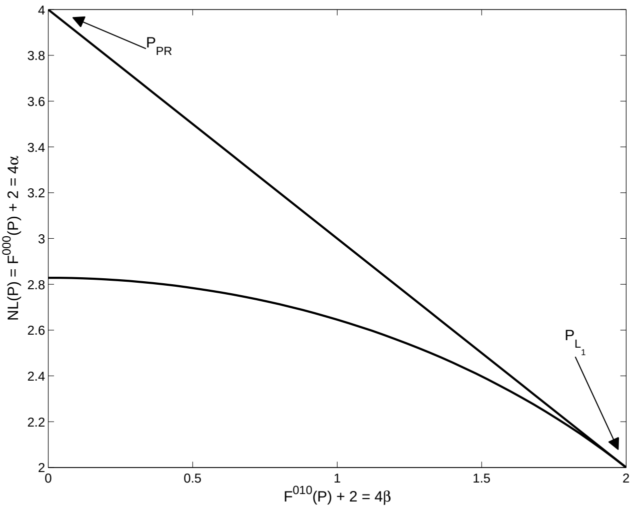

Figure 5.1: A two-dimensional slice of the non-signalling polytope $\mathcal{NS}_{\mathrm{CHSH}}$ is shown for systems of the form as given in (5.20). The straight line between the PR-box $P_{\mathrm{PR}} \equiv P_{\mathrm{PR}}^{000}$ and the local system $P_{L_1} := 1/2(P_{\mathrm{PR}}^{000} + P_{\mathrm{PR}}^{010})$ is a boundary line of the non-signalling polytope. The curved line corresponds to the boundaries $\partial\mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*}$, $\partial\mathcal{Q}_{\mathrm{CHSH}}^{1}$, $\partial\mathcal{Q}_{\mathrm{CHSH}}^{2}$ and $\partial\mathcal{Q}_{\mathrm{CHSH}}$.

The second slice consists of binary systems $P_{\alpha,\beta}^{2} \in \mathcal{NS}_{\mathrm{CHSH}}$ defined as

$$P_{\alpha,\beta}^{2} := \alpha \cdot P_{\mathrm{PR}}^{000} + \beta \cdot P_{L_2} + (1-\alpha-\beta) \cdot P_{\mathbb{I}} \,, \tag{5.21}$$

with $\alpha \in [1/2, 1]$, $\beta \in [0,1]$ and $P_{L_2}(0,0|x,y) = 1$ for all $x, y \in \{0,1\}$. The resulting boundaries are shown in Figure 5.2. For this slice the quantum boundary $\partial\mathcal{Q}_{\mathrm{CHSH}}$, which is upper bounded by the dashed line corresponding to $\partial\mathcal{Q}_{\mathrm{CHSH}}^{2}$, does not agree with the solid curved line corresponding to the boundaries of $\mathcal{Q}_{\mathrm{CHSH}}^{1}$ and $\mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*}$. Hence, for this slice it is suggested that $\mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*}$ is an upper bound to the quantum set, i.e., $\mathcal{Q}_{\mathrm{CHSH}} \subsetneq \mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*}$. In conclusion, we can say that we have strong numerical evidence that Conjecture 1 holds, i.e., that $\mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*} = \mathcal{Q}_{\mathrm{CHSH}}^{1}$.

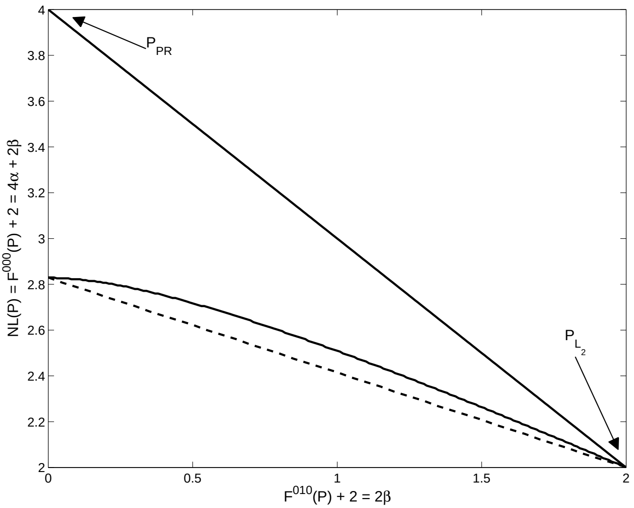

Figure 5.2: A two-dimensional slice of the non-signalling polytope $\mathcal{NS}_{\mathrm{CHSH}}$ is shown for systems of the form as given in (5.21). The straight line between the PR-box $P_{\mathrm{PR}} \equiv P_{\mathrm{PR}}^{000}$ and the local system $P_{L_2}(0,0|x,y) = 1$, for all $x, y \in \{0,1\}$, is a boundary line of the non-signalling polytope. The solid curved line corresponds to the boundaries $\partial\mathcal{R}_{\mathrm{CHSH}}^{\gamma_2^*}$ and $\partial\mathcal{Q}_{\mathrm{CHSH}}^{1}$ whereas the dashed line corresponds to $\partial\mathcal{Q}_{\mathrm{CHSH}}^{2}$ and is, therefore, located above (or on) the quantum boundary $\partial\mathcal{Q}_{\mathrm{CHSH}}$.

5.2 Sets of Systems Closed Under Wirings

5.2.1 Introduction

In the previous section we have seen different convex sets of bipartite systems which are induced by tensor norms. The only property we investigated was the geometry of these sets, i.e., which bipartite systems belong to a given set and which do not. In this section, we will investigate another property by introducing *wirings* into the picture.

First, note that we will consider the set $\mathcal{R}_m^\alpha \cap \mathcal{NS}_m$ instead of $\mathcal{R}_m^\alpha$ since *signalling* systems are not compatible, in the sense that causality would be violated, with the definition of wirings introduced in Section 2.4. Hence, for the remaining part of this chapter the set $\mathcal{R}_m^\alpha$ is actually defined as

$$\mathcal{R}_m^\alpha := \{P \,:\, \alpha(P) \leq 1 \wedge P \in \mathcal{NS}_m\} \,.$$

Now, assume we are given a set $\mathcal{R}_m^\alpha$ and Alice and Bob want to use the bipartite systems which are elements of $\mathcal{R}_m^\alpha$ as resources for some problem they want to solve. They are not allowed to communicate; however, they can apply a *wiring* (see Section 2.4 and Section 3.4) on their bipartite systems in order to obtain a new bipartite system in $\mathcal{NS}_m$. If this new bipartite system is *not* an element of $\mathcal{R}_m^\alpha$ any more, it seems that $\mathcal{R}_m^\alpha$ is not the right object to study if we investigate settings where bipartite systems are used as resources. Hence, the canonical object we should study is the *closure of* $\mathcal{R}_m^\alpha$ *under wirings*.

Let us make the notion of *closure under wirings* formal. We say that $\mathcal{R}_m^\alpha$ is *closed under wirings* if for all wirings $\mathcal{W} : \mathcal{NS}_m^{\times n} \rightarrow \mathcal{NS}_m$ and all collections of bipartite systems $P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n} \in \mathcal{R}_m^\alpha$, it holds that

$$\mathcal{W}(P_{A_1B_1}, P_{A_2B_2}, ..., P_{A_nB_n}) \in \mathcal{R}_m^\alpha \,.$$

If $\mathcal{R}_m^\alpha$ is not closed under wirings, we can consider the smallest subset of $\mathcal{NS}_m$ which contains $\mathcal{R}_m^\alpha$ and is closed under wirings. We denote this set by $cl(\mathcal{R}_m^\alpha) \subseteq \mathcal{NS}_m$ and it is called the *closure of* $\mathcal{R}_m^\alpha$ *under wirings* [ABL+09]. Note that if $\mathcal{R}_m^\alpha$ is closed under wirings we have of course $\mathcal{R}_m^\alpha = cl(\mathcal{R}_m^\alpha)$. Furthermore, we call a set $\mathcal{R}_m^\alpha \subseteq \mathcal{NS}_m$ a *theory* if it is closed under wirings. We have chosen this terminology because any (possibly yet unknown) physical theory will be closed under wirings [ABL+09].

The non-signalling, the quantum, and the local set of systems are all closed under wirings, i.e., $\mathcal{NS}_m = cl(\mathcal{NS}_m)$, $\mathcal{Q}_m = cl(\mathcal{Q}_m)$, and $\mathcal{L}_m = cl(\mathcal{L}_m)$. Furthermore, $\mathcal{L}_m$ is the smallest subset of $\mathcal{NS}_m$ which is closed under wirings and all other sets $\mathcal{R}_{m'}^\alpha$, with $m' \geq m$, which are closed under wirings have $\mathcal{L}_m$ as a subset. This follows immediately from the fact that all deterministic systems, which are, therefore, extremal local systems, can be generated by a trivial deterministic wiring [ABL+09].

Contribution

We prove sufficient conditions that a norm must fulfil such that it induces a convex set of bipartite systems which is closed under wirings. Furthermore, if the convex set is induced by a *tensor norm* then this tensor norm induces a theory if it fulfils a direct-product theorem. Using this result, we show that the projective tensor norm π and the dual Hilbertian tensor norm γ_2^* induce theories.

Related Works

Sets of bipartite systems which are closed under wirings have first been systematically studied in [ABL$^+$09], albeit not in the context of tensor norms.

The results of [DW08] imply that there exist infinitely many sets which are closed under wirings. However, all these theories are subsets of $\mathcal{Q}_{\mathrm{CHSH}}$. More precisely, for each isotropic quantum system P_λ which cannot be distilled[3] (see Section 6.5 for a definition of non-locality distillation) a corresponding theory could be constructed by computing $cl(\{P_\lambda\} \cup \mathcal{L}_m)$. In this theory, P_λ is maximally CHSH non-local since otherwise distillation for P_λ would be possible. Note that this result implies that the quantum set is *not* the smallest set closed under wirings which features non-locality and, therefore, refutes the conjecture of [ABL$^+$09].

The only theory beside $\mathcal{NS}_m$ that is known to be larger than $\mathcal{Q}_m$ is $\mathcal{Q}_m^1$ [NW10]. Note that it is not known whether the sets in this hierarchy, $\mathcal{Q}_m^i$ for $i \geq 2$, are closed under wirings as well (see also Section 2.7.1).

Applications

The results of this section are used in Section 6.4, which is about the universality of quantum systems, and Section 6.5, which is about the impossibility of non-locality distillation.

Open Problems

Are the sufficient conditions we provide in this section also *necessary* conditions for a norm to induce a theory? Furthermore, since every physical theory we will eventually discover yields a set of bipartite systems which is closed under wirings, the question arises whether the results of this chapter might be useful in order to put restrictions on possible future physical theories.

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[3] From [DW08] we only know that infinitely many λ's exist for which this is true. But we do not know their exact values.

5.2.2 Sufficient Conditions for Theories

We will present *sufficient* conditions a norm must fulfil in order to induce a set of systems which is closed under wirings. First, from the discussion of Section 3.4 we know that every wiring $\mathcal{W} : \mathcal{NS}_m^{\times n} \to \mathcal{NS}_m$ can be represented as a linear map $\mathcal{T}_\mathcal{W}$ on some tensor product space. Therefore, since for every theory $\mathcal{R}_m^\alpha$ it holds that

$$\mathcal{W}(P_{A_1B_1}, ..., P_{A_nB_n}) \in \mathcal{R}_m^\alpha \ ,$$

for all wirings $\mathcal{W}$ and all $P_{A_kB_k} \in \mathcal{R}_m^\alpha$, we obtain the equivalent condition

$$\mathcal{T}_\mathcal{W}(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \in \mathcal{R}_m^\alpha \ , \tag{5.22}$$

for all all wirings $\mathcal{T}_\mathcal{W}$ and all $P_{A_kB_k} \in \mathcal{R}_m^\alpha$. Recall that $\odot$ denotes the composition of systems as defined in Section 3.3. We are now ready to state the sufficient conditions:

Theorem 5.3. *Let $\alpha : \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n} \to \mathbb{R}$ be a norm. Then, the set $\mathcal{R}_m^\alpha$ is closed under wirings if α fulfils the following two conditions:*

1. *$\alpha(\mathcal{T}_\mathcal{W}(P)) \leq \alpha(P)$, for all $P \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$ and all wirings $\mathcal{T}_\mathcal{W}$.*

2. *$\alpha(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \leq \alpha(P_{A_1B_1}) \cdot ... \cdot \alpha(P_{A_nB_n})$, for all $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$ and where $P_{A_1B_1} \odot ... \odot P_{A_nB_n} \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$.*

Proof. The definition of the set $\mathcal{R}_m^\alpha$ and (5.22) imply that we need to prove that

$$\alpha\left(\mathcal{T}_\mathcal{W}(P_{A_1B_1} \odot ... \odot P_{A_nB_n})\right) \leq 1 \tag{5.23}$$

for all wirings $\mathcal{T}_\mathcal{W} : \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ and all $P_{A_kB_k} \in \mathcal{R}_m^\alpha$. Since $P_{A_kB_k} \in \mathcal{R}_m^\alpha$ implies $\alpha(P_{A_kB_k}) \leq 1$ (by definition), the two assumptions of the theorem imply (5.23) and, therefore, $\mathcal{R}_m^\alpha$ is closed under wirings. □

By using Theorem 3.3 we obtain the following corollary:

Corollary 5.1. *Let $\alpha : \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n} \to \mathbb{R}$ be a tensor norm. Then, the set $\mathcal{R}_m^\alpha$ is closed under wirings if*

$$\alpha(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \leq \alpha(P_{A_1B_1}) \cdot ... \cdot \alpha(P_{A_nB_n}) \ ,$$

for all $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$, and where $P_{A_1B_1} \odot ... \odot P_{A_nB_n} \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$.

5.2.3 Two Examples of Theories

Corollary 5.1, together with Theorem 3.1 and Theorem 3.2, imply that the projective and the dual Hilbertian tensor norms induce sets of bipartite systems which are closed under wirings. In other words, we have that

$$\mathcal{R}_m^{\pi} = cl(\mathcal{R}_m^{\pi}) , \quad (5.24)$$
$$\mathcal{R}_m^{\gamma_2^*} = cl(\mathcal{R}_m^{\gamma_2^*}) . \quad (5.25)$$

That $\mathcal{R}_m^{\pi}$ is closed under wirings was of course already clear after Section 5.1.3, where we proved that $\mathcal{R}_m^{\pi} = \mathcal{L}_m$. However, the fact that $\mathcal{R}_m^{\gamma_2^*}$ is closed under wirings is not trivial. As we have seen in Section 5.1.6, the sets induced by the dual Hilbertian tensor norm γ_2^* have an interesting structure. Namely, they are not comparable to the quantum set in the sense that there exist quantum systems which are not in $\mathcal{R}_m^{\gamma_2^*}$ and there are bipartite systems in $\mathcal{R}_m^{\gamma_2^*}$ which are not quantum. We will exploit this special property of the theory $\mathcal{R}_m^{\gamma_2^*}$ in Chapter 6. Furthermore, $\mathcal{R}_m^{\gamma_2^*}$ is the only known "non-trivial" theory (i.e., excluding $\mathcal{L}_m$, $\mathcal{Q}_m$ and $\mathcal{NS}_m$) besides $\mathcal{Q}_m^1$ which is explicitly defined.

Do there exist other tensor norms, beside π and γ_2^*, that induce theories? The answer turns out to be yes. There exist tensor norms, denoted by $\gamma_{p,q}$, with $1/p + 1/q = 1$ and $p, q \in [1, \infty]$ (see [DF93] on page 366) that obey direct-product theorems. The proof is similar to the one given in Theorem 3.2 but one uses a generalization of Lemma 3.3, namely that $\|A \otimes B\|_{p\to p} = \|A\|_{p\to p} \cdot \|B\|_{p\to p}$ holds [Ben77]. Hence, by using Corollary 5.1, it follows that $\mathcal{R}_m^{\gamma_{p,q}}$ are theories. Note that γ_2^* is a member of this family of tensor norms, namely it holds that $\gamma_2^* = \gamma_{2,2}$. However, it seems that only for the case where $p = q = 2$, one can find an algorithm (see Section 4.3) which computes $\gamma_{p,q}$. Hence, we do not know much about how these sets look like and how they compare to the quantum and local theories. The only fact we know is that the sets induced by $\gamma_{p,q}$ cannot be much larger than the quantum set since $\gamma_2^*(P) \leq K_G \cdot \gamma_{p,q}(P)$, for all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, and for any choice of p, q with $1/p + 1/q = 1$ (see for example [DF93]).

Chapter 6

Applications in Quantum Information Theory

6.1 Introduction

We will present four applications of the tensor-norm framework in quantum information theory. The first two applications are about properties of entangled two-prover games and the last two about the behaviour of bipartite systems under wirings.

In particular, we prove in Section 6.2 upper bounds on the entangled value of two-prover games. A parallel-repetition theorem for entangled XOR games is shown in Section 6.3. Section 6.4 investigates the question whether binary quantum systems can be seen as a universal resource for quantum theory. Finally, in Section 6.5 we provide arguments that support the impossibility of non-locality distillation for isotropic quantum systems.

6.2 Upper Bound on Entangled Values of Games

6.2.1 Introduction

How distinct are classical and quantum physics? There exist different ways to *measure* the "difference" between these two theories. The measure we will study in this section is based on two-prover games. We analyse how much larger the value of a two-prover game can be if the two provers can use entanglement instead of only shared randomness. In particular, we will compute an upper bound on the maximal ratio between the entangled and the classical value of two-prover games (see also Section 2.5.2):

$$v := \sup_G \left\{ \frac{\omega_{\mathcal{Q}}(G)}{\omega_{\mathcal{L}}(G)} \; : \; G = (\pi, V) \text{ two-prover game} \right\} .$$

We will see that this ratio is related to *Grothendieck's inequality* [Gro53]. This famous inequality establishes a connection between the injective and the dual Hilbertian tensor norm:

$$\gamma_2^*(G; \ell_1^{|\mathcal{X}|}, \ell_1^{|\mathcal{Y}|}) \le K_G \cdot \varepsilon(G; \ell_1^{|\mathcal{X}|}, \ell_1^{|\mathcal{Y}|}), \text{ for all } G \in \ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|} , \tag{6.1}$$

where $1.68 \lesssim K_G \lesssim 1.78$ is the *Grothendieck constant*. The exact value of Grothendieck's constant K_G is still unknown. The best known upper bound on K_G is given by $\frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.78$ [Kri79]. Recently, it has been shown that this upper bound is not tight [BMMN11]. The best lower bound is $K_G \ge 1.6770$ due to Reeds and Davis [Dav84, Ree91]. Finally, note that Grothendieck's inequality is a deep result in functional analysis and has been called by [DF93] "certainly the most exciting relationship between tensor norms".

Contribution

We prove an upper bound on the maximal ratio between the entangled and the classical value of arbitrary two-prover games (see Theorem 6.4). In order to achieve this, we introduce a generalization of the standard Grothendieck inequality given in (6.1) that allows us to study settings where Alice and Bob have multiple outputs. The results of this section have been published in [Duk11].

Related Work

Tsirelson has shown that v is upper bounded by the Grothendieck constant K_G for XOR games [Tsi87]. This result establishes the first connection between tensor norms and quantum information theory.

In [DKLR09], it has been shown that $v \le O(|\mathcal{A}| \cdot |\mathcal{B}|)$, independently of the input dimensions $|\mathcal{X}|$ and $|\mathcal{Y}|$. If we fix the dimension of the local Hilbert spaces to d in the computation of $\omega_{\mathcal{Q}}(G)$, it has been shown

[JPPG$^+$10b] that $v \leq O(d)$, independently of the input and output dimensions. Note that these two results also hold if one considers the more general setting of Bell inequalities instead of two-prover games.

Using tools from operator space theory, Junge and Palazuelos [JP11] study large violations of Bell inequalities. In order to prove that their results are almost tight, they also provide results corresponding to our Theorem 6.1 (the generalized Grothendieck inequality) and Theorem 6.4 (the upper bound on the entangled value of two-prover games). Note that their result is more general as it holds for Bell inequalities as well. This line of research is a continuation of [JPPG$^+$10a, JPPG$^+$10b], where it is shown that operator space theory is a natural framework to study arbitrary Bell inequalities. The authors of [BRSdW11] improve the work of [JP11] by providing explicit two-prover games in order to establish near optimal lower bounds on the ratio between the quantum and classical value of Bell inequalities.

Grothendieck's inequality has been generalized in different ways before. The latest generalization can be found in [BBT11], where references to other previous generalizations [Rie74, FR94, AMMN05] are provided.

The standard Grothendieck inequality and, in particular, the tensor norm γ_2 and its dual γ_2^*, have not only applications in quantum information theory, but are also used to prove lower bounds in communication complexity [LMSS07, LS07, LSS08]. Finally, Grothendieck's inequality serves as an inspiration to derive new semidefinite programs which can be used to approximate computationally hard problems [AN04, CW04].

6.2.2 Tensor Norms and Values of Two-Prover Games

The results of Chapter 5 immediately imply an upper bound on the *classical* value of two-prover games (see also Section 2.5.2 and Section 3.1.1):

$$\begin{aligned}
\omega_{\mathcal{L}}(G) \quad &:= \quad \sup_P \{|\langle G, P\rangle| \; : \; P \in \mathcal{L}_m\} \\
&\leq \quad \sup_P \{|\langle G, P\rangle| \; : \; \pi(P) \leq 1\} \\
&= \quad \varepsilon(G) \, ,
\end{aligned}$$

where we used Lemma 5.2 in the second line, and that ε is the dual tensor norm of π in the last line. However, it turns out that $\varepsilon(G)$ is also a lower bound on $\omega_{\mathcal{L}}(G)$.

Lemma 6.1. *Let* $G = (\pi, V)$ *be an arbitrary two-prover game with* $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. *Then*

$$\omega_{\mathcal{L}}(G) = \varepsilon(G) .$$

Proof. Let $\langle G, e_{x,a} \otimes e_{y,b} \rangle = \pi(x,y) \cdot V(a,b,x,y)$. By the definition of the injective tensor norm we have that

$$\begin{aligned}
\varepsilon(G) &= \sup_{P_A, P_B} \{ |\langle G, P_A \otimes P_B \rangle| \ : \ \|P_A\|_{\infty(1)} \leq 1, \|P_B\|_{\infty(1)} \leq 1 \} \\
&= \sup_{P_A, P_B} \left| \sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b,x,y) \cdot \langle f_{x,a}, P_A \rangle \cdot \langle f_{y,b}, P_B \rangle \right| ,
\end{aligned}$$

where supremum is over $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$. Thus, since $\pi(x,y) \cdot V(a,b,x,y) \geq 0$, it holds that $\langle f_{x,a}, P_A \rangle \geq 0$ and $\langle f_{y,b}, P_B \rangle \geq 0$ for the optimal case. Furthermore, the optimum is achieved when $\langle f_{x,a}, P_A \rangle$ and $\langle f_{y,b}, P_B \rangle$ are as large as possible, meaning that $\sum_a \langle f_{x,a}, P_A \rangle = 1$ and $\sum_b \langle f_{y,b}, P_B \rangle = 1$ for all $1 \leq x \leq |\mathcal{X}|$ and $1 \leq y \leq |\mathcal{Y}|$, respectively. But this implies that P_A and P_B correspond to systems of Alice and Bob, respectively, and ,therefore, the injective tensor norm of G is the same as the classical value of the game G. □

Similarly as in the case for the classical value of two-prover games we can upper bound the *entangled* value of two-prover games by a tensor norm.

Lemma 6.2. *Let* $G = (\pi, V)$ *be an arbitrary two-prover game with* $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$. *Then*

$$\omega_{\mathcal{Q}}(G) \leq \gamma_2^*(G) .$$

Proof. The statement follows from

$$\begin{aligned}
\omega_{\mathcal{Q}}(G) &:= \sup_P \{ |\langle G, P \rangle| \ : \ P \in \mathcal{Q}_m \} \\
&\leq \sup_P \{ |\langle G, P \rangle| \ : \ \gamma_2(P) \leq 1 \} \\
&= \gamma_2^*(G) ,
\end{aligned}$$

where we used Theorem 5.1 in the second line and that γ_2 is the dual of γ_2^* in the third line. □

6.2.3 A Generalized Grothendieck Inequality

We have stated in Section 3.2.3 that α is a cross norm if and only if it attains values between the injective tensor norm ε and the projective tensor norm π. Since γ_2 is a cross norm, we therefore have

$$\gamma_2(S; X, Y) \leq \pi(S; X, Y) ,$$

for all $S \in X \otimes Y$, and all pairs of finite-dimensional normed vector spaces (X, Y). Furthermore, in Section 2.1.2, we have seen that norms on finite-dimensional spaces are always *equivalent*, i.e., that there exists a constant $c(X, Y) \in \mathbb{R}$ depending on the normed vector spaces X and Y such that

$$\pi(S; X, Y) \leq c(X, Y) \cdot \gamma_2(S; X, Y) , \tag{6.2}$$

for all $S \in X \otimes Y$. The goal of this section is to find the smallest $c(X, Y) \in \mathbb{R}$ such that (6.2) is true given the local normed vector spaces $X := \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $Y := \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

If we consider the special case of correlation systems (see Section 2.3.4) and, therefore, of $X := \ell_\infty^{|\mathcal{X}|}$ and $Y := \ell_\infty^{|\mathcal{Y}|}$, the *standard Grothendieck inequality* [Gro53] (in tensor form) tells us that

$$\pi(P; \ell_\infty^{|\mathcal{X}|}, \ell_\infty^{|\mathcal{Y}|}) \leq K_G \cdot \gamma_2(P; \ell_\infty^{|\mathcal{X}|}, \ell_\infty^{|\mathcal{Y}|}) , \tag{6.3}$$

for all $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$. This is the dual version of (6.1). Note that this upper bound is independent of the input alphabet sizes $|\mathcal{X}|$ and $|\mathcal{Y}|$.

Before we can prove the generalized Grothendieck inequality in tensor form we need an additional result:

Lemma 6.3 (Alon & Naor [AN04]). *For any sets $\{x_i\}_{1\leq i\leq n}$ and $\{y_j\}_{1\leq j\leq m}$ of real unit vectors in a Hilbert space $\mathcal{H}$, there are sets $\{\tilde{x}_i\}_{1\leq i\leq n}$ and $\{\tilde{y}_j\}_{1\leq j\leq m}$ of real unit vectors in a Hilbert space $\tilde{\mathcal{H}}$, such that*

$$\langle x_i, y_j \rangle = \frac{\pi}{2\ln(1+\sqrt{2})} \int_{\tilde{\mathcal{H}}} sign\langle \tilde{x}_i, z\rangle \cdot sign\langle \tilde{y}_j, z\rangle \gamma(dz) ,$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$, where $\gamma(dz)$ is the normalized Gauss measure on $\tilde{\mathcal{H}}$.

Theorem 6.1 (Generalized Grothendieck Inequality in Tensor Form). *For any $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, it holds that*

$$\pi(P) \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|} \cdot \gamma_2(P) .$$

Proof. Let us assume that $\gamma_2(P) = 1$ for some $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Showing that $\pi(P) \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|}$ proves the theorem. Since $\gamma_2(P) = 1$, we can conclude according to Lemma 4.6 in Section 4.4.1, that there exist real vectors $\{m_{x,a}\}$ and $\{n_{y,b}\}$ in ℓ_2 with

$$\|(m_{x,a})\|_{1(\infty)\to 2} \leq 1 \; , \; \|(n_{y,b})^T\|_{2\to\infty(1)} \leq 1 \; ,$$

such that

$$P = \sum_{x,y,a,b} \langle m_{x,a}, n_{y,b}\rangle \cdot e_{x,a} \otimes e_{y,b} \; . \tag{6.4}$$

Applying the second part of Lemma 4.7 yields

$$\sum_{a\in\mathcal{A}} \|m_{x,a}\|_2^2 \leq 1 \; , \; \sum_{b\in\mathcal{B}} \|n_{y,b}\|_2^2 \leq 1 \; ,$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Let $K := \frac{\pi}{2\ln(1+\sqrt{2})}$. Applying Lemma 6.3 on the vectors $\{m_{x,a}\}$ and $\{n_{y,b}\}$ yields

$$\left\langle \frac{m_{x,a}}{\|m_{x,a}\|_2}, \frac{n_{y,b}}{\|n_{y,b}\|_2} \right\rangle = K \int_{\tilde{\mathcal{H}}} sign\langle \tilde{m}_{x,a}, z\rangle \cdot sign\langle \tilde{n}_{y,b}, z\rangle \gamma(dz) \; , \tag{6.5}$$

where $\tilde{m}_{x,a}$ and $\tilde{n}_{y,b}$ are unit vectors for all $x \in \mathcal{X}, y \in \mathcal{Y}, a \in \mathcal{A}$ and $b \in \mathcal{B}$. Combining (6.4) and (6.5) gives

$$\begin{aligned} P \;\; = \;\; & K \int_{\tilde{\mathcal{H}}} \left(\sum_{x,a} \|m_{x,a}\|_2 \cdot sign\langle \tilde{m}_{x,a}, z\rangle \cdot e_{x,a} \right) \\ \otimes \;\; & \left(\sum_{y,b} \|n_{y,b}\|_2 \cdot sign\langle \tilde{n}_{y,b}, z\rangle \cdot e_{y,b} \right) \gamma(dz) \; . \end{aligned}$$

Since π is a norm, we can apply the triangle inequality and get

$$\begin{aligned} \pi(P) \;\; \leq \;\; & K \cdot \sup_z \left\| \sum_{x,a} \|m_{x,a}\|_2 \cdot sign\langle \tilde{m}_{x,a}, z\rangle \cdot e_{x,a} \right\|_{\infty(1)} \\ \cdot \;\; & \left\| \sum_{y,b} \|n_{y,b}\|_2 \cdot sign\langle \tilde{n}_{y,b}, z\rangle \cdot e_{y,b} \right\|_{\infty(1)} \; , \end{aligned}$$

where we also used that π is a cross norm and, therefore, $\pi(P_A \otimes P_B) = \|P_A\|_{\infty(1)} \cdot \|P_B\|_{\infty(1)}$. Furthermore, by using the definition of the $\infty(1)$-norm, we have that

$$\left\| \sum_{x,a} \|m_{x,a}\|_2 \cdot sign\langle \tilde{m}_{x,a}, z\rangle \cdot e_{x,a} \right\|_{\infty(1)} \leq \max_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} \|m_{x,a}\|_2 ,$$

for any z. Using that $\sum_{a\in\mathcal{A}} \|m_{x,a}\|_2^2 \leq 1$ implies $\sum_{a\in\mathcal{A}} \|m_{x,a}\|_2 \leq \sqrt{|\mathcal{A}|}$ (by the Cauchy-Schwarz inequality) finishes the proof. □

By applying Lemma 2.1 we get the following dual theorem:

Theorem 6.2 (Generalized Grothendieck Inequality in Dual Tensor Form)**.** *For any $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ it holds that*

$$\gamma_2^*(G) \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|} \cdot \varepsilon(G) .$$

As the standard Grothendieck inequality is usually stated in matrix form, we will also give a matrix representation of our generalization. Since the $\|\cdot\|_{2\to\infty(1)}$ and $\|\cdot\|_{1(\infty)\to 2}$ operator norms will appear in the following theorem, it might be helpful for the reader to have a look at Lemma 4.7 again, which gives an alternative representation of these two operator norms.

Theorem 6.3 (Generalized Grothendieck Inequality in Matrix Form)**.** *For any set of real numbers $\{\alpha_{x,y}^{a,b}\}$, with $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$, it holds that*

$$\begin{aligned} & \sup \left| \sum_{x,y,a,b} \alpha_{x,y}^{a,b} \cdot \langle m_{x,a}, n_{y,b}\rangle \right| \\ \leq \quad & \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|} \cdot \sup \left| \sum_{x,y,a,b} \alpha_{x,y}^{a,b} \cdot \langle f_{x,a}, P_A\rangle \cdot \langle f_{y,b}, P_B\rangle \right| , \end{aligned}$$

where the first supremum is over vectors $m_{x,a}, n_{y,b} \in \ell_2$ such that $\|(m_{x,a})\|_{1(\infty)\to 2} \leq 1$ and $\|(n_{y,b})^T\|_{2\to\infty(1)} \leq 1$ and the second supremum over vectors $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ such that $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$, respectively.

Proof. Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ with

$$\langle G, e_{x,a} \otimes e_{y,b} \rangle := \alpha_{x,y}^{a,b} \,, \tag{6.6}$$

and $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Computing the injective tensor norm of G yields

$$\begin{aligned} \varepsilon(G) &= \sup |\langle G, P_A \otimes P_B \rangle| \\ &= \sup \left| \sum_{x,y,a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \cdot \langle f_{x,a} \otimes f_{y,b}, P_A \otimes P_B \rangle \right| \\ &= \sup \left| \sum_{x,y,a,b} \alpha_{x,y}^{a,b} \cdot \langle f_{x,a}, P_A \rangle \cdot \langle f_{y,b}, P_B \rangle \right| \,, \end{aligned} \tag{6.7}$$

where the supremums are over P_A and P_B with $\|P_A\|_{\infty(1)} \leq 1$ and $\|P_B\|_{\infty(1)} \leq 1$, respectively. On the other hand, using Lemma 4.6, we obtain

$$\begin{aligned} \gamma_2^*(G) &= \sup\{|\langle G, P \rangle| \;:\; \gamma_2(P) \leq 1\} \\ &= \sup\{|\langle G, P \rangle| \;:\; \|(m_{x,y})\|_{1(\infty)\to 2} \leq 1 \,,\; \|(n_{y,b})^T\|_{2\to\infty(1)} \leq 1\} \,, \end{aligned}$$

with $P = \sum_{x,y,a,b} \langle m_{x,a}, n_{y,b} \rangle \cdot e_{x,a} \otimes e_{y,b} \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Hence, by using (6.6), we obtain

$$\begin{aligned} \gamma_2^*(G) &= \sup \left| \sum_{x,y,a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \cdot \langle f_{x,a} \otimes f_{y,b}, P \rangle \right| \\ &= \sup \left| \sum_{x,y,a,b} \alpha_{x,y}^{a,b} \cdot \langle m_{x,a}, n_{y,b} \rangle \right| \,, \end{aligned}$$

where the supremum is over vectors $m_{x,a}, n_{y,b} \in \ell_2$ such that $\|(m_{x,a})\|_{1(\infty)\to 2} \leq 1$ and $\|(n_{y,b})^T\|_{2\to\infty(1)} \leq 1$. Equations (6.7) and (6.8), together with Theorem 6.2, yield the result. □

By setting $|\mathcal{A}| = |\mathcal{B}| = 1$ in Theorem 6.3 we obtain the standard Grothendieck inequality in matrix form:

$$\sup \left| \sum_{x,y} \alpha_{x,y} \cdot \langle m_x, n_y \rangle \right| \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sup \left| \sum_{x,y} \alpha_{x,y} \cdot s_x \cdot t_y \right| \,,$$

where the first supremum is over *unit* vectors $m_x, n_y \in \ell_2$ and the second supremum over $s_x, t_y \in \{-1, +1\}$. The reason for calling it "matrix form" stems from the fact that the real numbers $\alpha_{x,y}$ can be interpreted as a real $|\mathcal{X}| \times |\mathcal{Y}|$-matrix.

6.2.4 Result

Theorem 6.4. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ denote a two-prover game with fixed and finite input and output alphabets. Then*

$$\sup_G \frac{\omega_{\mathcal{Q}}(G)}{\omega_{\mathcal{L}}(G)} \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|}\ ,$$

independently of the input alphabet sizes $|\mathcal{X}|$ and $|\mathcal{Y}|$.

Proof. By using Lemma 6.1, Lemma 6.2, and the dual of the generalized Grothendieck inequality in tensor form given in Theorem 6.2, we obtain

$$\sup_G \frac{\omega_{\mathcal{Q}}(G)}{\omega_{\mathcal{L}}(G)} \leq \sup_G \frac{\gamma_2^*(G)}{\varepsilon(G)} \leq \frac{\pi}{2\ln(1+\sqrt{2})} \cdot \sqrt{|\mathcal{A}||\mathcal{B}|}\ ,$$

where the supremum is over two-prover games $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$.
□

6.2.5 Discussion

Theorem 6.4 can be seen as a generalization of Tsirelson's work [Tsi87]. In particular, an XOR game $G = (\pi, V)$ can be interpreted as an element of $\ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|}$ with the respective correlation systems $P \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ containing elements in $[-1, 1]$ corresponding to expectation values. Lemma 6.1 and Lemma 6.2 (in this case even with equality) can also be proven for this setting of correlation systems. Therefore, by using the standard Grothendieck inequality in dual tensor form (6.1), one obtains [Tsi87]

$$\sup_G \frac{\omega_{\mathcal{Q}}(G)}{\omega_{\mathcal{L}}(G)} = \sup_G \frac{\gamma_2^*(G)}{\varepsilon(G)} \leq K_G\ ,$$

for G an XOR game, independently of the input dimensions $|\mathcal{X}|$ and $|\mathcal{Y}|$.

We have shown that for a two-prover game $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, the injective tensor norm and classical value of the game are equal, i.e., that $\varepsilon(G) = \omega_{\mathcal{L}}(G)$ (see Lemma 6.1). For Bell inequalities there is no equality relation any more. It only holds that $\omega_{\mathcal{L}}(G) \leq \varepsilon(G)$ for all Bell inequalities G. The reason for losing the equality stems from the fact that, in contrast to two-prover games, a Bell inequality can have *negative* entries. Furthermore, the fact that $\omega_{\mathcal{L}}(G)$ is not equal to $\varepsilon(G)$ for Bell inequalities G is the reason for our proof of Theorem 6.4 not going through for Bell inequalities.

6.3 Parallel Repetition of Entangled Games

6.3.1 Introduction

Assume the setting where a two-prover game $G = (\pi, V)$ is repeated N times (see also Section 2.5.2). Either the game is repeated *sequentially*, i.e., a full round is completed before a new round is started, or in *parallel*. In the latter case, N mutually independent pairs of inputs (x_i, y_i) are chosen according to the distribution π and sent to the provers. The provers then compute outputs $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$, respectively. Finally, the predicate V is evaluated for all tuples (a_i, b_i, x_i, y_i) separately. This N-fold repetition of a game G can be seen as a new game, denoted by $G^{\odot N}$, where this new game is only won if *all* N rounds are won.

For sequential composition, this probability is obviously equal to the probability of winning a single game taken to the power of N. However, for parallel composition the problem gets more involved as it is generally not true that $\omega_{\mathcal{L}}(G^{\odot N})$ is equal to $\omega_{\mathcal{L}}(G)^N$, as shown in [For89]. The same is true for entangled games, i.e., there exist games such that $\omega_{\mathcal{Q}}(G^{\odot N}) > \omega_{\mathcal{Q}}(G)^N$ [KR10]. Note that $\omega_{\mathcal{L}}(G^{\odot N}) \geq \omega_{\mathcal{L}}(G)^N$ and $\omega_{\mathcal{Q}}(G^{\odot N}) \geq \omega_{\mathcal{Q}}(G)^N$ is obviously true for all games G. However, $\omega_{\mathcal{L}}(G^{\odot N})$ cannot become arbitrarily large. It has been shown that the quantity $\omega_{\mathcal{L}}(G^{\odot N})$ decreases exponentially fast in N. A first proof of this fact, also known as the *Parallel Repetition Theorem*, has been given in [Raz95]. Raz's proof has been simplified in [Hol07] and extended to the case of provers using arbitrary non-signalling systems. It is not known whether there exists a parallel-repetition theorem for entangled games as well.

Contribution

We provide an alternative proof of the perfect parallel-repetition theorem for entangled XOR games (see Theorem 6.5) which was first shown in [CSUU07]. The results of this section have been published in [Duk11].

Related Work

Parallel repetition theorems are known for entangled unique and entangled XOR games. Quantitatively, it is known that if $G = (\pi, V)$ is a two-prover game then, for all $N \geq 1$, it holds that

$$\omega_{\mathcal{Q}}(G^{\odot N}) = \omega_{\mathcal{Q}}(G)^N \ ,$$

if G is an XOR game [CSUU07], and

$$\omega_{\mathcal{Q}}(G^{\odot N}) \leq \left(1 - \frac{(1-\omega_{\mathcal{Q}}(G))^2}{16}\right)^N \ ,$$

if G is a unique game [KRT08].

6.3.2 Dual Hilbertian Tensor Norm and XOR Games

Let us first prove a tight connection between the entangled value of XOR games and the dual Hilbertian tensor norm.

Lemma 6.4. *Let $G = (\pi, V)$ be an XOR game with $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and $|\mathcal{A}| = |\mathcal{B}| = 2$. Then*

$$\omega_{\mathcal{Q}}(G) = \gamma_2^*(G) \ .$$

Proof. That $\omega_{\mathcal{Q}}(G) \leq \gamma_2^*(G)$ follows from Lemma 6.2. So let us show that $\omega_{\mathcal{Q}}(G) \geq \gamma_2^*(G)$. Let $\mathcal{A} = \mathcal{B} = \{1,2\}$, $P_{x,y}^{a,b} := \langle f_{x,a} \otimes f_{y,b}, P\rangle$, with $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, and $\langle G, e_{x,a} \otimes e_{y,b}\rangle = \pi(x,y) \cdot V(a,b,x,y)$. Then

$$\begin{aligned}
\gamma_2^*(G) &= \sup\{|\langle G, P\rangle| \ : \ \gamma_2(P) \leq 1\} \\
&= \sup\left\{\sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b,x,y) \cdot P_{x,y}^{a,b} \ : \ \gamma_2(P) \leq 1\right\} \\
&= \sup\left\{\sum_{x,y} \pi(x,y) \big(V(a=b,x,y) \cdot (P_{x,y}^{1,1} + P_{x,y}^{2,2})\right. \\
&+ \left. V(a \neq b, x,y) \cdot (P_{x,y}^{1,2} + P_{x,y}^{2,1})\big) \ : \ \gamma_2(P) \leq 1\right\} \ ,
\end{aligned} \tag{6.8}$$

where we used the fact that G is an XOR game, i.e., we have that $V(a = b, x, y) := V(1,1,x,y) = V(2,2,x,y) \in \{0,1\}$ and $V(a \neq b, x, y) := V(1,2,x,y) = V(2,1,x,y) \in \{0,1\}$. We do not have to take the absolute value since $\gamma_2(P) = \gamma_2((-1) \cdot P)$. Lemma 4.6 and Lemma 4.7 imply, together with $\gamma_2(P) \leq 1$, the following constraints (see also Theorem 4.2):

$$\|m_{x,1} \pm m_{x,2}\|_2 \leq 1 \ , \ \|n_{y,1} \pm n_{y,2}\|_2 \leq 1 \ , \ P_{x,y}^{a,b} = \langle m_{x,a}, n_{y,b}\rangle \ , \tag{6.9}$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$. We cannot hope to simulate $P_{x,y}^{a,b}$ in (6.8) by some measurements on a quantum state, since these values can be negative and do not have to correspond to valid probabilities. But what we can do is to show that there exists a quantum state $|\Psi\rangle$ and observables $A_1, ..., A_{|\mathcal{X}|}$ and $B_1, ..., B_{|\mathcal{Y}|}$ with binary outcomes, such that

$$\begin{aligned} \Pr[a = b|A_x, B_y, |\Psi\rangle] &\geq P_{x,y}^{1,1} + P_{x,y}^{2,2} && (6.10) \\ \Pr[a \neq b|A_x, B_y, |\Psi\rangle] &\geq P_{x,y}^{1,2} + P_{x,y}^{2,1} \, , && (6.11) \end{aligned}$$

where a is the outcome of Alice's measurement A_x and b the outcome of Bob's measurement B_y. So, if we assume that (6.10) holds, we get

$$\begin{aligned} \gamma_2^*(G) &= \sup \Bigg\{ \sum_{x,y} \pi(x,y)(V(a=b,x,y) \cdot (P_{x,y}^{1,1} + P_{x,y}^{2,2}) \\ &+ V(a \neq b, x, y) \cdot (P_{x,y}^{1,2} + P_{x,y}^{2,1})) \ : \ \gamma_2(P) \leq 1 \big\} \\ &\leq \sum_{x,y} \pi(x,y) \left(V(a=b,x,y) \Pr[a=b|A_x, B_y, |\Psi\rangle] \right. \\ &+ \left. V(a \neq b, x, y) \Pr[a \neq b|A_x, B_y, |\Psi\rangle] \right) \\ &\leq \omega_{\mathcal{Q}}(G) \, . \end{aligned}$$

It remains to be shown that (6.10) and (6.11) hold. First, note that (6.10) can be rewritten as

$$\begin{aligned} \Pr[a = b|A_x, B_y, |\Psi\rangle] &\geq \langle m_{x,1}, n_{y,1} \rangle + \langle m_{x,2}, n_{y,2} \rangle \, , && (6.12) \\ \Pr[a \neq b|A_x, B_y, |\Psi\rangle] &\geq \langle m_{x,1}, n_{y,2} \rangle + \langle m_{x,2}, n_{y,1} \rangle \, , && (6.13) \end{aligned}$$

by using (6.9). Second, we set $m_x := m_{x,1} - m_{x,2}$ and $n_y := n_{y,1} - n_{y,2}$, apply the second part of Lemma 5.4 (which we are allowed to use because of the constraints given in (6.9)) and get observables $A_1, ..., A_{|\mathcal{X}|}$ and $B_1, ..., B_{|\mathcal{Y}|}$ with eigenvalue ± 1 and a quantum state $|\Psi\rangle$ such that

$$\langle m_x, n_y \rangle = \langle \Psi | A_x \otimes B_y | \Psi \rangle \, .$$

Since $\langle \Psi | A_x \otimes B_y | \Psi \rangle$ is the expectation value when measuring the observables A_x and B_y with eigenvalues ± 1, we have that

$$\begin{aligned} \Pr[a = b|A_x, B_y, |\Psi\rangle] &= \frac{1 + \langle \Psi | A_x \otimes B_y | \Psi \rangle}{2} = \frac{1 + \langle m_x, n_y \rangle}{2} \, , && (6.14) \\ \Pr[a \neq b|A_x, B_y, |\Psi\rangle] &= \frac{1 - \langle \Psi | A_x \otimes B_y | \Psi \rangle}{2} = \frac{1 - \langle m_x, n_y \rangle}{2} \, . && (6.15) \end{aligned}$$

By straightforward calculations (6.14) implies (6.12), where the conditions $\|m_{x,1}+m_{x,2}\|_2 \leq 1$ and $\|n_{y,1}+n_{y,2}\|_2 \leq 1$ of (6.9) are used. And similarly, we prove that (6.13) holds as well. $\square$

6.3.3 Introducing New Local Norms

In the proof of the main result of this section we will need Theorem 5.1 which establishes a connection between quantum systems and the Hilbertian tensor norm γ_2. Note, however, that Theorem 5.1 only applies in the setting where the local normed vector spaces are $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $\ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$, and, hence, it does not support the treatment of multiple systems. The goal of this section is to introduce the tools that allow us to consider Theorem 5.1 in a setting where the local normed vector spaces are $E_{\infty(1)}^{A^n}$ and $E_{\infty(1)}^{B^n}$ (which we have introduced in Section 3.3.2).

First, let us introduce new local normed vector spaces, denoted by $\Lambda_{\infty(1)}^{A^n}$ and $\Lambda_{1(\infty)}^{A^n}$, which are generalizations of the normed vector spaces $\ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|})$, respectively, to several systems:

$$\begin{aligned}
\Lambda_{\infty(1)}^{A^n} &:= \left(\ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|}), \|\cdot\|_{\infty(1)^n}\right) , \\
\Lambda_{1(\infty)}^{A^n} &:= \left(\ell_1^{|\mathcal{X}_1|}(\ell_\infty^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_1^{|\mathcal{X}_n|}(\ell_\infty^{|\mathcal{A}_n|}), \|\cdot\|_{1(\infty)^n}\right) ,
\end{aligned}$$

where the $\infty(1)^n$ and $1(\infty)^n$-norms are defined as

$$\begin{aligned}
\|P_A\|_{\infty(1)^n} &:= \max_{x_1,\ldots,x_n} \sum_{a_1,\ldots,a_n} |\langle f_{x_1,a_1} \otimes ... \otimes f_{x_n,a_n}, P_A\rangle| , \\
\|G_A\|_{1(\infty)^n} &:= \sum_{x_1,\ldots,x_n} \max_{a_1,\ldots,a_n} |\langle G_A, e_{x_1,a_1} \otimes ... \otimes e_{x_n,a_n}\rangle| ,
\end{aligned}$$

with $P_A \in \Lambda_{\infty(1)}^{A^n}$ and $G_A \in \Lambda_{1(\infty)}^{A^n}$. Note that $1(\infty)^n$ is the dual norm of $\infty(1)^n$.

Now, let us define $\mathcal{X}^n := \mathcal{X}_1 \times ... \times \mathcal{X}_n$ and $\mathcal{A}^n := \mathcal{A}_1 \times ... \times \mathcal{A}_n$. Then, we can prove the following result which shows that the normed vector spaces $\ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|})$ and $\Lambda_{\infty(1)}^{A^n}$ are essentially the same mathematical object:

Lemma 6.5. *Let the linear map $\mathcal{P}_A : \Lambda_{1(\infty)}^{A^n} \to \ell_1^{|\mathcal{X}^n|}(\ell_\infty^{|\mathcal{A}^n|})$ be defined as*

$$\mathcal{P}_A(f_{x_1,a_1} \otimes ... \otimes f_{x_n,a_n}) := f_{x_1} \otimes ... \otimes f_{x_n} \otimes f_{a_1} \otimes ... \otimes f_{a_n} =: f_{x^n} \otimes f_{a^n} ,$$

for all $x^n \in \mathcal{X}^n$ and all $a^n \in \mathcal{A}^n$ and with $x^n := (x_1, ..., x_n)$ and $a^n := (a_1, ..., a_n)$. Then, it holds that

$$\|\mathcal{P}_A^T(P_A)\|_{\infty(1)^n} = \|P_A\|_{\infty(1)} ,$$

for all $P_A \in \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|})$ and, therefore,

$$\|\mathcal{P}_A^T\|_{\infty(1)\to\infty(1)^n} \leq 1 ,$$

where $\mathcal{P}_A^T : \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}) \to \Lambda_{\infty(1)}^{A^n}$ is the transposed map of $\mathcal{P}_A$.

Proof. First, note that $\mathcal{P}_A$ is a permutation. Then, by the definition of the $\infty(1)^n$-norm we obtain

$$\begin{aligned}
\|\mathcal{P}_A^T(P_A)\|_{\infty(1)^n} &= \max_{x_1,\dots,x_n} \sum_{a_1,\dots,a_n} |\langle f_{x_1,a_1} \otimes ... \otimes f_{x_n,a_n}, \mathcal{P}_A^T(P_A)\rangle| \\
&= \max_{x_1,\dots,x_n} \sum_{a_1,\dots,a_n} |\langle \mathcal{P}_A(f_{x_1,a_1} \otimes ... \otimes f_{x_n,a_n}), P_A\rangle| \\
&= \max_{x^n} \sum_{a^n} |\langle f_{x^n} \otimes f_{a^n}, P_A\rangle| \\
&= \|P_A\|_{\infty(1)} ,
\end{aligned}$$

for all $P_A \in \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|})$. This immediately implies the fact that $\|\mathcal{P}_A^T\|_{\infty(1)\to\infty(1)^n} \leq 1$. ☐

The map $\mathcal{P}_A^T$ is bijective and linear and, therefore, because of Lemma 6.5, one says that the normed vector space $\ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|})$ is *isometrically isomorphic* to $\Lambda_{\infty(1)}^{A^n}$. Hence, these two spaces can be considered identical for all practical purposes.

Furthermore, let us establish a connection between the $\infty(1)^n$-norm and the multipartite version of the injective tensor norm $\varepsilon_{\infty(1)}^{A^n}$ we have introduced in Section 3.3.2.

Lemma 6.6. *If $P_A \in \ell_\infty^{|\mathcal{X}_1|}(\ell_1^{|\mathcal{A}_1|}) \otimes ... \otimes \ell_\infty^{|\mathcal{X}_n|}(\ell_1^{|\mathcal{A}_n|})$ then*

$$\varepsilon_{\infty(1)}^{A^n}(P_A) \leq \|P_A\|_{\infty(1)^n} .$$

Proof. By definition of the $\varepsilon^{A^n}_{\infty(1)}$-norm, we have that

$$
\begin{aligned}
& \varepsilon^{A^n}_{\infty(1)}(P_A) \\
= & \sup_{G_{A_1},\dots,G_{A_n}} |\langle G_{A_1} \otimes \dots \otimes G_{A_n}, P_A\rangle| \\
\leq & \sup \sum_{\substack{a_1,..,a_n\\x_1,..,x_n}} |\langle G_{A_1}, e_{x_1,a_1}\rangle|..|\langle G_{A_n}, e_{x_n,a_n}\rangle||\langle f_{x_1,a_1} \otimes \dots \otimes f_{x_n,a_n}, P_A\rangle| \\
\leq & \sup_{x_1,..,x_n} \sum_{a_1,..,a_n} |\langle f_{x_1,a_1} \otimes \dots \otimes f_{x_n,a_n}, P_A\rangle| \\
= & \|P_A\|_{\infty(1)^n} \ ,
\end{aligned}
$$

where the supremum is over $G_{A_k} \in \ell_1^{|\mathcal{X}_k|}(\ell_\infty^{|\mathcal{A}_k|})$ such that $\|G_{A_k}\|_{1(\infty)} \leq 1$, for all $1 \leq k \leq n$, and where we used the triangle inequality in the first inequality and that $\|G_{A_k}\|_{1(\infty)} \leq 1$, for all $1 \leq k \leq n$, in the second inequality. □

6.3.4 Composite Two-Prover Games

Let $G = (\pi, V)$ be a two-prover game. Recall that we defined the corresponding tensor element $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ as (see Section 2.5.2)

$$\langle G, e_{x,a} \otimes e_{y,b}\rangle := \pi(x,y) \cdot V(a,b,x,y) \ ,$$

for all $x \in \mathcal{X}, a \in \mathcal{A}, y \in \mathcal{Y}$ and $b \in \mathcal{B}$. The composite game $G^{\odot n}$ is then an element of $\Pi^{A^n}_{1(\infty)} \otimes \Pi^{B^n}_{1(\infty)}$ (see Section 3.3.3). Now, let us define $f_{x^n,a^n} := (f_{x_1} \otimes \dots \otimes f_{x_n}) \otimes (f_{a_1} \otimes \dots \otimes f_{a_n})$ with $x^n := (x_1, \dots, x_n)$ and $a^n := (a_1, \dots, a_n)$. Hence, $x^n \in \mathcal{X}^n := \mathcal{X} \times \dots \times \mathcal{X}$ and $a^n \in \mathcal{A}^n := \mathcal{A} \times \dots \times \mathcal{A}$. The entangled value of the two-prover game $G^{\odot n}$ is then computed as

$$
\begin{aligned}
& \omega_{\mathcal{Q}}(G^{\odot n}) \qquad\qquad (6.16)\\
:= & \sup_P \sum_{\substack{x^n \in \mathcal{X}^n\\ y^n \in \mathcal{Y}^n}} \pi^n(x^n,y^n) \sum_{\substack{a^n \in \mathcal{A}^n\\ b^n \in \mathcal{B}^n}} V^n(a^n,b^n,x^n,y^n) \cdot \langle f_{x^n,a^n} \otimes f_{y^n,b^n}, P\rangle \ ,
\end{aligned}
$$

with $\pi^n(x^n,y^n) := \prod_i \pi(x_i,y_i)$, $V^n(a^n,b^n,x^n,y^n) := \prod_i V(a_i,b_i,x_i,y_i)$ and where the supremum is over all quantum systems $P \in \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}) \otimes \ell_\infty^{|\mathcal{Y}^n|}(\ell_1^{|\mathcal{B}^n|})$. Note that the quantum system P has input alphabets $\mathcal{X}^n$, $\mathcal{Y}^n$ and output alphabets $\mathcal{A}^n$, $\mathcal{B}^n$. We can then prove the following lemma:

Lemma 6.7. *Let $G^{\odot n} \in \Pi_{1(\infty)}^{A^n} \otimes \Pi_{1(\infty)}^{B^n}$ be a composite two-prover game. Then*

$$\omega_{\mathcal{Q}}(G^{\odot n}) \leq \gamma_2^*(G^{\odot n}; \Pi_{1(\infty)}^{A^n}, \Pi_{1(\infty)}^{B^n}) \ .$$

Proof. Consider the map $\mathcal{P}_A : \Lambda_{1(\infty)}^{A^n} \to \ell_1^{|\mathcal{X}^n|}(\ell_\infty^{|\mathcal{A}^n|})$ given in Lemma 6.5. Applying it on the composite game $G^{\odot n} \in \Pi_{1(\infty)}^{A^n} \otimes \Pi_{1(\infty)}^{B^n}$ yields $(\mathcal{P}_A \otimes \mathcal{P}_B)(G^{\odot n}) \in \ell_1^{|\mathcal{X}^n|}(\ell_\infty^{|\mathcal{A}^n|}) \otimes \ell_1^{|\mathcal{Y}^n|}(\ell_\infty^{|\mathcal{B}^n|})$. Therefore, by using (6.16) the entangled value of the composite game $G^{\odot n}$ can be written as

$$\omega_{\mathcal{Q}}(G^{\odot n}) = \sup_P |\langle (\mathcal{P}_A \otimes \mathcal{P}_B)(G^{\odot n}), P \rangle| \ ,$$

where the supremum is over all quantum systems $P \in \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}) \otimes \ell_\infty^{|\mathcal{Y}^n|}(\ell_1^{|\mathcal{B}^n|})$. Using Theorem 5.1 implies then

$$\begin{aligned}
& \omega_{\mathcal{Q}}(G^{\odot n}) \\
\leq \ & \sup_P \left\{ |\langle (\mathcal{P}_A \otimes \mathcal{P}_B)(G^{\odot n}), P \rangle| \ : \ \gamma_2(P; \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}), \ell_\infty^{|\mathcal{Y}^n|}(\ell_1^{|\mathcal{B}^n|})) \leq 1 \right\} \\
= \ & \sup_P \left\{ |\langle G^{\odot n}, (\mathcal{P}_A^T \otimes \mathcal{P}_B^T)(P) \rangle| \ : \ \gamma_2(P; \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}), \ell_\infty^{|\mathcal{Y}^n|}(\ell_1^{|\mathcal{B}^n|})) \leq 1 \right\} \\
\leq \ & \sup_P \left\{ |\langle G^{\odot n}, \mathcal{P}^T(P) \rangle| \ : \ \gamma_2(\mathcal{P}^T(P); \Lambda_{\infty(1)}^{A^n}, \Lambda_{\infty(1)}^{B^n}) \leq 1 \right\} \\
\leq \ & \sup_P \left\{ |\langle G^{\odot n}, P \rangle| \ : \ \gamma_2(P; \Lambda_{\infty(1)}^{A^n}, \Lambda_{\infty(1)}^{B^n}) \leq 1 \right\} \ ,
\end{aligned}$$

where we defined $\mathcal{P}^T := \mathcal{P}_A^T \otimes \mathcal{P}_B^T$ and where we used in the second inequality that

$$\gamma_2(P; \ell_\infty^{|\mathcal{X}^n|}(\ell_1^{|\mathcal{A}^n|}), \ell_\infty^{|\mathcal{Y}^n|}(\ell_1^{|\mathcal{B}^n|})) \geq \gamma_2((\mathcal{P}_A^T \otimes \mathcal{P}_B^T)(P); \Lambda_{\infty(1)}^{A^n}, \Lambda_{\infty(1)}^{B^n}) \ ,$$

which follows from the fact that γ_2 is a tensor norm, Lemma 3.1 and Lemma 6.5. Then, using Lemma 6.6 and the definition of the Hilbertian tensor norm γ_2 yields

$$\begin{aligned}
\omega_{\mathcal{Q}}(G^{\odot n}) \ & \leq \ \sup_P \left\{ |\langle G^{\odot n}, P \rangle| \ : \ \gamma_2(P; E_{\infty(1)}^{A^n}, E_{\infty(1)}^{B^n}) \leq 1 \right\} \\
& = \ \gamma_2^*(G^{\odot n}; \Pi_{1(\infty)}^{A^n}, \Pi_{1(\infty)}^{B^n}) \ ,
\end{aligned}$$

where we used the norm duality of (3.10) in the last line. □

6.3.5 Result

Let us provide an alternative proof for the parallel-repetition theorem for entangled XOR games given in [CSUU07]. The proof of [CSUU07] contains two parts. In the first part, they show that the sum of XOR games obeys a perfect product rule by using semidefinite programming techniques and then, in a second step, they use Fourier analysis to get a perfect parallel-repetition theorem for entangled XOR games. The first part corresponds to applying the direct-product result for the γ_2^* tensor norm on a game $\tilde{G} = (\pi, \tilde{V})$, where $\tilde{V}$ has range $\{-1, +1\}$ instead of $\{0, 1\}$ and $\tilde{G}$ is interpreted as an element of $\ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|}$. By using Lemma 5.5 in Section 5.1.5, it is not difficult to show that $\gamma_2^*(\tilde{G}) = 2 \cdot \omega_{\mathcal{Q}}(\tilde{G}) - 1$. Hence, $\gamma_2^*(\tilde{G})$ is the *quantum bias* of an XOR game, denoted by $\varepsilon_q(\tilde{G})$ in [CSUU07]. The second part is required because $\omega_{\mathcal{Q}}(\tilde{G})$ is a rescaling of $\gamma_2^*(\tilde{G})$, which is due to the fact that $\tilde{V}$ has range ± 1.

The crucial idea in our alternative proof is to interpret the XOR game G as an element of $\ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, with $|\mathcal{A}| = |\mathcal{B}| = 2$, instead of $\ell_1^{|\mathcal{X}|} \otimes \ell_1^{|\mathcal{Y}|}$.

Theorem 6.5. *Let $G = (\pi, V)$ be an XOR game with $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and $|\mathcal{A}| = |\mathcal{B}| = 2$. Then*

$$\omega_{\mathcal{Q}}(G^{\odot N}) = \omega_{\mathcal{Q}}(G)^N \ .$$

Proof. It is clear that $\omega_{\mathcal{Q}}(G^{\odot N}) \geq \omega_{\mathcal{Q}}(G)^N$ by executing the rounds individually. For the other direction, by Lemma 6.7, we have

$$\omega_{\mathcal{Q}}(G^{\odot N}) \leq \gamma_2^*(G^{\odot N}) \ .$$

Applying the direct-product result of Theorem 3.2 and using Lemma 6.4 yields

$$\omega_{\mathcal{Q}}(G^{\odot N}) \leq \gamma_2^*(G^{\odot N}) \leq \gamma_2^*(G)^N = \omega_{\mathcal{Q}}(G)^N \ .$$

□

6.4 On the Universality of Binary Quantum Systems

6.4.1 Introduction

Let us consider the following setting: Alice and Bob share entangled quantum states on which they can perform local measurements on their respective parts. However, due to experimental difficulties they can only apply measurements with binary outcomes. Nevertheless, they would like to have a bipartite quantum system at their disposal that has many inputs and outputs (for example, to play a two-prover game or to perform some two-party computations). The question arises whether they can obtain (by applying wirings) such arbitrary quantum systems from quantum systems with *binary* outcomes.

Formally, we will analyse the following question in this section: does for every quantum system $P \in \mathcal{Q}_m$ and any $\epsilon > 0$, exist a wiring $\mathcal{W}$, an integer $n \geq 1$, and a collection of quantum systems $P^1, ..., P^n$ with *binary* outcomes such that

$$\|P - \mathcal{W}(P^1, ..., P^n)\| \leq \epsilon ,$$

where $\| \cdot \|$ denotes an arbitrary norm on the set of systems? In other words, can we approximate any quantum system P arbitrarily well by applying a wiring on quantum systems with binary outcomes? If this is the case, we say that quantum systems with binary outcomes are *universal*.

Contribution

We provide partial answers to the above question. First, we prove that *isotropic* quantum systems are *not* universal, i.e., we show that there exists a quantum system (with ternary inputs and binary outputs) which cannot be approximated by isotropic quantum systems (see Theorem 6.6). Second, we show, based on a well-supported conjecture, that *binary* quantum systems are not universal, either (see Theorem 6.7).

Related Work

There is an alternative argument that shows that isotropic quantum systems are not universal. First, there exist Bell inequalities that are not optimally violated by maximally entangled quantum states [JP11, VW11, LVB11]. Second, the isotropic quantum system $P_{1/\sqrt{2}}$ is obtained by measurements on a maximally entangled quantum state. And third, joint measurements on several systems on Alice's and Bob's side, respectively, are more general operations than local strategies (since every classical operation can also be seen as a quantum operation). These three facts imply the result, too.

It has been shown in [FW09, FW11] that the PR-box P_{PR}, which is an *isotropic* system, is universal for bipartite *non-signalling* systems: for any bipartite non-signalling system $P \in \mathcal{NS}_m$ and any $\epsilon > 0$ there exists an $n \geq 1$ and a wiring $\mathcal{W} : \mathcal{NS}_{\mathrm{iso}}^{\times n} \to \mathcal{NS}_m$ such that

$$\|P - \mathcal{W}(P_{\mathrm{PR}}, ..., P_{\mathrm{PR}})\| \leq \epsilon \, .$$

Open Problems

Are quantum systems with binary outcomes and *arbitrary* inputs universal? Assuming this is indeed the case, it would be desirable to have an *efficient* algorithm which takes as input an arbitrary quantum system and outputs the wiring and the input systems which yield this particular quantum system.

6.4.2 System-Expectation Map

Let us introduce a linear map $\mathcal{E} : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}) \to \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$, with $\mathcal{A} = \mathcal{B} = \{1, 2\}$, called *system-expectation map*, which when given a system with binary outputs as input it computes the *expectation value* of the outputs for each input pair, where we have a value of $+1$ if the outputs are the same and -1 if they are different, i.e.,

$$\begin{aligned}\langle f_x \otimes f_y, \mathcal{E}(P)\rangle \quad &:= \quad \langle f_{x,1} \otimes f_{y,1}, P\rangle + \langle f_{x,2} \otimes f_{y,2}, P\rangle \\ &- \quad \langle f_{x,1} \otimes f_{y,2}, P\rangle - \langle f_{x,2} \otimes f_{y,1}, P\rangle \, , \end{aligned} \tag{6.17}$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ and all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$.

The map $\mathcal{E}$ has the property that it can be written as the tensor product of two local maps:

$$\mathcal{E}(P) = (\mathcal{E}_A \otimes \mathcal{E}_B)(P) \, , \tag{6.18}$$

with $\mathcal{E}_A : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \to \ell_\infty^{|\mathcal{X}|}$ and $\mathcal{E}_B : \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|}) \to \ell_\infty^{|\mathcal{Y}|}$. Let us prove this fact. First, we define the linear maps $\mathcal{E}_A$ and $\mathcal{E}_B$ as

$$\langle f_x, \mathcal{E}_A(P_A)\rangle \quad := \quad (+1) \cdot \langle f_{x,1}, P_A\rangle + (-1) \cdot \langle f_{x,2}, P_A\rangle \, , \tag{6.19}$$

$$\langle f_y, \mathcal{E}_B(P_B)\rangle \quad := \quad (+1) \cdot \langle f_{y,1}, P_B\rangle + (-1) \cdot \langle f_{y,2}, P_B\rangle \, , \tag{6.20}$$

with $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$ and $P_B \in \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$. Then, due to linearity, the application of the map $\mathcal{E}_A \otimes \mathcal{E}_B$ on some $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ can be written as

$$\langle f_x \otimes f_y, (\mathcal{E}_A \otimes \mathcal{E}_B)(P)\rangle = \sum_{i=1}^{n} \langle f_x, \mathcal{E}_A(P_A^i)\rangle\langle f_y, \mathcal{E}_B(P_B^i)\rangle \ , \tag{6.21}$$

where $P = \sum_i P_A^i \otimes P_B^i$ is an arbitrary decomposition. Plugging (6.19) and (6.20) into (6.21), yields, by some straightforward calculations, (6.17).

Lemma 6.8. *Let $\mathcal{E}_A : \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \to \ell_\infty^{|\mathcal{X}|}$ be given as in (6.19). Then*

$$\|\mathcal{E}_A(P_A)\|_\infty \leq \|P_A\|_{\infty(1)} \ ,$$

for all $P_A \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|})$.

Proof. By using the definition of $\mathcal{E}_A$ given in (6.19) we can write the correlation system $\mathcal{E}_A(P_A) \in \ell_\infty^{|\mathcal{X}|}$ as

$$\mathcal{E}_A(P_A) = \left(\begin{array}{c} (+1)\cdot\langle f_{1,1}, P\rangle + (-1)\cdot\langle f_{1,2}, P_A\rangle \\ \vdots \\ (+1)\cdot\langle f_{|\mathcal{X}|,1}, P\rangle + (-1)\cdot\langle f_{|\mathcal{X}|,2}, P_A\rangle \end{array} \right) ,$$

and, therefore, obtain

$$\begin{aligned} \|\mathcal{E}_A(P_A)\|_\infty &= \max_{x\in\mathcal{X}} |(+1)\cdot\langle f_{x,1}, P_A\rangle + (-1)\cdot\langle f_{x,2}, P_A\rangle| \\ &\leq \max_{x\in\mathcal{X}} |\langle f_{x,1}, P_A\rangle| + |\langle f_{x,2}, P_A\rangle| \\ &= \|P_A\|_{\infty(1)} \ , \end{aligned}$$

where we used the triangle inequality in the second line. □

6.4.3 Dual Hilbertian Tensor Norm and Ternary Systems

We have shown in Section 5.1.6 that there exists a quantum correlation system with ternary inputs which has dual Hilbertian tensor norm strictly larger than one (see Lemma 5.7). Let us now show that this implies that there also exists a quantum system with binary outcomes and ternary inputs for which this is the case. By using the system-expectation map of Section 6.4.2, we can prove the following result.

Lemma 6.9. *There exists a quantum system $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ with ternary inputs and binary outputs such that $\gamma_2^*(P) \geq 1.125$.*

Proof. From Lemma 5.7, we know that there exists a quantum correlation system $\tilde{P} \in \ell_\infty^{|\mathcal{X}|} \otimes \ell_\infty^{|\mathcal{Y}|}$ with ternary inputs such that $\gamma_2^*(\tilde{P}) \geq 1.125$. By using Lemma 5.4, we can conclude that there exists a quantum system $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ with ternary inputs and binary outputs such that $\mathcal{E}(P) = \tilde{P}$, with the system-expectation map $\mathcal{E}$ as given in (6.17). Since γ_2^* is a tensor norm and, therefore, Lemma 3.1 holds, Lemma 6.8 implies

$$\begin{aligned}
\gamma_2^*(P) &\geq \gamma_2^*((\mathcal{E}_A \otimes \mathcal{E}_B)(P)) \\
&= \gamma_2^*(\mathcal{E}(P)) \\
&= \gamma_2^*(\tilde{P}) \\
&\geq 1.125\,,
\end{aligned}$$

where we used (6.18) in the second line and Lemma 5.7 in the last line. □

6.4.4 Isotropic Quantum Systems are not Universal

Since $P_1 \equiv P_{\mathrm{PR}} \in \mathcal{NS}_{\mathrm{iso}}$ is a unit of bipartite non-locality [FW09], it is natural to ask whether the maximal isotropic *quantum* system $P_{1/\sqrt{2}} \in \mathcal{Q}_{\mathrm{iso}}$ is a universal resource for bipartite *quantum* systems. The answer turns out to be negative.

Theorem 6.6. *There exists a quantum system $P \in \mathcal{Q}_3$ with ternary inputs and binary outputs such that for any $n \geq 1$ there exists no wiring $\mathcal{W} : \mathcal{Q}_{\mathrm{iso}}^{\times n} \to \mathcal{Q}_3$ such that*

$$\left\| P - \mathcal{W}(P_{1/\sqrt{2}}, ..., P_{1/\sqrt{2}}) \right\| \leq \epsilon\,,$$

for some constant $\epsilon > 0$.

Proof. First note that $\|\cdot\|$ denotes an arbitrary norm on the set of systems. We will show that there exists a quantum system $P \in \mathcal{Q}_3$ with ternary inputs and binary outputs such that

$$\gamma_2^*\left(P - \mathcal{W}(P_{1/\sqrt{2}}, ..., P_{1/\sqrt{2}})\right) \geq 0.125\,, \tag{6.22}$$

for any $n \geq 1$ and all wirings $\mathcal{W} : \mathcal{Q}_{\mathrm{iso}}^{\times n} \to \mathcal{Q}_3$. Then, because γ_2^* is a norm defined over a *finite*-dimensional vector space it follows that

$$C \cdot \|P\| \leq \gamma_2^*(P) \leq D \cdot \|P\|\,, \tag{6.23}$$

for all $P \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ and some positive real constants C and D (see Section 2.1.2). Therefore, (6.22) and (6.23) imply the theorem.

According to Theorem 5.2 we can conclude that $\gamma_2^*(P_{1/\sqrt{2}}) = 1$ and, therefore, $P_{1/\sqrt{2}} \in \mathcal{R}_3^{\gamma_2^*}$ (see Chapter 5). Then, since the convex set induced by γ_2^* is closed under wirings (see (5.25) in Section 5.2.3) we obtain $\mathcal{W}(P_{1/\sqrt{2}}, ..., P_{1/\sqrt{2}}) \in \mathcal{R}_3^{\gamma_2^*}$. But this implies by definition of the set $\mathcal{R}_3^{\gamma_2^*}$, that

$$\gamma_2^* \left(\mathcal{W}(P_{1/\sqrt{2}}, ..., P_{1/\sqrt{2}}) \right) \leq 1 \,.$$

Together with Lemma 6.9 we obtain

$$\left| \gamma_2^*(P) - \gamma_2^* \left(\mathcal{W}(P_{1/\sqrt{2}}, ..., P_{1/\sqrt{2}}) \right) \right| \geq 0.125 \,,$$

which, by the reverse triangle inequality, implies (6.22). □

Hence, isotropic quantum systems are not universal. But what about arbitrary binary quantum systems? If we assume that Conjecture 1 of Section 5.1.6 holds, the answer is no again. Therefore, we can prove a stronger result than Theorem 6.6. Namely, even if an arbitrary collection of binary quantum systems is given as input to the wiring there exists a quantum system which cannot be obtained the wiring.

Theorem 6.7 (Based on Conjecture 1)**.** *There exists a quantum system $P \in \mathcal{Q}_3$ with ternary inputs and binary outputs such that for any $n \geq 1$ and any collection $P^1, P^2, ..., P^n \in \mathcal{Q}_{\mathrm{CHSH}}$, there exists no wiring $\mathcal{W} : \mathcal{Q}_{\mathrm{CHSH}}^{\times n} \to \mathcal{Q}_3$ such that*

$$\left\| P - \mathcal{W}(P^1, P^2, ..., P^n) \right\| \leq \epsilon \,,$$

for some constant $\epsilon > 0$.

Proof. By Conjecture 1, we can conclude that $\gamma_2^*(P^i) = 1$ for all $P^i \in \mathcal{Q}_{\mathrm{CHSH}}$. The proof is then basically the same as the proof of Theorem 6.6. □

6.5 Towards the Impossibility of Non-Locality Distillation

6.5.1 Introduction

We will analyse a special class of wirings in this section, called *non-locality distillation protocols*. Non-locality distillation is possible for a certain non-local system if there exists a wiring on several copies of this given system such that the output system is more non-local than the initial systems. The interest in non-locality distillation protocols stems from the fact that typically, the more non-local a system is, the more useful it is as a resource. Let us provide two examples which support this claim.

First, the security of the key generated in an entanglement based quantum key-distribution protocol relies on the fact that the outputs of the shared non-local quantum states are not fully pre-determined as classical information and, therefore, results in a lack of knowledge by any possible adversary. Typically, the more non-local a system is, the less knowledge the adversary has about the outputs of Alice and Bob and, therefore, about the key [BHK05, CR08, HRW10, MPA11]. However, this is not a necessary condition as has been recently shown in [AMP11].

Second, the communication complexity of distributed functions asks for the minimum number of bits Alice and Bob have to communicate in order to compute a distributed function. Providing Alice and Bob with non-local systems as resources can considerably reduce the number of communicated bits. For example, when Alice and Bob share PR-boxes, they can compute every distributed function with just a single bit of communication [vD05]. Note that these systems are not available in quantum mechanics.

Non-locality distillation protocols are closely related to *entanglement distillation protocols* [BBS96]. But there are some crucial differences. First, entanglement distillation allows Alice and Bob to apply local *quantum operations* on the quantum states "inside" a given system. In particular, these quantum operations can act on *several* systems simultaneously whereas in non-locality distillation protocols Alice and Bob are only given systems with a *classical* input/output behaviour. Second, entanglement distillation protocols allow Alice and Bob to use *classical* communication which stands in contrast to non-locality distillation protocols which are *non-interactive*.

Non-locality distillation has so far mainly been investigated in the case of non-local quantum systems with *binary* inputs and outputs for Alice and Bob, respectively. They are the simplest non-trivial example of non-local correlations and, therefore, are the best understood and most investigated ones. And furthermore, they can be obtained by experiments in the lab [AGR81, TBZG98]. The hardest instances with respect to distillability are known as *isotropic* systems. If a certain isotropic system can be distilled, then all binary systems of the same non-locality can be distilled as well. This follows from the fact that there exists a depolarization wiring [MAG06, Sho09] which transforms every non-signalling system to an isotropic one while preserving the non-locality (see also Lemma 3.4 in Section 3.4.2). Understanding isotropic systems is, therefore, crucial in the analysis of non-locality distillation protocols.

Formally, we have the following definition of non-locality distillation. Assume we are given several copies of the same binary non-local system. *Non-locality distillation* is possible for a binary non-local system $P \in \mathcal{NS}_{\mathrm{CHSH}} \backslash \mathcal{L}_{\mathrm{CHSH}}$ if there exists a wiring $\mathcal{W} : \mathcal{NS}_{\mathrm{CHSH}}^{\times n} \to \mathcal{NS}_{\mathrm{CHSH}}$, for some $1 \leq n < \infty$, such that

$$NL(\mathcal{W}(P, P, ..., P)) > NL(P) \ .$$

We call such a $\mathcal{W}$ a *non-locality distillation protocol*.

Contribution

We construct a single-parameter family of cross norms which continuously interpolates between the projective and dual Hilbertian tensor norm while preserving certain useful properties of them. Then, based on two conjectures we can show the following result: non-locality distillation is impossible for isotropic quantum systems (see Theorem 6.8). The proof idea is as follows: we show that for every isotropic quantum system there exists a convex set containing it, such that all other systems in this particular set are not more non-local than the given isotropic system. Then, the impossibility result follows by using that the convex sets are closed under wirings and, therefore, the output of any non-locality distillation protocol must again be an element of the given set.

We also provide sufficient conditions on a family of tensor norms which would imply the impossibility of non-locality distillation for non-local isotropic systems (see Theorem 6.9).

Related Work

It has been shown in [DW08] that at most limited distillability is possible for isotropic quantum systems. That *two* isotropic copies cannot be distilled is shown in [Sho09]. Furthermore, in [For11] numerical bounds for non-locality distillation protocols on a few systems are provided. On the other hand, it has been shown in [FWW09] that certain non-isotropic (quantum) systems can be distilled. Improved protocols, which can distil a PR-box from almost local systems, have then been provided in [BS09, ABL+09]. The optimality of distillation protocols has been further analysed in [HR10].

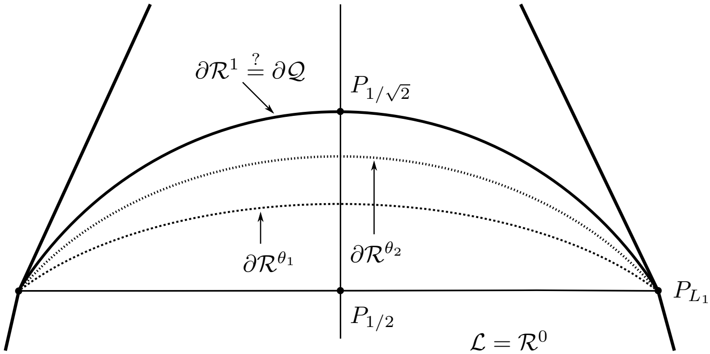

Figure 6.1: A zoomed-in version of Figure 2.1 in Section 2.6.1 is shown. See also Figure 5.1 in Section 5.1.6 for more details about this particular slice of the non-signalling polytope. The sets $\mathcal{R}_m^{\theta_1}$ and $\mathcal{R}_m^{\theta_2}$, with according boundaries $\partial\mathcal{R}_m^{\theta_1}$ and $\partial\mathcal{R}_m^{\theta_2}$, are closed under wirings by Lemma 6.10. For each set $\mathcal{R}_{\mathrm{CHSH}}^{\theta}$, an isotropic system is farthest away from the local polytope (see Lemma 6.11), i.e., the boundary curves reach their "highest" point at isotropic systems. Numerical simulations suggest (see Section 5.1.6) that $\mathcal{R}_{\mathrm{CHSH}}^1 = \mathcal{Q}_{\mathrm{CHSH}}$ and, therefore, $\partial\mathcal{R}_{\mathrm{CHSH}}^1 = \partial\mathcal{Q}_{\mathrm{CHSH}}$ for this particular slice of the non-signalling polytope.

Open Problems

Are Conjecture 2 and Conjecture 3 indeed true? Is it possible to extend our proof to *all* non-local isotropic systems?

Section Outline

In Section 6.5.2 we provide the results (without proof) we need in order to prove our main result, Theorem 6.8, which is stated in Section 6.5.3. The proofs for the results of Section 6.5.2 are then given in Section 6.5.4. Finally, in Section 6.5.5 we discuss the results of this section and provide sufficient conditions for the impossibility of non-locality distillation.

6.5.2 A Continuous Hierarchy of Non-Local Theories

Let us introduce a single-parameter family of *cross norms* defined over the space of bipartite systems. Members of this family are denoted by χ_θ :

$\mathcal{NS}_m \to \mathbb{R}_0^+$, for $0 \leq \theta \leq 1$, and they interpolate between the projective tensor norm π and the dual Hilbertian tensor norm γ_2^*. The associated convex sets are then defined as

$$\mathcal{R}_m^\theta := \{P \ : \ \chi_\theta(P) \leq 1 \wedge P \in \mathcal{NS}_m\} \ , \tag{6.24}$$

with $\mathcal{R}_{\rm iso}^\theta$ and $\mathcal{R}_{\rm CHSH}^\theta$ denoting the restrictions to isotropic and binary non-signalling systems, respectively. In order to prove that the sets $\mathcal{R}_m^\theta$ are closed under wirings we need the following conjecture:

Conjecture 2. Let $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$ for $1 \leq k \leq n$. Then

$$\chi_\theta(P_{A_1B_1} \odot \ldots \odot P_{A_nB_n}) \leq \chi_\theta(P_{A_1B_1}) \cdot \ldots \cdot \chi_\theta(P_{A_nB_n}) \ ,$$

with $P_{A_1B_1} \odot \ldots \odot P_{A_nB_n} \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$.

First, note that there is no wiring involved in this conjecture. And second, for $\theta = 0$ and $\theta = 1$, we know by Theorem 3.1 and Theorem 3.2, respectively, that this conjecture is true. Then, we can prove the following result (see Figure 6.1 and Section 6.5.4 for a proof):

Lemma 6.10 (Based on Conjecture 2)**.** *The set $\mathcal{R}_m^\theta$ is closed under wirings for all $0 \leq \theta \leq 1$ and satisfies the following conditions:*

1. $\mathcal{L}_m \subseteq \mathcal{R}_m^{\theta_1} \subseteq \mathcal{R}_m^{\theta_2} \subseteq \mathcal{NS}_m$ *for all* $0 \leq \theta_1 \leq \theta_2 \leq 1$.
2. $\mathcal{L}_m = \mathcal{R}_m^0$, $\mathcal{L}_{\rm iso} = \mathcal{R}_{\rm iso}^0$, *and* $\mathcal{Q}_{\rm iso} = \mathcal{R}_{\rm iso}^1$.
3. $\mathcal{L}_{\rm iso} \subseteq \mathcal{R}_{\rm iso}^{\theta_1} \subseteq \mathcal{R}_{\rm iso}^{\theta_2} \subseteq \mathcal{Q}_{\rm iso} \subseteq \mathcal{NS}_{\rm iso}$ *for all* $0 \leq \theta_1 \leq \theta_2 \leq 1$.

We want the sets $\mathcal{R}_m^\theta$ to *continuously* interpolate between the local set and the quantum set. For this we need the following result: for each non-local isotropic system $P_\lambda \in \mathcal{Q}_{\rm iso} \backslash \mathcal{L}_{\rm CHSH}$, there exists θ such that P_λ is located on the boundary of the set $\mathcal{R}_{\rm CHSH}^\theta$ and the isotropic systems $P_{\lambda+\delta}$, for $\delta > 0$, are not elements of $\mathcal{R}_{\rm CHSH}^\theta$. In other words (see Section 6.5.4 for a proof):

Lemma 6.11 (Based on Conjecture 3)**.** *For each $P_\lambda \in \mathcal{Q}_{\rm iso} \backslash \mathcal{L}_{\rm CHSH}$ there exists $\theta \equiv \theta(\lambda) \in [0,1]$ such that $P_\lambda \in \mathcal{R}_{\rm CHSH}^\theta$ and $NL(P_\lambda) \geq NL(P)$ for all $P \in \mathcal{R}_{\rm CHSH}^\theta$.*

Conjecture 3, which is stated in Section 6.5.4, is about the solution of a certain SDP which involves computing the Hilbertian tensor norm γ_2. We have very strong numerical evidence (see Figure 6.2) that this conjecture is indeed true.

6.5.3 Impossibility of Non-Locality Distillation

We have now all the tools we need in order to prove the main result of this section. It says that it is impossible to distil non-locality for isotropic quantum systems.

Theorem 6.8 (Based on Conjecture 2 and Conjecture 3)**.** *Let P_λ be a non-local isotropic quantum system. Then, for any $n \geq 1$ and all wirings $\mathcal{W} : \mathcal{NS}_{\mathrm{iso}}^{\times n} \to \mathcal{NS}_{\mathrm{CHSH}}$ it holds that*

$$NL(\mathcal{W}(P_\lambda, P_\lambda, ..., P_\lambda)) \leq NL(P_\lambda) \ .$$

Proof. Let $\theta \equiv \theta(\lambda)$ be such that $P_\lambda \in \mathcal{R}^\theta_{\mathrm{CHSH}}$ and $NL(P_\lambda) \geq NL(P)$ for all $P \in \mathcal{R}^\theta_{\mathrm{CHSH}}$. Such a θ always exists according to Lemma 6.11. Then, since $\mathcal{R}^\theta_{\mathrm{CHSH}}$ is closed under wirings due to Lemma 6.10 we can conclude that $\mathcal{W}(P_\lambda, P_\lambda, ..., P_\lambda) \in \mathcal{R}^\theta_{\mathrm{CHSH}}$. But since P_λ is maximally non-local in the set $\mathcal{R}^\theta_{\mathrm{CHSH}}$ due to Lemma 6.11, we obtain $NL(P_\lambda) \geq NL(\mathcal{W}(P_\lambda, P_\lambda, ..., P_\lambda))$. □

6.5.4 Proofs

Interpolating Between π and γ_2^*

The goal of this section is to prove Lemma 6.10. We will first introduce a new cross norm, denoted by χ_θ, which continuously interpolates between the projective tensor norm and the dual Hilbertian tensor norm.

Lemma 6.12. *The function $\chi_\theta : \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)} \to \mathbb{R}$ defined as*

$$\chi_\theta(P) := \sup_G \left\{ |\langle G, P \rangle| \ : \ \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^\theta \leq 1 \right\} \ ,$$

with $G \in E^{A^n}_{1(\infty)} \otimes E^{B^n}_{1(\infty)}$, is a cross norm with the following properties:

- *$\chi_0(P) = \pi(P)$ for all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.*
- *$\chi_1(P) = \gamma_2^*(P)$ for all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.*
- *$\chi_{\theta_2}(P) \leq \chi_{\theta_1}(P)$ for all $0 \leq \theta_1 \leq \theta_2 \leq 1$ and all $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.*

Proof. We have $\chi_0(P) = \pi(P)$ and $\chi_1(P) = \gamma_2^*(P)$ immediately from the definition of χ_θ and the fact that π is the dual norm of ε and γ_2^* is the dual norm of γ_2. The third item is implied by the following steps:

$$\begin{aligned}
\chi_{\theta+\delta}(P) &= \sup_G \left\{ |\langle G, P\rangle| \ : \ \varepsilon(G)^{1-\theta-\delta} \cdot \gamma_2(G)^{\theta+\delta} \leq 1 \right\} \\
&= \sup_G \left\{ |\langle G, P\rangle| \ : \ \left(\frac{\gamma_2(G)}{\varepsilon(G)}\right)^{\delta} \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \leq 1 \right\} \\
&= \sup_G \left\{ \left(\frac{\varepsilon(G)}{\gamma_2(G)}\right)^{\delta} \cdot |\langle G, P\rangle| \ : \ \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \leq 1 \right\} \\
&\leq \sup_G \left\{ |\langle G, P\rangle| \ : \ \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \leq 1 \right\} \\
&= \chi_\theta(P) \,,
\end{aligned}$$

where we used that $\varepsilon(G) \leq \gamma_2(G)$ (see Lemma 3.2) and $\delta \geq 0$ in the fourth line.

Let us now prove that χ_θ is a norm. That $\chi_\theta(P) = 0$ if and only if $P = 0$ follows from the fact that $\chi_\theta(P) \leq \pi(P)$ (by the first and third item) and that π is a norm itself. For the triangle inequality, we have

$$\begin{aligned}
\chi_\theta(P^1 + P^2) &= \sup_G \left\{ |\langle G, P^1 + P^2\rangle| \ : \ \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \leq 1 \right\} \\
&\leq \sup_G \left\{ |\langle G, P^1\rangle| + |\langle G, P^2\rangle| \ : \ \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \leq 1 \right\} \\
&\leq \chi_\theta(P^1) + \chi_\theta(P^2) \,.
\end{aligned}$$

Finally, $\chi_\theta(c \cdot P) = |c| \cdot \chi_\theta(P)$ follows immediately from the definition of χ_θ. That χ_θ is indeed a cross norm follows from the fact that it is a norm, that $\gamma_2^*(P) \leq \chi_\theta(P) \leq \pi(P)$ (by the first, second and third item) and Lemma 3.2. □

Lemma 6.13. *Let $\mathcal{T}_\mathcal{W} : \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n} \to \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ be an arbitrary wiring. Then*

$$\chi_\theta\left(\mathcal{T}_\mathcal{W}(P)\right) \leq \chi_\theta\left(P\right) \,,$$

for all $P \in \Pi_{\infty(1)}^{A^n} \otimes \Pi_{\infty(1)}^{B^n}$ and all $0 \leq \theta \leq 1$.

Proof. Since π and γ_2 are tensor norms, we obtain by using Theorem 3.3 that

$$\varepsilon(\mathcal{T}_\mathcal{W}^T(G))^{1-\theta} \cdot \gamma_2(\mathcal{T}_\mathcal{W}^T(G))^{\theta} \leq \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^{\theta} \,, \tag{6.25}$$

for all $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and where $\mathcal{T}_\mathcal{W}^T : \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}) \to E_{1(\infty)}^{A^n} \otimes E_{1(\infty)}^{B^n}$ is the transposed matrix of $\mathcal{T}_\mathcal{W}$. Then, we obtain

$$\begin{aligned}
\chi_\theta(\mathcal{T}_\mathcal{W}(P)) &= \sup_G \left\{ |\langle G, \mathcal{T}_\mathcal{W}(P)\rangle| \;:\; \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^\theta \le 1 \right\} \\
&= \sup_G \left\{ |\langle \mathcal{T}_\mathcal{W}^T(G), P\rangle| \;:\; \varepsilon(G)^{1-\theta} \cdot \gamma_2(G)^\theta \le 1 \right\} \\
&\le \sup_G \left\{ |\langle \mathcal{T}_\mathcal{W}^T(G), P\rangle| \;:\; \varepsilon(\mathcal{T}_\mathcal{W}^T(G))^{1-\theta} \cdot \gamma_2(\mathcal{T}_\mathcal{W}^T(G))^\theta \le 1 \right\} \\
&\le \sup_{\tilde{G}} \left\{ |\langle \tilde{G}, P\rangle| \;:\; \varepsilon(\tilde{G})^{1-\theta} \cdot \gamma_2(\tilde{G})^\theta \le 1 \right\} \\
&= \chi_\theta(P)\,,
\end{aligned}$$

with $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ and $\tilde{G} \in E_{1(\infty)}^{A^n} \otimes E_{1(\infty)}^{B^n}$, and where we used (6.25) in the third line. □

Hence, we are now ready to provide a proof of Lemma 6.10 stated in Section 6.5.2.

Lemma 6.10 (Based on Conjecture 2). *The set $\mathcal{R}_m^\theta$ is closed under wirings for all $0 \le \theta \le 1$ and satisfies the following conditions:*

1. *$\mathcal{L}_m \subseteq \mathcal{R}_m^{\theta_1} \subseteq \mathcal{R}_m^{\theta_2} \subseteq \mathcal{NS}_m$ for all $0 \le \theta_1 \le \theta_2 \le 1$.*
2. *$\mathcal{L}_m = \mathcal{R}_m^0$, $\mathcal{L}_{\text{iso}} = \mathcal{R}_{\text{iso}}^0$, and $\mathcal{Q}_{\text{iso}} = \mathcal{R}_{\text{iso}}^1$.*
3. *$\mathcal{L}_{\text{iso}} \subseteq \mathcal{R}_{\text{iso}}^{\theta_1} \subseteq \mathcal{R}_{\text{iso}}^{\theta_2} \subseteq \mathcal{Q}_{\text{iso}} \subseteq \mathcal{NS}_{\text{iso}}$ for all $0 \le \theta_1 \le \theta_2 \le 1$.*

Proof. That $\mathcal{R}_m^\theta$ is closed under wirings follows immediately from Theorem 5.3, Lemma 6.13, and Conjecture 2.

1. $\mathcal{L}_m \subseteq \mathcal{R}_m^{\theta_1}$ follows from Lemma 6.12 and Lemma 5.2. The third item of Lemma 6.12 implies $\mathcal{R}_m^{\theta_1} \subseteq \mathcal{R}_m^{\theta_2}$. By definition of $\mathcal{R}_m^\theta$ we have $\mathcal{R}_m^\theta \subseteq \mathcal{NS}_m$.

2. By Lemma 6.12 and Lemma 5.2 we have $\mathcal{L}_m = \mathcal{R}_m^0$ and $\mathcal{L}_{\text{iso}} = \mathcal{R}_{\text{iso}}^0$. By Theorem 5.2 and Lemma 6.12 we get $\mathcal{Q}_{\text{iso}} = \mathcal{R}_{\text{iso}}^1$.

3. That $\mathcal{L}_{\text{iso}} \subseteq \mathcal{R}_{\text{iso}}^{\theta_1} \subseteq \mathcal{R}_{\text{iso}}^{\theta_2} \subseteq \mathcal{NS}_{\text{iso}}$ follows as a special case from the first item. By the second item we have $\mathcal{Q}_{\text{iso}} = \mathcal{R}_{\text{iso}}^1$ and, therefore, by Lemma 6.12, we have $\mathcal{R}_{\text{iso}}^\theta \subseteq \mathcal{Q}_{\text{iso}}$, for all $\theta \in [0,1]$. □

Computing the Interpolation Norm

The goal of this section is to prove Lemma 6.11. In order to achieve this, we will calculate an explicit expression for the value $\chi_\theta(P_\lambda)$, where P_λ is a non-local isotropic quantum system (see Lemma 6.16).

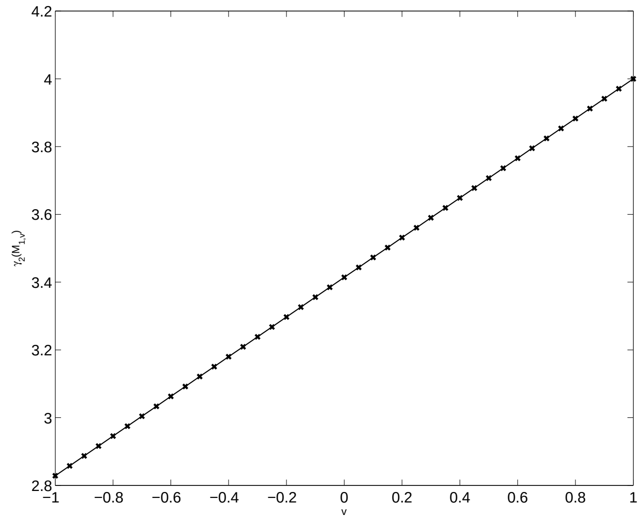

Figure 6.2: The solid line is $(2+\sqrt{2})+v(2-\sqrt{2})$ and the $\times$ correspond to the numerical solution of the SDP associated with $\gamma_2(G_{1,v})$, as given in Conjecture 3.

Let us first define the element $G_{u,v} \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$, with $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathcal{B} = \{1,2\}$, as follows:

$$\langle G_{u,v}, e_{x,a} \otimes e_{y,b} \rangle := \begin{cases} u \ , \ \textit{if } (x-1) \wedge (y-1) = (a-1) \oplus (b-1) \\ v \ , \ \textit{otherwise} \end{cases} ,$$

for $x, y, a, b \in \{1,2\}$ and with $u, v \in \mathbb{R}$. Note that $c \cdot G_{u,v} = G_{cu,cv}$, for $c \in \mathbb{R}$, and that $G_{+1,-1}$ corresponds to the canonical CHSH Bell inequality (see also Example 3 in Section 2.5.1) and, therefore, $|\langle G_{+1,-1}, P\rangle| \leq 2$, for all $P \in \mathcal{L}_{\text{CHSH}}$ and $|\langle G_{+1,-1}, P\rangle| \leq 2\sqrt{2}$, for all $P \in \mathcal{Q}_{\text{CHSH}}$. Furthermore, it holds that $\langle G_{1/4,1/4}, P\rangle = 1$ for all bipartite systems P and $\alpha(G_{u,u}) = 4|u|$ for all cross norms $\alpha : \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}) \to \mathbb{R}$. This follows from

$$\alpha(G_{u,u}) = \alpha(u \cdot \mathbb{I}_A \otimes \mathbb{I}_B) = |u| \cdot \|\mathbb{I}_A\|_{1(\infty)} \cdot \|\mathbb{I}_B\|_{1(\infty)} ,$$

with $\langle \mathbb{I}_A, e_{x,a}\rangle = 1$ and $\langle \mathbb{I}_B, e_{y,b}\rangle = 1$ for all $a, b, x, y \in \{1,2\}$ and $\|\mathbb{I}_A\|_{1(\infty)} = 2$ and $\|\mathbb{I}_B\|_{1(\infty)} = 2$. Computing the inner product between

$G_{u,v}$ and an isotropic system P_λ yields

$$\langle G_{u,v}, P_\lambda \rangle = u(2+2\lambda) + v(2-2\lambda) \ . \tag{6.26}$$

Lemma 6.14. *The injective cross norm $\varepsilon : \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}) \to \mathbb{R}$ is computed for $G_{1,v}$ as follows:*

$$\varepsilon(G_{1,v}) = \max(|3+v|, |3v+1|) \ .$$

Proof. The injective cross norm of $G_{1,v}$ is computed by

$$\varepsilon(G_{1,v}) = \sup_{P_A, P_B} \{ |\langle G_{1,v}, P_A \otimes P_B \rangle| \ , \ \|P_A\|_{\infty(1)} \leq 1 \ , \ \|P_B\|_{\infty(1)} \leq 1 \} \ .$$

Setting $\langle f_{1,1}, P_A \rangle = \langle f_{2,1}, P_A \rangle = \langle f_{1,1}, P_B \rangle = \langle f_{2,1}, P_B \rangle = 1$ and all other values to zero, yields $\varepsilon(G_{1,v}) \geq |3+v|$. On the other hand, setting $\langle f_{1,2}, P_A \rangle = \langle f_{2,2}, P_A \rangle = \langle f_{1,1}, P_B \rangle = \langle f_{2,1}, P_B \rangle = 1$ and all other values to zero yields $\varepsilon(G_{1,v}) \geq |1+3v|$. By iterating over all possible extremal settings of P_A and P_B, one can show that $\varepsilon(G_{1,v})$ is indeed upper bounded by $\max(|3+v|, |3v+1|)$ (see also Section 4.2). $\square$

Next, we state a conjecture about the value of the Hilbertian tensor norm for the element $G_{1,v}$.

Conjecture 3. The Hilbertian tensor norm $\gamma_2 : \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|}) \to \mathbb{R}$ is computed for $G_{1,v}$, with $v \in [-1, 1]$, as follows:

$$\gamma_2(G_{1,v}) = (2+\sqrt{2}) + v(2-\sqrt{2}) \ .$$

Let us provide arguments that support this conjecture. First, we can prove that $\gamma_2(G_{1,v}) \geq (2+\sqrt{2}) + v(2-\sqrt{2})$. By norm duality we have that

$$\gamma_2(G_{1,v}) = \sup_P \{ |\langle G_{1,v}, P \rangle| \ : \ \gamma_2^*(P) \leq 1 \} \ .$$

Since $\gamma_2^*(P_{1/\sqrt{2}}) = 1$ by Theorem 5.2, we obtain that

$$\begin{aligned} \gamma_2(G_{1,v}) &\geq |\langle G_{1,v}, P_{1/\sqrt{2}} \rangle| \\ &= (2+\sqrt{2}) + v(2-\sqrt{2}) \ , \end{aligned}$$

where we used (6.26) in the second line.

In order to prove the upper bound one can guess an analytical expression for the positive semidefinite matrix M associated with the SDP given in Theorem 4.1 (see also the corresponding proof) such that the constraints are fulfilled and the attained value of the SDP is indeed $(2+\sqrt{2})+v(2-\sqrt{2})$. We were not able to do this, but numerically solving the SDP of Theorem 4.1 (see Figure 6.2) indicates that Conjecture 3 is indeed true.

In order to obtain an explicit expression for $\chi_\theta(P_\lambda)$ we need the following result which says that the optimal $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ in the definition of χ_θ (see Lemma 6.12) is given by $G_{u,v}$, for some $u, v \in \mathbb{R}$.

Lemma 6.15. *Let $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$ with $\mathcal{X} = \mathcal{Y} = \mathcal{A} = \mathcal{B} = \{1,2\}$. Then there exist $u, v \in \mathbb{R}$ with $|u| \geq |v|$ such that $\varepsilon(G_{u,v}) \leq \varepsilon(G)$, $\gamma_2(G_{u,v}) \leq \gamma_2(G)$ and*

$$|\langle G_{u,v}, P_\lambda\rangle| \geq |\langle G, P_\lambda\rangle| \,,$$

for all $0 \leq \lambda \leq 1$.

Proof. By analysing the depolarization wiring given in [MAG06] (see also Lemma 3.4 in Section 3.4.2) one can see that

$$8 \cdot \mathcal{W}_{\mathrm{iso}}(e_{x,a} \otimes e_{y,b}) = \begin{cases} G_{1,0} \,, & \textit{if } (x-1) \wedge (y-1) = (a-1) \oplus (b-1) \\ G_{0,1} \,, & \textit{otherwise} \end{cases} . \tag{6.27}$$

Note that, in particular, we have $\mathcal{W}_{\mathrm{iso}}(P_\lambda) = P_\lambda$. An analogous result holds for the transposed depolarization wiring $\mathcal{W}_{\mathrm{iso}}^T$:

$$8 \cdot \mathcal{W}_{\mathrm{iso}}^T(f_{x,a} \otimes f_{y,b}) = \begin{cases} G_{1,0} \,, & \textit{if } (x-1) \wedge (y-1) = (a-1) \oplus (b-1) \\ G_{0,1} \,, & \textit{otherwise} \end{cases} . \tag{6.28}$$

This can be seen as follows. First, assume that $(x-1) \wedge (y-1) = (a-1) \oplus (b-1)$ and that $(\tilde{x}-1) \wedge (\tilde{y}-1) = (\tilde{a}-1) \oplus (\tilde{b}-1)$. We then get

$$\begin{aligned} \langle \mathcal{W}_{\mathrm{iso}}^T(f_{x,a} \otimes f_{y,b}), e_{\tilde{x},\tilde{a}} \otimes e_{\tilde{y},\tilde{b}}\rangle &= \langle f_{x,a} \otimes f_{y,b}, \mathcal{W}_{\mathrm{iso}}(e_{\tilde{x},\tilde{a}} \otimes e_{\tilde{y},\tilde{b}})\rangle \\ &= \frac{1}{8}\langle f_{x,a} \otimes f_{y,b}, G_{1,0}\rangle \\ &= \frac{1}{8}\langle G_{1,0}, e_{\tilde{x},\tilde{a}} \otimes e_{\tilde{y},\tilde{b}}\rangle \,, \end{aligned}$$

where we used (6.27) in the second line. A similar analysis can be done for the case $(x-1) \wedge (y-1) = (a-1) \oplus (b-1)$ and $(\tilde{x}-1) \wedge (\tilde{y}-1) \neq (\tilde{a}-1) \oplus (\tilde{b}-1)$ and, hence, we have $8 \cdot \mathcal{W}_{\mathrm{iso}}^T(f_{x,a} \otimes f_{y,b}) = G_{1,0}$ for $(x-1) \wedge (y-1) = (a-1) \oplus (b-1)$. Analogously, one can analyse the case where $(x-1) \wedge (y-1) \neq (a-1) \oplus (b-1)$ and obtain $8 \cdot \mathcal{W}_{\mathrm{iso}}^T(f_{x,a} \otimes f_{y,b}) = G_{0,1}$.

By using (6.28), we get for an arbitrary $G \in \ell_1^{|\mathcal{X}|}(\ell_\infty^{|\mathcal{A}|}) \otimes \ell_1^{|\mathcal{Y}|}(\ell_\infty^{|\mathcal{B}|})$

$$\begin{aligned} \mathcal{W}_{\text{iso}}^T(G) &= \sum_{x,y,a,b} \langle G, e_{x,a} \otimes e_{y,b} \rangle \cdot \mathcal{W}_{\text{iso}}^T(f_{x,a} \otimes f_{y,b}) \\ &= \sum_i \alpha_i \cdot G_{u_i,v_i} \\ &= G_{u,v} \ , \end{aligned} \tag{6.29}$$

with $\alpha_i, u_i, v_i, u, v \in \mathbb{R}$ and $u = \sum_i \alpha_i \cdot u_i$ and $v = \sum_i \alpha_i \cdot v_i$.

Since π and γ_2^* are tensor norms and the duals of ε and γ_2, respectively, we obtain by Theorem 3.3 that

$$\begin{aligned} \varepsilon(G) &\geq \varepsilon(\mathcal{W}_{\text{iso}}^T(G)) = \varepsilon(G_{u,v}) \ , \\ \gamma_2(G) &\geq \gamma_2(\mathcal{W}_{\text{iso}}^T(G)) = \gamma_2(G_{u,v}) \ . \end{aligned}$$

Furthermore, since P_λ is invariant under the depolarization wiring, we get

$$\langle G, P_\lambda \rangle = \langle G, \mathcal{W}_{\text{iso}}(P_\lambda) \rangle = \langle \mathcal{W}_{\text{iso}}^T(G), P_\lambda \rangle = \langle G_{u,v}, P_\lambda \rangle \ . \tag{6.30}$$

What remains to be shown is that $u, v \in \mathbb{R}$ can be chosen in such a way that $|u| \geq |v|$ for any G. If, after applying $\mathcal{W}_{\text{iso}}^T$ on G, we have u and v such that $|u| \geq |v|$, then we are done. So assume that we have $\mathcal{W}_{\text{iso}}^T(G) = G_{u,v}$ with $|u| < |v|$.

First, let us denote the local strategy of Alice which flips her output a by $\mathcal{T}_A^a$ and the identity map of Bob by $\mathbb{I}_B$. Note that $\mathcal{T}_A^a$ is a symmetric matrix and, hence, $(\mathcal{T}_A^a)^T = \mathcal{T}_A^a$. It is easy to see that $(\mathcal{T}_A^a \otimes \mathbb{I}_B)^T(G_{u,v}) \equiv (\mathcal{T}_A^a \otimes \mathbb{I}_B)(G_{u,v}) = G_{v,u}$. Therefore, applying Theorem 3.3 again yields $\varepsilon(G) \geq \varepsilon(G_{v,u})$ and $\gamma_2(G) \geq \gamma_2(G_{v,u})$. Finally, that $|\langle G_{v,u}, P_\lambda \rangle| \geq |\langle G, P_\lambda \rangle|$ follows from $|\langle G_{u,v}, P_\lambda \rangle| = |\langle G, P_\lambda \rangle|$ (see (6.30)) and $|u| < |v|$. □

Lemma 6.16 (Based on Conjecture 3)**.** *Let $P_\lambda \in \ell_\infty^{|\mathcal{X}|}(\ell_1^{|\mathcal{A}|}) \otimes \ell_\infty^{|\mathcal{Y}|}(\ell_1^{|\mathcal{B}|})$ with $1/2 \leq \lambda \leq 1/\sqrt{2}$. Then*

$$\chi_\theta(P_\lambda) = \max\left(1, \lambda \cdot 2^{1-\theta/2}\right) \ .$$

Proof. By using Lemma 6.15 and that $u \cdot G_{1,v} = G_{u,uv}$, we obtain

$$\begin{aligned} \chi_\theta(P_\lambda) &= \sup_G \{ |\langle G, P_\lambda \rangle| \ : \ \varepsilon(G)^{1-\theta} \gamma_2(G)^\theta \leq 1 \} \\ &= \sup_{u,v} \{ |\langle u \cdot G_{1,v}, P_\lambda \rangle| \ : \ \varepsilon(u \cdot G_{1,v})^{1-\theta} \gamma_2(u \cdot G_{1,v})^\theta \leq 1 \} \\ &= \sup_{u,v} \{ |u| \cdot |\langle G_{1,v}, P_\lambda \rangle| \ : \ |u| \cdot \varepsilon(G_{1,v})^{1-\theta} \gamma_2(G_{1,v})^\theta = 1 \} \ , \end{aligned}$$

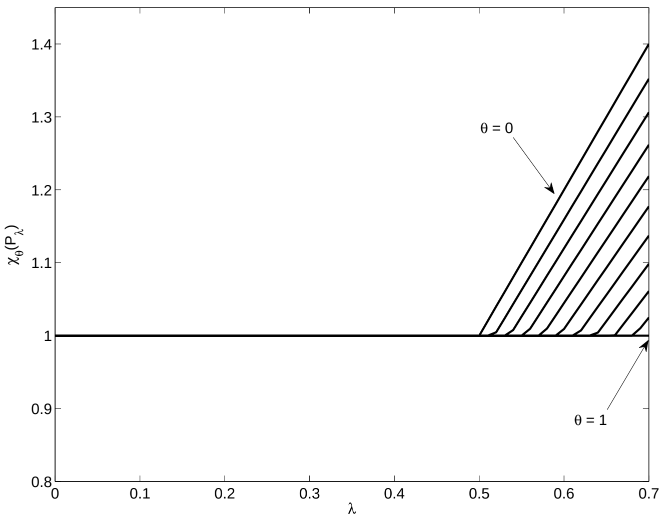

Figure 6.3: The cross norm $\chi_\theta(P_\lambda)$, as given in Lemma 6.16, is plotted for different values of $\theta \in [0,1]$, with a step size of 0.1. The top curve corresponds to $\theta = 0$ and, hence, to $\pi(P_\lambda)$, and the bottom curve (which is a straight line) corresponds to $\theta = 1$, and, hence, to $\gamma_2^*(P_\lambda)$.

where $|v| \leq 1$ since $|u| \geq |uv|$ due to Lemma 6.15. Solving for u yields then

$$\chi_\theta(P_\lambda) = \sup_{v\in[-1,1]} \frac{|\langle G_{1,v}, P_\lambda\rangle|}{\varepsilon(G_{1,v})^{1-\theta}\gamma_2(G_{1,v})^\theta} .$$

Using (6.26), Lemma 6.14 (together with $|v| \leq 1$) and Conjecture 3 yields

$$\chi_\theta(P_\lambda) = \sup_{v\in[-1,1]} \frac{(2+2\lambda)+v(2-2\lambda)}{((2+\sqrt{2})+v(2-\sqrt{2}))^\theta \cdot (3+v)^{1-\theta}} =: f(v) . \quad (6.31)$$

See Figure 6.4 for a plot of the function $f(v)$ for $\theta = 0.3$ and different values of λ and Figure 6.5 for a plot of the function $f(v)$ for $\theta = 0.3$ and $\lambda = 2^{\theta/2-1}$.

Showing that $f(v)$ attains its maximum for either $v = 1$ or $v = -1$ proves the lemma since $f(1) = 1$ and $f(-1) = \lambda \cdot 2^{1-\theta/2}$. In order to show this, it is sufficient to prove the following claims:

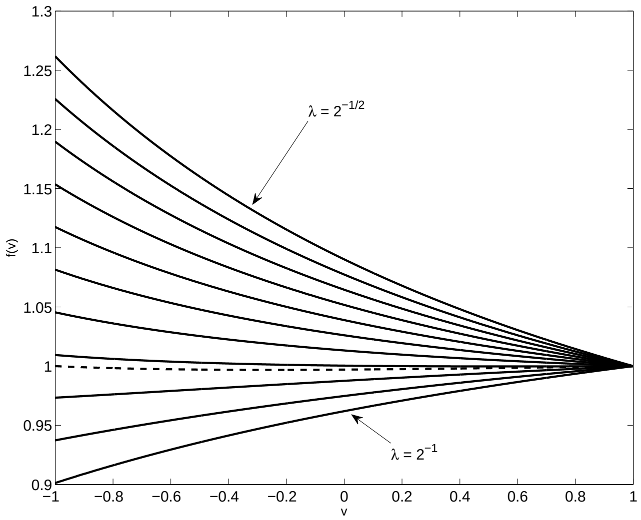

Figure 6.4: The function $f(v)$ given in (6.31) is plotted for $\theta = 0.3$ and $\lambda \in [1/2, 1/\sqrt{2}]$ with steps of 0.02. The top curve corresponds to $\lambda = 1/\sqrt{2}$ and the the bottom curve to $\lambda = 1/2$. The dotted curve corresponds to the case where $\theta = 0.3$ and $\lambda = 2^{0.3/2-1}$. A zoomed-in version of this dotted curve is shown in Figure 6.5.

1. $f(v)$ is convex for $\lambda \geq 2^{\theta/2-1}$. Note that for $\lambda = 2^{\theta/2-1}$, we have $f(-1) = f(1) = 1$.
2. $f(1) = 1$ for all θ and λ.
3. $\chi_\theta(P_\lambda) \leq \chi_\theta(P_{\lambda+\delta})$ for all λ and θ and any $\delta \geq 0$.

That these conditions are sufficient can be seen as follows. First, for $\lambda \geq 2^{\theta/2-1}$ we have that $f(v)$ is convex and, therefore, its maximum is achieved for either $v = 1$ or $v = -1$. Second, if $\lambda < 2^{\theta/2-1}$ we can conclude from the second and third item that the maximum of $f(v)$ is achieved for $v = 1$.

We prove the first item with the help of "Mathematica" by analytically computing the second derivative of $f(v)$ and then showing that $f''(v) \geq 0$ for all $-1 \leq v \leq 1$, $1/2 \leq \lambda \leq 1/\sqrt{2}$ and $0 \leq \theta \leq 1$ with $\lambda \geq 2^{\theta/2-1}$. The minimum of $f''(v)$ is equal to 0 and is achieved for $\lambda = 1/\sqrt{2}$, $\theta = 1$ and any $v \in [-1, 1]$. The second item is obvious, and for the third item one observes that

$$(2 + 2(\lambda + \delta)) + v(2 - 2(\lambda + \delta)) = (2 + 2\lambda) + v(2 - 2\lambda) + 2\delta(1 - v) \ ,$$

and $1 - v \geq 0$ since $v \in [-1, 1]$. □

We now have all the necessary tools in order to provide a proof of Lemma 6.11 in Section 6.5.2.

Lemma 6.11 (Based on Conjecture 3). *For each $P_\lambda \in \mathcal{Q}_{\text{iso}} \backslash \mathcal{L}_{\text{CHSH}}$ there exists $\theta \equiv \theta(\lambda) \in [0, 1]$ such that $P_\lambda \in \mathcal{R}^\theta_{\text{CHSH}}$ and $NL(P_\lambda) \geq NL(P)$ for all $P \in \mathcal{R}^\theta_{\text{CHSH}}$.*

Proof. We define θ as

$$\theta(\lambda) \equiv \theta := 2 \cdot \log_2(2\lambda) \ .$$

Note first, that by this definition we have $\theta \in [0, 1]$ for $1/2 \leq \lambda \leq 1/\sqrt{2}$. Let us consider a fixed λ and its corresponding $\theta \equiv \theta(\lambda)$. Then, since $\chi_\theta(P_\lambda) = \max\left(1, \lambda \cdot 2^{1-\theta/2}\right)$ by Lemma 6.16, we can conclude that $\chi_\theta(P_\lambda) = 1$ and $\chi_\theta(P_{\lambda+\delta}) > 1$ for any $\delta > 0$. But this also implies that $\chi_\theta(P) > 1$ for any P with $NL(P) > NL(P_\lambda)$ since the depolarization wiring brings any system P to an isotropic system without losing non-locality (see Lemma 3.4) and not increasing the χ_θ norm (by Lemma 6.13). Hence, we can conclude that $NL(P_\lambda) \geq NL(P)$ for all $P \in \mathcal{R}^\theta_{\text{CHSH}}$. □

6.5.5 Discussion

The definition of the set $\mathcal{R}^\theta_m$ given in (6.24) can be changed to

$$\mathcal{S}^\theta_m := \{P \ : \ \chi_\theta(P) \leq 1 \wedge P \in \mathcal{Q}_m\} \ .$$

It is easy to see that $\mathcal{S}^\theta_m \subseteq \mathcal{R}^\theta_m$ and that the sets $\mathcal{S}^\theta_m$ are closed under wirings as well. Hence, if we assume that Conjecture 1 in Section 5.1.6 is indeed true we would actually obtain a continuous hierarchy of non-local theories which, in the case of binary systems, interpolates between the local theory and quantum theory, and not only an approximation to quantum theory.

Let us have a closer look at the proof of Theorem 6.8 and the facts it is based on. Since the proof is heavily based on the actual definition of

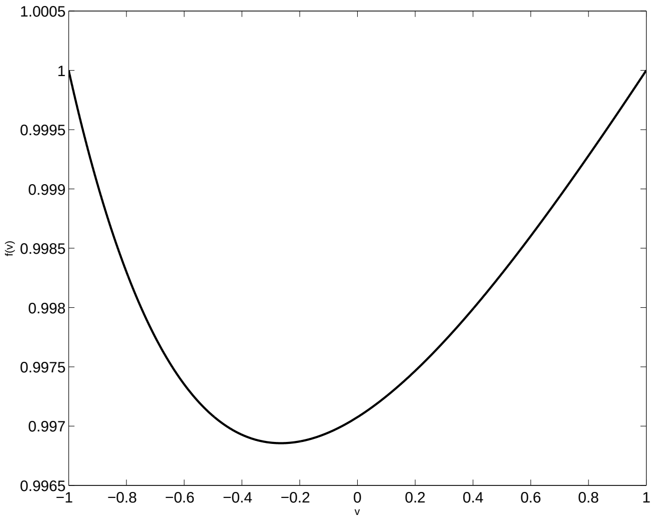

Figure 6.5: The function $f(v)$ given in (6.31) is plotted for $\theta = 0.3$ and $\lambda = 2^{0.3/2-1}$. Note that $f(-1) = f(1) = 1$ and f is convex.

the norm χ_θ, one might hope that there exists another family of norms for which one actually can prove Conjecture 2. This other norm would need to have the following sufficient properties: (1) it should not increase under wirings, (2) it should not increase under compositions, and (3) the associated convex sets should continuously interpolate between the extremal theories. Formally, we have:

Theorem 6.9. *Let $\alpha_\theta : \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)} \to \mathbb{R}$, for $0 \leq \theta \leq 1$, be a cross norm. Then, if α_θ fulfils the following conditions, non-locality distillation for non-local isotropic systems is impossible:*

1. $\alpha_\theta(\mathcal{T}_\mathcal{W}(P)) \leq \alpha_\theta(P)$, *for all* $P \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$ *and all wirings* $\mathcal{T}_\mathcal{W}$.

2. $\alpha_\theta(P_{A_1B_1} \odot ... \odot P_{A_nB_n}) \leq \alpha_\theta(P_{A_1B_1}) \cdot ... \cdot \alpha_\theta(P_{A_nB_n})$, *for all* $P_{A_kB_k} \in \ell_\infty^{|\mathcal{X}_k|}(\ell_1^{|\mathcal{A}_k|}) \otimes \ell_\infty^{|\mathcal{Y}_k|}(\ell_1^{|\mathcal{B}_k|})$ *and where* $P_{A_1B_1} \odot ... \odot P_{A_nB_n} \in \Pi^{A^n}_{\infty(1)} \otimes \Pi^{B^n}_{\infty(1)}$.

3. *For each* $P_\lambda \in \mathcal{NS}_{\mathrm{iso}} \backslash \mathcal{L}_{\mathrm{CHSH}}$, *there exists* $\theta \equiv \theta(\lambda) \in [0, 1]$ *such that*

$\alpha_\theta(P_\lambda) \leq 1$ *and* $NL(P_\lambda) \geq NL(P)$ *for all* $P \in \mathcal{NS}_{\mathrm{CHSH}}$ *with* $\alpha_\theta(P) \leq 1$.

Hence, finding a cross norm which fulfils the three properties of Theorem 6.9 would immediately imply that non-locality distillation is impossible for non-local isotropic systems. Note that if α_θ is actually a *tensor norm*, then the first property would already be satisfied (see Theorem 3.3 in Section 3.4).

Chapter 7

Conclusion and Outlook

7.1 Thesis Summary

In this thesis, we have developed a framework, based on the theory of tensor norms, that allowed us to study the properties of bipartite systems, two-prover games and Bell inequalities. The framework contains three main parts:

(1) The embedding of bipartite systems and two-prover games into tensor product spaces. This allowed us to evaluate different tensor norms on bipartite systems (yielding convex sets of systems) and two-prover games (yielding winning probabilities of two-prover games).

(2) The composition of bipartite systems and two-prover games. This allowed us to combine bipartite systems in order to obtain larger systems and to study parallel-repetition results for two-prover games.

(3) The representation of wirings as linear maps on tensor product spaces. This allowed us to study sets of bipartite systems that are closed under wirings.

In order to demonstrate the power and usefulness of the framework we proved four applications in quantum information theory:

(1) We proved an upper bound on the maximal winning probability of two-prover games where the provers have entanglement as resources. In order to prove this result we first derived a generalized version of Grothendieck's inequality that could deal with settings of arbitrary output alphabet sizes. And second, we established close connections between quantum systems and the Hilbertian tensor norm γ_2 and between local systems and the projective tensor norm π.

(2) We provided an alternative proof of the perfect parallel-repetition theorem for entangled XOR games. We proved this result by showing that the dual Hilbertian tensor norm γ_2^* obeys a direct-product theorem and that the winning probability of entangled XOR games can be computed by the dual Hilbertian tensor norm.

(3) We showed that there exist quantum systems that cannot be obtained by wirings of isotropic quantum systems. We, therefore, say that isotropic quantum systems are not universal for quantum theory. Based on a (numerically supported) conjecture, we prove that even arbitrary binary quantum systems are not universal for quantum theory. In order to prove these results, we used that the dual Hilbertian tensor norm γ_2^* induces a convex set of bipartite systems that is closed under wirings and that this convex set is closely related to the set of binary quantum systems.

(4) We proved sufficient conditions for tensor norms that imply the impossibility of non-locality distillation for isotropic systems. We then constructed a continuous hierarchy of cross norms and prove, based on two conjectures and the sufficient conditions, that non-locality distillation is impossible for isotropic quantum systems.

7.2 Future Directions

As a line of possible future work, it would be interesting to investigate whether our generalization of Grothendieck's inequality (see Theorem 6.1 and Theorem 6.3 in Section 6.2.3) and our direct-product theorems (see Theorem 3.1 and Theorem 3.2 in Section 3.3.3) have other applications in quantum information theory, communication complexity or approximation algorithms.

In Section 5.2.3, we have mentioned that the tensor norms $\gamma_{p,q}$, for $1/p + 1/q = 1$, obey direct-product theorems and, therefore, induce convex set of bipartite systems that are closed under wirings. Hence, the first two sufficient conditions that would imply the impossibility of non-locality distillation are already satisfied (see Theorem 6.9 in Section 6.5.5). It would, therefore, be interesting to understand the geometry of the convex sets of bipartite systems that are induced by $\gamma_{p,q}$ since this would directly imply impossibility results for non-locality distillation.

In this thesis we only investigated bipartite systems that were non-signalling. However, the framework we developed does not put any restriction on the bipartite systems that can be embedded into the tensor product spaces. For example, we can use signalling system to model the transmission of a single bit between the two parties and, therefore, the communication complexity of distributed function evaluation can be investigated with the help of our framework. Since all wirings can be represented as linear maps on tensor product spaces the question arises what properties the outputs of wirings on signalling systems have. For example, it is not even clear whether the outputs are valid bipartite systems or not (see also Remark 2 in Section 2.4).

Furthermore, it would be interesting to consider different local normed vector spaces. For example, one could study local normed vector spaces and tensor norms on them that arise from continuous and infinite input and output alphabets. Another possibility would be to consider the trace norm for operators on Hilbert spaces as the local norm. This would allow to study bipartite quantum states and not only bipartite systems (see also Rudolph's work [Rud00, Rud01a, Rud01b, DHR02, Rud03, Rud05]). Finally, it would be interesting to generalize the framework to multipartite settings (see Section 3.3.2 where we already introduced multipartite versions of the projective and injective tensor norms) in order to analyse problems in quantum information theory that involve entangled systems shared between more than two parties (see for example [PGWP$^+$08]).

Appendix A

Various Technical Results

A.1 Equivalence of γ_2 Definitions

We will show the following equality:

$$\begin{aligned} & \inf_{\hat{S}=W\cdot V} \|W\|_{2\to Y} \cdot \|V\|_{X^*\to 2} \\ = & \inf \left\{ \sup \left(\sum_i |\langle R_A, S_A^i\rangle|^2 \right)^{1/2} \sup \left(\sum_i |\langle R_B, S_B^i\rangle|^2 \right)^{1/2} \right\} \end{aligned} \tag{A.1}$$

with $S = \sum_{i=1}^n S_A^i \otimes S_B^i \in X \otimes Y$ and where the supremums are over $R_A \in X^*$ and $R_B \in Y^*$, respectively, such that $\|R_A\|_{X^*} \leq 1$ and $\|R_B\|_{Y^*} \leq 1$. This implies the equivalence of the two expression for the Hilbertian tensor norm γ_2 given in Section 3.2.2.

Let us first show that the right hand side of (A.1) is larger or equal to the left hand side. First, let $S = \sum_{i=1}^n S_A^i \otimes S_B^i \in X \otimes Y$ be the optimal decomposition on the right hand side of (A.1). Then, we define $W : \ell_2^n \to Y$ and $V : X^* \to \ell_2^n$ as follows:

$$\begin{aligned} W(\lambda) &:= \sum_{i=1}^n \langle \lambda, e_i\rangle \cdot S_B^i \ , \\ V(R_A) &:= \sum_{i=1}^n e_i \cdot \langle R_A, S_A^i\rangle \ . \end{aligned}$$

The operator $\hat{S} : X^* \to Y$ corresponding to S can be represented as

$$\hat{S}(R_A) = \sum_{i=1}^{n} \langle R_A, S_A^i \rangle \cdot S_B^i \; . \tag{A.2}$$

That $\hat{S} = W \cdot V$ holds follows then by

$$(W \cdot V)(R_A) = W(V(R_A)) = W\left(\sum_i e_i \cdot \langle R_A, S_A^i \rangle\right) = \sum_i \langle R_A, S_A^i \rangle \cdot S_B^i \; . \tag{A.3}$$

We then get

$$\begin{aligned} \|V\|_{X^* \to 2} &= \sup_{\|R_A\|_{X^*} \leq 1} \|V(R_A)\|_2 \\ &= \sup_{\|R_A\|_{X^*} \leq 1} \left\| \sum_i e_i \cdot \langle R_A, S_A^i \rangle \right\|_2 \\ &= \sup_{\|R_A\|_{X^*} \leq 1} \left(\sum_i |\langle R_A, S_A^i \rangle|^2 \right)^{1/2} \; . \end{aligned} \tag{A.4}$$

On the other hand, using the duality relation between norms, we have

$$\begin{aligned} \|W\|_{2 \to Y} &= \sup_{\|\lambda\|_2 \leq 1} \|W(\lambda)\|_Y \\ &= \sup_{\|\lambda\|_2 \leq 1} \left\| \sum_{i=1}^{n} \langle \lambda, e_i \rangle \cdot S_B^i \right\|_Y \\ &= \sup_{\|\lambda\|_2 \leq 1} \sup_{\|R_B\|_{Y^*} \leq 1} \left| \left\langle R_B, \sum_{i=1}^{n} \langle \lambda, e_i \rangle \cdot S_B^i \right\rangle \right| \\ &= \sup_{\|R_B\|_{Y^*} \leq 1} \sup_{\|\lambda\|_2 \leq 1} \left| \sum_{i=1}^{n} \langle \lambda, e_i \rangle \cdot \langle R_B, S_B^i \rangle \right| \; . \end{aligned}$$

By setting $\langle \mu, e_i \rangle := \langle R_B, S_B^i \rangle$, with $\mu \in \ell_2^n$, and using that ℓ_2^n is self dual, we get

$$\begin{aligned} \|W\|_{2 \to Y} &= \sup_{\|R_B\|_{Y^*} \leq 1} \sup_{\|\lambda\|_2 \leq 1} \left| \sum_{i=1}^{n} \langle \lambda, e_i \rangle \cdot \langle R_B, S_B^i \rangle \right| \\ &= \sup_{\|R_B\|_{Y^*} \leq 1} \sup_{\|\lambda\|_2 \leq 1} |\langle \lambda, \mu \rangle| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|R_B\|_{Y^*}\leq 1} \|\mu\|_2 \\
&= \sup_{\|R_B\|_{Y^*}\leq 1} \left(\sum_i |\langle \mu, e_i\rangle|^2 \right)^{1/2} \\
&= \sup_{\|R_B\|_{Y^*}\leq 1} \left(\sum_i |\langle R_B, S_B^i\rangle|^2 \right)^{1/2} ,
\end{aligned} \tag{A.5}
$$

which finishes the first part of the proof.

Let us now show that the right-hand side of (A.1) is smaller or equal to the left-hand side. Let $\hat{S} = W \cdot V$ be the optimal factorization of $\hat{S}$ on the left-hand side of (A.1). Then there exist $S_A^i \in X$ and $S_B^i \in Y$ such that $W(\lambda) = \sum_{i=1}^n \langle \lambda, e_i\rangle \cdot S_B^i$ and $V(R_A) = \sum_{i=1}^n e_i \cdot \langle R_A, S_A^i\rangle$, respectively. Hence, $\sum_i S_A^i \otimes S_B^i$ is a valid representation of S (see also (A.2) and (A.3)). Using (A.4) and (A.5) finishes the proof.

A.2 Duality Relation Between Operator Norms

Lemma A.1. *Let $X := (\mathbb{R}^n, \|\cdot\|_X)$ be a finite-dimensional normed vector space and $\mathcal{T} : \ell_2^m \to X$ a linear operator. Then*

$$
\|\mathcal{T}\|_{2\to X} = \|\mathcal{T}^T\|_{X^*\to 2} .
$$

Proof. Representing $\mathcal{T}$ as a row matrix, with rows $a_i \in \ell_2^m$, for $1 \leq i \leq n$, yields

$$
\begin{aligned}
\|\mathcal{T}\|_{2\to X} &= \sup_{\|\lambda\|_2\leq 1} \|\mathcal{T}(\lambda)\|_X \\
&= \sup_{\|\lambda\|_2\leq 1} \left\| \begin{pmatrix} \langle a_1, \lambda\rangle \\ \vdots \\ \langle a_n, \lambda\rangle \end{pmatrix} \right\|_X \\
&= \sup_{\|\lambda\|_2\leq 1} \sup_{\|\mu\|_{X^*}\leq 1} \left| \sum_{i=1}^n \mu_i \cdot \langle a_i, \lambda\rangle \right| .
\end{aligned} \tag{A.6}
$$

On the other hand, by using that the 2-norm is self dual, we obtain

$$
\begin{aligned}
\|\mathcal{T}^T\|_{X^* \to 2} &= \sup_{\|\mu\|_{X^*} \leq 1} \left\| \sum_{i=1}^{n} \mu_i \cdot a_i \right\|_2 \\
&= \sup_{\|\mu\|_{X^*} \leq 1} \sup_{\|\lambda\|_2 \leq 1} \left| \sum_{i=1}^{n} \mu_i \cdot \langle a_i, \lambda \rangle \right| , \qquad \text{(A.7)}
\end{aligned}
$$

and, therefore, $\|\mathcal{T}\|_{2 \to X} = \|\mathcal{T}^T\|_{X^* \to 2}$. □

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