Doctoral Thesis

Colorability properties of random graphs

Author(s):
Panagiotou, Konstantinos

Publication Date:
2008

Permanent Link:
https://doi.org/10.3929/ethz-a-005649827

Rights / License:
In Copyright - Non-Commercial Use Permitted
Colorability Properties of Random Graphs

A dissertation submitted to the

Swiss Federal Institute of Technology Zurich

for the degree of

Doctor of Sciences

presented by

Konstantinos Panagiotou

Diplom-Informatiker, Technische Universität München

born 16.06.1980

citizen of Greece

accepted on the recommendation of

Prof. Dr. Angelika Steger, examiner

Prof. Joel H. Spencer, co-examiner

2008
To my family
## Contents

Abstract vii

Zusammenfassung ix

Acknowledgements xi

Chapter 1. Introduction 1
  1.1. Motivation: Average-Case Analysis 1
  1.2. Graph Coloring 4
  1.2.1. Exact Algorithms 5
  1.2.2. Approximation Algorithms 5
  1.2.3. Heuristics & Random Graphs 6
  1.3. Overview 8

Chapter 2. The Chromatic Number of Sparse Random Graphs 11
  2.0.1. Techniques and Outline 12
  2.1. Technical Preliminaries 13
  2.2. Approaching the Values of the Chromatic Number 14
  2.3. Proof of the Main Result 16
  2.3.1. The sparse case \(n^{-1} \ll p \leq n^{-1+1/20}\) 17
  2.3.2. The dense case \(n^{-1+1/20} \leq p \leq n^{-3/4-\delta}\) 18
  2.3.3. Proof of Theorem 2.1 24

Chapter 3. Extremal Subgraphs of Random Graphs 25
  3.1. Preliminaries & Notation 28
  3.2. Finding a Near-Optimal Bipartition 31
  3.3. On Properties of (Near-)Optimal Bipartitions 41
  3.4. Proof of Theorem 3.3 48
  3.5. Generalizations & Open problems 53

Chapter 4. Random \(\ell\)-colorable and \(\ell\)-colored Graphs 63
  4.0.1. Tools & Techniques 64
  4.1. The Balancedness of \(\ell\)-Colored Graphs 67
  4.1.1. Preliminaries 67
  4.1.2. Proof of Theorem 4.3 – First Statement 69
  4.1.3. Proof of Theorem 4.3 – Second Statement 72
  4.1.4. \(\Theta(1)\)-colored Graphs 78
  4.1.5. \(\ell\)-colored Graphs with \(cn^2\) Edges 78
  4.2. Unique Colorability of Random \(\ell\)-colorable Graphs 80
  4.2.1. Almost All Very Balanced Graphs have a Unique Coloring 81
Abstract

This thesis is devoted to the study of colorability and extremal properties of random graphs. The theory of random graphs was founded by Paul Erdős and Alfréd Rényi in a series of papers between 1959 and 1968. Since then it has become a rapidly developing and a very fruitful branch of combinatorics, discrete mathematics, and probability theory, and has gained attention from many experts from mathematics as well as theoretical computer science.

The first topic we study is the chromatic number of the random graph $G_{n,p}$. Since the seminal work of Erdős and Rényi [ER59], computing the probable value of $\chi(G_{n,p})$ has been a fundamental problem in the theory of random graphs, and several papers have contributed to determining its asymptotic value and to understanding its concentration. From today's point of view, there are numerous impressive results stating that the chromatic number is sharply concentrated about a (known) value. We mention selectively the following two results. Achlioptas and Naor [AN05] proved for edge probabilities $p = \frac{\delta}{n}$ the remarkable result that for roughly half of all $c \in (0, \infty)$ one has $\chi(G_{n,p}) = \ell + 1$ with probability one when $n \to \infty$, and the slightly weaker statement $\chi(G_{n,p}) \in \{\ell, \ell + 1\}$ for the remaining $c$'s. Here the value of $\ell$ was given as an explicit function of $c$. Moreover, Alon and Krivelevich [AK97b] showed that for edge probabilities $p \leq n^{-1/2-\delta}$, where $\delta > 0$, the chromatic number is asymptotically almost surely concentrated in two consecutive values. However, their proof does not provide any clue on where this values are located.

The focus of the first part of the thesis is on the chromatic number $\chi(G_{n,p})$, where $p = p(n) \leq n^{-3/4-\delta}$, for every fixed $\delta > 0$. We prove that asymptotically almost surely the chromatic number of $G_{n,p}$ is $\ell$, $\ell + 1$, or $\ell + 2$, where $\ell$ is the maximum integer satisfying $2(\ell - 1)\log(\ell - 1) \leq (n - 1)p$. This improves the currently best bounds by Łuczak [Luc91], which located $\chi(G_{n,p})$ in an interval of unbounded size.

In our second result we investigate the typical structure of maximum subgraphs of $G_{n,p}$. A related result by Erdős, Kleitman, and Rothschild [EK76] states that large triangle-free graphs are bipartite (and equivalently, two-colorable) in a probabilistic sense. More precisely, they showed that if $T_n$ denotes a graph drawn uniformly at random from the set of all triangle-free graphs on $n$ labeled vertices, then the probability that $T_n$ is bipartite tends to one as $n$ tending to infinity. For a graph $G$, let $t(G)$ denote the maximum number of edges in a triangle-free subgraph of $G$, and let $b(G)$ be the maximum number of edges in a bipartite subgraph of $G$. Of course, we always have $t(G) \geq b(G)$.

Our general intuition — guided by the above result — suggests that, for dense enough graphs, these two parameters will typically be equal. Indeed, this was confirmed by
Babai, Simonovits, and Spencer [BSS90], who showed that asymptotically almost surely \( t(G_{n,p}) = b(G_{n,p}) \), whenever \( p \geq \frac{1}{2} - \delta \), for some fixed \( \delta > 0 \).

Babai et. al. asked in [BSS90] whether this result could be extended to cover edge probabilities \( p \) of the form \( n^{-c} \), for some positive constant \( c \). In the second chapter of this thesis we answer this question as follows. Let \( K_\ell \) denote the complete graph on \( \ell \) vertices. We prove that there is a constant \( c = c(\ell) > 0 \), such that whenever \( p \geq n^{-c} \), with probability tending to one when \( n \) goes to infinity, every maximum \( K_\ell \)-free subgraph of the binomial random graph \( G_{n,p} \) is \((\ell - 1)\)-partite. Hence, our result not only answers the 15-year old question of Babai, Simonovits and Spencer, but it also extends the statement to any fixed-size complete graph.

In our proof we derive a statement of independent interest: we show, among other results, that the maximum cut of almost all graphs with \( M \) edges, where \( M \gg n \) and \( \binom{n}{2} - M \gg n \), is “nearly unique”. Let \( M' = \min\{M, \binom{n}{2} - M\} \). Then, given any maximum cut \( C \) of \( G_{n,M} \), we can obtain all other maximum cuts by moving at most \( O(\sqrt{n^3/M'}) \) vertices between the parts of \( C \). This proves rigorously for almost all densities that any two maximum cuts of \( G_{n,M} \) are “very close” to each other.

In the last chapter of this work we investigate properties of the class of all \( \ell \)-colorable graphs with \( n \) vertices, where \( \ell = \ell(n) \) may depend on \( n \). Let \( G_{n,\ell} \) denote a random \( \ell \)-colorable graph, i.e., a graph drawn uniformly at random from this class. We show that for any \( \epsilon > 0 \), if \( \ell(n) \leq \frac{n}{2\log n + \sqrt{(2+\epsilon)\log n}} \), then \( G_{n,\ell} \) has a unique coloring (up to permutations of the colors) with probability tending to one when \( n \to \infty \).

The inherent difficulty in studying properties of random graphs from \( G_{n,\ell} \) is that the edges do not appear independently. We overcome this problem by investigating the more approachable model of colored graphs: the essential difference is that every graph in \( G_{n,\ell} \), which has \( t \) different colorings with \( \ell \) colors, counts exactly \( t \) times as a colored graph. In this context we prove that with high probability all color classes of a random \( \ell \)-colored graph with \( n \) vertices, where \( 1 \ll \ell(n) \ll \frac{n}{\sqrt{\log n}} \), deviate by roughly at most \( \sqrt{2\log \ell} \) from \( \left\lfloor \frac{n}{\ell} \right\rfloor \), and there is at least one color class with such deviation. In other words, we characterize precisely the maximum deviation of the size of the color classes in a “typical” coloring compared to a completely balanced coloring.
Zusammenfassung


Unser erstes Resultat ist über die chromatische Zahl des zufälligen Graphen $G_{n,p}$. Seit der bahnbrechenden Arbeit von Erdös und Rényi [ER59] ist die Bestimmung des voraussichtlichen Wertes von $\chi(G_{n,p})$ eines der fundamentalen Probleme in der Theorie der Zufallsgraphen, und viele Arbeiten haben dazu beigetragen, ihren asymptotischen Wert zu untersuchen und ihre Konzentration zu verstehen. Aus heutiger Sicht der Dinge gibt es bereits zahlreiche beeindruckende Resultate die aussagen, dass die chromatische Zahl sehr scharf konzentriert ist. Andererseits gibt es kein Resultat das aussagt, dass die Konzentration extrem scharf in allen nicht-trivialen Fällen ist. In diesem Zusammenhang haben Achlioptas und Naor [AN05] für Kantenwahrscheinlichkeiten der Form $p = \frac{c}{n}$, wobei $c \in \mathbb{R}_+$, das bemerkenswerte Resultat gezeigt, dass für ungefähr die Hälfte aller $c$'s “$\chi(G_{n,p}) = \ell + 1$” mit Wahrscheinlichkeit eins wenn $n \to \infty$, und die etwas schwächere Aussage “$\chi(G_{n,p}) \in \{\ell, \ell + 1\}$” für alle übrigen $c$'s. Hierbei haben die Autoren $\ell$ als eine explizite Funktion von $c$ angegeben. Im Gegensatz dazu, für Kantenwahrscheinlichkeiten $\frac{1}{n} \ll p \leq 1 - \omega(\frac{1}{n})$ gibt es keine vergleichbaren Konzentrationsresultate.

Der Schwerpunkt des ersten Teiles dieser Arbeit ist die Untersuchung der chromatischen Zahl $\chi(G_{n,p})$, wobei $p = p(n) \leq n^{-3/4-\delta}$ und $\delta > 0$ konstant ist. Wir zeigen dass asymptotisch fast sicher $\chi(G_{n,p}) \in \{\ell, \ell + 1, \ell + 2\}$, wobei $\ell$ die grösste ganze Zahl ist, für die $2(\ell - 1)\log(\ell - 1) \leq (n - 1)p$ gilt. Das verbessert die zur Zeit besten bekannten Schranken, die ein unbegrenztes Intervall angaben, aus dem $\chi(G_{n,p})$ Werte annehmen konnte.

Unser zweites Resultat beschäftigt sich mit der typischen Struktur von maximalen Subgraphen des $G_{n,p}$. Ein verwandtes Resultat von Erdös, Kleitman, and Rothschild [EKR76] besagt in einem probabilistischem Sinne, dass grosse dreiecksfreie Graphen bipartit (und somit auch 2-färbbar) sind. Genauer gesagt, falls $T_n$ ein Graph ist, der aus der Menge aller dreiecksfreien Graphen mit $n$ Knoten zufällig und gleichverteilt gezogen wird, so strebt die Wahrscheinlichkeit, dass $T_n$ bipartit ist, gegen eins wenn $n$
gegen Unendlich geht. Sei \( t(G) \) die maximale Anzahl von Kanten in einem dreiecksfreien Subgraphen von \( G \), und sei \( b(G) \) die maximale Anzahl Kanten in einem bipartiten Subgraph von \( G \). Offensichtlicherweise gilt \( t(G) \geq b(G) \). Unsere Intuition suggeriert – vor allem wegen des gerade erwähnten Resultates – dass diese zwei Parameter typischerweise gleich sein werden. Wir bestätigen dies folgendermassen. Sei \( \mathcal{K}_\ell \) der vollständige Graph mit \( \ell \) Knoten. Wir beweisen dass eine Konstante \( c = c(\ell) > 0 \) existiert, so dass für alle \( p = p(n) \geq n^{-c} \), mit Wahrscheinlichkeit gegen eins strebend falls \( n \to \infty \), alle maximalen \( \mathcal{K}_\ell \)-freie Subgraphen des \( G_{n,p} (\ell - 1) \)-partit sind. Dies beantwortet eine mehr als 15 Jahre alte Frage von Babai, Simonovits und Spencer [BSS90].

Als wichtiges Hilfsmittel in unserem Beweis verwenden wir eine Aussage von unabhängigerem Interesse: wir zeigen unter anderem, dass der maximale Schnitt (maximum cut) von fast allen Graphen mit \( M \) Kanten, wobei \( M \gg n \) und \( \binom{n}{2} - M \gg n \), fast eindeutig ist. Sei \( M' = \min \{ M, \binom{n}{2} - M \} \). Falls \( C \) irgendein maximaler Schnitt des \( G_{n,M} \) ist, so erhalten wir mit hoher Wahrscheinlichkeit alle anderen maximalen Schnitte durch das Verschieben von höchstens \( \mathcal{O}(\sqrt{n^2/M'}) \) Knoten zwischen den Teilen von \( C \). Das beweist für fast alle möglichen Dichten rigoros dass die Menge der maximalen Schnitte des \( G_{n,M} \) eine Häufung um einen einzig maximalen Schnitt aufweist.

Im letzten Kapitel dieser Arbeit befassen wir uns mit der Klasse aller \( \ell \)-färbbaren Graphen, wobei diesmal \( \ell = \ell(n) \) auch eine Funktion der Anzahl der Knoten sein darf. Sei \( G_{n,\ell} \) ein zufälliger \( \ell \)-färbbarer Graph mit \( n \) Knoten. Wir zeigen für jedes \( \varepsilon > 0 \) und \( \ell(n) \leq \frac{n}{2 \log n + \sqrt{(2 + \varepsilon) \log n}} \) dass asymptotisch fast sicher \( G_{n,\ell} \) eine eindeutige Färbung mit \( \ell \) Farben hat (bis auf die \( \ell! \) Permutationen der Farben).

Eine inhärente Schwierigkeit die beim Studium von Eigenschaften des \( G_{n,\ell} \) auftaucht ist die Tatsache dass das Vorhandensein der Kanten nicht durch unabhängige Ereignisse bestimmt wird. Wir umgehen dieses Problem indem wir die Klasse der gefärbten Graphen untersuchen: der entscheidende Unterschied hier ist, dass jeder Graph in \( G_{n,\ell} \), der \( t \) unterschiedliche Färbungen hat, genau \( t \) mal in der Klasse der \( \ell \)-gefärbten Graphen vorkommt. In diesem Zusammenhang beweisen wir, dass ein zufälliger \( \ell \)-färbbarer Graph \( G \) die folgende Eigenschaft mit hoher Wahrscheinlichkeit hat: in jeder Färbung von \( G \) weicht die Anzahl der Knoten in jeder Farbklasse um höchstens \( \sqrt{2 \log \ell} \) von \( \frac{n}{\ell} \) ab, und es gibt mindestens eine Farbklasse mit so grosser Abweichung. Mit anderen Worten, wir charakterisieren die maximale Abweichung in der Größe einer Farbklasse im Vergleich zur einer vollständig balancierten Färbung.
Acknowledgements

First and most important, I want to express my very deep gratitude to everybody who supported me during the years as a PhD student. I am especially most grateful to my supervisor Angelika Steger, who not only gave me the unique opportunity to join her excellent research group in Zurich, but who also shared her impressive knowledge and intuition with me. We spent countless hours working together, where she revealed me the beauty of mathematics and theoretical computer science. I am very much obliged for everything that I have learned from her.

I sincerely appreciate the collaboration and the work with my co-authors Graham Brightwell, Amin Coja-Oghlan, Nicla Bernasconi, Stefanie Gerke, Julian Lorenz, Justus Schwartz, Alexander Souza, and Andreas Weiβl. In particular, I want to deeply thank Graham Brightwell for inspiring me and supporting this work tremendously. Special thanks for many fruitful and inspiring discussions go also to Amin Coja-Oghlan. Thank you all so much for sharing your knowledge with me and enjoying so much the world of mathematics!

I am grateful to Joel Spencer, one of the leading experts in the theory of random graphs and combinatorics, who agreed to co-examine my thesis.

The familiar and supportive atmosphere at Angelika’s group at ETH made me feel very pleasant during the time as a student. I want to thank all my colleagues and co-workers for their invaluable help, endless excellent conversations on several topics, puzzle solving, coding, hiking, and many more. Thank you for being there, and for encouraging me on my way to here.

Last but not least, I am deeply grateful to my family for their love and invaluable support they gave me during the last years.

Konstantinos Panagiotou
Zurich, 26.03.2008
CHAPTER 1

Introduction

1.1. Motivation: Average-Case Analysis

One of the core disciplines in computer science is the development and the analysis of algorithms. The main purpose of the analysis of an algorithm is to quantify its performance with respect to some well-defined measure; such measures typically include running time, memory consumption, or quality of approximation.

During the past decades research on algorithm analysis has focused almost exclusively on worst-case analysis. In the worst-case paradigm, as the name immediately suggests, solely the worst behavior of an algorithm is taken into account in order to characterize its quality, and the goal is to determine its performance on the most ill-behaved input instances. This is done for good reason: worst-case bounds on performance measures provide a guarantee that under no circumstances the algorithm will need more resources than estimated, or perform worse than predicted by the provided performance bounds.

However, worst-case analysis often has a major drawback: from a practical point of view it may lead to far too pessimistic results, which do not capture the real behavior of the algorithm in question. The reason for this phenomenon is usually the fact that the algorithm fails on “few” exceptional instances, but performs well in “most” other cases. Nevertheless, for the worst-case analysis only the few bad instances are relevant, and the algorithm’s behavior in “typical” scenarios is not considered.

One of the most striking examples that illustrates the discrepancy between theoretical and practical performance is the well-known SIMPLEX algorithm, which is used to solve linear programs. A common technique for dealing with hard combinatorial optimization problems is to derive a linear program, such that its solution is close to the solution of the given problem. The SIMPLEX method, proposed by Dantzig in the late 40’s (his monograph [Dan98] contains a very detailed treatment of the topic) is the oldest linear programming algorithm. It can safely be declared as one of the most important algorithms that were discovered in the twentieth century, and probably it still remains the linear programming algorithm most widely used in practice. Since its discovery, numerous variants (pivot rules) have been proposed (see e.g. [TZ93] and references therein), and many of them seem to work very well in practice. However, no variant is known to be polynomial or even close to polynomial in the worst case, and for many variants there exist carefully designed instances on which exponential running times are achieved. On the other hand, it is known that linear programs can be solved in polynomial time with the ellipsoid method (see [BGT81] for a detailed survey) or the algorithm by Karmakar [Kar84a]. Nevertheless, these algorithms do not seem to perform nearly as efficient as SIMPLEX in practical scenarios.
Hence, it is natural to ask for an explanation for the huge gap between the worst-case and the empirically observed performance of SIMPLEX. This question had been open for more than 25 years, when it was finally answered by Borgwardt in 1987: he proved through probabilistic analysis that the so-called SHADOWVERTEX variant of the SIMPLEX algorithm has polynomial expected running time (where he assumed that the input polytope is chosen at random in a suitable model), and gave sharp asymptotic bounds for the expected number of pivot steps. His results are summarized in the book [Bor87].

A second example that illustrates the discrepancy between worst-case and practical performance is the running time of the well-known QUICKSORT algorithm, which was first proposed by Hoare [Hoa62]. QUICKSORT takes a sequence of numbers as input, and outputs them in a sorted order. To achieve this, it selects a pivot element, and divides the sequence into elements that are smaller than the pivot, and elements that are larger than or equal to the pivot element. Then it recursively proceeds to sort the two subsequences; the recursion stops when a (sub-)sequence of length at most one has to be sorted. It can be shown that for every deterministic pivot selecting strategy which may only investigate constantly many elements of the sequence, there exist input sequences consisting of \( n \) elements, on which QUICKSORT performs \( \Theta(n^2) \) comparisons, see e.g. [MJ96] and [McI99]. Compared to other sorting algorithms, as for instance MERGESORT or HEAPSORT, which have a worst-case running time of \( \mathcal{O}(n \log n) \) [CLR90], QUICKSORT appears to be an inefficient algorithm, which should not be used in practical scenarios. However, QUICKSORT is considered as one of the best comparison-based algorithms, and is widely used in practice. This empirical observation was supported by a probabilistic analysis (see [MR95] for a general discussion): if we assume that all \( n! \) input sequences of length \( n \) are equally probable, it can be shown that the expected running time is \( \Theta(n \log n) \), and that it is very sharply concentrated around this value.

The examples mentioned above demonstrate the importance and the potential of the average-case analysis of algorithms in an impressive way. Unfortunately, the stated algorithms are among the few examples for which an average case analysis could be performed successfully. Additionally, sorting (and selection) belong to a class of problems for which – at least from a practical point of view – efficient algorithms exist; the situation is different for most combinatorial optimization problems.

Since the fundamental work of Cook, Karp, Garey and Johnson [Coo71, Kar72a, GJ79a] about \( \mathcal{NP} \)-completeness, appropriate tools for characterizing the complexity of a specific problem are available. Even though proving that a problem is \( \mathcal{NP} \)-hard classifies that problem as being very difficult (in the sense that the existence of worst-case polynomial time algorithms is considered unlikely), from a practical point of view this is rather unsatisfactory: combinatorial optimization problems have many important applications with immense economic potential, and finding solutions to instances that actually occur is therefore inevitable. However, it might be possible to design efficient algorithms that determine satisfactory solutions very quickly in practical scenarios.
Over the last decades several approaches were pursued to deal with $\mathcal{NP}$-complete problems. One of them, which is nowadays very well understood from the theoretical viewpoint, is the design and the analysis of approximation algorithms. An approximation algorithm is an algorithm that calculates a solution which is close to the optimal solution, and where precise statements about its worst-case “proximity” to the optimum (or equivalently, its approximation ratio) can be made.

At first glance, approximation algorithms are a valuable resource, as they allow the use of efficient and thus in practice applicable algorithms with a guaranteed approximation ratio. In the last decade, immense progress was made in the theory of approximation algorithms, see for instance the book “Complexity and Approximation” [APMS+99] by Ausiello et al. Especially the characterization of the class $\mathcal{NP}$ through probabilistically checkable proofs and the related non-approximability results as well as the developed theory for the complexity class $\mathcal{APX}$ allow us nowadays to characterize the approximability of many optimization problems very precisely. Unfortunately, the answer is often disappointing, as many problems turned out to be $\mathcal{APX}$-complete or worse. As in the theory of $\mathcal{NP}$-completeness, proving that a problem is $\mathcal{APX}$-hard essentially terminates the search for efficient approximation algorithms, and thus does not provide any solution for the underlying practical problems.

To cope with the situation that most combinatorial optimization problems are $\mathcal{NP}$-hard and even hard to approximate, many heuristic approaches were developed and are now used for solving such problems in practice. The emphasis of heuristic approaches is on algorithms which perform empirically very well. Theoretical statements about the quality of the applied methods are merely existent, but especially in the last years more effort has been directed towards discovering the relevant underlying structures. From today’s viewpoint it seems futile to try to design deterministic, polynomial time algorithms for $\mathcal{NP}$-complete problems. However, as in the case of the Simplex or QuickSort algorithm, it may be possible to design polynomial time algorithms which (provably) calculate good solutions in the “normal” or “typical” case. That is, we are looking for algorithms with good expected performance (i.e., they perform well in average), or if this is not possible, for algorithms that perform well on most/typical instances.

From the above discussion it follows that average-case analysis may provide an attractive alternative to worst-case analysis, especially if we focus on the practical solvability of certain problems. Of course, if we want to perform such an analysis, we have to define suitable models: what exactly is the “average case”? Perhaps the most natural way to define this notion is to assume that the input to an algorithm is drawn according to a probability distribution defined on all possible input instances. That is, we want to analyze certain performance measures of an algorithm, given the fact that its input is random.

It is obvious that the structure of a “typical” random instance is strongly related to the design and the analysis of algorithms with good average-case behavior. Having the
above discussion in mind, the key challenges in average-case analysis can be summarized as follows:

- Find a natural probability distribution on the class of all possible inputs that captures the main characteristics of the intended application.
- Determine the typical structure of a random input instance, and relate it to the parameters which are crucial to the performance measures of the algorithm in question.

The main contribution of this thesis are several results which can be considered as progress in this direction for a specific class of problems, namely coloring problems defined on graphs. As we shall elaborate later in more detail, the focus is mainly on understanding important properties of random instances, and simultaneously developing adequate tools for dealing with similar problems.

The next sections contain precise definitions of the considered problems as well as a glance at their history and related results. This chapter concludes with a high-level overview of the results obtained in this thesis.

### 1.2. Graph Coloring

Let \( G = (V,E) \) be a graph with vertex set \( V \) and edge set \( E \subseteq \binom{V}{2} \). An \( \ell \)-coloring \( c \) of \( G \) is a mapping from \( V \) to \( \{1, \ldots, \ell\} \) such that for all edges \( \{u,v\} \in E \) we have \( c(u) \neq c(v) \). That is, we want to color the vertices of \( G \) with \( \ell \) colors such that adjacent vertices receive distinct colors. The minimal \( \ell \) such that \( G \) admits an \( \ell \)-coloring is called the chromatic number of \( G \), and is commonly denoted by \( \chi(G) \). The problem of determining the chromatic number of a graph \( G \), or similarly, determining an optimal coloring, is not only one of the most important problems in the theory of graphs, and has lead to numerous beautiful and deep results, but it also appears frequently at the core of several optimization problems within practical applications. Many problems can be reformulated in terms of finding a coloring with a small number of colors, or calculating the chromatic number of a graph exactly or approximately.

Deciding whether the chromatic number of a given graph is smaller or equal to two is an easy computational problem that can be solved in time proportional to \(|V| + |E|\) with a simple depth-first search on \( G \). On the other hand, very shortly after Cook had published his fundamental paper on complexity theory [Coo71], Karp proved in his work [Kar72b] that it is \( \mathcal{NP} \)-complete to decide whether \( \chi(G) \leq \ell \) for any \( \ell \geq 3 \). The problem even remains \( \mathcal{NP} \)-complete on planar graphs with maximum vertex degree 4, as shown by Garey and Johnson [GJ79b]. On the other hand, on planar graphs it is trivial already for \( \ell > 3 \) (this result is an immediate consequence of the famous Four Color Theorem [AH77, AHK77] proved by Appel, Haken and Koch, which says that every planar graph \( G \) satisfies \( \chi(G) \leq 4 \)).

But the situation is even worse: Feige and Kilian [FK98] proved that no polynomial time algorithm approximates \( \chi(G) \) within a factor of \(|V|^{1-\epsilon}\) unless \( \mathcal{NP} = \mathcal{ZPP} \) (here \( \epsilon \) denotes an arbitrarily small positive constant). Hence, from a current perspective,
there is only little hope of finding efficient algorithms that perform well on *every* graph coloring instance, even if we require only *very* approximate solutions. Altogether, the situation does not appear particularly encouraging, from both the theoretical and practical point of view.

1.2.1. Exact Algorithms. Despite the fact that the coloring problem is very difficult from the complexity theory point of view, there is an enormous amount of work devoted to designing exact algorithms for it. One of the earliest results in this area is a 1976 paper by Lawler [Law76]. It essentially contains three results: first, an algorithm for finding a 3-coloring of a graph (given that the graph is 3-chromatic) in time $O(3^{n/3}) \approx O(1.4422^n)$. Second, an $O(2^n)$ algorithm for finding a 4-coloring of a 4-chromatic graph. Finally and most importantly, it presents an algorithm for finding the chromatic number of an arbitrary graph in time $O((1 + 3^{1/3})^n) \approx O(2.4422^n)$. Lawler's algorithm follows a dynamic programming approach, in which the chromatic number not just of $G$ is computed, but of all its induced subgraphs. For each induced subgraph $S$ of $G$, $\chi(S)$ is determined by generating all maximal independent sets $I \subseteq S$, adding one to the chromatic number of $S \setminus I$, and taking the minimum of all those values. The claimed running time of this technique then follows readily from an upper bound of $3^{n/3}$ on the number of maximal independent sets in any graph with $n$ vertices, due to Moon and Moser [MM65] (observe that this bound is tight when the given graph is a disjoint union of triangles).

Since then, this area of research has grown substantially. Lawler's 3-coloring algorithm was improved in the sequence of papers [Sch94, Epp01, Bys03, BE05], with the most recent algorithm needing time $O(1.3289^n)$. Moreover, some of the algorithms developed could be extended to color $\ell$-chromatic graphs with $\ell$ colors, where $\ell$ is a small constant.

Lawler's general chromatic number algorithm was also improved in the series of papers [Bys03, Epp03, BH06, Koi06, BH]. The currently fastest algorithm is by Koivisto [Koi06] and has a running time of $O(\text{poly}(n)2^n)$, but unfortunately needs exponential space. On the other hand, the algorithm by Björklund and Husfeldt [BH] is slower (its running time is bounded by $O(2.2461^n)$), but requires only a polynomial (in the size of the graph) amount of memory.

1.2.2. Approximation Algorithms. As discussed previously, it is well-known that from an algorithmic point of view even the approximation of the chromatic number in polynomial time within (almost all) non-trivial factors is very hard. Thus, researchers have tried to develop fast algorithms for several variants of the problem. More precisely, research has focused on the following natural question: Given a $\ell$-colorable graph with $n$ vertices, how many colors are needed in the worst case so that it is possible to color it in polynomial time? Here $\ell$ is assumed to be any fixed constant greater than two.

For graphs that have a constant chromatic number the first non-trivial worst-case approximation algorithm is due to Wigderson [Wig83]. He described an algorithm which colors any 3-chromatic graph with $O(\sqrt{n})$ colors, and more generally, any $\ell$-chromatic...
graph with $O(n^{1-1/(\ell-1)})$ colors. Wigderson’s algorithm is based on a simple but beautiful idea, which we describe here briefly for the case $\ell = 3$. If the graph has maximum vertex degree at most $\sqrt{n}$, then the problem is easy: just color the vertices in any order with the smallest currently available color. This process will use at most $\sqrt{n} + 1$ colors, as every time a vertex $v$ is considered, at most $\sqrt{n}$ colors will be “blocked” because they were already assigned to the neighbors of $v$. On the other hand, suppose that there is a vertex $v$ with degree larger than $\sqrt{n}$. In any case the neighborhood of $v$ is easily seen to be a bipartite graph. Indeed, suppose that this is not the case. Then there is a cycle of odd length somewhere in the neighborhood of $v$. But then we need three colors to color the vertices of that cycle, and then (at least) four colors to color $v$ and its neighborhood, which is a contradiction to the assumption that the original graph was 3-chromatic. Hence, $v$ and its neighborhood can be colored in polynomial time with three colors, and the whole algorithm is restarted in the remaining graph with a fresh set of colors. To conclude, the graph can be colored always with $O(\sqrt{n})$ colors, as the maximum number of vertices with degree larger than $\sqrt{n}$ that were encountered during the process is clearly at most $\sqrt{n}$.

The result of Wigderson was improved by Berger and Rompel [BR90], who showed that $O\left(\left\lceil \frac{n}{\log n} \right\rceil^{1-1/(\ell-1)}\right)$ colors also suffice. This result was an important step, as it proved that $\sqrt{n}$ is by no means a lower bound for the number of colors needed to color 3-chromatic graphs in polynomial time, which was believed by parts of the community till then. Shortly after, Blum [Blu94] improved this bound further with a completely different approach, and proved that for the case $\ell = 3$ already $O(n^{3/8}\text{polylog}(n))$ colors are sufficient. Karger, Motwani, and Sudan [KMS98] improved Blum’s guarantee to roughly $O(n^{1/4})$, using ideas of Goemans and Williamson [GW95], who exploited semidefinite programming to design approximation algorithms for $\text{MaxCut}$ and $\text{MaxSat}$. This result was then again improved with similar techniques by Blum and Karger [BK97] and Arora, Chlamtac and Charikar [ACC06] to $O(n^{3/14})$ and $O(n^{0.211})$ respectively.

For general graphs of arbitrary chromatic number, the best algorithmic result known to date is due to Halldórsson [Hal93]. Halldórsson’s algorithm has a performance guarantee of $O\left(\frac{n\log\log n}{\log^2 n}\right)^2$. This result is based upon an algorithm by Boppana and Halldórsson [BH90], which finds an independent set within a $\frac{n}{(\log n)^2}$ factor of the maximum.

1.2.3. Heuristics & Random Graphs. The plethora of worst-case $\mathcal{NP}$-hardness results for the graph coloring problem motivates the study of heuristics, which give “reasonable” results for “most” instances “efficiently”, where the terms “reasonable”, “most”, and “efficiently” are usually not well-defined (if defined at all!), and are typically estimated in an empirical way. The main idea behind the design of heuristics is to relax the requirement that the algorithm is fast or performs well on all instances.

In order to compare graph coloring heuristics it is important to consider meaningful benchmark instances, and to check which heuristic usually performs better. However, although empirical results may be informative, we seek for more rigorous measures for
evaluating heuristics. Among the test instances that are in common use are various types of random graphs, and the major goal is to perform an average-case analysis of the proposed algorithms.

In the context of graph coloring, the $G_{n,p}$ model, which was introduced by Erdős and Rényi in the 60’s [ER60], is nowadays one of the most studied random graph models, and provides very interesting and – for certain values of the parameters – extremely “difficult” benchmark instances. The random graph $G_{n,p}$ is defined as follows: the vertex set is $V = \{1, \ldots, n\}$, and each of the $\binom{n}{2}$ possible edges among two vertices in $V$ is present with probability $0 < p = p(n) < 1$ independently. In other words, $G_{n,p}$ is a product probability space of $\binom{n}{2}$ Bernoulli variables with success probability $p$. Thus, the expected number of edges in $G_{n,p}$ is $\frac{1}{2} p$, and the expected degree of a vertex is $(n - 1)p$.

In addition to the $G_{n,p}$ model several other models of random graphs for studying the performance of coloring heuristics were proposed. Another well-studied model is $G_{n,\ell}$ (and variations thereof), where a graph is chosen uniformly at random from all $\ell$-colorable graphs with $n$ vertices. Moreover, there are many models of semi-random graphs. Semirandom settings allow for an adversary to change the outcome of the random experiment to a certain degree. In this way they capture a larger class of input distributions and have been used to bridge the gap between average-case and worst-case analysis. Indeed, by assuming that the underlying instance is completely random we may exclude pathological counter examples. By allowing an adversary to change the random graph in some arbitrary even though partially restricted way, we require the algorithm to be able to cope with inputs which are unlikely in the given (purely random graph) model. Such models were investigated e.g. in [AKS98, FK01, CO02].

The focus in this thesis is solely on the $G_{n,p}$ and $G_{n,\ell}$ models. The theory of random graphs is one of the most rapidly developing areas of combinatorics and discrete mathematics, with already thousands of papers devoted to the subject. We certainly do not intend to cover it here, and refer the interested reader to the excellent books [Bo101] and [JLR00] for a very general treatment of the topic. Moreover, [AS00] provides a thorough survey of several important tools and results. Within the theory of random graphs, (algorithmic) graph coloring has become one of the most prominent problems, and the obtained results often combine tools and ideas from many fields, as computer science, combinatorics, and probability theory. Krivelevich [Kri02] provides a sound survey of these results, and gives pointers to the most influential papers in this area.

In the next section we give a high-level overview of the results described in this thesis. We assume that the reader has some background in graph theory and random structures.
1.3. Overview

In this section we briefly present the main results of this work. The presentation is rather concise; every chapter of the thesis contains one result, together with a detailed introduction to the considered problem.

Chapter 2 - On the Chromatic Number of Random Graphs

Since the seminal work of Erdős and Rényi [ER59], computing the probable value of \( \chi(G_{n,p}) \) has been a fundamental problem in the theory of random graphs. Bollobás [Bol88] was the first to obtain an asymptotically tight result: he showed that if \( 0 < p < 1 \) is fixed, then

\[
\chi(G_{n,p}) \sim \frac{n \log(1 - p)}{2 \log(np)} \quad \text{a.a.s.} \tag{1.1}
\]

Moreover, Łuczak [Luc91] extended (1.1) to the regime \( \frac{1}{n} \ll p = o(1) \), proving that

\[
\left| \chi(G_{n,p}) - \frac{np}{2 \log(np)} \right| = O\left( \frac{np \cdot \log \log np}{\log^2 np} \right) \quad \text{a.a.s.} \tag{1.2}
\]

Thus, while (1.2) shows that \( \chi(G_{n,p}) \sim \frac{np}{2 \log(np)} \), the additive error term \( O\left( \frac{np \cdot \log \log np}{\log^2 np} \right) \) is unbounded if the average degree \( np \) tends to infinity.

In addition to its probable value also the concentration of \( \chi(G_{n,p}) \) has received considerable attention. Alon and Krivelevich [AK97b] proved that the chromatic number is two point concentrated under the assumption \( p \ll n^{-1/2} \), which is best possible in the sense that there are \( p = p(n) \) for which \( \chi(G_{n,p}) \) is not concentrated on one value. However, their proof does not yield the specific values on which \( \chi(G_{n,p}) \) is concentrated.

In this context, Achlioptas and Naor [AN05] proved for edge probabilities \( p = \frac{c}{n} \) the remarkable result that for roughly half of all \( c \in (0, \infty) \) one has \( \chi(G_{n,p}) = \ell + 1 \) a.a.s., and the slightly weaker statement \( \chi(G_{n,p}) \in [\ell, \ell + 1] \) for the remaining \( c \)'s. Here the value of \( \ell \) is given by an explicit function of \( c \).

In this chapter our main focus is on the chromatic number \( \chi(G_{n,p}) \), where \( p = p(n) \leq n^{-3/4 - \delta} \), for every fixed \( 0 < \delta \leq \frac{1}{7} \). We prove that a.a.s. \( \chi(G_{n,p}) \) is \( \ell \), \( \ell + 1 \), or \( \ell + 2 \), where \( \ell \) is the maximum integer satisfying \( 2(\ell - 1) \log(\ell - 1) \leq (n - 1)p \). This improves the currently best bounds for random graphs with unbounded average degree, as it locates \( \chi(G_{n,p}) \) for the first time in an interval of constant size.

The material of this chapter will appear in [COPS]; an extended abstract appeared already in [COPS07].

Chapter 3 - Extremal Subgraphs of Random Graphs

In our second result we investigate the typical structure of maximum subgraphs of \( G_{n,p} \).

Let \( K_{\ell} \) denote the complete graph on \( \ell \) vertices. We prove that there is a constant \( c = c(\ell) > 0 \), such that whenever \( p \geq n^{-c} \), with probability tending to one when \( n \) goes to infinity, every maximum \( K_{\ell} \)-free subgraph of the binomial random graph \( G_{n,p} \) is \((\ell - 1)\)-partite. This result answers a 15-year old question of Babai, Simonovits
and Spencer [BSS90] about maximum triangle-free subgraphs of $G_{n,p}$, and extends the statement to any fixed-size complete graph.

In our proof we derive a statement of independent interest: we show, among other results, that the maximum cut of almost all graphs with $M$ edges, where $M \gg n$ and $\binom{n}{2} - M \gg n$, is “nearly unique”. Set $M' = \min(M, \binom{n}{2} - M)$, and let $G_{n,M}$ be a graph drawn uniformly at random from the set of all such graphs. Let $C$ be any of the maximum cuts of $G_{n,M}$. Then with high probability we obtain all other maximum cuts of $G_{n,M}$ by moving at most $O(\sqrt{n^3/M'})$ vertices between the parts of $C$. This proves rigorously for almost all densities that the set of maximum cuts of $G_{n,M}$ has a single-cluster behavior, i.e., all maximum cuts share a very large common core.

An extended abstract of this work appeared in [BPS07].

Chapter 4 - Random $\ell$-colorable Graphs

In the last chapter of this work we investigate properties of the class of all $\ell$-colorable graphs on $n$ vertices, where $\ell = \ell(n)$ may depend on $n$. Let as before $G_{n,\ell}$ denote a random $\ell$-colorable graph. We show that if $\ell(n) \leq \frac{n}{2\log n + \sqrt{(2+\epsilon) \log n}}$, for any $\epsilon > 0$, $G_{n,\ell}$ has a unique coloring (up to permutations of the colors) with probability tending to one as $n \to \infty$.

The inherent difficulty in studying properties of random $\ell$-colorable graphs is that the edges do not appear independently. We overcome this problem by investigating the more approachable model of colored graphs: the essential difference is that every $\ell$-colorable graph that has $t$ different colorings with $\ell$ colors counts exactly $t$ times as a colored graph. In this context we prove that with high probability all color classes of a random $\ell$-colored graph with $n$ vertices, where $1 \ll \ell(n) \ll \frac{n}{\sqrt{\log n}}$, deviate by roughly at most $\sqrt{2\log \ell}$ from $\left\lfloor \frac{n}{\ell} \right\rfloor$, and there is at least one color class with such deviation. In other words, we characterize precisely the maximum deviation of the size of the color classes in a “typical” coloring compared to a completely balanced coloring.

This chapter is based on the manuscript [PS].
CHAPTER 2

The Chromatic Number of Sparse Random Graphs

In this chapter we study the chromatic number of random graphs $G_{n,p}$, in which each possible edge is included in the graph independently with probability $p$. Since the seminal work of Erdős and Rényi [ER59], computing the probable value of $\chi(G_{n,p})$ has been a fundamental problem in the theory of random graphs. Bollobás [Bol88] was the first to obtain an asymptotically tight result: he showed that if $0 < p < 1$ is fixed, then

$$\chi(G_{n,p}) \sim -\frac{n \log(1-p)}{2 \log(np)} \text{ a.a.s.} \tag{2.1}$$

Łuczak [Luc91] extended (2.1) to the regime $\frac{1}{n} \ll p = o(1)$, proving that

$$\left| \chi(G_{n,p}) - \frac{np}{2 \log(np)} \right| = O\left( \frac{np \cdot \log \log np}{\log^2 np} \right) \text{ a.a.s.} \tag{2.2}$$

Thus, while (2.2) shows that $\chi(G_{n,p}) \sim \frac{np}{2 \log(np)}$, the additive error term $O(\frac{np \log \log np}{\log^2 np})$ is unbounded if the average degree $np$ tends to infinity.

In 2005 Achlioptas and Naor [AN05] proved for edge probabilities $p = \frac{c}{n}$ the remarkable result that for roughly half of all $c \in (0, \infty)$ one has “$\chi(G_{n,p}) = \ell + 1$” a.a.s., and the slightly weaker statement “$\chi(G_{n,p}) \in \{\ell, \ell + 1\}$” for the remaining $c$’s. Here the value of $\ell$ is given by an explicit function of $c$. Note that this result of Achlioptas and Naor deals with random graphs of bounded average degree, and yields the value of $\chi(G_{n,p})$ up to an additive error of at most one in this case.

For larger values of $p$, i.e., random graphs with non-constant average degree, the concentration of $\chi(G_{n,p})$ also received considerable attention but is still much less understood. Shamir and Spencer [SS87] proved that $\chi(G_{n,p})$ is concentrated on $O(\sqrt{n})$ integers for any sequence $p = p(n)$ of edge probabilities. Furthermore, they showed that $\chi(G_{n,p})$ is concentrated in an interval of constant length for $p \ll n^{-1/2}$. Moreover, Łuczak [Luc91] proved that $\chi(G_{n,p})$ is concentrated on two consecutive integers if $p \ll n^{-5/6}$. Finally, Alon and Krivelevich [AK97b] proved that two point concentration actually holds under the weaker assumption $p \ll n^{-1/2}$, which is best possible in the sense that there are $p = p(n)$ for which $\chi(G_{n,p})$ is not concentrated on one value. However, none of these papers [AK97b, Luc91, SS87] yields the specific values on which $\chi(G_{n,p})$ is concentrated. For instance, while Alon and Krivelevich show that for each $p = p(n)$ there exists a sequence $\ell = r(n,p)$ such that a.a.s. it holds $\chi(G_{n,p}) \in \{r, r+1\}$, the proof does not yield any clue on what the value of $r$ is.

The main result of the chapter is the following theorem, which determines the chromatic number as an explicit function of $n$ and $p$. 

11
Theorem 2.1. Let $0 < \delta \leq \frac{1}{4}$, $\frac{1}{n} \leq p = p(n) \leq n^{-3/4-\delta}$, and let $\ell$ be given by

$$\ell = \ell(n,p) = \max\{l \in \mathbb{N} : 2(l-1)\log(l-1) \leq p(n-1)\}. \quad (2.3)$$

Then $\chi(G_{n,p}) \in \{\ell, \ell+1, \ell+2\}$ a.a.s. Furthermore, for every fixed $\varepsilon > 0$, if $p(n-1) \in ((2\ell-1)\log \ell + \varepsilon, 2\ell \log \ell)$, then $\chi(G_{n,p}) \in \{\ell+1, \ell+2\}$ a.a.s.

Hence, Theorem 2.1 yields the value of $\chi(G_{n,p})$ up to an additive error of at most two for random graphs of average degree up to $n^{1/4-\delta}$, and is a natural extension of the main theorem of [AN05].

2.0.1. Techniques and Outline. The proof of Theorem 2.1 builds on and extends some of the techniques from [AN05, AK97b, Luc91, SS87]. Suppose that $n^{-1} \ll p \leq n^{-3/4-\delta}$, and let $\ell$ be given as in (2.3). To bound $\chi(G_{n,p})$ from below, we just verify that the expected number of $(\ell-1)$-colorings is $o(1)$, so that Markov’s inequality yields that $\chi(G_{n,p}) \geq \ell$ a.a.s.

Following Achlioptas and Naor [AN05], we employ the second moment method to bound $\Pr[\chi(G_{n,p}) \leq \ell + 1]$ from below. That is, we estimate the second moment $\mathbb{E}[X^2]$ of the number $X$ of $(\ell+1)$-colorings of $G_{n,p}$; this estimate employs a general result from [AN05] on optimizing certain functions over stochastic matrices (Theorem 2.6 below). Since $\Pr[X > 0] \geq \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]}$, the upper bound on $\mathbb{E}[X^2]$ gives us a lower bound for the probability that $\chi(G_{n,p}) \leq \ell + 1$. More precisely, in Section 2.2 we shall prove that

$$\Pr[\chi(G_{n,p}) \leq \ell + 1] \geq e^{-\Theta((pn)^2)} \cdot n^{-\ell^2}. \quad (2.4)$$

Now, the obvious problem is that the right-hand side of the “lower bound” in the expression above actually tends to 0 as $n \to \infty$. This problem does not occur in the sparse regime considered in [AN05], where $np = c$ is constant. Indeed, in this particular case it can be shown that $\Pr[\chi(G_{n,p}) \leq \ell + 1] \geq \alpha(c)$, where $\alpha(c)$ remains bounded away from 0 as $n \to \infty$ (of course, this does not follow from (2.4)). Therefore, Achlioptas and Naor can boost this lower bound using a sharp threshold result of Achlioptas and Friedgut [AF99], thus concluding that actually $\chi(G_{n,p}) \leq \ell + 1$ a.a.s.

However, in the case that $np \to \infty$, which is the main focus of the present work, we cannot bound $\Pr[\chi(G_{n,p}) \leq \ell + 1]$ away from 0 uniformly as $n \to \infty$. In addition, the sharp threshold result [AF99] does not apply. Nevertheless, adapting arguments from Shamir and Spencer [SS87], in Section 2.3 we shall prove that a.a.s. $G = G_{n,p}$ admits a set $U$ of vertices of size $|U| \leq n^{3/2}p \log n$ such that $\chi(G \setminus U) \leq \ell + 1$. Thus, to prove that $\chi(G_{n,p}) \leq \ell + 2$ a.a.s. we just need to show that any such partial $(\ell+1)$-coloring can be modified, such that by spending one additional color, we can construct a $(\ell+2)$-coloring of the entire graph a.a.s.

To this end, we consider two cases. If $np \leq n^{1/20}$, say, then a slight variation of Łuczak’s argument [Luc91] yields that $\chi(G) \leq \ell + 2$ a.a.s. By contrast, the case $n^{1/20} \leq np \ll n^{1/4}$ requires new ideas: extending tools developed by Alon and Krivelevich [AK97b], we show the following. Suppose that an $(\ell+1)$-coloring of $G \setminus U$ is given. Then one can recolor a few vertices in $G \setminus U$ with one new color so that the resulting $(\ell+2)$-coloring
of $G \setminus U$ can be extend to an $(\ell + 2)$-coloring of all of $G$. Thus, to color the vertices in $U$ we recolor a few vertices of $G \setminus U$ and reuse some of the “old” colors to color $U$.

The above argument relates to the proof of Alon and Krivelevich as follows. In \cite{AK97b}, it is assumed that there is a set $W \subset V$ of size $|W| \leq \sqrt{\pi} \ln n$ such that $\chi(G \setminus W) \leq k$ for a certain number $k$, and the goal is to prove that then $\chi(G) \leq k + 1$. By contrast, in the present paper the “exceptional” set $U$ has size $n^{3/2} p \log n \gg \sqrt{\pi} \ln n$. Thus, we need to extend the coloring to a significantly larger number of “exceptional” vertices and study the combinatorial structure of $G$ more precisely.

The remainder of the chapter is structured as follows. We introduce some notational and technical facts in Section 2.1, that will be used extensively in the sequel. In Section 2.2 we exploit the tools developed in \cite{AN05} to prove (2.4). Section 2.3 is divided into two parts: in Section 2.3.1 we prove the main theorem for the case $np \leq n^{1/20}$, and the remaining cases are treated in Section 2.3.2.

### 2.1. Technical Preliminaries

Let $G = (V, E)$ be a graph, and $X, Y \subseteq V$. We denote by $e(X, Y)$ the number of edges in $G$ with one endpoint in $X$ and one endpoint in $Y$. Furthermore, for every $v \in V$ we will denote by $\Gamma(v)$ the neighbors of $v$ in $G$, and by $\Gamma(X) := \bigcup_{v \in X} \Gamma(v)$. Moreover, we will write $x^\mu$ for the product $x \cdot (x-1) \cdots (x-y+1)$.

Before we prove our main result in the following sections we introduce a few important facts from probability theory. An extensive treatment can be found e.g. in \cite{AS00} or \cite{JLR00}. The first lemma is a well-known estimate for the tail of the binomial distribution.

**Lemma 2.2** (Chernoff bounds). Let $X$ be a binomially distributed variable with $\lambda := \mathbb{E}[X]$. For every $t \geq 0$ it holds

$$\Pr[X \geq \lambda + t] \leq e^{-\frac{t^2}{2\lambda + 3t}}, \quad \text{and} \quad \Pr[X \leq \lambda - t] \leq e^{-\frac{t^2}{2\lambda}}.$$

The next lemma is a special case of a far more general result based on martingale inequalities. We present this version here, as it suffices for our intended application.

**Lemma 2.3** (Azuma/Hoeffding’s inequality). Let $f$ be a function on graphs, such that $|f(G) - f(G')| \leq 1$, whenever $G$ and $G'$ differ only in edges incident to the same vertex. Then the random variable $X = f(G_{n,p})$ with $\lambda := \mathbb{E}[X]$ satisfies

$$\Pr[X \geq \lambda + t] \leq e^{-\frac{t^2}{2\lambda n}}, \quad \text{and} \quad \Pr[X \leq \lambda - t] \leq e^{-\frac{t^2}{2\lambda n}}.$$

Finally, we are going to exploit the following version of the Lovász Local Lemma, which provides us with a lower bound for the probability of the non-occurrence of certain events.

**Lemma 2.4** (Lovász Local Lemma, symmetric version). Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event $A_i$ is mutually independent of a set of all other events $A_j$ but of at most $d$, and that $\Pr[A_i] \leq p$ for all $1 \leq i \leq n$. If $cp(d + 1) \leq 1$, then $\Pr[\bigwedge_{i=1}^n \overline{A_i}] > 0$. 


2.2. Approaching the Values of the Chromatic Number

Let $G_{n,m}$ be a random graph on $n$ labeled vertices and $m$ edges, drawn uniformly at random from the set of all such graphs. In this section we shall derive a lower bound for $\chi(G_{n,m})$, which holds with high probability, and an upper bound for $\chi(G_{n,p})$, which holds with some probability, that we can bound from below.

**Proposition 2.5.** Let $\ell : \mathbb{N} \to \mathbb{N}$ be a function such that $2 \leq \ell(n) \leq n^{1/2} \log n$. For every fixed $0 < \varepsilon < 1$, if $d \geq \frac{\log \ell}{\log \ell - \log(\ell - 1)} + \varepsilon$, then $G_{n,dn}$ is a.a.s. not $\ell$-colorable.

**Proof.** We show the statement for $d_0 := \frac{\log \ell}{\log \ell - \log(\ell - 1)} + \varepsilon$, as the property of not being $\ell$-colorable is increasing. Observe that

$$\log \ell - \log(\ell - 1) = -\log \left(1 - \frac{1}{\ell}\right) \geq \frac{1}{\ell},$$

and hence $d_0 \leq \log \ell + \varepsilon$. \hspace{1cm} (2.5)

The number of edges connecting vertices in different parts of a $\ell$-partition of the vertex set is at most $\frac{\ell - 1}{2} n^2$. Thus, the probability that a given $\ell$-partition is a valid coloring of $G_{n,d_0 n}$ can be estimated from above by

$$\frac{\left(\frac{\ell - 1}{2} n^2\right)^{d_0 n}}{\left(\frac{n}{2}\right)^{d_0 n}} \leq \left(1 - \frac{1}{\ell}\right)^{d_0 n} \cdot \left(1 + \frac{1}{n - 1}\right)^{d_0 n} \leq \left(1 - \frac{1}{\ell}\right)^{d_0 n} \cdot e^{2d_0}.$$

With this, the expected value of the number $X$ of $\ell$-partitions, which are colorings of $G_{n,d_0 n}$ can be estimated with

$$\mathbb{E}[X] \leq \ell^n \cdot e^{2d_0} \cdot \left(1 - \frac{1}{\ell}\right)^{d_0 n} \leq \exp \left\{ n \left( \log \ell + \frac{2d_0}{n} + d_0 \log \left(1 - \frac{1}{\ell}\right) \right) \right\} \leq \exp \left\{ n \left( \frac{2d_0}{n} + \varepsilon \log \left(1 - \frac{1}{\ell}\right) \right) \right\} \leq \exp \left\{ n \left( \frac{2\ell \log \ell}{n} - \frac{\varepsilon}{\ell} \right) \right\}.$$

It is easily seen that the exponent is negative for sufficiently large $n$, with our assumption on $\ell$. The proof now completes with the first moment method ($\Pr[X > 0] \leq \mathbb{E}[X]$). \hfill $\square$

Observe that for $\ell \to \infty$, it holds $\frac{\log \ell}{\log \ell - \log(\ell - 1)} = (\ell - \frac{1}{2}) \log \ell + o(1)$. Before we proceed with the proof of the upper bound for $\chi(G_{n,p})$, let us introduce a tool which plays a crucial role in our arguments. Let $S_\ell$ denote the set of $\ell \times \ell$ row-stochastic matrices, i.e., matrices from $[0,1]^{\ell \times \ell}$, such that the values of the rows sum up to one. For $M \in S_\ell$, let

$$\mathcal{H}(M) := -\frac{1}{\ell} \sum_{1 \leq i,j \leq \ell} m_{ij} \log m_{ij} \quad \text{and} \quad \mathcal{E}(M) := \log \left(1 - \frac{2}{\ell} + \frac{1}{\ell^2} \sum_{1 \leq i,j \leq \ell} m_{ij}^2\right), \quad (2.6)$$

and define the function $g_d(M) := \mathcal{H}(M) + d\mathcal{E}(M)$. In our proof we will exploit the following general result by Achlioptas and Naor [AN05].

**Theorem 2.6.** Let $\ell \in \mathbb{N}$ and $I_\ell$ be the constant $\ell \times \ell$ matrix, whose entries are all equal to $\frac{1}{\ell}$. If $d \leq (\ell - 1) \log(\ell - 1)$, then for all $M \in S_\ell$ it holds $g_d(I_\ell) \geq g_d(M)$.
Proposition 2.7. Let \( \ell : \mathbb{N} \to \mathbb{N} \) be a function such that \( 2 \leq \ell \leq \frac{n^{1/2}}{\log n} \). Let \( C_{\ell,m} \) denote the random variable, which counts the number of colorings of \( G_{n,m} \) with \( \ell \) colors. If \( \frac{m}{n} < (\ell - 1) \log(\ell - 1) \), then \( \Pr \left[ C_{\ell,m} > 0 \right] \geq e^{-17 \left( \frac{m}{n} \right)^2} \cdot n^{-\epsilon^2} \).

Proof. In order to prove the statement, we use similar ideas as in [AN05]. An important difference here is that \( \ell \) is a function of \( n \) (instead of being constant), and our contribution is that we take into account how it modifies the involved constants (which now become functions of \( \ell \)) in the original proof. Furthermore, we are working directly with the uniform random graph \( G_{n,m} \) which does not have any multiple edges or loops.

Let \( d := \frac{m}{n} \) and denote by \( B_\ell \) the number of “balanced” colorings of \( G_{n,dn} \), where balanced means that the sizes of all color classes are either \( \lfloor \frac{n}{\ell} \rfloor \) or \( \lceil \frac{n}{\ell} \rceil \). For the sake of exposition, we shall omit in the remainder floors and ceilings. As the number of edges not connecting vertices in the same color class is \( \frac{\ell-1}{2\ell} n^2 \), by using \( 1 - x \geq e^{-2x} \), valid for small \( x \), the probability that a balanced partition is a valid coloring is for sufficiently large \( n \)

\[
\frac{\left( \frac{\ell-1}{2\ell} n^2 \right) \cdot \left( 1 - \frac{1}{\ell} \right)^{dn}}{\left( \frac{n}{2} \right)^{dn}} \geq \left( 1 - \frac{1}{\ell} \right)^{dn} \cdot e^{-\epsilon^2} \cdot \left( 1 - \frac{1}{\ell} \right)^{dn} \cdot e^{-8d^2}.
\]

By applying Stirling’s formula \( 1 \leq \frac{n!}{(\frac{n}{e})^n \sqrt{2\pi n}} \leq 2 \) we obtain easily

\[
E[B_\ell] \geq \frac{n!}{(\frac{n}{e})^n} \cdot \left( 1 - \frac{1}{\ell} \right)^{dn} \cdot e^{-8d^2} \geq (2\pi n)^{-\frac{\ell-1}{2\ell}} \cdot \ell^\ell \cdot \left[ \ell \cdot \left( 1 - \frac{1}{\ell} \right)^{\frac{d}{\ell}} \right]^n \cdot e^{-8d^2}.
\]

In the following we are going to argue that

\[
E[B^2_\ell] \leq e^{8d} \cdot n^{\ell^2} \cdot \left[ \ell \cdot \left( 1 - \frac{1}{\ell} \right)^{\frac{d}{\ell}} \right]^{2n}.
\]

For large \( n \) this will complete the proof with plenty of room to spare, as due to Cauchy-Schwartz we have that

\[
\Pr \left[ B_\ell > 0 \right] \geq \frac{E[B_\ell]^2}{E[B^2_\ell]} \geq e^{-16d^2} \cdot \left( \frac{\ell}{2\pi} \right)^{\ell-1} \cdot n \geq e^{-17d^2} n^{\ell^2}.
\]

In order to calculate \( E[B^2_\ell] \), it is sufficient to consider pairs of balanced partitions, and to bound the probability that both are simultaneously valid colorings. Let \( \Pi = (V_1, \ldots, V_\ell) \) and \( \Pi' = (V'_1, \ldots, V'_\ell) \) be two partitions, and define \( d_{ij}(\Pi, \Pi') := |V_i \cap V'_j| \). The probability that an edge is bichromatic in both \( \Pi \) and \( \Pi' \) is proportional to the number of edges, that do not join vertices in one of the color classes \( V_1, \ldots, V_\ell \) or \( V'_1, \ldots, V'_\ell \). The number of such edges is precisely \( \left( \frac{n}{2} \right) - 2\ell \left( \frac{n}{2\ell} \right) + \sum \left( \frac{d_{ij}(\Pi, \Pi')}{} \right) \). Hence, the probability that both \( \Pi \) and \( \Pi' \) are valid colorings is for sufficiently large \( n \)

\[
\frac{\left( \frac{n}{2} - 2\ell \left( \frac{n}{2\ell} \right) + \sum \left( \frac{d_{ij}(\Pi, \Pi')}{} \right) \right)^{dn}}{\left( \frac{n}{2} \right)^{dn}} \leq \frac{\left( \frac{n}{2} - 2\ell \left( \frac{n}{2\ell} \right) + \sum \left( \frac{d_{ij}(\Pi, \Pi')}{} \right) \right)^{dn}}{\left( \frac{n}{2} \right)^{dn}} = \left( 1 - \frac{2\left( \frac{n}{\ell} - 1 \right)}{n - 1} + \sum \frac{d_{ij}(d_{ij} - 1)}{n(n - 1)} \right)^{dn},
\]
which reduces with \( \frac{n^\ell}{n^2} \geq 1 - \frac{2}{\ell} \) and \( \frac{d_{ij}}{n} \leq \frac{d_{ij}}{n^2} \) to at most \( e^{8d}q^{dn} \), where we abbreviated \( q := 1 - \frac{2}{\ell} + \sum \left( \frac{d_{ij}}{n} \right)^2 \) (note that \( q \geq \frac{1}{4} \)). Let \( D \) be the set of matrices with non-negative integer entries, where all rows and columns sum up to \( \frac{n}{2} \). By using \( (\frac{n}{2})^x \leq x! \leq 10\sqrt{x}(\frac{x}{2})^x \), we obtain

\[
\mathbb{E}[B^2] \leq e^{8d} \cdot \sum_{D \in \mathcal{D}} \frac{n!}{\prod_{i=1}^{\ell} d_{ij}!} q^{dn} \leq e^{8d} \cdot 10\sqrt{n} \cdot \sum_{D \in \mathcal{D}} e^{n \left( \left( H\left( \frac{\ell}{n} \right) + d\mathcal{E} \left( \frac{\ell}{n} \right) \right) \right)} ,
\]

where \( H \) and \( \mathcal{E} \) are defined in (2.6). By applying Theorem 2.6 we obtain for all matrices \( D \in \mathcal{D} \)

\[
\mathcal{H} \left( \frac{\ell}{n} \right) + d\mathcal{E} \left( \frac{\ell}{n} \right) \leq \mathcal{H}(J_\ell) + d\mathcal{E}(J_\ell) = \log \ell + d\log \left( 1 - \frac{2}{\ell} + \frac{1}{\ell^2} \right) .
\]

As the number of matrices in \( \mathcal{D} \) is at most \( n^{(\ell-1)^2} \), we obtain

\[
\mathbb{E}[B^2] \leq e^{8d} \cdot 10\sqrt{n} \cdot n^{(\ell-1)^2} \cdot e^{n \left( 2\log \ell + d\log((1-\frac{1}{\ell})^2) \right)} \leq e^{8d} \cdot n^{2 - \ell} \cdot (1 - \frac{1}{\ell})^{2dn} .
\]

\[\square\]

From the above proposition we obtain easily the following lemma for the binomial random graph, as the models \( G_{n,p} \) and \( G_{n,m} \) behave similarly when \( m \approx p \binom{n}{2} \).

**Lemma 2.8.** Let \( 0 < \delta \leq \frac{1}{2} \) and \( p = p(n) \leq n^{-1/2 - \delta} \). Then the following statement is true for sufficiently large \( n \). Let \( C_{\ell,p} \) be the number of colorings of \( G_{n,p} \) with \( \ell \) colors. If \( \ell \) is the maximum integer satisfying \( 2(\ell - 2)\log(\ell - 2) \leq p(n - 1) \), then

\[
\Pr[C_{\ell,p} > 0] \geq e^{-6(pn)^2} \cdot n^{-2\ell^2} . \tag{2.8}
\]

**Proof.** Let \( C_{\ell,m} \) denote the number of colorings of \( G_{n,m} \) with \( \ell \) colors. It is straightforward to show that the number of edges of \( G_{n,p} \) is precisely \( \left\lfloor p \binom{n}{2} \right\rfloor \) with probability at least \( n^{-2} \). Hence, with \( m = \left\lfloor p \binom{n}{2} \right\rfloor \) we obtain

\[
\Pr[C_{\ell,p} > 0] \geq \Pr[C_{\ell,m} > 0] \cdot n^{-2}
\]

\[
\Pr[C_{\ell,m} > 0] \geq e^{-17\left( \left\lfloor p \binom{n}{2} \right\rfloor / n^2 \right)^2} \cdot n^{-\ell^2} \cdot n^{-2} \geq e^{-6(pn)^2} \cdot n^{-2\ell^2} .
\]

\[\square\]

**2.3. Proof of the Main Result**

In this section we are going to show that whenever \( \ell \) is the maximum integer such that \( 2(\ell - 2)\log(\ell - 2) \leq p(n - 1) \), then \( G_{n,p} \) is a.a.s. colorable with \( \ell + 1 \) colors. Due to technical reasons we shall consider two ranges of \( p \), namely \( p \leq n^{-1+1/20} \) and \( n^{-1+1/20} < p \leq n^{-3/4 - \delta} \), separately in the Sections 2.3.1 and 2.3.2. Finally, in Section 2.3.3 we are going to show how this implies Theorem 2.1.
2.3.1. The sparse case ($n^{-1} \ll p \leq n^{-1+1/20}$). Our first lemma follows directly from Fact 2 in [Luc91], and we state it without proof.

**Lemma 2.9.** Let $n^{-1} \ll p \leq n^{-1+1/20}$. Every subset $U$ of the vertex set of $G_{n,p}$, such that $|U| \leq n^{3/4}$, spans a.a.s. less than $(\frac{3}{2} - \frac{1}{12})|U|$ edges.

The next lemma states essentially that a.a.s. $G_{n,p}$ is almost $\ell$-colorable, if we choose the value of $\ell$ carefully. A similar argument can be found in [Luc91].

**Lemma 2.10.** Let $n^{-1} \leq p \leq n^{-3/4}$ and let $\ell$ be the maximum integer satisfying $2(\ell - 2) \log(\ell - 2) \leq p(n - 1)$. Then a.a.s. there is a subset $U_0 = U_0(G_{n,p})$ of the vertex set of $G_{n,p}$ of size at most $n^{3/2}p \log n$, such that $G_{n,p} \setminus U_0$ is $\ell$-colorable.

**Proof.** According to Lemma 2.8, from the definition of $\ell$ we obtain
\[
\Pr[\chi(G_{n,p}) \leq \ell] \geq e^{-6(pn)^2} \cdot n^{-2\ell^2} := \varepsilon.
\]
Let $X$ be the random variable which counts the minimal number of vertices that have to be deleted from $G_{n,p}$, such that an $\ell$-colorable graph remains ($X$ is then precisely $|U_0|$). In the sequel we are going to show that
\[
\Pr[X \geq 4\sqrt{\log(\varepsilon^{-1})}n] < \varepsilon,
\]
from which the claim follows directly for sufficiently large $n$.

For every pair of graphs $G$ and $G'$ on $n$ vertices, which differ solely in edges that are adjacent to the same vertex, we clearly have $|X(G) - X(G')| \leq 1$. Hence, $X$ satisfies a Lipschitz condition, and by Lemma 2.3 we have for every $\lambda > 0$
\[
\Pr[|X - \mathbb{E}[X]| \geq \lambda] < 3e^{-\frac{\lambda^2\varepsilon}{n}}.
\]
But since the event “$\chi(G_{n,p}) \leq \ell$” implies the event “$X = 0$”, we may estimate
\[
\varepsilon \leq \Pr[\chi(G_{n,p}) \leq \ell] \leq \Pr[X = 0] \leq \Pr[|X - \mathbb{E}[X]| \geq \lambda] < 3e^{-\frac{2\mathbb{E}[X]}{n}},
\]
which yields $\mathbb{E}[X] < \sqrt{n \log(\varepsilon^{-1})}$. Having this, by setting $\lambda := \sqrt{2n \log(\varepsilon^{-1})}$ in (2.10), we see that (2.9) immediately follows, with plenty of root spare.

The above lemma states that if we choose $\ell$ as prescribed, then all vertices of $G_{n,p}$ are colorable with the colors $\{1, \ldots, \ell\}$, except for a small set $U_0$. In the remainder we shall assume that $G = G_{n,p}$ satisfies the assertions of Lemma 2.9 and 2.10 and we are going to argue that by using only one additional color, we can color $U_0$ such that we obtain a valid coloring for the whole graph.

To achieve this, we construct a set $U \supseteq U_0$ of size at most $n^{3/4}$, such that $I = \Gamma(U) \setminus U$ is stable. For such a set $U$, we can color $G$ with $\ell + 1$ colors. Indeed, $U$ is 3-colorable, as due to Lemma 2.9 all its subsets have average degree less than 3. Now color $U$ with the colors $\{1, 2, 3\}$, and $I$ with a fresh color $\ell + 1$ (c.f. Figure 1).

To obtain $U$, we begin with $U_0$, and extend it by two vertices $x, y$ in its neighborhood, if $(x, y) \in G$. Now observe that this process stops with $|U| \leq n^{3/4}$, as otherwise the number of edges joining vertices in $U$ would be for sufficiently large $n$ greater
than $3^{\frac{|U|-|U_0|}{2}} \geq (\frac{3}{2} - \frac{1}{12})|U|$, contradicting Lemma 2.9. Hence, we obtain the following result:

**Lemma 2.11.** Let $n^{-1} \leq p = p(n) \leq n^{-1+1/20}$ and let $\ell$ be the maximum integer satisfying $2(\ell - 2) \log(\ell - 2) \leq p(n - 1)$. Then a.a.s. $\chi(G_{n,p}) \leq \ell + 1$.

**2.3.2. The dense case ($n^{-1+1/20} \leq p \leq n^{-3/4-\delta}$).** In this section we will assume that $n^{-1+1/20} \leq p = p(n) \leq n^{-3/4-\delta}$, where $\delta > 0$. In order to deal with edge probabilities $p$ in this regime, and thus with denser random graphs than in the previous section, we have to extend the above argument significantly. Let $\ell = \ell(p, n)$ be the maximum integer satisfying $2(\ell - 2) \log(\ell - 2) \leq p(n - 1)$, and note that a straightforward calculation yields $\ell \geq \frac{np}{3 \log(np)}$.

A graph $G$ with vertex set $V$ is called $d$-choosable, if for every family $C = \{S_v \in \mathbb{N}^d \mid v \in V\}$ of sets of colors for the vertices of $G$, there exists a proper vertex coloring $c : V \to \mathbb{N}$, such that $c(v) \in S_v$ for all $v$. The lemma below is a well-known fact and states that every not too large subset of vertices of the $G_{n,p}$ spans very few edges, if $p$ is substantially smaller than $n^{-3/4}$; furthermore, every such subset is t-choosable, for a small constant $t$.

**Lemma 2.12.** Let $n^{-1+1/20} \leq p = p(n) \leq n^{-3/4-\delta}$, where $\delta > 0$. $G_{n,p}$ has a.a.s. the following properties. Every subset $U$ of the vertices of $G_{n,p}$, such that $|U| \leq n^{3/4}$ spans at most $(\lceil \frac{1}{5} \rceil - 1)|U|$ edges. Furthermore, every $U$ is $2\lceil \frac{1}{5} \rceil$-choosable.

**Proof.** Let $\beta := \lceil \frac{1}{5} \rceil - 1$. The expected number of sets $U$, that span more than $\beta|U|$ edges can be estimated as follows. The number of ways to choose $|U|$ is $\binom{n}{|U|}$, and the number of ways to choose the edges is $\binom{n}{|U|\beta}$. Using $\binom{n}{s} \leq \left(\frac{en}{s}\right)^s$, the desired probability is at most

$$\sum_{s=1}^{n^{3/4}} \binom{n}{s} \left(\frac{s}{\beta^s}\right)^{p^{\beta^s}} \leq \sum_{s=1}^{n^{3/4}} \left(\frac{e^{c+1}}{(2\beta)^\beta} \cdot n \cdot s^{\beta-1} \cdot p^\beta\right)^s \leq \sum_{s=1}^{n^{3/4}} \left(\Theta(1) \cdot n^{1+(\beta-1)^{\frac{1}{2}-\beta(\frac{1}{2}+\delta)}\beta^s}\right)^s = o(1).$$

Now, the choosability follows immediately from the fact that in every $U' \subseteq U$ we find a vertex of degree at most $2(\lceil \frac{1}{5} \rceil - 1)$. \qed
Similarly as before, we shall see that a.s.a. the random graph $G = G_{n,p}$ admits a set $U \subset V$ of size $|U| \leq 2n^{3/2}p \log n$, such that $\chi(G \setminus U) \leq \ell$, and such that $U$ is only “sparsely connected” to $V \setminus U$. However, we cannot guarantee that $\Gamma(U) \setminus U$ is a stable set. Instead, we shall recolor a few vertices in $V \setminus U$ with a new “joker color” in such a way that we can then reuse some of the colors of the $\ell$-coloring of $G \setminus U$ to color the vertices in $U$. Hence, we will just need one additional color, so that $\chi(G) \leq \ell + 1$ a.a.s.

We now elaborate the details of this approach. As a first step, we show that there is a subset $U$ of the vertex set of size $|U| \leq 2n^{3/2}p \log n$, such that a.a.s. $\chi(G \setminus U) \leq \ell$, and every vertex in $V \setminus U$ has at most a constant number of neighbors in $U$.

**Lemma 2.13.** Let $n^{-1+1/20} \leq p = p(n) \leq n^{-3/4-\delta}$, where $\delta > 0$, and let $\ell$ be the maximum integer satisfying $2(\ell - 2) \log(\ell - 2) \leq p(n - 1)$. $G_{n,p}$ has a.a.s. the following property. There is a set of vertices $U = U(G_{n,p})$ of size $< 2n^{3/2}p \log n$ such that $\chi(G \setminus U) \leq \ell$, and every vertex in $V \setminus U$ has at most $\xi := 4\left[\frac{1}{5}\right]$ neighbors in $U$.

**Proof.** According to Lemma 2.10, there exists a.a.s. a set $U_0$ of size at most $n^{3/2}p \log n$, such that $\chi(G_{n,p} \setminus U_0) \leq \ell$. Moreover, according to Lemma 2.12, every subset $X$ of the vertices of size at most $n^{3/4}$ spans a.a.s. at most $\left\lceil \frac{1}{5} \right\rceil |X|$ edges. We assume that $G = G_{n,p}$ has these properties. To obtain $U$, we start with $U_0$, and enhance it iteratively with vertices, which have more than $\xi$ neighbors in the current set $U$. This process stops with $|U| < 2n^{3/2}p \log n$, as otherwise we would get a subset $U$ of the vertices of $G$ with $|U| = 2n^{3/2}p \log n \leq n^{3/4}$, that spans more than $(|U| - |U_0|)\xi \geq 4\left[\frac{1}{5}\right]n^{3/2}p \log n > \left[\frac{1}{5}\right]|U|$ edges — a contradiction. \hfill \Box

Let $c : V \setminus U \to \{1, \ldots, \ell\}$ be a $\ell$-coloring of $V \setminus U$. In order to obtain a $(\ell + 1)$-coloring $c^* : V \to \{1, \ldots, \ell + 1\}$ of the entire graph, we shall recolor some of the vertices in $V \setminus U$ with the new color $\ell + 1$, so that we can reuse the old colors $1, \ldots, \ell$ to color the vertices in $U$. More precisely, our strategy is as follows. As every subset of vertices of size at most $n^{3/4}$ is a.a.s. $\eta := 2\left[\frac{1}{5}\right]$-choosable, we shall assign to each vertex $u \in U$ a list $L_u \subset \{1, \ldots, \ell\}$ of colors, such that $|L_u| = \eta$ and the following holds. Let

$$\Gamma_u := \{v \in \Gamma(u) \setminus U : c(v) \in L_u\}$$

be the set of all neighbors in $V \setminus U$ of $u \in U$, whose color lies in $L_u$. Then $\Gamma_u = \bigcup_{u \in U} \Gamma_u$ is a stable set.

If we could exhibit such lists $(L_u)_{u \in U}$, then it would be easy to obtain a $(\ell + 1)$-coloring $c^*$ of $G$: color the vertices in $\Gamma_u$ with the “joker color” $\ell + 1$, thereby making the colors in $L_u$ available for $u \in U$. Then, color each vertex in $u \in U$ with a color from $L_u$, which is possible due to Lemma 2.12. For an illustration, see Figure 2.

Hence, the remaining task is to show that a.a.s. there exist lists $(L_u)_{u \in U}$ with the desired property. In this context, Alon and Krivelevich [AK97b] proved the following.

**Lemma 2.14.** Let $n^{-1+1/20} \leq p = p(n) \leq n^{-3/4-\delta}$, where $\delta > 0$. $G_{n,p}$ has a.a.s. the following property. Assume that $R \subset V$ is such that $G_{n,p} \setminus R$ has a $t$-coloring $c : V \setminus R \to \{1, \ldots, t\}$. Moreover, suppose that $Q \subset R$ is a set of size $|Q| \leq \sqrt{n}$. Then
there exist lists \((L_u)_{u \in Q}\) of colors 1, \ldots, \(t\) such that \(|L_u| \geq \eta := 2\lceil \frac{1}{\delta} \rceil\) for all \(u \in Q\) and \(\Gamma_Q = \{v \in \Gamma(u) \setminus R : c(v) \in L_u, u \in Q\}\) is stable.

Due to the previous discussion, the above lemma implies that if \(|U| \leq \sqrt{n}\) (i.e., if we could set \(R = Q = U\) in the previous lemma), then \(G_{n,p}\) would be \((t + 1)\)-colorable. In fact, Proposition 3.1 of [AK97b] claims just this consequence. The version that we stated above follows directly from their proof – they show that the claim of Lemma 2.14 holds, and proceed as just described.

Lemma 2.14 is unfortunately not strong enough for our intended application. By applying Lemma 2.13 we get only a set \(U\) of size \(\leq 2n^{3/2}p \log n\), whereas Lemma 2.14 requires that \(|Q| \leq \sqrt{n} \ll 2n^{3/2}p \log n\). Therefore, to construct the lists \((L_u)_{u \in U}\), we extend the approach of Alon and Krivelevich as follows. We shall show that up to a small “exceptional set” \(Z \subset U\) of size at most \(\sqrt{n}\), all vertices in \(v \in U \setminus Z\) have the property that their neighborhood \(\Gamma(v) \setminus U\) outside of \(U\) is only sparsely connected to the neighborhoods \(\Gamma(U) \setminus U\) of the remaining vertices in \(U\). This will enable us to apply the Lovász Local Lemma to prove in a probabilistic fashion that such lists \(L_u\) for \(u \in U \setminus Z\) exist. Furthermore, the exceptional vertices in \(Z\) will be considered separately: as \(|Z| \leq \sqrt{n}\), we can just apply Lemma 2.14 with \(Q = Z\) and \(R = U\) to obtain the lists \(L_u\) for \(u \in Z\).

Let \(e(A,B)\) be the set of edges having one endpoint in \(A\) and the other endpoint in \(B\), i.e., \(e(A,B) := \{\{x,y\} \in E_G \mid x \in A, y \in B\}\), where \(E_G\) denotes the edge set of \(G\). Moreover, let \(e(A) := e(A,A)\). The following lemma yields the desired small exceptional set \(Z\).

**Lemma 2.15.** Let \(n^{-1+1/20} \leq p = p(n) \leq n^{-3/4-\delta}\), where \(\delta > 0\). \(G_{n,p}\) has the following property a.a.s. For every subset \(U\) of the vertices of size \(1 \ll |U| \leq 2n^{3/2}p \log n\), such that every \(v \notin U\) has at most \(\xi := 4\lceil \frac{1}{\delta} \rceil\) neighbors in \(U\), there exists \(Z \subset U\) of
size $|Z| \leq n^{-1/2}p^{-1}(\log n)^6$ such that every $v \in U \setminus Z$ satisfies
\[
e(\Gamma(v) \setminus U, \Gamma(U \setminus Z) \setminus \Gamma(U) \setminus U) \leq \xi^{-7} \left( \frac{np}{\log(np)} \right)^2, \tag{2.11}
\]
\[
e(\Gamma(v) \setminus U, \Gamma(Z) \setminus U) \leq \xi^{-7} \frac{np}{\log(np)}. \tag{2.12}
\]

**Proof.** Denote by $D$ the set of graphs, which have the property that all their vertices have degree at most $pn \log n$. It is easy to see that a.a.s. $G_{n,p} \in D$—we omit the details.

For every subset $U$ of the vertices denote by $D_U$ the event that all vertices in $U$ have degree at most $pn \log n$, and by $N_U$ the event that all $v \notin U$ have at most $\xi$ neighbors in $U$. Furthermore, denote by $A_U$ the set of graphs having property (2.11), but with $Z$ having size at most $z := \frac{1}{2}n^{-1/2}p^{-1}(\log n)^6$. We will show that for all $U$
\[
q_U := \Pr[G_{n,p} \not\in A_U \mid G_{n,p} \in D_U \cap N_U] \leq n^{-2|U|}.
\]

By combining this with the fact that there are at most $n^{|U|}$ ways to choose $U$ we obtain as follows the first statement of the lemma:
\[
\Pr[\exists U : |U| \leq 2n^{3/2}p \log n \text{ and } G_{n,p} \not\in A_U \text{ and } G_{n,p} \in N_U]
\leq \Pr[G_{n,p} \not\in D] + \sum_{|U| \leq 2n^{3/2}p \log n} \Pr[G_{n,p} \not\in A_U \text{ and } G_{n,p} \in D \cap N_U]
\leq o(1) + \sum_{|U| \leq 2n^{3/2}p \log n} \Pr[G_{n,p} \not\in A_U \mid G_{n,p} \in D \cap N_U] \leq o(1) + \sum_{i \geq 1} n^i n^{-2i} = o(1).
\]

Now we prove $q_U \leq n^{-2|U|}$. Let $U$ be fixed, and assume that $G_{n,p} \in D_U \cap N_U$. In the sequel we are going to argue that $G_{n,p} \not\in A_U$ implies that there is a set $B \subseteq U$ of size $z$ such that $e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) \geq pn^{3/2} \log n)^3 := \mu$. To see this, let $B_0 := \emptyset$, and define $B_{i+1} := B_i \cup \{v_{i+1}\}$, where $v_{i+1} \in U \setminus B_i$, and $e(\Gamma(v_{i+1}) \setminus U, \Gamma(U \setminus B_i) \setminus U) > \xi^{-7} \left( \frac{np}{\log(np)} \right)^2$ (if there are more vertices with this property, choose any of them). Note that if $i + 1 \leq z$ it is guaranteed that such a vertex exists, as otherwise we would have $G_{n,p} \in A_U$. With this notation $B$ is then just $B_z = \{v_1, \ldots, v_z\}$. It follows that
\[
e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) = \left| \bigcup_{i=1}^{z} E(\Gamma(v_i) \setminus U, \Gamma(U) \setminus U) \right|
\geq \left| \bigcup_{i=1}^{z} E(\Gamma(v_i) \setminus U, \Gamma(U \setminus B_{i-1}) \setminus U) \right|.
\]

Observe that we can estimate the right-hand side of the above inequality from below by at least $z \cdot \xi^{-2} \cdot \xi^{-7} \left( \frac{np}{\log(np)} \right)^2$, as due to the event $G_{n,p} \in N_U$ each edge is counted in the disjoint union of the sets $E(\Gamma(v_i) \setminus U, \Gamma(U \setminus B_{i-1}) \setminus U)$ at most $\xi^2$ times. With $z = \frac{1}{2}n^{-1/2}p^{-1}(\log n)^6$ the statement $e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) \geq pn^{3/2} \log n)^3 = \mu$ follows then with plenty of room to spare for large $n$.

The above discussion results in the inequality
\[
q_U \leq \Pr[\exists B \subseteq U : |B| = z \text{ and } e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) \geq \mu \mid G_{n,p} \in D_U \cap N_U]. \tag{2.13}
\]
To estimate the latter probability observe that all edges between the neighborhoods of vertices in \( U \), except of those which have (at least) one endpoint in \( U \), are included in \( G_{n,p} \) independently with probability \( p \), since we have conditioned only on the edges between \( U \) and \( V \setminus U \). Hence the quantity \( e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) \) is binomially distributed, and its expected value is for large \( n \) at most

\[
\binom{|\Gamma(B)|}{2} \cdot p + |\Gamma(B)||\Gamma(U)| \cdot p \leq (z \cdot p n \log n)^2 p + n^2 p^2 (\log n)^8 \\
\leq 2n^2 p^2 (\log n)^8 := \lambda_1. 
\]

(The first term above is due to edges connecting vertices in \( \Gamma(B) \setminus U \), and the second due to edges having one endpoint in \( \Gamma(B) \setminus U \), and the other in \( \Gamma(U) \setminus (U \cup \Gamma(B)) \).) It is straightforward to see that \( \lambda_1 = o(\mu) \). Since there are at most \( 2^{|U|} \) ways to choose \( B \) we obtain from (2.13) by applying Lemma 2.2

\[
q_U \leq \sum_{B \subseteq U, |B| = z} \Pr\{ e(\Gamma(B) \setminus U, \Gamma(U) \setminus U) \geq \mu \mid G_{n,p} \in D_U \cap N(U) \} \\
\leq e^{-\Theta(\mu)} \langle |U| \leq 2n^{3/2} p \log n \rangle \ll n^{-2|U|}. 
\]

In the remainder we show the existence of a set \( Z \) of size at most \( n^{-1/2} p^{-1} (\log n)^6 =: s \), which in addition satisfies (2.12). In order to achieve this, let \( Z \) be the exceptional set guaranteed to exist by the first statement of the lemma (having size at most \( \xi^2 \)), and as long as there is a vertex \( v \in U \) violating (2.12), set \( Z := Z \cup \{v\} \). Suppose now that this process stops with a set of size \( \geq s \). As every vertex not in \( U \) has at most \( \xi \) neighbors in \( U \), this implies that there is in \( G_{n,p} \) a subset of the vertices of size \( s \), which has the property that the neighborhoods of its vertices span at least \( \xi^{-2} \cdot \xi^{-7} \frac{np}{\log|np|} \cdot \frac{s}{2} =: \kappa \) edges. To complete the proof, we show that the expected number of such sets in \( G_{n,p} \) is \( o(1) \).

Let \( S \) be any set of vertices of size \( s \). Obviously,

\[
\Pr\{ \exists S : e(\Gamma(S)) \geq \kappa \} \leq \Pr\{ G_{n,p} \notin D \} + \sum_{S : |S| = s} \Pr\{ e(\Gamma(S)) \geq \kappa \mid G_{n,p} \in D_S \}. 
\]

In the remainder we are going to show \( \Pr\{ e(\Gamma(S)) \geq \kappa \mid G_{n,p} \in D_S \} = o(n^{-s}) \), which will complete the proof, as the total number of sets \( S \) is at most \( \binom{n}{s} \leq n^s \). Note that the quantity \( e(\Gamma(S)) \), conditioned on the event \( G_{n,p} \in D_S \), is binomially distributed with expected value at most \( \binom{|\Gamma(S)|}{2} \cdot p \leq (s \cdot p n \log n)^2 \cdot p \leq p n \log n \) \( |D_S| =: \lambda_2 \), as conditioning on \( D_S \) only affects the edges joining \( S \) with \( V \setminus S \). It is now easy to verify that \( \lambda_2 \ll \kappa \), and by applying again Lemma 2.2 we obtain

\[
\Pr\{ e(\Gamma(S)) \geq \kappa \mid G_{n,p} \in D_S \} \leq e^{-\Theta(\kappa)} = e^{-\Theta(n^s \log(\frac{np}{s})} = o(n^{-s}). 
\]

Now, by applying Lemma 2.14 to \( Q = Z \) and \( R = U \) we obtain lists \((L_u)_{u \in Z} \subset \{1, \ldots, \ell\}\) of colors, such that \( |L_u| \geq \eta \), and such that the set \( \Gamma_z \) of neighbors \( v \in V \setminus U \) of vertices \( u \in Z \) whose color \( c(v) \) belongs to \( L_u \) is stable.
As a final step, we assign lists \( L_u \) to the vertices \( u \in U \setminus Z \). For each vertex \( u \in U \setminus Z \) we consider the set

\[
F_u := \{ c(v) : v \in \Gamma_Z \text{ is adjacent to some } w \in \Gamma(u) \setminus U \}.
\]

Hence, \( F_u \) consists of all colors in \( \{1, \ldots, \ell\} \), that we do not want to include in \( L_u \); otherwise we would generate a conflict with a vertex \( v \in \Gamma(Z) \setminus U \), that will have the “joker color” in \( c^* \) due to the choice of the lists for the vertices in \( Z \). Note that (2.12) implies

\[
|F_u| \leq \frac{np}{\xi^2 \log(np)} \leq \frac{\ell}{2} \quad \text{for all } u \in U \setminus Z.
\]

Now, independently for all \( u \in U \setminus Z \) we choose a list \( L_u \subset \{1, \ldots, \ell\} \setminus F_u \) of size \( |L_u| = \eta \) uniformly at random.

Letting \( \Gamma_u = \{ v \in \Gamma(u) : c(v) \in L_u \} \), \( \Gamma_u = \bigcup_{u \in U} \Gamma_u \) and \( \Gamma_{U \setminus Z} = \bigcup_{u \in U \setminus Z} \Gamma_u \) we complete the proof by showing that with positive probability \( \Gamma_u \) is stable. The crucial ingredient is the following lemma.

**Lemma 2.16.** With the above assumptions, the probability that \( \Gamma_{U \setminus Z} \) is stable (taken over the choice of the random lists \( (L_u)_{u \in U \setminus Z} \)) is positive.

Lemma 2.16 implies that there is a way to choose the lists \( (L_u)_{u \in U \setminus Z} \) such that \( \Gamma_{U \setminus Z} \) is stable, because a randomly chosen list of sets has this property with positive probability. Further, we already know that \( \Gamma_Z = \bigcup_{u \in Z} \Gamma_u \) is stable (by Lemma 2.14). Moreover, \( G \) contains no \( \Gamma_{U \setminus Z} \times \Gamma_Z \) edges, because in the construction of the lists \( (L_u)_{u \in U \setminus Z} \) we have forbidden all colors \( F_u \) that might yield such edges. Thus, we have established that \( \Gamma_U = \Gamma_{U \setminus Z} \cup \Gamma_Z \) is stable, which implies that we can color \( G \) with \( \ell + 1 \) colors. Putting all together, we have proved the following statement:

**Lemma 2.17.** Let \( n^{-1+1/20} \leq p = p(n) \leq n^{-3/4 - \delta} \), where \( \delta > 0 \). Furthermore, let \( \ell \) be the maximum integer satisfying \( 2(\ell - 2) \log(\ell - 2) \leq p(n-1) \). Then a.a.s. \( \chi(G_{n,p}) \leq \ell + 1 \).

**Proof of Lemma 2.16.** We consider an auxiliary graph \( H = (V_H, E_H) \), whose vertex set

\[
V_H := \bigcup_{u \in U \setminus Z} \Gamma(u) \setminus U \times \{u\}
\]

is the disjoint union of the neighborhoods \( \Gamma(u) \setminus U \) for \( u \in U \setminus Z \). Moreover, for pairs of elements \( (v, u), (w, u') \in V_H \) the edge \( \{(v, u), (w, u')\} \) is in \( E_H \) iff \( v \) and \( w \) are adjacent in \( G \). Furthermore, for each edge \( f = \{(x, u), (y, u')\} \) of \( H \) we let \( A_f \) be the event that \( c(x) \in L_u \) and \( c(y) \in L_{u'} \). Thus, if none of the events \( A_f \) for \( f \in E_H \) occurs, then \( \Gamma_{U \setminus Z} \) is a stable set. To estimate this probability we shall employ Lemma 2.4 (Lovasz Local Lemma).

Thus, we need to derive an upper bound on the probability of each individual event \( A_f \), where \( f = \{(x, u), (y, u')\} \in E_H \). The list \( L_u \) of \( u \in U \setminus Z \) is obtained by choosing \( \eta \)}
2.3.3. Proof of Theorem 2.1. Let \( \ell : \mathbb{N} \to \mathbb{N} \) be a function such that \( 2 \leq \ell(n) \leq \frac{n^{1/2}}{\log n} \). From Proposition 2.5 and the asymptotic equivalence of the models \( G_{n,p} \) and \( G_{n,m} \) (see e.g. Proposition 1.12 in [JLR00]) we obtain that whenever \( p(n-1) > u(\ell) \), it holds \( \Pr[\chi(G_{n,p}) \geq \ell + 1] = 1 - o(1) \). Indeed, the number of edges of \( G_{n,p} \) deviates from \( \binom{n}{2}p \) by at most \( \log n \sqrt{n^2p} \) with probability \( 1 - o(1) \). Now choose \( n \) so large, such that \( \binom{n}{2}p - \log n \sqrt{n^2p} \geq (1 - \varepsilon/2)\binom{n}{2}p \). By conditioning on \( G_{n,p} \) having so many edges, we immediately obtain the claimed statement.

Furthermore, Lemmas 2.11 and 2.17 imply that whenever \( p \) satisfies \( p(n-1) < c(\ell-1) \), then \( \Pr[\chi(G_{n,p}) \leq \ell + 1] = 1 - o(1) \). Putting everything together yields

- if \( p(n-1) \in (c(\ell-1), u(\ell)) \), then a.a.s. \( \chi(G_{n,p}) \in [\ell, \ell + 1, \ell + 2] \);
- if \( p(n-1) \in (u(\ell), c(\ell)) \), then a.a.s. \( \chi(G_{n,p}) \in [\ell + 1, \ell + 2] \).

This completes the proof.
CHAPTER 3

Extremal Subgraphs of Random Graphs

It is well-known that, in many different contexts, large triangle-free graphs are bipartite. For example, Mantel [Man07] proved that the maximum triangle-free subgraph of a complete graph on \( n \) vertices is a complete bipartite graph with \([n/2]\) vertices in one class and \([n/2]\) vertices in the other class. Erdős, Kleitman, and Rothschild [EK76] proved that such a statement is also true in a probabilistic sense. More precisely, they showed that if \( T_n \) denotes a graph drawn uniformly at random from the set of all triangle-free graphs on \( n \) labeled vertices, then the probability that \( T_n \) is bipartite tends to 1 for \( n \) tending to infinity. Recently, this result was generalized independently by Steger [Ste05] and Osthus, Prömel and Taraz [OPT03] to the case that, in addition to the number of vertices, also the number of edges is prescribed. The following result is from [OPT03].

**Theorem 3.1.** Let \( T_{n,m} \) denote a graph drawn uniformly at random from the set of all triangle-free graphs on \( n \) labeled vertices and \( m \) edges. Then for any \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \Pr \left[ T_{n,m} \text{ is bipartite} \right] = \begin{cases} 
1, & \text{if } m = o(n) \\
0, & \text{if } \frac{n}{2} \leq m \leq (1 - \varepsilon)\frac{\sqrt{3}}{4}n^{1/2}\sqrt{\log n} \\
1, & \text{if } m \geq (1 + \varepsilon)\frac{\sqrt{3}}{4}n^{1/2}\sqrt{\log n}.
\end{cases}
\]

For a graph \( G \), let \( t(G) \) denote the maximum number of edges in a triangle-free subgraph (not necessarily induced) of \( G \), and let \( b(G) \) be the maximum number of edges in a bipartite subgraph of \( G \). So \( b(G) \) is just the maximum size of a cut in \( G \). Of course, we always have \( t(G) \geq b(G) \). Our general intuition — guided by the above results — suggests that, for dense enough graphs, these two parameters will typically be equal.

In 1990, Babai, Simonovits and Spencer [BSS90] studied these parameters for the binomial random graph \( G_{n,p} \), which was introduced by Erdős and Rényi in [ER60]. They proved, among others, the following result.

**Theorem 3.2.** There is a positive constant \( \delta \) such that, for \( p \geq \frac{1}{2} - \delta \),

\[
\lim_{n \to \infty} \Pr \left[ t(G_{n,p}) = b(G_{n,p}) \right] = 1.
\]

Perhaps what is most striking about this result is its domain of validity. It seems unlikely that the property “\( t(G_{n,p}) = b(G_{n,p}) \)” has a threshold for constant \( p \); indeed, Babai et. al. asked in [BSS90] whether this result could be extended to cover edge probabilities \( p \) of the form \( n^{-c} \), for some positive constant \( c \).

As far as we know, Theorem 3.2 could hold whenever \( p = p(n) \geq n^{-1/2+\varepsilon} \), for arbitrary \( \varepsilon > 0 \). The property does not hold for example when \( p_0(n) = \frac{1}{\log n} (\log n)^{1/2}n^{-1/2} \), as an
easy calculation shows that the random graph $G_{n,p}$ asymptotically almost surely has
an induced 5-cycle $H$ such that no other vertex has more than one neighbour in $H$:
any maximum-size triangle-free subgraph then includes all the edges of $H$, and is not
bipartite.

In this paper, we answer affirmatively the 16-year old question of Babai, Simonovits
and Spencer: we prove that Theorem 3.2 holds whenever $p = p(n) \geq n^{-c}$, for some
fixed $c > 0$. In fact, we prove the following stronger result.

**Theorem 3.3.** There is a positive constant $c$ such that, if $p = p(n) \geq n^{-c}$, then
\[
\lim_{n \to \infty} \Pr \left[ \text{every maximum triangle-free subgraph of } G_{n,p} \text{ is bipartite} \right] = 1.
\]

It should be noted that Theorem 3.1 cannot be used directly to prove a theorem of
this type. For given $p$, the result does imply that there is an $m = m(n)$ such that
the expected number of non-bipartite triangle-free subgraphs of $G_{n,p}$ with $m$ edges
is $o(1)$, while the expected number of bipartite subgraphs with $m$ edges tends to
infinity. However, the events that particular bipartite subgraphs exist in the graph are very far
from being independent, so this certainly does not prove that there is asymptotically
almost surely a bipartite subgraph of $G_{n,p}$ with this number $m$ of edges.

Let us indicate our general strategy for proving Theorem 3.3, and explain the main
points of difficulty. We need a little notation first. Let $[n] := \{1, \ldots, n\}$ and $p = p(n) \geq n^{-c}$, where $c > 0$ is some fixed and small constant.
For a bipartition $\Pi = (A, B)$ of $[n]$, and a graph $G$ with vertex set $[n]$, we let $E(G; \Pi)$ denote the set of edges of $G$ with one
endpoint in each part. The edges of $E(G; \Pi)$ are said to go across $\Pi$; the other edges
of $G$ are said to be inside the (parts of the) partition. A $d$-perturbation of $E(G; \Pi)$ is
a triangle-free subgraph of $G$ obtained by adding at most $d$ edges, which are inside $\Pi$,
and removing any number of edges from $E(G; \Pi)$.

An adaptation of the proof of Babai et. al. [BSS90] enables us to restrict our attention
to triangle-free subgraphs of the $G_{n,p}$ that are “almost bipartite”, specifically that
are $p^{-c}$-perturbations of some bipartite subgraph, for some positive constant $c$. One
of the new ingredients here is that we have to use the sparse regularity lemma and a
probabilistic embedding lemma (see Section 3.1) in order to cover cases where $p = o(1)$.

If we now fix a partition $\Pi$ with the additional constraint that the two classes $A$ and $B$ are
roughly equal, it is not too hard to show that, with reasonably high probability, no $p^{-c}$-
perturbation of $E(G_{n,p}; \Pi)$ has more edges than $E(G_{n,p}; \Pi)$. However, for fixed $G = G_{n,p}$,
this is certainly not true simultaneously for all partitions $\Pi$: for instance, if $(x, y)$ is an
edge of $G$, and $x$, $y$ and all their common neighbours are in $A$, then $E(G; \Pi)$ could be
enlarged by adding the edge $(x, y)$, keeping the graph triangle-free.

On the other hand, we only need to consider partitions $\Pi$ in which $E(G; \Pi)$ is optimal,
i.e., has the maximum number of edges among all bipartite subgraphs, or nearly so. By
definition, such partitions have more edges going across them than typical partitions
do, so it seems plausible that a fixed near-optimal $E(G; \Pi)$ is still unlikely to have
a $p^{-c}$-perturbation with more edges. We are able to explicitly confirm this intuition.
However, the calculations we are making will not work if there are too many near-optimal partitions \( \Pi \), as then it becomes too likely that one of them could be improved by a \( p^{-c} \)-perturbation. The final ingredient of our proof is to show that this is unlikely to be the case: in the range we consider, a random graph typically has relatively few bipartitions that are optimal or near-optimal.

Before making this statement more precise, we need some more notation. The *distance* of two bipartitions/cuts \( \Pi = (A, B) \) and \( \Pi' = (A', B') \) of \([n]\) is defined as the number of vertices in which they differ, i.e.

\[
\text{dist}(\Pi, \Pi') := \min \{|A' \cap A| + |B' \cap B|, |A' \cap B| + |B' \cap A|\}.
\] (3.1)

Observe that due to \( n = (|A' \cap A| + |B' \cap B|) + (|A' \cap B| + |B' \cap A|) \) we have for all pairs \( \Pi \) and \( \Pi' \) that \( \text{dist}(\Pi, \Pi') \leq \frac{n}{2} \). We denote by \( \text{dist}(G; \Pi) \) the *minimum* distance of \( \Pi \) to an optimal bipartition of \( G \) (other than \( \Pi \) itself, if \( \Pi \) is optimal). Furthermore, we say that two cuts have *gap* \( g \), if the difference of their sizes is precisely \( g \), i.e., if

\[
gap(G; \Pi, \Pi') := |E(G; \Pi)| - |E(G; \Pi')| = g.
\] (3.2)

Finally, we say that \( \Pi \) has gap \( g \) if its number of edges differs from an optimal bipartition by exactly \( g \), that is we let \( \text{gap}(G; \Pi) := b(G) - |E(G; \Pi)| \). Our result for near-optimal bipartitions is as follows. We state it for the uniform random graph \( G_{n,M} \).

**Theorem 3.4.** There is a constant \( C > 1 \) such that the following is true for sufficiently large \( n \). Let \( n^{-1} \ll p = p(n) \leq \frac{1}{2} \) and \( M = M(n) := \binom{n}{2} \). Furthermore, let \( r = r(n) \geq 1 \) satisfy \( r \ll (pn)^{1/8} \) and \( \omega = \omega(n) \gg 1 \), and define

\[
s_0 := C \cdot \omega \cdot r^d \cdot \sqrt{np}^{-1}.
\]

Then

\[
\Pr \left[ \exists \Pi : \text{gap}(G_{n,M}; \Pi) = r - 1 \text{ and } \text{dist}(G_{n,M}; \Pi) \geq s_0 \right] \leq \omega^{-1}.
\]

As this result may be of independent interest, we also state the following corollary, which is a result for *optimal* bipartitions, i.e., maximum cuts, of the binomial random graph.

**Corollary 3.5.** Suppose that \( n^{-1} \ll p \leq \frac{1}{2} \) and \( \omega = \omega(n) \gg 1 \). Then, asymptotically almost surely, all maximum cuts \( \Pi \) and \( \Pi' \) of \( G_{n,p} \) satisfy \( \text{dist}(\Pi, \Pi') \leq \omega \sqrt{np}^{-1} \).

The proof is a straightforward application of Theorem 1.4 with \( r = 1 \). In other words, Corollary 3.5 says that, for most graphs, the maximum cut is *unique* up to movements of a small set \( U \) of vertices.

Although our main focus is on the most appealing case of triangle-free graphs, our methods extend easily to more general settings. Let \( K_\ell \) be the complete graph on \( \ell \) vertices. We have the following result, replacing the triangle by an arbitrary complete graph.
Theorem 3.6. Let $\ell \geq 3$. There is a $c = c(\ell) > 0$ such that, whenever $p = p(n) \geq n^{-c}$,

$$\lim_{n \to \infty} \Pr \left[ \text{every maximum } K_\ell\text{-free subgraph of } G_{n,p} \text{ is } (\ell-1)\text{-partite} \right] = 1.$$ 

We believe that a similar result is true not only for complete graphs, but also for many other graphs as well. In a graph $H$ with chromatic number $\chi := \chi(H)$, a colour-critical edge is an edge $e$ such that the graph with edge set $E(H) \setminus \{e\}$ has chromatic number $\chi - 1$. It is known that, if $H$ has a colour-critical edge, then the maximum number of edges in an $H$-free graph is the Turán number, i.e., the largest $H$-free graph is the same as the largest $\chi$-partite graph. If $H$ does not have a colour-critical edge, then this fails, as adding one edge to the Turán graph does not create a copy of $H$.

We expect that Theorem 3.6 is true of any fixed $H$ that has at least one color-critical edge. On the other hand, such a result automatically fails for any graph $H$ without a colour-critical edge, as adding a single edge to the largest $\ell$-partite graph does not create a copy of $H$. Babai et. al. [BSS90] discuss what can be proved for graphs without colour-critical edges. We treat neither case here.

Outline of the Chapter. The chapter is structured as follows. In Section 3.1 we introduce some notation and state a few facts from the theory of random graphs. Let $T(G)$ denote the set of maximum triangle-free subgraphs of a given graph $G$. In Section 3.2 we prove that for every $T \in T(G_{n,p})$ there exists a.a.s. a bipartition $\Pi_T$ such that $T$ is a $p^{-12} \log^2 n$-perturbation of $\Pi_T$. Next, in Sections 3.3 and 3.4 we present the proofs of Theorems 3.4 and 3.3. Finally, Section 3.5 demonstrates how we can adapt our proofs to prove Theorem 3.6.

3.1. Preliminaries & Notation

In this section we will present some basic facts from the theory of random graphs and from probability theory, which we will use frequently in the remainder of the paper. Without further reference we will often use the following estimates for the tail of the binomial distribution, which can be found for instance in [JLR00].

Lemma 3.7. Let $X$ be a random variable that is binomially distributed with parameters $n$ and $p$, and set $\lambda := E(X) = np$. For any $t \geq 0$, it holds that

$$\Pr \left[ X \geq \lambda + t \right] \leq e^{-\frac{t^2}{2(\lambda + t)\lambda}} \quad \text{and} \quad \Pr \left[ X \leq \lambda - t \right] \leq e^{-\frac{t^2}{2\lambda}}.$$ 

Before we continue, let us introduce some notation. We denote by $G_n$ the set of all graphs with vertex set $[n]$. Furthermore, let $G \in G_n$ and $X, Y \subseteq [n]$ be two subsets of its vertices. In the remainder, we will denote by $E(G)$ the edge set of $G$, by $E(G; X)$ the set of edges between vertices in $X$ and by $E(G; X, Y)$ the set of edges of $G$ joining a vertex of $X$ and a vertex of $Y$. Furthermore, $e(G; X) := |E(G; X)|$ and $e(G; X, Y) := E(G; X, Y)$, where edges inside $X \cap Y$ are counted only once.

Applying the above tail bounds to edge sets in random graphs, we easily obtain the following statement. Unless stated otherwise, logarithms are always to the base 2.
Proposition 3.8. Let \( p \gg \frac{\log n}{n} \) and define

\[
B_{n,p} := \left\{ G \in \mathcal{G}_n \mid \exists X, Y \subseteq [n] \text{ such that} \right. \\
X \cap Y = \emptyset, |X| \geq |Y| \geq 10p^{-1}\log n \text{ and } |e(G; X, Y) - p|X||Y|| \geq \frac{1}{2}p|X||Y| \left. \right\}.
\]

Then \( \Pr [G_{n,p} \in B_{n,p}] = o(1) \).

The proposition above is not best possible, but it suffices for our purposes. In the subsequent proofs we will often exploit the “equivalence” of the binomial random graph model \( G_{n,p} \) and the uniform random graph model \( G_{n,\lambda} \), when \( p = \lambda/\binom{n}{2} \). More precisely, we will use Pittel’s inequality (see e.g. [JLR00]) which states that for any property \( Q \) of graphs

\[
\Pr [G_{n,\lambda} \not\in Q] \leq 3\sqrt{\lambda} \Pr [G_{n,p} \not\in Q], \text{ where } p = \lambda/\binom{n}{2}.
\] (3.3)

Now let us turn our attention to optimal bipartitions of the uniform random graph. Recall that for a given graph \( G \), we denote by \( b(G) \) the number of edges in an optimal bipartition of \( G \); the following proposition provides bounds for \( b(G_{n,\lambda}) \), which hold with high probability.

Proposition 3.9. Let \( M \gg n \). For sufficiently large \( n \)

\[
\Pr \left[ \frac{M}{2} \leq b(G_{n,\lambda}) \leq \frac{M}{2} + \sqrt{4nM} \right] \geq 1 - e^{-n}.
\]

Proof. The inequality \( b(G_{n,\lambda}) \geq M/2 \) is a well-known fact that holds for all graphs with \( M \) edges. In order to show the upper bound, we prove the analogous result for the binomial random graph \( G_{n,p} \) and exploit inequality (3.3), which relates both models.

Let \( p := \lambda/\binom{n}{2}, L := \frac{p}{2} \binom{n}{2} \) and \( \Delta := \sqrt{2pn^2(n - 1)} \). In the sequel we will show

\[
\Pr [b(G_{n,p}) \geq L + \Delta] \ll n^{-1}e^{-n},
\] (3.4)

which, together with (3.3), proves the proposition. To see (3.4), define for every partition \( \Pi = (A, B) \) of the vertex set of \( G_{n,p} \) the random variable

\[
X_\Pi = \begin{cases} 
1, & e(G_{n,p}; \Pi) \geq L + \Delta \\
0, & \text{otherwise.}
\end{cases}
\]

The number of edges of \( G_{n,p} \) across \( \Pi \) is binomially distributed with parameters \(|A||B|\) and \( p \); with Lemma 3.7 we obtain for sufficiently large \( n \)

\[
\mathbb{E} [X_\Pi] = \Pr [\text{Bin} (|A||B|, p) \geq L + \Delta] \leq e^{-\frac{\Delta^2}{21p|A||B|(1 + \frac{1}{p})}} \leq e^{-\frac{2\Delta^2}{21(L + \frac{1}{p})}} \leq e^{-2n}.
\]

Therefore, if we let \( X = \sum_\Pi X_\Pi \), we readily obtain \( \Pr [X = 0] \leq 2^n e^{-2n} \ll n^{-1}e^{-n} \). This proves (3.4).

Recall that we say that a bipartition has gap \( g \), if the number of the edges joining vertices in different parts differs from the size of an optimal partition by exactly \( g \). The next proposition states that a.a.s. all bipartitions of \( G_{n,\lambda} \) with small gap are “balanced”.
Proposition 3.10. Let $M \gg n$, $p := M/\binom{n}{2}$ and $\lambda = \lambda(n) \geq 0$. Furthermore, let

$$B_{n,M} := \left\{ G \in G_{n,M} \mid \forall \Pi = (A, B) such that gap(G; \Pi) \leq \lambda \ it \ holds\right.$$

$$\left| A \right| - \frac{n}{2} \leq 3n^\frac{3}{4}p^{\frac{1}{4}} + \lambda^\frac{1}{2}p^{\frac{1}{2}} and \left| B \right| - \frac{n}{2} \leq 3n^\frac{3}{4}p^{\frac{1}{4}} + \lambda^\frac{1}{2}p^{\frac{1}{2}} \right\}.$$

For sufficiently large $n$ we have that $\Pr \left[ G_{n,M} \in B_{n,M} \right] \geq 1 - e^{-n}$.

Proof. We show the analogous result for the binomial random graph $G_{n,p}$ and use inequality (3.3) to prove the statement. Let $\Pi = (A, B)$ be a partition of the vertex set and write $\left| A \right| = \frac{n}{2} + d$ and $\left| B \right| = \frac{n}{2} - d$. Now assume that $|d| > 3n^{3/4}p^{-1/4} + \lambda^{1/2}p^{-1/2}$. The number of possible edges across $\Pi$ is

$$\left| A \right| \cdot \left| B \right| = \left( \frac{n}{2} + d \right) \left( \frac{n}{2} - d \right) = \frac{n^2}{4} - d^2 \leq \frac{n^2}{4} - \left( 9n^{3/2}p^{-1/2} + \lambda p^{-1} \right).$$

The number $C_\Pi$ of edges across $\Pi$ is binomially distributed with parameters $\left| A \right| \left| B \right|$ and $p$. Let us assume that $\Pi$ is a bipartition of $G_{n,p}$ with gap at most $\lambda$. With Lemma 3.7 we obtain that, whenever $n$ is sufficiently large, with probability larger than $1 - e^{-\frac{1}{4}n}$, the number of edges in $G_{n,p}$ is at least $p\left( \binom{n}{2} \right) - \sqrt{2n^{3/2}p^{1/2}}$. This implies with Proposition 3.9 that every optimal bipartition of $G_{n,p}$ contains for sufficiently large $n$ at least $\frac{n^2}{4} - n^{3/2}p^{1/2}$ edges. Hence, for sufficiently large $n$, the probability that $\Pi$ has indeed gap less than $\lambda$ is at most

$$\Pr \left[ C_\Pi \geq \frac{pn^2}{4} - n^{3/2}p^{1/2} - \lambda \right]$$

$$\leq \Pr \left[ C_\Pi \geq E \left[ C_\Pi \right] + \left( 9n^{3/2}p^{-1/2} + \lambda p^{-1} \right) \cdot p - \left( n^{3/2}p^{1/2} + \lambda \right) \right]$$

$$\leq \Pr \left[ C_\Pi \geq E \left[ C_\Pi \right] + 8n^{3/2}p^{1/2} \right] \leq e^{-2n},$$

where the last step is again due to Lemma 3.7. Therefore,

$$\Pr \left[ G_{n,p} \notin B_{n,p} \right] \leq e^{-\frac{1}{4}n} + 2n \cdot e^{-2n} \ll n^{-1}e^{-n},$$

which completes the proof with Pittel’s inequality (3.3).

Finally, we state bounds for the number of non-edges across any optimal bipartition of the random graph $G_{n,M}$. The following corollary is a straightforward consequence of Propositions 3.9 and 3.10.

Corollary 3.11. Let $M \gg n$ and define

$$\overline{b}(G) = \min \left\{ \left| A \right| \left| B \right| - e(G; \Pi) \mid \Pi = (A, B) \ is \ an \ optimal \ bipartition \ of \ G \right\}.$$

There is a constant $C > 0$ such that for sufficiently large $n$

$$\Pr \left[ \overline{b}(G_{n,M}) \geq 1 \right.$$

$$\left. \frac{1}{2} \left( \binom{n}{2} - M \right) - \sqrt{\frac{Cn^5}{M}} \right] \geq 1 - 2e^{-n}.$$
deduce that with probability at least $1 - 2e^{-n}$, the minimum number of non-edges across any optimal bipartition is at least
\[
\left( \frac{n}{2} - 3n^{\frac{3}{4}}p^{-\frac{1}{4}} \right) \left( \frac{n}{2} + 3n^{\frac{3}{4}}p^{-\frac{1}{4}} \right) - \frac{M}{2} - \sqrt{4nM}
\]
and the claim follows from $p = M/(\binom{n}{2})$ and $M \leq n^2$. \hfill \square

### 3.2. Finding a Near-Optimal Bipartition

Suppose we have $p = p(n) \geq n^{-c}$ for some positive (small) constant $c$. For a graph $G$, we denote by $T(G)$ the set of maximum triangle-free subgraphs of $G$. In this section we will prove that a.a.s. every $T \in T(G_{n,p})$ is “almost” bipartite. More precisely, our proof consists of two parts:

- In Lemma 3.20 we mimic the proof of [BSS90] to show that there is a bipartition $\Pi = \Pi_T = (A, B)$ with the property that at most $o(n^2)$ edges of $T$ do not go across $\Pi$, i.e., connect vertices in $A$ or in $B$. The new ingredient here is an application of the sparse version of Szemerédi’s regularity lemma and a probabilistic embedding lemma, see below.

- Second, in Lemma 3.23 we show that in fact there is a bipartition $\Pi'$ with the property that at most $p^{-12\log^2 n}$ edges of $T$ do not go across $\Pi'$. This proof uses similar ideas as in [BSS90], but differs from the original proof in most details.

Before we continue with our proof let us first introduce a variant of Szemerédi’s regularity lemma which can be meaningfully applied to sparse graphs, that we are going to exploit. Before we state it formally, we need a few technical definitions.

**Definition 3.12.** A bipartite graph $B = (V_1 \cup V_2, E)$ is called $(\varepsilon, p)$-regular if for all $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$ with $|V'_1| \geq \varepsilon |V_1|$ and $|V'_2| \geq \varepsilon |V_2|$,\[
\left| \frac{e(B; V'_1, V'_2)}{|V'_1||V'_2|} - \frac{|E|}{|V_1||V_2|} \right| \leq \varepsilon p.
\]

**Definition 3.13.** Let $G = (V, E)$ be a graph and $\varepsilon > 0$. A partition $(C_i)_{i=0}^k$ of $V$ is called an equitable partition with exceptional class $C_0$ if $|C_1| = |C_2| = \ldots = |C_k|$ and $|C_0| \leq \varepsilon |C_1|$. An $(\varepsilon, p)$-regular partition is an equitable partition $(C_i)_{i=0}^k$ such that with the exception of at most $\varepsilon k^2$ pairs, the pairs $(C_i, C_j)$ $1 \leq i \leq j \leq k$ are $(\varepsilon, p)$-regular.

**Definition 3.14.** Let $G = (V, E)$ be a graph and let $0 < \eta \leq 1$, $0 < p \leq 1$ and $b \geq 1$. We say that $G$ is $(\eta, b, p)$-upper-uniform if, for all disjoint sets $X$ and $Y$ with $|X|, |Y| \geq \eta |V|$,\[
\frac{e(G; X, Y)}{|X||Y|} \leq bp.
\]

We now state the sparse variant of Szemerédi’s regularity lemma; see [Koh97] and [KR03].
Theorem 3.15. For any $0 < \varepsilon < 1/2$ and $b, m_0 \geq 1$, there are constants $\eta = \eta(\varepsilon, b, m_0) > 0$ and $M_0 = M_0(\varepsilon, m_0) \geq m_0$ such that for any $p > 0$, any $(\eta, b, p)$-upper-uniform graph with at least $m_0$ vertices has an $(\varepsilon, p)$-regular partition $(C_i)_{i=0}^k$ such that $m_0 \leq k \leq M_0$.

A further tool which we will need in our proofs is an embedding lemma, which essentially states that almost every graph that can be partitioned so that all pairs of classes are suitably dense and $(\varepsilon, p)$-regular contains a copy of any fixed graph $H$. We need one further definition before we make this result precise.

Definition 3.16. For a graph $H = (V_H, E_H)$ with vertex set $V_H$ and edge set $E_H$ let $G(H, n, m, \varepsilon)$ be the class of graphs on vertex set $V = \bigcup_{x \in V_x} V_x$, where the $V_x$ are pairwise disjoint sets of size $n$, and edge set $E = \bigcup_{(x, y) \in E_H} E_{xy}$, where $E_{xy}$ is the edge set of an $(\varepsilon, m/n^2)$-regular bipartite graph with $m$ edges between $V_x$ and $V_y$.

Unfortunately, it can be shown that not all graphs in $G(H, n, m, \varepsilon)$ contain a copy of $H$. On the other hand, if $m$ is sufficiently large and $\varepsilon$ is sufficiently small, we can hope that only a tiny fraction of the graphs in $G(H, n, m, \varepsilon)$ do not contain a copy of $H$. This was conjectured by Kohayakawa, Łuczak and Rödl in [KLR97].

Conjecture 3.17. Let $H$ be a fixed graph. For any $\beta > 0$, there exist constants $\varepsilon_0 > 0$, $C > 0$, $n_0 > 0$ such that for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$ it holds

$$\|G \in G(H, n, m, \varepsilon) : H \text{ is not a subgraph of } G\| \leq \beta^n \left(\frac{n^2}{m}\right)^{\varepsilon(H)}.$$  

Here $d_2(H) := \max \left\{ \frac{e_F}{|V(F)|-2} \mid F \subseteq H, |V(F)| \geq 3 \right\}$ denotes the 2-density of a graph.

In this work we only need a weaker version of the above conjecture, which holds if $H$ is a complete graph and if the number of edges $m$ is slightly larger. The theorem below was proved by Gerke, Marciniszyn and Steger in [GMS05].

Theorem 3.18. Let $\ell \geq 3$. For all $\beta > 0$, there exist constants $n_0 \in \mathbb{N}$, $C > 0$, and $\varepsilon_0 > 0$ such that

$$\|G \in G(K_\ell, n, m, \varepsilon) : K_\ell \text{ is not a subgraph of } G\| \leq \beta^n \left(\frac{n^2}{m}\right)^{\ell \varepsilon_0},$$

provided that $m \geq Cn^{2-1/(\ell-1)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$.

In fact, in [GMS05] a much stronger counting version of the above theorem was proved. We do not need this strengthening here. Note also that the above theorem implies Conjecture 3.17 for $H = K_3$, which was proved already by Kohayakawa, Łuczak and Rödl in [KLR96].

A final ingredient in our proofs is the following lemma from [KRS04], which states that $(\varepsilon, p)$-regular graphs, whose edge number is only specified within bounds, contain a $(3\varepsilon, p)$-regular spanning subgraph with a given number of edges.
Lemma 3.19. Let $p \gg n^{-1}$. For every $\varepsilon > 0$, $\alpha > 0$, and $C > 1$ there exists an $n_0$ such that the following holds. If $B = (V_1 \cup V_2, E)$ is an $(\varepsilon, p)$-regular graph satisfying $|V_1|, |V_2| \geq n_0$ and $\alpha p|V_1||V_2| \leq \varepsilon (B; V_1, V_2) \leq C p|V_1||V_2|$, then there exists an $(3\varepsilon, p)$-regular graph $B' = (V_1 \cup V_2, E')$ with $E' \subseteq E$ and $|E| = \alpha p|V_1||V_2|$.

Now we proceed with our results. Recall that $T(G)$ denotes the set of maximum triangle-free subgraphs of the graph $G$.

Lemma 3.20. Let $\varepsilon > 0$. There exists $C > 0$ such that, for $p \geq C n^{-1/2}$, a random graph $G_{n,p}$ a.a.s. has the following property. For all $T \in T(G_{n,p})$ there is a partition $\Pi_T = \Pi = (A, B)$ of the vertex set such that all but at most $\varepsilon p n^2$ edges of $T$ go across $\Pi$. Furthermore, $\frac{n}{2} - \varepsilon n \leq |A|, |B| \leq \frac{n}{2} + \varepsilon n$.

Proof. The proof is similar to the proof of the analogous result in [BSS90] for constant density $p$. The new ingredients here are the sparse version of Szemerédi’s regularity lemma (Theorem 3.15) and the probabilistic embedding lemma (Theorem 3.18).

First we collect some properties of random graphs. Using Chernoff’s inequality it is easy to verify that, for every $c, \varepsilon \in (0, 1]$, a.a.s. every subset $U$ of the vertices of $G_{n,p}$ with $|U| \geq cn$ spans more than $(1-\varepsilon)\frac{1}{2} p |U|^2$ and less than $(1+\varepsilon)\frac{1}{2} p |U|^2$ edges. Similarly, we have that, for every $\xi, \varepsilon > 0$, a random graph $G_{n,p}$ a.a.s. is such that whenever $X$ and $Y$ are two disjoint subsets of the vertices with $|X|, |Y| \geq \xi n$ we have $|e(G_{n,p}; X, Y) - p |X||Y|| \leq \varepsilon p |X||Y|$. In particular, this implies that $G_{n,p}$ is a.a.s. $(\mu, (1+\varepsilon), p)$-upper-uniform, for all fixed $\mu > 0$. Hence, a.a.s. Theorem 3.15 applies to $G_{n,p}$ and all its spanning subgraphs.

Next we show how to choose the constant $C$. To do this we need some careful preparations. Let

$$F(n, m, \alpha) := \{G \in G(K_3, n, m, \alpha) : K_3 \text{ is not a subgraph of } G\}.$$

We apply Theorem 3.18 with $\beta := \frac{3}{c \varepsilon}$ to obtain the constants $n_\varepsilon, C_\varepsilon$ and $\varepsilon'$, which may depend on $\varepsilon$. Next we let $\varepsilon'' := \frac{1}{4} \min(\varepsilon, \varepsilon')$, $b := 1 + \varepsilon$, $m_0 := \varepsilon^{-1}$, and apply Theorem 3.15 for $\varepsilon''$, $b$ and $m_0$ to obtain constants $\eta$ and $M_0$. Finally, we let $\mu := \min(\eta, \frac{1-\varepsilon''}{2M_0})$, and $C := \frac{C_\varepsilon}{\varepsilon'\mu}$.

We claim that for all $p \geq C n^{-1/2}$ the random graph $G_{n,p}$ a.a.s. does not contain a graph from $\bigcup_{n \geq \mu n} F(\tilde{n}, \varepsilon p \tilde{n}^2, \varepsilon')$. To see this, let $X$ denote the number of such copies; we prove the claim by showing $E[X] = o(1)$. Let $M(\tilde{n}) := \varepsilon p \tilde{n}^2$ and observe that

$$E[X] \leq \sum_{n \geq \mu n} n^{3\tilde{n}} \cdot |F(\tilde{n}, M(\tilde{n}), \varepsilon')| \cdot p^{3M(\tilde{n})}.$$  \hspace{1cm} (3.5)

Now we recall that $n_\varepsilon, C_\varepsilon$, and $\varepsilon'$ were chosen in such a way that we can apply Theorem 3.18 with $\beta := \frac{c}{c \varepsilon}$ to obtain the bound $|F(\tilde{n}, m, \varepsilon')| \leq \beta m^{(\frac{\tilde{n}}{m})^3}$ for all $m \geq C_\varepsilon \tilde{n}^{3/2}$. We need to check that $M(\tilde{n})$ satisfies $M(\tilde{n}) \geq C_\varepsilon \tilde{n}^{3/2}$. This follows from our choice of $C = \frac{C_\varepsilon}{\varepsilon'\mu}$ and the assumption $p \geq C n^{-1/2}$:

$$M(\tilde{n}) = \varepsilon p \tilde{n}^2 \geq \varepsilon p (\mu n)^2 \geq \varepsilon C \mu^2 n^{3/2} \geq C_\varepsilon n^{3/2}.$$
Together with the inequality \((\binom{n}{i}) \leq (\frac{e\mu}{k})^k\) we thus obtain from (3.5) that
\[
\mathbb{E}[X] \leq \sum_{n \geq \mu n} n^{3\beta M(n)} \left( \frac{e}{\epsilon p} \right)^{3M(n)} p^{3M(n)} = \sum_{n \geq \mu n} n^{3\beta e^{-3M(n)}} = o(1),
\]
where the last two equalities follow from the choice of \(\beta\) and the fact that \(M(\tilde{n}) = \Omega(n^{3/2})\). This completes the proof of the claim.

Now consider a random graph \(G_{n,p}\) for \(p \geq Cn^{-1/2}\). The above discussion shows that \(G_{n,p}\) is a.a.s. a \((\mu, 1 + \epsilon, p)\)-upper-uniform graph, and that it does not contain a graph from \(\mathcal{F}(\tilde{n}, \epsilon p \tilde{n}^2, \epsilon')\), for all \(\tilde{n} \geq \mu n\). Furthermore, a.a.s. every subset \(U\) of the vertices of \(G_{n,p}\) with \(|U| \geq \mu n\) has the property \(e(G_{n,p}; U) - \frac{1}{2}p|U|^2 \leq \epsilon p|U|^2\), and for every two disjoint subsets \(X\) and \(Y\) of size at least \(\mu n\) it holds that \(|e(G_{n,p}; X, Y) - p|X||Y| \leq \epsilon p|X||Y|\).

In the remainder of the proof we assume that \(G_{n,p}\) has all these properties. Let \(T \subseteq T(G_{n,p})\) denote any maximum triangle-free subgraph of \(G_{n,p}\). We apply Theorem 3.15 to \(T\) with \(\epsilon''\), \(b = 1 + \epsilon\) and \(m_0 = \epsilon^{-1}\) to obtain an \((\epsilon'', p)\)-regular partition \((C_i)_{i=0}^k\), where \(m_0 \leq k \leq M_0\). Next we define the reduced graph \(R\) consisting of \(k\) labeled vertices corresponding to the classes \(C_1, \ldots, C_k\), and an edge between two vertices whenever the corresponding partition classes form an \((\epsilon'', p)\)-regular bipartite graph with at least \(\epsilon p|C_i|^2\) edges. Now we show that if \(R\) contains a triangle, then \(T\) would contain a triangle. To see this, observe first that we have \(|C_i| \geq \frac{\epsilon^{-1} m_0 n}{M_0} \geq \mu n\), for all \(1 \leq i \leq k\). Additionally, if \(R\) contained a triangle, by definition, there would exist three sets \(C_{i_1}, C_{i_2}, C_{i_3}\) that would induce three bipartite graphs that are \((\epsilon'', p)\)-regular and contain at least \(\epsilon p|C_i|^2\) edges. With Lemma 3.19 we deduce that these bipartite graphs have (spanning) subgraphs with \emph{exactly} \(\epsilon p|C_i|^2\) edges, which are \((3\epsilon'', p)\)-regular and therefore also \((\epsilon', p)\)-regular. That is, \(T\) contains a graph from \(\mathcal{G}(\tilde{K}_3, |C_1|, \epsilon p|C_1|^2, \epsilon')\). As \(G_{n,p}\) and hence also \(T \subseteq G_{n,p}\) does not contain a graph from \(\mathcal{F}(|C_1|, \epsilon p|C_1|^2, \epsilon')\) this implies that \(T\) contains a triangle, contradicting the fact that \(T\) is triangle-free. We conclude that \(R\) contains no triangle.

The remainder of the proof is essentially the same as the proof of the Main Lemma in [BSS90] – we only sketch roughly the details and refer the reader to [BSS90] for a more detailed proof. Since \(R\) contains no triangle, Turán’s theorem yields \(e(R) \leq \frac{k^2}{4}\). On the other hand, we can show \(e(R) \geq (1 - 40\epsilon)\frac{k^2}{\gamma^2}\). To see this, observe that the number of edges of \(T\) which join vertices of the same \(C_i\) or vertices of \(C_0\) to some other vertex, or correspond to a “low-density” or non-regular pair \((C_i, C_j)\) is at most \(7\epsilon p n^2\). Furthermore, the number of edges of \(T\) in a “high-density” regular pair \((C_i, C_j)\) is at most \((1 + \epsilon)p\left(\frac{n}{k}\right)^2\). Therefore we obtain
\[
e(T) \leq e(R) \cdot p \left(\frac{n}{k}\right)^2 + 8\epsilon p n^2.
\]
But since the \(G_{n,p}\) has the property that any two disjoint sets \(X, Y\) of size \(n/2\) satisfy a.a.s. \(e(G_{n,p}; X, Y) \geq (1 - \epsilon)p\frac{n^2}{4}\) we know that \(e(T) \geq (1 - \epsilon)p\frac{n^2}{4}\), from which the claimed lower bound for \(e(R)\) follows easily. Now, due to the stability lemma in [Sim68], there is a function \(\gamma \rightarrow 0\) (when \(\epsilon \rightarrow 0\)), such that we can find a bipartition \((A_R, B_R)\) of \(R\) with the property that at most \(\gamma k^2\) edges do not go across the parts, and \(|A_R|, |B_R| \leq \frac{k}{2} + \gamma k\).
This completes the proof of the lemma, as it can easily be seen that this implies the existence of a bipartition of \( T \) with the claimed properties. (Note that the bound \( \pm \gamma k \) suffices to obtain the claim of the lemma if we start with a sufficiently small \( \varepsilon > 0 \). We omit the details.) \( \square \)

Before we proceed with showing that we can find a much better bipartition than stated in the above lemma, we need two auxiliary tools, which will be used extensively in the sequel.

**Lemma 3.21.** Let \( k \geq 1 \) be an integer, \( p \geq n^{-\frac{1}{3}} \) and \( c \in (0, 1) \). Then the random graph on the vertex set \([n]\) has a.a.s. the following property: For every subset \( U \) of the vertices of size \(|U| > cn\) there exists a set \( Q_U \) of \( O \left( p^{-k} \right) \) vertices, such that every \( k \)-tuple of \([n] \setminus Q_U \) is completely joined to at least \( (1 - c)p^k|U| \) and at most \((1 + c)p^k|U|\) vertices in \( U \).

**Proof.** The proof is very similar to the proof of the *randomness lemma* in [BSS90] and we omit some of the details. The important difference here is that the statement also holds for \( k \)-tuples which might have a non-empty intersection with \( U \), and also that an upper bound on the size of the common neighborhood is given.

We call a set of \( k \) vertices \( \{v_1, \ldots, v_k\} \) violating, if the number of their common neighbors in \( U \) is smaller that \((1 - c)p^k|U|\) or larger than \((1 + c)p^k|U|\). Assume there exist at least \( t := Cp^{-k} \) pairwise disjoint sets of \( k \) vertices that are violating, where \( C \) will be chosen later. Let \( X \) denote the union of these sets and let \( U' := U \setminus X \). Observe that \(|X| = kt = Ck p^{-k} = O \left( n^{1/3} \right) \ll p^k|U'| = \Omega \left( n^{2/3} \right) \). For each of these \( t \) sets, the number of joint neighbors in \( U' \) is either smaller than \((1 - c)p^k|U'| \leq (1 - c/2)p^k|U'|\) or larger than \((1 + c)p^k|U'| - |X| \geq (1 + c/2)p^k|U'|\). For each \( k \)-tuple, the probability that the number of joint neighbors in \( U' \) is so small or so big is by Chernoff’s theorem less than \( e^{-C'p^kn} \), for an appropriate constant \( C' \). As all these events are independent, we obtain that the probability that there exist \( t \) violating sets can be bounded from above by \( 2^n \) (the number of ways to choose \( U \)) times \( n^{kt} \) (the number of ways to choose the \( t \) sets of \( k \) vertices each) times \( e^{-C'p^kn} \); hence the probability that \( G_{n,p} \) does not satisfy the claim of the lemma at most

\[
2^n \cdot n^{kt} \cdot e^{-C'p^kn} = o(1),
\]

if \( C \) is chosen appropriately. \( \square \)

Observe that the above lemma captures only cases where the set \( U \) has linear size. As at some points in our subsequent proofs we will need to consider the special case \( k = 1 \) and subsets \( U \) which have sublinear size, we also state the following lemma.

**Lemma 3.22.** Let \( c \in (0, 1) \). Then there exists a constant \( C = C(c) \) such that the \( G_{n,p} \) has a.a.s. the following property: for every subset \( U \) of the vertices with \( p|U| > C \log n \), the number of vertices having more than \((1 + c)p|U|\) or less than \((1 - c)p|U|\) neighbors in \( U \) is \( O \left( p^{-1} \log n \right) \).

The proof is a straightforward application of Lemma 3.7 and is omitted.
Lemma 3.23. For \( p \geq n^{-1/15} \) the following holds a.a.s.: for every \( T \in T(G_{n,p}) \) there is a partition \( \Pi_T = \Pi = (A, B) \) of the vertex set such that all but \( p^{-12} \log^2 n \) edges of \( T \) go across \( \Pi \). Furthermore, \( |A| = \frac{n}{2} + o(n) \) and \( |B| = \frac{n}{2} + o(n) \).

Proof. We restrict our proof to the case \( p = o(1) \), because for the remaining cases the statement follows directly from [BSS90]. Note that the assumption \( p = o(1) \) implies that, for \( n \) sufficiently large, \( p^{-1} \) is larger than any given constant. Within the proof we will often use this fact in order to keep the formulas simpler. A more careful handling of the inequalities would easily lead to an improvement on the bound \( p \geq n^{-1/15} \) in the statement of the lemma. As however our bound in the second part of the proof (Sections 3.3 and 3.4) is even weaker, we put emphasis on readability of the proof instead of optimizing the constant.

Before we proceed, we collect some properties of the \( G_{n,p} \). First, we apply Lemma 3.21 with \( k = 2 \) and \( c = \frac{1}{4} \), which yields that \( G_{n,p} \) has a.a.s. the property that for every set \( U \) of size at least \( \frac{n}{4} \) except for \( O\left(p^{-2}\right) \) vertices (which may depend on \( U \)) all pairs of vertices have at least \( \frac{1}{2}p^2|U| \) and at most \( \frac{1}{4}p^2|U| \) common neighbors in \( U \). Furthermore, we apply Lemma 3.22 with \( c = \frac{1}{2} \) and obtain that the \( G_{n,p} \) a.a.s. fulfills the condition that for every sufficiently large subset \( U \) of the vertex set the number of vertices having more than \( \frac{1}{4}p|U| \) or less than \( \frac{1}{4}p|U| \) neighbors in \( U \) is at most \( O\left(p^{-1}\log n\right) \). Finally, we apply Proposition 3.8 and obtain that \( G_{n,p} \) has a.a.s. the property that for any two sufficiently large disjoint subsets of the vertex set, the number of edges between them is approximately the expected number of edges between them.

Let \( 0 < \varepsilon < \frac{1}{1000} \). We apply Lemma 3.20 using \( \varepsilon^5 \) in place of \( \varepsilon \). It follows that for sufficiently large \( n \) there is a.a.s. a partition \( \Pi_T = (A_T, B_T) \) of any \( T \in T(G_{n,p}) \) such that

\[
e(T; A_T) + e(T; B_T) \leq \varepsilon^5pn^2,
\]

and

\[
\frac{n}{2} - \varepsilon^5n \leq |A_T|, |B_T| \leq \frac{n}{2} + \varepsilon^5n.
\]

Moreover, we apply Proposition 3.10 with \( \lambda = \varepsilon^5pn^2 \) which yields that every bipartition \( \Pi' = (A', B') \) with gap at most \( \varepsilon^5pn^2 \) satisfies

\[
|A'|, |B'| \geq \frac{n}{2} - n^{3/4}p^{-1/4} - \lambda^{1/2}p^{-1/2} \geq (1 - \varepsilon)\frac{n}{2},\text{ and similarly, }|A'|, |B'| \leq (1 + \varepsilon)\frac{n}{2}.
\]

In the remainder of the proof we assume that the \( G_{n,p} \) has all the above properties.

Let \( T \in T(G_{n,p}) \). We shall call a partition \( \Pi = (A, B) \) optimal with respect to \( T \), if \( e(T; \Pi) \) attains its maximum over all possible partitions. Recall that \( b(G) \) denotes the size of an optimal bipartition of \( G \). From our assumptions we know that if \( \Pi \) was optimal (with respect to \( T \)) we would have

\[
e(T; \Pi) \geq e(T) - \varepsilon^5pn^2 \geq b(G_{n,p}) - \varepsilon^5pn^2,
\]

which implies that \( \Pi \) has gap at most \( \varepsilon^5pn^2 \). We deduce that all optimal bipartitions \( \Pi = (A, B) \) of \( T \) have the properties \( (1 - \varepsilon)\frac{n}{2} \leq |A|, |B| \leq (1 + \varepsilon)\frac{n}{2} \) and \( e(T; A) + e(T; B) \leq \varepsilon^5pn^2 \). In the remainder we fix such an optimal bipartition \( \Pi = (A, B) \).

Before we continue, let us introduce some notation. For a graph \( G \), a vertex \( v \) and a subset \( S \) of the vertices of \( G \) let

\[
d(G; v, S) := |\Gamma(G; v) \cap S|,
\]
where $\Gamma(G;v)$ denotes the set of neighbors of $v$ in $G$. We call an edge \emph{horizontal} if it is in $T$ and joins two vertices in $A$ or two vertices in $B$. The \emph{horizontal degree} of a vertex $v \in A$ is given by

$$d_H(v) := d(T;v,A).$$

We call an edge \emph{missing}, if it is in $G_{n,p}$ joining two vertices in $A$ and $B$, \emph{but is not} in $T$. The number of missing edges at a vertex $v \in A$ (with respect to the partition $\Pi$) is thus

$$d_M(v) := d(G_{n,p};v,B) - d(T;v,B).$$

In the remainder of the proof we will repeatedly use the following strategy. We will assume that the horizontal edges satisfy some property. We then will use the assumptions on $G_{n,p}$, and the assumption that $\Pi$ is optimal with respect to $T$, in order to expose more missing edges than horizontal edges. However, this will clearly contradict the maximality of $T$, as we could delete all horizontal edges from $T$, and add all missing edges to it in order to obtain a larger triangle-free graph.

In order to formalize the idea, let us first define some sets of \emph{exceptional} vertices and discuss a few of their properties. For this, let for a subset $U$ of the vertex set of $G_{n,p}$

$$B_1(U) := \left\{ v \in V(G_{n,p}) : \left| \Gamma(G_{n,p};v) \cap U \right| - p|U| \geq \frac{p|U|}{4} \right\}.
$$

and $$B_2(U) := \left\{ v \in V(G_{n,p}) : \exists u \in V(G_{n,p}) : \left| \Gamma(G_{n,p};v) \cap \Gamma(G_{n,p};u) \cap U \right| - p^2|U| \geq \frac{p^2|U|}{4} \right\}.$$

With the above definition, let

$$X_1^A = (B_1(A) \cup B_1(B) \cup B_2(A) \cup B_2(B)) \cap A,$$

$$X_2^A = \left\{ v \in A \mid d_H(v) \geq \varepsilon pn \right\} \setminus X_1^A,$$

$$X_3^A = \left\{ v \in A \mid d_M(v) \geq d_H(v) + 5\varepsilon pn \right\} \setminus (X_1^A \cup X_2^A).$$

Furthermore, let $A_0 = A$ and $A_i = A_{i-1} \setminus X_i^A$ for $i \in \{1,2,3\}$ and define similarly the above sets with $A$ replaced by $B$. Finally, let $X_i = X_i^A \cup X_i^B$ for $i \in \{1,2,3\}$.

Observe that due to our assumptions we have $|X_1| = O(p^{-2})$. Moreover, we can estimate the number of vertices in $X_3$ as follows. The number $m$ of missing edges in $T$ incident to at least one vertex in $X_3^A \cup X_3^B$ is

$$m \geq \frac{1}{2} \sum_{v \in X_3} d_M(v) \geq \frac{1}{2} \left| X_3^A \right| + \left| X_3^B \right| \cdot 5\varepsilon pn.$$

But $m$ is no greater than $\varepsilon^5pn^2$—we deduce $|X_3| \leq \varepsilon n$, with room to spare.

With the above assumptions we now show the following statements, which gather “self-improving” information on the number of horizontal edges.

(i) $X_2$ is small, i.e., $|X_2| \leq \varepsilon p^{-2}$.

(ii) Set $H_3 := E(T;A_3) \cup E(T;B_3)$. For large $n$ we have that $|H_3| \leq p^{-2}n \log n$.

(iii) $|X_3| \leq p^{-4} \log n$ (i.e. the actual number of exceptional vertices in $X_3$ is much smaller than the above calculation allows).

(iv) There are no vertices in $A_3$ or $B_3$ with horizontal degree in $H_3$ greater than $p^{-6} \log n$. 
(v) We improve the bound on $|H_3|$: $|H_3| \leq \frac{1}{2} p^{-12} \log^2 n$.

(vi) Finally, $e(T; A) + e(T; B) \leq p^{-12} \log^2 n$.

First we prove (i). Let $X_2^A = \{v_1, \ldots, v_t\}$, where $t = |X_2^A|$, and observe that for every $v \in X_2^A$ we have $d_H(v) \leq d(T; v, B)$, as otherwise $\Pi$ would not be optimal. Furthermore, note that all edges between the sets $\Gamma(T; v) \cap A$ and $\Gamma(T; v) \cap B$ in $G_{n, p}$ must be missing, as otherwise $T$ would contain a triangle. In the following we show a lower bound for the total number of missing edges $m$, which will immediately translate into an upper bound for $t$.

We write

$$\mathcal{F}(v_i) := E(G_{n, p}; \Gamma(T; v_i) \cap A, \Gamma(T; v_i) \cap B),$$

i.e., $\mathcal{F}(v_i)$ is the set of edges in $G_{n, p}$ which are missing “due to” $v_i$. Observe that we can estimate $f_i := |\mathcal{F}(v_i)| \geq \frac{p}{2} \cdot (\epsilon p n)^2$, as we assumed that the $G_{n, p}$ satisfies Proposition 3.8.

Let $t_0 = \left( \frac{1}{2p} \right)^2$ and suppose that $t$ satisfies $t \geq t_0$. Now define the quantity $m_0 := |\bigcup_{i=1}^{t_0} \mathcal{F}(v_i)|$ and observe that $m \geq m_0$. In order to bound $m_0$, we apply the inclusion-exclusion principle:

$$m_0 \geq \sum_{i=1}^{t_0} f_i - \sum_{1 \leq i < j \leq t_0} |\mathcal{F}(v_i) \cap \mathcal{F}(v_j)| \geq t_0 \cdot \frac{p}{2} \cdot (\epsilon p n)^2 - \left( \frac{t_0}{2} \right) \max_{i < j} |\mathcal{F}(v_i) \cap \mathcal{F}(v_j)|. \quad (3.9)$$

Recall that the size of the common neighborhood of any two vertices $v, w \in X_2^A$ in $A$ as well as in $B$ is at most $p^2 n$, by the definition of $X_2$. So $|\mathcal{F}(v) \cap \mathcal{F}(w)|$ is at most the maximum size of the set of edges of $G_{n, p}$ between two disjoint vertex sets of this size. Hence the assumption that $G_{n, p}$ satisfies Proposition 3.8 implies

$$|\mathcal{F}(v) \cap \mathcal{F}(w)| \leq 2p \cdot (p^2 n)^2.$$

From (3.9) we obtain $m_0 \geq \frac{\epsilon^4}{16} p^2 n^2$ due to the definition of $t_0$. Therefore, whenever $|X_2^A| \geq t_0$ we achieve a contradiction, as $m_0$ is at most $\epsilon^5 p n^2$.

Now we prove (ii). Recall that $A_3$ and $B_3$ denote the sets of vertices which are not exceptional. We may assume $e(T; A_3) \geq e(T; B_3)$, as otherwise we could interchange the roles of $A$ and $B$. Our objective is to derive upper and lower bounds for the number $N$ of instances of a configuration called a “chord”; these bounds will immediately imply a bound on $|H_3|$ that will show (ii). A chord consists of three vertices $x, y \in A_3$ and $z \in B$ with the property that $x$ and $y$ are connected by an edge in $H_3$, and $z$ is connected in $G_{n, p}$ and the edge $\{x, z\}$ is missing (i.e., it is in $G_{n, p}$ but not in $T$).

Consider an edge $\{x, y\} \in E(T; A_3)$. Then every vertex $z$ in the common neighborhood of $x$ and $y$ in $B$ forms a triangle in $G_{n, p}$. Hence, one of the edges $\{x, z\}$ or $\{y, z\}$ must be a missing edge (as otherwise there would be a triangle in $T$), which means that the three vertices $x, y$ and $z$ are a chord. Recall that due to the definition of $X_1$ we know that the common neighborhood of any two vertices in $A_3$ in $B$ is at least $\frac{2}{3} p^2 |B|$. Therefore, a lower bound for the number of chords is

$$N \geq e(T; A_3) \cdot \frac{3}{4} p^2 |B| \geq \frac{1}{2} |H_3| \cdot \frac{p^2 n}{4} = \frac{H_3 \cdot p^2 n}{8}. \quad (3.10)$$
In the sequel we derive an upper bound for $N$. For this, we first bound the number of chords that contain a vertex $x \in A_3$ such that $d_{H_3}(x) \leq C p^{-1} \log n$, where $C$ is the constant from Lemma 3.22 for $c = \frac{1}{4}$. We bound the number of such chords containing $x$ by their horizontal degree times an upper bound on the size of a common neighbourhood of any two vertices in $A_3$ in $G_{n,p}$. Using again the definition of $X_A^*$, we deduce that the common neighborhood of any two vertices in $A_3$ in $B$ is of size at most $\frac{5}{4}p^2|B|$. Hence we get

$$\sum_{x \in A_3: \ d_{H_3}(x) \leq C p^{-1} \log n} \sum_{w \in \Gamma(G_n; x) \cap A_3} |\Gamma(G_n; x) \cap \Gamma(G_n; w) \cap B|$$

$$\leq |A_3| \cdot C p^{-1} \log n \cdot \frac{5}{4}p^2|B| \leq C p n^2 \log n. \quad (3.11)$$

The same bound holds for vertices $x \in B_3$ with small horizontal degree. Now consider the vertices $x \in A_3 \cup B_3$ such that $d_{H_3}(x) \geq C p^{-1} \log n$. The number of chords containing such a vertex $x$ can be bounded as follows. Consider the set of missing edges at $x$, i.e., $S := (\Gamma(G_n; x) \cap B) \setminus \Gamma(T; x)$, and note that $|S| = \delta_M(x)$. As we assumed that $G_{n,p}$ satisfies Lemma 3.22 with $c = \frac{1}{4}$, all but $C'p^{-1}\log n$ vertices in $S$ have in $G_{n,p}$ at most $\frac{5}{4}p d_{H_3}(x)$ neighbors in the set $\Gamma(T; x) \cap A_3$. Here $C'$ is an appropriately chosen constant. Furthermore, note that $\delta_M(x) \leq d_{H_3}(x) + 5 \varepsilon n \leq 6 \varepsilon n$, as $x \in A_3 \cup B_3$. We deduce that the number of chords containing a vertex of high horizontal degree can be bounded as follows:

$$\sum_{x: \ d_{H_3}(x) \geq C p^{-1} \log n} \left( \delta_M(x) \cdot \frac{5}{4}p d_{H_3}(x) + C'p^{-1} \log n \cdot d_{H_3}(x) \right)$$

$$\leq \sum_{x \in A_3: \ d_{H_3}(x) \geq C p^{-1} \log n} \left( \frac{30}{4} \varepsilon p^2 n + C'p^{-1} \log n \right) \cdot d_{H_3}(x) \leq 16 \varepsilon p^2 n \cdot |H_3|. \quad (3.12)$$

Now by combining (3.10), (3.11) and (3.12) we obtain for sufficiently large $n$

$$|H_3| \cdot p^2 n \left( \frac{1}{8} - 16 \varepsilon \right) \leq 2 C p n^2 \log n,$$

from which the bound on $|H_3|$ follows with room to spare from our assumptions on $p$ and $\varepsilon$.

We continue by proving (iii). Let $H = E(T; A) \cup E(T; B)$ and recall that $H_3$ is the subset
of $H$ restricted to $A_3$ and $B_3$. Our strategy is as follows. We estimate $|H|$ from above and the number of missing edges from below. Comparing the two bounds will yield a contradiction, unless $|X_3| \leq p^{-4} \log n$. First observe that due to (i) and (ii) the number of horizontal edges is at most

$$\sum_{v \in X_1 \cup X_2 \cup X_3} d_H(v) + |H_3| \leq |X_1 \cup X_2| \cdot n + \sum_{v \in X_3} d_H(v) + |H_3|$$

$$\leq 2p^{-2} n \log n + \sum_{v \in X_3} d_H(v). \quad (3.13)$$

On the other hand, observe that for all vertices $v \in X_3$ we have $d_H(v) \leq \epsilon pn$, due to the definition of $X_2$. Therefore, the number of missing edges is at least

$$\frac{1}{2} \sum_{v \in X_3} d_M(v) \geq \frac{1}{2} \sum_{v \in X_3} (d_H(v) + 5\epsilon pn)$$

$$\geq \frac{1}{2} \sum_{v \in X_3} (2d_H(v) + 4\epsilon pn) \geq \sum_{v \in X_3} d_H(v) + 2|X_3|\epsilon pn. \quad (3.14)$$

Now replace all horizontal edges from $T$ with all missing edges in order to obtain a different triangle-free graph. Comparing (3.13) and (3.14) yields that to avoid a contradiction $X_3$ must satisfy

$$2p^{-2} n \log n \geq 2|X_3|\epsilon pn \Rightarrow |X_3| \leq \epsilon^{-1} p^{-3} \log n \leq p^{-4} \log n,$$

whenever $n$ is sufficiently large.

Next we show (iv). Let $v$ be a vertex in $A_3$ with $d_{H_3}(v) \geq p^{-6} \log n$ – we handle vertices in $B_3$ analogously. Recall that $A_3 = A \setminus (X_1 \cup X_2 \cup X_3)$. The definitions of the sets $X_i$ thus imply together with the fact that $|B| \geq (1-\epsilon)\frac{n}{2}$ that the number of neighbours of $v$ in $B$ is at least

$$d(T; v, B) = d(G_{n,p}; v, B) - d_M(v) \geq \frac{pn}{8} - 6\epsilon pn \geq \frac{pn}{9}.$$

Furthermore, note that the edges between the vertex sets $\Gamma(T; v) \cap B$ and $\Gamma(T; v) \cap A_3$ would form triangles in $T$. We deduce with Proposition 3.8 that the number of missing edges in $T$ is at least

$$e(G_{n,p}; \Gamma(T; v) \cap B, \Gamma(T; v) \cap A_3) \geq \frac{p}{2} \cdot \frac{pn}{9} \cdot p^{-6} \log n \geq \frac{1}{20} p^{-4} n \log n. \quad (3.15)$$

Finally observe that the number of horizontal edges in $T$ is due to (i)-(iii) at most

$$|H_3| + |X_1| \cdot n + |X_2 \cup X_3| \cdot \frac{5}{4}pn \leq 2p^{-3} n \log n. \quad (3.16)$$

But this contradicts with (3.15) the maximality of $T$ – we conclude that there are no vertices in $A_3 \cup B_3$ with horizontal degree at least $p^{-6} \log n$, i.e., (iv) is shown.

Now we show (v). For this, let $R$ be a matching of maximum cardinality in $H_3$, and let $m$ be the number of missing edges in $T$. As in the previous proofs, the central idea is to derive a lower bound on $m$ which contradicts the maximality of $T$. 


Note that from (iv) we have \(|R| \geq \frac{1}{2}|H_3|p^6(\log n)^{-1}\), as we can obtain a matching by greedily removing edges from \(H_3\). On the other hand, let \(e = \{u, v\} \in R\) for two vertices \(u, v \in A_3\). As \(u\) and \(v\) are not in \(X_1\) they have at least \(\frac{3}{4}p^2|B| \geq \frac{n^2}{4}\) common vertices in \(B\). Hence, for every \(e \in R\) we have at least \(\frac{n^4}{4}\) missing edges. As these missing edges are distinct for different edges in \(R\) we deduce that

\[
m \geq |R| \cdot \frac{p^2n^4}{4} \geq |H_3| \cdot \frac{p^6}{2\log n} \cdot \frac{p^2n^4}{4} \geq |H_3| \cdot \frac{p^8}{8\log n} n.
\]

As in (3.16) we obtain that the number of horizontal edges is at most \(2p^{-3}n\log n\). In order to avoid a contradiction \(|H_3|\) must hence satisfy

\[
|H_3| \cdot \frac{p^8}{8\log n} n \leq 2p^{-3}n\log n,
\]

from which the claimed bound on \(|H_3|\) follows for sufficiently large \(n\).

Finally we show (vi). Let \(X = X_1 \cup X_2 \cup X_3\). From our assumptions and (i)-(v) we know that \(|X| \leq 2p^{-3}\log n\), if \(n\) is sufficiently large, and the number of horizontal edges in \((A \cup B) \setminus X\) in \(T\) is less than \(\frac{1}{2}p^{-12}\log^2 n\).

To prove the statement, we replicate the argument from (iv). Let \(d\) be the maximum degree of a vertex in \(X_1\) and suppose \(d \geq p^{-3}\log n\). Furthermore, suppose that the maximum is attained at a vertex \(v \in A\). Observe that \(v\) has at least \(d\) neighbors in \(B\), as otherwise \(\Pi\) would not have been maximal. It follows that the number of missing edges is at least \(\frac{pd^2}{2}\), because we assumed that the \(G_{n,p}\) satisfies Proposition 3.8. On the other hand, the number of horizontal edges is at most \(|X| \cdot d + \frac{1}{2}p^{-12}\log^2 n\). Therefore, \(d\) must satisfy

\[
\frac{pd^2}{4} \leq 2p^{-4}\log n \cdot d + \frac{1}{2}p^{-12}\log^2 n
\]

which can only hold if \(d \leq 3p^{-13/2}\log n\). This concludes the proof. \(\square\)

### 3.3. On Properties of (Near-)Optimal Bipartitions

Before we proceed with the proof of Theorem 3.4 we introduce two auxiliary tools. The first lemma is a statement about the number of non-edges between sufficiently large parts of the vertex set of \(G_{n,M}\). More precisely, for an ordered partition \(\Pi\) of the vertex set in two pairs of (sufficiently large) parts, we want to bound the probability that the number of non-edges between the parts of the pairs is not near its expected value. The result is not best possible, but it suffices for our purposes and keeps the calculations short.

**Lemma 3.24.** Let \(n^{1/2} \leq s \leq \frac{n}{2}\) and \(\Pi = (V_1, W_1, V_2, W_2)\) be a partition of \([n]\) such that \(|V_1| + |V_2| = s\) and \(|W_1|, |W_2| \geq n/7\). Furthermore, for a graph \(G\) with vertex set \([n]\) let

\[
\overline{e}(G; \Pi) := |\Pi| - e(G; V_1, W_1) - e(G; V_2, W_2), \text{ where } |\Pi| := |V_1||W_1| + |V_2||W_2|.
\]

\[\tag{3.17}\]

\[\text{Lemma 3.24.} \quad \overline{e}(G; \Pi) \leq \frac{1}{2}n^{3/2}. \]
Let $n \ll M \leq \frac{1}{2} \binom{n}{2}$ and $p := M/\binom{n}{2}$. Then there is a constant $C > 0$ such that

$$\Pr \left[ \exists \Pi : |\mathcal{E}(G_{n,M}, \Pi) - (1 - p)| \Pi | \geq C \cdot s^{-1/2} \cdot (1 - p) | \Pi | \right] \leq e^{-n}.$$ 

**Proof.** We show the analogous result for the $G_{n,p}$ and use Pittel’s inequality (3.3) to conclude the proof. For a partition $\Pi$ with the above properties define the event

$$\mathcal{E}_\Pi := |\mathcal{E}(G_{n,p}, \Pi) - (1 - p) \Pi | \geq C \cdot s^{-1/2} \cdot (1 - p) \Pi | \Pi |.$$ 

Observe that according to our assumptions we have $| \Pi | \geq \frac{5n}{7}$. A straightforward application of Lemma 3.7 then yields that we can choose $C$ such that $\Pr | \mathcal{E}_\Pi | \leq 2^{-n}$. As there are at most $4^n$ ways for choosing $\Pi$, the desired probability can be bounded by

$$\Pr \left[ \bigcup_{\Pi} \mathcal{E}_\Pi \right] \leq 4^n \cdot 2^{-n} \ll n^{-1} e^{-n},$$

which completes the proof with (3.3).

The next proposition is a technical estimate for the probability that a trinomially distributed random variable of a special form deviates from its expectation. The bound is tight up to the determined constant.

**Proposition 3.25.** Let $\alpha < \frac{1}{2}$ and $d \leq \min(\sqrt{\alpha N}, \sqrt{(1 - 2\alpha)N})$ such that $2\alpha N + d < N$. There is a constant $C > 0$ such that

$$\left( \begin{array}{c} N \\ \alpha N, \alpha N + d \end{array} \right) \alpha^{2\alpha N + d} (1 - 2\alpha)^{(1 - 2\alpha)N - d} \geq \frac{C}{\alpha N}.$$ (3.18)

**Proof.** We obtain with Stirling’s formula $1 \leq x!/(x^e \sqrt{2\pi x}) \leq 2$ that there is a constant $C' > 0$ such that, if $d \leq \sqrt{\alpha N}$,

$$\left( \begin{array}{c} N \\ \alpha N, \alpha N + d \end{array} \right) = \frac{N!}{(\alpha N)!((\alpha N + d)!(1 - 2\alpha)N - d)!}$$

$$\geq \frac{1}{16\pi} \frac{\sqrt{N}}{\sqrt{\alpha N(\alpha N + d)(N - 2\alpha N - d)}} \cdot \frac{\alpha^{N N}}{(\alpha N(\alpha N + d))^{\alpha N + d}(1 - 2\alpha)^{(1 - 2\alpha)N - d}}$$

$$\geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d} \left( 1 + \frac{d}{\alpha N} \right)^{-\alpha N - d} (1 - 2\alpha)^{(1 - 2\alpha)N + d} \left( 1 - \frac{d}{(1 - 2\alpha)N} \right)^{(1 - 2\alpha)N + d}.$$ 

Recall that $d \leq \min(\sqrt{\alpha N}, \sqrt{(1 - 2\alpha)N})$. The inequality $1 + x \leq e^x$ implies with this fact that we can choose $C > 0$ such that

$$\left( \begin{array}{c} N \\ \alpha N, \alpha N + d \end{array} \right) \geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d} (1 - 2\alpha)^{(1 - 2\alpha)N + d} \cdot e^{-\frac{d + \alpha N}{\alpha N}} e^{-d \frac{1}{(1 - 2\alpha)N}}$$

$$\geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d} (1 - 2\alpha)^{(1 - 2\alpha)N + d} \cdot e^{-d \frac{1}{\alpha N}} e^{-d \frac{1}{(1 - 2\alpha)N}}$$

$$\geq \frac{C}{\alpha N} \cdot \alpha^{-2\alpha N - d} (1 - 2\alpha)^{(1 - 2\alpha)N + d}.$$ 

Substituting this bound in the left-hand side of (3.18) yields immediately the statement.

With the above tools we are ready to prove the main result of this section.
**Proof of Theorem 3.4.** Recall that for two bipartitions $\Pi$ and $\Pi'$ of the vertex set of a graph $G$ we denote by $\text{dist}(\Pi, \Pi')$ the number of vertices in which $\Pi$ and $\Pi'$ differ, and we say that $\Pi$ has gap $g$ (i.e., $\text{gap}(G; \Pi) = g$), if the number of edges across the parts of $\Pi$ is exactly $g$ less than the number of edges across an optimal bipartition. Note that for every pair of partitions $\Pi$ and $\Pi'$ it holds $\text{dist}(\Pi, \Pi') \leq \frac{n}{2}$.

The central idea in our proof is to consider how the random variable $b(G_{n,M})$ changes, as the uniform random graph $G_{n,M}$ evolves. More precisely, let $t = t(n) > 0$ and consider the expected change

$$
\mathbb{E}[b(G_{n,M+t}) - b(G_{n,M})].
$$

(3.19)

In order to obtain bounds for the above expression, we look at it from two different points of view: either removing $t$ edges uniformly at random from $G_{n,M+t}$ or adding $t$ edges uniformly at random to $G_{n,M}$.

Suppose that we delete $t$ edges uniformly at random from $G_{n,M+t}$ in order to obtain a graph with $M$ edges, and let $\Pi^*$ be any optimal bipartition of $G_{n,M+t}$. Observe that the size of the optimal bipartition decreases by at most the number of edges among the $t$ deleted that go across $\Pi^*$ (as the size of an optimal bipartition decreases only if edges are removed from all optimal bipartitions). Hence, the expected decrease of the size of the optimal bipartition can be bounded from above with the proportion of edges of $G_{n,M+t}$ across $\Pi^*$. Now we apply Proposition 3.9, which yields that $G_{n,M+t}$ has with probability $1 - e^{-n}$ the property that all its optimal bipartitions have size at most $\frac{1}{2}(M + t) + \sqrt{4n(M + t)}$; we obtain

$$
\mathbb{E}[b(G_{n,M+t}) - b(G_{n,M})] \leq t \cdot \frac{\frac{1}{2}(M + t) + \sqrt{4n(M + t)}}{M + t} + e^{-n}
$$

$$
\leq t \cdot \left( \frac{1}{2} + \sqrt{\frac{5n}{M}} \right).
$$

(3.20)

On the other hand, let $G$ be a graph and $\Pi^*(G)$ denote a canonical optimal bipartition of $G$, which is uniquely determined by $G$ (note that we can always induce a canonical ordering on the set of all partitions; $\Pi^*$ is then simply the minimum optimal partition with respect to this ordering). Furthermore, let $i^* := i^*(n)$ be the minimum integer such that $2^{i^*} s_0 \geq \frac{n}{2}$. For $0 \leq i \leq i^*$ define the event

$$
P_i := \exists \Pi : \text{gap}(G_{n,M}; \Pi) = r - 1 \text{ and } 2^{i} s_0 \leq \text{dist}(\Pi, \Pi^*(G_{n,M})) \leq 2^{i+1} s_0,
$$

and set $p_i := \Pr[P_i]$. As for every partition $\Pi$ and every graph $G$ we trivially have that $\text{dist}(G; \Pi) \leq \text{dist}(\Pi, \Pi^*(G))$, we deduce that the desired probability $p_{r,s_0}$ can be bounded from above by the probability of the union of the events $P_i$. In the main part of the proof we will show that whenever $n$ is sufficiently large we have $p_i \leq \frac{1}{2^{i+1} \omega}$ for all $0 \leq i \leq i^*$. From this, the statement of the theorem follows immediately:

$$
p_{r,s_0} \leq \sum_{i=0}^{i^*(n)} p_i \leq \frac{1}{2\omega} \sum_{i\geq 0} 2^{-i} \leq \frac{1}{\omega}.
$$
In the remainder we assume that \( 0 \le i \le i^* \) is fixed. Set \( s_i := 2^i s_0 \) and \( t_i := r^2 \frac{n(n-1)}{s_i(n-s_i)} \).

If we add \( t_i \) edges uniformly at random to \( G_{n,M} \), one of the following two events can occur:

(i) \( \Pi^* = \Pi^*(G_{n,M}) = (A^*, B^*) \) remains one of the optimal partitions

(ii) a bipartition different from \( \Pi^* \) “overtakes” \( \Pi^* \), i.e., its size is larger than the size of \( \Pi^* \) after having added \( t_i \) edges to \( G_{n,M} \).

Denote by \( I_{t_i} \) the increase of the size of \( \Pi^* \), and by \( I_O \) the indicator variable of the event that a bipartition \( \Pi \) different from \( \Pi^* \) (if it exists in \( G_{n,M} \)) overtakes \( \Pi^* \). Due to linearity of expectation, we obtain

\[
E [b(G_{n,M+t_i}) - b(G_{n,M})] \ge E [I_{t_i}] + E [I_O].
\]  

(3.21)

We first bound \( E [I_{t_i}] \). For this, observe that \( \Pi^* \) increases by the number of edges among the \( t_i \) added that go across \( \Pi^* \); therefore, the expected increase of \( e(G_{n,M}; \Pi^*) \) is \( t_i \) times the proportion of non-edges across \( \Pi^* \) in \( G_{n,M} \). Let \( C' > 0 \) be the constant that is guaranteed by Corollary 3.11. We apply Corollary 3.11 to obtain bounds for the minimum number of non-edges across any maximum bipartition of \( G_{n,M} \), which hold with exponentially high probability; furthermore, due to our assumption \( M \le \frac{1}{2} \binom{n}{2} \) we obtain that \( \binom{n}{2} - M \ge \frac{1}{4} n^2 \). These facts imply that, for sufficiently large \( n \), it holds

\[
E [I_{t_i}] \ge (1 - 2e^{-n}) \cdot t_i \cdot \frac{\frac{1}{2} \binom{n}{2} - M - \sqrt{C'M^2}}{\binom{n}{2} - M} \ge t_i \cdot \left( \frac{1}{2} - \sqrt{\frac{20C'n}{M}} \right).
\]  

(3.22)

Next we bound \( E [I_O] \) from below. Suppose that there is a bipartition \( \Pi = (A, B) \) in \( G_{n,M} \) with the properties \( \text{gap}(G_{n,M}; \Pi) = r - 1 \) and \( \text{dist}(\Pi, \Pi^*) =: s \) with \( s_i \le s \le 2s_i \).

Denote by \( \mathcal{E}_{\text{Bal}} \) the event that

\[
|A^*|, |B^*|, |A|, |B| \ge \frac{n}{2} - 4n^\frac{1}{4} p^{-\frac{1}{4}}
\]  

(3.23)

and observe that Proposition 3.10 implies \( \Pr [G_{n,M} \in \mathcal{E}_{\text{Bal}}] \ge 1 - e^{-n} \), with room to spare. In order to avoid ambiguities, we chose the notation without loss of generality such that \( |A^*| \ge |B^*| \) and

\[
s = |A \cap B^*| + |A^* \cap B| \le |A \cap A^*| + |B \cap B^*|.
\]

Note that if we add \( t_i \) edges uniformly at random to \( G_{n,M} \), edges added between \( A^* \cap B \) and \( A^* \cap A \), and between \( B^* \cap A \) and \( B^* \cap B \), contribute to \( e(G_{n,M}; \Pi) \), while edges added between \( A^* \cap B \) and \( B^* \cap B \), and between \( B^* \cap A \) and \( A^* \cap A \), contribute to \( e(G_{n,M}; \Pi^*) \). All other added edges contribute simultaneously to both \( \Pi^* \) and \( \Pi \), or to none of them. This motivates the definition of two ordered partitions

\[\Pi_X := (A^* \cap B, A^* \cap A, B^* \cap A, B^* \cap B) \quad \text{and} \quad \Pi_Y := (A^* \cap B, B^* \cap B, B^* \cap A, A^* \cap A).\]

Additionally, let \( |\Pi_X|, \sigma(G_{n,M}; \Pi_X) \), and similarly \( |\Pi_Y| \) and \( \sigma(G_{n,M}; \Pi_Y) \), be defined as in Lemma 3.24, see (3.17). As discussed above, the motivation for this definition is that the event \( O \) will occur, if out of the \( t_i \) edges added to \( G_{n,M} \), the number of edges added among those counted by \( \sigma(G_{n,M}; \Pi_X) \) is at least \( r \) plus the number of edges added among those counted by \( \sigma(G_{n,M}; \Pi_Y) \). We shall denote this event by \( \mathcal{E}_{XY} \).
In subsequent steps of the proof we will need bounds for the number of non-edges in $\Pi_\chi$ and $\Pi_\psi$, i.e., for the quantities $\overline{c}(G_{n,M}; \Pi_\chi)$ and $\overline{c}(G_{n,M}; \Pi_\psi)$. Let $\hat{C}$ be the constant guaranteed by Lemma 3.24 and define the event
\[
E_{\text{edges}} := \bigcap_{S \in \{\chi, \psi\}} \left\{ G \in G_{n,M} : |\overline{c}(G; \Pi_S) - (1 - p)|\Pi_S| \leq \hat{C}s^{-1/2}(1 - p)|\Pi_S| \right\}. \tag{3.24}
\]
Observe that due to our assumptions we have
\[
|A^* \cap B| + |B^* \cap A| = s \geq s_0 \geq r^4 \sqrt{np^{-1}} \geq \sqrt{n}. \tag{3.25}
\]
Furthermore, due to (3.23) we have for sufficiently large $n$ that
\[
|A^* \cap B|, |A \cap B^*| \leq \frac{s}{2} + 4n^{3/4}p^{-1/4} \left( s \leq \frac{n}{2} \right) \leq \frac{n}{4} + 4n^{3/4}p^{-1/4} \leq \frac{n}{3}. \tag{3.26}
\]
Hence, whenever $n$ is sufficiently large, we obtain that
\[
|A \cap A^*| = |A| - |A \cap B^*| \geq \frac{n}{2} - 4n^{3/4}p^{-1/4} - \frac{n}{3} \geq \frac{n}{7},
\]
and similarly, $|B \cap B^*| \geq \frac{n}{7}$. This implies with (3.25) that $|\Pi_\chi|$ and $|\Pi_\psi|$ fulfill the conditions of Lemma 3.24. Therefore $\Pr[G_{n,M} \in E_{\text{edges}}] \geq 1 - e^{-n}$.

Recall that $p_i$ denotes the probability that there is a bipartition $\Pi \neq \Pi^*$ with gap $r - 1$ and distance between $2^i s_0$ and $2^{i+1} s_0$ (but at most $\frac{n}{2}$) from $\Pi^*$, and denote by $E_i$ the corresponding event in $G_{n,M}$. The above discussion yields
\[
\mathbb{E}[1_{E_i}] \geq \Pr[E_i \cap E_{\text{Bal}} \cap E_{\text{edges}}] \cdot \Pr[E_{\chi, \psi} | E_i \cap E_{\text{Bal}} \cap E_{\text{edges}}]. \tag{3.27}
\]
Note that due to Proposition 3.10 and Lemma 3.24, as discussed above, we can estimate
\[
\Pr[E_i \cap E_{\text{Bal}} \cap E_{\text{edges}}] = p_i - \Pr[E_i \setminus (E_{\text{Bal}} \cap E_{\text{edges}})] \geq p_i - \Pr[E_{\text{Bal}} \cap E_{\text{edges}}] \geq p_i - 2e^{-n}.
\]
Our aim is to show that there is a constant $c_1 > 0$ such that
\[
\Pr[E_{\chi, \psi} | E_i \cap E_{\text{Bal}} \cap E_{\text{edges}}] \geq c_1 \cdot r^{-2}. \tag{3.28}
\]
This will complete the proof of the theorem, as from (3.20), where we set $t = t_i$, (3.21), (3.22), (3.27) and the above estimate we obtain
\[
t_i \cdot \left( \frac{1}{2} - \sqrt{\frac{20Cn}{M}} \right) + (p_i - 2e^{-n}) \cdot \frac{c_i}{r^2} \leq t_i \cdot \left( \frac{1}{2} + \sqrt{\frac{5n}{M}} \right),
\]
which implies with $s \leq \frac{n}{2}$ and hence $t_i = r^2 \frac{n(n-1)}{2s} \leq \frac{2r^2 n}{s_i}$ that we can choose a constant $c_2$ such that for sufficiently large $n$
\[
p_i \leq c_2 \cdot t_i \cdot \sqrt{\frac{n}{M}} \cdot r^2 \leq 4c_2 \cdot \frac{r^4 n}{s_i} \sqrt{\frac{1}{pn}} \leq \frac{8c_2 \cdot r^4}{2^{i+1} \cdot C \cdot \omega \cdot r^4 \cdot \sqrt{np^{-1}} \sqrt{p}} \leq \frac{1}{2^{i+1} \omega},
\]
if we set $C := 8c_2$.

Let $q := \Pr[E_{\chi, \psi} | E_i \cap E_{\text{Bal}} \cap E_{\text{edges}}]$; to complete the proof we estimate $q$ so as to obtain (3.28). Let for brevity $\overline{c}_\chi := \overline{c}(G_{n,M}; \Pi_\chi)$ and $\overline{c}_\psi := \overline{c}(G_{n,M}; \Pi_\psi)$. Furthermore, let $\overline{E}_\chi$ denote the set of edges counted in $\overline{c}_\chi$, and define similarly $\overline{E}_\psi$. In order to obtain a lower bound for $q$, it suffices to consider the event that precisely $x_i := \frac{s(n-s)}{n(n-1)} t_i$ edges out of $\overline{E}_\chi$ are added to $G_{n,M}$ and $x + r$ out of $\overline{E}_\chi$. Suppose that we add the $t_i$ edges one after the other – then there are $(x_{t_i})$ ways to choose the points in time
at which those edges are taken of \( E_X \) and \( E_Y \). Furthermore, the probability that an edge out of \( E_X \) is added to \( G_{n,M} \) is at least \( \frac{\xi_{x_i} - t_i}{(\frac{n}{2}) - M} \) and similarly, the probability that an added edge is contained in \( E_Y \) can be bounded from below by \( \frac{\xi_{y_i} - t_i}{(\frac{n}{2}) - M} \). Moreover, the probability that an added edge belongs neither to \( E_X \) nor \( E_Y \) is at least \( 1 - \frac{\xi_{x_i} + \xi_{y_i}}{(\frac{n}{2}) - M - i} \).

Putting all together yields

\[
q \geq \left( \frac{t_i}{x_i, x_i + r} \right) \left( \frac{\xi_{x_i} - t_i}{(\frac{n}{2}) - M} \right)^{x_i + r} \left( \frac{\xi_{y_i} - t_i}{(\frac{n}{2}) - M} \right)^{x_i} \left( 1 - \frac{\xi_{x_i} + \xi_{y_i}}{(\frac{n}{2}) - M - t_i} \right)^{t_i - 2x_i - r}. \tag{3.29}
\]

We now simplify the above expression so as to be able to apply Proposition 3.25 to obtain (3.28). As a first preparation, we derive tight bounds for the quantities \(|\Pi_X|\) and \(|\Pi_Y|\). Let \( s^{(1)} := |A^* \cap B|, s^{(2)} := |B^* \cap A| \) and note that \( s = s^{(1)} + s^{(2)} \). Furthermore, define the quantity \( \delta := s^{(1)} - \frac{n}{2} \). Observe that

\[
|\Pi_Y| = s^{(1)}(|B^*| - s^{(2)}) + s^{(2)}(|A^*| - s^{(1)}) = \frac{s(n - s)}{2} - \delta(|A^*| - |B^*| - 2\delta). \]

Due to the event \( E_{\text{Sat}} \) we have \(|A^*| - |B^*| \leq 8n^{3/4}p^{-1/4} \) and \(|s^{(1)} - s^{(2)}| \leq 8n^{3/4}p^{-1/4}\). This implies with the above definitions

\[
|\delta| \leq \min \left\{ \frac{s}{2}, 4n^{3/4}p^{-1/4} \right\} \quad \text{and hence} \quad |\Delta| \leq 32 \min \left\{ sn^{3/4}p^{-1/4}, n^{3/2}p^{-1/2} \right\}.
\]

Due to \( s \geq s_0 = C \omega r^4 \sqrt{np^{-1}} \) we can deduce

\[
|\Pi_Y| = \frac{s(n - s)}{2} \left( 1 + O \left( \min \left\{ (pn)^{-1/4}, \omega^{-1}r^{-4} \right\} \right) \right) \left( r \in (\frac{1}{2}n)^{1/8} \right) \frac{s(n - s)}{2} (1 + o(r^2)). \tag{3.30}
\]

Now note that \(|\Pi_X| + |\Pi_Y| = s(n - s)\) implies \(|\Pi_X| = \frac{s(n - s)}{2} + \Delta\); hence the above statement is also valid for \(|\Pi_X|\).

Before we proceed let us make some auxiliary calculations, which we will need in the remainder. First,

\[
x_i = \frac{s(n - s)}{n(n - 1)}, \quad t_i = \frac{s(n - s)}{n(n - 1)}, \quad r^2 \frac{n(n - 1)}{s_i(n - s_i)} \quad (s_i \leq s \leq 2s_i) \leq 2r^2.
\]

Due to (3.30) we may assume for sufficiently large \( n \) \(|\Pi_X|, |\Pi_Y| \geq \frac{sn}{17}\), which implies with the event \( E_{\text{edges}} \) that we have \( \xi_{x_i} \geq (1 - \hat{C}s^{-1/2})(1 - p)|\Pi_X| \geq \frac{sn}{11} \). With \( s_i \leq s \leq \frac{n}{2} \) and \( s_i \geq s_0 \geq \omega r^4 \sqrt{np^{-1}} \) we can deduce if \( n \) is large enough

\[
t_i x_i \leq \frac{11}{sn} \frac{r^2 n(n - 1)}{s_i(n - s_i)} \leq \frac{22r^2}{s_i^2} = o(1) \quad \text{and} \quad s^{-1}x_i \leq \frac{2r^2}{(\omega r^4 \sqrt{np^{-1}})^{1/2}} \leq n^{-\delta}. \tag{3.31}
\]

Now we proceed with simplifying (3.29). With the facts \( x_i + r \leq 2x_i \) and \( 1 - y \geq e^{-2y} \), which is valid for sufficiently small \( y \), the third term in right-hand side of (3.29) can be
estimated for sufficiently large \( n \) with

\[
\left( \frac{\mathcal{E}_{\text{edges}}}{(\binom{n}{2} - M)} \right)^{x_i} \geq \left( \frac{\mathcal{E}_{\text{edges}}}{\binom{n}{2} - M} \right)^{x_i} \left( 1 - \frac{t_i}{\mathcal{E}_{\text{edges}}} \right)^{x_i} e^{-\frac{2t_i}{\mathcal{E}_{\text{edges}}}}.
\]

\[ (3.31) \]

\[
\geq \left( \frac{\mathcal{E}_{\text{edges}}}{(\binom{n}{2})} \right)^{x_i} \left( 1 - \tilde{C}s^{-1/2} \right)^{x_i} \cdot \frac{1}{2}.
\]

\[ (3.32) \]

\[
\geq \frac{1}{4} \left( \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i}.
\]

\[ (3.33) \]

Precisely the same calculation yields that the second term of the right-hand side of (3.29) is for sufficiently large \( n \) at least

\[
\left( \frac{\mathcal{E}_{\text{edges}}}{(\binom{n}{2} - M)} \right)^{x_i + r} \geq \frac{1}{4} \left( \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i + r}.
\]

\[ (3.33) \]

In order to simplify the last term in (3.29), first recall that \( |\Pi_{\mathcal{X}}| + |\Pi_{\mathcal{Y}}| = s(n-s) \). Due to the event \( \mathcal{E}_{\text{edges}} \) it holds that \( \mathcal{E}_{\mathcal{X}} + \mathcal{E}_{\mathcal{Y}} \leq (1 + \tilde{C}s^{-1/2})(1 - p)s(n-s) \). Abbreviate \( m := (\binom{n}{2} - M = (1-p)(\binom{n}{2}) \), and observe that we have \( m \geq \frac{1}{2}n^2 \). This yields

\[
\left( 1 - \frac{\mathcal{E}_{\mathcal{X}} + \mathcal{E}_{\mathcal{Y}}}{m - t_i} \right)^{x_i + r} \geq \left( 1 - \tilde{C}s^{-1/2} \right)^{x_i + r} \cdot \frac{s(n-s)}{2(\binom{n}{2})}.
\]

\[ (3.34) \]

Note that due to \( r \ll (pn)^{1/8} \leq n^{1/8} \) we have \( 2^{-1} \leq m^{-1} \leq \frac{1}{n} \), which implies \( s^{1/2}(1 - \frac{5m}{n^2})^{-1} \leq 2s^{-1/2} \), whenever \( n \) is sufficiently large. Thus, using that for sufficiently small \( y \) the estimate \( 1 - y \geq e^{-2y} \) is true, we obtain for large \( n \) that

\[
\left( 1 - \frac{\mathcal{E}_{\mathcal{X}} + \mathcal{E}_{\mathcal{Y}}}{m - t_i} \right)^{x_i + r} \geq \left( 1 - \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i + r} \cdot \left( 1 - \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i + r} \cdot \left( 1 - \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i + r}.
\]

\[ (3.34) \]

By combining (3.32), (3.33), and (3.34) we obtain from (3.29) that

\[
q \geq \frac{1}{32} \left( \frac{t_i}{x_i + r} \right) \left( \frac{s(n-s)}{2(\binom{n}{2})} \right)^{2x_i + r} \left( 1 - \frac{s(n-s)}{2(\binom{n}{2})} \right)^{x_i + r}.
\]

Now it can be easily checked that we can apply Proposition 3.25 to estimate the above expression, where we set \( N := t_i, \alpha := \frac{s(n-s)}{n(n-1)}, d := r \). Indeed, \( \alpha \) reaches its maximum value when \( s = \frac{n}{2} \), and we obtain for \( n \geq 4 \) that \( \alpha \leq \frac{1}{3} < \frac{1}{2} \); furthermore, \( t_i = r^2 \frac{n(n-1)}{s(n-s)} \).
implies \( d \leq \sqrt{\min(\alpha, 1 - 2\alpha)}N \). Hence, if we denote by \( C'' \) the constant defined by Proposition 3.25, we obtain due to \( s_i \leq s \leq 2s_i \)

\[
q \geq \frac{1}{64} \cdot \frac{C''}{\chi_i} \geq \frac{C''}{64} \cdot \frac{1}{r^2} \frac{s_i(n - s_i)}{s(n - s)} \geq c_1 r^{-2},
\]

which is precisely (3.28), if we choose \( c_1 \) appropriately. This completes the proof. \( \square \)

### 3.4. Proof of Theorem 3.3

Let \( n^{-1/250} \leq p \leq \frac{1}{2} \). Furthermore, define the functions

\[
r_0 = r_0(p, n) := p^{-12} \log^2 n \quad \text{and} \quad s(r) := n^{2/3} \cdot (r + 1)^4,
\]

and abbreviate \( s_0 := s(2r_0) \). Before we continue, we need to introduce some notation. Let \( G \in \mathcal{G}_n \) be a graph with vertex set \([n] := \{1, \ldots, n\}\). A bipartition of \([n]\) is said to be **balanced**, if it is contained in the set

\[
\mathcal{B} \mathcal{a} \mathcal{l}_n := \left\{ (A, B) \mid |A| - \frac{n}{2} \leq \frac{n}{100} \quad \text{and} \quad \left| |B| - \frac{n}{2} \right| \leq \frac{n}{100} \right\}.
\]

Recall that \( T(G) \) denotes the set of maximum cardinality triangle-free subgraphs of a graph \( G \). Define the two “bad” events

\[
B_1 := \left\{ G \in \mathcal{G}_n \mid \exists \mathcal{T} \in T(G) : \text{there is no balanced bipartition} \ \Pi \text{ such that at most} \ r_0 \ \text{edges of} \ \mathcal{T} \ \text{are not across} \ \Pi \right\}, \quad (3.35)
\]

\[
B_2 := \left\{ G \in \mathcal{G}_n \mid \exists \Pi : \text{gap}(G; \ \Pi) \leq 2r_0 \quad \text{and} \quad \text{dist}(G; \ \Pi) \geq \frac{s(\text{gap}(G; \ \Pi))}{3} \right\}.
\]

Moreover, let \( B = B_1 \cup B_2 \) and define the “good” event \( \overline{B} = \mathcal{G}_n \setminus B \).

In the sequel we estimate the probability that there is a \( \mathcal{T} \in T(G_{n,p}) \) that is **not** bipartite. Observe that this implies the existence of a bipartition \( \Pi_{\mathcal{T}} = \Pi = (A, B) \) of the vertex set with the property that we can obtain \( \mathcal{T} \) if we consider the subgraph \( E(G_{n,p}, \ \Pi) \) of the random graph, remove \( t > 0 \) edges from it and add **at least** \( t + \text{gap}(G_{n,p}, \ \Pi) \) edges from \( E(G_{n,p}) \setminus E(G_{n,p}; \ \Pi) \) to it. Accordingly, for a fixed \( \Pi \) and any set \( S \) of edges not across \( \Pi \), let

\[
\mathcal{E}(\Pi, S) := \left\{ G \in \mathcal{G}_n \mid \exists X \subseteq E(G; \ \Pi) : E(G; \ \Pi) \setminus X \cup S \text{ is triangle-free} \right. \}
\]

\[
\text{and} \quad |S| - |X| \geq \text{gap}(G; \ \Pi) \right\}. \quad (3.36)
\]

Now let us assume that \( G_{n,p} \in \overline{B} \). Then the event that there is a \( \mathcal{T} \in T(G_{n,p}) \) which is not bipartite, implies the existence of a balanced partition \( \Pi = (A, B) \), which can be “enhanced” by at least one and at most \( r_0 \) edges, with both endpoints in \( A \) or \( B \), and possibly by removing any number \( \leq r_0 \) of edges with one endpoint in \( A \) and one in \( B \), such that we obtain a triangle-free graph. The above definition of the event \( \mathcal{E}(\Pi, S) \) thus implies that we have

\[
\text{Pr}[\exists \mathcal{T} \in T(G_{n,p}) : \mathcal{T} \text{ is not bipartite}] \leq \text{Pr}[G_{n,p} \in B] + \sum_{\Pi = (A, B) \in \mathcal{B} \mathcal{a} \mathcal{l}_n} \text{Pr}[G_{n,p} \in (\mathcal{E}(\Pi, S) \cap \overline{B})]. \quad (3.37)
\]
Let \( C \) be the constant guaranteed to exist by Theorem 3.4. By applying Lemma 3.23 and Theorem 3.4, for all \( r \in [1, 2r_0 + 1] \) and \( \omega := \frac{3}{2}n^{1/6 - 1/250} \), the first term on the right hand side of (3.37) can be bounded as follows for sufficiently large \( n \):

\[
\Pr \left[ G_{n,p} \in \mathcal{B} \right] \leq \Pr \left[ G_{n,p} \in \mathcal{B}_1 \right] + \Pr \left[ G_{n,p} \in \mathcal{B}_2 \right]
\]

\[
\leq o(1) + \sum_{r=1}^{2r_0+1} \frac{1}{\omega} = o(1) + \frac{2n^{12/250}\log^2 n + 1}{\omega} = o(1).
\] (3.38)

In the remainder of the proof we will bound the sum on the right hand side of (3.37). In particular, we show that the probability of the joint event \((G_{n,p} \in \mathcal{E}(\Pi, S)) \land (G_{n,p} \in \mathcal{B})\) can be estimated by the probability that two appropriately defined events \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) occur. This will allow us to use the FKG inequality in order to get a sufficient upper bound for the sum in (3.37). For a fixed \( \Pi = (A, B) \in \mathcal{B}al_n \) and a non-empty set \( S \subseteq \binom{A}{2} \cup \binom{B}{2} \) of edges, let

\[
\mathcal{E}_1(\Pi) = \left\{ G \in \mathcal{G}_n \mid \text{gap}(G; \Pi) \leq r_0 \text{ and } \forall \Pi' : \left( \text{gap}(G, \Pi') \leq r_0 \Rightarrow \text{dist}(\Pi, \Pi') \leq s_0 \right) \right\},
\]

\[
\mathcal{E}_2(\Pi, S) = \left\{ G \in \mathcal{G}_n \mid \exists X \subset E(G; \Pi) : (E(G; \Pi) \setminus X) \cup S \text{ is } \triangle\text{-free, and } |X| \leq |S| \right\}.
\] (3.39)

Note that the definition of \( \mathcal{E}_1 \) seems overly complicated, and one could think that an event of the type “there is no partition \( \Pi \) with \( \text{gap}(G; \Pi) \leq 2r_0 \) and \( \text{dist}(G; \Pi) \geq s_0 \)”, which would follow directly from the definition (3.35) of \( \mathcal{B}_2 \), could be sufficient as well. It turns out (see Proposition 3.27), that this is in fact the most delicate point of our proof: we need to relax the event, so that it becomes an increasing function in an appropriately defined partial ordering of all graphs with \( M \) edges. As a consequence, we may use the FKG inequality to get a sufficient estimate.

First we show that in fact we can bound the probability from (3.37) with the joint probability of the events \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \).

**PROPOSITION 3.26.** **For all** \( \Pi = (A, B) \in \mathcal{B}al_n \) **and** \( S \subseteq \binom{A}{2} \cup \binom{B}{2} \) **with** \( 1 \leq |S| \leq r_0 \)

\[
\Pr \left[ (G_{n,p} \in \mathcal{E}(\Pi, S)) \land (G_{n,p} \in \mathcal{B}) \right] \leq \Pr \left[ (G_{n,p} \in \mathcal{E}_1(\Pi)) \land (G_{n,p} \in \mathcal{E}_2(\Pi, S)) \right].
\] (3.40)

**PROOF.** Suppose that the event \((G_{n,p} \in \mathcal{E}(\Pi, S)) \land (G_{n,p} \in \mathcal{B})\) occurs. From (3.36) we deduce that \( \text{gap}(G_{n,p}; \Pi) \leq |S| \leq r_0 \); hence, \( G_{n,p} \not\in \mathcal{B}_2 \) implies that \( \text{dist}(G_{n,p}; \Pi) \leq \frac{s(r_0)}{3} \). Now consider any partition \( \Pi' \) different from \( \Pi \), which fulfills \( \text{gap}(G_{n,p}; \Pi, \Pi') \leq r_0 \). Clearly, \( \text{gap}(G_{n,p}; \Pi') \leq 2r_0 \), and consequently \( G_{n,p} \not\in \mathcal{B}_2 \) implies that \( \text{dist}(G_{n,p}; \Pi') \leq \frac{s(2r_0)}{3} \). Furthermore, due to the event \( G_{n,p} \in \mathcal{B} \) any two maximum bipartitions of \( G_{n,p} \) have distance at most \( \frac{s(0)}{3} \). As the distance of \( \Pi \) and \( \Pi' \) is at most the sum of their distances from any maximum bipartition plus the largest distance of any two maximum bipartitions, we obtain

\[
\text{dist}(\Pi, \Pi') \leq \text{dist}(G_{n,p}; \Pi) + \text{dist}(G_{n,p}; \Pi') + \frac{s(0)}{3} \leq s(2r_0) = s_0.
\]

That is, we have \( G_{n,p} \in \mathcal{E}_1(\Pi) \). The event \( \mathcal{E}_2(\Pi, S) \) is easily seen to hold simultaneously, as it is a relaxation of the condition in (3.36). \( \square \)
The next proposition states that we can estimate the probability from (3.40) with the product of the probabilities of the events $\mathcal{E}_1$ and $\mathcal{E}_2$. Its proof consists of the definition of an appropriate distributive lattice on graphs with respect of bipartions of the vertex set, and a subsequent application of the FKG inequality.

**Proposition 3.27.** For all $\Pi = (A, B) \in \mathcal{B} \mathcal{A} l_n$ and $S \subset \binom{A}{2} \cup \binom{B}{2}$ with $1 \leq |S| \leq r_0$

$$\Pr \left( \left( G_{n,p} \in \mathcal{E}_1(\Pi) \right) \wedge \left( G_{n,p} \in \mathcal{E}_2(\Pi, S) \right) \right) \leq \Pr \left[ G_{n,p} \in \mathcal{E}_1(\Pi) \right] \cdot \Pr \left[ G_{n,p} \in \mathcal{E}_2(\Pi, S) \right].$$

**Proof.** For the proof we need a variant of the FKG inequality which we are going to state now; a far more general treatment of the topic can be found in [AS00]. A lattice is a partially ordered set $(S, \leq)$ (with ground set $S$ and a partial order $\leq$ on $S$) in which every two elements $x$ and $y$ have a unique minimal upper bound and and a unique maximal lower bound, which we denote by $x \lor y$ and $x \land y$ respectively. $\mathcal{L}$ is called distributive, if for all $x, y, z \in \mathcal{L}$ we have

$$x \land (y \lor z) = (x \land y) \lor (x \land z). \quad (3.41)$$

Next, we need the concepts of log-supermodular, increasing, and decreasing functions. A function $f : S \to \mathbb{R}^+$ is called log-supermodular, if for all $x, y \in S$ it holds

$$f(x)f(y) \leq f(x \lor y)f(x \land y). \quad (3.42)$$

A function $f$ is called increasing if $x \leq y$ implies $f(x) \leq f(y)$. Similarly, $f$ is called decreasing if $x \leq y$ implies $f(x) \geq f(y)$. With these definitions we can state a probabilistic version of the well-known FKG inequality: if $\mathcal{L} = (\Omega, \leq)$ is a finite distributive lattice, $f : \Omega \to \mathbb{R}^+$ an increasing function, $g : \Omega \to \mathbb{R}^+$ a decreasing function, and $\mu : \Omega \to \mathbb{R}^+$ a log-supermodular probability measure on $\Omega$, then we have

$$\mathbb{E}[f \cdot g] \leq \mathbb{E}[f] \cdot \mathbb{E}[g]. \quad (3.43)$$

We now prove Proposition 3.27. In order to do so fix some $\Pi$ and $S$. As we want to apply inequality (3.43) we first define an appropriate partial ordering on graphs as follows. For two graphs $G$ and $H$ let

$$G \leq_{\Pi} H :\iff E(G; \Pi) \subseteq E(H; \Pi) \text{ and } (E(G) \setminus E(G; \Pi)) \supseteq (E(H) \setminus E(H; \Pi)). \quad (3.44)$$

Intuitively, a graph $G$ is “smaller” than a graph $H$ with respect to $\leq_{\Pi}$, if it has fewer edges across $\Pi$ and simultaneously more edges in the parts of $\Pi$. One easily checks that for any pair of graphs $G$ and $H$ the unique minimal upper bound of $G$ and $H$ is given by

$$G \lor H = (E(G; A) \cup E(H; A)) \cup (E(G; B) \cup E(H; B)) \cup (E(G; \Pi) \cap E(H; \Pi)),$$

while the unique maximal lower bound is given by

$$G \land H := (E(G; A) \cap E(H; A)) \cup (E(G; B) \cap E(H; B)) \cup (E(G; \Pi) \cup E(H; \Pi)).$$

As probability space we use the $G_{n,p}$, i.e., for any $G \in \mathcal{G}_n$ $\mu(G) := \Pr[G_{n,p} = G] = p^{e(G)}(1-p)^{\binom{n}{2} - e(G)}$. An easy calculation yields that

$$\mu(G)\mu(H) = p^{e(G)+e(H)}(1-p)^{2\binom{n}{2} - e(G) - e(H)}$$

$$= p^{e(G\lor H) + e(G\land H)}(1-p)^{2\binom{n}{2} - e(G\lor H) - e(G\land H)} = \mu(G \lor H)\mu(G \land H),$$
i.e. \( \mu \) is log-supermodular. Furthermore, it can easily be verified that the above defined operators are distributive, i.e., they fulfill (3.41). We leave the details to the reader.

Let \( \mathcal{E}_1 := \mathcal{E}_1(\Pi) \) and \( \mathcal{E}_2 := \mathcal{E}_2(\Pi, S) \). For \( i = 1, 2 \) we denote for a graph \( G \) by

\[
f_i(G) := \begin{cases} 
1, & \text{if } G \in \mathcal{E}_i \\
0, & \text{otherwise}
\end{cases}
\]

the indicator function for the event \( \mathcal{E}_i \). In the sequel we shall show that \( f_2 \) is decreasing with respect to \( \leq_{\Pi} \), and that \( f_1 \) is increasing. This will conclude the proof, as the above discussion yields that the conditions for (3.43) are fulfilled.

First we prove that \( f_2 \) is decreasing. For this it obviously suffices to show that if \( G \leq_{\Pi} H \), then \( H \in \mathcal{E}_2 \) implies that \( G \in \mathcal{E}_2 \). To see this observe that \( H \in \mathcal{E}_2 \) implies that there is a set \( X \subset E(H; \Pi) \) such that

\[
(E(H; \Pi) \setminus X) \cup S \text{ is triangle-free, and } |X| \leq |S|.
\]

Due to \( G \leq_{\Pi} H \) we have \( E(G; \Pi) \subseteq E(H; \Pi) \), and let \( X' := X \cap E(G; \Pi) \). Clearly, \( |X'| \leq |S| \), and as \( (E(H; \Pi) \setminus X) \cup S \) is triangle free, so is \( (E(G; \Pi) \setminus X') \cup S \). But this means \( G \in \mathcal{E}_2 \), as desired.

Finally, we prove that \( f_1 \) is increasing. For this, we show that if \( G \leq_{\Pi} H \), then \( G \in \mathcal{E}_1 \) implies \( H \in \mathcal{E}_1 \). Observe that by transitivity it is sufficient to consider the case that \( G \) and \( H \) differ in exactly one edge \( e \). The event \( G \in \mathcal{E}_1 \) implies

\[
gap(G; \Pi) \leq r_0 \text{ and } \forall \Pi' \text{ such that } \gap(G; \Pi, \Pi') \leq r_0 : \text{dist}(\Pi, \Pi') \leq s_0. \tag{3.45}
\]

Now we make a case distinction. First assume that \( e \) joins two vertices in \( A \) or two vertices in \( B \). Then, due to (3.44), we have \( H = G \setminus \{e\} \). Note that this implies that the size of a maximum bipartition satisfies \( b(G) - 1 \leq b(H) \leq b(G) \). As furthermore \( E(G; \Pi) = E(H; \Pi) \) we thus have \( \gap(H; \Pi) \leq \gap(G; \Pi) \leq r_0 \). Now let \( \Pi' \) be a bipartition which has the property \( \gap(H; \Pi, \Pi') \leq r_0 \). Observe that \( e(H; \Pi') \leq e(G; \Pi') \). We easily deduce

\[
\gap(G; \Pi, \Pi') = e(G; \Pi) - e(G; \Pi') \leq e(H; \Pi) - e(H; \Pi') = \gap(H; \Pi, \Pi') \leq r_0,
\]

which implies with (3.45) that \( \text{dist}(\Pi, \Pi') \leq s_0 \).

Now assume that \( e \) joins a vertex in \( A \) with a vertex in \( B \). Then \( H = G \cup \{e\} \). In this case we have \( b(G) \leq b(H) \leq b(G) + 1 \) and \( E(H; \Pi) = E(G; \Pi) \cup \{e\} \). This immediately implies

\[
\gap(H; \Pi) = b(H) - e(H; \Pi) \leq b(G) + 1 - (e(G; \Pi) + 1) = \gap(G; \Pi) \leq r_0.
\]

Now let again \( \Pi' \) be a bipartition with \( \gap(H; \Pi, \Pi') \leq r_0 \). Note that \( e(G; \Pi') \geq e(H; \Pi') - 1 \), as the edge \( e \) does not necessarily join two vertices in different parts of \( \Pi' \). This implies

\[
\gap(G; \Pi, \Pi') = e(G; \Pi) - e(G; \Pi') \leq (e(H; \Pi) - 1) - (e(H; \Pi') - 1)
\]
\[
\leq \gap(H; \Pi, \Pi') \leq r_0.
\]

Hence, (3.45) implies \( \text{dist}(\Pi, \Pi') \leq s_0 \), as desired. This completes the proof. \( \square \)
As a final ingredient in our proof we need estimates for the probabilities $\Pr [G_{n,p} \in \mathcal{E}_2(\Pi)]$ and $\Pr [G_{n,p} \in \mathcal{E}_2(\Pi, S)]$. These are given by the next proposition.

**Proposition 3.28.** For all $\Pi = (A, B) \in \text{Bal}_n$ and $S \subseteq \binom{A}{2} \cup \binom{B}{2}$ with $1 \leq |S| \leq r_0$ it holds for sufficiently large $n$

\[
\Pr [G_{n,p} \in \mathcal{E}_2(\Pi, S)] \leq e^{-\frac{n^2}{12n}} \tag{3.46}
\]

and

\[
\sum_{\Pi \in \text{Bal}_n} \Pr [G_{n,p} \in \mathcal{E}_1(\Pi)] \leq n\left(\frac{n}{s_0}\right) \tag{3.47}
\]

**Proof.** We first bound the probability for the event $\mathcal{E}_2$. Fix an edge $e \in S$ and note that as $\Pi$ is balanced there exist at least

\[
\min(|A|, |B|) \geq \frac{n}{2} - \frac{n}{100} = \frac{49}{100}n
\]

pairwise vertex-disjoint possible triangles across $\Pi$ which contain the edge $e$.

Denote by $Y$ the random variable which counts the number of those triangles in $G_{n,p}$.

With the definition of $\mathcal{E}_2$ in (3.39) we deduce

\[
\Pr [G_{n,p} \in \mathcal{E}_2(\Pi, S)] \leq \Pr [Y \leq |S|].
\]

The probability that a triangle (with $e$) is contained in $G_{n,p}$ is $p^2$; hence, $\mathbb{E}[Y] \geq \frac{49}{100}p^2n$.

On the other hand, observe that $|S| \ll \mathbb{E}[Y]$ with our assumptions on $p$. A simple application of Lemma 3.7 yields for sufficiently large $n$

\[
\Pr [Y \leq |S|] \leq \Pr \left[ Y \leq \frac{\mathbb{E}[Y]}{10} \right] \leq e^{-\frac{\mathbb{E}[Y]}{4}},
\]

which implies (3.46).

Next we show (3.47). Trivially, we have

\[
\sum_{\Pi \in \text{Bal}_n} \Pr [G_{n,p} \in \mathcal{E}_1(\Pi)] = \sum_{\Pi \in \text{Bal}_n} \sum_{G \in \mathcal{E}_1(\Pi)} \Pr [G_{n,p} = G]
\]

Now we want to interchange the order of summation, such that the first sum goes over (a carefully chosen subset of) all graphs in $G_n$. To achieve this, observe that the number of times the probability of a graph $G$ is counted above is equal to the number of balanced partitions $\Pi$ with the properties

\[
gap(G; \Pi) \leq r_0, \text{ and } \forall \Pi' \text{ such that } \gap(G; \Pi, \Pi') \leq r_0 : \text{dist}(\Pi, \Pi') \leq s_0. \tag{3.48}
\]

In the following we argue that we can construct all such partitions $\Pi$ by choosing an arbitrary partition $\Pi^*$ such that $\gap(G; \Pi^*) \leq r_0$, and modifying the parts of $\Pi^*$ in at most $s_0$ vertices. To see this, assume that there is a partition $\Pi$ fulfilling (3.48) with $\text{dist}(\Pi, \Pi^*) > s_0$. But then we have $\gap(\Pi, \Pi^*) \leq r_0$, as the partitions satisfy $\gap(G; \Pi) \leq r_0$ and $\gap(G; \Pi^*) \leq r_0$, which implies $\text{dist}(\Pi, \Pi^*) \leq s_0$ – a contradiction.
Hence, as there are precisely $\sum_{t\leq s_0} \binom{n}{t}$ ways to choose at most $s_0$ vertices, which change the class they belong to, and for sufficiently large $n$ the inequality $s_0 \leq \frac{n}{2}$ holds, we can conclude
\[
\sum_{\Pi \in \mathcal{B}_1} \Pr \left[ G_{n,p} \in \mathcal{E}_1(\Pi) \right] \leq \sum_{G \in \mathcal{G}_n} \left( \Pr \left[ G_{n,p} = G \right] \cdot \sum_{\Pi \in \mathcal{B}_1} 1 \right) \leq n \binom{n}{s_0}.
\]

**Proof of Theorem 3.3.** Recall that $n^{-1/250} \leq p \leq \frac{1}{2}$, which implies for sufficiently large $n$ the bounds $\tau_0 \leq n^{13/250}$ and $s_0 \leq n^{9/10}$. As there are at most $\binom{n}{s_0}$ ways to choose a set $S$ of edges out of all possible edges, the proof of the theorem can be completed with inequality (3.37) and Propositions 3.26, 3.27, and 3.28 as follows:
\[
\Pr \left[ \exists T \in \mathcal{T}(G_{n,p}) : T \text{ not bip.} \right] \leq \Pr \left[ \mathcal{B} \right] + \sum_{\Pi \in \mathcal{B}_1} \Pr \left[ G_{n,p} \in \mathcal{E}_1(\Pi) \right] \cdot \Pr \left[ G_{n,p} \in \mathcal{E}_2(\Pi, S) \right]
\leq o(1) + e^{-\frac{p^2 n}{12}} \cdot \left( \frac{n^2}{\tau_0} \right) \cdot \sum_{\Pi \in \mathcal{B}_1} \Pr \left[ G_{n,p} \in \mathcal{E}_1(\Pi) \right]
\leq o(1) + \exp \left\{ -\frac{p^2 n}{12} + 2\tau_0 \log n + (s_0 + 1) \log n \right\} = o(1),
\]
whenever $n$ is chosen sufficiently large. \(\square\)

### 3.5. Generalizations & Open problems

Let $\ell \geq 2$. Here we show how the proofs of the previous sections can be adapted in order to prove Theorem 3.6. As a tool, we will use a “higher-dimensional” variant of Theorem 3.4, see Theorem 3.29 below. Before we state it, we need to define the notion of distance for two $\ell$-partitions, which is a straightforward generalization of the case $\ell = 2$:
\[
\text{dist}(\Pi, \Pi') := \min_{\pi : [\ell] \rightarrow [\ell]} \sum_{1 \leq i < j \leq \ell} \left| V_i \cap V_{\pi(j)} \right|.
\]

The notion of the gap of two $\ell$-partitions is defined in the obvious way. The following theorem is a statement about the structure of the set of (near-)optimal $\ell$-partitions of the uniform random graph.

**Theorem 3.29.** Let $\ell \geq 2$. There are constants $C = C(\ell) > 0$ and $\varepsilon_0 = \varepsilon_0(\ell) > 0$ such that the following holds for sufficiently large $n$. Let $0 \leq \varepsilon \leq \varepsilon_0$, $n^{-1/\ell} \ll p \leq \frac{1}{2}$ and $M = p(n)$. Furthermore, let $r \geq 1$ satisfy $r \ll (pn)^{1/8}$ and $\omega \gg 1$, and define
\[
s_0 := C \cdot \omega \cdot r^4 \cdot \sqrt{np^{-1}}.
\]

Then
\[
\Pr \left[ \exists \Pi : \text{gap}(G_{n,M}; \Pi) = r - 1 \text{ and } \text{dist}(G_{n,M}; \Pi) \geq s_0 \right] \leq \omega^{-1}.
\]

The proof of Theorem 3.4 can easily be adapted to show the above theorem. The sole difference is that instead of considering bipartitions we have to consider $\ell$-partitions. In
this case it is easily seen that (3.21) is still valid, i.e., the expected increase of the size of a maximum \( \ell \)-partition can be estimated from below by

\[
\mathbb{E}[b(G_{n,M} + t_i) - b(G_{n,M})] \geq \mathbb{E}[\mathcal{I}_t] + \mathbb{E}[\mathcal{I}_O],
\]

where \( b \) denotes the size of maximum \( \ell \)-partition, \( \mathcal{I}_t \) the increase of the size of a fixed maximum \( \ell \)-partition \( \Pi^* \) of \( G_{n,M} \) and \( \mathcal{I}_O \) the indicator variable for the event that a partition \( \Pi \) with gap \( r - 1 \) becomes one of the optimal bipartition after adding \( t_i \) random edges to \( G_{n,M} \). \( \mathbb{E}[\mathcal{I}_t] \) can be routinely estimated from below. Moreover, to estimate \( \mathbb{E}[\mathcal{I}_O] \) we proceed exactly as in the proof of Theorem 3.4: we estimate the probability that the number of edges added which resulted in an increase of the size of \( \Pi \) is at least \( r \) plus the number of edges which increased the size of \( \Pi^* \). We leave the solely technical but straightforward details to the reader.

With the above observations, it is not very surprising that the proof of Theorem 3.3 can be adapted in order to prove Theorem 3.6 – in fact, it turns out that it does not make a difference for our proofs if we consider \( \ell \)-partitions instead of bipartitions of the \( G_{n,p} \). However, some details are significantly more tedious than it is above the case, and we shall elaborate more on this issue.

We proceed in three steps, mimicking the proof Theorem 3.3. Let \( \mathcal{F}(G_{n,p}) \) denote the set of the maximum \( \mathcal{K}_r \)-free subgraphs of \( G_{n,p} \), and let \( F \in \mathcal{F}(G_{n,p}) \). First, we prove a statement similar to that of Lemma 3.20: for every \( \varepsilon > 0 \) we can find a.a.s. a partition \( \Pi = (V_1, \ldots, V_{\ell-1}) \) such that all but \( \varepsilon n^2 \) edges of \( F \) go across \( \Pi \), and all parts of \( \Pi \) have approximately the same size. The proof is, as in the case of maximum triangle-free graphs, an application of the sparse version of Szemerédi’s regularity lemma (Theorem 3.15) and the probabilistic embedding lemma (Theorem 3.18), followed by an application of the stability lemma which results in the desired \( (\ell - 1) \)-partition. The calculations differ only in minor technical details, which are again left to the reader.

Second, we show that in fact we can find a better \( (\ell - 1) \)-partition, i.e., we prove a general version of Lemma 3.23.

**Lemma 3.30.** For \( \ell \geq 4 \) and \( p \geq n^{-100\ell^2} \) the following holds a.a.s. For every \( F \in \mathcal{F}(G_{n,p}) \) there is a partition \( \Pi_\ell = \Pi = (V_1, \ldots, V_{\ell-1}) \) of the vertices set such that all but \( 2p^{-5\ell^2} \log^2 n \) edges of \( F \) go across \( \Pi \). Furthermore, \( |V_i| = \frac{n}{\ell - 1} + o(n) \) for \( 1 \leq i \leq \ell - 1 \).

Before we give the proof details let us state an auxiliary result that we will use several times, which is a statement about the number of pairwise edge-disjoint copies of complete graphs in subgraphs of the \( G_{n,p} \).

**Proposition 3.31.** \( G_{n,p} \) has for every \( k \geq 3 \) a.a.s. the following property. There are constants \( c = c(k), c' = c'(k) \) such that for every \( k \) disjoint subsets of its vertices \( V_1, \ldots, V_k \), where \( |V_i| = s \geq c p^{-k^2} \log n \), the number of pairwise edge-disjoint \( \mathcal{K}_k \)'s with one vertex in each \( V_i \) is \( \geq c' s^2 \) and \( \leq k^2 ps^2 \).

**Proof.** The upper bound is easy to obtain, as the number of edges between any \( k \) sets \( V_1, \ldots, V_k \) of the given sizes is a.a.s. at most \( k^2 ps^2 \). For the lower bound, let us
fix $V_1, \ldots, V_k$. We apply Lemma 3.22 $k^2$ times, with $c = 1/2$, and where we assign to $k$ from that lemma the values $1, \ldots, k$, and with $U = V_1, \ldots, V_k$. This defines $k^2$ exceptional classes $X_{i,j}$ such that all $j$-tuples of vertices out of $|n| \setminus X_{i,j}$ have at most $\frac{3}{2}p^j$'s and at least $\frac{1}{p}j$'s common neighbors in $V_i$, and $|X_{i,j}| = o(p^{-i \log n})$. Let $V'_i := V_i \setminus (\bigcup_{j \neq i} X_{i,j})$, and note that for sufficiently large $n$ we have $|V'_i| \geq \frac{1}{2}|V_i|$.

The proof of the lower bound proceeds in two steps. First, we show that the number of $K_k$'s with one vertex in each $V'_i$ is at least $\frac{1}{2}(\frac{e}{2})^{k^2}p^{|V'_i|}$, with probability $\geq 1 - n^{-k s}$. Then we show that for every edge joining (without loss of generality) $V'_i$ to $V'_j$, $G_{n,p}$ is a.a.s. such that the number of $K_k$'s containing that edge (and having one vertex in each of the sets $V'_i, \ldots, V'_j$) is $\leq C_ks^{k-2}p^{|V'_i|}$, by combining the above statements the lower bound is proved.

To see the first claim, let $V_1, \ldots, V_k$ be disjoint subsets of the vertices of size $s$, and let for $1 \leq i \leq k$ $W_i \subseteq V_i$ be subsets of them such that $s/2 \leq |W_i| \leq s$. Denote by $X$ the number of $K_k$'s with one vertex in each of the $W_i$'s. Clearly, $\mathbb{E}[X] \leq \left(\frac{s}{2}\right)^k \cdot p^{|W_i|} \cdot \mathbb{E}[X] \leq s^k \cdot p^{|W_i|}$. We show that $X$ is sharply concentrated by using a variant of Azuma's inequality, see [JLR00, Corollary 2.27]. Let $E$ be the union of all possible edges between any two sets $W_i$ and $W_j$, and denote by $G_{E}$ the set of all graphs with subsets of $E$ as edges. The inequality says that $\Pr\{|X - \mathbb{E}[X]| \geq t\} \leq 2e^{-t^2/2S}$, where $S = \sum_{e \in E} E_{e}$, and $c_e \geq \max_{G \in G_{E}} |X(G) - X(G \setminus e)|$. Clearly, for every edge $e \in E$ we have $c_e \leq s^{k-2}$, which yields $2S \leq 2|E|s^{2(k-2)} \leq Cs^{2(k-2)}$, for a suitably chosen constant $C = C(k)$. With $c := 15k^22^{k+1}C$ we obtain for sufficiently large $n$

$$\Pr\{|X - \mathbb{E}[X]| \geq \mathbb{E}[X]/2\} \leq 2e^{-2k^2s^2(\frac{e}{2})^2k^2} \leq 2e^{-2k^2s^2(\frac{e}{2})^2k^2} \leq e^{-3ks\log n}. \quad (3.49)$$

The number of ways to choose the sets $W_1, \ldots, W_k$ is at most $n^{ks} \cdot 2^{k^2} \leq e^{2ks\log n}$, comparing this with the above bound yields that the $G_{n,p}$ has the property that for every $k$ disjoint sets of vertices $V_1, \ldots, V_k$ of size $s$, the number of $K_k$'s, which have one endpoint in each of the subsets $W_1 \subseteq V_1, \ldots, W_k \subseteq V_k$, such that $|W_i| \geq \frac{s}{2}$, is at least $\frac{1}{2}(\frac{e}{2})^{k^2}p^{|V'_i|}$, with probability $\geq 1 - n^{-k s}$.

Next we show the second claim. Let $e$ be an edge that joins a vertex in $V'_i$ to one in $V'_j$. The number $\tau_3$ of triangles with $e$ and a vertex $v_2 \in V'_j$ is at most $\frac{3}{2}p^{i-2}s$, as the common neighborhood of the endpoints of $e$ in $V_3$ has at most this size. Similarly, the number $\tau_i$ of $K_i$'s with $e$, and vertices $v_3 \in V'_j, \ldots, v_i \in V'_i$ is at most $\tau_{i-1}$ times the maximum size of the common neighborhood of $i - 1$ non-exceptional vertices in $V'_i$. It follows

$$\tau_i \leq \tau_{i-1} \cdot \frac{3}{2}p^{i-1}s \leq \cdots \leq \left(\frac{3}{2}\right)^{i-2} \cdot \tau_3 \cdot p^{|V'_i|} \cdot s^{i-3} \leq \left(\frac{3}{2}\right)^{i-2} \cdot p^{|V'_i|} \cdot s^{i-3} \cdot \left(\frac{3}{2}\right)^{i-2} \cdot p^{|V'_i|} \cdot s^{i-2}.$$

Hence, $\tau_k \leq C_ks^{k-2}p^{|V'_i|}$, and the proof is completed. \hfill \Box

The next result is a generalized version of the above statement. Its proof follows the same lines as the proof of Proposition 3.31, but the details are slightly more tedious.

**Proposition 3.32.** $G_{n,p}$ has for every $k \geq 3$ a.a.s. the following property. There are constants $c = c(k), c' = c'(k)$ such that for every $k$ disjoint subsets of its
vertices $V_1, \ldots, V_k$, with $|V_i| := s \geq cp^{-k^2} \log n$ and for $i \geq 2$ $|V_i| = s' \geq |V_1|$, the number of pairwise edge-disjoint $K_k$’s with one vertex in each $V_i$ is $\geq c'pss'$ and $\leq k^2pss'$.

Proof of Lemma 3.30. The main strategy is very similar to the strategy used in the proof of Lemma 3.23, but we have to make some important modifications. We will sketch only the relevant steps, and the missing details can easily be filled in by considering the respective steps in the original proof.

First of all, note that due to the discussion before Lemma 3.30 there is a.a.s. a partition $(V_1^F, \ldots, V_{\ell-1}^F)$ of $F$ such that

$$\sum_{i=1}^{\ell-1} e(F; V_i^F) \leq \epsilon^5pn^2, \text{ and } |V_i^F| = \frac{n}{\ell-1} \pm \epsilon^5n,$$

for any sufficiently small positive $\epsilon$. Moreover, with similar arguments as in the proof of Lemma 3.23 it can be shown that every $(\ell-1)$-partition $(P_1, \ldots, P_{\ell-1})$ of $G_{n,p}$ with gap at most $\epsilon^5pn^2$ satisfies a.a.s. $(1-\epsilon)\frac{n}{\ell-1} \leq |P_i| \leq (1+\epsilon)\frac{n}{\ell-1}$. We will assume that $G_{n,p}$ has all those properties simultaneously, and make also all other additional assumptions on $G_{n,p}$ made in Lemma 3.23.

In accordance with Lemma 3.23, we call a partition $\Pi$ optimal with respect to $F$, if $e(F; \Pi)$ attains its maximum over all possible partitions. By exploiting our assumptions we see that all optimal $(\ell-1)$-partitions $\Pi = (V_1, \ldots, V_{\ell-1})$ of $F$ satisfy $(1-\epsilon)\frac{n}{\ell-1} \leq |V_i| \leq (1+\epsilon)\frac{n}{\ell-1}$ and $\sum_{i=1}^{\ell-1} e(F; V_i) \leq \epsilon^3pn^2$.

Fix any $\Pi = (V_1, \ldots, V_{\ell-1})$ be an optimal $(\ell-1)$-partition of $F$. A horizontal edge is an edge in $F$ that joins two vertices in the same $V_i$, and a missing edge joins in $G_{n,p}$ two vertices in different $V_i$’s, but is not in $F$. As in Lemma 3.23 we can then define the horizontal degree and missing degree of any vertex $v$ with respect to $F$.

Next, for all $2 \leq i \leq \ell-1$ we denote by $B_i(U)$ the (minimal) set of vertices, such that for every $v_1, \ldots, v_i \not\in B_i(U)$ we have $|\bigcap_{j=1}^i \Gamma(G_{n,p}; v_j) \cap U - p^i|U|| \leq \frac{1}{4}p^i|U|$, and let

$$X_i^{V_i} = \left( \bigcup_{j=1}^{i-1} B_1(V_j) \cup \cdots \cup B_{i-1}(V_j) \right) \cap V_i, \text{ and } X_2^{V_i}, X_3^{V_i} \text{ remain as in Lemma 3.23.}$$

Moreover, define $V_i^0 := V_i$, and $V_i^j := V_i^{j-1} \setminus X_j^{V_i}$, and $X_i := \bigcup_{k=1}^{\ell-1} X_k^{V_i}$. Note that due to Lemma 3.22 we have $|X_i| = O(p^{-\ell+1})$, and that the number of missing edges in $F$ incident to at least one vertex in $|X_3|$ is at least $\frac{2}{3}|X_3|\epsilon pn$. From this we readily obtain that $|X_3| \leq \epsilon n$, as the number of missing edges is at most the number of horizontal edges (otherwise $F$ would not be a maximum $K_r$-free graph).

We now proceed with the following steps, mimicking the proof of Lemma 3.23.

(i) We first show that $X_2$ is small, i.e., $|X_2| \leq \epsilon p^{-2}$.

(ii) Set $H_3 := \bigcup_{i=1}^{\ell-1} E(F; V_i^0)$. We show that $|H_3| \leq p^{-\ell^2} n \log n$.

(iii) We show $|X_3| \leq p^{-\ell^2-2} \log n$.

(iv) We use (ii) to show that for all $v \in \bigcup_{i=1}^{\ell-1} V_i^0$ we have $d_{H}(v) \leq p^{-2\ell^2} \log n$.

(v) Then we show that in fact $|H_3| \leq p^{-5\ell^2} \log^2 n$. 

(vi) Finally, we show \( \sum_{i=1}^{\ell-1} e(F; V_i) \leq 2p^{-5\ell^2} \log^2 n. \)

We now show how the original proof has to be modified to prove this more general statements. To see (i), first note that \( d_H(v) \leq d(F; v, V_i) \) for all \( v \in X_2^V \) and all \( 2 \leq i \leq \ell - 1 \), i.e. the neighborhoods of \( v \) in all \( V_i \) have size at least \( \varepsilon pn \). Moreover, suppose that \( |X_2^V| \geq t_0 = c_\ell (\varepsilon p)^2 \), where \( c_\ell \) will be specified later. The number of missing edges can be bounded from below by the maximal number of pairwise edge-disjoint \( K_{t_i}'s \), which have one endpoint in each of \( V_2 \cap \Gamma(G_{n,p}, v), \ldots, V_{t-1} \cap \Gamma(G_{n,p}, v) \), and two endpoints in \( V_i \), namely \( v_i \) and one of its neighbors in \( V_i \). Let

\[
\mathcal{F}(v) := \{ \text{maximum set of pairwise edge-disjoint } K_{t_i}'s \text{ in } G_{n,p}, \text{ which contain } v \text{ and one of its neighbors in each } (V_j)_{j=1}^{t-1}, \}
\]

and set \( m_0 = |\bigcup_{v \in X_2^V} \mathcal{F}(v)|. \) Note that \( |\mathcal{F}(v)| \) equals the number of pairwise edge-disjoint \( K_{t_i}'s \) between the neighborhoods of \( v \), which are all of size at least \( \varepsilon pn \). By applying Proposition 3.31 we see that a.a.s. \( f_i \geq C_\ell p \cdot (\varepsilon pn)^2 \), for some \( C_\ell > 0. \) Then we perform a similar calculation as in (3.9), and it remains to estimate \( |\mathcal{F}(v) \cap \mathcal{F}(w)|. \)

Observe that \( |\mathcal{F}(v) \cap \mathcal{F}(w)| \) can be crudely bounded from above by the number of edges between the common neighborhoods of \( v \) and \( w \) in \( V_2, \ldots, V_{t+1}. \) But this neighborhoods have size at most \( 2p^2 n \) (as a.a.s. no two vertices have a larger common neighborhood in \( G_{n,p} \)), and the number of edges between any two sets of vertices of at most this size in \( G_{n,p} \) is a.a.s. \( \leq 2p(2p^2 n)^2 \), which implies \( |\mathcal{F}(v) \cap \mathcal{F}(w)| \leq 4\ell^2 p(p^2 n)^2. \) By choosing \( c_\ell = \frac{C_\ell}{8p^2} \) we see that \( m_0 \geq \frac{16}{16c_\ell} \varepsilon^4 pn^2, \) which contradicts the fact \( m_0 \leq \varepsilon^5 pn^2, \) whenever \( \varepsilon \) is sufficiently small.

Next we show how (ii) can be proved. Assume that the number of horizontal edges is maximized in \( V_3. \) We adapt the definition of a “chord”: here, a chord consists of \( \ell \) vertices \( V = \{x, y, v_2, \ldots, v_{t-1}\} \), such that \( \{x, y\} \) is a horizontal edge in \( V_3 \), and \( v_i \in V_3. \) Additionally, one of the edges joining vertices in \( V \) (except for \( \{x, y\} \)) is a missing edge, i.e., it is contained in \( G_{n,p} \) but not in \( F \), and all other edges are in \( G_{n,p}. \) See Figure 3.5 for an illustration.

Our objective is to derive an upper and a lower bound for the number \( N \) of chords, which will immediately imply the bound on \( |H_3| \) claimed in (ii). We begin with the lower bound. Note that the number of triangles in \( G_{n,p} \) connecting \( x, y, \) and a vertex \( v_2 \) in \( V_2 \setminus X \) is due to our assumptions for sufficiently large \( n \) at least \( \frac{3p^2 (|V_2| - |X|)}{4} \geq \frac{3p^2 n}{4\ell}, \) except for at most \( O\left( \frac{p^{-2}}{2} |X| \right) \) pairs \( x, y \) (as due to Lemma 3.21 there is a set \( Z \) of \( O(p^{-2}) \) vertices such that every other pair in \( V_3 \) has at least the claimed number of
neighbors). Hence, the number of triangles connecting any edge in $V^3_i$ with a vertex $v_2 \in V^2_2$ is at least

$$\left( \left( \frac{|H_3|}{\ell} - O(p^{-2}n) \right) \right)^{3p^2n} \geq \frac{|H_3| \cdot p^{2n}}{4\ell^2},$$

(3.50)
as we may assume that $|H_3| > p^{-2}n \log n$. Similarly, the number of $K^3_s$'s connecting $x, y, v_2$ with a $v_3 \in V^3_3$ is at least $\frac{3p^3n}{4\ell^2}$, except for at most $O(\left( \frac{p^2}{2} \right) + p^{-3}n^2)$ triples $x, y, v_2$. Exploiting (3.50) we obtain that the number of $K^3_s$'s with an edge in $V^3_i$, a vertex in $V^2_2$, and a vertex in $V^3_3$ is at least

$$\left( \left( \frac{|H_3| \cdot p^{2n}}{2\ell^2} - O(p^{-3}n^2) \right) \right)^{3p^2n} \geq \frac{|H_3| \cdot p^5n}{2^2\ell^3}. $$

This process can be continued to count $K^3_s$'s, $K^3_s$'s and so on. We obtain inductively with plenty of room to spare that the number of $K^3_s$'s is at least

$$|H_3| \cdot \frac{n^{\ell-2}p^{\ell-1}}{(2\ell)^\ell},$$

(3.51)

which provides us with the desired lower bound for the number of chords, as in every counted copy of the $K^3_s$ there is a missing edge in $F$.

To obtain an upper bound for $N$ we partition the set of chords into three classes and derive upper bounds for each one separately.

(A) Chords where $d_{H_3}(x) \leq p^{-2}\log n$, and there is a $2 \leq i \leq \ell - 1$ such that $(x, v_i)$ is missing. We count those as follows. Let $x \in V^3_i$ and $y \in \Gamma(F; x) \cap V^3_i$. Then an upper bound for the number of chords with $x$ and $y$ is given by $\ell^2$ multiplied with the number of $K^3_s$'s in $G_{n,p}$, that have one vertex in each $V^3_i$ ($2 \leq i \leq V^3_{\ell-1}$) and $x, y$. To estimate this number we use a similar argument as above: the number of $K^3_s$'s with $x, y$ and a vertex $v_2 \in V_2$ is at most $2p^3n$, as no two vertices have a larger neighborhood in $G_{n,p}$. Similarly, the number of $K^3_s$'s with $x, y, v_2$ and $v_3 \in V_3$ is at most $2p^3n$, and so on. Putting everything together yields that the number of (A)-chords is at most

$$\sum_{x \in V^3_i} \sum_{y \in \Gamma(F; x) \cap V^3_i} (2p^2n) \cdot (2p^3n) \cdots (2p^{\ell-1}n)$$

$$\leq |V^3_i| \cdot p^{-2}\log n \cdot (2n)^{\ell-2}p^{\ell-1} \leq (2n)^{\ell-2}p^{\ell-3}\log n.$$

(B) Chords where $d_{H_3}(x) \geq p^{-2}\log n$, and there is a $2 \leq i \leq \ell - 1$ such that $(x, v_i)$ is missing. Let $S = \Gamma(G_{n,p}, x) \setminus (V^3_i \cup \Gamma(F; x))$ and note that $d_M(x) = |S|$. Due to our assumptions $G_{n,p}$ is such that except of at most $Cp^{-1}\log n$ vertices, every vertex in $S$ has at most $\frac{5}{4}pd_{H_3}(x)$ neighbors in $\Gamma(F; x) \cap V^3_i$. Hence, the number of $K^3_s$'s with $x$, a neighbor of $x$ in $V^3_i$ and a vertex adjacent to a missing edge at $x$ is at most

$$d_M(x) \cdot \frac{5}{4}pd_{H_3}(x) + Cp^{-1}\log n \cdot d_{H_3}(x) \left\{ d_M(x) \leq 6\epsilon n \right\} \leq 8\epsilon p^2n \cdot d_{H_3}(x).$$

With a similar argument as in (A) we easily see that the number of $K^3_s$'s with $x$, a neighbor of $x$ in $V^3_i$, a vertex adjacent to a missing edge at $x$, and a vertex in
any of \( V_3 \)'s (different from the ones where a vertex is already taken from) is at most \( 2p^3n \), as this is the maximum number of common neighbors of any three vertices in \( G_{n,p} \). In the same way we can count \( K_5 \)'s, \( K_6 \)'s, etc. To conclude, the number of (B)-chords is at most

\[
\sum_{x \in V_1^3, d_{H_3}(x) > p^{-2}\log n} 8\varepsilon p^2n \cdot d_{H_3}(x) \cdot (2p^3n) \cdot \ldots \cdot (2p^{\ell-1}n) \leq 8\varepsilon(2n)^{\ell-2}p^{\ell-1}_1 \cdot |H_3|.
\]

\( (C) \) Chords with the property that there are indexes \( 2 \leq i,j \leq \ell-1 \) such that \( \{v_i,v_j\} \) is missing. In the sequel we assume that \( G_{n,p} \) is such that for every subset \( \mathcal{U} \) of the vertex set of size at least \( p^{-2}\log^2 n \) there is a set \( Z\mathcal{U} \) of at most \( p^{-3}\log n \) vertices such that every vertex pair in \([n] \setminus (\mathcal{U} \cup Z\mathcal{U}) \) has more than \( \frac{1}{2}p^2|\mathcal{U}| \) and less than \( \frac{3}{4}p^2|\mathcal{U}| \) common neighbors in \( \mathcal{U} \) (this statement is very similar to Lemma 3.22, and the proof is omitted).

We count (C)-chords as follows. Let \( e = \{x,y\} \in H_3 \) and \( v \in V_3 \) be one of the common neighbors of \( x,y \) that is one of the endpoints of the missing edge in the chord. Note that there are at most \( 2p^2n \) candidates for \( v \). Moreover, let \( M_v \) be the set of non-exceptional missing neighbors of \( v \), i.e., the set of vertices incident to missing edges \( e = \{v,w\} \), where \( w \in V_3 \). Now let \( v' \in M_v \cap \Gamma(G_{n,p}; x) \cap \Gamma(G_{n,p}; y) \). Observe that due to our assumptions the following is true.

- If \( |M_v| \geq p^{-2}\log^2 n \), then there are at most \( \frac{3}{2}p^2|M_v| \) ways to choose \( v' \), except for at most \( (p^{-3}\log n) + n \cdot p^{-3}\log n \leq 2np^{-3}\log n \) pairs \( x,y \).
- Otherwise, there are at most \( p^{-2}\log^2 n \) ways to choose \( v' \).

Without loss of generality let \( x,y \in V_1^3, v \in V_2^3 \) and \( v' \in V_3^3 \). Having chosen \( x,y,v,v' \), the number of \( K_3 \)'s containing those vertices, and one vertex in each of \( V_1, \ldots, V_{\ell-1} \) is at most \( (2n)^{\ell-4}p^{\ell-6}_1 \) (this is seen by exactly the same counting argument that we have already used in (A) and (B)). Putting everything together yields that the number of (C)-chords is at most

\[
\sum_{e \in H_3} \sum_v \sum_{v'} (2n)^{\ell-4}p^{(\ell-6)}_1
\]

\[
\leq \sum_{e \in H_3} \sum_v \left( \frac{3}{2}p^2|M_v| + p^{-2}\log^2 n \right) \cdot (2n)^{\ell-4}p^{(\ell-6)}_1 + \sum_v \sum_{v'} 2np^{-3}\log n \cdot (2n)^{\ell-4}p^{(\ell-6)}_1
\]

\[
\leq |H_3| \cdot 6\varepsilon p^2n \cdot \left( \frac{3}{2}p^2 \cdot 6\varepsilon pn + p^{-2}\log^2 n \right) \cdot (2n)^{\ell-4}p^{(\ell-6)}_1
\]

\[
+ n \cdot 2p^2n \cdot 2np^{-3}\log n \cdot (2n)^{\ell-4}p^{(\ell-6)}_1
\]

\[
\leq |H_3| \cdot 5\varepsilon(2n)^{\ell-2}p^{(\ell-1)}_1 + (2n)^{\ell-1}p^{(\ell-9)}_1 \log n.
\]

By combining the results from (A), (B) and (C) with (3.51) we see that \( |H_3| \) satisfies

\[
|H_3| \cdot \frac{n^{\ell-2}p^{(\ell-1)}_1}{(2\ell)^\ell} \leq |H_3| \cdot 13\varepsilon(2n)^{\ell-2}p^{(\ell-1)}_1 + 2(2n)^{\ell-1}p^{(\ell-9)}_1 \log n,
\]
from which (ii) follows readily for large $n$.

The proof of (iii) is identical to the proof of the analogous statement in Lemma 3.23, and we omit a detailed exposition. To see (iv), observe that the total number of horizontal edges in $F$ is due to (i)-(iii) at most

$$|H| \leq |H_3| + |X_1 \cup X_2 \cup X_3| n < 2p^{-\ell^2-2}n \log n.$$ 

Suppose that there is a vertex $v$ in $V_1^3$ with $d_{H_3}(v) \geq p^{-2\ell^2} \log n$ — we handle vertices in the other sets $V_2^3$ analogously. We will show that this implies that the number of missing edges is at least $2p^{-\ell^2-2}n \log n$, which contradicts the bound on $|H|$ derived above.

In order to give a lower bound for the number of missing edges, we estimate from below the maximum number of edge-disjoint $K_\ell$'s in $G_{n,p}$, which contain $v$, one of the vertices counted in $d_{H_3}(v)$, and $\ell - 1$ vertices in $\Gamma(F; v, V \setminus V_1)$, such that there is precisely one vertex in each $V_2, \ldots, V_{\ell-1}$. For this we count the maximum number of edge-disjoint $K_{\ell-1}$'s between the sets of vertices $\Gamma(F; v, V_2), \ldots, \Gamma(F; v, V_{\ell-1})$, and $\Gamma(F; v, V_1^3)$ in $G_{n,p}$.

Note that for $i \geq 2$ $d(F; v, V_i) \geq c_i pn$, for some $c_i > 0$ depending only on $\ell$, and due to our assumption $\Gamma(F; v, V_1^3) \geq p^{-2\ell^2} \log n$. We apply Proposition 3.32 with $k = \ell-1$, $V_1 = \Gamma(T; v, V_1^3)$, and $V_i = \Gamma(T; v, V_i)$ (truncated to their first $c_i pn$ vertices), which yields that there is a constant $c > 0$ such that there are at least $c p^2 d_{H_3}(x) n$ pairwise edge-disjoint copies of $K_{\ell-1}$ with one endpoint in each $\Gamma(F; v, V_1^3)$ and in each $\Gamma(F; v, V_i)$. But then the number of missing edges is $\geq c p^{-2\ell^2+2} \log n$, which completes the proof of (iv).

Next we prove (v). Let $m$ be the number of missing edges in $F$. Our aim is to show that $m \geq |H_3| \cdot \frac{p^{3\ell^2}}{\log n} n$, and hence $|H_3|$ must satisfy

$$|H_3| \cdot \frac{p^{3\ell^2}}{\log n} n \leq 2p^{-\ell^2-2}n \log n, \quad \text{as } |H| \leq 2p^{-\ell^2-2}n \log n.$$

This completes the proof of (v). To show the claimed bound for $m$ let $R$ be a matching of maximum cardinality in $H_3$. Note that using (iv) we obtain that $|R| \geq \frac{1}{2} |H_3| p^{2\ell^2} (\log n)^{-1}$. We now proceed in two steps. First, we bound from below the number of $K_\ell$'s in $G_{n,p}$, which contain an edge in $R$, and a vertex in each of the sets $V_2^3, \ldots, V_{\ell-1}^3$. In the second step, we estimate from above the maximum number of $K_\ell$'s, that contain any edge in $R$, and an additional (fixed) edge $e'$ connecting any two vertices in the sets $V_1^3$ and $V_j^3$, where $1 \leq i < j \leq \ell - 1$. By dividing these two numbers (and by dividing the result by $(\ell)$) we readily obtain a lower bound for the number of missing edges.

To obtain the first goal note that the number of $\tau_3$ of triangles with $e \in R$ and a vertex $v_2 \in V_2^3$ is at least $\frac{3}{4} p^2 |V_2^3| - |X_1 \cup X_2 \cup X_3| \geq c_3 p^2 n$, for some $c_3 > 0$. Similarly, the number $\tau_i$ of $K_i$'s with $e$ and $i - 2$ vertices $v_2 \in V_2^3, \ldots, v_{i-1} \in V_{i-1}^3$ is at least

$$\tau_i \geq \tau_{i-1} \cdot \left( \frac{3}{4} p^{i-1} \cdot |V_{i-1}| - |X_1 \cup X_2 \cup X_3| \right) \geq \cdots \geq c_i \cdot n^{i-2} p^{(\ell)^{-1}},$$

where $c_i$ depends only on $c_3$ and on $i$. Setting $i = \ell$ yields that the number of $K_\ell$'s in $G_{n,p}$, which contain an edge in $R$, and a vertex in each of the sets $V_2^3, \ldots, V_{\ell-1}^3$, is at
least $|R|c_{\ell}p^{(\ell_1)}\cdot n^{\ell_2} \geq |H_3| \cdot p^{3\ell^2-1} \cdot (\log n)^{-1} \cdot n^{\ell-2}$ (here we assume that the number of horizontal edges in $H_3$ in maximized in $V^i_j$).

To obtain the second goal we distinguish two cases: either $e'$ has a common endpoint with one edge in $R$, or $i \geq 2$, i.e., $e'$ joins vertices in $V^i_3$ and $V^j_3$, where $2 \leq i < j \leq \ell - 1$. In the former case, let us denote by $e$ the edge of $R$, to which $e'$ is adjacent to, and observe that $e$ is unique. This means that 3 vertices of the $K_i$'s that we want to count are specified (the one endpoint of $e$, the intersection of $e$ and $e'$, and the other end of $e'$); hence, the number of $K_i$'s with $e$ and $e'$ is at most $n^{\ell-3}$. In the latter case we want to count $K_i$'s in $G_{n,p}$ which have a vertex in $V^i_3$ and $V^j_3$, where $2 \leq i < j \leq \ell - 1$. These $K_i$'s have exactly one vertex in each $V^3_x$, such that $x \not\in \{1, i, j\}$, and two vertices in $V^3_1$, which are endpoints of an edge in $R$. As the number of indexes $x$ is $\ell - 4$, the number of $K_i$'s with $e'$ is at most $|R| \cdot n^{\ell-4} \leq n^{\ell-3}$ (observe that trivially $|R| \leq n$, as $R$ is a matching). Putting all together, the number of $K_i$'s, that contain exactly one edge in $R$, and an additional edge $e'$ connecting any two vertices in the sets $V^3_i$ and $V^3_j$, where $1 \leq i < j \leq \ell - 1$, is at most $n^{\ell-3}$.

By combining the last two results we conclude that the number of $K_i$'s, which contain exactly one edge from $R$ and are otherwise edge-disjoint, is at least

$$\frac{|H_3| \cdot p^{3\ell^2-1} \cdot n^{\ell-2}}{\log n} \cdot \frac{1}{2n^{\ell-3}} \cdot \frac{1}{\binom{\ell}{2}} = \frac{|H_3| \cdot p^{3\ell^2}}{\log n} \cdot \frac{n}{2},$$

which is the desired lower bound for the number of missing edges.

To complete the proof we show (vi). Let $d$ be the maximum degree of a vertex $v \in X$, and suppose that $d \geq p^{-2\ell^2} \log n$. Without loss of generality we may assume that $v \in V_1$. Then the degree of $v$ in every $V_i$ is also at least $d$, as otherwise the chosen partition would have not been maximal. By applying Proposition 3.31 we readily obtain that the number of edge-disjoint $K_{i-1}$'s joining the neighborhoods of $v$ is at least $cpd^2$, for a $c > 0$, which implies that there are at least that many missing edges.

On the other hand, the total number of horizontal edges is at most $|X|d + |H_3|$. By exploiting (i)-(v) this is at most $2p^{-\ell^2-2}\log nd + p^{-5\ell^2} \log^2 n$. Hence, $d$ satisfies

$$cpd^2 \leq 2p^{-\ell^2-2}\log nd + p^{-5\ell^2} \log^2 n,$$

from which we deduce $d \leq p^{-3\ell^2} \log n$. This completes the proof.

Proof of Theorem 3.6 Theorem 3.6 can be proved in a completely analogous way as Theorem 3.3 (see Section 3.4). The definitions of several events, as well as the partial ordering of graphs with respect to partitions of the vertex set all generalize in an obvious and natural way from bipartitions to $(\ell-1)$-partitions. We leave the straightforward details to the reader.
CHAPTER 4

Random $\ell$-colorable and $\ell$-colored Graphs

One of the most important problems in graph theory is the graph-coloring problem: given a graph $G$, determine its chromatic number $\chi(G)$, i.e., find the smallest number $\ell$ such that there is an assignment of one of $\ell$ distinct colors to each vertex in $G$ having the property that no two vertices connected by an edge are given the same color. From a complexity theory point of view it is already for $\ell \geq 3 \sqrt[n]{\mathcal{P}}$-hard to color properly $\ell$-colorable graphs [Kar72b]. On the other hand, considering the importance of the problem for graph theory itself and as a core problem within several practical applications, many researches tried to find algorithms which are efficient “on average”, although they are exponential in the worst case.

The first approach in this direction was made by Wilf [Wil84]. He described an algorithm that decides the question “$\chi(G) \leq \ell$” in constant expected time, assuming the uniform distribution on the set of all graphs. The reason for this at first sight surprising result is that almost all graphs contain a large number of $(\ell + 1)$-cliques. Therefore, with extremely high probability it is sufficient to investigate a small part of the graph in order to find an $(\ell + 1)$-clique, which is a certificate that the graph is not $\ell$-colorable. Moreover, as the cases in which such a clique is not found are so unlikely one can afford to spend exponential time to check whether the graph is indeed (not) $\ell$-colorable.

One criticism to the above approach is that it is overly “optimistic” in the sense that it is a priori assuming (in a probabilistic sense) that the graph is not $\ell$-colorable. A meaningful way to overcome this problem is to consider classes consisting of graphs that are colorable with $\ell$ colors. In this context, Turner [Tur88] investigated the class $G_{n,\ell}$ of all $\ell$-colorable graphs on $n$ vertices. He developed an algorithm that colors almost every graph in $G_{n,\ell}$ in polynomial time as long as $\ell \leq (1 - \varepsilon) \log n$. This result was extended by Prömel and Steger [PS92], who designed an algorithm that colors random graphs from the class of $K_{\ell+1}$-free graphs in linear expected time, and by Kučera [Kuč89], whose algorithm colors almost all graphs from $G_{n,\ell}$, where $\ell$ may be as large as $\frac{1}{196} \sqrt{n \log n}$.

In this chapter we study further properties of the class of all $\ell$-colorable graphs and related models, as we hope that this may shed some new light on the problem of designing a polynomial time algorithm that colors almost all $\ell$-colorable graphs optimally, where $\ell = \Omega(\sqrt{n \log n})$. The ultimate goal in this direction is to make progress towards answering the famous question of Karp [Kar84b], who asked if there is an algorithm that colors a given random graph with the minimal number of colors in expected polynomial time. Till today, this question remains widely open.

First, we investigate graphs drawn uniformly at random from $G_{n,\ell}$, where $\ell = \ell(n)$ may depend on $n$. A crucial property of the graphs in $G_{n,\ell}$ that turned out to be the key
feature for designing the algorithms in the papers [Tur88, Kuč98, PS92] mentioned above is the fact that almost all these graphs have a unique coloring with \( \ell \) colors (up to permutations of the colors). Our first main result in this chapter is the following result.

**Theorem 4.1.** The function \( \ell = \ell(n) = \frac{n}{2\log n + \alpha \sqrt{2\log n}} \) is the threshold for the property that a random \( \ell \)-colorable \( G \) graph is uniquely \( \ell \)-colorable:

- if \( \alpha > 1 \), then \( G \) is a.a.s. uniquely colorable with \( \ell \) colors,
- if \( \alpha < 0 \), then \( G \) is a.a.s. not uniquely colorable with \( \ell \) colors.

The second statement in the above theorem is not difficult to prove. It follows immediately from known results in classical random graph theory. Bollobás showed in his celebrated paper [Bol88], among other results, that the chromatic number of almost all graphs is in an interval \((\ell_1, \ell_2)\), where \( \ell_i = \frac{n}{2\log n - c_i \log \log n} \) and \( 0 < c_1 < c_2 \). Let \( \ell \) be as in Theorem 4.1, where \( \alpha < 0 \). Then it is easily seen that for sufficiently large \( n \) we have \( \ell \geq \ell_2 \), and hence almost all graphs have chromatic number less than \( \ell \). Consequently, these graphs have many colorings with \( \ell \) colors.

The first statement is much harder to prove. Let \( U C_{n,\ell} \subseteq G_{n,\ell} \) be the set of all \( n \)-vertex graphs which have a unique coloring with \( \ell \) colors (up to permutations). Then the first statement above can be rephrased as follows.

**Theorem 4.2.** Let \( \varepsilon > 0 \) and \( \ell = \ell(n) \) be an integer function which satisfies

\[
\ell(n) \leq \frac{n}{2\log n + (1 + \varepsilon)\sqrt{2\log n}},
\]

and let \( G \) be a graph drawn uniformly at random from \( G_{n,\ell} \). Then

\[
\Pr[G \in UC_{n,\ell}] = 1 - o(1).
\]

As already remarked, the property of unique colorability of random \( \ell \)-colorable graphs was studied already in a series of papers, where a similar statement was proved for \( \ell \) being smaller than \((1 - \varepsilon)\log n\) ([Tur88], \( \Theta\left(\sqrt{\frac{n}{\log n}}\right) \) ([Kuč89]) and \( \frac{n}{(1 + \varepsilon)\log n} \) ([PS95]).

Our results extend with some additional technical work to random \( \ell \)-colorable graphs with \( cn^2 \) many edges, but we do not elaborate on the details here.

4.0.1. Tools & Techniques. The inherent difficulty in studying properties of random graphs from \( G_{n,\ell} \) is that the edges are dependent. This difficulty has led many researchers to investigate the more approachable model of colored graphs: the essential difference is that every graph in \( G_{n,\ell} \) which has \( \ell \) different colorings with \( \ell \) colors, counts exactly \( \ell \) times as a colored graph (again, here we do not count colorings which can be obtained by permuting the colors). Therefore, an \( \ell \)-colored graph can be viewed as a tuple \((G, C)\), where \( C \) is a proper \( \ell \)-coloring, i.e., a (unordered) partition of the vertex set of \( G \) in \( \ell \) non-empty parts.

This notion of \( \ell \)-colored graphs was first introduced by Kučera [Kuč89], who developed in the same paper the previously mentioned coloring algorithm for random \( \ell \)-colorable graphs. His proof was based on the fact that the two models (\( \ell \)-colored and \( \ell \)-colorable
graphs) have many similarities. In the following years several papers dealt with investigating properties and with the coloring of (several variations of the model of) random \( \ell \)-colored graphs. Here of course the “planted” coloring is not known to the algorithm, and the goal is to determine any proper coloring of the given graph. We selectively mention a few papers from this area. In \([\text{AK97a}]\) Alon and Kahale designed an algorithm that colors a particular type of random \( \ell \)-colored graphs with high probability with \( \ell \) colors in polynomial time. Here \( \ell = \Theta(1) \) is assumed. Böttcher [Böt05] extended the ideas and constructed an algorithm with expected polynomial running time for the same model. More recently, Coja-Oghlan, Krivelevich and Vilenchik [COKV07] designed an algorithm that colors \( \ell \)-colorable graphs with \( n \) vertices and \( cn \) edges, where \( c \) is greater than some sufficiently large positive constant. Large parts of their analysis is based on the fact that actually the two classes of graphs share many structural properties (such as e.g. the existence of a single “cluster” of colorings).

Of course the model of random \( \ell \)-colored graphs is not the most natural model of random instances for coloring problems. However, as outline above, it turns out that in many cases the uniform model, and the model of \( \ell \)-colored graphs behave similarly, and the understanding of the latter helps us also to understand the structure of the former. In fact, as we shall see later, our proof of Theorem 4.2 is heavily based on a very strong structural property of random \( \ell \)-colored graphs, which is described below.

In order to state our results concerning \( \ell \)-colored graphs we require some notation. In the following we denote by \( \text{Col}_{n,\ell} \) the set of all \( \ell \)-colored graphs on \( n \) labeled vertices. Intuitively, we can “group” the graphs in \( \text{Col}_{n,\ell} \) according to the deviation of each color class from a completely balanced coloring, where each color class has \( \lceil \frac{n}{\ell} \rceil \) or \( \lfloor \frac{n}{\ell} \rfloor \) vertices. For this, let

\[
\mathcal{D}(f, \ell, n) = \left\{ (d_1, \ldots, d_\ell) \mid \forall i : d_i \in \mathbb{Z} \text{ and } \forall i : |d_i| \leq f \text{ and } \forall i : 1 \leq \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \leq n \text{ and } \sum_{i=1}^{\ell} d_i = r \right\}, \tag{4.1}
\]

where \( r = r(n, \ell) \) is the remainder of \( n \) when divided by \( \ell \). With this notation, the number of \( \ell \)-colored graphs, which have the property that the sizes of all of their color classes deviate at most \( f \) from \( \left\lfloor \frac{n}{\ell} \right\rfloor \) is precisely

\[
\text{col}_{n,\ell}(f) := \frac{1}{\ell!} \sum_{d \in \mathcal{D}(f, \ell, n)} \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_1, \ldots, \left\lfloor \frac{n}{\ell} \right\rfloor + d_\ell \right) \cdot 2^{\sum_{1 \leq i < j \leq \ell} (\left\lfloor \frac{n}{\ell} \right\rfloor + d_i)(\left\lfloor \frac{n}{\ell} \right\rfloor + d_j)}. \tag{4.2}
\]

This is seen as follows: let \( \mathcal{C} \) be the set of partitions of the vertex set into \( \ell \) parts, such that the \( i \)th part has size \( n_i = \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \) and \( |d_i| \leq f \) for all \( 1 \leq i \leq \ell \). Clearly, \( |\mathcal{C}| \) equals to the value of the multinomial coefficient. Moreover, the number of edges connecting vertices in different parts of any partition in \( \mathcal{C} \) is \( \sum_{1 \leq i < j \leq \ell} n_i n_j \), and for each edge we can choose whether to include it into the graph or not. Finally, if we sum over all admissible partitions we have counted in this way each \( \ell \)-colored graph precisely \( \ell! \) times, which proves (4.2). It follows that the number of all \( \ell \)-colored graphs is given by \( |\text{Col}_{n,\ell}| = \text{col}_{n,\ell}(n - \left\lfloor \frac{n}{\ell} \right\rfloor) \) (this result was already known to Wright [Wri61]).
In the remainder of the chapter we shall abbreviate for a vector \( d \)
\[
b(d) := \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_1, \ldots, \left\lfloor \frac{n}{\ell} \right\rfloor + d_\ell \right)
\] (4.3)
and similarly
\[
\log \mathcal{E}(d) := \sum_{1 \leq i \leq \ell} \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \right) \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \right),
\] (4.4)
where in this chapter the logarithms, if not stated otherwise, are to the base 2.

Our main result concerning colored graphs is the following theorem, which states that the number of all colored graphs is asymptotically of the same order of magnitude as the number of "very balanced" colored graphs; more precisely, if we draw a colored graph \( G \) uniformly at random from \( \text{Col}_{n,\ell} \), then with high probability the size of every color class of \( G \) will deviate roughly at most \( \sqrt{2 \log \ell} \) from \( \left\lfloor \frac{n}{\ell} \right\rfloor \), and there will be at least one color class with such deviation.

**Theorem 4.3.** There exist constants \( c_1, c_2 > 0 \) such that for all integer functions \( \ell(n) = o\left( \frac{n}{\sqrt{\log n}} \right) \) and \( \ell(n) = \omega(1) \), it holds
\[
\text{col}_{n,\ell}(f) = \begin{cases} 
\frac{\text{col}_{n,\ell}}{\ell} & \text{if } f < \sqrt{2 \log \ell - c_1 \cdot (\log \ell)^{2/3}} \\
(1 - o(1)) |\text{col}_{n,\ell}| & \text{if } f > \sqrt{2 \log \ell + c_2 \cdot (\log \ell)^{2/3}}.
\end{cases}
\] (4.5)

Theorem 4.3 gives a precise characterization of the balancedness of colored graphs and is therefore a strong generalization of the results in [Bol81], [Kuč89] and [PS95], where only the upper bound \( \text{col}_{n,\ell}(f) = (1 - o(1)) |\text{col}_{n,\ell}| \) for \( f = \epsilon n, f = \sqrt{n/\log n} \) and \( f = \epsilon \log n \) was shown. Furthermore, note that Theorem 4.3 scales smoothly with the considered chromatic number: the smaller \( \ell \) is, the lesser is the deviation of each colors class from \( n/\ell \) with high probability. Observe also that if \( \ell \) is in the range of the chromatic number of the random graph \( G_{n,1/2} \) (i.e., \( \ell \sim \frac{n}{2 \log n} \), see [Bol88]), the above theorem says that almost surely the maximum deviation of each color class from \( n/\ell \sim 2 \log n \) is \( \sim \sqrt{2 \log n} \), which is asymptotically much smaller than the size of every color class.

For constant \( \ell \) the first statement of the above theorem is not true. Instead, from previous work of Wright [Wri61], see Section 4.1.4, we obtain easily the following fact.

**Theorem 4.4.** If \( \ell \) is constant, then for any function \( \omega_n = \omega(1) \)
\[
\text{col}_{n,\ell}(f) = \begin{cases} 
(1 + o(1)) \cdot g(c) \cdot |\text{col}_{n,\ell}| & \text{if } f = c \\
(1 - o(1)) |\text{col}_{n,\ell}| & \text{if } f > \omega_n,
\end{cases}
\] (4.6)
where the function \( g \) satisfies \( 0 < g(c) < 1 \) for all \( c \geq 1 \) and \( g(0) = 0 \) if \( n \mod \ell > 0 \), otherwise \( 0 < g(0) < 1 \).

Furthermore, our results generalize to colored graphs with a fixed number of edges. For this, let \( \text{Col}_{n,M,\ell} \) denote the set of all \( \ell \)-colored graphs with precisely \( M \) edges and
\[
\text{col}_{n,M,\ell}(f) = \frac{1}{\ell!} \sum_{d \in \mathbb{D}(k,f,n)} b(d) \tilde{\mathcal{E}}(d), \text{ where } \tilde{\mathcal{E}}(d) = \left( \sum_{1 \leq i \leq \ell} \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \right) \left( \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \right) \right)^{M}.
\] (4.7)
The threshold reads for this case

**Theorem 4.5.** Let \( \epsilon > 0 \) be any positive constant and let \( M = cn^2 \), where \( 0 < c < \frac{1}{2} \). Moreover, let \( f(c) := \frac{2}{-\log(1-2c)} \). For all integer functions \( \ell(n) \leq \frac{n \log \frac{1}{c}}{\log n} \) with \( \ell(n) = o(1) \)

\[
\text{col}_{n,M,\ell}(f) = \begin{cases} 
\omega(|\text{Col}_{n,M,\ell}|), & f < (1-\epsilon)\sqrt{\log n \cdot f(c)} \\
(1-o(1))|\text{Col}_{n,M,\ell}|, & f > (1+\epsilon)\sqrt{\log n \cdot f(c)}.
\end{cases} 
\] (4.8)

In the above theorem, we can replace the \( \epsilon \) with an appropriate function which goes to zero when \( \ell \to \infty \), as we did in Theorem 4.3. For the sake of a better presentation, we choose the above (slightly weaker) formulation.

The remainder of this chapter is structured as follows. In Section 4.1 we prove Theorems 4.5 and 4.6, and we outline the proof of Theorem 4.5. Using these results, Section 4.2 deals with the proof of Theorem 4.2.

### 4.1. The Balancedness of \( \ell \)-Colored Graphs

#### 4.1.1. Preliminaries

We begin with a simple estimate for multinomial coefficients. In the following, \( 0, \) stands for a vector in \( \mathbb{Z}^\ell \) which has the property that all its entries are zero, except of \( r \), which are equal to one; note that there are precisely \( \binom{n}{r} \) such vectors.

**Proposition 4.6.** For every integer function \( \ell(n) = o(n) \) and every sequence \( d \in D(n - \left\lfloor \frac{n}{r} \right\rfloor, \ell, n) \)

\[
b(d) \leq b(0, r) \leq 2^{p(n, \ell) + o(\ell)},
\]

where

\[
p(n, \ell) := \left( n + \frac{\ell}{2} \right) \log \ell - \frac{\ell - 1}{2} \log n - \frac{\ell - 1}{2} \log 2\pi.
\] (4.9)

**Proof.** Let \( \alpha := \left\lfloor \frac{n}{r} \right\rfloor \), and denote by \( x^\mathbb{N} = x(x-1)\ldots(x-y+1) \) and \( x^\mathbb{P} = x(x+1)\ldots(x+y-1) \). Using the definition of \( b \) we obtain

\[
\frac{b(0, r)}{b(d)} = \prod_{i=1}^{\ell} \frac{(\alpha + d_i)!}{(\alpha + 1)! \cdot (\alpha!)^{\ell-r}} = \frac{1}{(\alpha + 1)^r} \cdot \frac{\prod_{\{i : d_i > 0\}} (\alpha + 1)^{d_i}}{\prod_{\{i : d_i < 0\}} \alpha^{-d_i}}
\]

\[
\geq \frac{1}{(\alpha + 1)^r} \cdot \frac{\prod_{\{i : d_i > 0\}} (\alpha + 1)^{d_i}}{\prod_{\{i : d_i < 0\}} \alpha^{-d_i}} \geq \frac{1}{(\alpha + 1)^r} \cdot (\alpha + 1)^r = 1,
\]

as \( \sum_{\{i : d_i < 0\}} d_i + \sum_{\{i : d_i > 0\}} d_i = r \). The inequality \( 2^{p(n, \ell) + o(\ell)} \geq b(0, r) \) can be proved by a straightforward application of Stirling’s formula; we skip the calculations. \( \square \)

For a vector \( d = (d_1, \ldots, d_\ell) \) recall that \( E(d) \) denotes the number of \( \ell \)-colored graphs with a fixed coloring \( (V_1, \ldots, V_\ell) \), where \( |V_i| = \left\lfloor \frac{n}{r} \right\rfloor + d_i \). The next statement relates this number to the value of the vector product \( d^T d = \sum_{i=1}^{\ell} d_i^2 \), and is a fact that will be used several times.
Proposition 4.7. For every \( n \), every integer function \( \ell(n) \leq n \) and every integer sequence \( d \in D(n - \left\lfloor \frac{n}{\ell} \right\rfloor, \ell, n) \)

\[
\log E(d) = \frac{\ell - 1}{2\ell} n^2 + \frac{r^2}{2\ell} - \frac{1}{2} d^T d =: e(n, \ell) - \frac{1}{2} d^T d,
\]

(4.10) where \( \log E(d) \) is defined in (4.4) and \( r = r(n, \ell) \) is the remainder of \( n \) and \( \ell \).

Proof. Note that \( \left\lfloor \frac{n}{\ell} \right\rfloor = \frac{n - r}{\ell} \). By using the definition of \( E \) we obtain

\[
\log E(d) = \left( \frac{\ell}{2} \right) \left( \frac{n - r}{\ell} \right)^2 + \frac{n - r}{\ell} \sum_{1 \leq i \leq \ell} (d_i + d_j) + \sum_{1 \leq i \leq j \leq \ell} d_i d_j.
\]

Denote the first sum by \( S_1 \) and the the second one by \( S_2 \). By double counting we obtain

\[
2S_1 + 2 \sum_{i=1}^{\ell} d_i = \sum_{i,j} (d_i + d_j) = 2\ell r, \text{ which shows } S_1 = (\ell - 1)r. \text{ Furthermore, we obtain } 2S_2 + \sum_{i=1}^{\ell} d_i^2 = \sum_{i,j} d_i d_j = r^2, \text{ i.e., } S_2 = \frac{r^2}{2} - \frac{1}{2} d^T d. \text{ We conclude}
\]

\[
\log E(d) = \left( \frac{\ell}{2} \right) \left( \frac{n - r}{\ell} \right)^2 + \frac{n - r}{\ell} (\ell - 1)r + \frac{r^2}{2} - \frac{1}{2} d^T d = \frac{\ell - 1}{2\ell} n^2 + \frac{r^2}{2\ell} - \frac{1}{2} d^T d.
\]

\[
\square
\]

Recall that the number of \( \ell \)-colored graphs with a fixed coloring that has the property that the \( i \)-th color class has size \( \left\lfloor \frac{n}{\ell} \right\rfloor + d_i \), for \( 1 \leq i \leq \ell \), is given by \( b(d) E(d) \), where \( d = (d_1, \ldots, d_\ell) \). Hence, the above proposition states that the larger \( d^T d \) is, the fewer corresponding colored graphs exist. The next statement makes this more precise.

Proposition 4.8. For all integer functions \( \ell = \ell(n) \) satisfying \( \ell(n) = \omega(1) \) and \( \ell(n) = o(n) \)

\[\frac{1}{\ell!} \sum_{d \in D(n - \left\lfloor \frac{n}{\ell} \right\rfloor, \ell, n) \atop d^T d \geq 18\ell} b(d) E(d) = o \left( |\text{Col}_{n, \ell}| \right).\]

Proof. By combining the statements of Propositions 4.7 and 4.6 we obtain

\[
S_C \leq \frac{1}{\ell!} b(0) e^{e(n, \ell)} \cdot \sum_{d^T d \geq 18 \ell} 2^{\frac{1}{\ell} d^T d}.
\]

(4.11)

We now determine an upper bound for the number of vectors \( d \) with \( d^T d = x \), for every \( x \geq 18\ell \). For this we count the number of solutions of the equation \( \sum_{i=1}^{\ell} q_i = x \), where \( q_i \in \mathbb{N}_0 \). It is a well-known fact that the number of such solutions is \( \binom{x + \ell - 1}{x} \) (to see this, first observe that the number of solutions equals the number of solutions of the equation \( \sum_{i=1}^{\ell} p_i = x + \ell, \) where \( p_i \in \mathbb{N}_0 \); then write \( x + \ell \) as the sum “1 + 1 + · · · + 1” ones, and note that any solution to \( \sum_{i=1}^{\ell} p_i = x + \ell \) can be obtained by choosing without replacement \( \ell - 1 \) out of the \( x + \ell - 1 \) “+”’s). From these solutions we pick the ones with the property that all \( q_i \)’s are square numbers. Then we choose for every \( 1 \leq i \leq \ell \) the sign of the corresponding \( d_i \); there are \( 2^\ell \) ways of doing this. Hence,

\[
\frac{\ell! \cdot S_C}{b(0) e^{e(n, \ell)}} \leq 2^\ell \sum_{x \geq 18\ell} \binom{x + \ell}{x} 2^{\frac{1}{\ell} - x} \leq 2^\ell \sum_{x \geq 18\ell} \left( \frac{e(x + \ell)}{\ell} \right)^x 2^{\frac{1}{\ell} - x} \leq (2e)^\ell \sum_{x \geq 18\ell} 2^{\log \frac{x + \ell}{\ell} - x},
\]
Write \( x = f_x \cdot \ell \); the exponent of two in the sum becomes \( \ell (\log (f_x + 1) - f_x/2) \). It can be easily seen that this function is always smaller than \( -\ell f_x/4 \) whenever \( f_x \geq 18 \). Putting all together yields

\[
\frac{\ell! \cdot S_C}{b(0, r) 2^{e(n, \ell)}} \leq (2e)^\ell \sum_{x \geq 18\ell} 2^{-x/4} \leq (2e)^\ell \cdot 2^{-18\ell} \sum_{x \geq 0} 2^{-x/4} \leq 2^{-2\ell},
\]

where the last step is true for sufficiently large \( \ell \) due to \( \log(2e) - \frac{18}{4} < -2 \).

We obtain that \( S_C \leq \frac{1}{\ell!} b(0, r) 2^{e(n, \ell)} \cdot 2^{-2\ell} \); the proof completes with the fact \(|\text{Col}_n, \ell| > \frac{1}{\ell!} b(0, r) E(0, r) > \frac{1}{\ell!} b(0, r) 2^{e(n, \ell) - 1}\), where the last step is again due to Proposition 4.7. \( \square \)

The above statement excludes a large class of possible choices for the vector \( d \) – it suffices to look at colored graphs with the property \( \frac{1}{2} d^T d < 9\ell \). The following proposition states that these colored graphs are “almost” balanced, i.e., the number of “large” \( d_i \)’s is “small”. Let

\[
T_g(d) := \{ 1 \leq i \leq \ell : |d_i| \geq g \},
\]

where \( g \geq 1 \) is any function which may depend on \( \ell \) and \( n \). We will use this notation throughout the paper without further reference.

**Proposition 4.9.** Let \( d \in D \left( n - \left\lfloor \frac{n}{\ell} \right\rfloor, \ell, n \right) \) and \( d^T d < 18\ell \). For every \( g \geq 1 \)

\[
|T_g(d)| \leq 18\ell g^{-2}.
\]

**Proof.** We readily obtain

\[
18\ell > d^T d = \sum_{i=1}^{\ell} d_i^2 \geq |T_g(d)| \cdot g^2,
\]

which implies the statement. \( \square \)

Assume that \( \ell(n) = \omega(1) \). Due to the above statement, for every function \( g = g(n) = \omega(1) \), the number of elements in a given vector \( d \) that satisfies \( \frac{1}{2} d^T d \leq 9\ell \), which are larger than \( g \) in absolute value, is \( o(\ell) \). Therefore, provided that we restrict our attention to colored graphs with a ”small” value of \( \frac{1}{2} d^T d \), we find \((1 - o(1))\ell \) color classes, which deviate less than \( g \) from \( \lfloor n/\ell \rfloor \); this observation will be one of the main ingredients in the following proofs.

4.1.2. Proof of Theorem 4.3 – First Statement. In this section we will prove the first statement of Theorem 4.3. Let \( c_1 = 16\sqrt{60} \) and \( f_0 = \sqrt{2\log \ell - c_1 \cdot (\log \ell)^{1/2}} \); with this notation, the claimed statement is

\[
\text{col}_{n, \ell}(f_0) = o\left(|\text{Col}_{n, \ell}|\right).
\]

Let us describe informally the proof idea before we give the necessary technical details. Observe that \( \text{col}_{n, \ell}(f_0) \) is the sum over all vectors \( d \in D \left( f_0, \ell, n \right) \) of the terms \( b(d) \text{E}(d) \), divided by \( \ell! \). Our aim will be to collect for every such \( d \) a set of vectors \( m(d) \in D \left( n - \left\lfloor \frac{n}{\ell} \right\rfloor, \ell, n \right) \) such that \( m(d) \cap m(d') = \emptyset \) for distinct \( d, d' \). This mapping should have the additional property that \( b(d) \text{E}(d) \) is asymptotically negligible compared to the corresponding sum \( S(d) := \sum_{d \in m(d)} b(d) \text{E}(d) \). Having this, we can obtain an upper
bound for \( \text{col}_{n,\ell}(f_0) \) by summing over the union of the sets \( m(d) \), where \( d \in \mathcal{D}(f_0, \ell, n) \), and by multiplying the resulting sum by the maximum of the ratio \( \frac{b(d) E(d)}{S(d)} \), which is by construction \( o(1) \). As this last sum is at most the number of all \( \ell \)-colored graphs, the proof will be completed.

Recall the definitions of \( b \) and \( E \) from (4.3) and (4.4). In order to formalize the above idea, let

\[
\omega_\ell = 16c_{1/4}^{1-1/2}\ell^{2/3} \tag{4.14}
\]
denote an auxiliary function. We define a \textit{mapping} \( m \) which maps a given \( d \in \mathcal{D}(f_0, \ell, n) \) with \( d^T d < 18\ell \) onto a set of vectors \( m(d) \subseteq \mathcal{D}(n - [n/\ell], \ell, n) \), such that the following three properties are fulfilled for \( \ell \) sufficiently large:

(i) \( m \) is injective: for all admissible \( d, d' \) such that \( d \neq d' \) we have \( m(d) \cap m(d') = \emptyset \),

(ii) \( \lvert m(d) \rvert \geq \frac{\ell}{2} \), i.e., \( m(d) \) is large and

(iii) for all \( d \in m(d) \) it holds \( b(d) E(d) \geq b(d) E(d) \cdot 2^{-\frac{1}{2}f_0^2(1+\frac{8}{\omega_\ell})} \).

Let \( S(d) = \sum_{a \in m(d)} b(a) E(a) \). With (ii) and (iii) we obtain for all \( d \in \mathcal{D}(f_0, \ell, n) \)

\[
\frac{b(d) E(d)}{S(d)} \leq 2 \cdot 2^{-\frac{1}{2}f_0^2(1+\frac{8}{\omega_\ell})} \ell = o(1),
\]

where the last step is an immediate consequence of the definitions of \( f_0 \) and \( \omega_\ell \). The theorem can be proved with (4.2) and (i) as follows:

\[
\text{col}_{n,\ell}(f_0) = \frac{1}{\ell!} \cdot \sum_{d \in \mathcal{D}(f_0, \ell, n), d^T d \geq 18\ell} b(d) E(d) + \frac{1}{\ell!} \cdot \sum_{d \in \mathcal{D}(f_0, \ell, n), d^T d < 18\ell} b(d) E(d)
\]

Prop. 4.8

\[
\leq \left( o(1) + \max_{d \in \mathcal{D}(f_0, \ell, n), d^T d < 18\ell} \frac{b(d) E(d)}{S(d)} \right) \cdot |\text{col}_{n,\ell}| = o(|\text{col}_{n,\ell}|). \tag{4.15}
\]

This completes the proof of the theorem, provided that we find a mapping that satisfies (i)-(iii). The rest of this section is devoted to this task. Consider for a given vector \( d = (d_1, \ldots, d_\ell) \in \mathcal{D}(f_0, \ell, n) \) with \( d^T d < 18\ell \) the subset of indexes

\[
P(d) = \left\{ 1 \leq i \leq \ell - f_0 - 1 : \left\{ i \leq j \leq i + f_0 + 1 : |d_j| \geq \frac{f_0}{\omega_\ell} \right\} \subseteq \left\{ 1 \leq j \leq f_0 \right\} \right\}. \tag{4.16}
\]

Let \( g_k := \text{sign}(d_k) \cdot (f_0 + 1) \) and \( s_k := -\text{sign}(d_k) \), where we assume \( \text{sign}(0) = +1 \). \( m \) maps a given vector \( d \) onto the set of vectors

\[
\left\{ \ldots, d_k - 1, d_k + g_k, d_{k+1} + s_k, \ldots, d_k + |g_k| + s_k, d_{k+|g_k|+1}, \ldots \right\}
\]

\[
k \in P(d) \text{ and } |d_k| \leq \frac{f_0}{\omega_\ell}. \tag{4.17}
\]

It is easy to verify that for all pairs \( d, d' \in \mathcal{D}(f_0, \ell, n) \) of distinct vectors we have \( m(d) \cap m(d') = \emptyset \). To see this, let \( \tilde{d} \in \cup m(d) \), where the union is over all \( d \in \mathcal{D}(f_0, \ell, n) \) such that \( d^T d < 18\ell \). Note that \( \tilde{d} \) has a unique element which is larger than \( f_0 + 1 \) in absolute value. Let us assume that this element is at position \( i \), where \( i < \ell - f_0 \). To
invert the mapping, decrease $|\tilde{d}_i|$ by $f_0 + 1$, and modify the elements $\tilde{d}_{i+1}, \ldots, \tilde{d}_{i+f_0+1}$ by $+1/-1$ if $\tilde{d}_i < 0/\tilde{d}_i > 0$. Hence, no integer sequence is constructed more than once by $m$, which shows that property (i) is fulfilled by $m$. Furthermore, note that every element in $m(d)$ is an admissible integer sequence, in the sense that the invariant $\sum_{i=1}^{\ell} d_i = r(n, \ell)$ is preserved, where $r(n, \ell)$ stands for the remainder of $n$ when divided by $\ell$.

Next we show that (ii) is fulfilled for $m$, i.e., $|\{k \in P(d) \text{ and } |d_k| \leq \frac{f_0}{\omega}\}| \geq \ell/2$. Note that it suffices to show that $|P(d)| \geq \frac{2\ell}{3} - f_0$; we then can conclude with Proposition 4.9 that the number of entries of $d$, which are larger than $\frac{f_0}{\omega}$ in absolute value, fulfills for large enough $\ell$ the claimed bound.

**Proposition 4.10.** Let $d \in D(\omega, \ell, n)$ with $d^T d < 18\ell$. Then $|P(d)| \geq \frac{2\ell}{3} - f_0$.

**Proof.** We shall prove the statement by contradiction; for this we determine bounds for the quantity $|T_{f_0}/\omega\ell(d)| = \{|1 \leq i \leq \ell : |d_i| \geq f_0/\omega\ell\}$.

First, note that Proposition 4.9 guarantees that $|T_{f_0}/\omega\ell(d)| \leq 18\ell\omega^2/f_0^2$. On the other hand, let $B(d) := \{1, \ldots, \ell - f_0 + 1\} \setminus P(d)$ and assume that $|B(d)| > \frac{\ell}{3} = 20\ell\omega^2/f_0^2$. Let $x \in B(d)$; then due to the definition of $B$ and (4.16) there are at least $f_0/\omega\ell$ indexes $i$ in the interval $[x, \ldots, x + f_0 + 1]$ with $f_0 \geq |d_i| \geq f_0/\omega\ell$. With the notation of (4.12), we obtain with the above assumptions for sufficiently large $\ell$ the following lower bound for the number of entries in $d$, that are at least $f_0/\omega\ell$ in absolute value:

$$
\left|T_{f_0}/\omega\ell(d)\right| \geq \sum_{x \in B(d)} \frac{f_0}{\omega\ell} \geq \frac{19 |B(d)|}{20 \omega\ell} > \frac{19\ell\omega^2}{f_0^2},
$$

where the first inequality follows from the fact that we count each "large" $d_i$ at most $f_0 + 2$ times. But this is a contradiction to the fact $|T_{f_0}/\omega\ell(d)| \leq 18\ell\omega^2/f_0^2$.

We conclude that $|B(d)| \leq 20\ell\omega^2/f_0^2 = \frac{\ell}{3}$, which proves the statement. \hfill \Box

Finally we show that (iii) holds for the mapping $m$ defined above. Let for the moment $\tilde{d} \in m(d)$ be a vector which differs from the $d$ at the $k$-th position by $f_0 + 1$ and at the positions $k+1, \ldots, k+f_0+1$ by $-1$ (the symmetric case can be handled in the same way, where we negate all signs). With Proposition 4.7 and the definition of the multinomial coefficient we obtain

$$
b(\tilde{d})E(\tilde{d}) = b(d)E(d) \cdot \prod_{i=1}^{f_0+1} \left(\frac{n}{\ell} + d_{k+i} \right) \cdot \frac{1}{(\frac{n}{\ell} + d_k + f_0 + 1)^{f_0+1}} \cdot 2^{-\frac{1}{2}(f_0+1)\sum_{k=k+1}^{k+f_0+1} d_k - \frac{f_0^2}{2}}. \quad (4.18)
$$

Recall the definition (4.16) of $P(d)$. As $k \in P(d)$, there are at most $f_0/\omega\ell$ indexes $i_k$ in the interval $[k+1, k+f_0+1]$ with $f_0 \geq |d_{i_k}| \geq f_0/\omega\ell$. Therefore, for large $\ell$

$$
\sum_{x=k+1}^{k+f_0+1} d_x \geq \left((f_0 + 1) - \frac{f_0}{\omega\ell} \right) \cdot \left(-\frac{f_0}{\omega\ell} \right) + \frac{f_0}{\omega\ell} \cdot (-f_0) \geq -2f_0^2/\omega\ell.
$$

With this, the exponent of the power-of-2 term in (4.18) can be (crudely) bounded from below with $-\frac{1}{2}f_0^2(1 + \frac{2}{\omega\ell})$, as due to (4.17) we know that $|d_{i_k}| \leq f_0/\omega\ell$. In order to give
a lower bound for the term with the product in (4.18) observe that \( f_0 = o\left(\lceil n/\ell \rceil \right) \) due to the assumptions on \( \ell \); therefore, \( |d_i| = o\left(\lceil n/\ell \rceil \right) \) holds for all \( 1 \leq i \leq \ell \). We obtain

\[
\frac{\prod_{i=1}^{f_0+1} \left(\frac{n}{\ell} + d_{k+i}\right)}{(\frac{n}{\ell} + d_k + f_0 + 1)^{f_0+1}} \geq \left(\frac{3}{2} \frac{n}{\ell} \right)^{f_0+1} \geq 2^{-f_0+1}.
\]

Putting all together yields \( b(\tilde{d})E(\tilde{d}) \geq b(d)E(d) \cdot 2^{-\frac{1}{2} f_0 (1 + \frac{c_2}{\omega(n)})} \), i.e., \( m \) fulfills (iii).

### 4.1.3. Proof of Theorem 4.3 – Second Statement

In this section we will prove the second statement of Theorem 4.3. Let \( c_2 = 2^{14} \) and \( f_0 = \sqrt{2 \log \ell + c_2 (\log \ell)^{2/3}} \); with this notation, the claimed statement is

\[
\text{col}_{n, \ell}(f_0) = (1 - o(1)) |\text{Col}_{n, \ell}|.
\]

We write throughout \( \ell(n) \leq \frac{n}{\alpha(n)} \), where \( \alpha(n) = \omega(\sqrt{\log n}) \).

Before we proceed with the proof of Theorem 4.3 let us make an important preparation. Our aim is to exclude a large class of admissible vectors \( d \), i.e., we show that the number of corresponding \( \ell \)-colored graphs is negligible.

**Proposition 4.11.** Let \( d \in D(\lfloor n / \ell \rfloor, \ell, n) \). Then

\[
\left| \sum_{i : |d_i| \geq f_0} d_i \right| \geq \frac{19\ell}{f_0} \Rightarrow d^T d > 18\ell.
\]  

**Proof.** Consider the dot product \( d^T d \). We can readily obtain a lower bound for it with the assumption on magnitude of the sum of the \( d_i \)'s that are larger than \( f_0 \) in absolute value:

\[
d^T d \geq \sum_{i : |d_i| \geq f_0} d_i^2 \geq \sum_{i : |d_i| \geq f_0} d_i \cdot \min_{i : |d_i| \geq f_0} |d_i| \geq \frac{19\ell}{f_0} \cdot f_0 = 19\ell.
\]

Combining the above statement with Proposition 4.8 we easily see that the number of \( \ell \)-colored graphs, which correspond to integer sequences satisfying the precondition of (4.19), is \( o(|\text{Col}_{n, \ell}|) \).

In the remainder we prove the second part of Theorem 4.3. We partition the set of all integer sequences, which contain at least one value larger than \( f_0 \) according to the positions of that values and according to their magnitude. More precisely, recall (4.12), fix a non-empty index set \( T \subseteq \{1, \ldots, \ell \} \) and an integer sequence \( x = (x_1, \ldots, x_{|T|}) \) with \( |x_j| \geq f_0 \) for all \( 1 \leq j \leq |T| \), and consider the set

\[
D_{T,x} = \left\{ d \in D(\lfloor n / \ell \rfloor, \ell, n) : d^T d < 18\ell \text{ and } T_{f_0}(d) = T = \{i_1, \ldots, i_{|T|}\} \right\}.
\]

and \( \forall 1 \leq j \leq |T| : d_{i_j} = x_j \) and \( \forall j \not\in T : |d_j| < f_0 \).

The main tool in our proof is the following statement, which says that for every set of integer sequences \( D_{T,x} \) the set of corresponding colored graphs is “small”.
Lemma 4.12. Let $T \subset \{1, \ldots, \ell\}$ be such that $1 \leq |T| \leq 18\ell/f_0^2$, and set $d(\ell) := 2^{12f_0^{-2/3}}$. Then for every $x = (x_1, \ldots, x_{|T|})$ such that $|x_i| \geq f_0$ for all $1 \leq i \leq |T|$ and sufficiently large $n$

$$\frac{1}{\ell!} \sum_{d \in D_{T,x}} b(d) E(d) \leq 2 \frac{1-d(\ell)}{2} \left( \sum_{i=1}^{f_0} x_i^2 \right)^{\ell!} \cdot |\text{Col}_{n,\ell}|,$$

(4.21)

Proof of second statement of Theorem 4.3. Our strategy is to show that the number of colored graphs with at least one color class deviating at least $f_0$ in absolute value from $[n/\ell]$ is $o(|\text{Col}_{n,\ell}|)$. Note that due to Proposition 4.9 it is sufficient to consider colored graphs having fewer than $18\ell/f_0^2$ color classes that deviate more than $f_0$ from $[n/\ell]$. We count those by first choosing the number $t$ of entries which have a large deviation. Second, we choose their positions, i.e., the set $T$ in the lemma above. Third, we fix their values, i.e., the vector $x$. Finally, we sum up for all integer sequences in $D_{T,x}$ the number of corresponding colored graphs. We thus obtain with (4.2)

$$|\text{Col}_{n,\ell}| - \text{col}_{n,\ell}(f_0) \leq \frac{1}{\ell!} \sum_{d \in D(n,\ell,n)} b(d) E(d) + \frac{18\ell/f_0^2}{\ell!} \sum_{t=1}^{18\ell/f_0^2} \sum_{T:|T|=t} \sum_{x=(x_1, \ldots, x_1)} \sum_{|x_i| \geq f_0} \sum_{d \in D_{T,x}} b(d) E(d)$$

$$\leq o(|\text{Col}_{n,\ell}|) + |\text{Col}_{n,\ell}| \cdot \sum_{t=1}^{18\ell/f_0^2} \ell^t \cdot \sum_{x=(x_1, \ldots, x_1)} \sum_{|x_i| \geq f_0} \left( \sum_{|x| \geq f_0} 2^{1-d(\ell)} x_i^2 \right)^{t}$$

(4.22)

Let $z = \frac{1-d(\ell)}{2}$. The inner sum above can be estimated as follows:

$$\sum_{|x_i| \geq f_0} 2^{-zx^2} = 2 \cdot 2^{-zf_0^2} \sum_{x \geq 0} 2^{-z(x^2 + 2xf_0)} \leq 2 \cdot 2^{-zf_0^2} \sum_{x \geq 0} 2^{-zx^2}.$$  

The last sum converges rapidly:

$$\sum_{x \geq 0} 2^{-zx^2} \leq 1 + \sum_{x \geq 1} 2^{-\frac{x}{z}} \leq 1 + \sum_{x \geq 1} \frac{4}{x^2} = 1 + \frac{4}{6\pi^2} \leq 8.$$  

Recall the definition of $f_0$ at the beginning of this chapter. By putting all together we obtain from (4.22)

$$|\text{Col}_{n,\ell}^t| - \text{col}_{n,\ell}(f_0) \leq o(|\text{Col}_{n,\ell}|) + |\text{Col}_{n,\ell}| \cdot \sum_{t=1}^{18\ell/f_0^2} (8\ell)^t \cdot 2^{1-d(\ell)} f_0^2 + t$$

$$\leq o(|\text{Col}_{n,\ell}|) + |\text{Col}_{n,\ell}| \cdot \sum_{t \geq 1} 2^{-t \cdot 2/(\log t)^2/3} = o(|\text{Col}_{n,\ell}|),$$

which concludes the proof of the theorem.

□

The remainder of this section is devoted to the proof of Lemma 4.12. Before we proceed, observe that due to symmetry it suffices to consider sets $T$ such that $T = \{1, \ldots, t\}$. In
the sequel we will assume that \( t \) is a fixed value between 1 and \( 18\ell / f_0^2 \), and that \( x \) is a fixed vector such that for all \( 1 \leq i \leq t \) we have \(|x_i| \geq f_0\). For brevity we will write \( \mathcal{D} = \mathcal{D}_{[1, \ldots, t]} \).

The core idea is similar to the main idea in Section 4.1.2: we will define an injective mapping from the set \( \mathcal{D} \) of “unbalanced” integer sequences \( d \) (i.e. those which have \( t \) values larger than \( f_0 \)) to the set of “mainly balanced” integer sequences, which have the property that all their values are smaller than \( f_0 + 1 \). By exploiting this mapping we will transform the sum counting the unbalanced colored graphs into a weighted sum over all colored graphs, where the weights correspond to the “profit” we make by switching from unbalanced to balanced partitions. We then will show that these weights are “small enough”, i.e., inequality (4.21).

The mapping works in this case as follows. Define for \( d \in \mathcal{D} \) the “excess” \( X = \sum_{i=1}^{t} d_i = \sum_{i=1}^{t} x_i \) as the sum of its first \( t \) elements, and let \( s = X / |X| \) if \( X \neq 0 \) and \( s = 0 \) otherwise. Note that due to Propositions 4.11 and 4.9 we have that \( X < 19\ell / f_0 = o(\ell) \). The mapping splits up the excess of \( d \) into parts of size \( s \), and distributes them among the entries of \( d \) with indices larger than \( t \). More formally, the mapping is given by

\[
m(d) = (0, \ldots, 0, d_{t+1} + s, \ldots, d_{t+\lfloor X/s \rfloor} + s, d_{t+\lfloor X/s \rfloor+2}, \ldots, d_{\ell}).
\]

(4.23)

Note that whenever \( \ell \) is sufficiently large the mapping is always well-defined. Moreover, \( m(d) \in \mathcal{D}(n - \lfloor n/\ell \rfloor, \ell, n) \), as the invariant \( \sum_{i=1}^{t} (m(d))_i = r(n, \ell) \) is preserved (recall that \( r(n, \ell) \) denotes the remainder of \( n \) when divided by \( \ell \)).

Observe that every integer sequence in \( \mathcal{D}(n - \lfloor n/\ell \rfloor, \ell, n) \) is constructed at most once by \( m \), i.e., for distinct integer sequences \( d, d' \in \mathcal{D} \) we have \( m(d) \neq m(d') \). Hence, \( m \) is injective.

Intuitively, it can happen that the above defined mapping is sometimes in a specific sense “bad”. This case occurs if there are many entries among the first \( |X| \), which are modified, that are equal to \( f_0 \) in absolute value. In such a case \( m(d) \) can be even more “unbalanced” than \( d \) itself (i.e. have more entries that are larger than \( f_0 \) in absolute value), which would imply that the “profit” made by switching from \( d \) to \( m(d) \) is not large enough in order to yield (4.21).

On the other hand, given an integer sequence \( d \) with \( d^T d < 18\ell \), Proposition 4.9 states that the number of entries greater than some given \( g \) in absolute value is “small”, i.e., at most \( 18\ell / g^2 \). In the following, we show that this is in almost all cases also a local property of those integer sequences, namely, that within a subsequence of a given length we have with high probability only proportionally many “large” entries.

In order to make this more precise, observe that for any fixed \( x \) of size \( t \), we can group the integer sequences in \( \mathcal{D} \) according to the values of their remaining \( \ell - t \) entries. This grouping has the advantage that for any pair \( d \) and \( d' \) of integer sequences in such a group, the number of corresponding \( \ell \)-colored graphs is equal, i.e., \( b(d) E(d) = b(d') E(d') \), and therefore allows us to compare directly the number of graphs for which the mapping (4.23) is “good” with the number of graphs for which it is “bad”. The following definition formalizes the grouping idea.
Definition 4.13. An integer sequence \( d \in D \) is of type \( y = (y_{f_0+1},\ldots,y_{f_0-1}) \) if
\[
\left| \{ i \mid d_i = j \} \right| = y_j \text{ for all } -f_0 < j < f_0.
\]
The set of \( d \in D \) of type \( y \) is denoted by \( D(y) \). \( d \) is bad if
\[
\left| \{ i \in \{ t+1,\ldots,t+|X| \} \mid |d_i| > f_0^{1/3} \} \right| \geq \left| X \right| \cdot 256f_0^{-2/3}.
\] (4.24)

The following corollary is an immediate consequence of the above definition.

Corollary 4.14. Let \( d, d' \in D(y) \). Then \( b(d)E(d) = b(d')E(d') \), where \( b \) and \( E \) are defined in (4.3) and (4.4).

The lemma below states that at least half of all \( d \in D \) of a given type will not be bad.

Lemma 4.15. Let \( B(y) = \{ d \in D(y) \mid d \text{ is bad} \} \). For large \( n \)
\[
|B(y)| \leq \frac{1}{2} |D(y)|.
\]

Proof. First observe that \( |D(y)| = \frac{(|X| \cdot 256f_0^{-2/3})^z}{\ell} \), where \( \ell \) is the number of indices \( 1 \leq i \leq \ell \) with \( |d_i| > f_0^{1/3} \). Furthermore, due to (4.20), every \( d \in D(y) \) satisfies \( d^d < 18 \ell \); it follows from Proposition 4.9 that the number of indices \( 1 \leq i \leq \ell \) with \( |d_i| > f_0^{1/3} \) is at most \( 18 \ell f_0^{-2/3} \) and therefore
\[
\sum_{|d_i| > f_0^{1/3}} y_j \leq 18 \ell f_0^{-2/3}.
\]

Next we derive the claimed upper bound for \( |B(y)| \). We count the number of sequences \( d \) which are bad by choosing \( z = \left( \left| X \right| \cdot 256f_0^{-2/3} \right) \geq 1 \) positions \( p_1,\ldots,p_z \) out of the positions \( t+1,\ldots,t+|X| \). This can be done in \( \binom{|X|}{z} \) ways. Then we choose the values \( v_1,\ldots,v_z \) for these positions, such that \( |v_i| \geq f_0^{1/3} \) for all \( 1 \leq i \leq z \); there are at most \( (18 \ell f_0^{-2/3})^z \) ways to do that. Finally, we permute all the remaining \( (\ell-t)-z \) values. In this way we have constructed every sequence in \( B(y) \) exactly \( \text{aut}(y) \) times. Hence, using \( \binom{A}{B} \leq \left( \frac{eA}{B} \right)^B \) and \( z \geq 256 \left| X \right| f_0^{-2/3} \)
\[
\frac{|B(y)|}{|D(y)|} \leq \frac{\binom{|X|}{z} (18 \ell f_0^{-2/3})^z ((\ell-t)-z)!}{(\ell-t)!} \leq \frac{\ell^z \cdot (18e)^z}{(\ell-t)^z} \leq \frac{\ell^z}{(\ell-t)^z} \cdot 2^z(-8+8 \log 18e).
\]

Observe that for sufficiently large \( \ell \) we may assume that \( \ell-t-z \geq \frac{\ell}{2} \). Therefore
\[
\frac{\ell^z}{(\ell-t)^z} \leq \left( \frac{\ell}{\ell-t-z} \right)^z \leq 2^z.
\]
Putting all together yields \( |B(y)| \leq 2^{-z} \cdot |D(y)| \), which completes the proof. \( \square \)

The next two lemmas are the main tools used in the proof of Lemma 4.12, and state that whenever \( d \) is not bad, the ratios \( \frac{E(d)}{E(m(d))} \) and \( \frac{b(d)}{b(m(d))} \) are small. The proofs can be found at the end of the section.

Lemma 4.16.
\[
\max_{d \in D, \text{not bad}} \frac{E(d)}{E(m(d))} \leq 2 \cdot \frac{(1-z^2)2^{-2/3}}{z} \cdot \frac{\sum_{i=1}^{t} x_i^2}{t},
\]
Lemma 4.17. For large \( n \)

\[
\max_{d \in \mathcal{D}} \frac{b(d)}{b(m(d))} \leq 2^{f_0^{-2/3} \sum_{i=1}^{t} x_i^2}.
\]

Proof of Lemma 4.12. Lemma 4.15 together with Corollary 4.14 imply for every \( y \)

\[
\sum_{d \in D(y)} b(d)E(d) \leq 2 \cdot \sum_{d \not \in \mathcal{D}(y)} b(d)E(d). \tag{4.25}
\]

Having this, note that given any (admissible) \( x \) of size \( t \), we can write the set \( \mathcal{D} \equiv \mathcal{D}_{[1, \ldots, t], x} \) as the disjoint union over all integer sequences having different types \( y \), which implies that in order to prove the lemma it suffices to consider integer sequences which are not bad. With this observation, we can now prove (4.21) as follows. Recall the definition of \( m \) (4.23); we obtain due to the fact that \( m \) is injective

\[
\frac{1}{t!} \sum_{d \in \mathcal{D}} b(d)E(d) \leq \frac{2}{t!} \sum_{y \in \mathcal{D}(y)} \sum_{d \not \in \mathcal{D}(y)} b(d)E(d)
\]

\[
\leq \frac{2}{t!} \sum_{y \in \mathcal{D}(y)} \sum_{d \not \in \mathcal{D}(y)} \frac{b(d)E(d)}{b(m(d))} \frac{b(m(d))E(m(d))}{b(m(d))}
\]

\[
\leq 2 \cdot |\text{Col}_{n, l}| \cdot M,
\]

where

\[
M = \max_{d \in \mathcal{D}} \frac{E(d)}{b(m(d))} \frac{b(d)}{b(m(d))}. \tag{4.26}
\]

Applying Lemmas 4.16 and 4.17 we readily obtain \( M \leq \frac{1}{2} 12^{-\frac{1}{2} \cdot d(y)} \sum_{i=1}^{t} x_i^2 \), which completes the proof.

Proof of Lemma 4.16. Recall that \( X = \sum_{i=1}^{t} d_i = \sum_{i=1}^{t} x_i \) and let \( s \) be the sign of \( X \). By considering the logarithm of the left hand side of the statement and using (4.23) (the definition of \( m \)), we obtain by applying Proposition 4.7

\[
2(\log E(d) - \log E(m(d))) = \sum_{i=1}^{t} ((m(d))^2 - d_i^2)
\]

\[
= - \sum_{j=1}^{t} x_j^2 + \sum_{j=1}^{t} ((d_{t+j} + s)^2 - d_{t+j}^2) = - \sum_{j=1}^{t} x_j^2 + |X| + 2s \sum_{j=1}^{t} d_{t+j}.
\]

Now recall (4.24); as we consider only sequences that are not bad we can bound the last sum with

\[
2s \sum_{j=1}^{\lfloor |X| \rfloor} d_{t+j} \leq 2 \left( \frac{256}{f_0^{2/3}} |X| \cdot f_0 + \left( 1 - \frac{256}{f_0^{2/3}} \right) |X| \cdot f_0^{1/3} \right) \leq 2^{10} \cdot |X| f_0^{1/3}. \tag{4.27}
\]
Let \( \xi = 2^{11.5^2} \). Putting all together yields

\[
2(\log E(d) - \log E(m(d))) \leq -(1 - \xi) \sum_{j=1}^{t} x_j^2 - \xi \sum_{j=1}^{t} x_j^2 + 2^{11} \cdot |X| f_0^{1/3}
\]

\[
\leq -(1 - \xi) \sum_{j=1}^{t} x_j^2 - \xi |X| \cdot \min_{1 \leq i \leq t} |x_i| + 2^{11} \cdot |X| f_0^{1/3},
\]

which completes the proof, as \( |x_i| \geq f_0 \) for all \( 1 \leq j \leq t \).

\( \square \)

**Proof of Lemma 4.17.** Abbreviate \( \alpha = \lceil n/\ell \rceil \) and recall (4.3); using the definition (4.23) of \( m \) we obtain

\[
\frac{b(d)}{b(m(d))} = \frac{\prod_{j=1}^{t} (\alpha + (m(d))_j)!}{\prod_{j=1}^{t} (\alpha + d_j)!} = \frac{(\alpha!)^t}{\prod_{x_i \leq f_0} (\alpha + x_i)!} \leq \alpha^{-\sum_{x_i \leq f_0} x_i} \cdot \alpha^{-\sum_{x_i \geq f_0} x_i} = \alpha^{-X}.
\]

When \( s = 1 \), the second term can be estimated as follows:

\[
\prod_{j=1}^{t} (\alpha + d_{t+j}) \leq \alpha^X \prod_{j=1}^{t} \left( 1 + \frac{d_{t+j}}{\alpha} \right) \leq \alpha^X \cdot e^{\frac{X}{\alpha} \sum_{j=1}^{t} d_{t+j}}. \tag{4.28}
\]

If \( s = -1 \), observe that \( X < 0 \); we obtain with the inequality \( 1 - x \geq e^{-x-x^2} \), which is valid for small \( x \):

\[
\prod_{j=1}^{t} \frac{1}{\alpha + d_{t+j}} \leq \alpha^{-X} \cdot \prod_{j=1}^{t} \frac{1}{1 - \frac{d_{t+j}}{\alpha}} \leq \alpha^X \cdot e^{\frac{X}{\alpha} \sum_{j=1}^{t} d_{t+j}}. \tag{4.29}
\]

As we consider only sequences that are not bad, we can estimate with (4.24)

\[
\sum_{j=1}^{t} d_{t+j} \leq \sum_{j=1}^{t} |d_{t+j}| \leq 2^{10} \cdot |X| f_0^{1/3},
\]

where the derivation is essentially the same as in (4.27). Similarly, the additional error term in (4.29) can be crudely estimated from above with

\[
\frac{1}{\alpha^2} \sum_{j=1}^{t} d_{t+j}^2 \leq \frac{|X| f_0^2}{\alpha^2} = o(|X|),
\]

as \( |d_{t+j}| \leq f_0 \) and \( \alpha = \lceil n/\ell \rceil = \omega(\sqrt{\log n}) \). By putting all together we obtain with plenty of room to spare

\[
\max_{d \in \mathcal{P}, d \text{ not bad}} \frac{b(d)}{b(m(d))} \leq 2^{f_0^{1/3} |X|},
\]

and the statement follows from the immediate fact

\[
f_0^{1/3} |X| \leq f_0^{-2/3} \sum_{i=1}^{t} x_i^2.
\]

\( \square \)
4.1.4. \(\Theta(1)\)-colored Graphs. Let \(\ell > 1\) and \(c \geq 0\) be constants, and let \(r(n, \ell) = r = n \mod \ell\). Furthermore, let us for the moment assume \(c > 0\). Wright [Wri61, Thm. 1] showed that for sufficiently large \(n\)

\[
\text{col}_{n,\ell}(c) = \frac{1}{\ell!} \sum_{d \in D(c,\ell,n)} b(d)E(d) = (1 + o(1))\frac{1}{\ell!}2^{c(n,\ell)-p(n,\ell)} \cdot L(r, c),
\]

where

\[
L(r, c) = \sum_{d \in D(c,\ell,n)} 2^{-\frac{1}{\ell}d^Td}.
\]

\(L(r, c)\) is easily seen to be a constant; therefore, \(\text{col}_{n,\ell}(c)\) is a constant fraction of all \(\ell\)-colored graphs. If \(c = 0\) and \(r > 0\), then the sum in (4.31) is empty, i.e., the number of corresponding colored graphs is zero. Finally, if \(c = 0\) and \(r = 0\), the same argument as above yields that we get a constant fraction of all colored graphs. This proves the lower bound stated in Theorem 4.3 for constant \(\ell\).

The upper bound follows immediately from equations (4.30) and (4.31), and the fact that \(L(r, c)\) converges (rapidly) to a constant for \(c \to \infty\).

4.1.5. \(\ell\)-colored Graphs with \(cn^2\) Edges. In this section we are going to show how Theorem 4.5 can be proved. For this, observe that the number of \(\ell\)-colored graphs with a fixed number of edges is given by a very similar expression to the number of all \(\ell\)-colored graphs – the power-of-two term in (4.2) has to be replaced with the binomial coefficient defined in (4.7). But these two functions behave very similar – recall the approximation of the binomial coefficient

\[
\binom{n}{\alpha n} \approx 2^n H(\alpha),
\]

where \(H\) denotes the binary entropy function and \(0 < \alpha < 1\), see e.g. [Juk01]. Note that \(H(\alpha)\) is a constant, if \(\alpha\) is constant; it is therefore not surprising that the proof carries over if we add the additional restriction on the number of edges. In the sequel we will use the following estimates for fractions of binomial coefficients:

\[
\left(1 - \frac{\beta + x}{\alpha}\right)^x \leq \left(\frac{\alpha-x}{\beta}\right) \leq \left(1 - \frac{\beta - x}{\alpha-x}\right)^x.
\]

We now outline how the counterparts of the needed lemmas and propositions can be proved; we assume throughout that \(M = cn^2\) for \(0 < c < \frac{1}{2}\).

**Proposition 4.18 (Prop. 4.8).** Let \(C(c) = \frac{30}{\log(1-2c)} + \frac{1}{2}\) and define

\[
\mathcal{L}'(n, \ell) := \left\{ d \in D \left( n - \left[ \frac{n}{\ell} \right] , \ell, n \right) : \frac{1}{2}d^Td \geq C(c)\ell \right\}.
\]

For all integer functions \(\ell(n)\) with \(\ell(n) \to \infty\) for \(n \to \infty\), it holds

\[
\sum_{d \in \mathcal{L}'(n, \ell)} b(d)\tilde{E}(d) = o \left( |\text{Col}_{n,M,\ell}| \right).
\]
\textbf{Proof.} Recall (4.10) and (4.9); Let \( r \) be the remainder of \( n \) and \( \ell \) and 0, be the vector which has only zero entries except for the first \( r \), which are one. We first derive a lower bound for the number of graphs in \( \text{Col}_{n,M,\ell} \) by

\[
|\text{Col}_{n,M,\ell}| \geq \frac{1}{\ell!} \cdot b(0_r) \cdot \left( \frac{e(n, \ell) - \frac{r}{2}}{M} \right)
\]

First consider the case \( d^T d \geq \frac{2}{c} n \log n \). As the sum over all multinomial coefficients is \( \ell^n \), we can estimate with the inequality \( (\alpha \beta) \leq e^{-\alpha \beta} \cdot (\frac{\alpha}{\beta}) \)

\[
\frac{1}{\ell!} \sum_{d \in \mathcal{C}'(n,\ell)} b(d) \hat{E}(d) \leq \frac{\ell^n}{\ell!} \left( \frac{e(n, \ell) - \frac{r}{2}}{M} \right) \leq e^{-\frac{n}{2} \log n} = o \left( \frac{1}{\ell!} \left( \frac{e(n, \ell) - \frac{r}{2}}{M} \right) \right).
\]

Next we consider the case \( C(c) \ell \leq d^T d < \frac{2}{c} n \log n \). Observe that

\[
\sum_{d \in \mathcal{C}'(n,\ell)} b(d) \hat{E}(d) \leq \frac{1}{\ell!} b(0_r) \left( \frac{e(n, \ell) - \frac{r}{2}}{M} \right) \sum_{d^T d \geq 2C(c) \ell} \frac{e(n, \ell) - \frac{r}{2}}{M} = \sum_{d^T d \geq 2C(c) \ell} \frac{e(n, \ell) - \frac{r}{2}}{M} \cdot \left( 1 - (1 + \Theta(\ell^{-1})) \right) 2c \frac{1}{2} \sum_{d^T d \geq 2C(c) \ell} d^T d - \frac{r}{2}
\]

and the remainder of the proof remains due to the choice of \( C(c) \) the same (observe that the additional \( \frac{1}{2} \) in the definition of \( C(c) \) cancels the \( \frac{r}{2} \), which can be at most \( \frac{r}{2} \)). \( \square \)

Now we can mimic the proof of Proposition 4.9 and we obtain for every integer sequence \( d \) with the desired properties

\[
|T_g(d)| \leq C(c) \cdot \frac{\ell}{g^2},
\]

where \( T_g \) is defined in (4.12) and \( C \) is a constant which depends only on \( c \).

Let \( f_0 = (1 - \epsilon) \sqrt{\log \frac{\ell}{-\log(1 - 2c)}} \), where \( \epsilon > 0 \) is an arbitrarily small constant. The first statement of Theorem 4.5 can be proved with the same mapping as in Section 4.1.2. For this, observe that as above, the proof of Proposition 4.10 can be rewritten with different constants for the current case. The remainder of the proof remains the same – in (4.18) we replace the power-of-2-term by

\[
\tilde{E}(d) \leq \frac{1}{(1 + \Theta(\ell^{-1}))(2c)^{\frac{1}{2} (f_0 + 1)^2 + dk(f_0 + 1) - \frac{1}{2} \sum_{k=1}^{f_0 + 1} dk + \frac{f_0 + 1}{2}}},
\]

where we used again (4.32). The proof of the first statement then completes with the assumption on the value of \( f_0 \) and (4.15). Observe that we used the correction factor \( (-\log(1 - 2c))^{-\frac{1}{2}} \) in the threshold function in order to compensate for change in the basis function (from \( \frac{1}{2} \) to roughly \( (1 - 2c) \)).

To complete the proof, we show how the calculations in Section 4.1.3 can be modified to prove the second part of Theorem 4.5. Here we set \( f_0 = (1 + \epsilon) \sqrt{\log \frac{\ell}{-\log(1 - 2c)}} \), where
\( \varepsilon > 0 \) is again an arbitrarily small constant. As a first step, we show that the absolute value of the “large” entries cannot become large, i.e. we can assume
\[
\sum_{i \in T_{\ell}(d)} d_i = \mathcal{O}(\ell/f_0),
\]
where the constant in the \( \mathcal{O} \) depends only on \( c \); the proof is essentially the same as in Proposition 4.11. The remainder of the proof and especially the definition of the mapping (4.23) remain the same, as we make the same assumptions. A slight modification has to be made in (4.25) and (4.26), where we simply substitute the \( E \)-term with the appropriate binomial coefficient \( \tilde{E}(d) = \binom{e(n,\ell)-\frac{1}{2}d^Td}{M} \). Finally, in Lemma 4.16 we estimate with (4.32) the ratio
\[
\frac{\tilde{E}(d)}{E(m(d))} = \frac{\binom{e(n,\ell)-\frac{1}{2}d^Td}{M}}{\binom{e(n,\ell)-\frac{1}{2}(m(d))^Tm(d)}{M}} \leq (1 - (1 + \Theta(\ell^{-1}))2c)^{-\frac{1}{2}(d^Td-(m(d))^Tm(d))},
\]
where we set \( x = \frac{1}{2}(d^Td-(m(d))^Tm(d)) \) in (4.32) and used \( d^Td = \mathcal{O}(\ell) \). The remainder of the proof of the lemma follows the lines of (4.27). The proof of Lemma 4.17 does not change at all, as it is independent of \( E \). This concludes the proof.

### 4.2. Unique Colorability of Random \( \ell \)-colorable Graphs

This section deals with the proof of Theorem 4.2. Before we proceed with the details, let us make a brief note on the model. In remainder we will view a coloring/partition \( V = V_1 \cup \cdots \cup V_\ell \) of the vertex set \( V \) of a given graph \( G \) as an ordered partition \( \Pi = \{V_1, \ldots, V_\ell\} \) in \( \ell \) non-empty parts; observe that in this way we consider every coloring of \( G \) precisely \( \ell! \) times, and hence we have to normalise our calculations by this factor.

For a fixed partition \( \Pi = \{V_1, \ldots, V_\ell\} \) of the vertex set, let \( G_{\Pi} \) denote the set of all graphs with the property that no two vertices which belong to the same color class are connected by an edge. Furthermore, let \( G_{\Pi} \) denote a random graph with respect to \( \Pi \), i.e.,
\[
\Pr[G_{\Pi} = G] = \frac{1}{|G_{\Pi}|}.
\]
Note that one can obtain \( G_{\Pi} \) by connecting each pair of vertices contained in different classes of \( \Pi \) independently and with probability \( \frac{1}{2} \). Consequently, by conditioning on the partition when performing calculations allows us to work on a random graph, where each (admissible) edge is drawn uniformly at random with probability \( \frac{1}{2} \) and independently from the others.

In our proof only “very balanced” partitions \( \Pi \) will be relevant.

**Definition 4.19.** A partition \( \Pi = \{V_1, \ldots, V_\ell\} \) of \([n]\) in \( \ell \) non-intersecting subsets is called \( \varepsilon \)-balanced if and only if \( |V_i|/n - \frac{1}{\ell} \leq (1 + \varepsilon)\sqrt{2\log n} \) for \( 1 \leq i \leq \ell \). We denote by \( \text{Bal}_n,\ell(\varepsilon) \) the set of \( \varepsilon \)-balanced \( \ell \)-partitions of \([n]\).

The following corollary is a straightforward implication of Theorem 4.3.
Corollary 4.20. Let $\varepsilon > 0$ and $\ell = o\left(\frac{n}{\sqrt{\log n}}\right)$. Then

$$|\text{Col}_{n,\ell}| = (1 + o(1)) \cdot \frac{1}{\ell!} \cdot \sum_{\Pi \in \mathcal{B}_n,\ell} |G_{\Pi}|.$$ 

The main ingredient that we will exploit is the following statement.

Lemma 4.21. Let $\varepsilon > 0$ and $\ell = \ell(n)$ satisfy $\ell \leq \frac{n}{2 \log n + (1 + \varepsilon)\sqrt{2 \log n}}$. Let $\Pi = (V_1, \ldots, V_\ell)$ be an $\frac{\varepsilon}{\ell}$-balanced partition of $[n]$, and let $G$ be a graph drawn uniformly at random from $\hat{G}_\Pi$. Then

$$\Pr[\text{G is uniquely } \ell\text{-colorable}] = 1 - o(1).$$

Proof of Theorem 4.2. Let $G_{n,\ell}$ be a graph drawn uniformly at random from $G_{n,\ell}$, and recall that $\mathcal{UC}_{n,\ell}$ is the set of uniquely $\ell$-colorable graphs from $G_{n,\ell}$. First observe that (see below for an explanation)

$$\Pr[ G_{n,\ell} \in \mathcal{UC}_{n,\ell} ] = \frac{1}{\ell!} \sum_{\Pi} \Pr[ G_{n,\ell} \in \mathcal{UC}_{n,\ell} \mid G_{n,\ell} \in G_{\Pi}] \cdot \Pr[ G_{n,\ell} \in G_{\Pi}], \quad (4.33)$$

where the sum is over all possible partitions $\Pi$ in $\ell$ parts/(color) classes. In general, note that for any event $A$ and any set of not necessarily disjoint events $B_1, \ldots, B_x$, where $\Omega = B_1 \cup \cdots \cup B_x$ is the underlying probability space, the inequality $\Pr[ A ] \leq \sum_{i=1}^x \Pr[ A \mid B_i ] \Pr[ B_i ]$ holds; equality holds in general only if $B_i \cap B_j = \emptyset$ for all $i \neq j$. In our case, the partition according to $\Pi$ of the probability space $G_{n,\ell}$ of all $\ell$-colorable graphs does not have this property. Nevertheless, the equality is true, as in the sum the uniquely colorable graphs are counted exactly once (per definition, they consist only of a single valid partition of their vertex set).

The probability that $\Pi$ is a valid coloring for $G_{n,\ell}$ is equal to $\frac{|G_{\Pi}|}{|\text{Col}_{n,\ell}|}$, and by applying Lemma 4.21 we get

$$\Pr[ G_{n,\ell} \in \mathcal{UC}_{n,\ell} ] \geq \frac{1}{\ell!} \sum_{\Pi \in \mathcal{B}_n,\ell,\left(\frac{\varepsilon}{\ell}\right)} \Pr[ G_{n,\ell} \in \mathcal{UC}_{n,\ell} \mid G_{n,\ell} \in G_{\Pi}] \cdot \Pr[ G_{n,\ell} \in G_{\Pi}]$$

$$\geq (1 - o(1)) \cdot \frac{1}{\ell!} \sum_{\Pi \in \mathcal{B}_n,\ell,\left(\frac{\varepsilon}{\ell}\right)} \frac{|G_{\Pi}|}{|\text{Col}_{n,\ell}|} = 1 - o(1),$$

where the last step is due to Corollary 4.20. This completes the proof of the theorem, assuming that Lemma 4.21 holds. 

4.2.1. Almost All Very Balanced Graphs have a Unique Coloring. This section deals with the proof of Lemma 4.21. Let for the remainder $\varepsilon > 0$ be arbitrary but fixed and set $\varepsilon' := \frac{\varepsilon}{\ell}$. We will assume throughout that $\ell(n) \leq \frac{n}{\alpha(n)}$, where $\alpha(n) \geq 2 \log n + (1 + \varepsilon)\sqrt{2 \log n}$. Furthermore, for a given partition $\Pi = (V_1, \ldots, V_\ell)$ of $[n]$ we denote by $n_k$ the cardinality of the $k$-th part, i.e., $|V_k| = n_k$.

Note that for $\varepsilon'$-balanced partitions $\Pi$ we can obtain due to the assumptions on $\ell$ for sufficiently large $n$ the following bounds:

$$n_k \leq \max_{1 \leq i \leq \ell} n_j \leq (1 + \varepsilon')\frac{n}{\ell} \quad \text{and} \quad n_k \geq \min_{1 \leq i \leq \ell} n_j \geq \frac{n}{\ell} - (1 + \varepsilon')\sqrt{2 \log n} \geq 2 \log n + \varepsilon' \sqrt{\log n}.$$
We shall use these estimates without further reference.

Before proving Lemma 4.21 we investigate closer how a given graph $G \in \mathcal{G}_{\Pi}$ can be “re-colored”, in order to obtain a second valid coloring $\Pi' = (V_1', \ldots, V_\ell')$. Let us denote by $n_{ij}(\Pi, \Pi')$ the number of vertices which are common in the $i$-th color class before the re-coloring and the $j$-th color class after the re-coloring, i.e., $n_{ij}(\Pi, \Pi') = |V_i \cap V_j'|$; we write $n_{ij}$ for $n_{ij}(\Pi, \Pi')$ if it is clear from the context which partitions are considered. With this notation, let

$$\mathcal{P}_\Pi := \left\{ G \in \mathcal{G}_{\Pi} : \text{all valid re-colorings } \Pi' \text{ of } G \text{ satisfy}
\begin{align*}
    &n_{ix}(\Pi, \Pi') \cdot n_{jx}(\Pi, \Pi') \leq 4 \frac{n}{\ell} 
    \text{ for all } i \neq j, 1 \leq x \leq \ell
\end{align*}
\right\}.$$

The following proposition states that a.a.s. the products of the $n_{ij}$’s cannot become too “large”.

**Proposition 4.22.** Let $\Pi$ be an $\epsilon'$-balanced partition and let $G$ be a graph drawn uniformly at random from $\mathcal{G}_{\Pi}$. Then

$$\Pr [G \not\in \mathcal{P}_\Pi] = o(1).$$

**Proof.** $G \not\in \mathcal{P}_\Pi$ implies that there is a re-coloring $\Pi' = (V_1', \ldots, V_\ell')$ which has the property that there exist indexes $i, j$ and $x$ with $i \neq j$ such that $n_{ix} \cdot n_{jx} > 4 \frac{n}{\ell}$. Hence, $(V_i \cap V_j') \cup (V_j \cap V_i')$ is an independent set and therefore all $n_{ix} \cdot n_{jx}$ edges connecting a vertex in $(V_i \cap V_j')$ to another vertex in $(V_j \cap V_i')$ are missing. We obtain with $|V_i| \leq (1 + \epsilon') \frac{n}{\ell}$ and $\frac{n}{\ell} \geq 2 \log n + \sqrt{2 \log n}$

$$\Pr[G \not\in \mathcal{P}_\Pi] \leq \sum_{i,j} \sum_{a,b > 4\frac{n}{\ell}} \binom{|V_i|}{a} \binom{|V_j|}{b} 2^{-ab} \leq \ell^2 \cdot 2^{2(1+\epsilon')n/\ell} \cdot 2^{-4n/\ell} \leq n^2 \cdot 2^{-n/\ell} = o(1).$$

The above lemma states that it suffices to consider only re-colorings $\Pi'$ with the property $\mathcal{P}_\Pi$, i.e., $n_{ix} \cdot n_{jx} \leq 4 \frac{n}{\ell}$ for all $i \neq j$ and $x$. In the following we will show that we can restrict our considerations to an even smaller class of re-colorings.

**Definition 4.23.** Let $\Pi$ be $\epsilon'$-balanced partition and let $\Pi'$ be a valid re-coloring for a graph $G \in \mathcal{G}_{\Pi}$. $\Pi'$ is called standardized, if $n_{ij}(\Pi, \Pi') \leq \left\lfloor \frac{\ell n}{2} \right\rfloor + 1$ for all $i \neq j$.

The lemma below states that it suffices to consider only standardized re-colorings, provided that the considered graphs are in $\mathcal{P}_\Pi$.

**Lemma 4.24.** Let $\Pi$ be a $\epsilon'$-balanced partition. Then every non-standardized re-coloring $\Pi' = (V_1', \ldots, V_\ell')$ for a graph $G \in \mathcal{P}_\Pi$ can be transformed into a standardized re-coloring. More precisely, there is a permutation $\mu : [\ell] \to [\ell]$ such that $(V'_{\mu(1)}, \ldots, V'_{\mu(\ell)})$ is standardized.

**Proof.** The main idea behind the proof is that if a color class “moves” more than half of its vertices to any other color class, then it could as well keep those vertices and...
remove the remaining ones instead – this would result in the same re-coloring, where just the order of the color classes is different.

In order to formalize this idea we proceed as follows: a re-coloring can be viewed as a weighted directed graph $D(\Pi, \Pi')$ with $\ell$ vertices, where each vertex represents a color class in $\Pi$ and there is a directed edge connecting vertex $i$ and vertex $j$ with weight $n_{ij}$. We now derive from the graph $D(\Pi, \Pi')$ a graph $\tilde{D}(\Pi, \Pi')$ by deleting all “light” edges, i.e., those edges $(i, j)$, which have weight smaller than $\left\lfloor \frac{n_{ij}}{2} \right\rfloor + 2$. Observe that $\tilde{D}$ has a very simple structure: it consists of at most $\ell$ edges and no vertex has outdegree greater than 1. Furthermore, every vertex has indegree at most 1, because otherwise the property $P_\Pi$ would be violated. Therefore, its connected components are paths or directed cycles, which can be of length one, i.e., form a loop.

We define the permutation $\mu$ as follows: for every $e = (x_e, y_e) \in \tilde{D}$, $\mu$ maps $y_e$ to $x_e$; in case of a directed path, $\mu$ also maps the first to the last vertex of it. For all remaining values $x$, we set $\mu(x) = x$. We now show that $\Pi'' = (V'_1, \ldots, V'_{\mu(\ell)})$ is standardized. For this, assume that

$$n_{i\mu(j)}(\Pi, \Pi'') = n_{ij}(\Pi, \Pi') > \left\lfloor \frac{n_{ij}}{2} \right\rfloor + 1.$$ 

Hence, we know that $(i, j) \in \tilde{D}$ and therefore $\mu(j) = i$, by definition of $\mu$. Therefore, we have $n_{ij}(\Pi, \Pi') \leq \left\lceil \frac{n_{ij}}{2} \right\rceil + 1$ for all $i \neq j$, which shows that $\Pi''$ is indeed standardized. $\square$

With the above considerations our proof strategy for Lemma 4.21 is to estimate the probability that a random graph $G_\Pi \in G_\Pi$ has a second coloring $\Pi'$. We say that a subset $S$ of the vertex set of $G_\Pi$ is $k$-composed, if it is composed by vertices from exactly $k$ distinct color classes of $G_\Pi$, i.e., $|\{1 \leq i \leq \ell : S \cap V_i \neq \emptyset\}| = k$. Observe that we can distinguish among the following cases:

- $B_2$: there exists a standardized $\Pi'$ with the property that all color classes are either 1-composed or 2-composed.
- $B_m^3$ for $3 \leq m \leq \ell$: there exists a standardized $\Pi' = (V'_1, \ldots, V'_\ell)$ such that there is a $m$-composed color class $V'_i$, where $|V_i| \leq |V'_i|$.
- $B_2^2$: there exists a standardized $\Pi'$ with the property that there is a 2-composed color class $V'_i$, where $|V_i| < |V'_i|$.

Proposition 4.25. Let $\varepsilon' > 0$, and let $G$ be a random graph from $G_\Pi$, where $\Pi$ is a $\varepsilon'$-balanced partition. Then

$$\Pr[G \not\in \mathcal{UC}_{n,\ell}] \leq \Pr[G \in B_2] + \Pr[G \in B_2^2] + \sum_{m \geq 3} \Pr[G \in B_m^3] + \Pr[G \not\in P_\Pi].$$

Proof. Observe that $G \not\in \mathcal{UC}_{n,\ell}$ implies that there exists a $\Pi' = (V'_1, \ldots, V'_\ell)$ such that at least one of its classes is $m$-composed, for some $m \geq 2$, as otherwise all color classes would be 1-composed, i.e., we would have simply permuted the colors. $\Pi'$ is either standardized or not. In the former case, suppose that $\Pi'$ neither certifies the event $B_2$ nor the events $B_m^3$ for $m \geq 3$. This implies that all $m$-composed color classes, where $m \geq 3$, have the property that their size decreased due to the re-coloring. Consequently, there must be a 2-composed class, which became larger, i.e., the event $B_2^2$ will occur. $\square$
In the sequel we prove sufficient upper bounds for the probabilities of the events $B^2_n$, $B^2_{\geq}$, and $B^2_{=m}$. We will make use of the following estimate for binomial coefficients, which is a straightforward implication of Stirling’s formula.

**Lemma 4.26.** Let $b > 1$. For every sufficiently large natural number $n$ and $0 < k \leq n$

$$\log_b \binom{n}{k} \leq n \cdot H_b \left( \frac{k}{n} \right),$$

where $H_b(x) = -x \log_b x - (1-x) \log_b (1-x)$ denotes the entropy function with respect to the basis $b$. Furthermore, if $k = o(n)$ or $k = n - o(n)$

$$\log_b \binom{n}{k} \leq k \log_b \left( \frac{n}{k} \right) + \Theta(k).$$

**Proposition 4.27.** Let $\Pi = (V_1, \ldots, V_i)$ be an $\varepsilon'$-balanced partition and $G$ a graph drawn uniformly at random from $G_{\Pi}$. Then

$$\Pr \left[ G \in B^2_{\geq} \right] = o(1).$$

**Proof.** The event $B^2_{\geq}$ implies that there exist two indexes $i \neq j$ and an independent set $S$ with $|S| > |V_i|$ in $G$ such that $S$ is composed out of $1 \leq x \leq \ceil{|V_i|/2} + 1$ vertices from $V_i$ and the remaining $|S| - x$ vertices from $V_j$. Observe that for $x = 1$ this implies that there is (at least) one isolated vertex in the induced subgraph of $G$ defined by the union of the two color classes $V_i \cup V_j$, as otherwise $|S| \leq |V_i|$, a contradiction. More precisely, denote by $I$ the event

$$I := \left\{ G \in G_{\Pi} : \exists 1 \leq i, j \leq \ell : \forall v \in V_i : (i \neq j) \wedge (v \text{ is isolated in the induced subgraph of } G \text{ defined by } V_i \cup V_j) \right\}.$$

The probability for the event $I$ to occur is bounded from above by $n \cdot \ell \cdot 2^{-\min_{1 \leq i \leq \ell} |V_i|}$, as there are at most $n$ choices for the vertex $v$ and $\ell - 1$ choices for $j$. With $\ell \leq n$ and $|V_i| \geq 2 \log n + \varepsilon' \sqrt{n \log n}$ we deduce that this probability is at most $2^{-\varepsilon' \sqrt{n \log n}} = o(1)$.

Consequently, it remains to consider 2-composed independent sets $S$ which inherit $2 \leq x \leq \ceil{|V_i|/2} + 1$ vertices from $V_i$ and the remaining vertices from $V_j$. Note that $|S| > \frac{n}{\ell} - (1 + \varepsilon') \sqrt{2 \log n} := s_0$. For large $n$ we have

$$\Pr \left[ G \in B^2_{\geq} \right] \leq \Pr \left[ G \in I \right] + \Pr \left[ G \in (B^2_{\geq} \cap \neg I) \right]$$

$$\leq o(1) + \sum_{s > s_0} \sum_{i,j} \sum_{2 \leq x \leq \ceil{|V_i|/2} + 1} \left( \frac{|V_i|}{x} \right) \left( \frac{|V_j|}{s - x} \right) 2^{-x(s - x)},$$

which simplifies with $|V_k| \leq \bar{n} := \frac{n}{\ell} + (1 + \varepsilon') \sqrt{2 \log n}$ for all $k$ to

$$\Pr \left[ G \in B^2_{\geq} \right] \leq o(1) + \ell^2 \cdot \sum_{s > s_0} \sum_{2 \leq x \leq \ceil{\bar{n}/2} + 1} \left( \frac{\bar{n}}{x} \right) \left( \frac{\bar{n}}{\bar{n} - s + x} \right) 2^{-x(s - x)}$$

$$\leq o(1) + \ell^2 \cdot \sum_{s > s_0} \sum_{2 \leq x \leq \ceil{\bar{n}/2} + 1} 2^{(\bar{n} - s + 2x) \log n - x(s - x)}.$$
The exponent of 2 is convex in $\chi$; furthermore, it is easily seen that it becomes maximal when $x = 2$. With the fact $\tilde{n} - s \leq \tilde{n} - s_0 \leq 2(1 + \varepsilon')\sqrt{2\log \tilde{n}}$ we obtain that

$$\Pr_{\Pi} [B^2_{s}] \leq o(1) + 16 \cdot \ell^2 \cdot \tilde{n} \cdot \sum_{s > s_0} 2^{2(1+\varepsilon)\sqrt{2\log \tilde{n}}}.$$ 

Observe that $\sqrt{\log \tilde{n}} \log \tilde{n} \ll \frac{\varepsilon}{s}$. We conclude

$$\Pr_{\Pi} [B^2_{s}] \leq o(1) + 16 \cdot \ell^2 \cdot \tilde{n} \cdot \sum_{s > s_0} 2^{-2(1-o(1))s} = o(1).$$

\[ \square \]

**Proposition 4.28.** Let $\Pi = (V_1, \ldots, V_\ell)$ be an $\varepsilon'$-balanced partition and $G$ a graph drawn uniformly at random from $G_{\Pi}$. Then

$$\Pr [G \in B_2] = o(1).$$

**Proof.** Assume that there exists a standardized re-coloring $\Pi'$ with the property that all color classes are 2-composed or 1-composed. This implies either the event $B^2_{s'}$, or that there is an integer $2 \leq k \leq \ell$ and an integer $t \geq 1$ such that there are $k$ color classes $V_{i_0}, \ldots, V_{i_{k-1}}$ in $\Pi$, where the $i$th class “gives” $t \leq \lceil n_{i}/2 \rceil + 1$ nodes to the $i_{(j+1) \mod k}$th color class, i.e., we have $n_{i_{j, i_{j+1} \mod k}} (\Pi, \Pi') = t$.

In order to keep notation short, we write $n_{i_k} := n_{i_0}$. Clearly, for $0 \leq j \leq k - 1$, the $t + (n_{i_{j+1}} - t)$ vertices must form an independent set, as otherwise the resulting coloring would not be valid. Therefore the number of edges not included in $G$ is

$$F(k; i, t) := F(k; i_0, \ldots, i_{k-1}, t) = \sum_{j=0}^{k-1} t \cdot (n_{i_{j+1}} - t).$$

Furthermore, there are $B(k; i, t) := \prod_{j=0}^{k-1} \binom{n_{i_j}}{t}$ ways to choose the vertices which move. Define the function

$$E(k; i, t) := \sum_{j=0}^{k-1} n_{i_j} H \left( \frac{t}{n_{i_j}} \right) - \sum_{j=0}^{k-1} t(n_{i_{j+1}} - t),$$

where we assume that $H$ denotes the binary entropy function. Observe that

$$\log_2 B(k; i, t) - F(k; i, t) \leq E(k; i, t).$$

By elementary calculus we have that $\frac{\partial}{\partial t} n_{i_j} H \left( \frac{t}{n_{i_j}} \right) = \log \frac{n_{i_j} - t}{t}$. Then, by differentiating $E$ we obtain

$$\frac{\partial E}{\partial t} = \log \left( \frac{n_{i_j} - t}{t} \right) - n_{i_{j+1}} + t.$$

Due to $1 \leq t \leq \lceil n_{i_j}/2 \rceil + 1$ and $\log n_{i_j} = o(\min_{1 \leq x \leq \ell} n_x)$ we see that $\frac{\partial E}{\partial t} < 0$ for all admissible values of $t$, whenever $n$ is chosen sufficiently large. As $E$ becomes maximal in the direction of the gradient of $E$, we deduce that $E$ obtains its global maximum when $t$ is chosen to be as small as possible, i.e., $t = 1$. 
With the above observations we can bound the desired probability as follows. We first choose \( k \), then the indexes \( i = (i_0, \ldots, i_{k-1}) \) of the color classes, and finally \( t \). Hence,

\[
\Pr [G \in B_2] \leq \sum_{k=2}^{\ell} \sum_{i} \sum_{t} B(k; i, t) \cdot 2^{-F(k; i, t)} \leq \sum_{k=2}^{\ell} \varepsilon^k \cdot \max_{i_j} \{ n_{i_j} \} \cdot \max_{i,t} \{ 2^{E(k; i, t)} \}.
\]

Observe that \( \min_{1 \leq i \leq n} n_i \geq \frac{n}{\ell} - (1 + \varepsilon') \sqrt{2 \log n} \geq 2 \log n + \varepsilon' \sqrt{2 \log n} \). We obtain

\[
\max_{i,t} E(k; t, i) \leq \sum_{i,j=0}^{k-1} \log n_{i,j} - (n_{i,j} - 1) = - \sum_{j=0}^{k-1} (1 - o(1)) n_{i,j} \leq -(1 - o(1))2k \log n.
\]

With \( \max_{1 \leq i \leq \ell} n_i \leq (1 + \varepsilon') \frac{n}{\ell} \) we conclude

\[
\Pr [G \in B_2] \leq \sum_{k=2}^{\ell} \varepsilon^k \cdot (1 + \varepsilon') \frac{n}{\ell} \cdot 2^{-[1-o(1)]2k \log n} = \sum_{k=2}^{\ell} 2^{-k[1-o(1)] \log n} = o(1).
\]

\[\square\]

**Proposition 4.29.** Let \( \Pi = (V_1, \ldots, V_\ell) \) be an \( \varepsilon' \)-balanced partition and \( G \) a graph drawn uniformly at random from \( G_\Pi \). We have

\[
\sum_{m \geq 3} \Pr [G \in B_{\geq m}] = o(1).
\]

**Proof.** First observe that for \( m \geq 2 \log n - 2 \log \log n + C := m_0 \), where \( C \) is an absolute positive constant, there is no \( m \)-composed independent set whp. To see this note that otherwise \( G \) would contain an independent set of size \( m_0 \), where all \( m_0 \) vertices would belong to different color classes. The expected number of such sets is at most

\[
\left( \frac{n}{m_0} \right)^{2 - \left( \frac{m_0}{m} \right)} \leq \left( \frac{c n}{m_0} \right)^{m_0} 2^{-\left( \frac{m_0}{2} + m_0/2 \right)} \leq 2^{m_0 \log n - m_0 \log m_0 - m_0^2/2 + \Theta(m_0) = o(1)}.
\]

Next assume that \( m < m_0 \) and that there is an independent set of size \( s \geq \frac{n}{\ell} - (1 + \varepsilon') \sqrt{2 \log n} \) in \( G \), which is obtained by gathering \( x_1, \ldots, x_m \) nodes from color classes with sizes \( n_1, \ldots, n_m \). For any given \( x = (x_1, \ldots, x_m) \) the probability for the event to occur is at most

\[
P_m(x) := \prod_{i=1}^{m} \left( \begin{array}{c} n_i \\ x_i \end{array} \right) \cdot 2^{-\sum_{1 \leq i < j \leq k} x_i x_j} \leq 2 \left( \tilde{n} \sum_{i=1}^{m} H \left( \frac{x_i}{\tilde{n}} \right) - \sum_{1 \leq i < j \leq m} x_i x_j \right), \tag{4.34}
\]

where we abbreviated \( \tilde{n} = \frac{n}{\ell} + (1 + \varepsilon') \sqrt{2 \log n} \). In order to get an upper bound for the exponent, we maximize it with respect to the side condition \( \sum_{i=1}^{k} x_i = s \) by using Lagrange multipliers. We define the function \( P_m(x, \lambda) := P_m(x) + \lambda(\sum_{i=1}^{k} x_i - s) \) which has the partial derivatives

\[
\frac{\partial P_m}{\partial x_i} = H' \left( \frac{x_i}{\tilde{n}} \right) - \sum_{j \neq i} x_j + \lambda \quad \text{and} \quad \frac{\partial P_m}{\partial \lambda} = \sum_{i=1}^{m} x_i - s.
\]

Because of symmetry \( P_m \) obtains an extremal value when the \( x_i \)'s are as equal as possible. Moreover, \( P_m \) becomes extremal when \( x_i = 1 \) for all \( i \) and \( x_j = s - m + 1 \) for one index \( j \). It follows from (4.34)

\[
P_m(x) \leq 2 \left( (m - 1) \log \tilde{n} + \tilde{n} H \left( \frac{s - m + 1}{\tilde{n}} \right) + \frac{1}{2} m^2 - \frac{1}{2} m - s(m - 1) \right) =: P_m^{\text{max}}.
\]
As there are \( \binom{\ell}{m} \) ways to choose the color classes in \( G \) and \( \binom{s-1}{m-1} \) natural number solutions to the equation \( \sum_{i=1}^{m} x_i = s \), the overall probability for the event \( B_{\geq \ell}^m \) is bounded from above by

\[
\Pr [G \in B_{\geq \ell}^m] \leq \left( \frac{\ell}{m} \right) \cdot \left( \frac{s-1}{m-1} \right) \cdot p_{\max}^m \leq 2^{m \log \left( \frac{\ell}{m} \right) + \Theta(m)} \cdot \left( \frac{s-1}{m-1} \right) \cdot p_{\max}^m .
\]

We first investigate "small" \( m \). For \( m = 3, 4 \) observe that

\[
\hat{n} H \left( \frac{s - m + 1}{\hat{n}} \right) = o(\log n).
\]

Therefore, with \( \binom{s-1}{m-1} \leq S^{m-1} \) and (4.35)

\[
\log \Pr [G \in B_{\geq \ell}^m] \leq m \log n + m \log s + m \log \hat{n} - s(m-1) + o(\log n)
\]

\[
\leq m \cdot (\log n + \log s + \log \hat{n} - s) + s + o(\log n)
\]

\[
\leq -m \frac{s}{2} + s + o(s) = \Omega(\log n).
\]

Now assume that \( 5 \leq m \leq 2 \log n - 2 \log \log n + C \). With \( H(x) \leq 1 \) for \( 0 \leq x \leq 1 \) we obtain

\[
\log \frac{\Pr [G \in B_{\geq \ell}^m]}{\binom{s-1}{m-1}} \leq m \log \frac{\ell}{m} + \Theta(m) + (m-1) \log \hat{n} + \frac{m^2}{2} - \frac{m}{2} - S(m-1)
\]

\[
\leq m \cdot \left( \log \frac{\ell}{m} - \log S + \log \hat{n} + \frac{m}{2} - S + \Theta(1) \right) + m \log S + 3S,
\]

where we estimated \( \hat{n} \leq 2s \). Observe that with the definition of \( s \) and \( \hat{n} \)

\[
-\log S + \log \hat{n} = -\log \left( \frac{n}{\ell} - c \sqrt{\log n} \right) + \log \left( \frac{n}{\ell} + c \sqrt{\log n} \right) = O(1),
\]

as \( \sqrt{\log n} = o\left( \frac{n}{4} \right) \). Write now \( \ell = \frac{n}{2 \log n + d(n) + x} \), for some \( x \geq 0 \). We obtain

\[
\log \frac{\Pr [G \in B_{\geq \ell}^m]}{\binom{s-1}{m-1}} - m \log s - 3s \leq m \cdot \left( \log \frac{\ell}{m} + \frac{m}{2} - s + \Theta(1) \right),
\]

which is at most

\[
m \cdot \left( \frac{m}{2} - \log n - \log (2 \log n + d(n) + x) - \log m + \log \log n - x - \omega_n + \Theta(1) \right)
\]

\[
\leq m \cdot \left( \frac{m}{2} - \log n - x - \omega_n - \log m + \Theta(1) \right).
\]

First we investigate the case \( m \leq (2 - \epsilon) \log n \), where \( \epsilon \) is an arbitrarily small positive constant. We estimate \( \binom{s-1}{m-1} \leq 2^{(m-1) \log S} \) and obtain with (4.36)

\[
\log \Pr [G \in B_{\geq \ell}^m] \leq m \cdot \left( \frac{m}{2} - \log n - x - \omega_n - \log m + \Theta(1) \right) + 2m \log S + 3S.
\]

For \( 5 \leq m \leq (2 - \epsilon) \log n \) this function is convex, as the second derivative with respect to \( m \) is \( 1 - \frac{1}{m \ln 2} \), which is always positive. Therefore, the maximum is located at \( m_0 = 5 \) or \( m_0 = (2 - \epsilon) \log n \). By substitution we obtain

\[
\log \Pr [G \in B_{\geq \ell}^5] \leq 5 \cdot \left( -\log n - x - \omega_n + \Theta(1) \right) + 10 \log S + 3S = -\Omega(\log n),
\]

\[
\log \Pr [G \in B_{\geq \ell}^{(2 - \epsilon) \log n}] \leq (2 - \epsilon) \log n \cdot \left( -\frac{\epsilon}{2} \log n - x + 2 \log S + \Theta(1) \right) + 3S
\]

\[
= -\Omega(\log^2 n),
\]
and we conclude $\Pr [G \in B^m ] = n^{-\Omega(1)}$.

Finally, let $(2 - \epsilon) \log n \leq m \leq 2 \log n - \log \log n + C =: m^u$. We estimate $\binom{S-1}{m-1} \leq 2^{(S-m) \log S}$ and obtain with (4.36)

$$\log \Pr [G \in B^m ] \leq m \cdot \left( \frac{m}{2} - \log n - x - \omega_n - \log m + \Theta(1) \right) + S(\log S + 3)$$

This function is again convex in $m$. Therefore

$$\log \Pr [G \in B^{(2 - \epsilon \log n)} ]$$

$$\leq (2 \log n - \log \log n + \omega_n + x) (\log (2 \log n - \log \log n + \omega_n + x) + 3)$$

$$- (2 - \epsilon) \log n (\Theta(\log n) + x) + O(\log n)$$

$$\leq 2 \log n \log \log n + x \log \log n - \Omega(\log^2 n + x \log n) = -\Omega(\log^2 n),$$

and

$$\log \Pr [G \in B^{m^u} ] \leq S(\log S + 3) - (2 \log n - 2 \log \log n + C)(2 \log \log n + x + \omega_n)$$

$$\leq -\Omega(\log n \log \log n + x \log n).$$

The maximum is achieved at $m_0 = m^u$; putting all the above facts together completes the proof. \qed
Bibliography


References


[COPS] A. Coja-Oghlan, K. Panagiotou, and A. Steger, On the chromatic number of random graphs, Accepted for publication in Journal of Combinatorial Theory, Series B.


