Doctoral Thesis

Direct and large-eddy simulation of compressible rectangular jet flow

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DIRECT AND LARGE-EDDY SIMULATION OF COMPRESSIBLE RECTANGULAR JET FLOW

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2003
Abstract

Simulation results of a Mach 0.5 rectangular jet with a nozzle aspect ratio of five and a Reynolds number of 2000 based on the narrow nozzle width are presented. The focus is on the flow development in the near-nozzle region and on the flow-induced acoustic radiation from the jet. Perturbations based on most unstable modes of the laminar inflow profile predicted by linear stability theory are enforced at the inflow to initiate transition. A high-order simulation code was developed and validated and direct numerical simulations (DNS) as well as large-eddy simulations (LES) were performed. For the LES the approximate deconvolution model (ADM) is applied, which is based on an approximate inversion of the LES filter. DNS results show rapid development of the rectangular jet in the transitional regime towards an axisymmetric shape. The transition process of the jet is well reproduced by the LES and mean flow data show very good agreement with the filtered DNS data. Acoustic analysis of DNS data based on Lighthill’s analogy confirms typical azimuthal-directive behaviour of non-axisymmetric jets along the minor axis of the jet in the main radiation direction. Acoustic far-field prediction based on LES data reproduces the dominant lower frequencies up to a cutoff frequency.

Kurzfassung

Es werden Simulationsergebnisse für einen aus einer rechteckigen Düse austretenden Freistrahl mit Seitenverhältnis 1:5 vorgestellt. Die Machzahl beträgt 0.5 und die mit der kurzen Seite der Düse gebildete Reynoldszahl 2000. Der Schwerpunkt liegt auf der Untersuchung des düsennahen Bereiches sowie der strömungsinduzierten akustischen Abstrahlung des Freistrahlles. Im Einströmbereich werden Störungen basierend auf den instabilsten Moden einer linearen Stabilitätsanalyse des laminaren Strömungsprofils überlagert, um die Transition einzuleiten. Es wurde ein Simulationsprogramm basierend auf kompakten Differenzenverfahren hoher Ordnung entwickelt, und direkte numerische Simulationen (DNS) sowie Grobstruktursimulationen (LES) wurden durchgeführt. Für die Grobstruktursimulationen wird ein auf der approximativen Invertierung des LES-Filters beruhendes Model (ADM) verwendet. Die DNS-Ergebnisse zeigen, wie sich das mittlere Strömungsfeld des rechteckigen Freistrahl sehr schnell einer radialsymmetrischen
Chapter 1

Introduction

1.1 Non-axisymmetric jet flow

Free shear flows have been extensively studied in the past due to their great significance in practical applications like propulsion and combustion. Fundamental studies of mixing layers by Bradshaw (1966), the well-known experiments of Brown & Roshko (1974) and Papamouschou & Roshko (1988) and others aimed at the understanding of this important building block configuration. The existence of coherent structures at high Reynolds numbers was identified to be crucial for the mixing process in free shear flows. Comprehensive reviews have been provided by Liu (1989) and Hussain (1986). Increasing complexity, the study of plane and axisymmetric jet flow found then enormous interest, where a wealth of studies is also available. Early measurements of the plane jet were conducted by Sato (1960), Bradbury (1965) and extended later by many others e.g. Gutmark & Wygnanski (1976), Hussain & Clark (1977), Mumford (1982), Antonia et al. (1983), Namer & Ötügen (1988) and Huang & Hsiao (1999). Even more data is available on the axisymmetric configuration. We here only mention the experiments by Crow & Champagne (1971). The existence of coherent structures was confirmed for both plane and axisymmetric jet flow. Further insight, at least at moderate Reynolds numbers, was gained with computational studies of such flows. Recent simulations were performed for example by Stanley et al. (2002) for the plane jet and Freund et al. (2000a,b) and Freund (2001) for axisymmetric flow.

It was soon realized that non-axisymmetric jet configurations have profound advantages versus their round counterparts in certain respects. In particular, they can increase the mixing with surrounding fluid. Due to the presence of higher-order instability modes, non-axisymmetric jets are more unstable than round configurations. Non-azimuthal curvature of vortex rings leads to the important phenomenon of axis switching. This means that the cross-section of the evolving jet takes shapes similar to the one of the jet nozzle but with axes rotated versus the original nozzle. Streamwise vorticity created mainly by self-induction of the az-
Introduction

Non-axisymmetric jets are found to show increased entrainment when compared to round jets, and by far the largest entrainment is obtained using vortex generators (tabs). At a Mach number of 0.95 (not shown) Zaman (1999) finds less entrainment for the rectangular jet nozzle, however, assumes the reason for this to be a different nozzle-edge radius of the circular nozzle.

Figure 1.1: Figures reproduced from Zaman (1999): Total axial mass flux $\dot{m}$ over entering mass flux through the jet $\dot{m}_e$ with varying nozzle shapes of equal equivalent diameter $D_{equiv}$ (round, lobed, rectangular, elliptical, round with tabs, rectangular with tabs) at (a) Mach 0.3 and (b) Mach 1.63.

imathal vortex rings increases the mixing with ambient fluid enormously. This research was pioneered by Sforza et al. (1966) who conducted measurements for a variety of jet-nozzle shapes. Ho & Gutmark (1987) carried out experiments for a small aspect ratio elliptic jet and concluded that self-induction is the dominating effect that leads to increased spreading. Thorough reviews and summaries of data for different nozzle shapes are available from Gutmark et al. (1995), Gutmark & Grinstein (1999), and Zaman (1999). They also include a discussion of additional means to increase mixing in the form of vortex generators such as tabs (small protrusions placed at the nozzle exit), lobed and tapered nozzles, etc. In figure 1.1 two plots from Zaman (1999) are reproduced that show the jet spreading from different nozzle shapes of equal exit areas at two jet Mach numbers. The spreading is measured by the ratio of the total downstream mass flux $\dot{m}$ to the mass flux through the jet nozzle $\dot{m}_e$. Non-axisymmetric jets are found to show increased entrainment when compared to round jets, and by far the largest entrainment is obtained using vortex generators (tabs). At a Mach number of 0.95 (not shown) Zaman (1999) finds less entrainment for the rectangular jet nozzle, however, assumes the reason for this to be a different nozzle-edge radius of the circular nozzle.
Non-axisymmetric jets therefore can be an efficient means of passive flow control to improve large- and small-scale mixing. They can also reduce noise generation (see also section 1.3) compared to their round counterparts, a fact that holds for subsonic and supersonic cases. In particular, rectangular and square jets have found deeper attention e.g. by Trentacoste & Sforza (1967), Sfeir (1976, 1979), Krothapalli et al. (1981), Tsuchiya et al. (1986), Zaman (1996), and Lozanova & Stankov (1998). Kim & Samimy (1999) additionally examine trailing edge modifications of rectangular jets. Also an increasing number of computational studies is becoming available. Grinstein et al. (1995) performed a combined experimental/computational study of a square jet, and only recently Grinstein (2001) presented data from an thorough aspect ratio and Mach number dependency study of a rectangular jet.

It is generally agreed upon that the downstream development of a jet strongly depends on the initial conditions at the jet nozzle. Early work in that field was done by Bradshaw (1966) who examined the sensitivity of a mixing layer. Ho & Huerre (1984) give a review of perturbed shear layers. Recent publications (Lozanova & Stankov, 1998; Antonia & Zhao, 2001; Xu & Antonia, 2002) are in agreement with that finding. A very illustrative example about the influence of jet forcing is given in Reynolds et al. (2003), where the aspect of bifurcation, i.e. the separation of the initial jet into two jets caused by strong inflow forcing is summarized. Parameters which influence the downstream behaviour of a jet are the initial shear layer thickness, the shear layer thickness distribution (which strongly depends on the upstream curvature of the nozzle), the nozzle shape, Mach and Reynolds numbers, and the turbulence intensities at the inflow. Additionally, the presence of peaks in the excitation spectrum at the inflow is of more importance than simply the integral turbulent intensity (Hussain, 1986), since the mere presence of one frequency can trigger instabilities in the jet if the frequency matches the one of an unstable jet mode.

Almost always the collectivity of these parameters is not available from a particular experiment, the reason being that they are not documented or simply partially unknown. This makes it difficult to compare experimental data taken in different facilities and also computational data with experimental data. Inflow conditions of numerical computations can at best be matched to some degree and it is left to the creativity of the researcher how well an experiment is reproduced by a computation. A sound portion of uncertainty, however, always remains.
**1.2 LES modelling using approximate deconvolution**

Large-eddy simulation (LES) has been established as a widely used research tool in simulating high Reynolds number flow and a large variety of models is currently in use. Recent reviews are available from Lesieur & Métais (1996), Meneveau & Katz (2000) and Domaradzki & Adams (2002). Whereas the first two references deal with classical LES modelling approaches, i.e. direct modelling of the unclosed LES terms, the latter reference focuses on so called deconvolution-type models (also called velocity estimation models). Here the unclosed terms of the LES equations are not modeled but instead one tries to find an estimate for the unfiltered dependent variables from which the unclosed terms are computed directly.

The recently proposed approximate deconvolution model (ADM) (Stolz & Adams, 1999) falls in that category and we also employ this model in the present study. The model is based on explicit filtering and approximate defiltering (deconvolution) of the conservative variables. It was successfully applied to a number of canonical flows including incompressible turbulent channel flow (Stolz et al., 2001a), incompressible isotropic turbulence (Müller et al., 2002), compressible channel flow (von Kaenel et al., 2002a), supersonic compression corner flow (Stolz et al., 2001b; von Kaenel et al., 2002b) and to supersonic boundary layer cases (Stolz & Adams, 2003).

In the following the principle of ADM is briefly outlined. Consider a general vector-valued quantity of dependent variables \( \mathbf{u} \), which is for the conservative formulation of the Navier-Stokes equation composed of the density \( \rho \), the momentum components \( \rho u_i \) and the total energy \( E \) as \( \mathbf{u} = \{ \rho, \rho u_1, \rho u_2, \rho u_3, E \} \). Following the terminology of Domaradzki & Adams (2002) it can be split into a part that can be represented on a numerical grid \( \mathbf{u}^L \) and a part \( \mathbf{u}^S \) that cannot. Now, in addition to the grid projection a filter is introduced. A filtered quantity \( \mathbf{u}^L \) is defined by convolution with the filter kernel \( G^L \) as

\[
\mathbf{u}^L(x) = \int G^L(x-x') \mathbf{u}^L(x') dx'
\]

and is defined on the numerical grid. The bounds of the integral are determined by the compact support of the filter kernel. The vector-
valued quantity $\mathbf{u}$ is then split as

$$
\mathbf{u} = \mathbf{u}^L + (\mathbf{u}^L - \mathbf{u}^\ell) + \mathbf{u}^S.
$$

(1.2)

"filtered" "subfilter" "subgrid"

Applying the grid projection and the filter to a generic transport equation with the flux function $\mathbf{f}$ (Einstein summation rule applies)

$$
\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x_i} = 0
$$

(1.3)

one obtains

$$
\frac{\partial \mathbf{u}^L}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})^L}{\partial x_i} = 0.
$$

(1.4)

The filtered flux is a function of the unfiltered solution $\mathbf{u}$ resulting in the LES closure problem. In the approximate deconvolution model of Stolz & Adams (1999) the underlying idea is to find an approximation $\mathbf{u}^*$ for $\mathbf{u}$ from $\mathbf{u}^L$ rather than modelling the unclosed terms themselves. By using a defiltering operation, the subfilter part in (1.2) can be partially recovered and an approximation of the fluxes is obtained directly from the deconvolved fields and they are filtered afterwards.

No information from the subgrid scales can be retained. Therefore their effect is modeled by adding a so-called relaxation term on the right-hand side of the equation, using a high-order secondary filter $G_2$. This relaxation term acts only on the high-wavenumber region of the spectrum. The model equation then reads

$$
\frac{\partial \mathbf{u}^L}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u}^L)^L}{\partial x_i} = -\chi(1 - G_2) \ast \mathbf{u}^L.
$$

(1.5)

The inverse of the parameter $\chi$ has the dimension of time and can be determined dynamically during the computation (Stolz et al., 2001b). Further details of the formulation are given in section 2.3.

1.3 Jet-flow acoustics

The problem of aeroacoustically induced noise attracts increasing attention, in particular due to the enormous interest of reducing noise emission of aircraft frames and engines. Numerical methods are being developed
and used to tackle problems of noise propagation and more recently also of the noise generation caused by turbulent flows. An excellent review on the physical mechanisms involved is given by Crighton (1975).

In the prediction of noise the relative success of Reynolds-averaged-Navier-Stokes (RANS) models in fluids engineering has not extended, even when the only objective is to predict general trends, and it is not obvious how to improve their fidelity given the complexity of statistical noise sources. Since the inherent flow unsteadiness is the source of jet noise, time-accurate simulations hold more promise. Direct numerical simulation (DNS) has been successfully used to predict radiated noise from turbulent flows at low Reynolds numbers (Freund, 2001) and naturally LES is attractive for higher Reynolds number applications. Nevertheless, the computation of flow-induced noise has its unique difficulties, and several important issues arise when extending established LES capability to aeroacoustic problems. Additional questions such as the effects of subgrid-scale modelling and numerical errors must be addressed. Most easily LES can be applied if the bulk of the noise, at least at frequencies of interest, comes from scales that are retained in the simulation and do not have to be modeled. This is believed to be the case based on statistical analysis (Narayanan et al., 2002). Additionally, it seems that the low- and high-frequency noise does indeed come from the vicinity of the jet nozzle which can be included in numerical simulations.

There are several possibilities to compute the aeroacoustically generated noise (for recent reviews see e.g. Lele (1997), Wells & Renault (1997), Lilley (1995) and Goldstein (1995)). The simplest is its computation from first principles, that is solving the full compressible Navier-Stokes equations for the flow (DNS) and the acoustic far-field. Due to the different length scales of the acoustic and the flow field this is computationally prohibitive for most real applications. However, it can be applied for low Reynolds number cases. Recently, direct computation of the sound was applied using LES for the flow field of a high Reynolds number round jet with promising results (Bogey et al., 2003). Another possibility is to couple the noise-generating flow domain (solving the Navier-Stokes equations) with an exterior acoustic domain with less resolution restrictions and simpler equations (solving the Euler or wave equation) and to compute the radiation in time simultaneously. Here, the problem of interpolation from one grid to another has to be solved. Similarly, surface integral methods that solve the Kirchhoff integral over a shell surrounding the sound sources have the advantage that the far-
field can be computed at any desired location and no external grid is needed. The restriction on the choice of the surface around the noise sources is, however, that it has to be located sufficiently far away from non-linear fluid interactions and yet well inside the computational domain to avoid an influence by the boundary treatment of the simulation. A recent example is given by Zhao et al. (2001) for a subsonic circular jet.

For the present study we make use of another method, namely Lighthill’s acoustic analogy (Lighthill, 1952, 1954). The theory treats the non-linear fluid interactions as source terms for an acoustic far-field computation and is well suited for jet-noise computation. It has the advantage that it can be used in a post processing step to an available simulation (given sufficient temporal resolution of the data). Apart from providing information about the far-field sound of turbulent flow it also gives an indication where the dominant acoustic sources are located within the flow field.

1.4 Objectives and outline of the present work

The target of the present investigation is threefold and is centered around the transitional flow of rectangular subsonic jet.

1. First, the transition process of a non-axisymmetric jet of rectangular cross-section issuing into a quiescent ambient was studied using DNS. The focus was laid on the flow structure behind the nozzle. The aspect ratio was set to five which is in between the two limits of a plane and square jet-nozzle cross-section. The computational domain was chosen to be large enough to capture the beginning of the characteristic-decay region of the mean velocity at the jet center as defined by Sforza et al. (1966), however, it does not extend into the subsequent axisymmetric-decay region. For this purpose a high-order Cartesian-grid DNS code was developed and extensively validated. It was furthermore optimized for the use on a NEC SX5 parallel vector machine, where most of the computations have been performed. Particular attention was paid to the definition of inflow disturbances for the laminar jet and the inflow base profile. Idealized inflow data were imposed which match jet-nozzle outflow conditions in experiments only approximately. No attempt was made to exactly match an existing experiment for the large
uncertainty in documented inflow data. Additionally, to our best knowledge there are no low Reynolds number data of such flows available. An inviscid linear instability solver was used to compute the most unstable modes of the chosen artificial inflow profile and this mode was superimposed at the inflow with constant amplitude and frequency. Flow visualization of the transition process was carried out and statistical flow data were obtained.

2. Second, we investigated the capability of LES with the Approximate Deconvolution Model (ADM) to predict transitional free shear flows by a direct comparison of LES with filtered DNS data. The model was implemented in the previously developed DNS code. The flow conditions at the inflow were known from the DNS and could be imposed also in the LES calculations in order to separate errors from LES modelling from those of inflow data.

In an additional study we also assessed the influence of inflow conditions on the downstream behaviour of the jet. Turbulent duct flow data generated in a separate temporal simulation were used as inflow data for an LES simulation of a jet. Comparison with jet flow with a laminar forced inflow was carried out.

3. The third objective was to compute the radiated sound originating from the turbulence created by the rectangular jet. Lighthill’s theory in frequency-space formulation was used to compute the far-field noise from the jet flow predicted by DNS and LES simulations. The location of the noise sources was determined and the radiated far-field sound as well as far-field-sound spectra were computed. Comparison with acoustic radiation from a round jet (Freund, 2001) was made. One intention of this study was to assess how well LES can predict the louder, lower-frequency noise of the flow. Additionally, the question was assessed whether the particular formulation of the employed subgrid scale model ADM can be used to improve the noise prediction capability of the LES by computing the Lighthill source terms based on deconvolved rather than filtered quantities.

The outline of this report is as follows. The description of the numerical method, the boundary conditions, the inflow treatment as well as details of the LES model are covered in chapter 2. Results from the DNS simulation of a rectangular jet are presented in chapter 3 and of
1.4 Objectives and outline of the present work

the corresponding LES in chapter 4. Finally, the acoustic analysis of the jet flow is presented in chapter 5 and conclusions are drawn in chapter 6.
2.1 Physical model

The compressible Navier-Stokes equations in conservative form

\[
\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}_1}{\partial x_1} + \frac{\partial \mathbf{f}_2}{\partial x_2} + \frac{\partial \mathbf{f}_3}{\partial x_3} = 0
\]  

(2.1)

are the underlying equations for the simulation method. The conservative variables are \( \mathbf{u} = \{\rho, \rho u_1, \rho u_2, \rho u_3, E\} \), with the density \( \rho \), the velocity components \( u_i \), the total energy \( E = p/(\gamma - 1) + \rho/2(u_1^2 + u_2^2 + u_3^2) \), the pressure \( p \) and the temperature \( T \). The variables are non-dimensionalized by a reference length \( L_{\text{ref}} \), a reference velocity \( u_{\text{ref}} \), a reference density \( \rho_{\text{ref}} \), and a reference temperature \( T_{\text{ref}} \), where \( _d \) denotes a quantity with dimension. The flux function in the \( x_1 \)-coordinate direction is given by

\[
\mathbf{f}_1 = \begin{pmatrix}
\rho u_1 \\ \rho u_1^2 \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ (E + p)u_1 - u_1 \tau_{11} - u_2 \tau_{12} - u_3 \tau_{13} + q_1
\end{pmatrix},
\]

(2.2)

where the viscous stresses are defined as

\[
\tau_{11} = \frac{2}{3} \frac{\mu}{Re} \left(2 \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \frac{\partial u_3}{\partial x_3}\right) \quad \text{and} \quad \tau_{12} = \frac{\mu}{Re} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right). 
\]

(2.3)

The definitions for the fluxes \( \mathbf{f}_2, \mathbf{f}_3 \) as well as for \( \tau_{22}, \tau_{33}, \tau_{13} \) and \( \tau_{23} \) are analogous. The heat fluxes are defined as

\[
q_1 = -\frac{\mu}{(\gamma - 1)M^2 Re Pr} \frac{\partial T}{\partial x_1},
\]

(2.4)

with \( q_2 \) and \( q_3 \) again defined accordingly. \( M, Re \) and \( Pr \) are the Mach Reynolds and Prandtl numbers, based on the reference quantities \( _{\text{ref}} \).
defined above and $\mu$ is the dynamic viscosity normalized by $\mu^d_{ref} = \mu^d(T^d_{ref})$. We assume the equation of state for a perfect gas $\gamma M^2 p = \rho T$ to be valid and compute the viscosity using Sutherland’s law

$$\mu = T^d \frac{1 + S}{T + S},$$

with the Sutherland’s constant $S^d = 110.4 K/T^d_{ref}$.

### 2.2 Numerical Method

#### 2.2.1 Discretization

Non-equidistant meshes are required for most shear flow simulations, since the resolution of steep flow gradients is facilitated by a locally refined mesh. For this reason a mapping is introduced and for the discretization of (2.1) the physical space $\mathbf{x} = (x_1, x_2, x_3)$ is mapped onto the computational space $\xi(\mathbf{x}) = (\xi_1(\mathbf{x}), \xi_2(\mathbf{x}), \xi_3(\mathbf{x}))$. For computational efficiency we consider meshes only with grid stretching along each coordinate direction. The Jacobian of the mapping is diagonal. Equation (2.1) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \xi_i}{\partial x_i} \frac{\partial \mathbf{f}_i}{\partial \xi_i} = 0,$$

where $\frac{\partial \xi_i}{\partial x_i}$ are the diagonal terms of the Jacobian. Derivatives are determined in the computational space ($\xi_i$) and subsequently multiplied with the Jacobian. Partial derivatives that occur in the definition of the fluxes are also evaluated using derivatives in computational space and multiplying the corresponding metric terms. Second derivatives are computed by successively applying first derivatives.

Refinement in the cross-stream directions ($i = 2, 3$) toward the jet center is accomplished by a hyperbolic tangent coordinate mapping,

$$\xi_i \in [-0.5, 0.5] \quad \rightarrow \quad x_i \in [-B_i/2, B_i/2],$$

$$x_i(\xi_i) = \frac{1}{k_i} \text{atanh} \{2\xi_i \text{tanh}(k_i B_i/2))\}, \quad i = 2, 3$$

where $B_i$ is the dimension of the computational domain in the corresponding direction, and the parameters $k_i$ determine the amount of
2.2 Numerical Method

grid-point condensation. Metric terms are computed analytically as

\[
\frac{\partial \xi_i}{\partial x_i} = \frac{k_i}{2 \tanh(k_i B_i/2) \cosh^2(k_i x_i)}, \quad i = 2, 3
\]  

(2.8)

For the streamwise direction \( (i = 1) \) a constant metric term is used for the simulations, however, the implementation allows also for variable metric terms.

For spatial discretization a fifth-order compact upwind-biased scheme (Adams & Shariff, 1996, CUVB) is used for the convective terms and a sixth order (at interior points) compact central scheme (Lele, 1992) for the diffusive terms. Time integration is performed with a third order low-storage Runge-Kutta scheme (Williamson, 1980). The discretization schemes are described in detail in the respective references to which we refer the reader for further aspects.

2.2.2 Boundary conditions

For the simulation of turbulent flow in an open environment a proper formulation of the artificial boundary conditions at the computational-domain boundaries is crucial. On one hand, large vortical structures have to leave the computational domain at the outflow boundary without non-physical reflections, and on the other hand entrainment has to be allowed in the cross-stream directions. Additionally, acoustic waves originating from the transition region must be allowed to leave the domain without affecting the flow by backward reflection. The accurate and efficient formulation of such artificial boundary conditions for the Navier-Stokes equations in three space dimensions is still an unsolved problem. However, it is found in practical simulations (e.g. Adams (1998), Stanley & Sarkar (1999)) that a combination of one-dimensional non-reflecting boundary conditions (Thompson, 1987) and sponge layers, which are added at the far-field boundaries, gives reasonable results. In the sponge layers the governing transport equation for the vector of conservative variables \( \mathbf{u} \) is modified according to

\[
\frac{\partial \mathbf{u}}{\partial t} = \text{rhs}(\mathbf{u}) - \sigma \cdot (\mathbf{u} - \mathbf{u}_0),
\]  

(2.9)

where we have written the flux terms of the Navier-Stokes equations briefly as \( \text{rhs}(\mathbf{u}) = -\sum_i \mathbf{f}_i \), so that the solution is driven toward a given steady target solution \( \mathbf{u}_0 \). The coefficient \( \sigma = \sigma(\eta) \geq 0 \) is non-zero only
in the sponge layers and is of the form $\sigma(\eta) = \alpha \eta^\beta, 0 < \eta < 1$, where $\eta$ denotes a normalized coordinate across the sponge. This was validated by comparison with linear stability theory. Growth-rate comparisons for unstable eigensolutions of parallel jets have shown a very good agreement with linear theory, where the parameters $\alpha = 2, \beta = 3$ were used. We used these values also for the simulations (see section 2.2.3).

For non-reflecting boundary conditions the formulation of Thompson (1987) is implemented. The procedure is well known and documented and we only briefly reproduce the essentials here. For further details see e.g. Adams (1993). In evaluating the fluxes at the boundaries the convective and the diffusive fluxes are computed separately. Neglecting the diffusive terms, the convective fluxes are projected on the local characteristics. Based on this projection, the incoming characteristics are set to zero and the outgoing ones are computed. Back-projection onto the coordinate directions gives the convective fluxes at the boundaries. The diffusive fluxes are added unchanged and the result is used in the time-marching procedure.

At the inflow boundary all conservative quantities are specified. Away from the nozzle area an additional sponge zone is used. This sponge zone dampens backward directed acoustic waves and avoids their reflection at the inflow plane. The choice of the target solution $u_0$ in equation (2.9) will be discussed later. An overview of the applied boundary conditions is schematically shown in figure 2.1.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure2.1.png}
\end{center}

\textit{Figure 2.1: Schematics of the boundary treatment for the jet computations.}

### 2.2.3 Validation

In order to check the validity of the computational method and its implementation as a computer code it was subjected to a set of test cases,
which are described below.

**Taylor Green vortex**

The Taylor Green vortex consists of a periodic two-dimensional vortex array and is a standard test case for incompressible flow solvers for the validation of the diffusive-term treatment. It has an exact solution (Taylor, 1923) and the flow-field (at time instant) is given by

\[
\begin{align*}
  u_1 &= -\sqrt{A} \cos(2\pi x_1) \sin(2\pi x_2) e^{8\pi^2 t/Re} \\
  u_2 &= \sqrt{A} \sin(2\pi x_1) \cos(2\pi x_2) e^{8\pi^2 t/Re} \\
  p &= p_{ref} - \frac{A}{4} [\cos(4\pi x_1) + \cos(4\pi x_2)] e^{8\pi^2 t/Re}.
\end{align*}
\] (2.10)

The flow field decays with the exponential decay rate \(8\pi^2/Re\). We used the setup of Sandham & Reynolds (1989, p. 54) with \(Re = 1\), box dimensions \((x_1, x_2, x_3) \in [-0.5, 0.5]^3\), a reference pressure \(p_{ref} = 1/\gamma M^2\), a reference Mach number of \(M = 0.2\) and \(A = 0.0016\). The Mach number led to a local peak Mach number of 0.008 which was close to the incompressible limit. The decay of the numerical solution was compared with the exact vortex decay rate. On a grid of only \(12^3\) nodes the relative error at time \(t = 1/(8\pi^2)\) was less than 0.0229%.

**Temporally growing instabilities**

A rigorous two-dimensional test case is the comparison of growth rates for linearly unstable waves in a constant base flow. Two parallel base-

<table>
<thead>
<tr>
<th>profile</th>
<th>(u_1(x_3) = \tanh(2x_3))</th>
<th>(u_1(x_3) = 1 + 0.5 \exp(-4x_3^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid</td>
<td>30 \times 250</td>
<td>30 \times 250</td>
</tr>
<tr>
<td>domain ((x_1 \times x_3))</td>
<td>([0, 2\pi/\Re(\kappa_1)] \times [-15, 15])</td>
<td>([0, 2\pi/\Re(\kappa_1)] \times [-15, 15])</td>
</tr>
<tr>
<td>mapping parameter (k)</td>
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<td>1</td>
</tr>
<tr>
<td>initial amplitude (A_0)</td>
<td>(1 \times 10^{-5})</td>
<td>(1 \times 10^{-5})</td>
</tr>
<tr>
<td>(M)</td>
<td>0.01</td>
<td>0.5</td>
</tr>
<tr>
<td>(Re)</td>
<td>(10^6)</td>
<td>(10^6)</td>
</tr>
<tr>
<td>temperatures (T(x_3))</td>
<td>(T(\pm 15) = 1.0)</td>
<td>(T(\pm 15) = 1.0, T(0) = 2.25)</td>
</tr>
<tr>
<td>frequency (\omega)</td>
<td>(0.0 + i 0.3795)</td>
<td>(2.0723 + i 0.1525)</td>
</tr>
<tr>
<td>wavenumber (\kappa_1)</td>
<td>(0.8898 + i 0.0)</td>
<td>(1.790 + i 0.0)</td>
</tr>
</tbody>
</table>

*Table 2.1: Parameters for temporal linear stability tests.*
Figure 2.2: Eigenfunctions for the profile $u_1(x_3) = \tanh(2x_3)$, magnitude, real part, imaginary part.

Figure 2.3: Eigenfunctions for the profile $u_1(x_3) = 1 + 0.5\exp(-4x_3^2)$, magnitude, real part, imaginary part.
flow profiles, namely a parallel shear flow with a hyperbolic tangent profile \( u(x_3) = \tanh(2x_3) \) and a generic jet profile \( u(x_3) = 1 + 0.5 \exp(-4x_3^2) \) were tested. The base flow was parallel in the \( x_1 \)-direction and was only a function of the \( x_3 \)-direction. For the chosen profiles the temperature profiles were determined with the Crocco-Busemann relation (White, 1974) with temperature constraints as listed in table 2.1. An inviscid compressible temporal/spatial eigensolver (for details and notation see appendix A) was used to compute the most unstable mode for the given parameters. The eigenfunctions of this mode were superimposed onto the parallel mean flow (amplitude \( A_0 \)) and the theoretically predicted growth rates were compared with the ones given by the simulation code. 2D simulations using the mapping in equation (2.7) were performed. To prevent temporal mean-flow variation a forcing was introduced that cancels the mean-flow distortion. For small perturbations the result of the modified equation is equivalent to the exact equation for the perturbations. At the lateral boundaries non-reflecting boundary conditions were used. The parameters of the two cases and the results of the eigensolver are listed in table 2.1.

In figure 2.2 and 2.3 the eigenfunctions are shown for the two cases, where \( \hat{\cdot} \) denotes the complex eigenfunction for the respective variable. Normalization was achieved such that \( \max(|\hat{u}_1|) = 1 \). For the tests the box size was chosen so that it exactly contained one wavelength of the instability mode. The modal energy of this mode was computed as

\[
E(1,0) = u_1(n_{\kappa,1} = 1, n_{\kappa,3} = 0) \ u_1^*(n_{\kappa,1} = 1, n_{\kappa,3} = 0) \quad . \quad (2.11)
\]

Here, \( u_1(n_{\kappa,1}, n_{\kappa,3}) \) denotes the spatial Fourier transform of the downstream velocity (wavenumbers \( \kappa_1, \kappa_3, \ n_{\kappa,i} = \frac{\Re(\kappa_i)B_1}{2\pi} \)) and \( \cdot^* \) denotes the complex conjugate. The temporal growth of \( E(1,0) \) was used for comparison with linear theory. The modal energy had a relative error

\[
\frac{|E_{\text{computed}}(1,0)(t) - E_{\text{linear theory}}(1,0)(t)|}{E_{\text{linear theory}}(1,0)(t)} \quad (2.12)
\]

which was less than 0.1% over one dimensionless time unit \( (0 \leq t \leq 1) \).

**Spatially growing jet instability**

In order to check the validity of the outflow boundary condition formulation the spatial growth of an instability of a parallel shear flow profile
Figure 2.4: Comparison of the spatial growth rate for the profile $u_1(x_3) = 1 + 0.5 \exp(4x_3^2)$. (a) Parameter variation of the sponge: $\alpha = 0.2, \beta = 3$; $\alpha = 2.0, \beta = 3$; $\alpha = 10.0, \beta = 3$; no sponge; linear stability theory; (b) contours of the $u_3$ velocity component.

was analyzed. The variation of the spatial growth rate within the flow field is a sensitive indicator for the appropriateness of the boundary-condition formulation. The spatial eigenproblem (see appendix A) was solved for the same generic profile $u_1(x_3) = 1 + 0.5 \exp(4x_3^2)$ with a Mach number of 0.5. The spatially most unstable symmetric mode was superimposed onto the parallel mean flow with an initial amplitude of
A_0 = 1 \times 10^{-5} \text{ (at the inflow) and an initial downstream evolution according to linear theory. During the computation the parallel base flow was again retained using a forcing term. At the inflow the jet-profile with the superimposed time-dependent spatial eigenmode was enforced. The eigenfunctions were normalized such that max(|\tilde{u}_1|) = 1. At the outlet non-reflecting boundary conditions in combination with a sponge layer (eq. 2.9) were applied. The parameters are summarized in table 2.2. The spatial growth rates were computed sampling the u_1-velocity over 10 oscillation periods in time and evaluating the modal growth of the instability by a temporal Fourier transform of the signal. The spatial growth rates using various sponge parameters are plotted in figure 2.4 (a). A strong deviation from the theoretical value from linear stability is observed when no sponge at all is used. The influence of the sponge parameters \( \alpha \) and \( \beta \) is weak and the best results were obtained for the combination \( \alpha = 2 \) and \( \beta = 3 \). We used those values for the computations. In figure 2.4 (b) u_3 velocity contours are shown to visualize the spatial growth of the instability.}

### Table 2.2: Parameters for spatial linear stability test.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_0 )</td>
<td>( 1 \times 10^{-5} ) (at the inflow)</td>
</tr>
<tr>
<td>Grid</td>
<td>( 240 \times 241 )</td>
</tr>
<tr>
<td>Domain ((x_1 \times x_3))</td>
<td>([0, 12 \cdot 2\pi/R(k)_{1}] \times [-15, 15])</td>
</tr>
<tr>
<td>Mapping parameter ( k )</td>
<td>0.8</td>
</tr>
<tr>
<td>Initial amplitude ( A_0 )</td>
<td>( 1 \times 10^{-5} )</td>
</tr>
<tr>
<td>M</td>
<td>0.5</td>
</tr>
<tr>
<td>Re</td>
<td>( 10^6 )</td>
</tr>
<tr>
<td>Temperatures ( T(z) )</td>
<td>( T(0) = 2.25, T(\pm 15) = 1 )</td>
</tr>
<tr>
<td>Frequency ( \omega )</td>
<td>( 0.0 + 1, 1.9888 )</td>
</tr>
<tr>
<td>Wavenumber ( \kappa_1 )</td>
<td>1.7195241 - i, 0.1209497</td>
</tr>
</tbody>
</table>

### 2.3 LES modelling

The previously described DNS code was extended by a subgrid-scale model for large-eddy simulation in order to allow computations at higher Reynolds number not possible with DNS. As subgrid-scale model we used the Approximate Deconvolution Model (ADM) formulation for compressible flows. Details of the model are described by Stolz et al. (2001a,b). We reproduce the formulation here briefly and describe the
modifications that were used for the present simulations.

2.3.1 Approximate Deconvolution Model

We follow the notation introduced in section 1.2. In classical LES modelling the governing equations are formally filtered in space such that the unknown filtered variables $\mathbf{u}^L$ can be resolved on a given numerical grid. Non-closed terms appear as a result of the filtering of non-linear terms and closure is achieved by employing an appropriate subgrid-scale model. Often the filter is associated implicitly with the grid filter and is not explicitly applied in the computations. The underlying principle of ADM is the definition of a primary discrete filter (with kernel $G$) which is explicitly applied to the Navier-Stokes equations (2.1). Instead of using a model for the unknown terms in LES (the subgrid-scale stress tensor), an approximate deconvolution operator $Q_N$ (see below) of the filter $G$ is applied to the filtered variables $\mathbf{u}^L$ to approximate the unfiltered fields denoted by $\mathbf{u}^*$. In the following all dependent variables are considered to be grid functions and the superscript $^L$ is dropped. The essence of ADM is that the flux terms are computed directly using the approximately unfiltered fields $\mathbf{u}^* \approx \mathbf{u}$ and are subsequently filtered with the discrete filter. The approximately unfiltered quantities $\mathbf{u}^*$ are computed by convolving the deconvolution operator $Q_N$ with the filtered quantity $\mathbf{u}$

$$\mathbf{u}^* = Q_N \ast \mathbf{u} \quad .$$

The operator $Q_N$ is obtained by expanding the inverse of the filter $G$ as an infinite series and truncating it at finite $N$. This leads to a regularized approximate inverse operator

$$Q_N = \sum_{i=0}^{N} (I - G)^i \approx G^{-1} \quad ,$$

where $I$ is the identity operator. In numerical tests the deconvolution order $N = 5$ was determined to be sufficient (Stolz, 2001). For higher $N$ the results did not significantly improve. Using lower $N$ led to too much dissipation in the flow computations.

This deconvolution procedure has the following advantages, that are often lacking in conventional LES modelling: (i) the formulation is in physical space and is therefore in its very principle not only designed for
academic flow situations in contrast to spectral eddy viscosity formulations that require spectral code implementation, (ii) it defines a filter in physical space that allows consistent comparison with DNS data, (iii) no formal spatial filter commutation error occurs, since filtering is applied subsequently to the computation of the flux terms of the equation.

To model the transfer of energy to small scales a relaxation term is added, which drains energy from the subfilter scales beyond the filter cutoff. It has the form \((I - G_2) \ast \overline{\mathbf{u}}\) applied to a filtered quantity \(\overline{\mathbf{u}}\) and \(G_2\) is a secondary filter of high order. In ADM the secondary filter is constructed from the product of the deconvolution operator and the primary filter kernel \(G\):

\[
G_2 = Q_N \ast G. \tag{2.15}
\]

This relaxation term operates only in the high-wavenumber band (to be interpreted in a local sense (Vasilyev et al., 1998), since no spatial Fourier transform can be defined in complex geometries) and it serves as an energy sink in this band, where the term ‘energy’ has to be interpreted in a more general sense: The relaxation term operates on all conservative variables, the mass, momentum and energy equation. Although formally no modelling has to be employed in the conservation of mass (linearity), actual simulations show that simulations become unstable if no relaxation is used. No additional high-order filtering is applied, which otherwise is often used in compressible LES to stabilize computations (e.g. Rizetta et al. (2000), Bodony & Lele (2002)). In figure 2.5 the three operators \(G\), \(Q_N G\) and \(G_2 = 1 - Q_N G\) are plotted for an equidistant grid.

The filtered conservation equations (for mass, momentum and energy) modeled with ADM are then written as

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho}u_j)^*}{\partial x_j} = -\chi_1(\bar{\rho} - \bar{\rho}^*) \tag{2.16a}
\]

\[
\frac{\partial \bar{\rho} u_i}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{(\bar{\rho} u_i)^* (\bar{\rho} u_j)^*}{\rho^*} + \bar{p}^* \delta_{ij} - \bar{T}_{ij}^* \right) = -\chi_2 (\bar{\rho} u_i - (\bar{\rho} u_i)^*) \tag{2.16b}
\]

\[
\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{(\bar{E}^* + \bar{p}^*) (\bar{\rho} u_j)^*}{\rho^*} - \bar{T}_{ij}^* (\bar{\rho} u_i)^* (\rho^*) + \bar{q}_j^* \right) = -\chi_3 (\bar{E} - \bar{E}^*), \tag{2.16c}
\]

with \(i = 1, 2, 3\). \(\bar{\rho}, \bar{\rho} u_i\) and \(\bar{E}\) denote the filtered conservative variables, \(T\) the temperature and \(p\) the pressure. \(^*\) refers to a deconvolved variable
Figure 2.5: Transfer functions of the primary filter $G$, $Q_N G$, and $1 - Q_N G$.

and $\sim$ indicates that the respective quantities are computed according to their definition but with filtered conservative variables. Quantities denoted by $\sim^*$ are computed similarly but with deconvolved variables (e.g. $f(a, b) = ab$, $\bar{f} = \overline{\sigma b}$ and $\bar{f}^* = a^* b^*$).

The (spatially and temporally varying) relaxation coefficients $\chi_i$ are determined dynamically. The amount of energy contained in the high wavenumber band for each the mass, momentum, and the total energy is estimated locally using a structure function approximation. The relaxation parameter is computed based on the assumption that this amount of energy should approximately remain constant in order to ensure a well-resolved computation. To achieve this, a forward projection in time is performed for the high-pass filtered quantities using an Euler-forward step – once with and once without relaxation term. A new value for the relaxation parameter is then obtained by weighting the previous one with the ratio of produced to dissipated energy. This algorithm leads to a large spatial variation of each $\chi_i$, which is smoothed by filtering. An implicit Padé-type filter (Lele, 1992) with a cutoff wavenumber of $\kappa'_c = \pi/8 \approx \kappa_c/5$ is used for this purpose. The filtering is analogous to averaging procedures encountered for the dynamic Smagorinsky constant. The dynamic procedure is described in detail by Stolz et al. (2001b).

To prevent spurious oscillations of the relaxation parameter in regions of highly distorted grids (lateral sponge regions), we enforce the relaxation parameter to be zero in the sponge regions. Also, for sta-
2.3 LES modelling

bility reasons we require, as in Stolz et al. (2001b), that the relaxation parameter is positive and bounded by $1/\Delta t$, $\Delta t$ being the time step of the computation. Different from the DNS, for the LES we discretize the convective terms with a sixth order compact central finite-difference scheme (Lele, 1992) to avoid additional numerical diffusion.

2.3.2 Filter definition

The definition of a discrete filter is a key part of the ADM model. For the present LES simulations we used an explicit five-point stencil filter with filter width $\Delta = h\pi/\kappa_c$, where $h$ is the computational grid spacing and $\kappa_c$ the cutoff wavenumber in computational space. A filtered grid function (one-dimensional) $\Phi_i$ is then defined as

$$\Phi_i = \sum_{j=-\nu_l}^{\nu_r} \alpha_j \Phi_{i+j},$$

(2.17)

where $\alpha_j$ are the filter coefficients and $\nu_l, \nu_r$ the left and right bounds of the filter stencil. For the present simulations central five-point stencils with $\nu_l = \nu_r = 2$ were used. Accordingly, the transfer function of the filter, its Fourier transform in the equidistant computational space, reads

$$G(\kappa) = \sum_{j=-\nu_l}^{\nu_r} \alpha_j e^{i\kappa j}.$$  

(2.18)

The $k$th discrete moment of the filter at the grid point $i$ with coordinate $x_i$ can be defined as

$$M_i^k = \frac{1}{\Delta x^k} \sum_{j=-\nu_l}^{\nu_r} \alpha_j (x_i - x_{i+j})^k.$$  

(2.19)

We follow Stolz (2001, p. 18) and determine the filter coefficients with the following constraints:

1. Conservation property: The transfer function equals unity at $\kappa = 0$,

$$\sum_{j=-\nu_l}^{\nu_r} \alpha_j = 1.$$  

(2.20)
2. Vanishing transfer function for the Nyquist frequency $\kappa = \kappa_n = \pi$,

$$\sum_{j=-\nu_l}^{\nu_r} (-1)^j \alpha_j = 0. \quad (2.21)$$

3. Conservation of polynomials up to the order $K - 1$ in physical space, i.e. vanishing $k$th moments, $k = 1, \ldots, K - 1$ with $K = 3$,

$$M^k_i = 0 \ \forall \ i. \quad (2.22)$$

4. Minimal dispersion error, i.e. the imaginary part of the transfer function $\Im(G(\kappa))$ should be small which leads to

$$\int_0^\pi \Im(G(\kappa))^2 \kappa = \frac{\pi}{2} \sum_{j=-\nu_l}^{\nu_r} (\alpha_j - \alpha_{-j})^2 = \min . \quad (2.23)$$

For a three-dimensional domain this filter is applied for all three directions successively. At the boundaries the filter stencil becomes one-sided and a special treatment is needed. We apply at the second point of the inflow and outflow panes a one-sided four-point stencil with one vanishing moment (condition 3 only up to $K = 2$). In the lateral directions at the second point of the computational boundary, condition 4 is discarded and the coefficients are determined using $K = 3$ with also a one-sided four-point stencil. In boundary-normal direction at the boundary grid points no filtering is applied at all boundaries.
3.1 Flow description

A rectangular jet configuration with an aspect ratio of \( W = L_2/L_3 = 5 \), a Reynolds number based on the jet center velocity and \( L_2/2 \) of 5000 (corresponds to 2000 based on the height of the nozzle exit \( L_3 \)) and a Mach number of \( M_j = 0.5 \) was the subject of investigation. Unless specified otherwise, reference quantities are the jet center velocity \( u^d_j \), the dimension \( L_2^d/2 \), the density \( \rho^d_j \) and the temperature \( T^d_j \). The index \( \cdot_j \) denotes jet center quantities. The configuration and the coordinate system are illustrated in figure 3.1. The \( x_2 \)-axis will be denoted as the major and the \( x_3 \)-axis as the minor jet axis, and simulation parameters are summarized in table 3.1.

![Figure 3.1: Jet nozzle geometry and coordinate system.](image)

3.2 Inflow treatment

The transition process of jet flow is very sensitive to minor changes of the inflow conditions, since a small variation in the inflow perturbations can lead to an excitation of different instability modes in the initial jet profile. The jet acts as an amplifier for those small disturbances. It is very well known from experimental as well as numerical investigations that the inflow disturbances are crucial for the downstream jet development (see
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Re_{L_2/2,j})</td>
<td>5000</td>
</tr>
<tr>
<td>(M_j)</td>
<td>0.5</td>
</tr>
<tr>
<td>(Pr)</td>
<td>0.72</td>
</tr>
<tr>
<td>(T_{\infty}/T_j)</td>
<td>0.936143</td>
</tr>
<tr>
<td>(T_{\text{ref}}/T_j)</td>
<td>273K</td>
</tr>
<tr>
<td>(W = L_2/L_3)</td>
<td>5</td>
</tr>
<tr>
<td>Circular frequency (\Re(\omega))</td>
<td>2.7066</td>
</tr>
<tr>
<td>Amplitude of excitation</td>
<td>(2.5 \times 10^{-3})</td>
</tr>
<tr>
<td>Grid ((x_1, x_2, x_3))</td>
<td>(337 \times 229 \times 229)</td>
</tr>
<tr>
<td>Mapping parameters (k_2 = k_3)</td>
<td>0.46</td>
</tr>
<tr>
<td>Computational domain ((x_1, x_2, x_3))</td>
<td>([0:15] \times [-7:7] \times [-7:7])</td>
</tr>
<tr>
<td>Time step (\Delta t)</td>
<td>(\approx 1.4 \times 10^{-3})</td>
</tr>
<tr>
<td>Time samples for statistics</td>
<td>1284</td>
</tr>
<tr>
<td>Sampling time (in periods (2\pi/\Re(\omega)))</td>
<td>299 (128)</td>
</tr>
<tr>
<td>(\delta = \delta_i = \sqrt{2/(9 \sinh^{-1}(1))})</td>
<td>0.1783</td>
</tr>
</tbody>
</table>

Table 3.1: Parameters of the jet DNS simulation. The subscript \(j\) denotes jet-center quantities. Quantities are non-dimensionalized by \(\rho^d_j\), \(u^d_j\), \(T^d_j\) and \(L_2^d/2\).

also section 1.1). This is particularly true for non-axisymmetric jets that are more susceptible to instabilities than round configurations (Lozanova & Stankov, 1998; Zaman, 1999).

In experimental investigations it is very difficult to determine the inflow conditions accurately, \(i.e.\) the initial profile with shear-layer thicknesses and the level, shape and spectrum of perturbations at the nozzle that trigger the jet. When not using an explicit forcing, the upstream disturbances, which are different in every experimental facility, are mainly responsible for the initialization of instabilities. Therefore very often jets are forced at a particular frequency at which they react most strongly to the forcing. However, in doing so only the frequency is locked and the actual response of the jet still depends on the particular form of the forcing (fluid actuators, loudspeakers, nozzle actuator, etc.).

For the present investigation the intention was to set up a reference simulation with clearly defined inflow disturbances for later validation of a large-eddy simulation. By merely enforcing a laminar profile, transition would have been triggered by numerical roundoff errors and therefore would have rendered the simulation results grid-dependent, which was undesirable. An instability-mode forcing was therefore used for the present simulation as described in the following. The nozzle of the jet itself was not modeled within the computational domain, but instead
3.2 Inflow treatment

![Figure 3.2: Eigenfunctions for the profile along the minor axis of the jet, magnitude, real part, imaginary part.](image)

an inflow profile with velocity half-widths adapted to the nozzle dimensions was prescribed. This was defined as a laminar smoothed top-hat inflow profile for the streamwise velocity component according to Yu & Monkewitz (1990),

\[
u(\eta, \zeta) = \left\{ (1 + \sinh[|\eta|\sinh^{-1}(1)]^{2n_2}) (1 + \sinh[|\zeta|\sinh^{-1}(1)]^{2n_3}) \right\}^{-1},
\]

where \(\eta\) and \(\zeta\) are the cross-stream coordinates normalized by the corresponding nozzle half-width \(L_i/2, i = 2, 3\). The parameters \(n_2 = 9\) and \(n_3 = n_2L_3/L_2\) were chosen to ensure equal vorticity thicknesses \(\delta = \delta_i = L_i/(\sqrt{2} n_i \sinh^{-1}(1))\) in both directions. The cross-stream velocity components of the laminar inflow profile were set to zero. Based on the velocity profile (eq. 3.1) and assuming parallel two-dimensional flow we performed an inviscid spatial linear stability analysis to obtain the most unstable non-oblique spatial modes, which are the sinuous (antisymmetric) and varicose (symmetric) modes, respectively. The latter mode, which had a slightly smaller growth rate than the symmetric mode, was arbitrarily chosen as inflow disturbance (see Appendix A for
details). In figure 3.2 the eigenfunctions of the streamwise and cross-stream velocity as well as the density and pressure perturbations for the spatially most amplified varicose mode are shown. They were computed on a grid of 120 nodes (grid parameter $k_3 = 0.3$) with the $\zeta-$coordinate spanning the domain $\zeta \in [-7; 7]$. The eigenfunctions were then converted to conservative variables based on the base flow profile, scaled and interpolated onto the DNS grid. This disturbance was superimposed onto the laminar inflow profile (eq. 3.1) along the center of the rectangular inflow profile ($|x_2| < L_2/2$) where the flow is mainly two-dimensional. At the edges of the inflow profile and away from the nozzle ($|x_2| \geq L_2/2$) the modal excitation was smoothly ramped to zero by multiplication with

$$\vartheta(\eta) = \exp\{-(1.2\eta)^8\}.$$ (3.2)

Here again $\eta$ is equal to the $x_2-$coordinate normalized with the nozzle half-width $L_2/2$. An amplitude of the inflow disturbance of $\max(|\hat{u}_1|) = 2.5 \times 10^{-3}$ at the inflow was applied where $|\hat{u}_1|$ denotes the amplitude of the eigenfunction of the streamwise velocity variable $u_1$. The circular frequency $\Re(\omega)$ predicted by linear stability theory and confirmed by DNS was 2.7066 (normalized with $L_2/2$ and the jet center velocity at the inflow).

### 3.3 Outflow sponge solution

The target solution $u_0$ in equation (2.9) for the sponge layers was chosen to be equal to the quiescent ambient flow, except at the outflow where it was a two-dimensional Gaussian velocity profile with velocity half-widths $h_{\perp,2}$ and $h_{\perp,3}$ estimated from the experimental data of Tsuchiya et al. (1986) as $h_{\perp,2} = 2.0$ and $h_{\perp,3} = 3.2$ in dimensionless units. The mass flux through the outflow plane is unknown a priori since ambient fluid is entrained by the jet. For the formulation of an appropriate outflow boundary condition the outflow sponge target solution was approximated by matching the mass flux through the outflow plane to the influx through the jet at inflow. Entrainment was still possible since the sponge formulation does allow the solution at the outflow to develop. The sponge domain ranges from $x_1 = 13.7$ to $x_1 = 15$ and from about $x_i = \pm 4.2$ to $x_i \pm 7$ for $i = 2, 3$. It was found that this simple boundary treatment has only a weak upstream influence. Improvement could
be achieved by enforcing a (a priori known or estimated) entrainment velocity field in the sponges, however, to some extent the computed entrainment of the jet always depends on the far-field flow boundaries.

3.4 Results

The simulation was started using the laminar inflow profile throughout the entire computational domain on a coarse grid of $242 \times 161 \times 161$ points in a reduced-size computational domain $[0 : 14] \times [-5 : 5] \times [-5 : 5]$ with inflow disturbance switched on. After an initial transient the grid was refined using high-order spline interpolation and the computational domain was enlarged to the values given in table 3.1. In the enlarged region the values of the solution at the boundaries of the smaller domain were used as initial data. Sampling was started after a brief transient caused by the interpolation and by the domain enlargement had faded. Statistical flow data were obtained using temporal and spatial averaging as follows. Because of the three-dimensionality of the flow no homogeneous coordinate direction was present to improve statistical sampling. However, the sector symmetry over the four quadrants of the rectangular profile could be exploited. Quantities averaged in this way and in time are denoted by $\langle \cdot \rangle$ in the following. A total of 1284 samples from a dimensionless time interval of 299 (128 periods of the harmonic excitation) spanned by roughly $2.1 \times 10^5$ time steps was analyzed. About 1600 CPU hours on 4 processors of a NEC SX5 parallel vector computer were used for the statistical averaging and a runtime memory of about 5000 Mega Bytes. Optimization achieved about 45% of the theoretical peak performance.

In the following we first present instantaneous flow data to visualize the flow topology and second we show time- and sector-averaged data.

3.4.1 Instantaneous data

In figure 3.3 the instantaneous flow topology of the transition process is visualized by a density iso-surface. We observe a strong growth of the linear instability mode in the laminar region. Highly three-dimensional disturbances of the initially largely two-dimensional instability mode grow at the lateral edges of the jet. An $x_1$-symmetric vortex structure develops but breaks up almost immediately into small-scale turbulence. To visualize this transitional vortical structure, in figure 3.4 a close-up of a
pressure iso-surface in the transition region is depicted. Vortices symmetric to the \((x_1, x_3)\)-plane can be identified which deform and finally breakup into disordered structures. Additionally, figure 3.5 displays density contours in two cross-sections through the jet center, the major and minor axis planes of the jet. The excited varicose mode develops towards the antisymmetric sinuous mode which is followed by a rapid transition to turbulence.

The observed flow structure and transition location are of course the result of the imposed inflow disturbances. Since the most unstable eigenmode would also develop naturally, a similar transition topology and location would be obtained using a broad-banded disturbance spectrum and adapting the amplitude levels. However, small-scale contributions of random noise render the perturbations grid dependent, which is undesirable and gives rise to systematic differences between LES and DNS.

### 3.4.2 Statistically averaged data

To assess the mean properties of the analyzed flow, sector- and time-averaged flow data at 9 equidistantly spaced downstream stations (each 36th grid plane is shown which corresponds to a distance between the stations of \(1.61L_2/2\)), where the first station represents the inflow plane. The data are plotted along the major and the minor axes of the jet. The
3.4 Results

Figure 3.4: Snapshot of pressure iso-surface in the transition region. Side view (a) and top view (b) at the same time instant, flow from left to right.

last station is still well before the beginning of the sponge zone at the end of the computational domain.

In figure 3.6 and 3.7 the density and streamwise velocity profiles $\langle u_1 \rangle$ are shown. Ahead of the transition region the jet profile spreads only marginally due to diffusion. After transition has taken place one observes a strong spreading of the profiles away from the major axis, so that the entire profile transforms towards an elliptical shape. In the transition region itself a non-monotonic profile is observed, which is caused by correlated large scale structures present in this region. In figure 3.8 we plot downstream velocity half-width contours $\langle u_1 \rangle (x_1, x_2, x_3) = 1/2 \langle u_1 \rangle (x_1, 0, 0)$ at the same 9 downstream stations. The different spreading behaviour of the jet with respect to its two sym-
Figure 3.5: Snapshot of density contours in the major (a) and minor (b) jet-plane at the same time instant.
3.4 Results

metry axes can clearly be identified. Whereas a strong expansion of the jet along its minor axis takes place immediately, even a slight contraction is observed along its major axis. Only towards the end of the computational domain the velocity half-width of the inflow profile is recovered. Qualitatively this corresponds to experimental observations (Tsuchiya et al., 1986). From the same experiments for non-axisymmetric jets it is also known that in the transition region a so called saddle-back profile occurs along the major axis of the mean flow. Here, the mean centerline streamwise velocity drops below the value reached in the surrounding shear layers. Our simulation is qualitatively in good agreement with these findings (see figure 3.7 (a), 5th and 6th station).

In figure 3.9 (a) the decay of the jet centerline velocity is shown, which indicates the closing of the potential core at \( x_1 \approx 6 \). Further downstream the velocity drops rapidly between \( x_1 = 6 \) and \( x_1 = 7.5 \), a region in which large symmetrical structures dominate the flow behaviour. Even further downstream the deterministic structures breakup and an almost linear decay of the centerline velocity is observed. The far-field limit of an inverse dependency on the downstream coordinate of the centerline-velocity decay is not reached within the limitations of the computational domain. Figure 3.9 (b) depicts the downstream development of the velocity half-widths \( h_{\frac{1}{2},2}(x_1) \) and \( h_{\frac{1}{2},3}(x_1) \) in the major and minor plane of the jet. They are defined by the location where the \( \langle u_1 \rangle \)-component reaches half its value \( \langle u_1 \rangle (x_1,0,0) \) in the jet center at the corresponding \( x_1 \). The spreading of the jet along the minor axis can clearly be identified, whereas along the major axis the velocity half-width increases only marginally throughout the domain. The sharp jump in the latter curve is caused by the local side lobes in the downstream velocity profile at the edges of the rectangular jet, that can also be seen in figure 3.8.

In figure 3.10 the transverse velocity components \( \langle u_2 \rangle \) and \( \langle u_3 \rangle \) are plotted. The averaged velocity components in the direction orthogonal to the jet center planes vanish due to symmetry and are not shown. In the transition region one observes a peak of the cross-stream velocity components, which is roughly as high along the major axis as along the minor axis, whereas in the turbulent regime the profiles flatten out again.

For an investigation of turbulent statistics we examine the development of the turbulent kinetic energy \( tke \) defined by \( \frac{1}{2} \langle \rho \rangle u''_k u''_k \), the Reynolds normal stress \( \langle \rho u''_1 u''_1 \rangle \) and the Reynolds shear stresses \( \langle \rho u''_1 u''_2 \rangle \) and \( \langle \rho u''_1 u''_3 \rangle \). \( \sim \) denotes a Favre-averaged quantity and \( \cdot'' \) the corre-
sponding deviations. The number of samples computed is obviously not sufficient to obtain smooth profiles especially in the developed turbulent regime. The roughness of the profiles in the transition region, however, is caused by the deterministic inflow forcing used in this simulation. It is evident, and was also repeatedly shown in experiments, that the turbulence levels increase rapidly when the jet undergoes transition. However, this process depends strongly on parameters such as the shape of the nozzle, initial turbulence intensity, Reynolds number etc. This makes it difficult to compare computational and experimental data, since experimental inflow and lateral conditions are usually not known with sufficient accuracy. In this simulation we find a rapid growth of the Reynolds stresses beginning roughly at a distance of $10L_3$ downstream of the nozzle, (see figures 3.11 and 3.12). They reach peak values at the jet edges along the major axis, from where three-dimensional disturbances of the transition process originate. The same holds for the turbulent kinetic energy which is plotted in figure 3.13.
3.4 Results

Figure 3.6: Mean density profiles $\langle \rho \rangle$ at different downstream stations along major (a) and minor (b) axis.

Figure 3.7: Mean streamwise velocity profiles $\langle u_1 \rangle$ at different downstream stations along major (a) and minor (b) axis.
Figure 3.8: Downstream development of the velocity half-width contours $\langle u_1 \rangle (x_1, x_2, x_3) = 1/2 \langle u_1 \rangle (x_1, 0, 0)$ at the same downstream stations as in figure 3.7.

Figure 3.9: (a) decay of the centerline velocity of the jet and (b) development of the velocity half-widths $h_{1,2}$ in the major and $h_{1,3}$ in the minor jet plane. The beginning of the outflow sponge region is indicated by the vertical line.
3.4 Results

Figure 3.10: Mean cross-stream velocity profiles at different downstream stations \( \langle u_2 \rangle \) along major (a) and \( \langle u_3 \rangle \) along minor (b) axis.

Figure 3.11: Reynolds normal stress \( \langle u''_1 u''_1 \rangle \) at different downstream stations along major (a) and minor (b) axis.
Figure 3.12: Reynolds shear stresses at different downstream stations \( \langle \rho u''_1 u''_2 \rangle \) along major (a) and \( \langle \rho u''_1 u''_3 \rangle \) along minor (b) axis.

Figure 3.13: Turbulent kinetic energy \( tke \) at different downstream stations; along major (a) and minor (b) axis.
In this chapter results from the LES computation of the rectangular jet configuration are presented. First, the LES data are compared with the previously obtained DNS data. Second, we assess the behaviour of the downstream jet-breakup when subjected to different inflow conditions. A comparison is made between a laminar forced inflow and a fully turbulent inflow profile, results of which are presented in section 4.2.

### 4.1 The transitional rectangular jet

#### 4.1.1 Flow parameters

The configuration was precisely the same as in the corresponding DNS simulation (see figure 3.1 and table 3.1). The LES grid was composed of $113 \times 77 \times 77$ grid points, which is one third of the DNS resolution in each coordinate direction. Accordingly, only about 4.5% of runtime memory was used for the LES (including the modelling effort) and the computational time of the LES (per time unit) was about 1.6% of that of the DNS.

The grid in the lateral directions was fine enough to resolve the velocity gradients in the initial shear layers of the jet at the inflow. We again superimposed the same spatially evolving unstable two-dimensional eigensolution of the inviscid linear stability analysis for the planar laminar profile along the minor axis of the jet at the center of the rectangular profile. The LES simulation was initialized using a fully developed DNS field filtered on the LES grid using the same filter as in the LES simulations. Sampling was started after an initial transient of the order of one flow-through time during which the dynamics of the filtered quantities developed.

#### 4.1.2 Inflow treatment

The formulation of the inflow condition had to be adapted for the LES simulation. It was found that merely applying Dirichlet conditions at the inflow as in the DNS did not lead to stable computations. The boundary
condition was therefore enforced by a sponge-layer formulation as in equation (2.9) using a temporally and spatially varying instability mode within a narrow region at the inflow. The spatial dimension of the inflow sponge was 10 grid points in the streamwise direction and the sponge parameters were the same as at the outflow. The target profile $u_0$ for the inflow sponge was chosen to be the laminar inflow profile with the superimposed instability wave with a spatial evolution over the inflow sponge region as predicted by linear stability theory. In the $x_2$-direction the excitation was ramped to zero again according to eq. (3.2).

In figure 4.1 the spatial evolution of the centerline velocity at fixed instants in time with approximately the same phase angle of excitation is plotted. For comparison, also the theoretical growth envelope from inviscid linear stability theory is plotted. The initial linear viscously damped growth of the instability is well represented in DNS and LES, confirming that the inflow sponge treatment is formulated appropriately.

![Figure 4.1: Evolution of the centerline velocity $u_1(x_1,0,0)$, $\text{--- LES}$, $\text{----- DNS}$, $\text{-------- linear stability theory.}$]
4.1 The transitional rectangular jet

4.1.3 Comparison of DNS and LES results

Instantaneous results

In the following we first compare instantaneous flow data to check how well the overall flow topology of the LES matches the DNS results. The transition process is initially governed by the growth of the linear instability in the parallel region of the jet, which rapidly develops a three-dimensional character. The breakdown of the laminar jet is initiated in the shear layers at the edges of the rectangular profile. The vortex rollers originating from the shear layer of the jet bend and finally breakup into two opposing vortices, which are oriented at an angle of about 45° with respect to the main jet axis. Subsequently these structures breakup into irregular structures with increasingly fine scale. In figure 4.2 a close-up of the transition region is shown for the DNS (a) and the LES (b) at the same phase angle of excitation. We plot a density iso-surface to visualize the vortex structures as well as density isolines in the jet center plane ($x_3 = 0$). It can be seen that the LES captures the early transitional flow structure quite well.

Figure 4.2: Density iso-surface ($\rho \approx 1.0035$) in the transition region viewed from positive $x_3$-direction and density contours in the jet midplane: DNS (a) and LES (b).
Statistical results

For comparison with LES the DNS database shown in section 3.4 was filtered with the same filter as used for the LES simulations. With the LES a total of about 4000 samples were taken, spanning a time interval of about 1100 (500 periods of the excitation frequency).

Again we plot mean the flow data in the jet center-planes ($x_2 = 0$ and $x_3 = 0$) at 9 different equidistant downstream stations, where the inflow plane corresponds to the first graph shown. The stations are identical with the ones used in section 3.4 and the distance between the stations is again $1.61L_2/2$. Due to the mean flow symmetry it is sufficient to plot the profiles only on the positive $x_2$ and $x_3$ semi-axes.

In figures 4.3 to 4.5 the mean profiles of the density and the downstream and cross-stream velocity components are shown. The density profiles of the two simulations match rather well. The velocity profiles of the LES capture the main characteristics of the DNS data quite well. Slight discrepancies occurred only for the profiles along the $x_3$-axis further downstream. The same behaviour was observed for the cross-stream component $u_3$ with somewhat larger deviations. For an investigation of turbulent statistics we computed the evolution of the turbulent kinetic energy $tke = \frac{1}{2} \langle \rho u_k u_k'' \rangle$ and the Reynolds normal stress $\langle \rho u'_i u'_1'' \rangle$ in the flow domain. Those are shown in figures 4.6 to 4.8. The strong increase of turbulent kinetic energy in the shear layers of the transition region was very well reproduced by the LES. Also the downstream development of $tke$ and $\langle \rho u'_i u'_1'' \rangle$ was in good agreement between DNS and LES. Obviously, the turbulent kinetic energy was dominated by the Reynolds normal stress $\langle \rho u'_i u'_1'' \rangle$.

In figure 4.9 (a) the development of the jet-center velocity $u_1(x_1, 0, 0)$ is plotted for the DNS and LES results. The breakdown of the potential core of the jet was slightly delayed in the LES simulation, but the downstream behaviour matched that of the DNS well. Figure 4.9 (b) shows the development of the jet half-widths $h_{x,2}(x_1)$ and $h_{x,3}(x_1)$ which is defined by the location where the $u_1$-mean-component reaches half its value in the jet center at corresponding $x_1$. Here, downstream discrepancies between DNS and LES could also be observed, however, the overall agreement is acceptable. The same holds for the velocity half-width contours shown in figure 4.10.
4.1 The transitional rectangular jet

Figure 4.3: Mean density profiles $\langle \rho \rangle$ at different downstream stations along major (a) and minor (b) axis, LES, --- filtered DNS.

Figure 4.4: Mean streamwise velocity profiles $\langle u_1 \rangle$ at different downstream stations along major (a) and minor (b) axis, LES, --- filtered DNS.
Figure 4.5: Mean cross-stream velocity profiles at different downstream stations $\langle u_2 \rangle$ along major (a) and $\langle u_3 \rangle$ along minor (b) axis, --- LES, ---- filtered DNS.

Figure 4.6: Reynolds normal stress $\langle pu''u'' \rangle$ at different downstream stations along major (a) and minor (b) axis, --- LES, ---- filtered DNS.
4.1 The transitional rectangular jet

Figure 4.7: Reynolds shear stresses at different downstream stations \(\langle pu''_1u''_2\rangle\) along major (a) and \(\langle pu''_1u''_3\rangle\) along minor (b) axis, \(\text{LES}\), \(\text{filtered DNS}\).

Figure 4.8: Turbulent kinetic energy tke at different downstream stations along major (a) and minor (b) axis, \(\text{LES}\), \(\text{filtered DNS}\).
Figure 4.9: (a) Decay of the centerline velocity of the jet: \( \text{LES} \) and \( \text{filtered DNS} \) (b) development of the velocity half-widths \( h_{\frac{1}{2},2} \) \( \text{filtered DNS} \), \( \text{LES} \), and \( h_{\frac{1}{2},3} \) \( \text{filtered DNS} \), \( \text{LES} \). The beginning of the outflow sponge region is indicated by the vertical line.

Figure 4.10: Downstream development of the velocity half-width contours \( \langle u_1 \rangle (x_1, x_2, x_3) = 1/2 \langle u_1 \rangle (x_1, 0, 0) \) at the same downstream stations. (a) \( \text{LES} \), (b) \( \text{filtered DNS} \). The arrow indicates increasing downstream coordinate.
4.2 Turbulent rectangular jet

In this section we compare two different inflow conditions to assess the jet’s inflow sensitivity. As outlined in section 1.1, jet flow is very susceptible to minor disturbances at its inflow, since they are the inherent trigger for the transition of the jet. Whereas in a laminar jet profile the most unstable modes develop and finally lead to the breakdown of the jet, a turbulent inflow contains a wide range of modes. In order to quantitatively assess the effect of broad-band inflow disturbances, we compared the behaviour for a fully developed turbulent inflow with that for an inflow disturbed only by a laminar instability mode.

In the following first the inflow generation using a separate turbulent duct flow computation is detailed. Then results of linear stability analysis of an artificial mean profile adapted to the turbulent mean profile of the duct computation are described to define an inflow forcing. Comparison of the results is made afterwards.

4.2.1 Inflow generation

Fully turbulent inflow

The inflow data for the turbulent jet computations were generated in a separate temporal simulation of a turbulent isothermal duct. Cross-sectional data slices sampled from that simulation were fed into the jet simulation at the inflow plane as is detailed below. By this procedure the effect of upstream influence of the jet into the nozzle was neglected, which is justified by the fact that the inflow Mach number is in the high subsonic range and the influence of upstream travelling waves should be small. The parameters of the inflow simulation were chosen to match the jet simulation of Freund (2001) for a Mach 0.9 transitional round jet and are listed in table 4.1. Here, the shorter dimension of the rectangular-jet nozzle corresponded to the diameter of the round jet since it is this length that governs the initial shear layers in the rectangular jet. The parameters are defined based on the bulk quantities $u_B = \frac{1}{V} \int_V u \rho dV$, $\rho_B = \frac{1}{V} \int_V \rho dV$, the constant wall temperature and $L_2/2$. $V$ is the volume of the computational domain.

The average temperature within the flow developed according to the production of thermal energy through dissipation because of the isothermal walls. Therefore the Reynolds number and the Mach number based on the averaged center quantities in the duct are a result of the sim-
ulation itself and they could only be matched approximately, e.g. by iterating towards the desired values (see table 4.1). Freund (2001) computed a cold jet adapted to the experiments of Stromberg et al. (1980). The corresponding temperature profile cannot be enforced in a temporal simulation, where the flow is developed and statistically stationary. For the present simulations we used the temperature profile given by the isothermal duct simulations, in order to avoid an additional uncertainty by introducing an energy sink to reproduce the cold temperature profile. For details of the duct simulation and definition of all the parameters see appendix B. We first present results of the inflow computation and afterwards describe the algorithm used to feed the duct data into the jet simulation.

Figure 4.11 (a) shows the normalized downstream velocity profiles along the two center axes of the duct. We plot three profiles along the centerline of the duct, one normalized with the average friction velocity, one normalized with the local friction velocity and the van-Driest-transformed profiles which are also normalized with the local friction velocity. The wall units are scaled accordingly. Additionally the law of the wall ($u^+_1 = 2.5 \ln(x^+_3 + 5.1)$) and a linear profile at the wall is shown for reference. We observe a deviation of the profiles compared to classical data from the law of the wall for channel inflow. Similar results were also found for higher Reynolds number cases in a systematic aspect ratio study (see appendix B). For the given aspect ratio the influence of the confining sidewalls (the two shorter walls in the duct) is obviously not negligible and the mean flow is displaced towards the center of the duct. Additionally, the Reynolds number was rather low in that simulation to expect fully developed logarithmic wall behaviour. For reference we also show the mean profile in physical coordinates in 4.11 (b).

The skin friction along the duct walls is plotted in figure 4.12. It was clearly observed that the higher gradient in the velocity profile along the walls at $x_3 = \text{const.}$ resulted in much higher skin friction along these walls. The corner vortices in the mean flow appear as local peaks in the skin friction. For the jet inflow the Reynolds stresses are relevant and they are displayed in 4.12 (b). The maximum of $\langle \rho u''_1 u''_1 \rangle$ is close to the wall in the middle of the longer sidewall of the duct (at $x_2 = 0$) and therefore exactly at the position of the unstable shear layers of the jet.

To give an overview of the flow topology one slice perpendicular to the mean flow direction of an instantaneous flow field is shown in figure 4.13 (a). The flow vectors denote the projected velocity within this slice and
4.2 Turbulent rectangular jet

Figure 4.11: Mean flow: Downstream velocity component in (a) wall units: $u_1^{+,w}(x_3^+)$ normalized with average friction velocity, $u_1^+(x_3^+)$ normalized with local friction velocity, $u_{1,D}(x_3^+)$ normalized with local friction velocity, log law: $2.5 \ln(x_2^+)+5.1$ and (b) $\langle u_1 \rangle / u_B$.

Figure 4.12: (a) Wall friction $\tau/\tau^w$ and (b) Reynolds stresses along the duct centerlines: $\langle pu_1'u_2' \rangle^{\frac{1}{2}} / (u_\tau \sqrt{\rho B})$, $\langle pu_1'u_2'' \rangle^{\frac{1}{2}} / (u_\tau \sqrt{\rho B})$, $\langle pu_1'u_3'' \rangle^{\frac{1}{2}} / (u_\tau \sqrt{\rho B})$.

Figure 4.13: Flow field: (a) instantaneous field, (b) time- and sector- averaged flow field enlarged by a factor of 2 (vectors are in the $x_2$-$x_3$-cross-plane, contours denote the downstream $u_1$ component, increment 0.1 from 0.0 to 1.4 (a) and to 1.2 (b)).
LES results

The turbulent duct data were used as inflow condition for a jet computation. A large number, 17200, of cross-stream data slices was stored on disk and subsequently read in during the jet simulation. No explicit coupling was used and the timesteps of both simulations were not identical. The duct grid was interpolated to the jet grid using fourth order spline interpolation and outside of the duct the values at the inflow were set to the wall quantities which correspond to the ambient values of the jet simulation. The grid resolution near the jet center is coarser than for the duct computation and therefore an implicit filtering due to the coarsening of the duct data to the jet grid was involved. However, large structures of the duct flow that are dominant for the transition process in the jet are still represented on the LES grid for the jet. Inflow data were renormalized by the jet-center quantities. The reference temperature is kept to be the wall temperature of the duct, i.e. the ambient temperature of the jet. Since only values at certain time steps were stored on disk a temporal interpolation was used to interpolate the inflow data from a biased 4-point stencil in time at the desired instant in time (i.e. third order spline interpolation, Neville-Aitken algorithm) as was also used in Adams (2000). The previously applied sponge formulation for the boundary condition could not be used here because data were only available at the inflow plane. The obtained data were therefore enforced in the jet computation at the inflow plane by adjusting the flux derivatives in that plane after their computation. This was done in each Runge-Kutta substep before the explicit filtering of the flux terms in the LES algorithm in such a way that the updated filtered solution was obtained after projection in time. It was found that this led to a stable enforcement of the inflow condition.

Laminar inflow

The growth rates of instabilities strongly depend on the shear-layer thickness of the jet. In order to compare laminar with the turbulent data, the shear-layer thickness in the minor plane of the laminar profile was therefore adapted to the one of the mean profile of the turbulent sim-
4.2 Turbulent rectangular jet

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<tr>
<td>no. of slices (per time unit)</td>
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Table 4.1: Parameters of the inflow duct computation. Non-dimensionalization is done by the bulk quantities and $L_2/2$. The index $j$ denotes quantities normalized with the center-quantities and $L_2/2$.

The velocity profile was chosen according to equation (3.1) with the parameters $n_2 = 20$ and $n_3 = n_2 L_3 / L_2$ to ensure equal vorticity thicknesses $\delta = \delta_i = L_i / (\sqrt{2} n_i \sinh^{-1}(1))$ in each cross-stream direction. The cross-stream velocity components of the laminar inflow profile were set to zero. The non-dimensional temperature profile was adjusted to the mean profile from the turbulent simulation using

$$T(\zeta) = (T_j - T_\infty)u(\zeta) + T_\infty,$$

where $\zeta$ is the normalized cross-stream coordinate. This ad hoc definition ensures that the jet-center temperature $T_j$ and the ambient temperature $T_\infty$ match the one from the turbulent simulation and vary smoothly in between. Computing the temperature profile with the Crocco-Busemann relation would have resulted in temperature peaks in the shear layers which was undesirable. Linear stability analysis (see appendix A) was used to compute the most unstable non-oblique varicose mode and this mode was superimposed onto the mean profile defined above with an amplitude $\max(|\hat{u}_1|) = 2.5 \times 10^{-3}$. Again the eigenfunctions were normalized by the maximum amplitude of the streamwise component $\max(|\hat{u}_1|)$. The eigenfunctions were interpolated to the LES grid using fourth-order
spline interpolation. The simulation parameters are summarized in table 4.2.

4.2.2 Results

Unlike in the DNS and previous LES simulations for the present comparison the target solution for the outflow sponge was not estimated from experimental data but instead computed during the simulation. The flow data in the outflow sponge region were averaged in the $x_1$-direction and the cross-stream velocity components were set to zero. Additionally possible backflow in the $x_1$-direction was clipped to zero. The so-obtained target solution was filtered in space and additionally filtered in time. It was found that this led to a very stable and physically meaningful outflow target solution, since it adapted itself to the entrained fluid of the jet.

For both cases the computations were started using a down-interpolated fully turbulent flow field from the previously studied DNS jet simulation (see section 3). The internal energy was adjusted to the present Mach number and the fields were filtered on the actual LES grid. Statistical sampling was only started after an initial transient of Reynolds number and Mach number adjustment had decayed. We summarize the parameters for both cases in table 4.2. In the following first a visualization of instantaneous data is presented, and second a statistical data comparison is made.

**Instantaneous data**

Caused by the smaller vorticity thickness compared to the previously computed jets of section 3 and 4.1, the growth rate of the laminar instabilities is larger by a factor of almost 2.5. That means that the effect of the increase in vorticity thickness of the present profile overwhelms the effect of decreasing growth rates with increasing Mach number (Sandham & Reynolds, 1989). The linear growth predicted by inviscid linear stability theory is reduced in the actual rectangular jet computation due to viscous effects and due to the finite width of the profile. However, it still leads to a significantly faster breakup of the jet compared to the lower Mach number and higher vorticity thickness case in chapter 3 and 4.1.

Figure 4.14 displays density iso-contours in the major planes of the jet for the two cases. In the streamwise direction the entire computa-
4.2 Turbulent rectangular jet

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<td>$T_{ref}$</td>
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</tbody>
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| grid             |          |
| computational domain $(x_1, x_2, x_3)$ | $[0 : 20] \times [-8 : 8] \times [-8 : 8]$ |

| laminar case     |          |
| circular frequency $\Re(\omega)$ | 4.753    |
| $\kappa_1$      | 9.0784 - 4.214 |
| amplitude of excitation | $2.5 \times 10^{-3}$ |
| time samples for statistics | 2342 |
| sampling time (in periods)   | $\approx 224$ (180) |

| turbulent case    |          |
| time samples for statistics | 2100 |
| sampling time     | $\approx 204$ |

Table 4.2: Parameters for the laminar/turbulent inflow comparison. Parameters are non-dimensionalized by the jet center quantities $u_j, \rho_j$ and the wall temperature (ambient temperature) $T_\infty$ and $L_2/2$.

tional domain is shown including the outflow sponge region, in the lateral direction only the relevant part of the domain is shown. In the laminar case about one wavelength of the symmetric instability can be observed before the potential core closes. The turbulent inflow case clearly shows almost instantaneous breakup of the turbulent profile. For both cases one observes a strong spreading along the minor jet axis, whereas in the direction of the major axis the spreading is much less.

Statistically averaged data

Statistically averaged flow data for the two cases are presented here. The averaging operator $\langle \cdot \rangle$ implies again averaging over the four quadrants of the jet profile and time. The data are plotted along the minor ($x_3$) and the major ($x_2$) axes at equidistantly spaced downstream stations with spacing $5L_3$. The last station is well before the beginning of the outflow sponge region. Both cases are plotted in the same graphs for better comparison. The number of statistical samples as well as sampling time is listed in table 4.2.

The faster jet breakup of the turbulent case, as already seen in the
LES results

Figure 4.14: Density contours in major (a,b) and minor (c,d) jet planes: (a,c) laminar case, (b,d) turbulent case, increment 0.02 from 0.95 to 1.15.

Instantaneous data, expresses itself in the faster spreading of the downstream velocity profiles as shown in figure 4.16. Whereas the laminar profile remains stable up to about $x_1 = 2.5$ the potential core of the turbulent jet ends already around $x_1 = 1$, which is confirmed by the centerline decay in figure 4.15(a). This effect is caused by the presence of initial turbulence intensity in the turbulent inflow. Further downstream, however, the jet behaviour is very similar for the two cases. In figures 4.17 to 4.19 the Reynolds-normal stresses, the Reynolds-shear stresses and the turbulent kinetic energy are plotted. The peaks of the turbulent initial profile are located exactly in the shear layers of the jet and initiate their breakdown, whereas in the laminar case the disturbances have to develop from the linear stability. The shear stresses rapidly rise for both cases after the inflow and peak around the closing of the potential core. Again, further downstream the profiles for the two cases become more and more similar and the influence of the different inflow conditions is only expressed by a shift in $x_1$-direction. The different spreading of the two jets can be seen clearly in figure 4.15(b).
where the velocity half-widths $h_{1,2}$ and $h_{1,3}$ are plotted. Along the major axis $h_{1,2}$ remains almost constant throughout the entire domain but a strong increase of $h_{1,3}$ is observed for both cases. This leads to elliptically-shaped velocity half-width profiles in the cross-stream planes as shown in figure 4.20. The downstream stations of the contours are identical with the ones in figures 4.16 to 4.19.
Figure 4.15: (a) Decay of the centerline velocity of the jet: \( \text{--- laminar and \, \text{----- turbulent case} \)} (b) development of the velocity half-widths \( h_{1,2} \) \( \text{--- turbulent and \, \text{----- laminar case, and} \, h_{2,3} \, \text{--- turbulent and \, \text{----- laminar case.} \) The beginning of the outflow sponge region is indicated by the vertical line.

Figure 4.16: Mean streamwise velocity profiles \( \langle u_1 \rangle \) at different downstream stations along major (a) and minor (b) axis, \( \text{--- laminar case, \, \text{----- turbulent case.} \)}
4.2 Turbulent rectangular jet

Figure 4.17: Reynolds normal stress $\langle p u''_1 u''_1 \rangle$ at different downstream stations along major (a) and minor (b) axis, \textcolor{black}{laminar case,} \textcolor{red}{turbulent case.}

Figure 4.18: Reynolds shear stresses at different downstream stations $\langle p u''_1 u''_2 \rangle$ along major (a) and $\langle p u''_1 u''_3 \rangle$ along minor (b) axis, \textcolor{black}{laminar case,} \textcolor{red}{turbulent case.}
Figure 4.19: Turbulent kinetic energy $tke$ at different downstream stations along major (a) and minor (b) axis, \(\cdash\) laminar case, \(\cdash\cdash\) turbulent case.

(a) (b)

Figure 4.20: Downstream development of the velocity half-width contours \(\langle u_1 \rangle (x_1, x_2, x_3) = 1/2 \langle u_1 \rangle (x_1, 0, 0)\) at the same downstream stations. (a) laminar case, (b) turbulent case. The arrow indicates increasing downstream coordinate.

(a) (b)
Chapter 5

Acoustic analysis

In the present chapter the acoustic analysis of the transitional rectangular jet in chapter 3 is presented and compared with analogous results from a corresponding LES simulation. We used Lighthill’s acoustic analogy for the evaluation of the acoustic source terms and the far-field computation.

5.1 Far-field sound computation

Lighthill’s famous equation (Lighthill, 1952, 1954; Goldstein, 1976; Blake, 1986) reads

$$\frac{\partial^2 \rho}{\partial t^2} - c_\infty^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j},$$  (5.1)

with the source tensor

$$T_{ij}(x, t) = \rho u_i u_j + \delta_{ij}\{p - p_\infty - c_\infty^2 (\rho - \rho_\infty)\} - \tau_{ij}. \quad (5.2)$$

Here $\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)$ is the viscous stress tensor and $p_\infty, \rho_\infty$ are ambient quantities. Taking $\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ as the equivalent noise source, the far-field sound at point $x$ is evaluated using the free-space Green’s function and the divergence theorem as density fluctuations

$$\rho'(x, t) = \rho(x, t) - \rho_\infty = \frac{1}{4\pi c_\infty^4} \int_V \frac{R_i R_j}{R^3} \frac{\partial^2}{\partial t^2} T_{ij}(y, t - \frac{R}{c_\infty}) \, dy, \quad (5.3)$$

where $R = |x - y|$ and $R_i = x_i - y_i$, respectively. Evaluation is simplified by reformulating (5.3) in frequency space, in dimensionless form, as

$$\rho'(x, \omega) = \frac{M_\infty^4}{4\pi} \int_V \frac{R_i R_j}{R^3} \omega^2 T_{ij}(y, \omega) e^{-1M_\infty R\omega} \, dy, \quad (5.4)$$

since this avoids the interpolation needed to compute retarded times in equation (5.3). Both formulations correctly represent the quadrupole nature of the source, and have been shown to facilitate an accurate
Acoustic analysis

numerical evaluation (Bastin et al., 1997). The variables are non-
dimensionalized using jet exit quantities and $L_2/2$ defined in figure 3.1. $M_\infty$ is the Mach number based on the ambient speed of sound $c_\infty$. In evaluating (5.4) we neglected $\tau_{ij}$ in (5.2) which is typically small (Goldstein, 1976). The second term in the tensor is also small for nearly isentropic flow, and so was also expected to be small here. Thus for the present analysis we retained only $pu_iu_j$ in the source tensor. Since for the radiation to the far-field only fluctuations of $T_{ij}$ in time are significant (see eq. (5.3)), the temporal average can be subtracted and in the following $T_{ij}$ denotes these fluctuations.

5.2 DNS results

5.2.1 Database description

A total of 552 samples in time (dimensionless time interval 78.8) of the full DNS fields were used to compute the sound sources. This set was subdivided into sub-intervals of five overlapping sets of 192 samples each to increase statistical sampling, which is a standard procedure in signal processing. The length of the interval was chosen as to well resolve the expected lower frequencies (in particular the forcing frequency of the jet). The spectral resolution was given by the temporal resolution of the data. The mean-value corrected Lighthill stress tensor was computed in each interval and time transformed using discrete Fourier transform. The samples were windowed in time using a function constructed from a half period of a cosine function, which ramped the signal smoothly to zero over 50 samples at the beginning and end of each window to reduce spurious high frequencies introduced by the finite sample length. Figure 5.1 shows a windowed fluctuation pressure signal at $x_2 = x_3 = 0$, $x_1 = 6.2$. The data was unfortunately stored at slightly non-equidistant intervals, but the change of the interval length was less than one percent, and the associated error in the evaluation of Fourier sums was expected to have negligible consequence for the frequencies of interest.

The far-field sound spectra were obtained by numerical evaluation of equation (5.4) using trapezoidal rule quadrature. The source terms $T_{ij}$ go to zero at large $x_2$ and $x_3$ and at the inflow boundary, which allows integrating all the way to the domain boundaries. At the outflow boundary they are also small compared to their size in the transition area, though not zero since turbulent structures leave the domain.
5.2 DNS results

We therefore smoothly ramped down the source terms close to the outflow boundary in order to avoid affecting the volume integral by leaving sources. It was found that the influence on the particular form of the smoothing function is weak and therefore the same window function as for the Fourier transform in time was applied.

5.2.2 Acoustic analysis

In order to visualize the location and structure of the sound sources in figure 5.2 we plot the contours of the dominant source tensor components $T_{11}, T_{12}, T_{13}$ in the two jet-center planes. Here the flow is from left to right and in the streamwise direction only the physical domain is shown. In the lateral directions only the part with significant source terms is shown. From this visualization we see that the maximum source terms are obtained at the edges of the jet around the location where the potential core closes. The location of the peak of the source does not necessarily coincide with the virtual origin of the radiated sound, since most of the components do not radiate to the far-field. This was shown by Freund (2001), who actually filtered the source terms in order to only obtain the radiating components. However, this visualization does give the indication that the dominant noise-producing structures are located in the transitioning shear layers. Far-field spectra and intensities were computed on two arcs in the $x_2 = 0$ and $x_3 = 0$ planes at a radius of

Figure 5.1: Windowed pressure fluctuation signal at sensor point on the jet-center-axis in the transition region of the DNS field.
Figure 5.2: Instantaneous contours of $T_{11}$ (a,b), $T_{12}$ (c,d) and $T_{13}$ (e,f) in the major (a, c, e) and minor (b, d, f) jet-plane, ⊕ origin of the far-field arcs.
60L_2/2 and with angle \( \theta \) measured from the downstream axis. The center of the arcs \( x_1 = 5, x_2 = 0, x_3 = 0 \) is labeled in figure 5.2. In figure 5.3 the radiated sound-pressure-level, \( \text{SPL} = 20 \log_{10}(p_{\text{rms}}/p^d_{\text{ref}}) \), is plotted for the two arcs, taking the standard dimensional reference condition \( p^d_{\text{ref}} = 2 \times 10^{-5}\text{Pa} \) and assuming standard atmospheric pressure for the computation of the speed of sound. The anticipated directivity based on a Doppler scaling (Crighton, 1975)

\[
\text{SPL} \sim \frac{1}{(1 - M_c \cos(\theta))^5}
\] \hspace{1cm} (5.5)

with assumed convective Mach number \( M_c = 0.6 M \) is also plotted.

We observe a directivity of the jet peaking near \( \theta = 35^\circ \). At this angle, the SPL in the major jet-plane is 5dB lower than that in the minor jet-plane. This effect is reversed at \( \theta = 75^\circ \). This trend is consistent with experiments of non-axisymmetric jets which typically show that noise is more directive in the minor plane (Kinzie & McLaughlin, 1999), although we note that those observations are for supersonic jets. The overall directivity matches well with the expected directivity for uniformly moving sound sources (eq. (5.5)).
In figure 5.4 the dilatation of the velocity field

\[
\frac{\partial u_i}{\partial x_i} = -\frac{\rho_j}{\rho_\infty} \frac{\partial \rho}{\partial t}
\]  

(5.6)

is shown in the major and minor axis plane of the jet. It was computed on a $40 \times 35$ equidistant grid within the plotting range by retransforming $\rho'(x, \omega)$ from equation (5.4) to physical space. The base wavelength caused by the forcing frequency of the jet can be clearly seen. The signals appear noisy since for this plot only one sample of the DNS time
Figure 5.5: Far-field spectra for $\theta = (a) 5^\circ$, (b) $30^\circ$, (c) $60^\circ$, (d) $90^\circ$, (e) $120^\circ$, (f) $150^\circ$, on the arc at radius $60L_2/2$ in the major and minor jet-planes. Also plotted are the spectra from a round-jet DNS (Freund, 2001) and experiments of Stromberg et al. (1980) at $\theta = 30^\circ$. 
series was used to compute the spectra $T_{ij}(x, \omega)$.

Figure 5.5 shows noise spectra, $(pp^*)^{1/2}$, at six angles for each of the two arcs. The spectra are highly peaked, as expected for a forced low-Reynolds-number jet. At $\theta = 5^\circ$, the two spectra nearly coincide, which is not surprising since the observation points are very close at low angles to the jet. For $\theta = 30^\circ$, the spectrum in the minor plane is slightly higher at most frequencies. By $\theta = 150^\circ$, the spectra are flatter and now not dominated by the forced instability frequency. We observe a slight increase of the far-field spectrum at the high-frequency end in this case, which is unphysical. This might be caused by the non-equidistant time steps which were neglected in the sound analysis. For comparison we also plot for all angles the far-field spectrum (arbitrary level) of the DNS data of Freund (2001) and experiments of Stromberg et al. (1980) at an angle of 30 degrees. If the Strouhal number is scaled such that $L_3$ corresponds to the jet diameter, the drop-off in the spectra matches the present data.

5.3 LES results

5.3.1 LES database

The physical parameters of the LES simulation match those of the DNS. The flow was computed using the approximate deconvolution model as described in Stolz et al. (2001b) and Rembold et al. (2001) (see also chapter 2.3) on a $141 \times 77 \times 77$ mesh, which corresponds to one-third the DNS mesh in each direction. We found it, however, necessary to increase the size of the downstream absorbing sponge layer over that used in the DNS and the one used in the previously presented LES in section 4.1 to suppress reflections from the outflow plane (streamwise extent of the sponge in DNS is $1.6L_2/2$, compared with $4.8L_2/2$ in present LES). The simulation parameters are listed in table 5.1. Note that for the present LES simulation we used constant time steps and the temporal sampling rate was approximately 1.5 times that of the DNS. A total of 3000 samples spanning a non-dimensional time interval of 270 was analyzed. This time interval was subdivided again into 13 overlapping sub-intervals of 600 samples each and the results were obtained by ensemble-averaging over the sub-intervals. Temporal windowing of the same form as in the DNS was applied to each sub-interval over the first and last 50 samples. In figure 5.6 we show the analogous pressure fluctuation signal at a sen-
5.3 LES results

Figure 5.6: Windowed pressure fluctuation signal at sensor point on the jet-center-axis in the transition region of the LES field.

sensor point in the transition region of the jet with the temporal window function applied. Also the same spatial windowing of the source terms was applied. The spectra and far-field sound were computed exactly the same way as for the DNS.

5.3.2 Acoustic analysis

Naturally, in LES only filtered quantities are directly available for computation of the Lighthill source, although deconvolution potentially offers a means of improving this. In the approximate deconvolution model, an approximate inverse of the filter is used to partially recover unfiltered data, which is then used to model subgrid-scale effects on the filtered flow field, $\overline{\mathbf{u}}^c$. The overbar denotes filtered quantities and the super-

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid $(x_1, x_2, x_3)$</td>
<td>$141 \times 77 \times 77$</td>
</tr>
<tr>
<td>computational domain</td>
<td>$[0 : 18] \times [-7 : 7] \times [-7 : 7]$</td>
</tr>
<tr>
<td>computational time</td>
<td>270</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>$4.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>number of samples</td>
<td>3000</td>
</tr>
<tr>
<td>sampling interval</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters of the jet LES database (other parameters are identical with the LES in section 4.1).
Figure 5.7: Instantaneous contours of $T_{11}$ (a,b), $T_{12}$ (c,d) and $T_{13}$ (e,f) in the major (a,c,e) and minor (b,d,f) jet-planes computed from the LES database, $\oplus$ origin of the far-field arcs.
Figure 5.8: Sound pressure level along two arcs in the major o and minor • jet-plane of radius 60L₂/2 around the transition area; expected Doppler-scaling.

Script L is again used for represented quantities on the LES grid. u* is an approximation for the unfiltered field. Thus, it is tempting to ask whether parts of the sound spectrum can be also recovered in an LES.

The source tensor $T_{ij}$ can be decomposed into a part that can be represented on the LES grid ($T_{ij}^L$) and a part that cannot ($T_{ij}^S$), just as the velocity field $u$ is decomposed into a filtered field represented on the grid $\overline{u}^L$ plus two error terms:

$$T_{ij} = T_{ij}^L(u) + T_{ij}^S(u), \quad u = \overline{u}^L + (u^L - \overline{u}^L) + (u - u^L)$$

(5.7)

We used the same five-point explicit filter in computational space with two vanishing moments in physical space, which was used for the previous LES. The source tensor components beyond the grid cutoff $T_{ij}^S(u)$ cannot be recovered and must be modelled. These were not considered here. The represented part of the tensor $T_{ij}^L(u)$ is a function of $u$ but can only be evaluated in LES using the filtered field $\overline{u}^L$. Seror et al. (2001) have shown that the acoustic spectra computed from incompressible homogeneous isotropic turbulence can be improved when the Lighthill stress tensor based on resolved velocities is supplemented with
Figure 5.9: Far-field spectra at for the (a) $5^\circ$, (b) $30^\circ$, (c) $60^\circ$, (d) $90^\circ$, (e) $120^\circ$, (f) $150^\circ$, for the arc at radius $60L_2/2$ in the major and minor jet-plane. Also plotted are the spectra from a round-jet DNS (Freund, 2001) $\circ$ and experiments of Stromberg et al. (1980) $\cdot\cdot\cdot$ at $\theta = 30^\circ$.
5.3 LES results

the subgrid-scale contribution. In the present study we investigated if the far-field spectral prediction can be improved when $\mathbf{u}^L$ is replaced by $\mathbf{u}^* \approx \mathbf{u}^L$, as it is done in the LES flow computation. First, we computed the far-field sound using $T_{ij}^L(\mathbf{u}^L)$ and then compared it with the prediction using deconvolved quantities $T_{ij}^L(\mathbf{u}^*)$. The difference was found to be insignificant, as will be discussed below. The following results are based on the first formulation of the source term.

The instantaneous source distribution, plotted in figure 5.7, is seen to be very similar to the distribution in the DNS case (see figure 5.2) with maximum values at the edges of the transitional jet. Quantitative analysis in the far-field, however, revealed significant discrepancies between the DNS far-field and the LES data (see figure 5.3 vs. figure 5.8). Whereas the directivity in the main radiation direction (at angles around 30 degrees) was well recovered, we observed spurious radiation of the LES jet at higher angles. The far-field spectra show that the spurious noise is at high frequencies, where a rapid increase in the acoustic intensity in the far-field was found, particularly at high angles. For reference the data of Freund (2001) and Stromberg et al. (1980) at $\theta = 30$ degrees are plotted again. An examination of the origin of these spurious waves is presented in the next section.

The same analysis was repeated with the source tensor based on the deconvolved velocities to include the subfilter-scale contribution. We found no significant difference between the results using the two source formulations.

5.3.3 Analysis of spurious noise predicted by LES

When analyzing the origin of this spurious noise one faces two interwoven phenomena that are difficult to separate. First, the LES flow simulation could actually be polluted by unphysical high-frequency noise, which can be caused e.g. by the LES model. This is discussed below. Second, the acoustic analysis procedure itself has its limits up to which prediction is reliable. For one thing, the computation of $T_{ij}$ in frequency space can introduce errors due to finite window length. However, this influence was found to be negligible. Furthermore, the evaluation of the volume integral (5.4) in computing the far-field intensities and spectra is a limiting factor. While this is not a problem in evaluating DNS results due to sufficient grid resolution, in the LES it can be limiting.

In order to remain within the accuracy of the discrete volume integral,
the spatial Nyquist frequency must not be exceeded for the integrand. The change of the argument of the exponential term in (5.4) must therefore be limited from grid point to grid point as $\pi > M_\infty \omega \Delta R$ which leads to a maximal frequency of $\omega \approx 40$, which is not limiting. Assuming that the spatial variation of $T_{ij}(y, \omega)$ at a given $\omega$ is convection-dominated, this variation can be connected to some mean convection velocity $u_c$. For jets a good estimate is $u_c = 0.6 u_j$ and with a given Nyquist wavelength $\lambda_n$ given by the grid in the $x_1$-direction a maximum temporal frequency of $\omega = \frac{2\pi u_c}{\lambda_n} \approx 14$ can be estimated. Beyond that frequency the prediction cannot be expected to be accurate and errors amplified by the $\omega^2$ term in (5.4) may dominate. This effect can be demonstrated using artificial spatial noise for the $T_{ij}$ fields convected with $u_c = 0.6$ in the $x_1$-direction. This represents a convected frozen turbulence field which in theory does not radiate at all (Crighton, 1975). The physically meaningless far-field of such a field with $\Re(T_{ij})$ and $\Im(T_{ij})$ chosen randomly from the interval $[0.005 : 0.005]$ is shown in figure 5.10. (This corresponds to typical amplitudes for the present jet case at a frequency of $\omega = 12$.) The unphysical increase in the spectra can be clearly observed. For the following analysis we therefore considered only frequencies up to a cut-off frequency of $\omega_c = 12$ up to which numerical integration should be reliable (figures 5.12 and 5.13).
The LES results can be themselves affected by high-frequency noise present in the simulation. If the intention is to compute aeroacoustically induced sound directly from an LES flow simulation on the same grid, one has to suppress the generation of any meaningless spurious waves in the LES since no spectral separation can be made. In order to assess the effect of the LES simulation itself on the creation of spurious noise, we investigated the following model variants, which essentially introduce different levels of dissipation at high wave numbers. Additionally, a standard ADM simulation (ADMHI) was performed on a more refined grid but with the same filter width as the original LES. Thus we considered:

1. M1: The deconvolution in the ADM procedure aims at recovering scales up to the filter cutoff frequency. It, however, also amplifies noise in the unphysical high wave number band above the filter cutoff in the LES simulation. A reduced deconvolution order simulation with $N = 2$ was carried out to assess the effect of noise generation by the deconvolution.

2. M2: To eliminate model influences, high-order filtering after every time step without any LES model is applied. Here sufficient dissipation is ensured solely by filtering all conservative variables using a high-order filter at each time step of the simulation. As filter we used the operator $(G_2)^2 = (Q_N G)^2$.

3. ADMHI: A higher-resolved LES with the unaltered ADM model was performed, however, with the same physical filter width as in the original LES simulation. The grid was refined here to $183 \times 101 \times 101$. Spurious noise of the order of the Nyquist wavenumber should be less dominant and also volume integration more accurate.

<table>
<thead>
<tr>
<th>case</th>
<th>$N_\omega$ of sub-intervals</th>
<th>$N_\omega$ of samples</th>
<th>$\Delta t_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DNS</td>
<td>5</td>
<td>192</td>
<td>0.14</td>
</tr>
<tr>
<td>ADM</td>
<td>13</td>
<td>600</td>
<td>0.09</td>
</tr>
<tr>
<td>M1</td>
<td>1</td>
<td>600</td>
<td>0.09</td>
</tr>
<tr>
<td>M2</td>
<td>1</td>
<td>600</td>
<td>0.09</td>
</tr>
<tr>
<td>ADMHI</td>
<td>1</td>
<td>800</td>
<td>0.07</td>
</tr>
</tbody>
</table>

*Figure 5.11: Parameters of the acoustic analysis for the various cases. The number of samples is per sub-interval.*
In table 5.11 the number of samples used for the present analysis are listed for the various cases.

We begin presenting acoustic results from the LES up to the cutoff frequency $\omega_c$. It can be readily observed from figure 5.12 that the overall prediction was very much improved. Accordingly, the far-field dilatation levels for frequencies up to $\omega_c$ of the LES (see figure 5.13) compare well with the ones from DNS (note that the phase angle of excitation is different), they simply lack the higher wavenumbers. This agreement is due to the fact that the dominant noise sources are at resolved low frequencies.

The reduced deconvolution order simulation (M1) as well as the simulation with high-order filtering (M2) consistently showed a too rapid drop-off in the spectra, see figures 5.14 and 5.15. This is particularly true for the M2 case. Note that the spectra are much noisier due to little statistical sampling, which however does not alter the conclusions. Both simulations catch the transition of the jet reasonably well and correctly predict the low frequencies especially at the forcing frequency. However, beginning around $\omega \approx 10$ the spectral drop-off compared to the original LES and DNS is too rapid. The M2 simulation is definitely free of spurious noise in the high wave number region, however the same increase in the spectra at upstream angles can be seen as in figure 5.9.
Figure 5.13: LES (ADM) results (frequencies up to $\omega_c$): Velocity dilatation contours in the major and minor jet planes. Dotted contours denote negative dilatation.

This confirms that the acoustic analysis procedure lacks accuracy in this frequency region. Additionally, this increase in the spectra depends on the Doppler shift, since numerical errors are expressed at different spectral locations due to different observation angles with respect to the mean convection velocity of sound sources. Evaluation of the acoustic far-field based on the ADMHI data showed essentially the same far-field prediction as the original lower resolution LES, however the spectra reproduced the correct drop-off up to a higher cutoff frequency according
to the more refined grid.

The unphysical increase in the far-field spectra at high frequencies, which resulted in an overprediction of the sound pressure level, can therefore be mainly attributed to numerical accuracy limitations of the acoustic analysis procedure. Up to the estimated cutoff frequency the LES prediction is accurate. The numerical tests also suggest that for aeroacoustic prediction of jet noise by LES the exact form of the subgrid-scale modelling is not of importance given that enough dissipation is ensured to dampen spurious noise (grid to grid oscillations) and given that the LES model introduces not an exceeding amount of dissipation such that the higher frequencies are correctly predicted.
Figure 5.14: LES M1: Caption as in figure 5.9
Figure 5.15: LES M2: Caption as in figure 5.9
Chapter 6

Summary and conclusion

The present investigation focused on the near-nozzle flow development of subsonic rectangular jet flow. The intention was to first study the transitional and turbulent flow topology behind a 1:5 aspect ratio rectangular jet nozzle at Mach 0.5 using direct numerical simulation (DNS). An analysis was undertaken based on flow visualization and statistical data. The second aim was to investigate the capability of predicting such free shear flows with large-eddy simulations. In particular the applicability of the approximate deconvolution model (ADM) as subgrid-scale model was to be assessed for free shear flows. In previous studies this model was successfully applied to a number of wall-bounded both incompressible and compressible flows as well as homogeneous isotropic turbulence. In a third part the acoustic radiation from the transitional jet flow was addressed. On the one hand the intention was to predict the aeroacoustic noise induced by the turbulent flow from the rectangular jet nozzle using DNS data. On the other hand the question to be answered was how well an LES can reproduce the location of the sound sources and predict the acoustic far-field.

For the purpose of these investigations a highly accurate computer program that solves the conservative compressible Navier-Stokes equations on a Cartesian domain was developed and validated. The formulation of appropriate boundary conditions was implemented and tested by comparison with theoretical results from linear stability analysis.

For the jet simulations particular attention was paid to the definition of inflow conditions. To trigger transition in a controlled manner an instability mode of the employed artificial mean-flow profile at the inflow plane of the jet simulations was enforced. The most unstable eigenmodes of that profile were determined based on inviscid spatial linear stability theory for the parallel inflow profile. From the dominant modes, namely the varicose (symmetric) and the sinuous (flapping) mode, the varicose mode was used for the inflow forcing at a frequency where the spatial amplification rate was maximum. The simulation results showed how the imposed essentially two-dimensional disturbance rapidly becomes three-dimensional and initiates the breakdown of the jet. The non-axisymmetry of the profile causes a rapid spreading of the jet along
its minor axis, which is consistent with experimental findings. The mechanism leading to this is the deformation of the non-axisymmetric vortex rings from the jet shear layers. Statistical flow data were compiled to provide reference data for an evaluation of subsequent large-eddy simulations.

Second, the approximate deconvolution model (ADM) was implemented into the DNS code. The jet simulation was repeated using LES and applying the same instability mode forcing as in the DNS. The grid was composed of every third grid point of the DNS in each direction and the computational effort (CPU time) was approximately 1.6% of the DNS. We found that the LES is well able to predict the proper transition location and the principal transitional flow structures. Comparison of statistical flow quantities was made by filtering the DNS data with the same filter as was applied for the LES computations. The overall agreement of the LES results with the filtered DNS data is good, however, small differences between DNS and LES further downstream were observed. They essentially consist of a slight underprediction of the jet spreading by the LES. Additionally, the response of the jet to different inflow conditions was tested using LES. A Mach 0.9 rectangular jet with an aspect ratio of 1 : 5 was simulated for this purpose, once subject to a comparable laminar inflow with an instability mode forcing, and once with a precomputed fully turbulent inflow. For the latter we performed a separate temporal LES computation of an isothermal duct flow. Data slices from that simulation were subsequently fed into the jet simulation code at each time step. It was clearly confirmed that the transition point of the jet was shifted upstream for the turbulent inflow jet. However, besides an axial shift the downstream development was very similar for the two cases.

In the third part of the present work the acoustic radiation from the computed jet flow was predicted by applying Lighthill’s acoustic analogy to the DNS and LES results. Sound-source locations were identified and the acoustic far-field and spectra were computed. Comparison of the location of the sound sources showed relatively good agreement of the DNS and the LES data indicating that the major parts of the far-field sound should also be reproduced by the LES data. To evaluate this, the Lighthill source terms were computed in frequency space for the DNS and LES database. Far-field noise intensities and spectra were determined along two circular arcs in the major and minor jet planes using the free-space Green’s function. The overall directivity predicted
by the DNS was found to match theoretical prediction based on Doppler scaling and the spectra showed characteristics similar to computational data from a round jet simulation of Freund (2001) when properly scaled. The jet was found to be louder in the direction of its minor axis and to radiate less along its major axis. This behaviour is typically also found in experiments of non-axisymmetric jets.

When computing the radiation based on the LES data, it was found that the low frequency part of the far-field spectra was well reproduced by the LES. Source formulations based on both the filtered velocities and the approximately deconvolved velocities were examined. Spurious waves from the LES data resulted in an unphysical increase of the spectral level at higher frequencies. No improvement could be obtained by evaluating the source terms based on the deconvolved rather than filtered source terms. An analysis of the origin of the spurious contamination of the spectra based on the LES data showed that the main source of errors is due to the acoustic analysis procedure itself. Up to an estimated cutoff frequency the LES reproduces the acoustic results from DNS reasonably well. The analysis confirmed that for the studied flow case LES can be used to predict the dominating part of the acoustic radiation.

For future investigations direct computation of the sound fields is certainly desirable for more complex geometries where no Green’s function is easily available. This also eliminates the additional source of errors caused by the numerical evaluation of the source terms. To be able to apply LES for the direct computation of sound, the influence of subgrid-scale modelling on the radiation has to be evaluated in more detail. ‘Quiet’ subgrid-scale models have to be used so that the emitted sound can be computed directly from the simulation. In that respect also the use of dissipative schemes without any subgrid-scale model (e.g. MILES) might be worth studying in detail, since here no artificial noise from a subgrid scale model is introduced. Additionally, if the long-term objective is to study the detailed mechanisms that actually produce the acoustic radiation, direct computation of the sound for simple fundamental flow cases where DNS is possible has certainly great potential. Control mechanisms have to be developed to influence the acoustic radiation. Early studies for two-dimensional turbulence indicate that already by very subtle changes of the flow field its acoustic radiation can be reduced dramatically (Wei & Freund, 2002). For flow control purposes, also the sound scattering caused by the time-dependent turbulence was shown to be of significance (Cervinõ et al., 2002).
Appendix A

Description of the linear stability eigensolver

An eigensolver for the parallel compressible base flow was implemented according to inviscid linear stability analysis, following Sandham & Reynolds (1989, p. 97). The results were used for code validation and for inflow instability forcing in the jet simulations.

A.1 Implementation

We start from the Euler equations formulated with the primitive variables \( \mathbf{u} = \{\rho, u_1, u_2, u_3, T\} \).

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} &= 0 \\
\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) &= -\frac{1}{\gamma M^2} \frac{\partial p}{\partial x_i} \quad i = 1, 2, 3 \\
\rho \left( \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} \right) &= -p(\gamma - 1) \frac{\partial u_i}{\partial x_i} .
\end{align*}
\]

(A.1)

The variables are non-dimensionalized using \( u_{r e f}^d, \rho_{r e f}^d, p_{r e f}^d \). Assuming only small perturbations \( \mathbf{u}' \) from the base flow \( \mathbf{u}^B \) the flow field is decomposed as \( \mathbf{u} = \mathbf{u}^B + \mathbf{u}' \) and equation (A.1) can be linearized. The base flow is assumed to be in the \( x_1 \)-direction and only depends on the cross-stream coordinate \( x_3 \). Using a normal-mode ansatz for the perturbations \( \mathbf{u}'(x, t) = \hat{\mathbf{u}}(x_3) \exp \{i(\kappa_1 x_1 + \kappa_2 x_2 - \omega t)\} \) and inserting into (A.1) one obtains after linearization the eigenvalue problem for \( \hat{\mathbf{u}} \)

\[
\begin{align*}
\mathbf{i} \hat{\rho} (u_1^B \kappa_1 - \omega) + \hat{u}_3 \mathbf{D} \hat{\rho}^B + \rho^B \{i[\kappa_1 \hat{u}_1 + \kappa_2 \hat{u}_2] + \mathbf{D} \hat{u}_3\} &= 0 \\
\rho^B \{-i \omega \hat{u}_1 + u_1^B i \kappa_1 \hat{u}_1 + \hat{u}_3 \mathbf{D} u_1^B\} &= -\frac{1}{\gamma M^2} \kappa_1 \mathbf{i} \hat{\rho} \\
\rho^B \{-i \omega \hat{u}_2 + u_1^B i \kappa_1 \hat{u}_2\} &= -\frac{1}{\gamma M^2} \kappa_2 \mathbf{i} \hat{\rho} \\
\rho^B \{-i \omega \hat{u}_3 + u_1^B i \kappa_1 \hat{u}_3\} &= -\frac{1}{\gamma M^2} \mathbf{D} \hat{\rho} .
\end{align*}
\]
\[
\rho^B \left\{ i(\kappa_1 u_1^B - \omega) \hat{T} + \hat{u}_3 D^B \right\} = - (\gamma - 1) p^B \cdot \\
(\hat{u}_1 i\kappa_1 + \hat{u}_2 i\kappa_2 + D\hat{u}_3). 
\] (A.2)

The non-dimensionalized pressure in the base profile is taken to be constant \( p^B = 1 \) and the equation of state for a perfect gas reads \( \rho^B / T^B = 1 \). The operator \( D \) represents the derivative in the \( x_3 \) direction and it was discretized using 2nd order finite differences. The derivatives of the mean profiles were evaluated analytically on the mesh. A hyperbolic mapping according to equation (2.7) was implemented. In matrix notation (A.2) can be written as

\[
A_t(u^B, \kappa_1, \kappa_2)\hat{u} = \omega \hat{u} \quad \text{(temporal problem)} \\
A_s(u^B, \omega, \kappa_2)\hat{u} = \kappa_1 B_s(u^B, \omega)\hat{u} \quad \text{(spatial problem)} , 
\] (A.3) (A.4)

where for the temporal eigenvalue problem \( \omega \) is separated and \( A_t \) depends only on the (given) mean profile and \( \kappa_1, \kappa_2 \). For the spatial problem \( \kappa_1 \) is separated and a general eigenvalue problem has to be solved with \( A_s \) and \( B_s \) depending on \( u^B, \kappa_2 \) and \( \omega \). These eigenvalue problems were solved using standard library routines (lapack, blas) applying one-sided stencils at the domain boundaries for computing the derivatives.

### A.2 Validation

The eigensolver was validated by the temporal problem for the incompressible mixing layer profile \( u(x_3) = 0.5(1 + \tanh(x_3)) \), which is well studied in the literature (Michalke, 1964; Sandham & Reynolds, 1989). A comparison was made for \( \Re(\kappa_1) = 0.4 \), a Mach number of 0.01, temperature constraints \( T(\pm\infty) = 1 \) on a grid with 181 nodes (mapping parameter \( k = 0.4 \)). The eigenvalue obtained by the direct solver is \( \Im(\omega) = 9.4097 \times 10^{-2} \) which compares well with \( \Im(\omega) = 9.409 \times 10^{-2} \) (Sandham and Reynolds) and \( \Im(\omega) = 9.410 \times 10^{-2} \) (Michalke).

Another validation test case was the temporal problem for the shear layer profile \( u(x_3) = \tanh(2x_3) \) considered in section 2.2.3. The parameters for this case are listed in table 2.1. The temporal growth rate \( \Im(\omega) \) is plotted in figure A.1 as a function of \( \kappa_1 \). The growth rates and also the eigenfunctions (see figure 2.2) compare well with the ones in Sandham & Reynolds (1989, p. 115, p. 126f).
Figure A.1: Amplification rate for the profile $u(x_3) = \tanh(2x_3)$. 
Appendix B

Duct flow simulations

B.1 Flow setup

Turbulent flow in a rectangularly shaped duct with dimensions $L_2$ and $L_3$ in the lateral and $L_1$ in the streamwise direction (see figure B.1) was considered. The aspect ratio is defined as $W = L_2/L_3$ and normalization of the equations is done based on the bulk quantities

$$u_{1,B} = \frac{1}{V} \int_V \frac{\rho u_1}{\rho_B} \, dV, \quad \rho_B = \frac{1}{V} \int_V \rho \, dV,$$  \hspace{1cm} (B.1)$$

the constant wall temperature and $L_2/2$. The volume $V$ denotes the domain of the duct $V = L_1 L_2 L_3$. Isothermal wall boundary conditions and periodic boundary conditions in the streamwise direction were employed. The average temperature within the flow develops according to the production of thermal energy through dissipation. We define the local wall shear stress as

$$\tau := \langle \mu \frac{\partial u_1}{\partial n} \rangle_{wall},$$  \hspace{1cm} (B.2)$$

where the derivative with respect to $n$ denotes the wall-normal derivative and the operator $\langle \cdot \rangle$ implies averaging in time, in the streamwise direction and over the four quadrants of the duct. An average wall shear stress is then defined by additionally averaging over the walls denoted by the superscript $w$

$$\tau^w := \langle \mu \frac{\partial u_1}{\partial n} \rangle^w.$$  \hspace{1cm} (B.3)$$

The wall friction velocity is then defined as

$$u_\tau = \sqrt{\frac{\tau}{\langle \rho \rangle_{wall}}}, \quad u_{\tau}^w = \sqrt{\langle \tau^w \rangle^w}$$  \hspace{1cm} (B.4)$$

again in a local and wall-averaged sense. The friction Reynolds number can also be written locally and wall-averaged as

$$Re_\tau = \frac{u_\tau \langle \rho \rangle_{wall} L_3/2}{\langle \mu \rangle_{wall}}, \quad Re_\tau^w = \frac{u_{\tau}^w \langle \rho \rangle^w L_3/2}{\langle \mu \rangle^w}.$$  \hspace{1cm} (B.5)$$
Normalization was done using both local and wall-averaged quantities as denoted in the captions of the figures.

### B.2 Simulation method

The LES model and the numerical details were as described in section 2.3. In order to achieve grid point condensation close to the walls, an hyperbolic tangent coordinate mapping was introduced in the cross-stream direction that maps the physical space $\mathbf{x} = (x_1, x_2, x_3)$ onto the computational space $\xi(\mathbf{x}) = (\xi_1(\mathbf{x}), \xi_2(\mathbf{x}), \xi_3(\mathbf{x}))$.

$$\xi_i \in [-0.5, 0.5] \quad \rightarrow \quad x_i \in [-L_i/2, L_i/2], \quad (i = 2, 3)$$

$$x_i(\xi_i) = \frac{\tanh(k_i \xi_i)}{l_i \tanh(k_i/2)}$$

with $l_2 = 1$ and $l_1 = W$. \hspace{1cm} (B.6)

In the streamwise ($x_1$) direction an equidistant mesh was used. The metric terms were determined analytically. In order to compensate for the momentum loss due to skin friction a forcing term was introduced into the non-dimensional momentum equation:

$$f_{\rho u_1}(t) = \frac{1}{2Re} (\langle \tau_{12} \rangle_{1,3}^{1,3} |_{x_2=1} - \langle \tau_{12} \rangle_{1,3}^{1,3} |_{x_2=-1}$$

$$+ W \langle \tau_{13} \rangle_{1,3}^{1,2} |_{x_3=-\frac{1}{2}} - W \langle \tau_{13} \rangle_{1,3}^{1,2} |_{x_3=-\frac{1}{2}}).$$ \hspace{1cm} (B.7)
Table B.1: Parameters of the duct simulations. Unless noted otherwise, parameters are non-dimensionalized based on the bulk quantities (eq. (B.1)) and $L_2/2$.

Here we denote $\langle \cdot \rangle^{i,j}$ as an average quantity in the $i,j$-plane. This is equivalent to setting the forcing term such that the integral $x_1$-momentum within the duct vanishes.

$$
\int_V \frac{\partial \rho u_1}{\partial t} \, dV = \int_V \text{rhs}_{\rho u_1}(t) \, dV + V f_{\rho u_1}(t) = 0 \quad . \tag{B.8}
$$

Here we have abbreviated the right-hand side of the $x_1$-momentum equation with $\text{rhs}_{\rho u_1}(t)$. In our simulations we used the latter formulation to compute the forcing term $f_{\rho u_1}$. The volume integrals where evaluated using the trapezoidal quadrature rule. In figure B.5 (a) the temporal development of the forcing term is plotted for both formulations for the W1RE2205 case and it can be observed that they differ only slightly due to numerical errors of discrete integration and derivative. In order to avoid a temporal drift a second term is added, which ensures that the bulk momentum is unity.

$$
f_{\rho u_{1,2}}(t) = -\langle \rho u_1 \rangle - 1 \quad . \tag{B.9}
$$
We found that this second term is at least two orders of magnitude smaller than the one computed with equation (B.8). The momentum forcing enters the energy equation as

\[ f_E = u_1 \cdot f_{pu}(t) \quad (B.10) \]

The forcing terms were evaluated based on the deconvolved quantities and added to the fluxes before filtering. Due to the non-conservative property of the filtering operation at the wall (no filtering is performed at the wall point) a slow drift in the bulk density was suppressed by density rescaling after each time step.

### B.3 Results

Six different cases were considered, the parameters of which are listed in table B.1. For the first two cases (W1RE2205 and W1RE5160) a Mach number of 0.5 was chosen for validation against incompressible data of Gavrilakis (1992) and Huser & Biringen (1993).

A systematic variation of the aspect-ratio of the duct was performed for a Mach number of 0.7 and the aspect-ratios \( W = 1.0, 2.5, 5.0 \) and \( \infty \) (cases W1RE5000 to W\( \infty \)RE5000), the latter corresponding to canonical plane channel flow. Note that for the channel flow case also spanwise periodicity was introduced. For the aspect-ratio study the Reynolds number \( Re_{B,L_3/2} \) based on \( L_3/2 \) and the bulk velocity, was kept constant. The time step for all simulations was of the order \( \Delta t \cdot W = 1.2 \times 10^{-3} \).

We show mean-flow quantities and wall shear stresses in figures B.2 and B.3 for the Mach 0.5 cases. Averaging is again performed in the streamwise direction, over the four duct quadrants and time. In figure B.2(a) velocity profiles along the two duct-centerlines are plotted and additionally the law of the wall \( (u^+ = 2.5 \ln(x_3^+) + 5.1) \) as well as the duct-flow-adjusted law of the wall given by Gavrilakis (1992) \( (3.2 \ln(x_3^+) + 3.9) \) is shown. The results for the mean-flow data match available incompressible DNS data very well, a result which is also found by Salinas Vázquez & Métais (2002) for the same configuration using LES. For case W1RE5160 the profile does not exactly match the one from Huser & Biringen (1993, Run A), however theirs also drops toward the law of the wall with increasing number of grid points (their Run B, not shown here).

The wall stresses, shown in figure B.3 along the two sidewalls of the duct, are not fully statistically converged but they match the results from
the cited references relatively well. The corner vortices in the mean flow are clearly developed as can be seen in figure B.4. Only one quadrant of the cross-stream slice is shown.

The friction Reynolds number $Re^w$ for the cases W1RE2205 and W1RE5160 in the references is 150 and 300 respectively, which is somewhat lower than the values 175 and 333 we find in the present study (see table B.1). We point out that the listed $Re^w$ is based on wall quantities and when evaluating it based on the bulk density the values drop to 150 and 317, which is considerably closer to the documented values. Comparison of compressible with incompressible data, however, remains difficult due to the variable density.

In figure B.5 (b) the temporal development of the friction Reynolds number is plotted for the case W1RE2205. It is defined by taking the temporal averaging out of $Re^w$ which we only denote by the explicit dependency on the time in $Re^w(t)$. One observes a variation in time with the same frequencies also seen in the development of the forcing terms.

In the following the results for the Mach 0.7 cases (cases W1RE5000 to W5RE5000) are described. Figure B.6 shows the van-Driest-transformed normalized downstream velocity profiles $u_{1,v_D}^+$ along the shorter center-axes of the ducts, normalized with the local friction velocity. Additionally the well-known law of the wall is plotted. We observe a significant difference from the profiles compared to classical data from the law of the wall for the increasing aspect ratio. Obviously the aspect ratio $W = 5$ is not yet large enough to make the influence of the other sidewalls on the mean flow profile in the middle of the duct along the shorter axis negligible. At this point it is still unclear which aspect ratio is sufficient to obtain channel-like profiles in the center of a duct.

The wall shear stress $\tau/\tau^w$ is shown in figure B.8 for the cases W2.5RE5000 and W5.0RE5000. The corner vortices for both cases are expressed in local peaks in the profiles along the major axis of the duct close to the corners. However, these profiles are not fully converged. The present streaky structures in the boundary layer close to the wall last for a long time and compared to channel flow, where one usually averages over those structures, they here lead to a very slow convergence of the wall-stress profiles.

For visualization of the streaks close to the wall contours of the wall-normal vorticity at $x_3^+ \approx 10$ are shown in figure B.9. The streaks become more narrow close to the confining side walls of the duct, whereas in the
center of the duct they are very similar to the streaks in channel flow.

The Reynolds stresses along the centerlines are depicted in figure B.10 for all four $M = 0.7$ cases. The channel data are close to data from literature for a similar Reynolds number. With increasing aspect-ratio $W$ beginning with $W = 1$ one observes an increase in the Reynolds normal stresses along the minor axis of the duct. The maximum Reynolds normal stress for all duct cases is found close to the walls ($x_3 = const$) at $x_2 = 0$.

Figure B.7 compares the mean-flow distribution for the cases W1RE5000, W2.5RE5000 and W5RE5000. It is observed how the influence of the mean corner vortex pair becomes larger with increasing aspect-ratio, induces neighboring vortices, and the duct center ($x_2 = 0$) becomes less and less influenced by the confining side walls.
Figure B.2: Comparison with incompressible data: $Re = 2205$ (a) and $Re = 5160$ (b). Mean flow along centerlines: $u_1^+(x_2^+)$; $u_1^+(x_3^+)$; logarithmic law $\frac{1}{0.4}\ln(x_3^+) + 5.1$ (lower curve), $3.2\ln(x_3^+) + 3.9$ from Gavrilakis (1992) (upper curve) and linear law at the wall; DNS data of Huser & Biringen (1993, Run A).

Figure B.3: Comparison with incompressible data: $Re = 2205$ (a) and $Re = 5160$ (b). Wall stress along sidewalls. LES, DNS data of (a) Gavrilakis (1992) and of (b) Huser & Biringen (1993).
Figure B.4: Mean flow field: $Re = 2205$ (a) and $Re = 5160$ (b). Vectors denote flow in cross-flow plane, contours denote downstream component, increment 0.1, from 0 to 1.3 (a) and to 1.2 (b).

Figure B.5: (a) Forcing terms $- (B.7)$ and $- (B.8)$ and (b) $Re^w_w(t)$ (see text for explanation) for case W1RE2205 in time.
Figure B.6: Aspect-ratio study: Mean flow along the centerline (coordinate $x_3$ at $x_2 = 0$) of the duct for $W = 1.0$, $W = 2.5$, $W = 5.0$ and $W = \infty$ (channel), logarithmic law $\frac{1}{0.4} \ln(x_3^+) + 5.1$ and linear law at the wall.

Figure B.7: Aspect-ratio study: Mean flow field for the cases W1RE5000, W2.5RE5000, W5.0RE5000: vectors denote mean cross-flow, contours denote downstream component with increment 0.1, from 0 to 1.2.
Figure B.8: Aspect-ratio study: Wall shear stress along the two sidewalls for $W_{2.5RE5000}$ (a) and $W_{5.0RE5000}$ (b).

Figure B.9: Aspect-ratio study: Instantaneous wall normal vorticity contours at $x_3^+ \approx 10$ for the cases $W_{1RE5000}$, $W_{2.5RE5000}$, $W_{5.0RE5000}$.
Figure B.10: Aspect-ratio study: Reynolds stresses along centerlines for (a) $W \infty R E5000$, (b) $W1RE5000$ (c) $W2.5RE5000$ (d) and $W5.0RE5000$; \rule{8.5cm}{0.25pt}, \rule{8.5cm}{0.25pt}, $\langle \rho u_1''u_1'' \rangle / (\sqrt{\rho B u_1'^w})$ (for the channel case $u_1'^w$ denotes averaging over the two walls), \bullet incompressible data from Kim et al. (1987) for $Re_\tau = 392.24$, \rule{8.5cm}{0.25pt}, $\langle \rho u_2''u_2'' \rangle / (\sqrt{\rho B u_2'^w})$, \rule{8.5cm}{0.25pt}, $\langle \rho u_3''u_3'' \rangle / (\sqrt{\rho B u_3'^w})$. 
Bibliography


# Curriculum vitae

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