Diss. ETH N$^{\text{o}}$ 21600

ON EXPANSION AND SPECTRAL PROPERTIES OF SIMPLICIAL COMPLEXES

A dissertation submitted to

ETH Zürich

for the degree of

Doctor of Sciences

presented by

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2013

Dr. Enzian erforscht zur Zeit das Ungefähre.
Im Verein mit einem Doctor Cherry-Brander
(und nach dessen chemisch-zoologischer Erkältungslehre)
drängelt er durch eine lange Wasserleitungsröhre
alle drei Minuten einen Feuersalamander.

Währenddessen schreibt und rechnet Dr. Cherry-Brander,
dividiert die Zeit durch Wärmegrade, subtrahiert die Fehlerquellen –
dann vergleichen beide Forscher mittels einer Art Expander
ihre und des Salamanders Muskelkräfte miteinander,
und sie übertragen die gefund'nen Resultate auf Tabellen.

Dr. Enzian betreibt das Ganze zoologisch, Dr. Cherry chemisch.
Für die Praxis war bis jetzt noch kein Ergebnis festzustellen.
Die Versuche sind rein akademisch.

Peter Paul Althaus

Abstract

This thesis deals with simplicial complexes as higher-dimensional generalizations of graphs. This perspective, the combinatorial study of simplicial complexes, has attracted increasing attention in recent years. Its aim is to find higher-dimensional analogues of basic properties and results from graph theory. This thesis mostly focusses on the notion of graph expansion and higher-dimensional notions corresponding to it.

In the first part of the thesis we consider a model of *random complexes*, introduced by Linial and Meshulam, that is a higher-dimensional analogue of the Erdős-Rényi random graph $G(n, p)$. We present two results generalizing well-established results for random graphs. First, we consider the threshold for the property of containing a subdivision of any fixed complete complex. The second result on random complexes concerns the eigenvalues of higher-dimensional analogues of adjacency matrices and graph Laplacians. We show that for higher-dimensional random complexes the spectra of these matrices have a certain concentration behaviour that has also been observed for random graphs.

For graphs, one of the reasons why the eigenvalues of adjacency matrices and Laplacians are of interest is their close connection to expansion. The spectral gap of a graph can be seen as a measure of its edge expansion that is computable in polynomial time. In the second part of this thesis we consider approaches to finding higher-dimensional analogues of this phenomenon.

There are several higher-dimensional notions that generalize edge expansion. One, *combinatorial expansion*, is based on cohomological notions and was suggested independently by Gromov, Linial and Meshulam and Newman and Rabinovich. We show that for combinatorial expansion the direct analogue of the graph situation fails: Spectral expansion does not imply combinatorial expansion in higher dimensions. It does however imply a weaker expansion property, as was shown by

Parzanchevski, Rosenthal and Tessler. We present a strengthening of their proof that yields an intermediate expansion property.

We will then consider two other approaches to finding a lower bound for combinatorial expansion in higher dimensions that is polynomially computable. The first approach is to consider semidefinite relaxations of a polynomial program describing combinatorial expansion. The second considers higher-dimensional notions of quasirandomness that were introduced by Gowers.

The eigenvalues of a graph also express properties other than expansion. Recently, Trevisan established a close connection of the largest Laplacian eigenvalue to the bipartiteness ratio, measuring how close a graph is to having a bipartite component. We present a generalization of a part of his result to 2-dimensional simplicial complexes.

The last part of this thesis deals with maps of simplicial complexes into Euclidean spaces and with questions concerning the intersections of images of simplices under such maps. We establish a connection between the non-trivial eigenvalues of the Laplacian of a simplicial complex and the minimal number of crossings of image simplices under any affine map. In the last chapter, we consider the overlap number, the maximal number of image simplices sharing a common point. We study a structure, *pagodas*, introduced by Matoušek and Wagner in order to improve the known bounds for the overlap numbers of complete 3-complexes.

Zusammenfassung

Diese Arbeit befasst sich mit Simplizialkomplexen als Verallgemeinerung von Graphen. Diese Sichtweise, die auch als ,,kombinatorische Theorie der Simplizialkomplexe" bezeichnet wird, gewinnt zunehmend an Aufmerksamkeit. Das Ziel hierbei ist es, grundlegende Begriffe und Resultate der Graphentheorie in höhere Dimensionen zu überführen, oder zumindest zu sehen, in wie weit solche Verallgemeinerungen möglich sind. Diese Arbeit konzentriert sich vor allem auf höher-dimensionale Entsprechungen des Begriffes der Graphenexpansion.

Der erste Teil der Arbeit befasst sich mit einem Modell für zufällige Simplizialkomplexe, genauer mit einer Verallgemeinerung des Erdős-Rényi-Zufallsgraphenmodells $G(n, p)$, die von Linial und Meshulam eingeführt wurde. Es werden zwei Ergebnisse präsentiert, beides höherdimensionale Analoga bekannter Graphenresultate. Zunächst betrachten wir den Schwellenwert für die Eigenschaft eines Zufallskomplexes, eine Unterteilung eines festen vollständigen Komplexes zu enthalten. Das zweite Ergebnis behandelt die Eigenwerte von höher-dimensionalen Verallgemeinerungen der Adjazenz- und Laplace-Matrizen von Graphen. Wir zeigen, dass die Eigenwerte dieser Matrizen für Zufallskomplexe höherer Dimension bezüglich ihrer Konzentration dasselbe Verhalten aufweisen wie auch die Eigenwerte von Zufallsgraphen.

Das Interesse an den Eigenwerten solcher Matrizen für Graphen ist unter anderem in der engen Beziehung, die diese zu Expansionseigenschaften aufweisen, begründet. Der zweite Eigenwert der Laplace-Matrix eines Graphen kann als in polynomieller Zeit berechenbares Maß für seine Kantenexpansion angesehen werden. Der zweite Teil dieser Arbeit behandelt Ansätze, höher-dimensionale Entsprechungen dieses Phänomens zu finden.

Es bestehen verschiedene Ansätze für die Verallgemeinerung von Kantenexpansion in höheren Dimensionen. Ein auf kohomologischen Be-

griffen basierender Ansatz, als „kombinatorische Expansion” bezeichnet, wurde von Gromov vorgschlagen, tauchte aber unabhängig davon auch in Arbeiten von Linial und Meshulam und von Newman und Rabinovich auf. Wir zeigen, dass die direkte Verallgemeinerung des oben erwähnten Graphen-Phänomens für diesen Expansionsbegriff nicht gilt: Kombinatorische Expansion folgt in höheren Dimensionen nicht aus spektraler Expansion. Eine andere, schwächere Expansionseigenschaft folgt jedoch aus spektraler Expansion. Dies wurde von Parzanchevski, Rosenthal und Tessler untersucht. Wir zeigen, wie dieser Beweis gestärkt werden kann, um eine zwischen kombinatorischer Expansion und der schwächeren Expansionseigenschaft liegende Eigenschaft zu erhalten.

Desweiteren betrachten wir zwei andere Ansätze für eine in polynomieller Zeit berechenbare untere Schranke für kombinatorische Expansion in höheren Dimensionen. Der erste Ansatz besteht darin, semidefinite Relaxierungen eines polynomiellen Programms, das kombinatorische Expansion beschreibt, zu betrachten. Der zweite verwendet einen höherdimensionalen Quasizufälligkeitsbegriff, welcher von Gowers eingeführt wurde.

Neben Expansion werden auch andere kombinatorische Eigenschaften von den Eigenwerten eines Graphen beschrieben. Trevisan konnte eine Verbindung des größten Laplace-Eigenwertes mit einem Wert herstellen, der ausdrückt, wie weit entfernt der Graph davon ist, eine bipartite Zusammenhangskomponente zu haben. Wir zeigen eine partielle Verallgemeinerung seines Resultats für 2-dimensionale Komplexe.

Der letzte Teil dieser Arbeit beschäftigt sich mit Abbildungen simplizialer Komplexe in Euklidische Räume und mit Fragen zu Schnittpunkten verschiedener Simplizes unter solchen Abbildungen. Für affine Abbildungen zeigen wir eine Verbindung zwischen den nicht-trivialen Laplace-Eigenwerten eines Simplizialkomplexes und der minimalen Anzahl von sich schneidenden Paaren von Bildern von Simplizes. Im letzten Kapitel der Arbeit betrachten wir die maximale Anzahl von Bildern von Simplizes, die alle einen gemeinsamen Punkt enthalten. Wir untersuchen eine Struktur, die einer Pagode, die von Matoušek und Wagner eingeführt wurde, um bisher bekannte Schranken für diese Anzahl für vollständige Komplexe zu verbessern.

Acknowledgements

First and foremost, I would like to express my gratitude to my advisor Uli Wagner. He has taught me almost everything I know about the subjects of this thesis – often in private lectures lasting up to two hours – and has been very generous in sharing his ideas with me. I also want to thank Emo Welzl for being given the opportunity to be a "Gremo" and thus part of a lively and very stimulating research environment that allowed me to learn a lot. Further thanks goes to Michael Farber for reviewing this thesis, giving helpful comments and for being a very friendly co-examiner.

I want to thank former and current Gremos for creating an inspiring work environment: Dominik Scheder, Marek Sulovský, Tobias Christ, Martin Jaggi, Heidi Gebauer, Robin Moser, Yves Brise, Andrea Francke, Sebastian Stich, Timon Hertli, Vincent Kusters, Hemant Tyagi, Antonis Thomas, Manuel Wettstein, and May Szedlák, whom I had the honour to have as fellow PhD students, as well as Bernd Gärtner, Michael Hoffmann, and Andrea Salow. This list should include as well Christoph Krautz and Henning Thomas, who also helped to enrich my life at ETH.

In the past months, when writing up this thesis, I became indebted to many people. I thank May Szedlák for being a wonderful student and enabling me to experience the joys of supervising. I am grateful to Andrea and Erich for giving me the resemblance of a private life during lunch breaks and to the Remy family for lovely days on a boat, where – even though being mocked about it – I was allowed to work in peace.

Sincerest thanks go to my flatmates, Micha, Camilla and most of all Susanne for keeping me fed and clean-clothed during these months, for making me become human again each night when I came home and for a wonderful surprise picnic in CAB G 19.2. I owe deep thanks to my parents for always believing in me and being supportive, whether during late-night phone calls or on a wonderful work/hike trip. Last but by no means least I want to thank Fred – for many things: for proof-reading this thesis, for trying to teach me how to not worry too much, but most of all for being there for me when no one else knew how to.

Contents

Introduction

Graph theory is a well-established field of research in discrete mathematics, with abundant applications in practice and theory. A classical generalization of graphs are hypergraphs, where sets of more than two vertices are allowed to form an edge. A more specific way to generalize graphs is given by simplicial complexes. This notion traditionally arises in algebraic topology, where simplicial complexes are used to achieve combinatorial descriptions of topological spaces. From a combinatorial perspective, they can be considered as a special class of hypergraphs carrying a topological interpretation.

In recent years, the study of simplicial complexes as generalizations of graphs has attracted increasing attention and many graph-theoretic results have found their counterparts in the world of higher-dimensional simplicial complexes. This newly emerging field of research is, so far, mostly of theoretical nature. It is not yet directed towards practical applications but instead has the goal to develop a combinatorial theory of simplicial complexes, where basic questions and concepts still need to be settled. From a theoretical perspective, one reason why it is an appealing area of research is that it combines combinatorics with other fields of mathematics, especially with geometrical and topological aspects. Nevertheless, one can hope for future applications since simplicial complexes offer a way to model systems that have a more complicated structure than the pairwise interaction modelled by graphs.

This thesis has its focus in this developing field of research. One of its main topics is the generalization of the notion of expansion for graphs to higher dimensions.

Expansion in Higher Dimensions

Roughly speaking, a graph is an expander if it is sparse and at the same time well-connected. Such graphs have found various applications, in theoretical computer science as well as in pure mathematics. Expander graphs have, e.g., been used to construct certain classes of error correcting codes and in a proof of the PCP Theorem, a deep result in computational complexity theory. They also yield examples of metric spaces that require the largest distortion when embedded into Euclidean space, among all finite metric spaces. These and other applications are covered in the surveys [62] and [81]. The latter also describes applications to number theory and group theory.

The great success of this concept for graphs has inspired the search for a corresponding notion in higher dimensions. One analogue, going by the name of *combinatorial expansion*, is based on concepts from algebraic topology, more specifically on cohomological notions. It emerged in various contexts as a useful notion. Linial, Meshulam and Wallach [79, 92] used the combinatorial expansion of the complete complex, containing all possible simplices on a fixed set of vertices, to study the cohomological properties of random complexes. Gromov suggested this notion when examining more geometrical notions of expansion: Any expander graph possesses the following geometric overlap property. When mapped to the real line $\mathbb{R}$, it exhibits a point in $\mathbb{R}$ that is covered by the images of a lot of edges. The higher-dimensional analogue of this situation is captured by the *overlap number* of a simplicial complex. Gromov [56] showed that any combinatorially expanding complex has a large overlap number. Similar concepts to combinatorial expansion were also suggested by Newman and Rabinovich [94] who studied higher-dimensional analogues of finite metric spaces.

For graphs, the combinatorial notion of expansion is closely related to the spectra of certain matrices associated with the graph: the adjacency matrix and the Laplacian. A different approach to generalizing expansion is hence to consider higher-dimensional analogues of these matrices. Higher-dimensional Laplacians were first introduced by Eckmann [40] in the 1940s and have since then been used in various contexts. We just mention one example – Kalai's higher-dimensional generalization of Cayley's formula for the number of labeled trees [68], whose proof is based on a proof of Cayley's formula using properties of the Laplacian. An analogue of the adjacency matrix for graphs can be derived from these higher-dimensional Laplacians. Many results in this thesis are concerned with the spectra of these matrices.

The spectra of Laplacians in turn were connected to the overlap number in a recent preprint by Parzanchevski, Rosenthal and Tessler [97], where yet another, more combinatorial notion of expansion is suggested. All these different notions of expansion are analogues of graph properties. For graphs, except for the geometrical overlap property, all notions of expansion are basically equivalent. This is not the case for simplicial complexes. The goal in this area is to understand relations between these properties and to detect which one is – possibly depending on the context – the "correct" notion of expansion.

In contrast to the many applications of graph expansion, these higher-dimensional concepts of expansion have yet to show a similar profitability, but the many contexts in which they appeared give rise to hope in this respect. Also, some further connections to other areas of mathematics have been starting to show: The spectra of graphs are known to be related to random walks on graphs and via this also to approximation algorithms for hard counting problems. A connection between eigenvalues of higher-dimensional Laplacians and random walks on simplicial complexes is developed in [96], but it does not seem to yield similar direct applications. A connection to coding theory has been announced in [82].

Structure of the Thesis

The first two chapters cover basic notions. We first introduce simplicial complexes, cohomology and random complexes in Chapter 1. Chapter 2 gives a review of expansion in graphs and then presents the several notions of expansion for complexes discussed above.

Results on Random Simplicial Complexes. The $G(n, p)$ model for random graphs, introduced by Erdős and Rényi in the 1960s, is the basis for a wide area of research in discrete mathematics and has been studied extensively in the last decades. A random model $X^k(n, p)$ for simplicial complexes of arbitrary fixed dimension k which generalizes this random graph model was introduced recently by Linial and Meshulam [79] and has since then gained a lot of attention, see, e.g., [9, 11, 29, 73, 92, 116]. Chapters 3 and 4 present two results on this model of random complexes. Both are generalizations of results that are well-established for random graphs.

The first result, presented in Chapter 3, is independent of the discussion on higher-dimensional expansion. It is a generalization of a well-known result of Ajtai, Komlós and Szemerédi [3, 41], that the (sharp) threshold for containment of any complete graph of fixed size as a topological minor is $p = 1/n$. We show that for the random complex $X^2(n, p)$ the (coarse) threshold for containing a subdivision of any fixed complete 2-complex is at $p = \Theta(1/\sqrt{n})$. A part of the result also extends to random complexes of dimension $k > 2$. An upper bound of $n^{-1/2+\epsilon}$ on the threshold was previously shown by Cohen, Costa, Farber and Kappeler [29]. This chapter is based on the extended abstract [59], the presented results are joint work with Uli Wagner.

The second result, described in Chapter 4, returns to the question of higher-dimensional expansion, more specifically to spectral expansion. We study the concentration of the spectra of the adjacency matrix and the normalized Laplacian of random complexes $X^k(n, p)$. For both matrices we show that the eigenvalues are asymptotically almost surely concentrated around two values for large enough p. This is analogous to the situation for random graphs, which we will also discuss. Our proof reduces the problem to a question on the spectra of vertex links in the complex. As these are random graphs of the type $G(n, p)$ we can then rely on the corresponding results for graphs. The reduction to vertex links works via a result of Garland [53] on Laplacians of simplicial complexes. A further result of this chapter is an analogue of Garland's result for the adjacency matrix, which enables us to pursue the same strategy also for this matrix. The results of this chapter are joint work with Uli Wagner; they were partly published in the extended abstract [58].

Connecting Different Notions of Expansion. For graphs, the close relation of the spectrum of the Laplacian and combinatorial expansion properties like the *edge expansion* is expressed, e.g., by the discrete Cheeger inequality and the Expander Mixing Lemma – results that we will discuss in some detail in Chapter 2. In Chapters 5 and 6, we explore the natural question whether there are higher-dimensional analogues of such results.

The discrete Cheeger inequality in particular implies that a graph is combinatorially expanding if and only if it is spectrally expanding. This is not true in higher dimensions. In Chapter 5, we give a probabilistic construction that shows that, in higher dimensions, spectral expansion does not imply combinatorial expansion. This result was published in [58]; it is joined work with Uli Wagner.

A second result in this chapter concerns the relation of spectral expansion to other expansion properties. In [97], Parzanchevski, Rosenthal and Tessler show an analogue of one part of the discrete Cheeger inequality for a more combinatorially inspired notion of expansion that is weaker than combinatorial expansion. We show an extension of their result to an intermediate expansion property.

The discrete Cheeger inequality can be seen as a tool to approximate the (hard to compute) edge expansion of a graph by a polynomially computable quantity, the second eigenvalue of the Laplacian. Chapter 6 presents two other basic approaches to finding a computable lower bound for combinatorial expansion in higher dimensions. This chapter describes ongoing research and is more focused on presenting ideas than on results. It is based on joint work with Uli Wagner.

Combinatorial Expansion can be phrased as a polynomial optimization problem in 0/1-variables. The first approach we consider in this chapter is to consider semidefinite relaxations of this program.

The second approach considers generalizations of *quasirandomness* properties for graphs [26, 27], which give an additional connection between spectral and expansion properties of graphs. A graph is quasirandom if it satisfies one of several equivalent conditions, all of which are expected to hold in a random graph of the same density. Next to spectral and expansion properties, a third quasirandom property involves the number of 4-cycles in the graph. In [55], Gowers extends several graph theoretic results connected to quasirandomness to 2-dimensional complexes. In particular he shows that the number of octahedra, which can be seen as generalizations of 4-cycles in graphs, is related to a property that has similarities with the notion of cohomological expansion. We explore the prospect of finding a lower bound using this notion.

The Largest Laplacian Eigenvalue. There is a strong and well-studied relation between the second eigenvalue of the Laplacian of a graph and its edge expansion, attested, e.g., by the discrete Cheerger inequality. Very recently, similar results have been achieved for the eigenvalue at the other end of the spectrum. In [109], Trevisan studies the connection between the largest eigenvalue of the normalized Laplacian of a graph G and a parameter, the *bipartiteness ratio* of G, that measures how far G is from having a bipartite connected component. He shows a result that can be seen as an analogue to the discrete Cheeger inequality. In Chapter 7, we consider possible analogues for bipartiteness for 2-dimensional simplicial complexes and find a partial generalization of Trevisan's result.

Mapping Simplicial Complexes into Euclidean Space. In Chapter 8, we consider maps of simplicial complexes into Euclidean spaces and study questions concerning the intersections of images of simplices under such maps. The results in this chapter are joint work with Uli Wagner.

The first question concerns the number of crossings of image simplices under an affine map. We apply a result of Parzanchevski, Rosenthal and Tessler from [97], a higher-dimensional analogue of the Expander Mixing Lemma, to establish a connection between the non-trivial eigenvalues of the Laplacian of a simplicial complex and the minimal number of crossings.

In the second section, we consider a question concerning the overlap number of a simplicial complex, which, as mentioned above, can also be seen as a measure of higher-dimensional expansion. The overlap numbers of complete complexes have been in the focus of active research. It is known that, asymptotically, the overlap number of a complete complex does not depend on the number of vertices but only on the dimension of the complex. The size of the constant in terms of the dimension, however, is not known precisely. We study a structure, *pagodas*, that was introduced by Matoušek and Wagner in [91] in order to improve the known bounds for the overlap numbers of complete 3-complexes.

Chapter 1
Preliminaries

The mathematical concept central to this thesis is the notion of a simplicial complex. In this chapter, we give a short introduction to simplicial complexes and to simplicial cohomology and collect some basic terminology. We furthermore introduce random complexes and elementary probabilistic tools for their study.

1.1 Simplicial Complexes

From a topological perspective, simplicial complexes provide a way of describing spaces in a combinatorial manner, by means of triangulations. It should be remarked that not all topological spaces permit a triangulation, but for those that do, it is a very convenient and concrete description. We will study simplicial complexes from a more combinatorial perspective, regarding them as a particular class of hypergraphs that allow a topological interpretation. In what follows, we focus on finite simplicial complexes.

Abstract Simplicial Complexes. A (finite abstract) *simplicial complex* X is a finite set system that is closed under taking subsets, i.e., $F \subset H \in X$ implies $F \in X$. The sets $F \in X$ are called *faces* of X. The *dimension* of a face F is $\dim(F) = |F| - 1$. The dimension of X is the maximal dimension of any face. A k-dimensional simplicial complex will also be called a *k-complex*. The empty set is considered as the unique (-1)-dimensional face of any simplicial complex. A k-dimensional simplicial complex is *pure* if all maximal simplices in X have dimension k.

Pure k-dimensional simplicial complexes are thus essentially the same as $(k+1)$-uniform hypergraphs.

We denote the set of i-dimensional faces by X_i. The *number of i-dimensional faces* is denoted by $f_i(X) := |X_i|$. The *i-skeleton* of X is the simplicial complex $X_{-1} \cup X_0 \cup \ldots \cup X_i$. A *vertex* of X is a 0-dimensional face $\{v\}$, the singleton set will be identified with its element v. The set of vertices X_0, also denoted by $V = V(X)$, is called the *vertex set* of X. Graphs can be considered as 1-dimensional simplicial complexes. Corresponding to the notation $G = (V, E)$ for graphs, we sometimes write 2-complexes as $X = (V, E, T)$ with $E = X_1$ and $T = X_2$.

A very basic example of a k-dimensional simplicial complex is the *complete k-complex* K_n^k that has vertex set $V = [n]$ and $X_i = \binom{[n]}{i+1}$ for all $i \leq k$. The complex K_n^{n-1} is also known as the *$(n-1)$-dimensional simplex* or $(n-1)$-*simplex*. So, in other words, K_n^k is the k-skeleton of the $(n-1)$-simplex.

For a face $F \in X$ of a k-dimensional complex X, the *link* $\mathrm{lk}(F) = \mathrm{lk}_X(F)$ of F is the complex $\{H \in X : H \cup F \in X\}$. We define the *degree* of F as $\deg(F) = \deg_X(F) = |\{G \in X_k : F \subseteq G\}|$, the number of k-dimensional faces containing F.

Geometric Simplicial Complexes. In order to associate simplicial complexes with toplogical spaces, one considers a more geometric notion: A *geometric simplicial complex* is a finite collection Δ of geometric simplices in $\mathbb{R}^m$ satisfying two conditions: If σ is in Δ and τ is a face of σ, then τ is also in Δ. Furthermore, the intersection of any two simplices in Δ is a common face of both, or empty. Here, a *geometric simplex* σ is the convex hull of a set of affinely independent points, the *vertices* of σ, in some Euclidean space $\mathbb{R}^m$. A *face* of σ is the convex hull of a subset of the vertices of σ.

A geometric simplicial complex Δ defines a topological space, its *polyhedron*, the union of all its simplices: $\|\Delta\| = \bigcup_{\sigma \in \Delta} \sigma \subset \mathbb{R}^m$. It carries the subspace topology inherited from the ambient Euclidean space $\mathbb{R}^m$. We call Δ a *triangulation* of $\|\Delta\|$. Any geometric simplicial complex Δ gives rise to an abstract complex X in a straight-forward way: A set of vertices forms an (abstract) simplex in X if and only if it is the vertex set of a geometric simplex in Δ. The geometric complex Δ is then called a *geometric realization* of X, or of any abstract complex isomorphic to it. Here, two simplicial complexes are *isomorphic* if there is a face-preserving bijection between their vertex sets.

Any abstract complex X has a geometric realization, e.g., as a subcomplex of a simplex of sufficiently high dimension. We denote by $\|X\|$ the polyhedron of any geometric realization of X. This is well-defined because the polyhedra of (geometric realizations of) two isomorphic complexes are homeomorphic (see, e.g., [90]). Also the abstract complex X is called a triangulation of $\|X\|$.

Subcomplexes and Subdivisions. A *subcomplex* of X (or Δ) is a subset $Y \subset X$ (Δ) that is itself a simplicial complex. A *subdivision* of a geometric simplicial complex Δ is a complex Δ' with $\|\Delta\| = \|\Delta'\|$ such that every simplex of Δ' is contained in some simplex of Δ. For an abstract complex X, a complex X' is a subdivision of X if there exist geometric realizations Δ and Δ' of X and X' such that Δ' is a subdivision of Δ. A subdivision of an abstract 2-complex X can be seen as a 2-complex X' that is obtained by replacing the edges of X with internally-disjoint paths and the triangles of X with internally-disjoint triangulated disks such that for every triangle the subdivision of the triangle agrees with the subdivisions of its edges; see Figure 1.1 for an illustration.

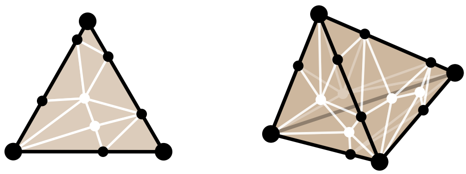

Figure 1.1: Subdivisions of K_3^2 (a *triangulated disk*) and of K_4^2. Vertices and edges internal to triangles are drawn in white.

1.2 Cohomology

Using simplicial complexes it is possible to assign a collection of groups to each topological space that admits a triangulation. These groups are assigned in a manner such that spaces that are homeomorphic are assigned the same collection of groups, making this a means to distinguish topological spaces. We now describe one way to associate such groups to a simplicial complex, namely cohomology groups.

For a graph, connectivity is equivalent to having a certain trivial cohomology group, and expansion in some sense measures how well-connected a graph is. We will later see that measuring how far a space is from having non-trivial cohomology can be considered as a notion for higher-dimensional expansion. We only describe the basic definition of cohomology, for a thorough introduction see, e.g., [93].

Orientations and Incidence Numbers. Let X be a simplicial complex. In order to define the cohomology groups of X, for each face of X we need to fix an *orientation*, i.e., a linear ordering of its vertices. We do this by fixing a linear ordering on the whole vertex set V and then consider each face with the induced orientation. Formally, let $F = \{v_0, v_1, \ldots, v_i\} \in X_i$ be an i-dimensional face with $v_0 < v_1 < \ldots < v_i$. For an $(i-1)$-dimensional face $G \in X_{i-1}$, we then define the *oriented incidence number* $[F : G]$ as follows:

$$[F : G] := \begin{cases} (-1)^j & \text{if } G \subseteq F \text{ and } F \setminus G = \{v_j\}, \\ 0 & \text{if } G \not\subseteq F. \end{cases}$$

In particular, for every vertex $v \in V$ and the unique empty face $\emptyset \in X_{-1}$, we have $[v : \emptyset] = 1$.

Cohomology Groups. Cohomology groups can be defined with coefficients in any Abelian group. For our purposes it suffices to focus on fields.

For a field $\mathbb{F}$, denote by $C^i(X; \mathbb{F})$ the vector space $\mathbb{F}^{X_i}$ of functions from X_i to $\mathbb{F}$. Elements of $C^i(X; \mathbb{F})$ are called *i-dimensional cochains of X with coefficients in* $\mathbb{F}$. Note that we have $C^i(X; \mathbb{F}) = 0$ if $i \notin \{-1, 0, \ldots, \dim X\}$ and, since $X_{-1} = \{\emptyset\}$, we have $C^{-1}(X; \mathbb{F}) \cong \mathbb{F}$.

For $-1 \leq i \leq \dim X$, the characteristic functions e_F of faces $F \in X_i$, called *elementary cochains*, form a basis of $C^i(X; \mathbb{F})$. On these basis elements, we define the linear *coboundary map* $\delta_i \colon C^i(X; \mathbb{F}) \to C^{i+1}(X, \mathbb{F})$ for $-1 \leq i < \dim X$ by $\delta_i e_F(H) = [H : F]$. Thus,

$$(\delta_i f)(H) := \sum_{F \in X_i} [H : F] \cdot f(F)$$

for $f \in C^i(X; \mathbb{F})$. If $i \notin \{-1, 0, \ldots, \dim X - 1\}$ we let $\delta_i = 0$. If the dimension is clear from the context, we omit the index and simply write δ. If we want to emphasize the underlying space X, we write δ_X.

Define

$$B^i(X;\mathbb{F}) := \operatorname{im}\delta_{i-1} \quad \text{and} \quad Z^i(X;\mathbb{F}) := \ker\delta_i.$$

We call $B^i(X;\mathbb{F})$ the *space of i-dimensional coboundaries* and $Z^i(X;\mathbb{F})$ the *space of i-dimensional cocycles*. A crucial, yet simple observation at this point is that $\delta_i \circ \delta_{i-1} = 0$, and hence $B^i(X;\mathbb{F}) \subseteq Z^i(X;\mathbb{F})$.

The i-th (reduced) *cohomology group* of X with coefficients in $\mathbb{F}$ is then the quotient group $\tilde{H}^i(X;\mathbb{F}) := Z^i(X;\mathbb{F})/B^i(X;\mathbb{F})$. Strictly speaking, this definition gives us $\tilde{H}^i(X;\mathbb{F})$ as a vector space. However, since this is not relevant to our purposes, we stick to the more common notion of $\tilde{H}^i(X;\mathbb{F})$ as a group. For a cocycle $f \in Z^i(X;\mathbb{F})$ the equivalence class $f + B^i(X;\mathbb{F})$ is called the *cohomology class* of f; it is denoted by $[f]$. While different choices of orientation might yield different coboundary maps and different coboundary and cocycle spaces, the group $\tilde{H}^i(X;\mathbb{F})$ does not depend on the choice of orientation.

Cohomology of graphs. As an example, we consider why a graph G is connected if and only if $\tilde{H}^0(G;\mathbb{F}) = 0$. The cochain spaces of G are non-trivial in three dimensions: $C_1(G;\mathbb{F}) = \mathbb{F}^E$, $C_0(G;\mathbb{F}) = \mathbb{F}^V$ and $C_{-1}(G;\mathbb{F}) \cong \mathbb{F}$. The coboundary map δ_0 maps a function $f \in \mathbb{F}^V$ to $\delta_0 f$ defined by $\delta_0 f(\{v_0, v_1\}) = f(v_1) - f(v_0)$, if $v_0 < v_1$. The coboundary map δ_{-1} maps $x \in \mathbb{F}$ to the constant function with value x.

Thus, $B^0(G;\mathbb{F})$ is the space of all constant functions in $\mathbb{F}^V$ and $Z_0(G;\mathbb{F})$ is the space of all functions that are constant on all connected components of G. Hence, we see that $\tilde{H}^0(G;\mathbb{F}) = 0$ if and only if G has exactly one connected component.

1.3 Random Complexes

In 2006, Linial and Meshulam [79] introduced a higher-dimensional analogue of the binomial Erdős-Rényi random graph model $G(n,p)$. In Chapters 3 and 4 we present two results on this model for random complexes.

The *random k-dimensional simplicial complex* $X^k(n,p)$ introduced in [79] has vertex set $V = [n]$, a *complete* $(k-1)$*-skeleton*, i.e., $X_i = \binom{[n]}{i+1}$ for $i < k$, and every possible k-dimensional face $F \in \binom{[n]}{k+1}$ is added to X independently with probability p, which may be constant or, more generally, a function $p(n)$ depending on n. Thus, $X^k(n,p)$ is a random variable taking values from the set of k-dimensional simplicial complexes with vertex set $[n]$ and a complete $(k-1)$-skeleton.

For a fixed such complex X, the probability that $X^k(n,p)$ has the value X is

$$\Pr[X^k(n,p) = X] = p^{f_k(X)}(1-p)^{\binom{n}{k+1}-f_k(X)}.$$

This model has been studied extensively, and threshold probabilities for several basic topological properties of $X^k(n,p)$ have been determined quite precisely. In particular,

(i) the sharp threshold for vanishing of the $(k-1)$-st cohomology $H^{k-1}(X^k(n,p); \mathbb{G})$ with coefficients in any fixed finite group $\mathbb{G}$ is at $p = \frac{k \log n}{n}$, see [79, 92];

(ii) the threshold for the vanishing of the $(k-1)$-st integral homology $H_{k-1}(X^k(n,p); \mathbb{Z})$ is also at $p = O(\frac{\log n}{n})$ see [61];

(iii) for $k = 2$, the threshold for vanishing of the fundamental group $\pi_1(X^2(n,p))$ is roughly at $p = 1/\sqrt{n}$, see [11];

(iv) the thresholds for collapsibility onto the $(k-1)$-skeleton and for vanishing of the top-dimensional homology $H_k(X^k(n,p))$ (with any group of coefficients) are at $p = \Theta(1/n)$, see [9, 29, 73];

(v) the threshold for embeddability of $X^k(n,p)$ into $\mathbb{R}^{2k}$ is at $p = \Theta(1/n)$, see [116].

We will consider the subdivision containment problem, which concerns finding a subdivision of a fixed complex in a random complex, as well as the spectral properties of random complexes.

1.4 Probabilistic Tools

For the analysis of the random complexes $X^k(n,p)$ we will need some basic notation and tools from probability theory, which we describe now. For a random variable X, we denote its expectation by $\mathbb{E}[X]$ and its variance by $\mathrm{Var}[X]$. We will use the following two results on the concentration of a random variable around its expectation:

Theorem 1.1 (Chebyshev's inequality, see, e.g., [67, p. 8])**.** *Let X be a random variable such that* $\mathrm{Var}[X]$ *exists. Then we have for any $t > 0$*

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

A classical result on the concentration of binomially distributed random variables is the following.

Theorem 1.2 (Chernoff bound, see, e.g., [66, Theorem 1], [67, Theorem 2.1])**.** *Let X be a binomially distributed random variable with parameters n and p. Then we have for any $t \geq 0$*

$$\Pr[X \geq \mathbb{E}[X] + t] \leq e^{-\frac{t^2}{2(np+t/3)}} \quad \textit{and} \quad \Pr[X \leq \mathbb{E}[X] - t] \leq e^{-\frac{t^2}{2np}}.$$

Chapter 2

Basics on Spectra and Expansion

One major topic of this thesis are Laplacian eigenvalues of simplicial complexes and their connections to expansion and other combinatorial or topological properties of the complex. In this chapter we introduce higher-dimensional generalizations of the adjacency matrix and the Laplacian of a graph and present several possible higher-dimensional analogues of graph expansion. We then discuss how these notions are related to (a natural concept of) connectivity in complexes. Finally, we describe basic properties of the higher-dimensional Laplacian and adjacency matrices.

2.1 Expansion and Spectra of Graphs

Before we describe the required notions for simplicial complexes, we discuss eigenvalues of graphs and how they are related to expansion properties of the graph. We consider only simple graphs, i.e., graphs without loops or multiple edges.

Matrices and their spectra. Let us begin by reviewing some basic linear algebraic notions. A symmetric real $(n \times n)$-matrix has a multiset of n real eigenvalues, called its *spectrum*, and $\mathbb{R}^n$ has an orthonormal basis of corresponding eigenvectors. More generally, this holds for matrices that are self-adjoint with respect to some (not necessarily the standard Euclidean) inner product. A matrix $M \in \mathbb{R}^{n \times n}$ is *self-adjoint* with respect to an inner product $\langle , \rangle$ on $\mathbb{R}^n$ if $\langle Mx, y\rangle = \langle x, My\rangle$ holds for all $x, y \in \mathbb{R}^n$.

We recall the variational characterization of eigenvalues for real vector spaces:

Theorem 2.1 (Courant-Fischer Theorem, see e.g. [64, Theorem 4.2.6])**.** *Let $M \in \mathbb{R}^{n\times n}$ be a matrix that is self-adjoint with respect to an inner product $\langle , \rangle$ and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be its eigenvalues. Let $k \in \{1, \ldots, n\}$ and let S denote a subspace of $\mathbb{R}^n$. Then*

$$\lambda_k = \min_{\{S:\dim(S)=k\}} \max_{\{x:0\neq x\in S\}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle}$$

and

$$\lambda_k = \max_{\{S:\dim(S)=n-k+1\}} \min_{\{x:0\neq x\in S\}} \frac{\langle Mx, x\rangle}{\langle x, x\rangle}.$$

This is a standard result for symmetric matrices that can easily be extended to self-adjoint matrices.

A self-adjoint matrix is *positive semidefinite* with respect to an inner product $\langle , \rangle$ if $\langle Mx, x\rangle \geq 0$ for all $x \neq 0$. An equivalent condition is that all of its eigenvalues are non-negative. For a matrix M we denote its *ℓ_2-norm*, or *spectral norm*, by $\|M\| = \max_{x\neq 0} \|Mx\|/\|x\|$, which for a symmetric matrix M is identical with the in absolute value largest eigenvalue of M.

Adjacency Matrix and Laplacians of Graphs. Given a graph $G = (V, E)$ on n vertices we define three $(n\times n)$-matrices. The *adjacency matrix* $A = A(G) \in \{0, 1\}^{V\times V}$ is given by $A_{u,v} = 1$ if and only if $\{u, v\} \in E$. The *combinatorial Laplacian* is the matrix

$$L = L(G) := D - A,$$

where $D = D(G) \in \mathbb{R}^{V\times V}$ is the diagonal matrix with entries $D_{v,v} = \deg_G(v)$, the *degrees* of the vertices. Both of these are *symmetric* matrices.

The eigenvalues of A and of L turn out to be quite sensitive to the maximum and minimum degree of G. For graphs with very non-uniform degree distributions, it is often more convenient to consider the *normalized Laplacian*, which is defined as

$$\Delta = \Delta(G) := D^{-1}L = I - D^{-1}A,$$

where $I \in \mathbb{R}^{V\times V}$ is the identity matrix. Strictly speaking, D^{-1} is defined only if there are no isolated vertices, i.e., if $\deg_G(v) > 0$ for all $v \in V$, which will be the case of primary interest to us. If there are isolated vertices, we adopt the convention that $D^{-1}_{v,v} = 0$ whenever $\deg_G(v) = 0$ and retain the definition $\Delta = D^{-1}L$. (The second equation $\Delta = I - D^{-1}A$ no longer holds in this case, since Δ has zero diagonal entries at isolated vertices.)

Sometimes, (e.g., in [25, 28, 31]) a slightly different matrix is referred to as the normalized Laplacian, namely $\mathscr{L} := I - D^{-1/2}AD^{-1/2}$. Assuming that there are no isolated vertices, Δ and $\mathscr{L}$ have the same spectra, since $\Delta x = \lambda x$ for some $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^V$ if and only if $\mathscr{L}y = \lambda y$, where $y = D^{1/2}x$.

The normalized Laplacian of a general graph is not symmetric, but it is self-adjoint with respect to the weighted inner product defined by

$$\langle x, y\rangle = \sum_{v\in V} \deg_G(v)x(v)y(v),$$

and so it also has n real eigenvalues.

It is an easy observation that the all-ones vector $\mathbf{1} = (1,\ldots,1)^T$ satisfies $L\mathbf{1} = \Delta\mathbf{1} = 0$. As both versions of the Laplacian are positive semidefinite with respect to their respective inner products, they have only non-negative eigenvalues. These are typically indexed in increasing order, i.e.,

$$0 = \lambda_1(L) \le \ldots \le \lambda_n(L) \text{ and } 0 = \lambda_1(\Delta) \le \ldots \le \lambda_n(\Delta).$$

For the adjacency matrix, the eigenvalues are in contrast to the above commonly listed in *decreasing* order as

$$\mu_1(A) \ge \ldots \ge \mu_n(A).$$

If G is *d-regular*, i.e., $\deg_G(v) = d$ for all $v \in V$, then $L = d\cdot I - A = d\cdot\Delta$, and so the spectra of A, L, and Δ are equivalent (up to scaling and linear shifts):

$$\lambda_i(L) = d \cdot \lambda_i(\Delta) \text{ and } \mu_i(A) = d - \lambda_i(L),$$

for $1 \le i \le n$. In this case we know that $\mu_1(A) = d$, corresponding to the trivial eigenvalue $L\mathbf{1} = \Delta\mathbf{1} = 0$ for the Laplacians. For general graphs, the adjacency matrix does not have a trivial first eigenvalue. Nevertheless, we define $\mu(G) := \max\{\mu_2(A), |\mu_n(A)|\}$.

Edge Expansion. It is a simple observation that a graph G is connected if and only if $\lambda_2(L) > 0$, or equivalently $\lambda_2(\Delta) > 0$. More generally, the multiplicity of 0 as an eigenvector of either Laplacian equals the number of connected components of G. As we will see in a moment, $\lambda_2(L)$, respectively $\lambda_2(\Delta)$, is closely related to the *edge expansion* of G, which in some sense measures how well-connected a graph is.

For $\varepsilon > 0$, we say that $G = (V, E)$ is ε-*edge expanding* if for every $S \subseteq V$,

$$\frac{|E(S, V \setminus S)|}{|E|} \geq \varepsilon \cdot \frac{\min\{|S|, |V \setminus S|\}}{|V|},$$

where $E(S, V \setminus S) = \{\{u, v\} \in E : u \in S, v \in V \setminus S\}$ is the set of edges across the cut $(S, V \setminus S)$. We call the best possible ε the *edge expansion* of G and denote it by $\varepsilon(G)$. Note that the inequality defining $\varepsilon(G)$ is equivalent to the more common condition

$$|E(S, V \setminus S)| \geq \frac{\varepsilon}{2} \cdot d \cdot |S|$$

for all $S \subseteq V$ with $|S| \leq |V|/2$, where $d = 2|E|/|V|$ is the average degree. Thus, $\varepsilon(G) = 2h(G)$, where

$$h(G) := \min\Big\{ \tfrac{|E(S,V \setminus S)|}{d|S|} : S \subseteq V, |S| \leq |V|/2 \Big\}$$

is the (normalized) *Cheeger constant* of G.

Yet another closely related notion is that of the *sparsest cut* of a graph G. The sparsity $\phi(S)$ of a cut $(S, V \setminus S)$ is defined by the relation

$$\tfrac{|E(S,V \setminus S)|}{|E|} = \phi(S) \cdot \tfrac{|S| \cdot |V \setminus S|}{\binom{|V|}{2}}.$$

Defining $\phi(G) := \min_{\emptyset \neq S \subsetneq V} \phi(S)$, it is not hard to see that

$$\frac{n}{n-1} \cdot \phi(G) \leq \varepsilon(G) \leq 2\phi(G).$$

Expander Graphs. For every connected graph there is of course some $\epsilon > 0$ such that G is ε-edge expanding. For a stronger condition we consider families of graphs. An infinite family of graphs $\{G_n : n \in \mathbb{N}\}$ is called a family of *expander graphs* if there exists a constant $\varepsilon > 0$ such that $\varepsilon(G_n) > \varepsilon$ for all $n \in \mathbb{N}$.

Especially interesting are such families that are additionally sparse, i.e., constant- or bounded-degree expanders: families of expander graphs that are d-regular or have degrees bounded by d where $d > 0$ is independent of n. Explicit constructions for such graph families have been the subject of active research for a long time, see, e.g., [51, 84, 88, 89, 102].

The Cheeger Inequality for Graphs. We now explain the connection of the edge expansion of a graph and the second-smallest eigenvalue of its Laplacian. This is established by the discrete Cheeger inequality, here stated in its simplest form, for d-regular graphs (due to Dodziuk [35], Alon and Milman [4, 6]; Cheeger [23] proved an analogous result for Laplacians on Riemannian manifolds.):

Theorem 2.2 (Discrete Cheeger Inequality). *Let $G = (V, E)$ be a d-regular graph, and let $\lambda_2 = \lambda_2(\Delta(G))$ be the second-smallest eigenvalue of its normalized Laplacian. Then the edge expansion $\varepsilon(G)$ satisfies*

$$\lambda_2 \leq \varepsilon(G) \leq \sqrt{8\lambda_2}.$$

The inequality on the left-hand side is proved fairly easily by expressing the characteristic function $\mathbf{1}_S \in \mathbb{R}^V$ of a subset $S \subseteq V$ as a linear combination of eigenvectors of the Laplacian Δ. We will refer to this as *"the easy part of the Cheeger inequality"*. We present a different proof in Chapter 6. The harder part is the inequality on the right-hand side. For a short proof see, e.g., [7].

Note that even the easy part of Cheeger's inequality is very useful. Computing the edge expansion of a graph is difficult, from the standpoint of complexity theory [16], but often also for explicit examples. In contrast, eigenvalues are computable in polynomial time. Essentially all explicit constructions of constant-degree expanders, e.g. [51, 84, 88, 89, 102], prove a lower bound on the edge expansion of the constructed graphs by analyzing their eigenvalues.

The Expander Mixing Lemma. Another result relating the eigenvalues of a graph to its expansion properties is the Expander Mixing Lemma. Informally, it states that for a d-regular graph, $\mu(G)$, the largest (in absolute value) non-trivial eigenvalue of the adjacency matrix $A(G)$, controls for each pair of vertex subsets $S, T \subset V$ how far the number of edges $E(S, T)$ between them diverges from the expected number in a random graph of corresponding density.

Theorem 2.3 (Expander Mixing Lemma [5]). *Let G be a d-regular graph with n vertices. Let $\mu(G) := \max\{\mu_2(A), |\mu_n(A)|\}$. Then for all $S, T \subset V$*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \mu(G) \cdot \sqrt{|S||T|}.$$

In Chapters 5 and 6 we explore possible higher-dimensional analogues of the discrete Cheeger inequality and the Expander Mixing Lemma.

2.2 Laplacians and Adjacency Matrices for Complexes

We now turn to the definitions of the higher-dimensional analogues of adjacency matrices and Laplacians of graphs. A generalization of the graph Laplacian was introduced by Eckmann [40] to study discrete boundary value problems on simplicial (or more general cell) complexes. Subsequently, combinatorial Laplacians were applied in a variety of contexts. Dodziuk [34] and Dodziuk and Patodi [36] showed how the continuous Laplacian of a Riemannian manifold can be approximated by the combinatorial Laplacians of a suitable sequence of successively finer triangulations of the manifold.

Kalai [68] used combinatorial Laplacians to prove a higher-dimensional generalization of Cayley's formula for the number of labeled trees, and further results in this direction, including a generalization of the Matrix-Tree Theorem, were obtained in [1, 39]. For further combinatorial applications, see, e.g., [37, 47, 48, 72]. For further background and references regarding combinatorial Laplacians, see also [63].

We also consider a normalized Laplacian that was used, e.g., in [53] and define, based on these two (related) versions of the Laplacian, a higher-dimensional analogue of the adjacency matrix.

Adjacency matrices. For a k-dimensional simplicial complex X the *adjacency matrix* $A_{k-1} = A_{k-1}(X)$ is a symmetric $(|X_{k-1}| \times |X_{k-1}|)$-matrix, defined as follows (with respect to the basis of elementary cochains):

$$(A_{k-1}(X))_{F,G} = \begin{cases} -[F \cup G : F][F \cup G : G] & \text{if } F \sim G, \\ 0 & \text{otherwise,} \end{cases}$$

where $F, G \in X_{k-1}$ and we write $F \sim G$ if F and G share a common $(k-2)$-face $F \cap G$ and $F \cup G \in X_k$.

It is not hard to see that for $F, G \in X_{k-1}$ with $F \sim G$ we have

$$-[F \cup G : F][F \cup G : G] = [F : F \cap G][G : F \cap G].$$

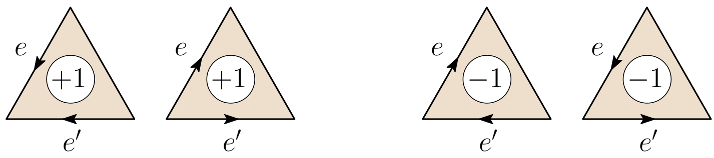

Figure 2.1: Signs of non-zero entries $A_1(X)_{e,e'}$. The arrows represent the orientations of edges.

For the case $k = 2$ this makes it easy to give an interpretation of these signs, see Figure 2.1 for an illustration. An entry $A_1(X)_{e,e'}$ is non-zero exactly if the two edges e and e' share a common vertex and the triangle $e \cup e'$ is contained in X. The sign of $A_1(X)_{e,e'}$ is then determined by the orientations of the two edges.

Note that for a graph G the matrix $A_0(G)$ agrees with the adjacency matrix because $[\{u, v\} : u][\{u, v\} : v] = -1$ for all vertices $u, v \in V$ (and any two vertices share the (-1)-dimensional face $\emptyset$). The motivation for the signs in higher dimensions will hopefully become clear later on.

Weighted Laplacians. For our purposes, the Laplacian of a k-dimensional simplicial complex X, just like the adjacency matrix, is a linear operator on $C^{k-1}(X; \mathbb{R})$. For the sake of a more complete account and because it comes at no extra cost, we introduce a general weighted Laplacian in an arbitrary dimension i, even though we will later focus on certain weight functions and on dimension $k - 1$ for a k-dimensional complex.

Suppose we are given a non-negative weight function w on the faces of a finite simplicial complex X. Define a weighted inner product on the space $C^i(X; \mathbb{R})$ by

$$\langle f, g\rangle := \sum_{F \in X_i} w(F) f(F) g(F).$$

The *weighted Laplacian* with respect to w is $\mathcal{L}_i := \mathcal{L}_i^{\mathrm{down}} + \mathcal{L}_i^{\mathrm{up}}$, where we define

$$\mathcal{L}_i^{\mathrm{down}} := \delta_{i-1}\delta_{i-1}^* \quad \text{and} \quad \mathcal{L}_i^{\mathrm{up}} := \delta_i^*\delta_i.$$

Here, $\delta_i^* \colon C^{i+1}(X; \mathbb{R}) \to C^i(X; \mathbb{R})$ denotes the *adjoint* of the coboundary map δ_i with respect to the given inner product. It is determined by the condition $\langle \delta_i^* f, g\rangle = \langle f, \delta_i g\rangle$ for $f \in C^{i+1}(X;\mathbb{R})$ and $g \in C^i(X;\mathbb{R})$. The value of $\delta_i^* f$ on $G \in X_i$ is

$$(\delta_i^* f)(G) = \sum_{F \in X_{i+1}} \frac{w(F)}{w(G)}[F : G]f(F).$$

By definition, all three maps $\mathcal{L}_i^{\mathrm{down}}, \mathcal{L}_i^{\mathrm{up}}, \mathcal{L}_i$ are *self-adjoint* and *positive semidefinite* (with respect to the given inner product). Furthermore, it is not hard to see that $\ker \mathcal{L}_i^{\mathrm{down}} = \ker \delta_{i-1}^*$, $\ker \mathcal{L}_i^{\mathrm{up}} = \ker \delta_i = Z^i(X;\mathbb{R})$ and $\ker \mathcal{L}_i = \ker \mathcal{L}_i^{\mathrm{down}} \cap \ker \mathcal{L}_i^{\mathrm{up}}$. Define

$$\mathcal{H}_i = \mathcal{H}_i(X;\mathbb{R}) := \ker \mathcal{L}_i = \ker \delta_{i-1}^* \cap Z^i(X;\mathbb{R}).$$

Then we have the *Hodge decomposition* of $C^i(X;\mathbb{R})$ into pairwise orthogonal subspaces

$$C^i(X;\mathbb{R}) = \mathcal{H}_i \oplus B^i(X;\mathbb{R}) \oplus \mathrm{im}(\delta_i^*),$$

(see [40, 63]); in particular, $\mathcal{H}_i \cong H^i(X;\mathbb{R})$. We focus on the operator $\mathcal{L}_i^{\mathrm{up}}$, more specifically we consider $\mathcal{L}_{k-1}^{\mathrm{up}}$ for k-dimensional complexes.

Spectra of $\mathcal{L}_{k-1}^{\mathrm{up}}$. Note that $B^{k-1}(X;\mathbb{R}) \subseteq Z^{k-1}(X;\mathbb{R}) = \ker \mathcal{L}_{k-1}^{\mathrm{up}}$. Every $f \in B^{k-1}$ is hence an eigenvector of $\mathcal{L}_{k-1}^{\mathrm{up}}$ with eigenvalue zero, a *trivial eigenvector*. As $\mathcal{L}_{k-1}^{\mathrm{up}}$ is self-adjoint, the remaining eigenvalues are the eigenvalues of the restriction of $\mathcal{L}_{k-1}^{\mathrm{up}}$ to the orthogonal complement (with respect to the given weighted inner product) $(B^{k-1}(X;\mathbb{R}))^{\perp}$. We call these the *non-trivial eigenvalues* of $\mathcal{L}_{k-1}^{\mathrm{up}}$. As eigenvalues of a positive semidefinite operator, they are non-negative. If the smallest non-trivial eigenvalue is zero, then by the Hodge decomposition we have $\mathcal{H}_{k-1}(X;\mathbb{R}) = 0$.

Even though the space of coboundaries $B^{k-1}(X;\mathbb{R})$ depends on the choice of orientation on the faces of X, its dimension does not. It is also not hard to see that the spectrum of $\mathcal{L}_{k-1}^{\mathrm{up}}$ does not depend on the choice of orientation, so neither does its non-trivial part.

Combinatorial Laplacians. The *combinatorial Laplacian*

$$L_i = L_i^{\mathrm{down}} + L_i^{\mathrm{up}}$$

corresponds to the case of *unit weights* $w(F) = 1$ for all $F \in X$, that is, to the choice of the standard inner product $\langle f, g\rangle = \sum_{f \in X_i} f(F)g(F)$.

The map δ_i^* is in this case also called the *boundary map*, often denoted by ∂_{i+1}. Thus, in this notation,

$$L_{k-1}^{\mathrm{up}} = L_{k-1}^{\mathrm{up}}(X) = \partial_k \delta_{k-1}.$$

With respect to the basis of elementary cochains, the coboundary map $\delta_i \colon C^i(X;\mathbb{R}) \to C^{i+1}(X;\mathbb{R})$ is represented by the following $(|X_{i+1}| \times |X_i|)$-matrix (also denoted by δ_i):

$$(\delta_i(X))_{F,G} = \begin{cases} [F:G] & \text{if } G \subsetneq F, \\ 0 & \text{otherwise.} \end{cases}$$

Its transpose δ_i^T corresponds to the boundary map ∂_{i+1}. Thus, with respect to this basis, the combinatorial Laplacian L_{k-1}^{up} can be expressed as the matrix $\delta_{k-1}^T \delta_{k-1}$.

We can now motivate the signs in the definition of the adjacency matrix $A_{k-1}(X)$: For a graph G it is not hard to see that $L_0^{\mathrm{up}} = L(G)$. Recall that the graph Laplacian satisfies $L(G) = D(G) - A(G)$. For a k-dimensional simplicial complex X, let $D_{k-1}(X)$ denote the diagonal matrix with entry

$$D_{k-1}(X)_{F,F} = \deg(F) = |\{H \in X_k : F \subset H\}|$$

for $F \in X_{k-1}$. Then we also have $L_{k-1}^{\mathrm{up}}(X) = D_{k-1}(X) - A_{k-1}(X)$.

Normalized Laplacians. Suppose that X is a pure k-dimensional simplicial complex. The *normalized Laplacian* $\Delta_i = \Delta_i^{\mathrm{down}} + \Delta_i^{\mathrm{up}}$ is the special case of the weighted Laplacian obtained by taking the weight function $w(F) := \deg(F)$. That is, the corresponding weighted inner product is

$$\langle f, g \rangle = \sum_{F \in X_i} \deg(F) f(F) g(F).$$

The adjoint δ_i^* of δ_i with respect to this weighted inner product is then defined by

$$(\delta_i^* f)(G) = \sum_{F \in X_{i+1}} \frac{\deg(F)}{\deg(G)} [F:G] f(F).$$

Note that we have $\deg(F) > 0$ for every $F \in X$, since we assume that X is pure. The normalized Laplacian is then

$$\Delta_{k-1}^{\mathrm{up}} = \Delta_{k-1}^{\mathrm{up}}(X) = \delta_{k-1}^* \delta_{k-1},$$

and with respect to the basis of elementary cochains, $\Delta^{\mathrm{up}}_{k-1}$ corresponds to the matrix

$$\Delta^{\mathrm{up}}_{k-1} = D^{-1}_{k-1} L^{\mathrm{up}}_{k-1} = I - D^{-1}_{k-1} A_{k-1}.$$

This finishes the part on higher-dimensional analogues of graph matrices. We now discuss higher-dimensional expansion properties and then return to the concepts considered here in Section 2.6, where we discuss some basic properties of these matrices.

2.3 Notions of Expansion for Complexes

There exist several approaches for defining a higher-dimensional analogue of graph expansion. We first give a very general definition of expansion for simplicial complexes that was introduced by Gromov [56] and then describe two specific cases: We study *combinatorial expansion*, which can be seen as a generalization of edge expansion in graphs and was considered independently by Linial, Meshulam and Wallach [79, 92] and also by Newman and Rabinovich [94]. The second, *spectral expansion*, is related to the spectra of the Laplacians introduced above.

We also consider a more combinatorially inspired notion of higher-dimensional expansion studied by Parzanchevski, Rosenthal and Tessler in [97].

Coboundary Expansion for Arbitrary Coefficients The notion of expansion described here was introduced by Gromov in [56] with a slightly different normalization and under the name *inverse (co)filling norm.* Like the Laplacian above, we introduce it for an arbitrary dimension i, even though we are mostly interested in the case $i = k-1$.

For a finite simplicial complex X and a field $\mathbb{F}$, assume that in every dimension i the space of cochains $C^i(X;\mathbb{F})$ is equipped with a *norm* $\|\cdot\|$. The basic idea underlying this notion of i-dimensional expansion is to provide a lower bound for the norm of the coboundary $\delta_{i-1}(f) \in C^i(X;\mathbb{F})$ of any $(i-1)$-dimensional cochain $f \in C^{i-1}(X;\mathbb{F})$.

One might wish to define such a bound in terms of the norm $\|f\|$ of f. However, recall that the space $B^{i-1}(X;\mathbb{F})$ is always contained in the kernel $Z^{i-1}(X;\mathbb{F})$ of $\delta = \delta_{i-1}$. Thus, there are always cochains $f \neq 0$ with $\delta f = 0$ which makes a bound in terms of $\|f\|$ impossible. Since the notion of expansion should detect whether $\widetilde{H}^{i-1}(X;\mathbb{F}) = 0$, i.e., whether $B^{i-1}(X;\mathbb{F}) = Z^{i-1}(X;\mathbb{F})$, the correct measure is instead the *distance* of the cochain f from $B^{i-1}(X;\mathbb{F})$, the trivial part of the kernel $Z^{i-1}(X;\mathbb{F})$. That is, we define, for $f \in C^{i-1}(X;\mathbb{F})$,

$$\|[f]\| := \min\{\|f + \delta_{i-2} g\| \colon g \in C^{i-2}(X;\mathbb{F})\}.$$

A cochain $f \in C^{i-1}(X;\mathbb{F})$ is called *minimal* if $\|[f]\| = \|f\|$. We say that X is *ε-expanding in dimension i* (with respect to the coefficients $\mathbb{F}$ and the given norms) if

$$\|\delta f\| \geq \varepsilon \cdot \|[f]\|$$

for all $f \in C^{i-1}(X;\mathbb{F})$. The best possible ε is called the *i-dimensional coboundary expansion* of X. Note that, in particular, $\tilde{H}^{i-1}(X;\mathbb{F}) = 0$ if X has i-dimensional expansion $\varepsilon > 0$.

Combinatorial Expansion We now focus on the case $\mathbb{F} = \mathbb{Z}_2$. For any weight function w with nonnegative real values on the simplices of X, we define the *weighted Hamming norm* on $C^i(X;\mathbb{Z}_2)$ by

$$\|f\| := \sum_{F \in X_i : f(F) = 1} w(F).$$

We use the weight function defined by $w(F) := 1/|X_i|$ for $F \in X_i$, such that the norm of a cochain $f \in C^{i-1}(X;\mathbb{Z}_2)$ is just the number of faces in the support of f, divided by the number of all $(i-1)$-faces of X. If X is ε-expanding in dimension i with respect to this norm, we say that X is *combinatorially ε-expanding in dimension i*.

We will focus on dimension $i = k-1$ and define the *combinatorial expansion* of a k-dimensional simplicial complex X as

$$\varepsilon(X) := \min_{\substack{f \in C^{k-1}(X), \\ f \notin B^{k-1}(X)}} \frac{\|\delta f\|}{\|[f]\|},$$

where $C^{k-1}(X) = C^{k-1}(X;\mathbb{Z}_2)$ and $B^{k-1}(X) = B^{k-1}(X;\mathbb{Z}_2)$.

Consider the case $k = 1$ of graphs. The space $B^0(X;\mathbb{Z}_2)$ of 0-dimensional coboundaries has only two elements, the constant functions $\mathbf{0}$ and $\mathbf{1}$ on V. Therefore, for a 0-dimensional cochain $f \in C^0(X;\mathbb{Z}_2)$ we have $\|[f]\| = \min\{|S|, |V \setminus S|\}/|S|$, where $S = \{v \in V \colon f(v) = 1\}$ is the support of f. As the support of δf is furthermore the set $E(S, V \setminus S)$, we see that 1-dimensional combinatorial expansion corresponds precisely to the definition of edge expansion discussed above.

Just as in the case of graphs, we call an infinite family $\{X_n : n \in \mathbb{N}\}$ of k-dimensional simplicial complexes (where k is fixed and independent of n) a *family of combinatorial expanders* if there is an $\varepsilon > 0$ such that $\varepsilon(X_n) > \varepsilon$ for all $n \in \mathbb{N}$.

The random complexes $X^k(n,p)$ that were introduced by Linial and Meshulam in [79] (and which we described in Section 1.3) show the existence of such families: At the end of this section, we consider the expansion properties of the complete complex and describe the combinatorial expansion of K_n^k in Proposition 2.4. Combined with standard Chernoff bounds this immediately implies that $X^k(n,p)$ is a.a.s. combinatorially expanding in dimension k if $p > C \log n/n$ for a suitable constant C. Much of the work in [79, 92] is devoted to refining this argument to obtain the optimal constant $C = k$ for the threshold.

The random complexes $X^k(n,p)$ are combinatorially expanding, but the degrees of $(k-1)$-faces in $X^k(n,p)$ are growing with n. The existence of bounded-degree expanders in the case of graphs motivates the question whether there are infinite families of complexes with bounded degree and bounded expansion. Recently, Lubotzky and Meshulam [83] proved the existence of an infinite family of 2-dimensional ϵ-expanders with maximum edge degree d, for some fixed $\epsilon > 0$ and d, using a different random model of simplicial complexes, based on *random Latin squares*. Neither higher-dimensional examples nor explicit constructions of such families are known to date.

An Analogue of the Sparsest Cut Problem. For a graph, its edge expansion $\varepsilon(G)$, which compares the number of elements in a cut to the size of the smaller side of the cut, is closely related to the *sparsest cut* problem, where we compare the size of a cut to its largest possible size. A similar observation can also be made for simplicial complexes and the notion of combinatorial expansion.

Let X be a k-dimensional simplicial complex with complete $(k-1)$-skeleton. Define, for the case of $\mathbb{Z}_2$-coefficients and the weighted Hamming norm as above,

$$\phi(X) := \min_{\substack{f \in C^{k-1}(X),\\ f \notin B^{k-1}(X)}} \frac{\|\delta_X f\|}{\|\delta_{K_n^k} f\|},$$

where $C^{k-1}(X) = C^{k-1}(X;\mathbb{Z}_2)$ and $B^{k-1}(X) = B^{k-1}(X;\mathbb{Z}_2)$. Using Proposition 2.4 on the combinatorial expansion of the complete complex, it is not hard to show that $\phi(X)$ is closely related to the combinatorial expansion of X. Indeed, we have:

$$\frac{n}{n-k} \cdot \phi(X) \leq \varepsilon(X) \leq (k+1)\phi(X).$$

Real Coefficients Now we want to consider coboundary expansion with coefficients in $\mathbb{R}$. As for the definition of the weighted Laplacian, we assume that we are given a weight function w with non-negative real values on the simplices of X and define a weighted inner product on $C^i(X;\mathbb{R})$ by $\langle f, g\rangle := \sum_{F\in X_i} w(F)f(F)g(F)$. We then consider the corresponding weighted ℓ_2-norm $\|f\| = \|f\|_2 := \sqrt{\langle f, f\rangle}$.

Suppose that X is ε-expanding in dimension i with respect to real coefficients and the given weighted ℓ_2-norms. By the variational definition of eigenvalues, Theorem 2.1, the minimal non-trivial eigenvalue of the weighted Laplacian $\mathcal{L}_i^{\mathrm{up}}(X)$ is given by

$$\min_{f\perp B^i(X;\mathbb{R})} \frac{\langle \mathcal{L}_i^{\mathrm{up}}(X)f, f\rangle}{\langle f, f\rangle} = \min_{f\perp B^i(X;\mathbb{R})} \frac{\|\delta_i f\|^2}{\|f\|^2} \geq \varepsilon^2.$$

Thus, we see that the minimal non-trivial eigenvalue of $\mathcal{L}_i^{\mathrm{up}}(X)$ is at least ε^2.

For the case of unit weights $w(F) = 1$ or the case of the weight function defined by $w(F) := \deg(F)$, we say that X is *spectrally expanding* in dimension i with respect to the combinatorial or the normalized Laplacian, respectively.

A more combinatorial approach

In [97] Parzanchevski, Rosenthal and Tessler consider a more combinatorially inspired notion of higher-dimensional expansion for k-complexes with a complete $(k-1)$-skeleton. For 2-complexes the underlying idea is the following: Instead of considering edge sets with their complements and the triangle sets spanned by them, i.e., the coboundary, as in the definition of combinatorial expansion, they consider 3-partitions of the vertex set and the sets of triangles spanned by such partitions.

More precisely, let X be k-dimensional simplicial complex X with complete $(k-1)$-skeleton, and consider a partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = V$ of the vertex set into non-empty sets.

Let $F(A_0, A_1, \ldots, A_k) := \{F \in X_k : |F \cap A_i| = 1, 0 \leq i \leq k\}$. Then we define

$$h(X) := \min_{\substack{V=\bigsqcup_{i=0}^k A_i,\\ A_i \neq \emptyset}} \frac{|V| \cdot |F(A_0, A_1, \ldots, A_k)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_k|}.$$

Note that this is an analogue of the notion of a sparsest cut for a graph G (for a sparse graph with $|E(G)| = O(|V|)$), which we denoted by $\phi(G)$, whereas $h(G)$ is the Cheeger constant of G. Nevertheless, we follow [97] and denote this parameter by $h(X)$.

Parzanchevski, Rosenthal and Tessler show a partial analogue of the discrete Cheeger inequality and a higher-dimensional Expander Mixing Lemma based on this notion of expansion, which we will encounter in Sections 5.2 and 8.1.

Let us consider how $h(X)$ compares to the combinatorial expansion $\varepsilon(X)$ and to $\phi(X)$, the analogue of the sparsest cut notion that we defined above. To this end, we rephrase the definition of $h(X)$ in terms of $\mathbb{Z}_2$-coboundaries.

For a given partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = V$, define a $\mathbb{Z}_2$-cochain $f_{A_0,A_1,\ldots,A_k} \in C^{k-1}(X, \mathbb{Z}_2)$ by

$$f_{A_0,A_1,\ldots,A_k}(F) = \begin{cases} 1 & \text{if } |F \cap A_i| = 1 \text{ for } 0 \leq i \leq k-1, \\ 0 & \text{otherwise,} \end{cases}$$

for $F \in X_{k-1}$. Then it is not hard to see that $|\delta_X f_{A_0,A_1,\ldots,A_k}| = |F(A_0, A_1, \ldots, A_k)|$ and that $|\delta_{K_n^k} f_{A_0,A_1,\ldots,A_k}| = |A_0| \cdot |A_1| \cdot \ldots \cdot |A_k|$. Therefore, we have

$$h(X) = \min_{\substack{V=\bigsqcup_{i=0}^k A_i,\\ A_i \neq \emptyset}} \frac{|V| \cdot |\delta_X f_{A_0,A_1,\ldots,A_k}|}{|\delta_{K_n^k} f_{A_0,A_1,\ldots,A_k}|}.$$

Comparing with the definition of $\phi(X)$, we see that for X with $|V| = n$ we have

$$\frac{f_k(X)}{\binom{n-1}{k}} \cdot \varepsilon(X) \leq \frac{n \cdot f_k(X)}{\binom{n}{k+1}} \cdot \phi(X) \leq h(X),$$

since the minimum in $h(X)$ is taken over a subset of the one defining $\phi(X)$. At the end of this section, we will see that there are complexes X with $h(X) > 0$ but $\varepsilon(X) = 0$.

2.4 Expansion of the Complete Complex

As an example and for further use we study the combinatorial and the spectral expansion of the complete complex K_n^k.

Combinatorial Expansion of the Complete Complex. A basic observation concerning combinatorial expansion, made independently by Gromov [56], by Linial, Meshulam and Wallach [79, 92] as well as by Newman and Rabinovich [94], is that complete complexes are combinatorially expanding in all dimensions.

Proposition 2.4. *The complete k-dimensional complex K_n^k on n vertices is combinatorially 1-expanding in dimension i for all $i = 0, 1, \ldots, k$. More precisely,*

$$\|\delta f\| \geq \frac{n}{n-i} \cdot \|[f]\|,$$

for all $f \in C^{i-1}(K_n^k; \mathbb{Z}_2)$.

Eigenvalues of the Complete Complex. In order to consider the spectral expansion of K_n^k, we first recall the following well-known (and easily verifiable) basic fact:

Lemma 2.5. *For a complex X with complete $(k-1)$-skeleton, the space $B^{(k-1)}(X) = \operatorname{im} \delta_{k-2}$ has dimension $\binom{n-1}{k-1}$. A basis is given by $\left\{\delta_{k-2} e_F : 1 \notin F \in \binom{[n]}{k-1}\right\}$. The space $\operatorname{im} \delta_{k-1}^*(X)$ is $\binom{n-1}{k}$-dimensional and has $\left\{\delta_{k-1}^* e_F : 1 \in F \in \binom{[n]}{k+1}\right\}$ as a basis.*

We get the following result on the spectra of the matrices $L_{k-1}^{\mathrm{up}}(K_n^k)$ and $\Delta_{k-1}^{\mathrm{up}}(K_n^k)$, as well as the spectrum of $A_{k-1}(K_n^k)$.

Lemma 2.6. *The eigenvalues of the combinatorial Laplacian $L_{k-1}^{\mathrm{up}}(K_n^k)$ are 0 with multiplicity $\binom{n-1}{k-1}$ and n with multiplicity $\binom{n-1}{k}$. The normalized Laplacian $\Delta_{k-1}^{\mathrm{up}}(K_n^k)$ has eigenvalues 0 with multiplicity $\binom{n-1}{k-1}$ and $\frac{n}{n-k}$ with multiplicity $\binom{n-1}{k}$. The eigenvalues of $A_{k-1}(K_n^k)$ are $n-k$ with multiplicity $\binom{n-1}{k-1}$ and $-k$ with multiplicity $\binom{n-1}{k}$.*

Proof. Note that $\Delta_{k-1}^{\mathrm{up}}(K_n^k) = 1/(n-k) L_{k-1}^{\mathrm{up}}(K_n^k)$ and $A_{k-1}(K_n^k) = (n-k)I - L_{k-1}^{\mathrm{up}}(K_n^k)$. It hence suffices to consider the spectrum of $L_{k-1}^{\mathrm{up}}(K_n^k)$. The following equality is contained implicitly in [68] and follows from a straightforward calculation using the matrix representations of the Laplacians:

$$L_{k-1}^{\mathrm{up}}(K_n^k) + L_{k-1}^{\mathrm{down}}(K_n^k) = nI.$$

Any non-zero element of $\ker L_{k-1}^{\mathrm{down}}(K_n^k) = \ker \delta_{k-2}^*(K_n^k) = \operatorname{im} \delta_{k-1}^*(K_n^k)$ is hence an eigenvector of L_{k-1}^{up} with eigenvalue n. Naturally, any non-zero element of $\ker L_{k-1}^{\mathrm{up}}(K_n^k) = Z^{k-1}(K_n^k) = B^{k-1}(K_n^k)$ is an eigenvector of L_{k-1}^{up} with eigenvalue 0. By Lemma 2.5 $\operatorname{im} \delta_{k-1}^*(K_n^k)$ and $B^{k-1}(K_n^k)$ have dimensions $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$, respectively. As these add up to $\binom{n}{k}$, the dimension of $C^{k-1}(K_n^k)$, we have determined the complete spectrum. □

2.5 Expansion vs. Connectivity

A very natural higher-dimensional generalization of connectivity for graphs can be defined as follows: In a k-dimensional simplicial complex X, a *k-path* in X is a pure k-dimensional simplicial subcomplex P of X such that there is an ordering $F_1, F_2, \dots, F_m$ of the k-simplices of P, such that any F_i and F_j with $|j-i| = 1$ share a common $(k-1)$-face. The complex X is called *hypergraph connected* if for any two k-simplices $F, F' \in X_k$ there is a k-path connecting F and F'. A *connected component* of a k-complex X with respect to hypergraph connectivity is a subcomplex C of X such that for every $(k-1)$-face F of C all k-faces containing F are also in C. Note that this condition does not need to hold for lower-dimensional simplices, so two distinct connected components can, e.g., have common vertices. Only the sets of $(k-1)$- and of k-faces have to be disjoint.

A graph G is ε-expanding for some $\varepsilon > 0$ if and only if it is connected. For each of the notions of higher-dimensional expansion we considered above, non-zero expansion implies hypergraph connectivity, as we will see in a moment. However, there is no equivalence between any of these properties and hypergraph connectivity.

Let X be a k-dimensional simplicial complex. For coboundary expansion with respect to a coefficient field $\mathbb{F}$ and any norm, there exists an $\varepsilon > 0$ such that X has expansion ε if and only if $H^{k-1}(X; \mathbb{F}) = 0$. For a graph G and any field $\mathbb{F}$, we saw that $\tilde{H}^0(G; \mathbb{F}) = 0$ is equivalent to G being connected. In higher dimensions, however, the analogous statement is not true and it is well-known that the vanishing of a cohomology group may depend on the choice of coefficients. A basic example for this is the real projective plane $\mathbb{R}P^2$, for which $H^1(\mathbb{R}P^2;\mathbb{R}) = 0$ but $H^1(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2$.

For the approach taken by Parzanchevski, Rosenthal and Tessler in [97], the notion corresponding to non-zero expansion is the following: A k-complex with complete $(k-1)$-skeleton satisfies $h(X) > 0$ if and only if $F(A_0, A_1, \ldots, A_k) \neq \emptyset$ for any partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = V$, $A_i \neq \emptyset$.

These properties satisfy the following relations:

Lemma 2.7. *Let X be a k-dimensional simplicial complex with complete $(k-1)$-skeleton.*

(a) $H^{k-1}(X;\mathbb{Z}_2)=0$ implies, but is not equivalent to $H^{k-1}(X;\mathbb{R})=0$.

(b) If $H^{k-1}(X;\mathbb{R})=0$, then $h(X) > 0$. The converse does not hold in general.

(c) If $h(X) > 0$, then X is hypergraph connected. The converse does not hold in general.

Proof. Assume that X is a k-complex with complete $(k-1)$-skeleton.

(a) That $H^{k-1}(X;\mathbb{Z}_2) = 0$ implies $H^{k-1}(X;\mathbb{R}) = 0$ follows from the universal coefficient theorem for cohomology, see, e.g., [93, Theorem 53.1]. This is a basic result in homological algebra, which connects cohomology groups with arbitrary coefficients to $\mathbb{Z}$-homology groups. The terminology required to repeat it here goes beyond the scope of this thesis.

A basic counterexample already mentioned above is the real projective plane $\mathbb{R}P^2$. We will show the existence of a family of counterexamples in Chapter 5 (Theorem 5.1).

(b) Parzanchevski, Rosenthal and Tessler show that $h(X) \geq \lambda(X)$, where $\lambda(X)$ is the smallest eigenvalue of the Laplacian $L^{\mathrm{up}}_{k-1}(X)$ on $(B^{k-1})^{\perp}$. We will discuss their result in more detail in Section 5.2. Here we only observe that it implies that $h(X) > 0$, if $H^{k-1}(X;\mathbb{R}) = 0$, i.e., if $\lambda(X) > 0$. They also give examples of complexes X with $\lambda(X) = 0$, but $h(X) > 0$.

(c) Assume that $h(X) > 0$. Parzanchevski, Rosenthal and Tessler [97, Proposition 4.1] show that for any $F \in X_{k-2}$ we have $h(\operatorname{lk} F) \geq (1 - \frac{k-1}{n})h(X)$. Hence, we see that $h(\operatorname{lk}(F)) > 0$, and, since the links of $(k-2)$-faces are graphs, that $\operatorname{lk}(F)$ is connected for every

$F \in X_{k-2}$. It is not hard to see that this implies hypergraph connectivity. For an example of a hypergraph connected complex with $h(X) = 0$, consider a k-path $X = P$ as above. Assume furthermore that every k-face F_i shares a common face only with its two neighbours F_{i-1} and F_{i+1} in the ordering along the path and with no other k-face of P. Any k-path is clearly hypergraph connected. If m is large enough we can see that there is a partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = V$ of the vertices such that $F(A_0, A_1, \ldots, A_k) = \emptyset$. Let, e.g., $F_1 = \{v_1, \ldots, v_{k+1}\}$, $F_2 = \{v_2, \ldots, v_{k+2}\}$ and so on: $F_i = (F_{i-1} \setminus \{v_{i-1}\}) \cup \{v_{k+i}\}$. The k-path P defined by this has the vertex set $\{v_1, \ldots v_{k+m}\}$ and for $k + m = l \cdot (k+1), l \geq 2$ we can, e.g., set $A_j = \{j \cdot l + 1, \ldots, (j+1) \cdot l\}$.

$\square$

2.6 Properties of Laplacian and Adjacency Matrices

We have already studied the connection between connectivity and spectral properties for graphs as well as for complexes. Now, we collect further properties of the spectrum of the generalized adjacency matrix $A_{k-1}(X)$ and the combinatorial and the normalized Laplacian $L^{\text{up}}_{k-1}(X)$ and $\Delta^{\text{up}}_{k-1}(X)$ of a k-dimensional simplicial complex X and compare these to corresponding results for graphs.

Extremal Values. Let G be a graph with maximal degree $d_{\max}$. It is known that the spectrum of the normalized Laplacian $\Delta(G)$ is contained in the interval $[0, 2]$ and the spectrum of the combinatorial Laplacian in the interval $[0, 2d_{\max}]$. For the adjacency matrix, all eigenvalues lie in the interval $[-d_{\max}, d_{\max}]$. An analogous statement holds for higher-dimensional complexes:

Lemma 2.8. *Let X be a k-dimensional simplicial complex. Suppose that* $\deg(F) \leq d_{max}$ *for every $F \in X_{k-1}$. Then every eigenvalue of A_{k-1} lies in the interval $[-kd_{max}, d_{max}]$. Furthermore, the spectrum of L^{up}_{k-1} is contained in $[0, (k+1)d_{max}]$ and the spectrum of Δ^{up}_{k-1} in $[0, k+1]$.*

Proof. To show the result on the adjacency matrix, we simplify notation and write A instead of A_{k-1}. Let μ be an eigenvalue of A. We first show that $|\mu| \leq kd_{\max}$.

Let f be a corresponding eigenvector so that $Af = \mu f$. Choose $F \in X_{k-1}$ such that $|f_F| \geq |f_{F'}|$ for every $F' \in X_{k-1}$ and scale f such that $f_F = 1$. Then

$$\begin{aligned}
|\mu| &= |\mu f_F| = |(Af)_F| \\
&= \left| \sum_{F' \in X_{k-1}} A_{F,F'} f_{F'} \right| = \left| \sum_{F \sim F'} -[F \cup F' : F][F \cup F' : F'] f_{F'} \right| \\
&\leq \sum_{F \sim F'} |f_{F'}| \leq |\{F' : F \sim F'\}| = k \deg(F) \leq k d_{\max}.
\end{aligned}$$

For the case of graphs, $k = 1$, this suffices. For $k > 1$, we proceed. Since f is an eigenvector with eigenvalue μ, we have $L^{\mathrm{up}}_{k-1} f = D_{k-1} f - \mu f$. Hence,

$$0 \leq f^T L^{\mathrm{up}}_{k-1} f = f^T (D_{k-1} - \mu I) f = \sum_{F \in X_{k-1}} (\deg(F) - \mu) f_F^2,$$

where the first inequality holds because L^{up}_{k-1} is positive semidefinite. Since $f \neq 0$, the sum has to have a non-negative entry, i.e., there is $F \in X_{k-1}$ with $f_F \neq 0$ and $\deg(F) \geq \mu$.

The upper bounds for the eigenvalues of the two Laplacian follow easily from the variational characterization of eigenvalues, see, e.g., [63, Theorem 3.2]. □

Interpretation of Extremal Values. For a graph G, the normalized Laplacian has an extremal maximal eigenvalue, i.e., $\lambda_n(\Delta) = 2$, if and only if G has a nontrivial bipartite connected component (see, e.g., [25, Lemma 1.7]).

Horak and Jost [63] give the following combinatorial criterion for complexes in which the spectrum $\Delta^{\mathrm{up}}_{k-1}$ contains the extremal value $k+1$:

Lemma 2.9 ([63, Theorem 7.1]). *Let X be a k-complex. The largest eigenvalue of the normalized Laplacian $\Delta^{\mathrm{up}}_{k-1}$ is $k+1$ if and only if there is a connected component C of X (w.r.t. hypergraph connectivity) and an orientation of the k-faces of X such that*

$$[H : F] = [H' : F] \text{ for all } F \subset H, H' \in C \text{ with } F \in X_{k-1}, H, H' \in X_k.$$

This condition is equivalent to C not containing any orientable k-circuits of odd length or non-orientable k-circuits of even length.

A pure k-dimensional simplicial complex Y is a *k-circuit of length* $(m-1)$ if there is an ordering of its k-simplices $F_1, F_2, \dots, F_m = F_1$, such that any F_i and F_j with $|j-i| = 1$ share a common $(k-1)$-face. It is *orientable* if it is possible to assign an orientation to all k-faces of Y in a way such that any two simplices sharing a common $(k-1)$-face induce a different orientation on this face.

They also show that a k-complex is fulfills this condition if the chromatic number of its 1-skeleton is $k+1$[63, Theorem 7.3]. In Chapter 7 we consider this and other possible analogues of bipartiteness for graphs.

For the adjacency matrix one can draw the same conclusions if $-kd_{\max}$ is an eigenvalue of A_{k-1}. If this is the case then for the corresponding eigenvector f, we see that for any $F \in X_{k-1}$ we have

$$(\Delta_{k-1}^{\mathrm{up}} f)_F = f(F) - (D^{-1}Af)_F = (1 + kd_{\max}/\deg(F))f(F) \geq (k+1)f(F).$$

Thus, we see that $k+1 \leq \langle \Delta_{k-1}^{\mathrm{up}} f, f\rangle / \langle f, f\rangle \leq \lambda_{\max}(\Delta_{k-1}^{\mathrm{up}}) \leq k+1$ and hence that $\lambda_{\max}(\Delta_{k-1}^{\mathrm{up}}) = k+1$.

Let us now turn to the other end of the spectrum of A_{k-1}. If for a graph G the adjacency matrix $A(G)$ has $d_{\max}$ as an eigenvalue, then G has a $d_{\max}$-regular component. For higher-dimensional complexes we have the following result:

Lemma 2.10. *If d_{max} is an eigenvalue of A_{k-1}, then any corresponding eigenvector f is a cocycle $f \in Z^{k-1}$ such that* $\deg(F) = d_{max}$ *for all F with $f_F \neq 0$.*

Proof. To simplify notation we write A instead of A_{k-1}. Let f such that $Af = d_{\max} f$. Then $L_{k-1}^{\mathrm{up}} f = D_{k-1} f - d_{\max} f$ and, as above,

$$0 \leq f^T L_{k-1}^{\mathrm{up}} f = \sum_{F \in X_{k-1}} (\deg(F) - d_{\max}) f_F^2 \leq 0.$$

Hence, $f^T L_{k-1}^{\mathrm{up}} f = 0$ and $\deg(F) = d_{\max}$ for all F with $f_F \neq 0$. It follows that $L_{k-1}^{\mathrm{up}} f = 0$, so $f \in Z^{k-1}$. □

Note that for graphs, a 0-cocycle is a function on the vertices of G that is constant on components. Thus, for a non-connected graph G with eigenvalue $d_{\max}$, the eigenvectors correspond to $d_{\max}$-regular components.

General Bounds. Recall that $\mu(G) = \max\{\mu_2(A), |\mu_n(A)|\}$ for a graph G. It is not hard to show that

$$\mu(G) \geq \sqrt{d \cdot (n-d)/(n-1)}$$

for every d-regular graph (see, e.g., [62, Claim 2.8]), hence $\mu(G) = \Omega(\sqrt{d})$ for $d \leq 0.99n$, say. For constant d, one has the sharper *Alon-Boppana bound* $\mu(G) \geq 2\sqrt{d-1} \cdot (1 - O(1/\log^2 n))$, see [46, 95].

A d-regular graph G is called a *Ramanujan graph* if it meets this bound for the spectral gap, that is, if $\mu(G) \leq 2\sqrt{d-1}$ and hence $\lambda_2(\Delta(G)) \geq 1 - 2\frac{\sqrt{d-1}}{d}$. By the discrete Cheeger inequality, Theorem 2.2, any family of Ramanujan graphs is a family of expanders – spectrally, these are "optimal expanders".

It is a deep result due to Lubotzky, Phillips and Sarnak [84] and independently to Margulis [89] that for every fixed number d with $d-1$ prime, there exist Ramanujan graphs on n vertices for infinitely many n (and moreover, these graphs can be explicitly constructed). Recently, the extistence of Ramanujan graphs for every fixed d has been shown by Marcus, Spielman and Srivastava [87]. (The graphs they consider are actually bipartite, so one has to allow that $\mu_n(A) = -d$ and consider $\mu(G)$ as the maximum absolute value of the remaining eigenvalues.)

A bound similar to the basic lower bound also holds for the eigenvalues of the higher-dimensional adjacency matrix. We call a k-dimensional simplicial complex X *d-regular* if $\deg(F) = d$ for all $F \in X_{k-1}$. For such a complex let

$$\mu(X) := \max\{|\mu| : \mu \text{ eigenvalue of } A_{k-1} \text{ on } (B^{k-1}(X))^{\perp}\}.$$

Note that, since X is d-regular, we have $A_{k-1} = dI - L^{\mathrm{up}}_{k-1}$ and all vectors in $B^{k-1}(X)$ are eigenvectors with eigenvalue d. This is why we are interested in the eigenvalues on $(B^{k-1}(X))^{\perp}$. The following lemma presents the basic bound for $\mu(X)$ for regular complexes X. We do not know of an analogue of the Alon-Boppana bound for $A_{k-1}(X)$.

Higher-dimensional analogues of Ramanujan graphs, so-called *Ramanujan complexes*, have been considered in [85, 86], see also the survey [82]. Their definition is based on algebraic concepts, they are finite quotients of certain Bruhat-Tits buildings that satisfy a different kind of spectral condition.

Lemma 2.11. *Let X be a d-regular k-dimensional simplicial complex, i.e.,* $\deg(F) = d$ *for all $F \in X_{k-1}$. Then:*

$$\mu(X) \geq \sqrt{d\frac{kf_{k-1}(X) - db^{k-1}(X)}{f_{k-1}(X) - b^{k-1}(X)}},$$

where $b^{k-1}(X) = \dim B^{k-1}(X)$*. If X furthermore has a complete $(k-1)$-skeleton then*

$$\mu(X) \geq \sqrt{dk \cdot (n-d)/(n-k)}.$$

Proof. To simplify notation we write A instead of A_{k-1} and b^{k-1} for $b^{k-1}(X)$. We use that the eigenvalues of A^2 are the squares of the eigenvalues of A. Furthermore, the diagonal entries of A^2 satisfy $A^2_{(F,F)} = k\deg(F) = kd$. Hence,

$$f_{k-1}(X)kd = \mathrm{trace}(A^2) = \sum_{\mu \text{ eigenvalue of } A} \mu^2.$$

Now, for $x \in B^{k-1}(X)$, we have $0 = L^{\mathrm{up}}_{k-1}x = dx - Ax$, so x is an eigenvector with eigenvalue d. Thus, the sum above is at most

$$d^2b^{k-1} + \mu(X)^2 \dim(B^{k-1}(X)^{\perp}) = d^2b^{k-1} + \mu(X)^2(f_{k-1}(X) - b^{k-1}).$$

For X with a complete $(k-1)$-skeleton, we have $f_{k-1}(X) = \binom{n}{k}$ and $b^{k-1} = \binom{n-1}{k-1}$. □

A further result by Horak and Jost is a lower bound of $\frac{k}{d_{\max}} + 1$ for the largest eigenvalue of the normalized Laplacian $\Delta^{\mathrm{up}}_{k-1}$ [63, Corollary 3.5]. For the adjacency matrix this yields an upper bound of $-k$ for the smallest eigenvalue. For a d-regular complex we hence see that $\mu(X) \geq k$, which for small values of d gives a better bound than Lemma 2.11.

Chapter 3
On the Subdivision Containment Problem

In this chapter, we leave the path we have been pursuing so far and consider a problem that is neither related to concepts of higher-dimensional expansion nor to spectra of higher-dimensional Laplacians. We study the subdivision containment problem on random complexes $X^k(n,p)$, a higher-dimensional analogue of the containment problem for topological minors in random graphs. We continue our investigation of higher-dimensional expansion in the following chapter where we consider the spectral properties of random complexes. The results of this chapter are joint work with Uli Wagner, it is based on the extended abstract [59].

A basic problem in graph theory is to determine whether a given graph G, which may be thought of as "large", contains a fixed graph H as a substructure. The most straightforward form of containment is that G contains a copy of H as a *subgraph*. Another important variant is that G contains some *subdivision* of H as a subgraph; in this case, one also says that G contains H as a *topological minor*.

For random graphs, the *containment problem* considers the probability that a binomial random graph $G(n,p)$ contains a copy of a given graph H. For subgraph containment, it is well-known [18] that this probability has a (coarse) threshold of $\Theta(n^{-1/m(H)})$, where $m(H)$ is the density of the densest subgraph of H. The (sharp) threshold for containment of any complete graph of fixed size as a topological minor is $p = 1/n$ by a well-known result of Ajtai, Komlós and Szemerédi [3, 41].

Subgraph containment admits a direct generalization to higher dimensions: We can ask whether a given simplicial complex X contains a fixed complex K as a subcomplex. The proof methods for random graphs extend directly to random 2-complexes, and the threshold probability for $X^2(n,p)$ to contain a fixed complex K as a subcomplex is given by the density (in terms of triangles versus vertices) of the densest subcomplex of K, see [11, 29].

For the question of topological minors, a natural higher-dimensional analogue is whether X contains some subdivision of K. Cohen, Costa, Farber and Kappeler [29] show that for any $\epsilon > 0$ and $p \geq n^{-1/2+\epsilon}$, the random complex $X^2(n,p)$ *asymptotically almost surely (a.a.s.)*, i.e., with probability tending to 1 as $n \to \infty$, contains a subdivision of any fixed K. Their method also extends to complexes of higher dimension.

We improve this upper bound on the threshold probability for 2-dimensional subdivision containment to $p = O(1/\sqrt{n})$ and show an upper bound of $O(n^{-1/k})$ for random k-complexes $X^k(n,p)$ with $k > 2$:

Theorem 3.1. *For every $k \geq 2$ and $t \geq k+1$ there is a constant $c_t > 1$ such that $X^k(n,p)$ with $p = \sqrt[k]{c_t/n}$ a.a.s. contains a subdivision of the complete k-complex K_t^k on t vertices.*

The result in [29] is proven by reduction to the subcomplex containment problem, by showing that a given 2-complex K can be subdivided to decrease its triangle density. For Theorem 3.1 we use a different approach, based on an idea going back at least to Brown, Erdős and Sós [21] and used also in [11]: For 2-complexes our proof is based on studying common links of pairs of vertices, which form random graphs of the type $G(n-2,p^2)$ and then uses results on the phase transition in random graphs. This approach also extends to complexes of dimension $k > 2$. For a (possibly more approachable) sketch of the proof for the case $k = 2$ we refer the reader to the extended abstract [59].

For 2-complexes we also show a corresponding lower bound and thus establish that the (coarse) threshold for containing a subdivision of any fixed complete 2-complex is at $p = \Theta(1/\sqrt{n})$:

Theorem 3.2. *There is a constant $c < 1$ such that for any $t \geq 10$ the random 2-complex $X^2(n,p)$ with $p = \sqrt{c/n}$ a.a.s. does not contain a subdivision of K_t^2.*

The somewhat technical proof of Theorem 3.2 is based on bounds on the number of triangulations of a fixed surface.

3.1 The Upper Bound on the Threshold

We first address Theorem 3.1. For fixed $t \geq k+1$, we aim to find a copy of a subdivision of K_t^k in $X^k(n,p)$. We would be allowed to subdivide faces of K_t^k of any dimension, but there will be no need for this: we find t vertices and take all faces of dimension at most $k-1$ spanned by these vertices to form the $(k-1)$-skeleton of our subdivision of K_t^k. We then show that the k-spheres (boundaries of k-simplices) spanned by the $(k-1)$-faces between any k of them can be filled with disjoint triangulated k-simplices.

Basic Set-Up. We will only consider k-complexes with vertex set $[n]$ and complete $(k-1)$-skeleton. For notational convenience, we assume without loss of generality that n is divisible by $2\binom{t}{k+1}$. Fix a partition of the vertex set $V = [n]$ into two sets U and W, each of size $\frac{n}{2}$. We will choose the t vertices of K_t^k from U, whereas the internal vertices for fillings will come from W. To ensure disjointness of the fillings of different k-spheres, we partition W into $\binom{t}{k+1}$ sets W_σ, $\sigma \in \binom{[t]}{k+1}$, each of size $n/(2\binom{t}{k+1})$, and choose the internal vertices of the filling for each $\sigma \in \binom{[t]}{k+1}$ from W_σ.

For a $(k-1)$-face $F \in \binom{[n]}{k}$, denote by G_F the graph $\bigcap_{H \in \binom{F}{k-1}} \mathrm{lk}(H)$, which has vertex set $[n] \backslash F$ and edge set

$$\left\{e \subset [n] \backslash F : e \cup H \in X \text{ for all } H \in \tbinom{F}{k-1}\right\}.$$

Denote by C_F^σ the largest connected component of $G_F[W_\sigma]$. If there are several components of maximum size, let C_F^σ be the one containing the smallest vertex.

The Main Idea. The basic idea of the proof is the following lemma, based on an idea going back at least to Brown, Erdős and Sós [21] that is also used in [11].

Lemma 3.3. *Let X be a k-complex with vertex set $[n]$ and complete $(k-1)$-skeleton. Suppose there is a set $A \subset U$, $|A| = t$ with a bijection $f{:}[t] \to A$ satisfying the following property: For every k-face $\sigma \in \binom{[t]}{k+1}$ of K_t^k there is a vertex $a \in f(\sigma)$ such that for $F = f(\sigma) \backslash \{a\}$ there are vertices $v, w \in C_F^\sigma$ with $F \cup \{v\} \in X$ and $\{a, w\} \in G_F$. Then X contains a subdivision of K_t^k.*

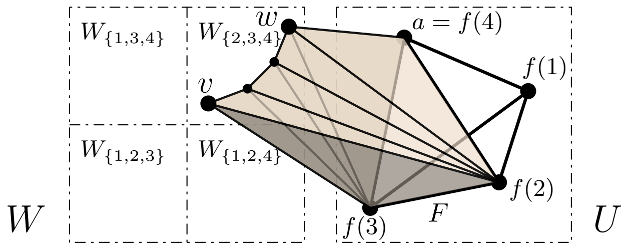

Figure 3.1: Part of a subdivision of a K_4^2: Filling of $f(\sigma)$ for $\sigma = \{2, 3, 4\}$

Proof. For $\sigma \in \binom{[t]}{k+1}$ and a, F, v, w as above there exists a path in $G_F[W_\sigma]$ between v and w which together with the k-faces $\{a, w\} \cup H$ for $H \in \binom{F}{k-1}$ and the k-face $F \cup \{v\}$, creates a k-dimensional disk filling the k-sphere (boundary of a k-simplex) created by the $(k-1)$-faces $F' \subset f(\sigma)$, $|F'| = k$. By choosing a distinct W_σ for each $\sigma \in \binom{[t]}{k+1}$ we ensure disjoint fillings. See Figure 3.1 for an illustration for the case $k = 2$. □

Random Complexes. We now proceed to show that a random complex $X^k(n, p)$ with $p = \sqrt[k]{c_t/n}$ for a suitable constant c_t a.a.s. satisfies the conditions of Lemma 3.3. We first give a criterion for complexes satisfying these conditions and then show that this criterion is satisfied a.a.s. by a random complex.

For fixed $F \subset U$, $|F| = k$ and $\sigma \in \binom{[t]}{k+1}$, call $u \in U \setminus F$ *connected to* C_F^σ if $\{u, w\} \in G_F$ for some $w \in C_F^\sigma$ and let

$$N_F^\sigma = \{u \in U \setminus F : u \text{ connected to } C_F^\sigma\}.$$

Let $\delta > 0$. Consider two families of k-complexes X with vertex set $[n]$ and complete $(k-1)$-skeleton:

- $\mathcal{A}_{F,\sigma} = \{X \subseteq \binom{[n]}{k+1} : \binom{[n]}{k} \subseteq X, \exists v \in C_F^\sigma \text{ with } F \cup \{v\} \in X\}$.
- $\mathcal{B}_{F,\sigma,\delta} = \{X \subseteq \binom{[n]}{k+1} : \binom{[n]}{k} \subseteq X, |N_F^\sigma| \geq (1-\delta)(|U| - k)\}$.

Lemma 3.4. *Let X be a k-complex with vertex set $[n]$ and complete $(k-1)$-skeleton. If there is a $\delta < 1/\left(\binom{t}{k+1}(k+1)\right)$ such that $X \in \mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}$ for all $F \subset U$, $|F| = k$ and $\sigma \in \binom{[t]}{k+1}$, then there exists a set $A \in \binom{U}{t}$ satisfying the conditions of Lemma 3.3.*

Proof. We need to show the existence of a set $A \subset U$, $|A| = t$ with a bijection $f:[t] \to A$ such that for every $\sigma \in \binom{[t]}{k+1}$ there is a vertex $a \in f(\sigma)$ such that for $F = f(\sigma)\setminus\{a\}$:

1. There is $v \in C_F^\sigma$ with $F \cup \{v\} \in X$.

2. The vertex a is connected to C_F^σ.

As $X \in \mathcal{A}_{F,\sigma}$ for all F and σ, the first condition holds for any choice of A,f,σ and a. So we only need to deal with the second condition. We consider tupels $(a_1, a_2, \dots, a_t)$ with $a_i \in U$ and all a_i pairwise distinct and let $A = \{a_1, a_2, \dots, a_t\}$. The function f is then determined by $f(i) = a_i$. We show that for a tupel chosen uniformly at random we have

$$\Pr\left[\exists\sigma \in \tbinom{[t]}{k+1}, a \in f(\sigma) : a \text{ not connected to } C^\sigma_{f(\sigma)\setminus\{a\}}\right] < 1.$$

Thus, there is a tuple that also satisfies the second condition. For fixed σ and $j \in \sigma$:

$$\begin{aligned}
&\Pr\left[f(j) \text{ not connected to } C^\sigma_{f(\sigma\setminus\{j\})}\right] \\
&= \sum\nolimits_{F\in\binom{U}{k}} \Pr\left[f(j) \notin N_F^\sigma \mid f(\sigma\setminus\{j\}) = F\right] \cdot \Pr\left[f(\sigma\setminus\{j\}) = F\right] \\
&\qquad\qquad\qquad\qquad\qquad = \tbinom{|U|}{k} \cdot \delta \cdot \tfrac{1}{\binom{|U|}{k}} = \delta
\end{aligned}$$

By a union bound, we hence have

$$\Pr\left[\exists\sigma, a \in f(\sigma) : a \text{ not connected to } C^\sigma_{f(\sigma)\setminus\{a\}}\right] \le \tbinom{t}{k+1}(k+1)\delta < 1.$$

$\square$

Lemma 3.5. *For every $k \ge 2$ and $t \ge k+1$ there is a constant $c = c(t,k) > 0$ such that for $p = \sqrt[k]{\frac{c}{n}}$ the random complex $X^k(n,p)$ asymptotically almost surely satisfies the conditions of Lemma 3.4.*

Proof. Let $k \ge 2$ and $t \ge k+1$. Let $T = \binom{t}{k+1}$. We show that there is $c > 2T$ and $\delta < 1/(T(k+1))$ such that $X^k(n,p)$ for $p = \sqrt[k]{c/n}$ a.a.s. satisfies the conditions of Lemma 3.4, i.e.,

$$X^k(n,p) \in \bigcap_{F,\sigma} \mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}.$$

Fix $F \subset U$, $|F| = k$ and $\sigma \in \binom{[t]}{k+1}$. The probability of the events $\mathcal{A} = \mathcal{A}_{F,\sigma}$ and $\mathcal{B} = \mathcal{B}_{F,\sigma,\delta}$ depends on the size of C_F^σ, the largest connected component of the graph $G_F[W_\sigma]$, which is a random graph of type $G(|W_\sigma|, p^k)$.

As $p^k = c/n = \frac{c/(2T)}{|W_\sigma|}$, for c large enough the graph $G_F[W_\sigma]$ *fails* to have a giant component of size linear in $|W_\sigma|$ with exponentially small probability: For every $\gamma > 0$ a random graph $G(n, \frac{1+\gamma}{n})$ has a connected component of size at least $\frac{\gamma^2 n}{5}$ with probability $1 - e^{-\kappa n}$ for some $\kappa = \kappa(\gamma) > 0$ (see e.g, [76]). So for any $c > 2T$ there are $\epsilon > 0$ and $\kappa > 0$ such that

$$\Pr[|C_F^\sigma| \geq (1-\epsilon)|W_\sigma|] \geq 1 - e^{-\kappa n}.$$

As we will later need that $\delta > e^{-\frac{c(1-\epsilon)}{2T}}$, we choose $c > 2T$ such that $e^{-\frac{c(1-\epsilon)}{2T}} < \frac{1}{T(k+1)}$ and then choose $\delta \in \left[e^{-\frac{c(1-\epsilon)}{2T}}, \frac{1}{T(k+1)}\right]$.

For $S \subset W_\sigma$ denote by $\Pr_S$ the conditional probability when conditioning on $C_F^\sigma = S$. Then $\Pr[\mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}]$ is at least

$$\sum_S \Pr_S\left[\mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}\right] \cdot \Pr\left[C_F^\sigma = S\right],$$

where the sum runs over all $S \subset W_\sigma$ with $|S| \geq (1-\epsilon)|W_\sigma|$.

As $\mathcal{A}_{F,\sigma}$ and $\mathcal{B}_{F,\sigma,\delta}$ depend on different kinds of k-faces and the presences of k-faces are decided independently, we have

$$\Pr_S\left[\mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}\right] = \Pr_S\left[\mathcal{A}_{F,\sigma}\right] \cdot \Pr_S\left[\mathcal{B}_{F,\sigma,\delta}\right].$$

We consider the two terms seperately:

$\Pr_S\left[\mathcal{A}_{F,\sigma}\right]$: Here we consider $\Pr\left[\exists v \in S \text{ with } F \cup \{v\} \in X\right]$. The number $f(X)$ of vertices $v \in S$ with $F \cup \{v\} \in X$ is a binomially distributed variable with parameters $|S|$ and p. Hence, its expectation is $|S|p$ and by Chernoff's inequality

$$\Pr\left[f(X) = 0\right] \leq e^{-\dfrac{|S|p}{2}} \leq e^{-(1-\epsilon)\frac{\sqrt[k]{c}}{4T} n^{1-1/k}}.$$

$\Pr_S\left[\mathcal{B}_{F,\sigma,\delta}\right]$: Call $u \in U \backslash F$ *connected to* S if $\{u, w\} \in G_F$ for some $w \in S$. Then we need to consider

$$\Pr\left[|\{u \in U \backslash F : u \text{ connected to } S\}| \geq (1-\delta)(|U| - k)\right].$$

For fixed $u \in U \setminus F$ the probablitiy *not* to be connected to S is $\lambda = (1-p^k)^{|S|} \leq e^{-p^k|S|} \leq e^{-\frac{c(1-\epsilon)}{2T}}$. For each u the decisions over the k-faces deciding whether u is connected to S are taken independently. Hence, also the number $g(X)$ of vertices $u \in U \setminus F$ that are connected to S is a binomially distributed variable with parameters $(|U| - k)$ and $(1 - \lambda)$. As we chose $\delta > e^{-\frac{c(1-\epsilon)}{2T}} \geq \lambda$, we get by Chernoff's inequality for large enough n:

$$\begin{aligned}\Pr\left[g(X) < (1-\delta)(|U|-k)\right] \\ &= \Pr\left[g(X) < \mathbb{E}[g(X)] - (\delta-\lambda)(|U|-k)\right] \\ &\leq e^{-\frac{(\delta-\lambda)^2(|U|-k)}{2(1-\lambda)}} \leq e^{-\frac{(\delta-\lambda)^2}{5}n}.\end{aligned}$$

Notice that the probabilities $\Pr_S\left[\mathcal{A}_{F,\sigma}\right]$ and $\Pr_S\left[\mathcal{B}_{F,\sigma,\delta}\right]$ don't depend on S. Hence we can use $\sum_S \Pr\left[C_F^\sigma = S\right] = \Pr\left[|C_F^\sigma| \geq (1-\epsilon)|W_\sigma|\right]$ and get by the choice of c and ϵ:

$$\begin{aligned}\Pr[\mathcal{A}_{F,\sigma} \cap \mathcal{B}_{F,\sigma,\delta}] &\geq \left(1 - e^{-(1-\epsilon)\frac{\sqrt[k]{c}}{4T}n^{1-1/k}}\right)\left(1 - e^{-\frac{(\delta-\lambda)^2}{5}n}\right)\left(1 - e^{-\kappa n}\right) \\ &\geq 1 - e^{-\beta n^{1-1/k}}\end{aligned}$$

for some $\beta > 0$. Applying a union bound, we get for some $\alpha > 0$:

$$\begin{aligned}\Pr\left[\exists F \subset U, |F| = k, \sigma \in \binom{[t]}{k+1} : \neg\mathcal{A}_{F,\sigma} \cup \neg\mathcal{B}_{F,\sigma,\delta}\right] \\ \leq \binom{n/2}{k} \cdot T \cdot e^{-\beta n^{1-1/k}} \leq e^{-\alpha n^{1-1/k}}.\end{aligned}$$

□

3.2 The Lower Bound on the Threshold

We now turn to the proof of Theorem 3.2 on random 2-complexes $X^2(n,p)$. Our goal is to show the existence of a constant $c \leq 1$ such that for $p = \sqrt{c/n}$ the probability to find a subdivision of K_t^2 converges to zero. The proof bases on the following simple observation: If a complex contains a subdivision of K_t^2 with $t \geq 10$, it also contains a subdivision of a triangulation of Σ_2, the orientable surface of genus 2[1]. We then use that the number of triangulations of any surface with a fixed number l of vertices is known to be at most simply exponential in l.

[1]The smallest possible triangulation of Σ_2 has 10 vertices, see [65].

Bounds on the number of triangulations of a fixed closed surface can be drawn from the theory of enumeration of maps on surfaces which has its beginning in Tutte's famous results on the number of rooted maps on the sphere [110, 111, 112]. As the terminology in these references differs a lot from ours and as furthermore the classes of objects that are counted are not exactly the same, we first explain in detail the enumeration result we will use. We rely on [13, 14, 15, 52].

Maps on Surfaces. Let S be a connected compact 2-manifold without boundary. A *map* $M = (S, G, \Phi)$ on S is a graph G together with an embedding Φ of G into S such that each connected component of $S \setminus \Phi(G)$ is simply connected, i.e., each face is a disk. Graphs are unlabelled, finite and connected, loops and multiple edges are allowed.

A map is *rooted* if an edge, a direction along the edge and a side of the edge are distinguished. An edge is called *double* if its image belongs to the boundary of only one face. Any other, *single*, edge belongs to two faces. The *valency* of a face is the number of single edges in its boundary plus twice the number of double edges. A *triangular map* is a map such that each face has valency three.

Two maps (S, G, Φ) and (S', G', Φ') are considered *equivalent* if there is a homeomorphism $h : S \to S'$ and a graph isomorphism $g : G \to G'$ such that $h\Phi = \Phi' g$.

Triangular Maps vs. Triangulations. Let $M = (S, G, \Phi)$ be a triangular map such that the graph $G = (V, E)$ is simple, i.e., doesn't have loops or multiple edges. Then every face of M has a boundary consisting of exactly three edges. Define a 2-complex $X(M) = (V, E, T(M))$ by letting

$$T(M) := \{\{u, v, w\} : u, v, w \in V \text{ are the vertices of a face of } M\}.$$

Equivalent maps yield isomorphic complexes:

Lemma 3.6. *Let $M = (S, G, \Phi)$ be a triangular map such that the graph $G = (V, E)$ is simple and let $M' = (S', G', \Phi')$ be equivalent to M. Then $X(M)$ and $X(M')$ are isomorphic.*

Proof. Since M and M' are equivalent, there is a homeomorphism $h : S \to S'$ and a graph isomorphism $g : G \to G'$ such that $h\Phi = \Phi' g$. We show that g is also an isomorphism between $X(M)$ and $X(M')$. As g is a graph isomorphism, all we need to show is that g preserves

2-faces. Let $\{u, v, w\} \in T(M)$, so u, v, w are the vertices of a face of M. This face is mapped to some disk in S' by h. As $h\Phi = \Phi' g$, this disk is a face of M' with $g(u), g(v)$ and $g(w)$ as boundary vertices. Hence, $\{g(u), g(v), g(w)\} \in T(M')$. The same argument shows that any 2-face of $X(M')$ is mapped to a 2-face of $X(M)$. $\square$

For a 2-complex $X = (V, E, T)$ such that $\|X\|$ is homeomorphic to a surface S, define a triangular map $M(X) = (S, (V, E), \Phi)$, where Φ is the restriction of a homeomorphism $\|X\| \to S$ to the 1-skeleton of X. The following lemma shows that $M(X)$ is well-defined and that isomorphic complexes give rise to equivalent maps:

Lemma 3.7. *Let $X = (V, E, T)$ be a 2-complex such that $\|X\|$ is homeomorphic to a surface S and let $X' = (V', E', T')$ be isomorphic to X. Let furthermore $\varphi : \|X\| \to S$ and $\varphi' : \|X'\| \to S$ be homeomorphisms and define Φ and Φ' to be the restrictions of φ and φ' to the 1-skeleta of X and X', respectively. Then $(S, (V, E), \Phi)$ and $(S, (V', E'), \Phi')$ are equivalent.*

Proof. Let $f : V(X) \to V(X')$ be an isomorphism between X and X'. Then the affine extension $\|f\| : \|X\| \to \|X'\|$ is a homeomorphism (see, e.g., [90, Proposition 1.5.4]). So $\|X'\|$ is also homeomorphic to S. Choosing $g = f$ and $h = \varphi' \|f\| \varphi^{-1}$, we get $h\Phi = \Phi' g$. $\square$

Lemmas 3.6 and 3.7 show that there is a bijection between equivalence classes of triangular maps with simple underlying graph on a surface S and isomorphism classes of 2-complexes with polyhedron homeomorphic to S.

The Number of Triangulations. In [52] Gao gives an asymptotic enumeration result for rooted triangular maps on any closed surface.

Theorem 3.8. *Let $T_g(l)$ denote the number of l-vertex rooted triangular maps on the orientable surface of genus g.*[2] *There is a constant t_g, independent of l, such that for $l \to \infty$,*

$$T_g(l) \sim t_g l^{5(g-1)/2} (12\sqrt{3})^l.$$

We are interested in the number $\tau_g(l)$ of l-vertex triangulations of the orientable surface Σ_g of genus g, i.e., the number of 2-complexes $X = (V, E, T)$ such that $|V| = l$ and $\|X\|$ is homeomorphic to Σ_g. By

[2] Gao also considers non-orientable surfaces, in which we are not interested here.

the considerations above this is the number of triangular maps on Σ_2 with a simple underlying graph. As Gao's result also allows loops and multiple edges and makes a distiction between equivalent maps that are rooted in a different way, we get $\tau_g(l) \leq T_g(l)$ and hence:

Corollary 3.9. *Let $\tau_g(l)$ be the number of triangulations of Σ_g, the orientable surface of genus g, with l vertices. There is a constant $K_g > 0$, independent of l, such that $\tau_g(l) \leq K_g^l$.*

Proof of the Lower Bound on the Threshold. Now that we have established (Corollary 3.9) that the number of triangulations of any fixed surface with a fixed number l of vertices is at most simply exponential in l, we can turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. Fix $t \in \mathbb{N}$, $t \geq 10$ and let T_0 be a triangulation of Σ_2, the orientable surface of genus 2, with 10 vertices. As T_0 is a subcomplex of K_t^2, we have:

$$\begin{aligned}\Pr\left[X^2(n,p) \text{ contains a subdivision of } K_t^2\right]& \\ \leq \Pr\left[X^2(n,p) \text{ contains a subdivision of } T_0\right]&.\end{aligned}$$

We show that for sufficiently small p the latter probability tends to 0. Ignoring that we only consider subdivisions of T_0, we get:

$$\Pr\left[X^2(n,p) \text{ contains a subdivision of } T_0\right] \leq \sum_{l=1}^{n} \sum_{T \in \mathcal{T}_l} \Pr\left[T \subseteq X^2(n,p)\right],$$

where the second sum is over the set $\mathcal{T}_l$ of all triangulations of Σ_2 that have l vertices. Denote by $\tau_2(l) = |\mathcal{T}_l|$ the number of such triangulations and choose $K = K_2$ as in Corollary 3.9 such that $\tau_2(l)$ is at most K^l. Let $p = \sqrt{c/n}$ for some $c \leq 1/K$.

By Euler's formula, every triangulation T of the oriented surface Σ_g of genus g satisfies $f_2(T) = 2(|V(T)| - 2 + 2g) = 2(|V(T)| + 2)$, if $g = 2$, and $\Pr\left[T \subseteq X^2(n,p)\right] \leq n^{|V(T)|} p^{f_2(T)}$. Hence,

$$\begin{aligned}\sum_{l=1}^{n} \sum_{T \in \mathcal{T}_l} \Pr\left[T \subseteq X^2(n,p)\right] &\leq \sum_{l=1}^{n} t_l \cdot n^l p^{2(l+2)} \\ &\leq \left(\frac{c}{n}\right)^2 \sum_{l=1}^{n} (cK)^l = \left(\frac{c}{n}\right)^2 \left(\frac{1-(cK)^{n+1}}{1-cK} - 1\right),\end{aligned}$$

which clearly converges to zero as n goes to infinity. □

Concluding Remarks. For the random 2-complex $X^2(n,p)$, we have shown that the property of containing a subdivision of the complete complex K_t^2 has a coarse threshold $p = \Theta(1/\sqrt{n})$ for any $t \geq 7$. For dimensions > 2 we could show an upper bound of $O(n^{-1/k})$ for the threshold.

The corresponding lower bound for higher dimensional complexes is open. It doesn't seem likely that an approach as simple as the one presented here will work in higher dimensions. An essential ingredient of our proof is that the number of triangulations of any fixed surface with a fixed number l of vertices is simply exponential in l. The proof also depends on the fact that $f_2(T) = 2(|V(T)| + 2)$ for any triangulation T of Σ_2, the surface of genus 2. In higher dimensions, it is not clear which manifold could play the role of Σ_2. Furthermore, bounds on the numbers of triangulations that are simply exponential are not to be expected: The number of k-spheres with l vertices, e.g., is known to be at least $2^{\Omega(l^{\lfloor k/2 \rfloor})}$ for $k > 3$ [69] and $2^{\Omega(l^{5/4})}$ for $k = 3$ [98].

It is very likely that, just as for graphs, the threshold for complete subdivision containment is actually a sharp threshold. For the upper bound an approach towards proving sharpness might be to combine the basic idea used here with more sophisticated arguments on the random graphs involved.

Chapter 4

On the Spectra of Random Complexes

In this chapter, we continue to explore higher-dimensional Laplacians and adjacency matrices by considering the spectral properties of the random complexes $X^k(n, p)$. The behaviour of the eigenvalues of random graphs is well-studied. For large p, it is known that the eigenvalues of the adjacency matrix as well as of the Laplacian are concentrated in two clusters, defined in terms of $d = p(n-1)$, the expected degree of a vertex in the random graph $G(n, p)$:

Theorem 4.1 ([31, 42])**.** *For every $c > 0$ there exist constants $C > 0$ and $c' > 0$ such that for $p \geq C \cdot \log n/n$ and $d = p(n-1)$ the following statements hold with probability at least $1 - n^{-c}$:*

(i) $\mu_1(A(G(n,p))) \in [d - c' \cdot \sqrt{d}, d + c' \cdot \sqrt{d}]$ *and* $\mu(G(n,p)) \leq c' \cdot \sqrt{d}$*;*

(ii) $1 - \frac{c'}{\sqrt{d}} \leq \lambda_2(\Delta(G(n,p)) \leq \ldots \leq \lambda_n(\Delta(G(n,p)) \leq 1 + \frac{c'}{\sqrt{d}}$.

Our result is a higher-dimensional analogue of Theorem 4.1 for the generalizations of the adjacency matrix and the Laplacian that were presented in Chapter 1.

Theorem 4.2. *For all $c > 0$ and $k \geq 1$ there exists a constant $C = C(c, k) > 0$ with the following property: Assume $p \geq C\frac{\log(n)}{n}$ and let d be the expected degree of any $(k-1)$-face F in $X^k(n, p)$, i.e., $d := p(n-k)$. Then there exist $\gamma_A = O(\sqrt{d})$ and $\gamma_\Delta = O(1/\sqrt{d})$ such that the following statements hold with probability at least $1 - n^{-c}$:*

(i) *The largest* $\binom{n-1}{k-1}$ *eigenvalues of* $A_{k-1}(X^k(n,p))$ *lie in the interval* $[d-\gamma_A, d+\gamma_A]$*, and the remaining* $\binom{n-1}{k}$ *eigenvalues lie in the interval* $[-\gamma_A, +\gamma_A]$.

(ii) *The smallest* $\binom{n-1}{k-1}$ *eigenvalues of* $\Delta^{\mathrm{up}}_{k-1}(X^k(n,p))$ *are (trivially) zero, and the remaining* $\binom{n-1}{k}$ *eigenvalues are contained in the interval* $[1-\gamma_\Delta, 1+\gamma_\Delta]$*. In particular,* $\tilde{H}^{k-1}(X^k(n,p);\mathbb{R}) = 0$.

Both concentration results are achieved by reducing the higher-dimensional problem to estimates for the eigenvalues of random graphs, i.e., to Theorem 4.1. For the Normalized Laplacian this is done by applying a fundamental estimate due to Garland [53] (see Section 4.2). For the generalized adjacency matrix we develop a similar result to this estimate.

In the first section of this chapter we will give a more detailed account of the spectral properties of random graphs. We will then go on to review Garland's estimate and prove the analogous result for the adjacency matrix. The last section of this chapter contains the proof of Theorem 4.2. The results of this chapter are joint work with Uli Wagner, it is based on the extended abstract [58].

Related Work Chung [24] studies a higher Laplacian for hypergraphs that is closely related to the combinatorial Laplacian $L_{k-1} = L^{\mathrm{up}}_{k-1} + L^{\mathrm{down}}_{k-1}$. In [24, Section 7], she proves a weaker concentration result for eigenvalues of random hypergraphs, namely, essentially, that for *constant* p and any $\varepsilon > 0$, the eigenvalues of $L_{k-1}(X^k(n,p))$ are concentrated in an interval of width $O(n^{1/2+\varepsilon})$. She also states, without proof, that the proof methods for random graphs can be extended to yield the sharp bound of $O(\sqrt{pn})$.

After submitting the original manuscript of the extended abstract [58], we became aware of a preprint by Hoffman, Kahle and Paquette [60], who prove closely related results. Specifically, following the basic approach of [49], they show that for any $\varepsilon > 0$ and $p \geq (k+\varepsilon)\frac{\log n}{n}$, the second eigenvalue of the Laplacian satisfies $\lambda_2(\Delta(G(n,p))) > 1/2$ with probability $1 - o(n^{1-k})$. (Thus, compared to the known results, they trade precise information about the constant factor in front of $\log n/n$ for weaker concentration.) Using a result by Żuk, that is a strengthening of Garland's estimate, they obtain as an immediate corollary that for $p \geq (2+\varepsilon)\frac{\log n}{n}$, the fundamental group of the random

2-complex $X^2(n,p)$ a.a.s. has a certain property known as *Kazhdan's Property (T)*.

In another recent preprint, Lu and Peng [80] study a rather different kind of Laplacian for random complexes. Specifically, given a k-dimensional complex X on a vertex set V and a parameter $s \leq \frac{k+1}{2}$, they consider an auxiliary weighted graph on the vertex set $\binom{V}{s}$ in which $I, J \in \binom{V}{s}$ are connected by an edge of weight w if $I \cap J = \emptyset$ and I and J are contained in precisely w common k-faces of X. Lu and Peng study the normalized Laplacian of this auxiliary weighted graph. However, this Laplacian seems to capture the topology of X only in a limited way. For instance, in the case $k = 2$ and $s = 1$, any two 2-dimensional complexes on n vertices that have a complete 1-skeleton and are d-regular (every edge is contained in d triangles) yield the same auxiliary graph, even though the topologies of these complexes (as measured by real cohomology groups and the usual Laplacian, say) may be very different.

4.1 Eigenvalues of Random Graphs

Theorem 4.1 summarizes known results on the concentration of eigenvalues for random graphs $G(n,p)$. Here we explain the corresponding references in more detail. For the normalized Laplacian the situation is simple: Building on the results for the adjacency matrix and relating the spectrum of $\Delta(G(n,p))$ to that of $A(G(n,p))$, Coja-Oghlan [31] proved the result for the normalized Laplacian. For $p \gg (\log n)^2/n$ this was also shown by Chung, Lu and Vu [28].

For the adjacency matrix the situation in the literature is more involved: Different ranges of p are covered in several references. Füredi and Komlós [50] showed for constant p that *asymptotically almost surely* (*a.a.s.*), i.e., with probability tending to 1 as $n \to \infty$, $\mu(G(n,p)) = O(\sqrt{d})$, where $d = p(n-1)$ is the expected average degree. Their method of proof, the so-called *trace method*, can be adapted to cover the range $\frac{\ln(n)^7}{n} \leq p \leq 1 - \frac{\ln(n)^7}{n}$ [30]. Feige and Ofek [42] extended the result to values of p as small as $C \cdot \log n/n$, but their proof requires an upper bound on p. They used methods of Friedman, Kahn, and Szemerédi [49], who proved that $\mu(G) = O(\sqrt{d})$ holds a.a.s. for *random d-regular graphs* with constant d. Below, we explain the situation in yet more detail and give a more precise statement than the one of Theorem 4.1, which we will need for our proof of the corresponding statement on the generalized adjacency matrix for simplicial complexes (Theorem 4.2).

We remark that both parts of Theorem 4.1 can be extended to very sparse random graphs $G(n,p)$ with $p = \Theta(1/n)$ (for which they fail to hold as stated) by passing to a suitable large *core subgraph*, see [31, 42]. Moreover, analogous results are also known for other random graph models, including random d-regular graphs [49] and random graphs with prescribed expected degree sequences [28, 32].

Concentration Results for the Adjacency Matrix Concentration results on the spectrum of $A(G(n,p))$ are usually proven using one of the two following sufficient (and equivalent) conditions:

Lemma 4.3. *For $A = A(G(n,p))$ with $d := (n-1)p$ the following two conditions are equivalent:*

(i) There is $\gamma = O(\sqrt{d})$ such that for $u = \frac{1}{\sqrt{n}}\mathbf{1}$:

$$\langle Au, u\rangle \in [d-\gamma, d+\gamma] \text{ and } |\langle Aw, w\rangle|, |\langle Au, w\rangle| \leq \gamma$$

for all $w \perp \mathbf{1}$ with $\|w\| = 1$;

(ii) $\|pJ - A\| = O(\sqrt{d})$, where J is the all-ones matrix.

Both (i) and (ii) imply that there is $\gamma = O(\sqrt{d})$ such that

$$\mu_1(A) \in [d-\gamma, d+\gamma] \text{ and } \mu_2(A), \ldots, \mu_n(A) \in [-\gamma, \gamma]. \tag{4.1}$$

Proof. We first show that (ii) implies (i). Let $M = pJ - A$ and choose some $w \perp \mathbf{1}$ with $\|w\| = 1$. As $M\mathbf{1} = np\mathbf{1} - A\mathbf{1}$, we have $|\langle Au, u\rangle - np| = |\langle Mu, u\rangle| \leq \|M\|$ and $|\langle Au, w\rangle| = |\langle Mu, w\rangle| \leq \|M\|$. Furthermore, as $Jw = 0$, $|\langle Aw, w\rangle| = |\langle Mw, w\rangle| \leq \|M\|$.

To show that (ii) follws from (i), we fix some $x \neq 0$ with $\|x\| = 1$ and show $|\langle Mx, x\rangle| = O(\sqrt{d})$. We can find $\alpha, \beta \in [0,1]$ with $\alpha^2 + \beta^2 = 1$ and a $w \perp \mathbf{1}$ such that $x = \alpha u + \beta w$. Then

$$\begin{aligned}|\langle Mx, x\rangle| &= |\alpha^2\langle Mu, u\rangle + 2\alpha\beta\langle Mu, w\rangle + \beta^2\langle Mw, w\rangle| \\ &\leq \alpha^2|np\langle u, u\rangle - \langle Au, u\rangle| + 2\alpha\beta|\langle Au, w\rangle| + \beta^2|\langle Aw, w\rangle| \\ &= (\alpha^2 + 2\alpha\beta + \beta^2)O(\sqrt{d}) = O(\sqrt{d}).\end{aligned}$$

That (i) implies (4.1) is shown in [42, Lemma 2.1] of which we later show a generalization, Lemma 4.9. $\square$

We now argue why condition (ii) holds for $p \geq C \cdot \log n/n$:

Theorem 4.4. *For every $c > 0$ there exist constants $C > 0$ and $c' > 0$ with the following property: Suppose $p \geq C \cdot \log n/n$ and let $A = A(G(n,p))$ and $d = p(n-1)$. Then $\|pJ - A\| = O(\sqrt{d})$ with probability at least $1 - n^{-c}$. Here, J denotes the all-ones matrix.*

For $C\frac{\ln(n)}{n} \leq p \leq \frac{(n/\ln(n)^5)^{1/3}}{n}$, Feige and Ofek show that for all $c > 0$ there is $c' > 0$ such that condition (i) of Lemma 4.3 with $\gamma = c'\sqrt{d}$ holds with probability $1 - n^{-c}$.

For the range $\frac{\ln(n)^7}{n} \leq p \leq 1 - \frac{\ln(n)^7}{n}$, Coja-Oghlan [30], adapting the original proof by Füredi and Komlós [50], shows that

$$\|pJ - A\| \leq (2 + o(1))\sqrt{np(1-p)}$$

holds with probability $1 - O(n^{-4})$. Note that we ask for a probability of $1 - n^{-c}$ for a given $c > 0$ but only for a concentration of $O(\sqrt{d})$. Coja-Oghlan's proof can be adapted to yield this.

For $p \geq 1 - \frac{\ln(n)^7}{n}$, it is not hard to see that the desired concentration result holds in this range: For a graph G consider its complement graph $\bar{G} = (V, \binom{V}{2} \setminus E(G))$. Then

$$\begin{aligned}\|pJ - A(G)\| &= \|J - A(K_n) - (1-p)J + A(\bar{G})\| \\ &\leq \|J - A(K_n)\| + \|(1-p)J - A(\bar{G})\|.\end{aligned}$$

As $\|J - A(K_n)\| = \|I\| = 1$ we can hence consider $G(n, 1-p)$ instead and there show a concentration of $O(\sqrt{np})$. Thus, it suffices to prove Lemma 4.5 below.

Lemma 4.5. *Let $p \leq \frac{\ln(n)^7}{n}$. Then for all $c > 0$ there is $c' > 0$ such that $\|pJ - A\| \leq c'\sqrt{n - \ln(n)^7}$ with probability at least $1 - n^{-c}$.*

Proof. By a simple argument (or, alternatively, the Gershgorin circle theorem) any eigenvalue λ of $pJ - A$ satisfies

$$|\lambda| \leq np + \mathrm{maxdeg}(G)(1 - 2p) \leq \ln(n)^7 + \mathrm{maxdeg}(G).$$

It remains to show that with probability at least $1 - n^{-c}$ all vertex degrees are at most $c'\sqrt{n - \ln(n)^7}$ for some $c' > 0$. This is done by a straightforward application of Chernoff bounds and a union bound.

□

4.2 Garland's Estimate Revisited

In [53] Garland studies the normalized Laplacian $\Delta_i^{\mathrm{up}}(X)$. His main result regards a conjecture of Serre's on the cohomology of certain groups. As a technical lemma, he proves a bound for the nontrivial eigenvalues of $\Delta_i^{\mathrm{up}}(X)$ in terms of the eigenvalues of the Laplacian on links of lower-dimensional faces (see also [19] for a very clear exposition).

We state the result for the case of $\Delta_{k-1}^{\mathrm{up}}(X)$ and the links of $(k-2)$-dimensional faces $F \in X_{k-2}$. In this case, $\operatorname{lk} F = \operatorname{lk}(F, X)$ is a graph on $n-k+1$ vertices and the normalized Laplacian $\Delta_0^{\mathrm{up}}(\operatorname{lk} F)$ agrees with the usual normalized graph Laplacian $\Delta(\operatorname{lk} F)$. Furthermore, we show an analogous result for the generalized adjacency matrix $A_{k-1}(X)$.

Normalized Laplacian

Theorem 4.6 ([53], see also [19, Theorem 1.5,1.6]). *Let X be a pure k-dimensional complex and let $\Delta_{k-1}^{\mathrm{up}} = \Delta_{k-1}^{\mathrm{up}}(X)$ be its normalized Laplacian. Denote by $\langle , \rangle$ the weighted inner product on $C^{k-1}(X;\mathbb{R})$ defined by $\langle f, g\rangle = \sum_{F\in X_i} \deg(F) f(F) g(F)$. Assume that*

$$\lambda_{\min} \leq \lambda_2(\Delta(\operatorname{lk} F)) \leq \lambda_{n-k+1}(\Delta(\operatorname{lk} F)) \leq \lambda_{\max}$$

for all $F \in X_{k-2}$. Then for all $f \in B^{k-1}(X)^{\perp}$ (where the orthogonal complement is taken with respect to $\langle , \rangle$)

$$(1 + k\lambda_{\min} - k)\langle f, f\rangle \leq \langle \Delta_{k-1}^{\mathrm{up}} f, f\rangle \leq (1 + k\lambda_{\max} - k)\langle f, f\rangle.$$

Thus all non-trivial eigenvalues of $\Delta_{k-1}^{\mathrm{up}}$ lie in $[1+k\lambda_{\min}-k, 1+k\lambda_{\max}-k]$.

We remark that Garland only states the lower bound. The upper bound follows directly from the proof, which we reproduce here in our notation. The main idea of the proof is to present the normalized Laplacian as a sum of matrices each of which has non-zero entries only on the link of some $(k-2)$-face. These matrices then correspond to the Laplacians of the links.

For a pure k-dimensional simplicial complex X, fix a face $F \in X_{k-2}$ of dimension $k-2$. Let ρ_F be the diagonal $|X_{k-1}| \times |X_{k-1}|$-matrix defined by

$$(\rho_F)_{G,H} = \begin{cases} 1 & \text{if } G = H \text{ and } F \subset G, \\ 0 & \text{otherwise.} \end{cases}$$

We set $\Delta_{k-1}^{\mathrm{up},F}(X) := \rho_F \Delta_{k-1}^{\mathrm{up}}(X) \rho_F$ and for $f \in C^{k-1}(X)$ furthermore define $f_F \in C^0(\operatorname{lk} F)$ by $f_F(\{u\}) = [F \cup \{u\} : F] f(F \cup \{u\})$.

Lemma 4.7. *Let X be a pure k-dimensional complex.*

a) $\sum_{F\in X_{k-2}} \Delta^{\mathrm{up},F}_{k-1}(X) = \Delta^{\mathrm{up}}_{k-1}(X) + (k-1)I.$

b) For $u, v \in V(\operatorname{lk} F)$ let $F_u = F \cup \{u\}$ and $F_v = F \cup \{v\}$. Then

$$(\Delta^{\mathrm{up},F}_{k-1}(X))_{F_u,F_v} = [F_u : F][F_v : F](\Delta(\operatorname{lk} F))_{u,v}.$$

So, for $f \in C^{k-1}(X)$, $\langle \Delta^{\mathrm{up},F}_{k-1}(X)f, f\rangle = \langle \Delta(\operatorname{lk} F)f_F, f_F\rangle$.

c) If $f \in B^{k-1}(X)^{\perp}$ then $f_F \in \mathbf{1}^{\perp}$.

Proof. a) Observe that $\Delta^{\mathrm{up},F}_{k-1}(X)$ is obtained by replacing by 0 all entries of $\Delta^{\mathrm{up}}_{k-1}(X)$ that are contained in a row or column corresponding to some $G \notin \operatorname{lk} F$. The non-zero entries of $\Delta^{\mathrm{up}}_{k-1}(X)$ lie on the diagonal or correspond to faces $G, H \in X_{k-1}$ that share a common $(k-2)$-face and for which $G \cup H \in X_k$. Hence, every non-zero entry $(\Delta^{\mathrm{up}}_{k-1}(X))_{G,H}$ with $G \neq H$ is contained in exactly one summand and the diagonal entries, which are 1, are each contained in exactly k summands.

b) First consider $u \neq v$ with $F \cup \{u, v\} \in X$. Straightforward calculations show that $\deg_X(F_u) = \deg_{\operatorname{lk} F}(u)$ and that furthermore $[F_{u,v} : F_u][F_{u,v} : F_v] = -[F_u : F][F_v : F]$, where $F_{u,v}$ stands for $F \cup \{u, v\}$. Hence,

$$\begin{aligned}(\Delta^{\mathrm{up},F}_{k-1}(X))_{F_u,F_v} &= \frac{[F_{u,v} : F_u][F_{u,v} : F_v]}{\deg_X(F_u)} = -\frac{[F_u : F][F_v : F]}{\deg_{\operatorname{lk} F}(u)}\\ &= [F_u : F][F_v : F](\Delta(\operatorname{lk} F))_{u,v}.\end{aligned}$$

If $F \cup \{u, v\} \notin X$, the corresponding entry is 0 in both matrices. For the diagonal entries we get

$$(\Delta^{\mathrm{up}}_{k-1}(X))_{F_u,F_u} = 1 = [F_u : F][F_u : F]\Delta(\operatorname{lk} F)_{u,u}.$$

c) For $f \in B^{k-1}(X)^{\perp}$ we have $\sum_{G\in X_{k-1}} \deg(G)f(G)[G : F] = \langle f, \delta_{k-2}e_F\rangle = 0$ and therefore

$$\begin{aligned}\langle f^F, \mathbf{1}\rangle &= \sum_{v\in V(\operatorname{lk} F)} \deg_{\operatorname{lk} F}(v) f^F(\{v\})\\ &= \sum_{v\in V(\operatorname{lk} F)} \deg(F_v)[F_v : F]f(F_v) = 0.\end{aligned}$$

$\square$

The statements of Lemma 4.7 can easily be combined to prove Garland's estimate:

Proof of Theorem 4.6. Let $f \in B^{k-1}(X)^{\perp}$. Then

$$\langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{\mathrm{up},F}(X) f, f \rangle = \sum_{F \in \mathcal{F}_f} \langle \Delta(\mathrm{lk}\, F) f_F, f_F \rangle,$$

where $\mathcal{F}_f = \{F \in X_{k-2} | F \subset G \text{ for some } G \text{ with } f(G) \neq 0\}$. Now, since $f \in B^{k-1}(X)^{\perp}$, we have $f_F \in \mathbf{1}^{\perp}$ and $f_F \neq 0$ for $F \in \mathcal{F}_f$. As furthermore $\sum_{F \in \mathcal{F}_f} \langle f_F, f_F \rangle = k \langle f, f \rangle$,

$$k \lambda_{\min} \langle f, f \rangle \leq \langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{\mathrm{up},F}(X) f, f \rangle \leq k \lambda_{\max} \langle f, f \rangle.$$

By Lemma 4.7 we have furthermore

$$\langle \Delta_{k-1}^{\mathrm{up}}(X) f, f \rangle = \langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{\mathrm{up},F}(X) f, f \rangle - (k-1) \langle f, f \rangle,$$

which concludes the proof. □

Adjacency Matrix

We now turn to the generalized adjacency matrix $A_{k-1}(X)$. The same methods as above can be applied to achieve a result of similar nature (Proposition 4.10). However, this only enables us to cover vectors from $B^{k-1}(X)^{\perp}$. Controlling the behaviour on this space sufficed for the normalized Laplacian, where $B^{k-1}(X)$ is always a subspace of the eigenspace of zero. For the generalized adjacency matrix we know much less about its eigenspaces, in particular we don't know of any trivial eigenvalues.

This is analogous to the situation for graphs, where $\mathbf{1}$, the all-ones vector, which is known to be the first eigenvector of the Laplacian (with eigenvalue 0), is not necessarily an eigenvector of the adjacency matrix. In [42] Feige and Ofek, considering the adjacency matrix of random graphs $G(n, p)$, show that for p large enough the first eigenvector can in some sense be replaced by $\mathbf{1}$. Following their strategy, we show that controlling the behaviour of the generalized adjacency matrix $A_{k-1}(X)$ on the two spaces $B^{k-1}(X)$ and $B^{k-1}(X)^{\perp}$ suffices to give concentration results for the spectrum of $A_{k-1}(X)$.

The results of this section together will yield the following theorem which can be considered as an analogue of Garland's Theorem 4.6 for the generalized adjacency matrix $A_{k-1}(X)$.

Theorem 4.8. *Let X be a k-dimensional simplicial complex with n vertices and complete $(k-1)$-skeleton and let $A_{k-1} = A_{k-1}(X)$ be its generalized adjacency matrix. Fix a positive value d and let $u = (1/\sqrt{n-k+1})\mathbf{1}$. Suppose that we have for all $F \in X_{k-2}$:*

(i) $|\langle A(\operatorname{lk} F)u, u\rangle - d| \leq f(n)$,

(ii) $|\langle A(\operatorname{lk} F)u, w\rangle| \leq g(n)$ *for all* $w \perp \mathbf{1}$ *with* $\|w\| = 1$ *and*

(iii) $|\langle A(\operatorname{lk} F)w, w\rangle| \leq h(n)$ *for all* $w \perp \mathbf{1}$ *with* $\|w\| = 1$.

Let $\varphi(n) = f(n) + g(n) + h(n)$. *Then:*

(a) $|\langle A_{k-1}b, b\rangle - d| \leq k \cdot \varphi(n)$ *for all* $b \in B^{k-1}(X)$ *with* $\|b\| = 1$,

(b) $|\langle A_{k-1}b, z\rangle| \leq k \cdot \varphi(n)$ *for all* $z \in B^{k-1}(X)^{\perp}$ *and* $b \in B^{k-1}(X)$ *with* $\|b\| = \|z\| = 1$ *and*

(c) $|\langle A_{k-1}z, z\rangle| \leq k \cdot h(n)$ *for all* $z \in B^{k-1}(X)^{\perp}$ *with* $\|z\| = 1$.

Hence, the largest $\binom{n-1}{k-1}$ *eigenvalues of* A_{k-1} *lie in the interval*

$$[d - k\varphi(n), d + 2k\varphi(n) + kh(n)],$$

and the remaining $\binom{n-1}{k}$ *eigenvalues lie in* $[-k(\varphi(n) + h(n)), kh(n)]$.

The following lemma explains the connection of Conclusions (a), (b) and (c) with the spectrum of $A_{k-1}(X)$. It is a generalization of [42, Lemma 2.1], which gives the a corresponding statement for graphs and deals with a single vector u, here replaced by the subspace $\mathcal{B}$, and is then used with $u = \frac{1}{\sqrt{n}}\mathbf{1}$. We will use $\mathcal{B} = B^{k-1}(X)$. Note that $B^{k-1}(X) = B^{k-1}(K_n^k)$ if X has a complete $(k-1)$-skeleton.

Lemma 4.9. *Let X be a k-dimensional simplicial complex with n vertices and complete $(k-1)$-skeleton, let $A_{k-1} = A_{k-1}(X)$ be its generalized adjacency matrix and let $\mathcal{B}$ be an $\binom{n-1}{k-1}$-dimensional subspace of $C^{k-1}(X)$. Suppose we have:*

(i) $0 \leq f_1(n) \leq \langle A_{k-1}b, b\rangle \leq f_2(n)$ *for all* $b \in \mathcal{B}$ *with* $\|b\| = 1$,

(ii) $|\langle A_{k-1}b, z\rangle| \leq g(n)$ *for all* $z \in \mathcal{B}^{\perp}$ *and* $b \in \mathcal{B}$ *with* $\|b\| = \|z\| = 1$,

(iii) $|\langle A_{k-1}z, z\rangle| \leq h(n)$ *for all* $z \in \mathcal{B}^{\perp}$ *with* $\|z\| = 1$.

Then the largest $\binom{n-1}{k-1}$ eigenvalues of A_{k-1} are contained in the interval

$$[f_1(n), f_2(n) + g(n) + h(n)],$$

and the remaining $\binom{n-1}{k}$ eigenvalues lie in $[-(g(n) + h(n)), h(n)]$.

Proof of Lemma 4.9. Write $A = A_{k-1}$. Let v be an arbitrary unit vector. Then there are unit vectors $b \in \mathcal{B}$, $z \in \mathcal{B}^\perp$ and $0 \leq \alpha, \beta \leq 1$ such that $v = \alpha b + \beta z$ and $\alpha^2 + \beta^2 = 1$. Because A is symmetric, we get

$$\langle Av, v\rangle = \alpha^2 \langle Ab, b\rangle + 2\alpha\beta\langle Ab, z\rangle + \beta^2\langle Az, z\rangle.$$

Using (i),(ii) and (iii) as well as $\alpha\beta \leq 1/2$ and $0 \leq \alpha, \beta \leq 1$, we can conclude that

$$-g(n) - h(n) \leq \langle Av, v\rangle \leq f_2(n) + g(n) + h(n).$$

Hence, all eigenvalues of A lie in $[-g(n) - h(n), f_2(n) + g(n) + h(n)]$. Now, let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{\binom{n}{k}}$ be the eigenvalues of A. Applying (i) and (iii) we get

$$\lambda_{\binom{n-1}{k}} \leq \max_{\substack{z \in \mathcal{B}^\perp, \\ \|z\|=1}} \langle Az, z\rangle \leq h(n) \quad \text{and} \quad \lambda_{\binom{n-1}{k}+1} \geq \min_{\substack{b \in \mathcal{B}, \\ \|b\|=1}} \langle Ab, b\rangle \geq f_1(n),$$

by the variational characterization of eigenvalues (Theorem 2.1), since $\dim \mathcal{B}^\perp = \binom{n-1}{k}$. □

The proof of Theorem 4.8 makes up the remainder of this section and is divided into two parts. We first deal with Conclusion (c) and then turn to Conclusions (a) and (b).

Conclusion (c) - Behaviour on $B^{k-1}(X)^\perp$

We address Conclusion (c) with the same methods that we used to prove Garland's Theorem 4.6.

Proposition 4.10. *Let X be a k-dimensional simplicial complex and let $A_{k-1} = A_{k-1}(X)$ be its generalized adjacency matrix. Assume that for all $F \in X_{k-2}$ and for all $w \in C_0(\operatorname{lk} F)$ with $w \perp \mathbf{1}$*

$$|\langle A(\operatorname{lk} F)w, w\rangle| \leq h(n)\langle w, w\rangle.$$

Then for all $z \in B^{k-1}(X)^\perp$ (where the orthogonal complement is taken with respect to the standard, non-weighted inner product)

$$|\langle A_{k-1}z, z\rangle| \leq k \cdot h(n)\langle z, z\rangle.$$

Proof. For any face $F \in X_{k-2}$ set $A_{k-1}^F := \rho_F A_{k-1} \rho_F$, the matrix obtained from A_{k-1} by replacing all rows and columns corresponding to $(k-1)$-faces not containing F by all-zero rows/columns. Similar as in Lemma 4.7, straightforward calculations show:

a) $\sum_{F \in X_{k-2}} A_{k-1}^F = A_{k-1}$,

b) $(A_{k-1}^F)_{F\cup\{u\},F\cup\{v\}} = [F \cup \{u\} : F][F \cup \{v\} : F]A(\operatorname{lk} F)_{u,v}$ for $F \in X_{k-2}$ and $u, v \in V(\operatorname{lk} F)$ and hence $\langle A_{k-1}^F f, f\rangle = \langle A(\operatorname{lk} F) f_F, f_F\rangle$ for any $f \in C^{k-1}(X)$.

As $z \in B^{k-1}(X)^\perp$ implies $z_F \in \mathbf{1}^\perp$ also with respect to the non-weighted inner product, this proves the proposition:

$$\begin{aligned} |\langle A_{k-1} z, z\rangle| &= \Big| \sum_{F \in X_{k-2}} \langle A_{k-1}^F z, z\rangle \Big| \\ &\leq \sum_{F \in X_{k-2}} |\langle A(\operatorname{lk} F) z_F, z_F\rangle| \leq k \cdot h(n) \langle z, z\rangle. \end{aligned}$$

□

As explained above, in contrast to the Laplacian, for the adjacency matrix we are also interested in the behaviour on $B^{k-1}(X)$. For this space, we can't apply a proof similar to the one above because $f \in B^{k-1}(X)$ doesn't imply that f^F is constant for every $F \in X_{k-2}$. (For a k-dimensional complex with complete $(k-1)$-skeleton, the basis vectors $\delta_{k-2} e_F$ are a simple counterexample.)

Conclusions (a) and (b) - Behaviour on $B^{k-1}(X)$

For $b \in B^{k-1}(X)$ we have $A_{k-1}(X)b = D_{k-1}(X)b$. If the complex X was regular, i.e. all $(k-1)$-faces would have the same degree d, $B^{k-1}(X)$ would be a subspace of the eigenspace of d.

The random complex $X^k(n, p)$ is not regular but with high probability the degrees of all $(k-1)$-faces lie close to the expected average degree $d = p(n-1)$. For an arbitrary complex we can fix any positive value d and study the divergences of the degrees from d by considering the diagonal matrix $E(X) = D_{k-1}(X) - dI$ which has entries $E(X)_{F,F} = \deg_X(F) - d$. Then $A_{k-1}(X)b = E(X)b + db$ for $b \in B^{k-1}(X)$.

It will turn out that our main task is to control the behaviour of $\|E(X)b\|$ for all $b \in B^{k-1}(X)$. We manage to reduce this to a question on the links of $(k-2)$-faces: Proposition 4.11 relates $\|Eb\|$ for every $b \in B^{k-1}(X)$ to the values $\|E\delta_{k-2}e_F\|$ for $F \in X_{k-2}$, to the behaviour of E on the coboundaries of elementary cochains. These values in turn match the values $\|E(\operatorname{lk} F)\mathbf{1}\|$ on the corresponding links.

Proposition 4.11. *Let X be a k-dimensional complex with vertex set $[n]$ and complete $(k-1)$-skeleton. Fix some positive value d and let $E = E(X) = D_{k-1}(X) - dI$. Assume that for all $F \in X_{k-2}$ we have*

$$\|E\delta e_F\| \leq f(n)\|\delta e_F\| = f(n)\sqrt{n-k+1}.$$

Then for all $b \in B^{k-1}(X)$

$$\|Eb\| \leq k \cdot f(n)\|b\|.$$

Remark 4.12. Proposition 4.11 also holds if E is replaced by any diagonal $|X_{k-1}| \times |X_{k-1}|$-matrix.

The proof of Proposition 4.11 is deferred to the end of this section. Here is how we use it to address Conclusions (a) and (b).

Proposition 4.13. *Let X be a k-dimensional simplicial complex with n vertices and complete $(k-1)$-skeleton. Fix some postive value d and suppose that we have*

$$\sum\nolimits_{v\in V(\operatorname{lk} F)} (\deg_{\operatorname{lk}(F)}(v) - d)^2 = \|E(\operatorname{lk} F)\mathbf{1}\|^2 \leq f(n)^2(n-k+1)$$

for all $F \in X_{k-2}$. Then

(i) $|\langle A_{k-1}b, b\rangle - d| \leq k \cdot f(n)$ for all $b \in B^{k-1}(X)$ with $\|b\| = 1$ and

(ii) $|\langle A_{k-1}b, z\rangle| \leq k \cdot f(n)$ for all $b \in B^{k-1}(X)$, $z \in B^{k-1}(X)^\perp$ with $\|b\| = \|z\| = 1$.

Proof. As $\deg(F \cup \{v\}) = \deg_{\operatorname{lk} F}(v)$ for $v \notin F$, we have

$$\begin{aligned}\|E\delta e_F\|^2 &= \sum\nolimits_{H\supset F} (\deg(H) - d)^2 \\ &= \sum\nolimits_{v\notin F} (\deg_{\operatorname{lk} F}(v) - d)^2 \leq f(n)^2(n-k+1).\end{aligned}$$

By Proposition 4.11 this implies that we have $\|Eb\| \leq k \cdot f(n)\|b\|$ for all $b \in B^{k-1}(X)$. Now, let $b \in B^{k-1}(X)$ and $z \in B^{k-1}(X)^\perp$. As $A_{k-1}b = D_{k-1}b = db + Eb$, we get

$$|\langle A_{k-1}b, b\rangle - d\|b\|^2| \leq \|b\| \cdot \|Eb\| \leq k \cdot f(n)\|b\|^2$$

and

$$|\langle A_{k-1}b, z\rangle| \le |\langle Eb, z\rangle| \le \|z\| \cdot \|Eb\| \le k \cdot f(n)\|z\|\|b\|.$$

□

To conclude the proof of Theorem 4.8 we are missing a small lemma:

Lemma 4.14. *Let G be a graph with n vertices with adjacency matrix $A = A(G)$ and let $u = \frac{1}{\sqrt{n}}\mathbf{1}$. Fix a positive value d. Assume that*

(i) $|\langle Au, u\rangle - d| \le f(n)$,

(ii) $|\langle Au, w\rangle| \le g(n)$ *for all* $w \perp \mathbf{1}$ *with* $\|w\| = 1$ *and*

(iii) $|\langle Aw, w\rangle| \le h(n)$ *for all* $w \perp \mathbf{1}$ *with* $\|w\| = 1$.

Then $\|E(G)\mathbf{1}\|^2 = \sum_{v \in V}(\deg(v) - d)^2 \le (f(n) + g(n) + h(n))^2 n$.

Proof. We have

$$\|E(G)\mathbf{1}\| = \|(\frac{d}{n}J - A)\mathbf{1}\| \le \|\frac{d}{n}J - A\| \cdot \|\mathbf{1}\|.$$

Furthermore, the conditions above imply $\|\frac{d}{n}J - A\| \le f(n)+g(n)+h(n)$. This can be seen by arguments similar to the ones used in Lemma 4.3.

□

Proof of Proposition 4.11

The proof of Propositon 4.11 is based on the observations in the following lemma. Its proof will use the following simple consequence of the Cauchy-Schwarz inequality:

$$\Big(\sum_{i \in I} a_i\Big)^2 \le |I| \sum_{i \in I} a_i^2. \tag{4.2}$$

Lemma 4.15. *Let X be a k-complex with vertex set $[n]$ and complete $(k-1)$-skeleton and let $b \in B^{k-1}(X)$. For every $(k-2)$-face $F \in X_{k-2}$ define*

$$h_b(F) := \sum_{v \notin F} [F \cup \{v\} : F] b(F \cup \{v\}).$$

Then

a) $b(H) = \frac{1}{n}\sum_{F \subset H, F \in X_{k-2}} [H : F] h_b(F)$ *for* $H \in X_{k-1}$, *b)* $\langle Eb, Eb\rangle \leq \frac{k}{n^2}\sum_{F\in X_{k-2}} h_b(F)^2\langle E\delta e_F, E\delta e_F\rangle$,

c) $\sum_{F\in X_{k-2}} h_b(F)^2 \leq k(n-k+1)\langle b, b\rangle$.

Proof. a) As X has a complete $(k-1)$-skeleton, we have $B^{k-1}(X) = B^{k-1}(K_n^k)$ and hence $\delta_{k-1}(K_n^k)b = 0$. Thus, for any $H \in X_{k-1}$ and $v \notin H$:

$$\begin{aligned}0 &= (\delta_{k-1}(K_n^k)b)(H\cup\{v\})\\ &= [H\cup\{v\} : H]b(H) + \sum_{F\subset H}[H\cup\{v\} : F\cup\{v\}]b(F\cup\{v\}).\end{aligned}$$

Note that $-[H\cup\{v\} : H][H\cup\{v\} : F\cup\{v\}] = [H : F][F\cup\{v\} : F]$. Thus, we can rearrange:

$$\begin{aligned}b(H) &= -[H\cup\{v\} : H]\sum_{F\subset H}[H\cup\{v\} : F\cup\{v\}]b(F\cup\{v\})\\ &= \sum_{F\subset H}[H : F][F\cup\{v\} : F]b(F\cup\{v\}).\end{aligned}$$

Summing over all $v \notin H$ and adding additional multiples of $b(H)$, we get

$$\begin{aligned}n\cdot b(H) &= \sum_{v\notin H}\sum_{F\subset H}[H : F][F\cup\{v\} : F]b(F\cup\{v\}) + k\cdot b(H)\\ &= \sum_{F\subset H}[H : F]\sum_{v\notin F}[F\cup\{v\} : F]b(F\cup\{v\})\\ &= \sum_{F\subset H}[H : F]h_b(F).\end{aligned}$$

b) By a) and (4.2) and because $\langle E\delta e_F, E\delta e_F\rangle = \sum_{H\supset F} E(H)^2$ for $F \in X_{k-2}$ we get:

$$\begin{aligned}\langle Eb, Eb\rangle &= \sum_{H\in X_{k-1}} E(H)^2 b(H)^2\\ &= \frac{1}{n^2}\sum_{H\in X_{k-1}} E(H)^2\Big(\sum_{F\subset H}[H : F]h_b(F)\Big)^2\\ &\leq \frac{k}{n^2}\sum_{H\in X_{k-1}} E(H)^2\sum_{F\subset H} h_b(F)^2\\ &= \frac{k}{n^2}\sum_{F\in X_{k-2}} h_b(F)^2\langle E\delta e_F, E\delta e_F\rangle.\end{aligned}$$

c) Again by (4.2):

$$\begin{aligned}\sum_{F\in X_{k-2}} h_b(F)^2 &\leq \sum_{F\in X_{k-2}} (n-k+1)\cdot \sum_{v\notin F} b(F\cup\{v\})^2 \\ &= (n-k+1)\cdot \sum_{H\in X_{k-1}} k\cdot b(H)^2 \\ &= k(n-k+1)\langle b,b\rangle.\end{aligned}$$

□

The statements of Lemma 4.15 together yield Proposition 4.11:

Proof of Propositon 4.11. Let $b\in B^{k-1}(X)$. As $\|\delta e_F\| = \sqrt{n-k+1}$ for $F\in X_{k-2}$, by Lemma 4.15:

$$\begin{aligned}\langle Eb,Eb\rangle &\leq \frac{k}{n^2}\sum_{F\in X_{k-2}} h_b(F)^2\langle E\delta e_F, E\delta e_F\rangle \\ &\leq \frac{k}{n^2}\sum_{F\in X_{k-2}} h_b(F)^2 f(n)\langle \delta e_F,\delta e_F\rangle \\ &\leq k^2\cdot\frac{(n-k+1)^2}{n^2}\cdot f(n)\langle b,b\rangle \leq k^2\cdot f(n)\langle b,b\rangle.\end{aligned}$$

□

4.3 The Spectra of Random Complexes

In this section, we prove Theorem 4.2, the concentration result on the spectra of the normalized Laplacian and the generalized adjacency matrix of random complexes $X^k(n,p)$. The basic idea is to reduce the statement to a question on the links of $(k-2)$-faces by applying Theorems 4.6 and 4.8. Since for every $(k-2)$-face F, the link $\mathrm{lk}(F, X^k(n,p))$ is a random graph with the same distribution as $G(n-k+1,p)$, we can then apply results on the eigenvalues of random graphs. For convenience, we repeat Theorem 4.2:

Theorem 4.2. *For all $c>0$ and $k\geq 1$ there exists a constant $C = C(c,k) > 0$ with the following property: Assume $p\geq C\frac{\log(n)}{n}$ and let $d := p(n-k)$. Then there exist $\gamma_A = O(\sqrt{d})$ and $\gamma_\Delta = O(1/\sqrt{d})$ such that the following statements hold with probability at least $1-n^{-c}$:*

(i) *The largest* $\binom{n-1}{k-1}$ *eigenvalues of* $A_{k-1}(X^k(n,p))$ *lie in the interval* $[d-\gamma_A, d+\gamma_A]$, *and the remaining* $\binom{n-1}{k}$ *eigenvalues lie in the interval* $[-\gamma_A, +\gamma_A]$.

(ii) *The smallest* $\binom{n-1}{k-1}$ *eigenvalues of* $\Delta^{\mathrm{up}}_{k-1}(X^k(n,p))$ *are (trivially) zero, and the remaining* $\binom{n-1}{k}$ *eigenvalues are contained in the interval* $[1-\gamma_\Delta, 1+\gamma_\Delta]$. *In particular,* $\tilde{H}^{k-1}(X^k(n,p);\mathbb{R}) = 0$.

Observe that $B^{k-1}(K_n^k) \subseteq \ker \Delta^{\mathrm{up}}_{k-1}(X^k(n,p))$ because $X^k(n,p)$ has a complete $(k-1)$-skeleton, so the multiplicity of 0 as an eigenvalue of $\Delta^{\mathrm{up}}_{k-1}(X^k(n,p))$ is at least $\binom{n-1}{k-1}$.

Proof of Theorem 4.2. Let $c > 0$. For $F \in \binom{n}{k-1}$, the link $\operatorname{lk} F = \operatorname{lk}(F, X^k(n,p))$ is a random graph $G(n-k+1,p)$. By Theorems 4.1 and 4.4 we can hence choose constants $C > 0$ and $c', c'' > 0$ such that for $p \geq C\log(n)/n$ the following holds with probability at least $1-n^{-c-k+1}$: $\|pJ - A(\operatorname{lk} F)\| < c'\sqrt{d}$ and furthermore all nontrivial eigenvalues of $\Delta(\operatorname{lk} F)$ are contained in the interval $[1 - c''/(k\sqrt{d}), 1 + c''/(k\sqrt{d})]$.

We first focus on the adjacency matrix: A union bound yields that for $p \geq C\log(n)/n$

$$\Pr\left[\exists F \in X_{k-2} : \|pJ - A(\operatorname{lk} F)\| > c'\sqrt{d}\right] \leq n^{-c}.$$

By Lemma 4.3 this implies that the conditions of Theorem 4.8 with $f(n), g(n), h(n) = O(\sqrt{d})$, and hence the desired concentration bounds, are fulfilled with probability at least $1 - n^{-c}$.

Now, consider the normalized Laplacian $\Delta^{\mathrm{up}}_{k-1}(X^k(n,p))$. Again, with a union bound we get for $p \geq C\log(n)/n$ that with probability $1 - n^{-c}$ we have

$$\forall F \in X_{k-2}\ \forall i = 2, \ldots, n-k+1 : |\lambda_i(\Delta(\operatorname{lk} F)) - 1| \leq c'/(k\sqrt{d}).$$

For every $(k-1)$-face $H \in \binom{[n]}{k}$ of $X^k(n,p)$, the random variable $\deg(H)$ is binomially distributed with parameters $(n-k)$ and p. By making C slightly larger, if necessary, we can ensure that for $p \geq C \cdot \log n/n$, the complex $X^k(n,p)$ is pure with probability at least n^{-c}. Hence, also the conditions of Theorem 4.6 are fulfilled with probability at least $1 - n^{-c}$. □

Remark 4.16. Note that that the preceding proof works for any random distribution $\mathcal{X}_k(n,p)$ on k-dimensional simplicial complexes with n

vertices and complete $(k-1)$-skeleton with the property that the link $\mathrm{lk}(F, \mathcal{X}_k(n,p))$ of every $F \in \binom{[n]}{k-1}$ is a random graph with distribution $G(n-k+1,p)$.

Concluding Remarks We have given a brief survey of higher-dimensional expansion properties of simplicial complexes and shown concentration results for the eigenvalues of the Laplacian and of the adjacency matrix of random complexes in the Linial–Meshulam model. In order to prove the result for the adjacency matrix, we presented an analogue of Garland's estimate for adjacency matrices.

Chapter 5

On Higher-Dimensional Discrete Cheeger Inequalities

In this chapter we explore possible higher-dimensional analogues of the discrete Cheeger inequality, Theorem 2.2. The most straightforward attempt at such an inequality would be to relate combinatorial expansion and eigenvalue gaps of higher-dimensional Laplacians. We will see in the first section that this attempt fails: In higher dimensions, a large eigenvalue gap for the normalized Laplacian does not imply combinatorial expansion.

A higher-dimensional Cheeger inequality for a different, more combinatorial, notion of expansion (see Section 2.3) was proven in [97]. In the second section of this chapter we show an extension of this result for 2-dimensional simplicial complexes with complete 1-skeleton.

The result presented in the first section of this chapter is joint work with Uli Wagner and is based on the extended abstract [58].

5.1 Spectral vs. Combinatorial Expansion

We will now show, by a simple probabilistic construction, that the most straightforward attempt at a higher-dimensional Cheeger inequality fails, even for the "easy part" (see the discussion after Theorem 2.2). In higher dimensions, spectral expansion (an eigenvalue gap for the Laplacian) does not imply combinatorial expansion:

Theorem 5.1. *For every $k > 1$ there is an infinite family of k-dimensional complexes $\{Y_n : n \in \mathbb{N}\}$, where Y_n has n vertices, with the following properties:*

- *All non-trivial eigenvalues of* $\Delta^{\mathrm{up}}_{k-1}(Y_n)$ *are* $1 \pm o(1)$*. Thus,* Y_n *is spectrally expanding and* $H^{k-1}(Y_n;\mathbb{R}) = 0$*.*
- *Every* Y_n *has a non-trivial cohomology class* $[a] \in H^{k-1}(Y_n;\mathbb{Z}_2)$ *of normalized Hamming weight* $\|[a]\| \geq \frac{1}{2} - o(1)$*. So,* Y_n *is not combinatorially expanding,* $\varepsilon(X) = 0$*; in particular,* $H^{k-1}(Y_n;\mathbb{Z}_2) \neq 0$*.*

For a graph G and any field $\mathbb{F}$, we saw in Section 1.2 that $\tilde{H}^0(G;\mathbb{F}) = 0$ if and only if G is connected. In Section 2.3 we discussed that in higher dimensions, this does not hold: The vanishing of cohomology groups can depend on the choice of coefficients. An example distinguishing between $\mathbb{Z}_2$- and $\mathbb{R}$-coefficients is the real projective plane $\mathbb{R}P^2$, for which $H^1(\mathbb{R}P^2;\mathbb{R}) = 0$ but $H^1(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2$.

Thus, the aim of Theorem 5.1 is not to prove the existence of complexes with trivial real cohomology and non-trivial $\mathbb{Z}_2$-cohomology. Instead the point is to show that for infinite families of combinatorially non-expanding complexes the spectral expansion does not need to converge to zero, but can be bounded away from zero.

The question of the existence of a higher-dimensional Cheeger inequality was, e.g., raised explicitly by Dotterrer and Kahle [38].

The Construction. The examples in Theorem 5.1 are obtained using the following probabilistic construction: Choose a map $a\colon \binom{[n]}{k} \to \mathbb{Z}_2$ randomly by setting $a(F) = 1$ with probability p and $a(F) = 0$ otherwise, independently for each $F \in \binom{[n]}{k}$. Thus, the support of a has the same distribution as the set of $(k-1)$-faces of the Linial-Meshulam random complex $X^{k-1}(n,p)$.

We then denote by $Y^k(n,p)$ the random k-dimensional simplicial complex with vertex set $V = [n]$ and complete $(k-1)$-skeleton obtained as follows: For every $H \in \binom{V}{k+1}$, we add H as a k-face to $Y^k(n,p)$ if and only if H contains an *even* number of $(k-1)$-faces F with $a(F) = 1$.

Note that, by construction, a is a $\mathbb{Z}_2$-*cocycle* in the complex $Y^k(n,p)$, i.e., $a \in Z^{k-1}(Y^k(n,p);\mathbb{Z}_2)$. For simplicity, we concentrate on the case $p = 1/2$ from now on and write Y instead of $Y^k(n,1/2)$.

A Non-Trivial $\mathbb{Z}_2$-Cohomology Class. Note that the Hamming norm of $a + b$ for any fixed cochain $b \in C^{k-1}(Y;\mathbb{Z}^2)$ is binomially distributed with parameters $\binom{n}{k}$ and $1/2$. Hence, the expected normalized Hamming distance between b and the randomly chosen a is $1/2$.

Since there are fewer than $2^{\binom{n}{k-1}}$ *coboundaries* $b \in B^{k-1}(Y;\mathbb{Z}^2)$ and $\binom{n}{k}$ independent random choices for the entries of a, a straightforward application of the Chernoff bound (Theorem 1.2) in combination with a union bound over all coboundaries implies that

$$\Pr[\exists b \in B^{k-1}(Y;\mathbb{Z}^2) : \|a+b\| \leq 1/2 - t] \leq e^{\binom{n}{k-1} - t^2/\binom{n}{k}} = o(1)$$

for, say $t = \log(n) \cdot n^{-1/2}$. Hence, a.a.s., a has normalized Hamming distance $1/2 - o(1)$ from any coboundary, i.e., $\|[a]\| \geq 1/2 - o(1)$.

Spectral Gap. For $H \in \binom{V}{k+1}$, the probability that H is a k-face of Y equals $1/2$. However, in contrast to the model $X^k(n, 1/2)$, the decisions for different k-faces that share some $(k-1)$-face are not independent. Nevertheless, we can still easily analyze the links of $(k-2)$-faces in Y:

Lemma 5.2. *For every $(k-2)$-face $F \in Y_{k-2} = \binom{V}{k-1}$, the random graph $\mathrm{lk}_Y(F)$ has the distribution $G(n-k+1, 1/2)$.*

Proof. Let $U := V \setminus F$. For $e \in \binom{U}{2}$, consider the event that $e \in \mathrm{lk}_Y(F)$, i.e., that $F \cup e \in Y$. We need to show that these events are mutually independent.

To see this, choose and fix, for each $e \in \binom{U}{2}$, an arbitrary $(k-1)$-simplex G_e with $e \subseteq G_e \subseteq F \cup e$; we call these the "*undecided*" $(k-1)$-simplices, and let

$$\mathcal{D} := \binom{V}{k} \setminus \{G_e \colon e \in \binom{U}{2}\}$$

be the set of remaining, "*decided*" $(k-1)$-simplices. Note that, by construction, each k-simplex of the form $F \cup e$, $e \in \binom{U}{2}$, contains exactly one undecided $(k-1)$-simplex G_e and that these are pairwise distinct.

Fix a map $r \colon \mathcal{D} \to \mathbb{Z}_2$ and condition upon the event that r is the restriction of a to $\mathcal{D}$. For each $e \in \binom{U}{2}$, we have $e \in \mathrm{lk}_Y(F)$ if and only if $a(G_e) = \sum_{G \in \mathcal{D}, G \subset F \cup e} r(G)$. For a fixed r, the (conditional) probability of this happening is $1/2$, and the values $a(G_e)$ are mutually independent since the G_e are pairwise distinct. Thus, for any set of edges $e_1, \ldots, e_\ell \in \binom{U}{2}$ and for any fixed r, we get the conditional probability

$$\Pr[\forall i : e_i \in \mathrm{lk}_Y(F) \mid a|_{\mathcal{D}} = r] = (1/2)^\ell.$$

Since this holds for all choices of r, it also holds unconditionally, which proves the lemma. $\square$

By this lemma and Remark 4.16, we can proceed as in the proof of Theorem 4.2 to show that there exists a constant $c > 0$ such that a.a.s. the non-trivial part of the spectrum of $\Delta_{k-1}^{\mathrm{up}}(Y)$ lies in the interval $[1 - \frac{c}{\sqrt{n-k}}, 1 + \frac{c}{\sqrt{n-k}}]$. This completes the proof of Theorem 5.1.

Related Work. The probabilistic construction of the examples in Theorem 5.1 is well-known in the study of quasirandomness for hypergraphs, see, e.g., the discussion in [55, Section 5]. In [24, Section 8], it is asserted, but without proof, that the eigenvalues of the combinatorial Laplacian of these examples are concentrated in an interval of width $O(\sqrt{n})$, but we are not aware of a proof appearing in the literature.

A recent preprint by Steenbergen, Klivans and Mukherjee [104], published after the publication of the extended abstract [58], also presents a class of counterexamples for this attempt at a higher-dimensional Cheeger inequality. They give an explicit construction for an infinite family of simplicial k-balls X_n whose spectral expansion is bounded away from zero, but who (while being combinatorially expanding) satisfy $\lim_{n\to\infty} \varepsilon(X_n) = 0$. These examples are a bit stronger in the sense that they show that assuming that $\tilde{H}^{k-1}(X;\mathbb{Z}_2) = 0$ will not help. The paper furthermore contains a Cheeger-type inequality for a different set-up (for the chain complex as opposed to the cochain complex) for a certain class of simplicial complexes.

Using a different notion of combinatorial expansion, the one discussed in Section 2.3, but the same notion of Laplacian spectra we consider here, Parzanchevski, Rosenthal and Tessler show a higher-dimensional Cheeger inequality in their preprint [97]. We will look at their result in more detail in the upcoming section.

5.2 A Cheeger-type Inequality

In the previous section we observed that for higher-dimensional complexes, the topologically motivated notion of combinatorial expansion seems to have no direct connection to the spectrum of the Laplacian of a complex. In contrast, Parzanchevski, Rosenthal and Tessler [97] were able to show an analogue of the "easy part" of the discrete Cheeger inequality for their, more combinatorially inspired, notion of expansion (see Section 2.3).

Theorem 5.3 ([97, 105])**.** *Let X be a k-dimensional simplicial complex. Then*

$$\lambda(X) \leq h(X),$$

where $\lambda(X)$ is the smallest eigenvalue of the Laplacian $L^{\mathrm{up}}_{k-1}(X)$ on $(B^{k-1}(X))^{\perp}$.

In [97] this result appears only for k-complexes with complete $(k-1)$-skeleton. It was extended to general complexes by Szedlák [105], for an adapted definition of $h(X)$. Parzanchevski, Rosenthal and Tessler also present several other results, including an analogue of the Expander Mixing Lemma (Theorem 2.3), which we will present (and apply) in Chapter 8.

In this section we present a further extension of their analogue of the discrete Cheeger inequality. For any k-dimensional complex X, define its *k-dimensional completion* as

$$K(X) := X \cup \{F \in \tbinom{V}{k+1} \colon F \setminus \{v\} \in X \text{ for all } v \in F\}.$$

If X has a complete $(k-1)$-skeleton, we get $K(X) = K_n^k$, the complete k-dimensional complex on n vertices. For a partition $V = \amalg_{i=0}^k A_i$ let $F(A_0, A_1, \ldots, A_{k-1})$ be the set of $(k-1)$-dimensional faces of X with exactly one vertex in each set A_i, $i = 0, 1, \ldots, k-1$, We show:

Proposition 5.4. *Let X be a k-dimensional simplicial complex. Define*

$$h'(X) := \min_{\substack{V=\bigsqcup_{i=0}^{k-1} A_i,\, f\in C^{k-1}(X,\mathbb{Z}_2),\\ \operatorname{supp}(f)\subset F(A_0,A_1,\ldots,A_{k-1})}} \frac{|V|\cdot|\delta_X f|}{|\delta_{K(X)} f|}.$$

For $|\delta_{K(X)} f| = 0$, we define $\frac{|V|\cdot|\delta_X f|}{|\delta_{K(X)} f|} = \infty$. Then

$$\lambda(X) \leq h'(X),$$

where $\lambda(X)$ is the smallest eigenvalue of the upper Laplacian $L_{k-1}^{\mathrm{up}}(X)$ on $(B^{k-1}(X))^\perp$.

Recall that for a k-dimensional simplicial complexes X with a complete $(k-1)$-skeleton

$$h(X) = \min_{\substack{V=\bigsqcup_{i=0}^{k} A_i,\\ A_i\neq\emptyset}} \frac{|V|\cdot|\delta_X f_{A_0,A_1,\ldots,A_k}|}{|\delta_{K_n^k} f_{A_0,A_1,\ldots,A_k}|},$$

where for a partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_k = V$ we define the $\mathbb{Z}_2$-cochain $f_{A_0,A_1,\ldots,A_k} \in C^{k-1}(X,\mathbb{Z}_2)$ by $f_{A_0,A_1,\ldots,A_k}(F) = 1$ for an edge $F \in F(A_0, A_1, \ldots, A_{k-1})$ and $f_{A_0,A_1,\ldots,A_k}(F) = 0$ otherwise. So, $f_{A_0,A_1,\ldots,A_k}$ is the characteristic function of the set $F(A_0, A_1, \ldots, A_{k-1})$, and clearly $\operatorname{supp}(f_{A_0,A_1,\ldots,A_k}) \subset F(A_0, A_1, \ldots, A_{k-2}, A_{k-1} \cup A_k)$. We hence minimize over a larger set of cochains and have $h'(X) \leq h(X)$. The real projective plane shows that $h'(X) < h(X)$ is possible.

In the remainder of this section, since we consider real as well as $\mathbb{Z}_2$-cohomology, we denote the real coboundary operator by $\delta^{\mathbb{R}}$, the $\mathbb{Z}_2$-coboundary by $\delta^{\mathbb{Z}_2}$. The space of $\mathbb{Z}_2$-cochains is denoted by $C^{k-1}(X;\mathbb{Z}_2)$, the space of real cochains by $C^{k-1}(X)$ instead of $C^{k-1}(X;\mathbb{R})$. Also, $B^{k-1}(X)$ stands for $B^{k-1}(X;\mathbb{R})$.

The following lemma points out a special behaviour of the $\mathbb{Z}_2$-cochains appearing in the definition of $h'(X)$ that will be central to our argument: The size of the $\mathbb{Z}_2$-boundary of such a cochain agrees with the size of its real coboundary.

Lemma 5.5. *Let X be a k-dimensional simplicial complex with vertex set V. Let $A_0, A_1, \dots, A_{k-1} \subset V$ be pairwise disjoint and let $f \in C^{k-1}(X, \mathbb{Z}_2)$ such that* $\operatorname{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$. *Choose an orientation of the simplices of X by fixing a linear ordering on V such that for all $i < j \in \{0, 1, \dots, k-1\}$, $v \in A_i$, $w \in A_j$ we have $v < w$. Then, interpreting f also as an $\mathbb{R}$-cochain with values in $\{0,1\}$, we have*

$$\|\delta^{\mathbb{R}} f\|^2 = \langle L^{\mathrm{up}}_{k-1} f, f\rangle = |\delta^{\mathbb{Z}_2} f|.$$

Here, $\|\cdot\|$ denotes the ℓ_2-norm and $\langle,\rangle$ the standard Euclidean inner product, while $|\cdot|$ denotes the Hamming norm

Proof. Note that any k-face $H \in X_k$ can have at most two $(k-1)$-faces that are contained in $F(A_0, A_1, \dots, A_{k-1})$, and the same holds for $\operatorname{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$.

For $H \in X_k$ consider $\delta^{\mathbb{R}} f(H) = \sum_{F \subset H, F \in X_{k-1}} [H : F] f(F)$. If H has no faces in $\operatorname{supp}(f)$ this sum is empty. It is ± 1 if t has exactly one face in $\operatorname{supp}(f)$. Otherwise H has exactly two faces F and F' with $f(F) = f(F') = 1$. By our choice of orientations, we have $[H : F] = -[H : F']$ and hence $\delta^{\mathbb{R}} f(H) = 0$.

This shows that $\langle L^{\mathrm{up}}_{k-1} f, f\rangle = \|\delta^{\mathbb{R}} f\|^2$ equals the number of k-faces with exactly one face in $\operatorname{supp}(f)$. As $\operatorname{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$, this is $|\delta^{\mathbb{Z}_2} f|$. □

Before we come to the proof of Proposition 5.4, we give an upper bound for the eigenvalue $\lambda(X)$. By the variational characterization of eigenvalues, $\lambda(X)$ is the minimum over all $f \in C^{k-1}(X)$ of unit norm that are orthogonal to $B^{k-1}(X)$. The key observation here is that we can get rid of this orthogonality constraint.

Lemma 5.6. *Let X be a k-complex with n vertices and let $\lambda(X)$ be the smallest eigenvalue of the upper Laplacian $L^{\mathrm{up}}_{k-1}(X)$ on $(B^{k-1}(X))^{\perp}$.*

Then

$$\lambda(X) \le \min_{\substack{f \in C^{k-1}(X), \\ f \notin B^{k-1}(X)}} \frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K(X)) f, f\rangle}. \tag{5.1}$$

If $\langle L^{\mathrm{up}}_{k-1}(K(X)) f, f\rangle = 0$, *we define* $\frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K(X)) f, f\rangle} = \infty$. *For* X *with complete* $(k-1)$*-skeleton* (5.1) *holds with equality.*

Proof. First assume that X has a complete $(k-1)$-skeleton. The following equality is contained implicitly in [68] and follows from a straightforward calculation using the matrix representations of the Laplacians:

$$L^{\mathrm{up}}_{k-1}(K^k_n) + L^{\mathrm{down}}_{k-1}(K^k_n) = nI.$$

Hence, we have

$$n\langle f, f\rangle = \langle L^{\mathrm{up}}_{k-1}(K^k_n) f, f\rangle + \langle L^{\mathrm{down}}_{k-1}(K^k_n) f, f\rangle$$

for any $f \in C^{k-1}(X) = C^{k-1}(K^k_n)$. Combining this with the variational characterization of eigenvalues and the fact that for $f \perp B^{k-1}(X) = B^{k-1}(K^k_n)$ we have $L^{\mathrm{down}}_{k-1}(K^k_n) f = 0$, we get:

$$\lambda(X) = \min_{\substack{f \in C^{k-1}(X), \\ f \perp B^{k-1}(X)}} \frac{\langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle f, f\rangle} = \min_{\substack{f \in C^{k-1}(X), \\ f \perp B^{k-1}(X)}} \frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K^k_n) f, f\rangle}.$$

For $f \notin B^{k-1}(X)$ that is not orthogonal to $B^{k-1}(X)$, let b be the projection of f onto $B^{k-1}(X)$ and let $z = f - b$. Then $z \perp B^{k-1}(X)$ and it holds that $\langle L^{\mathrm{up}}_{k-1}(X) z, z\rangle = \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle$ as well as $\langle L^{\mathrm{up}}_{k-1}(K^k_n) z, z\rangle = \langle L^{\mathrm{up}}_{k-1}(K^k_n) f, f\rangle$. This shows that we can omit the orthogonality constraint.

Now, consider the general case of a k-complex X with an arbitrary $(k-1)$-skeleton. Let $f \in C^{k-1}(X)$. We extend f to $\tilde{f} \in C^{k-1}(K^k_n)$ defined by $\tilde{f}(F) = f(F)$ if $F \in X$ and $\tilde{f}(F) = 0$ otherwise.

A straightforward calculation demonstrates that $\tilde{f} \perp B^{k-1}(K^k_n)$ if $f \perp B^{k-1}(X)$. Hence, we can argue as above to see that for $f \perp B^{k-1}(X)$ we get

$$n\langle f, f\rangle = n\langle \tilde{f}, \tilde{f}\rangle = \langle L^{\mathrm{up}}_{k-1}(K^k_n)\tilde{f}, \tilde{f}\rangle \ge \langle L^{\mathrm{up}}_{k-1}(K(X)) f, f\rangle.$$

Thus,

$$\lambda(X) = \min_{\substack{f \in C^{k-1}(X), \\ f \perp B^{k-1}(X)}} \frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K^k_n)\tilde{f}, \tilde{f}\rangle} \le \min_{\substack{f \in C^{k-1}(X), \\ f \perp B^{k-1}(X)}} \frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X) f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K(X)) f, f\rangle}.$$

For $f \notin B^{k-1}(X)$ that is not orthogonal to $B^{k-1}(X)$, we again consider the projection b of f onto $B^{k-1}(X)$. Letting $z = f - b$ note that we have $z \perp B^{k-1}(X) = B^{k-1}(K(X))$ and $\langle L^{\mathrm{up}}_{k-1}(X)z, z\rangle = \langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle$ as well as $\langle L^{\mathrm{up}}_{k-1}(K(X))z, z\rangle = \langle L^{\mathrm{up}}_{k-1}(K(X))f, f\rangle$, which shows that also in this case we can omit the orthogonality constraint □

Now we can prove Proposition 5.4:

Proof of Proposition 5.4. Let $n = |V|$. Fix sets $A_0, A_1, \ldots, A_{k-1} \subset V$ and $f \in C^{k-1}(X, \mathbb{Z}_2)$ with $\operatorname{supp}(f) \subset F(A_0, A_1, \ldots, A_{k-1})$ such that

$$h'(X) = \frac{n \cdot |\delta^{\mathbb{Z}_2}_X f|}{|\delta^{\mathbb{Z}_2}_{K(X)} f|}.$$

If $|\delta^{\mathbb{Z}_2}_{K(X)} f| = 0$, we have $h'(X) = \infty$ and there is nothing to show. Otherwise, we apply Lemmas 5.5 and 5.6 as follows: Since the value of $\lambda(X)$ does not depend on the chosen orientations of the simplices of X, we are free to choose the orientations as in Lemma 5.5, i.e., we fix a linear ordering on V such that for all $i < j$, $v \in A_i$, $w \in A_j$ we have $v < w$. Then by Lemma 5.5 we get $\langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle = |\delta^{\mathbb{Z}_2}_X f|$ and $\langle L^{\mathrm{up}}_{k-1}(K(X))f, f\rangle = |\delta^{\mathbb{Z}_2}_{K(X)} f|$. As $|\delta^{\mathbb{Z}_2}_{K(X)} f| \neq 0$, we have $f \notin B^{k-1}(X)$ and can apply Lemma 5.6 to obtain

$$\lambda(X) \leq \frac{n \cdot \langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle}{\langle L^{\mathrm{up}}_{k-1}(K(X))f, f\rangle} = h'(X).$$

□

Chapter 6

Attempts at a Criterion for Combinatorial Expansion

In this chapter we continue to explore higher-dimensional analogues of the Cheeger inequality and also of the Expander Mixing Lemma for graphs. The previous chapter showed that in higher dimensions a large eigenvalue gap for the Laplacian does not imply combinatorial expansion. This means that the most straightforward attempt at a higher-dimensional Cheeger inequality fails.

The discrete Cheeger inequality for graphs can be seen as a tool to approximate the (hard to compute) expansion of a graph by a polynomially computable quantity, the second eigenvalue of the Laplacian. We now present two other basic approaches to finding a computable lower bound for combinatorial expansion in higher dimensions. In this chapter we restrict our attention to 2-dimensional complexes.

In the first section, we study semidefinite relaxations of a combinatorial optimization problem that describes combinatorial expansion.

In the second section we follow a different approach: For graphs an eigenvalue gap for the Laplacian is one of many quasirandomness properties – properties of random graphs that can be shown to be equivalent for dense (non-random) graphs. We try to relate combinatorial expansion to a different quasirandomness property, to *octahedral quasirandomness*, as introduced in [55].

This chapter describes ongoing research and is more focused on presenting ideas than on results. It is based on joint work with Uli Wagner.

6.1 SDP Relaxations

In this section we focus on the sparsest cut problem, which, as we have seen in Sections 2.1 and 2.3, is an essentially equivalent reformulation of expansion – for graphs as well as in higher dimensions.

One way of proving the "easy part" of the discrete Cheeger inequality for graphs is to show that the second eigenvalue of the Laplacian is the solution to a certain *semidefinite program* (SDP), and that this program is a natural *relaxation* of a quadratic 0/1-program describing the sparsest cut problem. We will discuss this in more detail in a moment. This idea has first appeared in [10], where the best currently known approximation algorithm for the sparsest cut problem (of order $O(\sqrt{\log(n)})$) was presented.

We consider the 2-dimensional analogue $\phi(X)$ of the sparsest cut problem and see that it can also be described as a polynomial program in either 0/1- or (-1)/1-variables. This can then be relaxed to a polynomial program that is of degree 3 or higher. The approach we describe here is to study semidefinite relaxations of this program. As already mentioned above, we describe ongoing research. We will only describe an SDP relaxation, which if it has a finite value, would give a lower bound for $\phi(X)$. So far we have no results guaranteeing that this program yields a useful lower bound.

Semidefinite programs. A semidefinite program (SDP) is a program of the form

$$\begin{aligned} &\inf_X \quad C \bullet X \\ &\text{subject to} \quad X \succeq 0, \quad A_j \bullet X = b_j \quad (j = 1, \ldots, n), \end{aligned}$$

where the variable X ranges over all symmetric $(n \times n)$-matrices, and $b \in \mathbb{R}^m$ as well as symmetric $(n \times n)$-matrices $C, A_1, \ldots, A_m$ are given. We write $X \succeq 0$ is X is positive semidefinite and $A \bullet B$ stands for $\sum_{i,j} A_{i,j} B_{i,j}$.

With the ellipsoid method, semidefinite programs can be solved approximately up to any fixed precision in polynomial time [57]. For a more precise statement see, e.g., [54, Chapter 2.6]. Even though this is the only method for solving SDPs that has been proven to run in polynomial time, it has a poor performance in practice. There are other algorithms that allow fast solving of SDPs for practical purposes. For more information on semidefinite programming we refer the reader, e.g., to [8], [54] or [78].

Proof of the easy direction of the Cheeger inequality via SDP To describe the approach via SDPs in the graph case, we follow the exposition in Trevisan's blog [107, 108]. Let G be a d-regular graph with n vertices. Then we obviously can rephrase the sparsest cut problem as the following quadratic 0/1-program:

$$\begin{aligned}\phi(G) &= \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, V \setminus S)|}{\frac{d}{n-1} \cdot |S||V \setminus S|} \\ &= \min_{x \in \{0,1\}^V} \frac{\sum_{u,v \in V} A_{u,v} \cdot (x(u) - x(v))^2}{\frac{d}{n-1} \cdot \sum_{u,v \in V} (x(u) - x(v))^2}.\end{aligned}$$

Here, $A_{u,v}$ is the entry of the adjacency matrix $A(G)$ corresponding to vertices u and v. We want to show that $\frac{n}{n-1} \cdot \phi(G) \geq \lambda_2(\Delta(G)) = \frac{1}{d}\lambda_2(L(G))$. Since $\varepsilon(G) \geq \frac{n}{n-1} \cdot \phi(G)$, this will prove the part of the discrete Cheeger inequality we are interested in.

Consider the quadratic form $\sum_{u,v \in V} A_{u,v} \cdot (x(u) - x(v))^2$ for an arbitrary $x \in \mathbb{R}^V$. It is not hard to see that

$$\sum_{u,v \in V} A_{u,v} \cdot (x(u) - x(v))^2 = 2dx^T x - 2x^T A(G)x = 2x^T L(G)x.$$

The same holds if we consider this expression for the complete graph K_n instead of G:

$$\sum_{u,v \in V} (x(u) - x(v))^2 = \sum_{u,v \in V} A(K_n)_{u,v} \cdot (x(u) - x(v))^2 = 2x^T L(K_n)x.$$

Now, recall the variational characterization of eigenvalues, Theorem 2.1. It tells us that

$$\lambda_2(L(G)) = \min_{x \perp \mathbf{1}} \frac{x^T L(G)x}{x^T x}.$$

It is not hard to see that every $x \perp \mathbf{1}$ is an eigenvector of $L(K_n)$ with eigenvalue n and hence $x^T L(K_n)x = n \cdot x^T x$. Combining everything we have observed so far, we get:

$$\lambda_2(L(G)) = \min_{x \perp \mathbf{1}} \frac{\sum_{u,v \in V} A_{u,v} \cdot (x(u) - x(v))^2}{\frac{1}{n} \sum_{u,v \in V} (x(u) - x(v))^2}.$$

Next, we want to see that we can ignore the condition that $x \perp \mathbf{1}$ and take the minimum over all $x \in \mathbb{R}^V$. To see that this is true, observe that adding a constant vector $c \cdot \mathbf{1}$ to x does not change the value of the term we minimize over. Furthermore, for every $x \in \mathbb{R}^V$ there is c such that $x - c \cdot \mathbf{1} \perp \mathbf{1}$. Hence, we see that

$$\lambda_2(L(G)) = \min_{x \in \mathbb{R}^V} \frac{\sum_{u,v \in V} A_{u,v} \cdot (x(u) - x(v))^2}{\frac{1}{n} \sum_{u,v \in V} (x(u) - x(v))^2} \le d \cdot \frac{n}{n-1} \phi(G).$$

This shows the inequality we wanted to prove, but we have yet to see any semidefinite program. In order to see that the program describing $\lambda_2(L(G))$ is an SDP, we first show that

$$\lambda_2(L(G)) = \min_{m, x_1, \ldots, x_n \in \mathbb{R}^m} \frac{\sum_{i,j \in V} A_{i,j} \cdot \|x_i - x_j\|^2}{\frac{1}{n} \sum_{i,j \in V} \|x_i - x_j\|^2}, \tag{6.1}$$

where we identify V with $[n]$. This minimization problem is clearly a relaxation of the one we considered before. We hence need to show that it attains its minimum for 1-dimensional vectors.

First observe that for any finite sequences $(a_s), (b_s)$ we obviously have $\left(\min_s \frac{a_s}{b_s}\right) \cdot \sum_s b_s \le \sum_s a_s$ and hence $\min_s \frac{a_s}{b_s} \le \frac{\sum_s a_s}{\sum_s b_s}$. This implies

$$\begin{aligned} \frac{\sum_{i,j \in V} A_{i,j} \cdot \|x_i - x_j\|^2}{\frac{1}{n} \sum_{i,j \in V} \|x_i - x_j\|^2} &= \frac{\sum_s \sum_{i,j \in V} A_{i,j} \cdot (x_i(s) - x_j(s))^2}{\frac{1}{n} \sum_s \sum_{i,j \in V} (x_i(s) - x_j(s))^2} \\ &\ge \min_s \frac{\sum_{i,j \in V} A_{i,j} \cdot (x_i(s) - x_j(s))^2}{\frac{1}{n} \sum_{i,j \in V} (x_i(s) - x_j(s))^2}, \end{aligned}$$

which shows the equality in (6.1). Now, for a collection of vectors $x_1, \ldots, x_n$ consider the matrix $X(i,j) = \langle x_i, x_j \rangle$. It is not hard to see that this matrix is positive semidefinite, and that every positive semidefinite matrix is of this form. Note that we have

$$\sum_{i,j \in V} A_{i,j} \cdot \|x_i - x_j\|^2 = \sum_{i,j \in V} A_{i,j} \cdot (X(i,i) - 2X(i,j) + X(j,j)) = 2L(G) \bullet X.$$

The program in (6.1) hence is equivalent to the following semidefinite program:

$$\begin{aligned} &\min L(G) \bullet X \\ &\text{subject to} \quad L(K_n) \bullet X = n, \quad X \succeq 0. \end{aligned}$$

The Program in Dimension 2

Now, let X be a 2-dimensional simplicial complex with complete 1-skeleton and vertex set $[n]$. We consider the analogue of the sparsest cut problem:

$$\phi(X) = \min_{\substack{f \in C^1(X),\\ f \notin B^1(X)}} \frac{\|\delta_X f\|}{\|\delta_{K_n^2} f\|}, \tag{6.2}$$

where cochains and coboundaries are considered with coefficients in $\mathbb{Z}_2$. Since X has complete 1-skeleton, the space $C^1(X) = C^1(X;\mathbb{Z}_2)$ can be identified with the set of all 2-subsets of the vertex set $[n]$.

An element $f \in C^1(X)$ corresponds to a set A_f of edges on the vertex set $[n]$. A triangle with vertices in $[n]$ belongs to $\delta_{K_n^d}(f)$ if an odd number of its edges are in A_f. It belongs to $\delta_X(f)$ if it is contained in $\delta_{K_n^d}(f)$ and in X.

We can interpret f as a $0/1$-vector, with entries indexed by the edges of X. For $x, y, z \in \{0,1\}$ let $\varphi(x,y,z) = 1$ if $\{x,y,z\}$ contains an odd number of ones, and 0 otherwise. There are several ways to express this function as a polynomial in x, y, z. Possible expressions are:

$$\begin{aligned}
\varphi_1(x,y,z) &= x^3+y^3+z^3 - x^2(y+z) - y^2(x+z) - z^2(x+y) + 4xyz,\\
\varphi_2(x,y,z) &= (x-y-z)^2(y-x-z)^2(z-x-y)^2\\
&= (x^3+y^3+z^3 - x^2(y+z) - y^2(x+z) - z^2(x+y) + 2xyz)^2,\\
\varphi_3(x,y,z) &= ((x-y)^2 - z)^2\\
\varphi_4(x,y,z) &= x+y+z-2(xy+xz+yz)+4xyz\\
&= x(yz+(1-y)(1-z)) + (1-x)((1-y)z+y(1-z)),
\end{aligned} \tag{6.3}$$

We will mainly consider the expressions φ_1 and φ_2. Using the function φ described by any of the polynomial expressions above, we can *reformulate* (and normalize) (6.2) as the following polynomial program:

$$\phi(X) = \min_{y \in \{0,1\}^{\binom{n}{2}}} \frac{\binom{n}{3} \cdot \sum_{u,v,w \in [n]} X(u,v,w) \cdot \varphi(y_{uv}, y_{vw}, y_{uw})}{f_2(X) \cdot \sum_{u,v,w \in [n]} K_n^2(u,v,w) \cdot \varphi(y_{uv}, y_{vw}, y_{uw})}, \tag{6.4}$$

where $X(u,v,w)$ is 1 if the triangle with vertices u, v, w is contained in X, and 0 otherwise. Likewise, $K_n^2(u,v,w)$ is the characteristic function of the set $\binom{[n]}{3}$. Alternatively, we can consider the $+1/-1$-vector x with entries $x_e = 1 - 2y_e$ for $e \in \binom{n}{2}$ and rewrite (6.2) as:

$$\phi(X) = \min_{x \in \{-1,1\}^{\binom{n}{2}}} \frac{\binom{n}{3} \cdot \sum_{u,v,w \in [n]} X(u,v,w)(1 - x_{uv}x_{vw}x_{uw})}{f_2(X) \cdot \sum_{u,v,w \in [n]} K_n^2(u,v,w) \cdot (1 - x_{uv}x_{vw}x_{uw})}.$$

We focus on the 0/1-program. Note that φ_1 is homogeneous of degree 3, i.e., $\varphi_1(\alpha \cdot x, \alpha \cdot y, \alpha \cdot z) = \alpha^3 \cdot \varphi_1(x,y,z)$, and that φ_2 is homogeneous of degree 6. Using one of these two polynomial expressions from (6.3), the homogeneity makes it possible to consider the following relaxation of (6.4):

$$\begin{aligned} &\inf_{y \in \mathbb{R}^{\binom{n}{2}}} \sum_{u,v,w \in [n]} X(u,v,w) \cdot \varphi(y_{uv}, y_{vw}, y_{uw}) \\ &\text{subject to} \quad \sum_{u,v,w \in [n]} K_n^2(u,v,w) \cdot \varphi(y_{uv}, y_{vw}, y_{uw}) = 1, \quad y \in \mathbb{R}^{\binom{n}{2}}. \end{aligned} \tag{6.5}$$

Note that depending on the choice of the polynomial expression for $\varphi(x,y,z)$ it is not clear whether (6.5) has a finite value: Since $\varphi = \varphi_2$ is non-negative, the optimum of (6.5) is finite for this choice. For $\varphi = \varphi_1$ however, arbitrary small values might be possible.

This is a polynomial optimization problem whose degree, depending on the choice of the polynomial expression for $\varphi(x,y,z)$, is at least 3. Polynomial optimization in general is NP-hard. There are several hierarchies of semidefinite relaxations for polynomial optimization problems. We focus on a method based on postive semidefinite moment matrices.

For constraint satisfaction problems (CSPs) with local constraints, an important and large class of optimization problems, there is a very general framework for approximation with semidefinite programs [99, 100, 101], see also [54]. The reader might wonder why this framework is not applied here. The problem of finding the smallest coboundary is such a CSP. Each triangle in X describes a local constraint and we wish to minimize the number of satisfied constraints, i.e. the number of triangles containing an odd number of chosen edges. However, the program (6.4) at the same time considers the size of the coboundary in the complete complex. We can work around this by considering several programs, one for each possible size ℓ of $\delta_{K_n^2} f$ where we only consider edge sets attaining this size. These programs, however, do not only contain local constraints, involving a constant number of edges, but also the *global* constraint $|\delta_{K_n^2} f| = \ell$. For such optimization problems no general framework is known.

Moment Relaxation for Polynomial Optimization

We now give a very short description of the moment relaxation of a polynomial optimization problem. This method was first proposed by Lasserre [77]. We follow the very thorough survey [78] by Laurent.

Multivariate Polynomials. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Let

$$p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$$

be a multivariate polynomial. The degree of a monomial x^α is

$$\deg(x^\alpha) = |\alpha| := \sum_{i=1}^{n} \alpha_i.$$

The degree of $p(x)$ is the maximal degree of a monomial x^α in $p(x)$ with $p_\alpha \neq 0$. We identify a multivariate polynomial $p(x)$ of degree d with its sequence of coefficients $p = (p_\alpha)_{|\alpha| \leq d}$.

For a sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$ its *moment matrix* is the infinite matrix $M(y) \in \mathbb{R}^{\mathbb{N}^n \times \mathbb{N}^n}$ defined by $M(y)_{\alpha,\beta} := y_{\alpha+\beta}$. For $t \in \mathbb{N}$ we let $\mathbb{N}_t^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq t\}$. Then for a finite vector $y = (y_\alpha)_{\alpha \in \mathbb{N}_{2t}^n}$ we consider the finite moment matrix $M_t(y) := (y_{\alpha+\beta})_{\alpha,\beta \in \mathbb{N}_t^n}$. For $n = 1$ and $t = 2$, so that $\mathbb{N}_{2t}^n = [4]$, consider as an example a vector $y = (y_\alpha)_{\alpha \in [4]}$. Its moment matrix $M_2(y)$ is the matrix

$$M_2(y) = \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix}.$$

For a multivariate polynomial $p(x)$ in n variables and a sequence $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$ we define the sequence $p * y := M(y)p \in \mathbb{R}^{\mathbb{N}^n}$, with entry $(p * y)_\alpha = \sum_\beta p_\beta y_{\alpha+\beta}$. Abusing notation, we denote by $p * y$ also the finite vector $((p * y)_\alpha)_{\alpha \in \mathbb{N}_t^n}$.

We will also consider moment matrices of such vectors. As an example, consider a univariate polynomial $p(x) = p_0 + p_1 x + p_2 x^2$ of degree 2. For a vector $y = (y_\alpha)_{\alpha \in [6]}$ the moment matrix of $p * y$ is

$$M_2(p*y) = \begin{pmatrix} p_0y_0+p_1y_1+p_2y_2 & p_0y_1+p_1y_2+p_2y_3 & p_0y_2+p_1y_3+p_2y_4 \\ p_0y_1+p_1y_2+p_2y_3 & p_0y_2+p_1y_3+p_2y_4 & p_0y_3+p_1y_4+p_2y_5 \\ p_0y_2+p_1y_3+p_24_2 & p_0y_3+p_1y_4+p_2y_5 & p_0y_4+p_1y_5+p_2y_6 \end{pmatrix}.$$

Moment Relaxation. For multivariate polynomials $p, h_1, \dots, h_n(X)$ let

$$K := \{x \in \mathbb{R}^n : h_1(x) \geq 0, \dots, h_n(x) \geq 0\}.$$

We then consider the program

$$p_{\min} = \inf_{x \in K} p(x). \tag{6.6}$$

For $t \geq \lceil \deg(p)/2 \rceil$ and $d_j := \lceil \deg(h_j)/2 \rceil$ consider the following semi-definite program, the *moment relaxation* of order t of (6.6):

$$\begin{aligned} p_t^* = &\inf p^T y \\ &\text{subject to} \quad && y_0 = 1,\ M_t(y) \succeq 0, \\ &&& M_{t-d_j}(h_j * y) \succeq 0 \quad (j = 1, \dots, m). \end{aligned} \tag{6.7}$$

It does not become clear from the (very short) exposition here, but (6.7) is feasible if (6.6) is, and furthermore $p_t^* \leq p_{t+1}^* \leq p_{\min}$. So p_t^* for any $t \geq \lceil \deg(p)/2 \rceil$ is a lower bound for $p_{\min}$. Of course, p_t^*, just as $p_{\min}$, could be $-\infty$. So in order to get a useful lower bound, it is crucial to determine the behaviour of p_t^*. There are several results, guaranteeing, under certain conditions, convergence of p_t^* to $p_{\min}$ or even that $p_t^* = p_{\min}$ for large t, see, e.g., [78, Chapter 6].

Moment Relaxation of Combinatorial Expansion

We now want to describe the moment relaxation of the program (6.5), where we choose the polynomial expression $\varphi = \varphi_1$ from (6.3):

$$\varphi_1(x, y, z) = x^3 + y^3 + z^3 - x^2(y + z) - y^2(x + z) - z^2(x + y) + 4xyz.$$

As remarked, we don't know whether for $\varphi = \varphi_1$ the program (6.5) has a finite value. We nevertheless study its relaxations here, as these are of a simpler structure than for $\varphi = \varphi_2$. We later also give the relaxation for $\varphi = \varphi_2$.

Using the polynomial expression φ_1. For $y \in \mathbb{R}^{\binom{n}{2}}$ define three polynomials as follows:

$$p(y) = \sum_{u,v,w\in[n]} X(u,v,w) \cdot \varphi_1(y_{uv}, y_{vw}, y_{uw}),$$
$$h_1(y) = \sum_{u,v,w\in[n]} K_n^2(u,v,w) \cdot \varphi_1(y_{uv}, y_{vw}, y_{uw}) - 1,$$
$$h_2(y) = 1 - \sum_{u,v,w\in[n]} K_n^2(u,v,w) \cdot \varphi_1(y_{uv}, y_{vw}, y_{uw}).$$

Then for $K := \{y \in \mathbb{R}^{\binom{n}{2}} : h_1(y) \geq 0, h_2(y) \geq 0\}$, the relaxation (6.5) of $\phi(X)$ is the program $p_{\min} = \inf_{y\in K} p(y)$. The sequences of coefficients of the polynomials involved in this program can easily be determined:

Lemma 6.1. *For $\alpha \in \mathbb{N}^{\binom{n}{2}}$ let $\operatorname{supp}(\alpha) = \{e \in \binom{n}{2} : \alpha_e > 0\}$. Then:*

$$(h_1)_\alpha = \begin{cases} -1 & \text{if } \alpha = (0\ldots,0), \\ 24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \quad \text{for } \{u,v,w\} \in \binom{[n]}{3}, |\alpha| = 3, \\ -6 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \quad \text{for } \{u,v,w\} \in \binom{[n]}{3}, |\alpha| = 3, \\ 6(n-2) & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\}\} \\ & \quad \text{for } u,v \in [n], u \neq v, |\alpha| = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $(h_2)_\alpha = -(h_1)_\alpha$. Furthermore,

$$p_\alpha = \begin{cases} 24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \quad \text{for } \{u,v,w\} \in X_2, |\alpha| = 3, \\ -6 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \quad \text{for } \{u,v,w\} \in X_2, |\alpha| = 3, \\ 6 \cdot \deg(\{u,v\}) & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\}\} \\ & \quad \text{for } u,v \in [n], u \neq v, |\alpha| = 3, \\ 0 & \text{otherwise.} \end{cases}$$

In order to describe the moment relaxation of (6.5), we define for

$u, v, w \in [n]$ the following sequences $\beta_{\{u,v\}}, \beta_{\{u,v,w\}}, \beta_{\{u,v\}\{v,w\}} \in \mathbb{N}_3^{\binom{n}{2}}$:

$$(\beta_{\{u,v\}})_e = \begin{cases} 3 & \text{if } e = \{u,v\}, \\ 0 & \text{otherwise.} \end{cases} \qquad (\beta_{\{u,v,w\}})_e = \begin{cases} 1 & \text{if } e \subset \{u,v,w\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$(\beta_{\{u,v\}\{v,w\}})_e = \begin{cases} 2 & \text{if } e = \{u,v\}, \\ 1 & \text{if } e = \{v,w\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.2. *The moment relaxation of order t for $t \geq 2$ of the program* (6.5)*, where we choose the polynomial expression $\varphi = \varphi_1$ from* (6.3)*, is the following program:*

$$\begin{aligned} p_t^* = &\inf p^T y \\ &\textit{subject to } y_0 = 1, \quad (y_{\alpha+\beta})_{\alpha,\beta \in \mathbb{N}_t^n} \succeq 0, \\ &\qquad y_\alpha = \sum_{u,v,w \in [n]} K_n^2(u,v,w) \cdot \psi_{\alpha,u,v,w}(y), \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \psi_{\alpha,u,v,w}(y) = &\; y_{\alpha+\beta_{\{u,v\}}} + y_{\alpha+\beta_{\{u,w\}}} + y_{\alpha+\beta_{\{v,w\}}} \\ &- y_{\alpha+\beta_{\{u,v\}\{v,w\}}} - y_{\alpha+\beta_{\{v,w\}\{u,v\}}} - y_{\alpha+\beta_{\{u,v\}\{u,w\}}} \\ &- y_{\alpha+\beta_{\{u,w\}\{u,v\}}} - y_{\alpha+\beta_{\{v,w\}\{u,w\}}} - y_{\alpha+\beta_{\{u,w\}\{v,w\}}} \\ &+ 4y_{\alpha+\beta_{\{u,v,w\}}}. \end{aligned}$$

Proof. As $(h_2)_\alpha = -(h_1)_\alpha$, we have $M := M_{t-2}(h_1 * y) = -M_{t-2}(h_2 * y)$ and the conditions that $M \succeq 0$ and $-M \succeq 0$ together give $M = 0$. The moment relaxation of (6.5) is hence:

$$\begin{aligned} p_t^* = &\inf p^T y \\ &\text{subject to} \quad y_0 = 1, \quad M_t(y) \succeq 0, \quad M_{t-2}(h_1 * y) = 0. \end{aligned}$$

Using Lemma 6.1, this can be seen to be equivalent to program (6.8). □

For fixed t, the program in Lemma 6.2 gives a lower bound for $\phi(X)$ that is polynomially computable (up to arbitrary precision). However, in order to say anything about the quality of this bound, further study is needed, such as closer investigation of the properties of the polynomials that are involved in the definition of the programs described above.

Using the polynomial expression φ_2. We finish this section with the description of the moment relaxation of (6.5) for the polynomial expression $\varphi = \varphi_2$ from (6.3):

$$\begin{aligned}\varphi_2(x,y,z) &= (x^3+y^3+z^3 - x^2(y+z) - y^2(x+z) - z^2(x+y) + 2xyz)^2 \\ &= x^6+y^6+z^6 \\ &\quad - 2(x^5y + x^5z + xy^5 + y^5z + xz^5 + yz^5) \\ &\quad - (x^4y^2 + x^4z^2 + x^2y^4 + y^4z^2 + x^2z^4 + y^2z^4) \\ &\quad + 4(x^3y^3 + x^3z^3 + y^3z^3) \\ &\quad + 6(x^4yz + zy^4z + xyz^4) \\ &\quad - 4(x^3y^2z + x^3yz^2 + x^2y^3z + xy^3z^2 + x^2yz^3 + xy^2z^3) \\ &\quad + 10x^2y^2z^2.\end{aligned}$$

As above, we define three polynomials $y \in \mathbb{R}^{\binom{n}{2}}$:

$$\begin{aligned} q(y) &= \sum\nolimits_{u,v,w\in[n]} X(u,v,w)\cdot\varphi_1(y_{uv},y_{vw},y_{uw}),\\ g_1(y) &= \sum\nolimits_{u,v,w\in[n]} K_n^2(u,v,w)\cdot\varphi_1(y_{uv},y_{vw},y_{uw}) - 1,\\ g_2(y) &= 1 - \sum\nolimits_{u,v,w\in[n]} K_n^2(u,v,w)\cdot\varphi_1(y_{uv},y_{vw},y_{uw}).\end{aligned}$$

Then for $L := \{y \in \mathbb{R}^{\binom{n}{2}} : g_1(y) \geq 0, g_2(y) \geq 0\}$, the relaxation (6.5) of $\phi(X)$ is the program $q_{\min} = \inf_{y\in L} q(y)$.

For the description of the moment relaxation, which we will give at the end of this section, define the following sequences in $\mathbb{N}_6^{\binom{n}{2}}$:

$$(\gamma_{\{u,v\}})_e = \begin{cases} 6 & \text{if } e = \{u,v\},\\ 0 & \text{otherwise.}\end{cases} \qquad (\gamma^{i_1,i_2,i_3}_{e_1,e_2,e_3})_e = \begin{cases} i_\ell & \text{if } e = e_\ell,\\ 0 & \text{otherwise.}\end{cases}$$

$$(\gamma^{i_1,i_2}_{e_1,e_2})_e = \begin{cases} i_l & \text{if } e = e_l,\\ 0 & \text{otherwise.}\end{cases}$$

We first specify the sequences of coefficients of the polynomials q, g_1 and g_2 in the following lemma and then present the moment relaxation of the program $q_{\min} = \inf_{y\in L} q(y)$.

Lemma 6.3. *The sequences of coefficients of q, g_1, g_2 are as follows:*

$$(g_1)_\alpha = \begin{cases} -1 & \text{if } \alpha = (0,\dots,0), \\ 60 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{2,2,2\}, \\ -24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,2,3\}, \\ 36 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,1,4\}, \\ 24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{3,3\}, \\ -6 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{2,4\}, \\ -12 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in \binom{[n]}{3}, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,5\}, \\ 6(n-2) & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\}\} \text{ for } u,v \in [n], |\alpha| = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $(g_2)_\alpha = -(g_1)_\alpha$. Furthermore,

$$q_\alpha = \begin{cases} 60 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{2,2,2\}, \\ -24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,2,3\}, \\ 36 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\},\{v,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,1,4\}, \\ 24 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{3,3\}, \\ -6 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{2,4\}, \\ -12 & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\},\{u,w\}\} \\ & \text{for } \{u,v,w\} \in X_2, \{\alpha_e : e \in \operatorname{supp}(\alpha)\} = \{1,5\}, \\ 6 \cdot \deg(\{u,v\}) & \text{if } \operatorname{supp}(\alpha) = \{\{u,v\}\} \text{ for } u,v \in [n], |\alpha| = 6, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed t these coefficients determine the moment relaxation of order t of the program (6.5) corresponding to the choice of polynomial expression $\varphi = \varphi_2$ from (6.3). Also this relaxation, described explicitly in Lemma 6.4, gives a lower bound for $\phi(X)$ that is polynomially computable (up to arbitrary precision). Even though in this case, where we choose $\varphi = \varphi_2$, we at least know that the original program (6.5) is of finite value, we do not know more about the quality of this bound. Also in this case further study is needed.

Lemma 6.4. *The moment relaxation of order t for $t \geq 3$ of the program* (6.5)*, where we choose the polynomial expression $\varphi = \varphi_2$ from* (6.3)*, is the following program:*

$$
\begin{aligned}
q_t^* = &\inf q^T y \\
&\text{subject to } y_0 = 1, \quad (y_{\alpha+\gamma})_{\alpha,\gamma \in \mathbb{N}_t^n} \succeq 0, \\
&y_\alpha = \sum\nolimits_{u,v,w \in [n]} K_n^2(u,v,w) \cdot \Big(y_{\alpha+\gamma_{\{u,v\}}} + y_{\alpha+\gamma_{\{u,w\}}} + y_{\alpha+\gamma_{\{v,w\}}} \\
&- 2y_{\alpha+\gamma^{5,1}_{\{u,v\}\{v,w\}}} - 2y_{\alpha+\gamma^{1,5}_{\{u,v\}\{v,w\}}} - 2y_{\alpha+\gamma^{5,1}_{\{u,v\}\{u,w\}}} \\
&- 2y_{\alpha+\gamma^{1,5}_{\{u,v\}\{u,w\}}} - 2y_{\alpha+\gamma^{5,1}_{\{v,w\}\{u,w\}}} - 2y_{\alpha+\gamma^{1,5}_{\{v,w\}\{u,w\}}} \\
&- y_{\alpha+\gamma^{4,2}_{\{u,v\}\{v,w\}}} - y_{\alpha+\gamma^{2,4}_{\{u,v\}\{v,w\}}} - y_{\alpha+\gamma^{4,2}_{\{u,v\}\{u,w\}}} \\
&- y_{\alpha+\gamma^{2,4}_{\{u,v\}\{u,w\}}} - y_{\alpha+\gamma^{4,2}_{\{v,w\}\{u,w\}}} - y_{\alpha+\gamma^{2,4}_{\{v,w\}\{u,w\}}} \\
&+ 4y_{\alpha+\gamma^{3,3}_{\{v,w\}\{u,w\}}} + 4y_{\alpha+\gamma^{3,3}_{\{u,v\}\{v,w\}}} + 4y_{\alpha+\gamma^{3,3}_{\{u,v\}\{u,w\}}} \\
&+ 6y_{\alpha+\gamma^{4,1,1}_{\{u,v\},\{v,w\},\{u,w\}}} + 6y_{\alpha+\gamma^{1,4,1}_{\{u,v\},\{v,w\},\{u,w\}}} + 6y_{\alpha+\gamma^{1,1,4}_{\{u,v\},\{v,w\},\{u,w\}}} \\
&- 4y_{\alpha+\gamma^{3,2,1}_{\{u,v\},\{v,w\},\{u,w\}}} - 4y_{\alpha+\gamma^{3,1,2}_{\{u,v\},\{v,w\},\{u,w\}}} - 4y_{\alpha+\gamma^{2,3,1}_{\{u,v\},\{v,w\},\{u,w\}}} \\
&- 4y_{\alpha+\gamma^{2,1,3}_{\{u,v\},\{v,w\},\{u,w\}}} - 4y_{\alpha+\gamma^{1,2,3}_{\{u,v\},\{v,w\},\{u,w\}}} - 4y_{\alpha+\gamma^{1,3,2}_{\{u,v\},\{v,w\},\{u,w\}}} \\
&+ 10y_{\alpha+\gamma^{2,2,2}_{\{u,v\},\{v,w\},\{u,w\}}} \Big).
\end{aligned}
$$

6.2 Quasirandomness

In this section we consider a second approach towards a connection between combinatorial expansion and a polynomially computable quantity for 2-complexes. Following Gowers' notion of quasirandomness for 3-uniform hypergraphs [55], we try to connect the number of octahedra in a complex to combinatorial expansion. Here we rather aim for statements analogous to the Expander Mixing Lemma (Theorem 2.3) than to the Cheeger inequality.

The Expander Mixing Lemma. Recall the statement of the Expander Mixing Lemma: For a d-regular graph and any $S, T \subset V$ we have

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \mu(G) \cdot \sqrt{|S||T|},$$

where $\mu(G) := \max\{\mu_2(A), |\mu_n(A)|\}$. In this section, we will present an attempt to find, for 2-complexes, a statement similar to the Expander Mixing Lemma. As in higher dimensions, in view of Theorem 5.1, we cannot hope for a bound in the eigenvalues of the adjacency matrix, we will aim for a different kind of error bound, related to quasirandomness. The Expander Mixing Lemma is typically proven in a way that strongly relies on the concept of eigenvalues. We present a proof below.

Proof of the Expander Mixing Lemma. Let $d = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be the eigenvalues of the adjacency matrix $A = A(G)$. Let $b_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ be an eigenvector for $\mu_1 = d$, and let $b_1, b_2, \ldots, b_n$ be an orthomormal basis of eigenvectors of A such that $Ab_i = \mu_i b_i$. For sets $S, T \subset V$ express the characteristic vectors $\mathbf{1}_S$ and $\mathbf{1}_T$ in this basis:

$$\mathbf{1}_S = \sum_{i=1}^{n} \alpha_i b_i \quad \text{and} \quad \mathbf{1}_T = \sum_{i=1}^{n} \beta_i b_i.$$

Then, as $\mathbf{1}_S^T A \mathbf{1}_T = |E(S,T)|$, we have for J the all-ones matrix:

$$\begin{aligned}
\left| |E(S,T)| - \frac{d|S||T|}{n} \right| &= \left| \mathbf{1}_S^T \left(A - \frac{d}{n} J \right) \mathbf{1}_T \right| \\
&= \left| \mathbf{1}_S^T (\sum_{i\geq 2} \mu_i b_i) \mathbf{1}_T \right| \\
&= \left| \sum_{i\geq 2} \mu_i \alpha_i \beta_i \right| \\
&\leq \mu \cdot \sum_{i\geq 2} |\alpha_i \beta_i| \\
&\leq \mu \cdot \sqrt{\sum_{i\geq 2} \alpha_i^2} \sqrt{\sum_{i\geq 2} \beta_i^2} \\
&\leq \mu \cdot \sqrt{|S||T|},
\end{aligned}$$

where in the second-to-last step we apply the Cauchy-Schwarz inequality. □

Quasirandomness for Graphs. A family of graphs G_n with n vertices is called *quasirandom* if for all vertex subsets $S, T \subset V(G_n)$,

$$|E(S,T)| = p|S||T| + o(pn^2) \tag{6.9}$$

for a parameter $p = p(n)$ which is typically taken as the density of G_n. This notion was introduced by Chung, Graham and Wilson in [27]. Earlier, Thomason defined a similar, but stricter notion, called *jumbledness* [106]. For a survey on different notions of quasirandomness we refer to [75].

Chung, Graham and Wilson in [27] give a definition involving only a single set U and the number of edges in $E(U)$, but this is easily seen to be equivalent to our condition. For constant p, there are in fact several different ways of defining quasirandomness, which were shown to be equivalent in their paper. All of these conditions can be shown to hold with high probability for the random graph $G(n,p)$. Here, we focus on the three conditions given in the following theorem:

Theorem 6.5 ([27]). *Let G be a graph on n vertices and let $p \in (0,1)$ be fixed. Then the following statements are equivalent:*

(i) $|E(G)| = p\binom{n}{2} + o(n^2)$ *and the number of labeled (not necessarily induced) copies of C_4 in G is at most $n^4p^4 + o(n^4)$.*

(ii) For all subsets $S, T \subset V$ of the vertices of G,

$$|E(S,T)| = p|S||T| + o(n^2).$$

(iii) $|E(G)| = p\binom{n}{2} + o(n^2)$ *and* $\mu_1(A) = (1 + o(1))np$, $|\mu_i(A)| = o(n)$ *for all $i \geq 2$.*

Chung, Graham and Wilson [27] only state this result for $p = 1/2$ but the generalization to arbitrary fixed p is not too hard. As p is constant, the graphs considered are dense in the sense that they have cn^2 edges. In [26] Chung and Graham consider the behaviour of these properties for sparse graphs with $o(n^2)$ edges.

Note that the condition (6.9) defining quasirandomness resembles the statement of the Expander Mixing Lemma in that it gives a bound on how far the number of edges in $E(S,T)$ diverges from the expected number in a random graph of corresponding density. Actually, for a d-regular graph, the Expander Mixing Lemma proves that the third condition implies the second, and for general graphs, both statements can be proven in a similar way.

A different Expander Mixing Lemma? Note that the condition on eigenvalues as well as the condition on the number of C_4's are verifiable in polynomial time. In this section, we will present an attempt to find, for 2-complexes, a statement similar to the Expander Mixing Lemma. Instead of bound in terms of the eigenvalues of the adjacency matrix, we aim for an error bound that for graphs would correspond to a bound phrased in terms of the number of C_4's. More precisely, we consider the analogue of the following term:

$$\mathrm{quad}(f_G) = \sum_{x_0,x_1} \sum_{y_0,y_1} \prod_{(i,j)\in\{0,1\}^3} f_G(x_i, y_j),$$

where $f_G := A(G) - d/nJ$ and J is the all-ones matrix and we denote an entry $(f_G)_{x,y}$ by $f_G(x,y)$. One readily checks that for a d-regular graph G it measures the deviation of the number of C_4's from the expected number in a graph of corresponding density: Observe that the spectrum of f_G for a d-regular graph is almost identical to that of A. The eigenvalue d with eigenvector $\mathbf{1}$ is replaced by 0, and all other eigenvalues stay the same. Thus,

$$\mathrm{quad}(f_G) = \mathrm{trace}(f_G^4) = \sum_{i\geq 2} \mu_i(A)^4 = \mathrm{trace}(A^4) - d^4.$$

The expected number of copies of C^4 in a random graph of density d/n is of the order d^4, and $\mathrm{trace}(A^4)$ is the number of (possibly degenerate) closed walks of length 4 in G. For general, non-regular graphs, one can consider the corresponding function $f_G = A - pJ$ where $p = |E|/\binom{n}{2}$, but the connection between $\mathrm{quad}(f_G)$ and the number of C_4's in G can not be drawn so easily.

By the observation above we have $\mathrm{quad}(f_G) = \sum_{i\geq 2} \mu_i(A)^4 \geq \mu(G)^4$. Hence, the Expander Mixing Lemma implies the following weaker estimate for all $S, T \subset V$:

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \sqrt[4]{\mathrm{quad}(f_G)} \cdot \sqrt{|S||T|}.$$

We now give an alternative proof of this statement, that avoids the use of eigenvalues and furthermore also applies to general, non-regular graphs. This proof is inspired by methods used by Gowers to prove higher-dimensional quasirandomness results, which we will discuss below.

Lemma 6.6. *For a graph G let $p = |E|/\binom{n}{2}$. Then for any $S, T \subset V$*

$$\big| |E(S,T)| - p|S||T| \big| \leq \sqrt[4]{\mathrm{quad}(f_G)} \cdot \sqrt{|S||T|}.$$

Proof. Let $S, T \subset V$ and consider the characteristic functions $\mathbf{1}_S$ and $\mathbf{1}_T$.

$$\big||E(S,T)| - p|S||T|\big| = \Big|\sum_{v\in V}\sum_{u\in V} f_G(u,v)\mathbf{1}_S(u)\mathbf{1}_T(v)\Big|.$$

We show that $\big||E(S,T)| - p|S||T|\big|^4 \leq \mathrm{quad}(f_G)\cdot |S|^2|T|^2$. By applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}
\Big(\sum_{v,u\in V} f_G(u,v)\mathbf{1}_S(u)\mathbf{1}_T(v)\Big)^4 &= \bigg(\Big(\sum_{v\in V}\mathbf{1}_T(v)\Big(\sum_{u\in V} f_G(u,v)\mathbf{1}_S(u)\Big)\Big)^2\bigg)^2 \\
&\leq \bigg(\langle\mathbf{1}_T,\mathbf{1}_T\rangle\cdot\sum_{v\in V}\Big(\sum_{u\in V} f_G(u,v)\mathbf{1}_S(u)\Big)^2\bigg)^2 \\
&= \langle\mathbf{1}_T,\mathbf{1}_T\rangle^2\cdot\bigg(\sum_{v\in V}\Big(\sum_{u\in V} f_G(u,v)\mathbf{1}_S(u)\Big)^2\bigg)^2
\end{aligned}$$

Considering the second factor and applying the Cauchy-Schwarz inequality once more, we get

$$\begin{aligned}
&\bigg(\sum_{v\in V}\Big(\sum_{u\in V} f_G(u,v)\mathbf{1}_S(u)\Big)^2\bigg)^2 \\
&\quad= \Big(\sum_{u,u'\in V}\mathbf{1}_S(u)\mathbf{1}_S(u')\sum_{v\in V} f_G(u,v)f_G(u',v)\Big)^2 \\
&\quad\leq \Big(\sum_{u,u'\in V}\mathbf{1}_S(u)^2\mathbf{1}_S(u')^2\Big)\bigg(\sum_{u,u'\in V}\Big(\sum_{v\in V} f_G(u,v)f_G(u',v)\Big)^2\bigg)^2 \\
&\quad= \langle\mathbf{1}_S,\mathbf{1}_S\rangle^2\,\mathrm{quad}(f_G).
\end{aligned}$$

□

Unfortunately, even for graphs we are not aware of a stronger bound in terms of $\mathrm{quad}(f_G)$. For 2-complexes, our results are also likely to be suboptimal. Nevertheless, in view of Gowers' results on higher-dimensional quasirandomness, which we discuss next, this seems to be an interesting approach.

Quasirandomness for Hypergraphs. In [55], Gowers introduces a notion of quasirandomness for 3-uniform hypergraphs, which we now apply to 2-dimensional complexes.

A 2-dimensional simplicial complex X with vertex set V and complete 1-skeleton can be identified with the characteristic function of its set of 2-simplices, which we denote by $X : V \times V \times V \to \{0,1\}$. We then consider the function

$$f_X := X - p \cdot K_n^2 : V \times V \times V \to [-1,1],$$

where $p = f_2(X)/\binom{n}{3}$. The results of [55] are phrased in a more general setting of $[-1,1]$-valued ternary functions. For any such $f : X \times Y \times Z \to [-1,1]$, define

$$\operatorname{oct}(f) := \sum_{x_0,x_1 \in X} \sum_{y_0,y_1 \in Y} \sum_{z_0,z_1 \in Z} \prod_{(i,j,k) \in \{0,1\}^3} f(x_i, y_j, z_k).$$

Gowers proves the following higher-dimensional analogue of the equivalence of the first two statements in Theorem 6.5.

Theorem 6.7 ([55]). *Let X, Y and Z be sets of sizes L, M, and N and let $f : X \times Y \times Z \to [-1,1]$. Then the following statements are equivalent.*

(i) $\operatorname{oct}(f) \leq c_1 L^2 M^2 N^2$.

(ii) For any three functions $u : X \times Y \to [-1,1]$, $v : Y \times Z \to [-1,1]$ and $w : X \times Z \to [-1,1]$ we have the inequality

$$\Big|\sum_{x,y,z} f(x,y,z) u(x,y) v(y,z) w(x,z)\Big| \leq c_2 LMN.$$

(iii) For any tripartite graph G with vertex sets X, Y and Z, the sum of $f(x,y,z)$ over all triangles xyz of G is at most $c_3 LMN$ in magnitude.

We will mostly be interested in the statement that the first condition implies the second (and thus also the third), as this establishes the connection we are interested in. It is proven via the following lemma, for which we observe that it has a structure similar to the Expander Mixing Lemma:

Lemma 6.8 ([55]). *Let X, Y and Z be sets of sizes L, M, and N, respectively, and consider a map $f : X \times Y \times Z \to [-1,1]$. Let furthermore $u : X \times Y \to [-1,1]$, $v : Y \times Z \to [-1,1]$ and $w : X \times Z \to [-1,1]$. Then:*

$$\Big(\sum_{x,y,z} f(x,y,z) u(x,y) v(y,z) w(x,z)\Big)^8 \leq M^6 N^6 L^6 \operatorname{oct}(f).$$

Application to Combinatorial Expansion. Gowers' results, when restricted to $\{0,1\}$-valued functions, deal with three sets of edges and the set of triangles having exactly one edge from each. For combinatorial expansion, we consider one set of edges and the set of triangles containing an odd number of edges from it – its coboundary.

For a cochain $u \in C^1(X;\mathbb{Z}_2)$, viewed as a map $u : V \times V \to \{0,1\}$, write $\bar{u}(x,y) = 1-u(x,y)$. For $x,y,z \in V$ we can then express the value of the coboundary $\delta u(\{x,y,z\})$ as

$$u(x,y)\bar{u}(x,z)\bar{u}(y,z) + \bar{u}(x,y)u(x,z)\bar{u}(y,z) \\ + \bar{u}(x,y)\bar{u}(x,z)u(y,z) + u(x,y)u(x,z)u(y,z),$$

a term to which we can apply the above results. For $f_X = X - pK_n^2$ for some $p = p(n)$, we have

$$\sum_{x,y,z} f_X(x,y,z)\delta u(\{x,y,z\}) = 3||\delta_X u| - p|\delta_{K_n^2} u||$$

and see that this term can be expressed as a sum of four terms of the type that appear in Lemma 6.8. Applying Lemma 6.8 to each term separately, we get for any p:

$$||\delta_X u| - p|\delta_{K_n^2} u|| \leq 4/3 \cdot n^{18/8} \operatorname{oct}(f_X)^{1/8}.$$

This statement is similar to the Expander Mixing Lemma for graphs in that it bounds the deviation of $|\delta_X u|$ from its expected value $p|\delta_{K_n^2} u|$ in a random complex of density p. We will see soon (Corollary 6.11) which conclusions on the combinatorial expansion we can draw.

In order to judge the quality of this criterion, we consider random complexes $X^2(n,p)$. The following lemma tells us that for $p = \Omega(\log(n)/n)$ the criterion is not tight. For every $u \in C^1(X;\mathbb{Z}_2)$ the value of $|\delta_X u|$ a.a.s. differs from its expectation by at most $O(n^2\log(n))$, so that $||\delta_X u| - p|\delta_{K_n^2} u||^8 = O(n^{16}\log(n)^8)$. At the same time, one can show that a.a.s. $n^{18}\operatorname{oct}(f_X) = \Omega(n^{20}\log(n)^2)$.

This is quite different to the situation for graphs: For random graphs $G(n,p)$ with $p = \Omega(\log(n)/n)$ one can show that a.a.s. for all S, T the value $|E(S,T)|$ is concentrated around the expected value $p|S||T|$, with an error of the order $O(\sqrt{np|S||T|})$ (see, e.g., the introduction of [71]). As furthermore a.a.s. $\mu(G) = O(\sqrt{np})$ (Theorem 4.1), the Expander Mixing Lemma in some sense achieves the "correct" order of concentration.

We will give a proof of Lemma 6.9 at the end of this section. Note that we use very simple methods that possibly lead to suboptimal bounds. Nevertheless these results suffice for our purpose.

Lemma 6.9. *For every $\varepsilon > 0$ there is $C > 0$ such that the random complex $X = X^2(n,p)$ with $p = C \cdot \frac{\log(n)}{n}$ a.a.s. satisfies that for all $u \in C^1(X;\mathbb{Z}_2)$*

$$||\delta_X u| - p|\delta_{K_n^2} u|| \leq \varepsilon \cdot p|\delta_{K_n^2 u}| = O(n^2 \log(n)).$$

Furthermore, for every $C > 0$ there is $c > 0$ such that for the random complex $X^2(n,p)$ with $p = C \cdot \frac{\log(n)}{n}$ and $f_X = X - pK_n^2$ we have a.a.s. $\operatorname{oct}(f_X) \geq cn^2 \log(n)^2$.

Adapting Gowers' proof of Lemma 6.8 to the special case where one considers coboundaries, it is possible to estimate all four terms at once. This allows us to remove a factor of 4, but unfortunately it does not seem to change the situation much and to date we have not found any other, more significant improvements.

Lemma 6.10. *Let X be a 2-dimensional simplicial complex with n vertices and complete 1-skeleton. For any $u \in C^1(X;\mathbb{Z}_2)$ and any map $f : V \times V \times V \to [-1,1]$ we have*

$$\Big(\sum_{x,y,z} f(x,y,z)\delta u(\{x,y,z\})\Big)^8 \leq n^{18} \operatorname{oct}(f).$$

In particular, we have for $u \in C^1(X;\mathbb{Z}_2)$ such that $\delta_{K_n^2} u \neq 0$ and $f_X = X - pK_n^2$:

$$||\delta_X u| - p|\delta_{K_n^2} u|| \leq 1/3 \cdot n^{18/8} \operatorname{oct}(f_X)^{1/8}.$$

We will give the proof of Lemma 6.10 later in this section. First, we consider its connection to combinatorial expansion. For dense complexes we see that a bound for $\operatorname{oct}(f_X)$ implies a coarse kind of expansion – we can guarantee expansion only for cochains with a sufficiently large distance from $B^1(X,\mathbb{Z}_2)$. More precisely, we call a k-dimensional simplicial complex *coarsely combinatorially (ε,δ)-expanding* if $\|\delta_X u\| \geq \epsilon \|[u]\|$ for all $u \in C^1(X,\mathbb{Z}_2)$ with $\|[u]\| \geq \delta$.

Corollary 6.11. *Let X be a 2-dimensional complex with n vertices and complete 1-skeleton such that $f_2(X) = c \cdot \binom{n}{3}$ and let $f = X - c \cdot K_n^2$.*

If $\operatorname{oct}(f_X) \le \delta^8 \cdot n^6$ *then* X *is coarsely combinatorially* $(\epsilon, c\delta/(1-\epsilon))$*-expanding for any* $\epsilon > 0$*, i.e.,*

$$\|\delta_X u\| \ge \epsilon \|[u]\|$$

for all $u \in C^1(X, \mathbb{Z}_2)$ *with* $\|[u]\| \ge \frac{7\delta}{c(1-\epsilon)}$.

Proof. Let $u \in C^1(X, \mathbb{Z}_2)$ with $\|[u]\| \ge \frac{7\delta}{c(1-\epsilon)}$. By Lemma 6.10 and the combinatorial expansion of the complete complex (Proposition 2.4) we have

$$\begin{aligned}
|\delta_X u| &\ge c|\delta_{K_n^2} u| - n^{18/8} \operatorname{oct}(f_X)^{1/8} \\
&\ge \frac{c}{3} \cdot n|[u]| - \delta n^3 \\
&\ge \left(\frac{c}{3} \cdot n - \frac{(1-\epsilon)cn^3}{7\binom{n}{2}}\right)|[u]|.
\end{aligned}$$

Hence,

$$\frac{|\delta_X u|}{f_2(X)} \ge \left(\frac{n\binom{n}{2}}{3\binom{n}{3}} - \frac{(1-\epsilon)n^3}{7\binom{n}{3}}\right) \frac{|[u]|}{\binom{n}{2}} \ge \epsilon \frac{|[u]|}{\binom{n}{2}}.$$

□

Lemma 6.10 has similarities to the Expander Mixing Lemma for graphs but there is a significant difference: The bounding term in the Expander Mixing Lemma is expressed in the size of the vertex sets $|S|$ and $|T|$, whereas the bound in Lemma 6.10 has no relation to the size of $\operatorname{supp}(u)$. Lemma 6.9 tells us that we could hope for a bounding term with the order of magnitude $O(p|\delta_{K_n^2 u}|)$, maybe even for a term with size $O\big(\sqrt{np|\delta_{K_n^2 u}|}\big)$. With a similar proof as the one of Lemma 6.10 it is possible to get Proposition 6.12 - a bound, which does not achieve this "correct" order of magnitude but at least depends on u.

The deviation from the bound one could hope for is the same as above: For $X^2(n,p)$ with $p = \Omega(\log(n)/n)$ we have

$$(p|\delta_{K_n^2 u}|)^8 = O\big((\log(n)/n)^8 |\delta_{K_n^2 u}|^8\big) = O(n^4 \log(n)^8 |\delta_{K_n^2 u}|^4),$$

while a.a.s. $n^6 \operatorname{oct}(f_X)|\delta_{K_n^2} u|^4 = \Omega(n^8 \log(n)^2 |\delta_{K_n^2} u|^4)$.

An additional difference between these statements and the Expander Mixing Lemma is of course that we consider only one set of edges and its coboundary, which for graphs corresponds to $|E(S, V \setminus S)|$ for a vertex set S, whereas the Expander Mixing Lemma gives a bound for $|E(S,T)|$ for any T. It is not quite clear what a corresponding statement for 2-complexes should be.

Proposition 6.12. *Let X be a 2-dimensional complex with n vertices and complete 1-skeleton. Let $u \in C^1(X;\mathbb{Z}_2)$ such that $\delta_{K_n^2} u \neq 0$. Then for $f_X = X - pK_n^2$:*

$$||\delta_X u| - p|\delta_{K_n^2} u|| \leq n^{10/8} \operatorname{oct}(f_X)^{1/8} \sqrt{|u|}.$$

Hence, plugging in the combinatorial expansion of the complete complex, we have for u minimal, i.e., such that $|u| = |[u]|$,

$$||\delta_X u| - p|\delta_{K_n^2} u|| \leq \sqrt{3} \cdot n^{6/8} \operatorname{oct}(f_X)^{1/8} \sqrt{|\delta_{K_n^2} u|}.$$

The remainder of this section contains the proofs of Lemma 6.10, Proposition 6.12 and of Lemma 6.9. The first two proofs are similar to the proof of Gowers' Lemma 6.8. The main idea is the repeated application of the Cauchy-Schwarz inequality, more precisely the following simple consequence:

$$\left(\sum_{i=1}^{N} a_i\right)^2 \leq N \cdot \sum_{i=1}^{N} a_i^2. \tag{6.10}$$

We adopt a product notation introduced by Gowers in [55]. We will only use it for functions g in three variables x, y, z. We write, e.g., $g_{x_0,x_1}(y,z)$ for $g(x_0,y,z)g(x_1,y,z)$ and use $g_{x_0,x_1,y_0,y_1,z_0,z_1}$ as abbreviation for the product $\prod_{\epsilon\in\{0,1\}^3} g(x_{\epsilon_1}, y_{\epsilon_2}, z_{\epsilon_3})$.

Proof of Lemma 6.10. For $x, y, z \in V$, express the value of $\delta u(\{x,y,z\})$ as follows:

$$u(x,y)\left(1 - (u(x,z) - u(y,z))^2\right) + (1 - u(x,y))\left(u(x,z) - u(y,z)\right)^2.$$

We consider the term $\left(\sum_{x,y,z} f(x,y,z)\delta u(\{x,y,z\})\right)^8$. As in Gowers' proof of Lemma 6.8, the main idea is to repeatedly apply the consequence (6.10) of the Cauchy-Schwarz inequality. We will write $\phi(x,y,z)$ for $(u(x,z) - u(y,z))^2$, $\bar{\phi}(x,y,z)$ for $1 - \phi(x,y,z)$ and we let

$\bar{u}(x,y) = 1 - u(x,y)$. We start by applying (6.10) and get

$$\begin{aligned}
&\Big(\sum_{x,y,z} f(x,y,z)\delta u(\{x,y,z\})\Big)^8 \\
&\quad = \Big(\sum_{x,y,z} f(x,y,z)\left(u(x,y)\left(1-\phi(x,y,z)\right) + (1-u(x,y))\phi(x,y,z)\right)\Big)^8 \\
&\quad \le \left(n^2\sum_{x,y}\Big(\sum_{z} f(x,y,z)\left(u(x,y)\bar{\phi}(x,y,z) + \bar{u}(x,y)\phi(x,y,z)\right)\Big)^2\right)^4 \\
&\quad = n^8\cdot\left(\sum_{x,y}\Big(\sum_{z} f(x,y,z)\left(u(x,y)\bar{\phi}(x,y,z) + \bar{u}(x,y)\phi(x,y,z)\right)\Big)^2\right)^4.
\end{aligned}$$

We focus on the second factor and show that it is at most $n^{10}\operatorname{oct}(f_X)$. Consider the inner term. As $u(x,y)\bar{u}(x,y) = u(x,y)(1-u(x,y)) = 0$, we have:

$$\begin{aligned}
&\Big(\sum_{z} f(x,y,z)\left(u(x,y)\bar{\phi}(x,y,z) + \bar{u}(x,y)\phi(x,y,z)\right)\Big)^2 \\
&\quad = u(x,y)^2\Big(\sum_{z} f(x,y,z)\bar{\phi}(x,y,z)\Big)^2 \\
&\qquad\qquad + \bar{u}(x,y)^2\Big(\sum_{z} f(x,y,z)\phi(x,y,z)\Big)^2 \\
&\quad \le \sum_{z_0,z_1} f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y) + \phi_{z_0,z_1}(x,y)\right),
\end{aligned}$$

Hence, we can continue as follows:

$$\begin{aligned}
&\left(\sum_{x,y}\Big(\sum_{z} f(x,y,z)\left(u(x,y)\bar{\phi}(x,y,z) + \bar{u}(x,y)\phi(x,y,z)\right)\Big)^2\right)^4 \\
&\quad \le \Big(\sum_{z_0,z_1}\sum_{x,y} f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y) + \phi_{z_0,z_1}(x,y)\right)\Big)^4 \\
&\quad \le n^6\cdot\sum_{z_0,z_1}\Big(\sum_{x,y} f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y) + \phi_{z_0,z_1}(x,y)\right)\Big)^4,
\end{aligned}$$

where the last step is a second application of (6.10). Applying (6.10) once more, we can estimate the inner term as follows:

$$\Big(\sum_{x,y} f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y)+\phi_{z_0,z_1}(x,y)\right)\Big)^4$$
$$\leq n^2\cdot\bigg(\sum_x\Big(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y)+\phi_{z_0,z_1}(x,y)\right)\Big)^2\bigg)^2.$$

So showing that

$$\sum_{z_0,z_1}\bigg(\sum_x\Big(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y)+\phi_{z_0,z_1}(x,y)\right)\Big)^2\bigg)^2\leq n^2\operatorname{oct}(f_X)$$

will finish the proof. As $\phi(x,y,z)=u(x,z)\bar{u}(y,z)+\bar{u}(x,z)u(y,z)$ and $\bar{\phi}(x,y,z)=1-\phi(x,y,z)=\bar{u}(x,z)\bar{u}(y,z)+u(x,z)u(y,z)$, we have that

$$\begin{aligned}\bar{\phi}_{z_0,z_1}(x,y)&+\phi_{z_0,z_1}(x,y)\\&=\bar{\phi}(x,y,z_0)\bar{\phi}(x,y,z_1)+\phi(x,y,z_0)\phi(x,y,z_1)\\&\qquad=\bar{\varphi}(x,z_0,z_1)\bar{\varphi}(y,z_0,z_1)+\varphi(x,z_0,z_1)\varphi(y,z_0,z_1),\end{aligned}$$

where we set $\varphi(w,z_0,z_1)=u(w,z_0)\bar{u}(w,z_1)+\bar{u}(w,z_0)u(w,z_1)$ and then have that

$$\bar{\varphi}(w,z_0,z_1)=1-\varphi(w,z_0,z_1)=u(w,z_0)u(w,z_1)+\bar{u}(w,z_0)\bar{u}(w,z_1).$$

We can hence estimate the inner term

$$\Big(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y)+\phi_{z_0,z_1}(x,y)\right)\Big)^2$$

as follows:

$$\begin{aligned}&\Big(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y)+\phi_{z_0,z_1}(x,y)\right)\Big)^2\\&=\Big(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\varphi}(x,z_0,z_1)\bar{\varphi}(y,z_0,z_1)+\varphi(x,z_0,z_1)\varphi(y,z_0,z_1)\right)\Big)^2\\&=\bar{\varphi}(x,z_0,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)\bar{\varphi}(y,z_0,z_1)\Big)^2\\&\qquad+\varphi(x,z_0,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)\varphi(y,z_0,z_1)\Big)^2\\&\leq\sum_{y_0,y_1} f_{y_0,y_1,z_0,z_1}(x)\left(\bar{\varphi}_{y_0,y_1}(z_0,z_1)+\varphi_{y_0,y_1}(z_0,z_1)\right).\end{aligned}$$

Hence, we can proceed as follows:

$$
\begin{aligned}
&\left(\sum_x \left(\sum_y f_{z_0,z_1}(x,y)\left(\bar{\phi}_{z_0,z_1}(x,y) + \phi_{z_0,z_1}(x,y)\right)\right)^2\right)^2 \\
&\qquad \le \left(\sum_x \sum_{y_0,y_1} f_{y_0,y_1,z_0,z_1}(x)\left(\bar{\varphi}_{y_0,y_1}(z_0,z_1) + \varphi_{y_0,y_1}(z_0,z_1)\right)\right)^2 \\
&\qquad \le n^2 \sum_{y_0,y_1} \left(\sum_x f_{y_0,y_1,z_0,z_1}(x)\left(\bar{\varphi}_{y_0,y_1}(z_0,z_1) + \varphi_{y_0,y_1}(z_0,z_1)\right)\right)^2 \\
&\qquad = n^2 \sum_{y_0,y_1} \left(\bar{\varphi}_{y_0,y_1}(z_0,z_1) + \varphi_{y_0,y_1}(z_0,z_1)\right)^2 \sum_{x_0,x_1} f_{x_0,x_1,y_0,y_1,z_0,z_1} \\
&\qquad \le n^2 \sum_{y_0,y_1} \sum_{x_0,x_1} f_{x_0,x_1,y_0,y_1,z_0,z_1}.
\end{aligned}
$$

Putting everything together, we get:

$$
\begin{aligned}
\left(\sum_{x,y,z} f(x,y,z)\delta u(\{x,y,z\})\right)^8 &\le n^{18} \sum_{z_0,z_1} \sum_{y_0,y_1} \sum_{x_0,x_1} f_{y_0,y_1,z_0,z_1,x_0,x_1} \\
&= n^{18} \operatorname{oct}(f).
\end{aligned}
$$

□

The upcoming proof of Proposition 6.12 is very similar to the proof of Lemma 6.10, which we just saw. Recall that we expressed the value of $\delta u(\{x,y,z\})$ for $x,y,z \in V$ as

$$
u(x,y)\left(1 - (u(x,z) - u(y,z))^2\right) + (1 - u(x,y))\left(u(x,z) - u(y,z)\right)^2 .
$$

Now, instead of $\sum_{x,y,z} f_X(x,y,z)\delta u(\{x,y,z\}) = 3 \cdot ||\delta_X u| - p|\delta_{K_n^2} u||$, we consider the smaller term

$$
S(f) := \sum_{x,y,z} f_X(x,y,z) u(x,y)\left(1 - (u(x,z) - u(y,z))^2\right),
$$

which counts every triangle containing exactly one edge in $\operatorname{supp}(u)$ once, whereas every triangle with three edges in $\operatorname{supp}(u)$ is counted three times. Thus, $1/3 \cdot S(f) \le ||\delta_X u| - p|\delta_{K_n^2} u|| \le S(f)$ and $S(f)$ is a good estimate for $||\delta_X u| - p|\delta_{K_n^2} u||$.

Proof of Proposition 6.12. To simplify notation, we write f for f_X. We show that

$$
\left(\sum_{x,y,z} f(x,y,z) u(x,y)\left(1 - (u(x,z) - u(y,z))^2\right)\right)^8 \le \langle u,u\rangle^4 \cdot n^{10} \cdot \operatorname{oct}(f).
$$

The main difference to the proof of Lemma 6.10 is that we start by applying the actual Cauchy-Schwarz inequality. This gives us:

$$\begin{aligned}
&\Big(\sum_{x,y,z} f(x,y,z)u(x,y)\left(1-(u(x,z)-u(y,z))^2\right)\Big)^8 \\
&\leq \Bigg(\sum_{x,y} u(x,y)^2 \cdot \sum_{x,y}\Big(\sum_{z} f(x,y,z)\left(1-(u(x,z)-u(y,z))^2\right)\Big)^2\Bigg)^4 \\
&= \langle u,u\rangle^4 \cdot \Bigg(\sum_{x,y}\sum_{z_0,z_1} f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1)\Bigg)^4,
\end{aligned}$$

where we set $\psi(x,y,z) = \left(1-(u(x,z)-u(y,z))^2\right)$. We proceed with the second factor and show that it is at most $n^{10}\operatorname{oct}(f_X)$. From here on we argue similarly as in the proof of Lemma 6.10. By applying the consequence (6.10) of the Cauchy-Schwarz inequality three times, we get:

$$\begin{aligned}
&\Bigg(\sum_{x,y}\sum_{z_0,z_1} f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1)\Bigg)^4 \\
&\qquad\leq n^8 \cdot \sum_{z_0,z_1}\Bigg(\sum_{x}\Big(\sum_{y} f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1)\Big)^2\Bigg)^2.
\end{aligned}$$

We again concentrate on the second factor and show that it is at most $n^2\operatorname{oct}(f_X)$. Considering the inner sum we get, since $(1-(a-b)^2) = ab + (1-a)(1-b)$ for $a,b \in \{0,1\}$,

$$\begin{aligned}
&\sum_{y} f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1) \\
&= \sum_{y} f_{z_0,z_1}(x,y)\left(1-(u(x,z_0)-u(y,z_0))^2\right)\left(1-(u(x,z_1)-u(y,z_1))^2\right) \\
&= (1-u(x,z_0))(1-u(x,z_1))\sum_{y} f_{z_0,z_1}(x,y)(1-u(y,z_0))(1-u(y,z_1)) \\
&\quad + (1-u(x,z_0))u(x,z_1)\sum_{y} f_{z_0,z_1}(x,y)(1-u(y,z_0))u(y,z_1) \\
&\quad + u(x,z_0)(1-u(x,z_1))\sum_{y} f_{z_0,z_1}(x,y)u(y,z_0)(1-u(y,z_1)) \\
&\quad + u(x,z_0)u(x,z_1)\sum_{y} f_{z_0,z_1}(x,y)u(y,z_0)u(y,z_1).
\end{aligned}$$

Thus, writing $\bar{u}$ for $1-u$, we have:

$$
\begin{aligned}
&\Big(\sum_y f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1)\Big)^2\\
&= \bar{u}(x,z_0)^2\bar{u}(x,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)\bar{u}(y,z_0)\bar{u}(y,z_1)\Big)^2\\
&\qquad + \bar{u}(x,z_0)^2u(x,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)\bar{u}(y,z_0)u(y,z_1)\Big)^2\\
&\qquad + u(x,z_0)^2\bar{u}(x,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)u(y,z_0)\bar{u}(y,z_1)\Big)^2\\
&\qquad + u(x,z_0)^2u(x,z_1)^2\Big(\sum_y f_{z_0,z_1}(x,y)u(y,z_0)u(y,z_1)\Big)^2\\
&\le \sum_{y_0,y_1} f_{y_0,y_1,z_0,z_1}(x)\cdot\phi(y_0,y_1,z_0,z_1),
\end{aligned}
$$

where $\phi(y_0,y_1,z_0,z_1)$ is defined as

$$\bar{u}_{y_0,y_1,z_0,z_1}+\bar{u}_{y_0,y_1}(z_0)u_{y_0,y_1}(z_1)+u_{y_0,y_1}(z_0)\bar{u}_{y_0,y_1}(z_1)+u_{y_0,y_1,z_0,z_1}.$$

Plugging in what we have so far and applying (6.10) once again we can proceed as follows:

$$
\begin{aligned}
&\sum_{z_0,z_1}\Big(\sum_x\Big(\sum_y f_{z_0,z_1}(x,y)\psi(x,y,z_0)\psi(x,y,z_1)\Big)^2\Big)^2\\
&\qquad\le n^2\sum_{z_0,z_1}\sum_{y_0,y_1}\Big(\phi(y_0,y_1,z_0,z_1)\sum_x f_{y_0,y_1,z_0,z_1}(x)\Big)^2\\
&\qquad\qquad\le n^2\sum_{z_0,z_1}\sum_{y_0,y_1}\sum_{x_0,x_1} f_{x_0,x_1,y_0,y_1,z_0,z_1}=n^2\operatorname{oct}(f).
\end{aligned}
$$

□

We finish this section with the proof of Lemma 6.9 on random complexes $X^2(n,p)$.

Proof of Lemma 6.9. We first address the deviation of the sizes of coboundaries from their expected size. Let $\varepsilon>0$ and let $C>18/\epsilon^2$. For $X=X^2(n,p)$ with $p\ge C\cdot\frac{\log(n)}{n}$ we consider

$$P:=\Pr[\exists u\in C^1(X;\mathbb{Z}_2)\text{ minimal }:||\delta_X u|-p|\delta_{K_n^2}u||>\varepsilon p|\delta_{K_n^2}u|]$$

Fix $u \in C^1(X;\mathbb{Z}_2)$ minimal, i.e., such that $|u| = |[u]|$, we have by the combinatorial expansion of the complete complex (Proposition 2.4) $|\delta_{K_n^2} u| \geq n/3|u|$. Then by the Chernoff bound (Theorem 1.2):

$$\begin{aligned}\Pr[||\delta_X u| - p|\delta_{K_n^2} u|| > \varepsilon p|\delta_{K_n^2} u|] &\leq e^{-\frac{\varepsilon^2}{3} p|\delta_{K_n^2} u|} \\ &\leq e^{-\frac{\varepsilon^2}{9} pn|u|} \leq e^{-\frac{\varepsilon^2 \cdot C}{9} \log(n)|u|}.\end{aligned}$$

Applying a union bound, and estimating the number of $u \in C^1(X;\mathbb{Z}_2)$ that are minimal and satisfy $|u| = a$ by $\binom{\binom{n}{2}}{a}$ we get

$$P \leq \sum_{a=1}^{\binom{n}{2}} \binom{\binom{n}{2}}{a} e^{-\frac{\varepsilon^2 \cdot C}{9} \log(n) a} \leq \sum_{a=1}^{\binom{n}{2}} e^{(2-\frac{\varepsilon^2 \cdot C}{9}) \log(n) a} = \sum_{a=1}^{\binom{n}{2}} n^{-\alpha a}.$$

As $\alpha := \frac{\varepsilon^2 \cdot C}{9} - 2 > 0$ by the choice of C, this sum converges to zero. This proves the first statement.

Now, we look at $\operatorname{oct}(f_X)$. Recall that we have for $f_X = X - p \cdot K_n^2$:

$$\operatorname{oct}(f_X) = \sum_{x_0,x_1 \in V} \sum_{y_0,y_1 \in V} \sum_{z_0,z_1 \in V} \prod_{(i,j,k) \in \{0,1\}^3} f_X(x_i, y_j, z_k).$$

Let us first look at the expected value for terms of the type

$$\prod_{(i,j,k) \in \{0,1\}^3} f_X(x_i, y_j, z_k).$$

If there are indices $i, j \in \{0,1\}$ such that $x_i = y_j$, $x_i = z_j$ or $y_i = z_j$, the corresponding term is obviously 0. Otherwise, depending on the choices of x_0, x_1, y_0, y_1 and z_0, z_1, such a term is of the following form:

$$\prod_{\substack{(i,j,k) \\ \in \{0,1\}^3}} f_X(x_i, y_j, z_k) = \prod_{l=1,\ldots,s} f_X(x^{(l)}, y^{(l)}, z^{(l)})^{a_l}$$

where each of the s factors corresponds to a distinct triangle, and the triangle $\{x^{(l)}, y^{(l)}, z^{(l)}\}$ appears a_l times in $\prod_{(i,j,k) \in \{0,1\}^3} f_X(x_i, y_j, z_k)$. Because the existence of triangles in $X^2(n,p)$ is decided independently, the expected value is then

$$\mathbb{E}[f_X(x,y,z)^{a_0}] \cdot \mathbb{E}[f_X(x',y',z')^{a_1}] \cdot \ldots \cdot \mathbb{E}[f_X(x^{(s)}, y^{(s)}, z^{(s)})^{a_s}].$$

Consider $\mathbb{E}[f_X(x,y,z)^a]$ where $a \geq 1$ and $x,y,z \in V$ define a triangle in K_n^2. For the case $a=1$, clearly $\mathbb{E}[f_X(x,y,z)] = 0$. Otherwise, if a is even, which is the only case we will encounter,

$$\mathbb{E}[f_X(x,y,z)^a] = p(1-p)^a + (1-p)(-p)^a \leq p + p^a \leq 2p.$$

Now, consider $\mathbb{E}[\prod_{(i,j,k)\in\{0,1\}^3} f_X(x_i,y_j,z_k)]$. If all x_i, y_j and z_k are distinct, the expectation is zero. Otherwise every triangle $\{x_i,y_j,z_k\}$ appears an even number of times and we get

$$\mathbb{E}\Big[\prod_{\substack{(i,j,k)\\ \in\{0,1\}^3}} f_X(x_i,y_j,z_k)\Big] \leq 2^s p^s,$$

where s is the number of distinct triangles of type $\{x_i,y_j,z_k\}$. Distinguishing the possible cases, this yields

$$\mathbb{E}[\mathrm{oct}(f_X)] = O(n^5p^4 + n^4p^2 + n^3p).$$

We want to use Chebyshev's inequality (Theorem 1.1) to bound the deviation of $\mathrm{oct}(f_X)$ from its expectation and hence also consider the variance of $\mathrm{oct}(f_X)$. Write $(\bar{x},\bar{y},\bar{z})$ for $(x_0,x_1,y_0,y_1,z_0,z_1)$ and let $Y(\bar{x},\bar{y},\bar{z}) = \prod_{(i,j,k)\in\{0,1\}^3} f_X(x_i,y_j,z_k)$. Then $\mathrm{Var}[\mathrm{oct}(f_X)]$ can be expressed as

$$\sum_{(\bar{x},\bar{y},\bar{z})\in V^6} \mathrm{Var}\big[Y(\bar{x},\bar{y},\bar{z})\big] + \sum_{\substack{(\bar{x},\bar{y},\bar{z})\neq\\ (\bar{x}',\bar{y}',\bar{z}')\in V^6}} \mathrm{Cov}\big[Y(\bar{x},\bar{y},\bar{z}), Y(\bar{x}',\bar{y}',\bar{z}')\big],$$

where $\mathrm{Cov}[Y,Y']$ denotes the covariance of two random variables Y and Y'. We first consider $\mathrm{Var}\big[Y(\bar{x},\bar{y},\bar{z})\big]$ for $(\bar{x},\bar{y},\bar{z})$ fixed. We have:

$$\mathrm{Var}\big[Y(\bar{x},\bar{y},\bar{z})\big] = \mathbb{E}\big[Y(\bar{x},\bar{y},\bar{z})^2\big] - \mathbb{E}\big[Y(\bar{x},\bar{y},\bar{z})\big]^2 \leq \mathbb{E}\big[Y(\bar{x},\bar{y},\bar{z})^2\big].$$

Similar considerations as above show that

$$\sum_{(\bar{x},\bar{y},\bar{z})\in V^6} \mathbb{E}\big[Y(\bar{x},\bar{y},\bar{z})^2\big] = O(n^6p^8 + n^5p^4 + n^4p^2 + n^3p).$$

Now, we fix $(\bar{x},\bar{y},\bar{z}) \neq (\bar{x}',\bar{y}',\bar{z}')$ and write $Y = Y(\bar{x},\bar{y},\bar{z})$ and $Y' = Y(\bar{x}',\bar{y}',\bar{z}')$. If $(\bar{x},\bar{y},\bar{z})$ and $(\bar{x}',\bar{y}',\bar{z}')$ have no common triangle, then Y and Y' are independent and hence $\mathrm{Cov}\big[Y,Y'\big] = 0$. Otherwise, we have

$$\mathrm{Cov}\big[Y,Y'\big] = \mathbb{E}\big[Y\cdot Y'\big] - \mathbb{E}\big[Y\big]\cdot\mathbb{E}\big[Y'\big] \leq \mathbb{E}\big[Y\cdot Y'\big],$$

as the discussion above shows that $\mathbb{E}[Y], \mathbb{E}[Y'] \geq 0$. Just as above, we have $\mathbb{E}\big[Y \cdot Y'\big] = 0$ if there are i and j such that $x_i = y_j$, $x_i = z_j$ or $y_i = z_j$, or $x'_i = y'_j$, $x'_i = z'_j$ or $y'_i = z'_j$. The same holds if $(\bar{x}, \bar{y}, \bar{z})$ or $(\bar{x}, \bar{y}, \bar{z})$ contains 6 distinct vertices and $\{\bar{x}, \bar{y}, \bar{z}\} \neq \{\bar{x}, \bar{y}, \bar{z}\}$. Otherwise, following similar arguments as above we have $\mathbb{E}\big[Y \cdot Y'\big] = 2^s p^s$ where s is the number of distinct triangles of type $\{x_i, y_j, z_k\}$ or $\{x'_i, y'_j, z'_k\}$. Considering all possible choices for $(\bar{x}, \bar{y}, \bar{z})$ and $(\bar{x}', \bar{y}', \bar{z}')$ in these remaining cases, we see that

$$\sum_{\substack{(\bar{x},\bar{y},\bar{z})\neq \\ (\bar{x}',\bar{y}',\bar{z}') \\ \in V^6}} \operatorname{Cov}\big[Y(\bar{x}, \bar{y}, \bar{z}), Y(\bar{x}', \bar{y}', \bar{z}')\big] = O(n^6p^8 + n^7p^6 + n^5p^3 + n^4p^2 + n^3p).$$

Together, these results give:

$$\operatorname{Var}[\operatorname{oct}(f_X)] = O(n^8p^6 + n^7p^6 + n^5p^3 + n^4p^2 + n^3p).$$

Now, let $p = C \cdot \frac{\log(n)}{n}$. The considerations above tell us that

$$\mathbb{E}[\operatorname{oct}(f_X)] = O(n^2 \log(n)^2) \text{ and } \operatorname{Var}[\operatorname{oct}(f_X)] = O(n^2 \log(n)^6).$$

Combining this with Chebyshev's inequality, we see that there are $c_0 = c_0(C)$ and $c_1 = c_1(C)$ such that

$$\Pr\big[|\operatorname{oct}(f_X) - c_0 \cdot n^2 \log(n)^2| \geq n \log(n)^4\big] \leq \frac{c_1 \cdot n^2 \log(n)^6}{n^2 \log(n)^8}.$$

Hence, there is $c = c(C)$ such that we have a.a.s. $\operatorname{oct}(f_X) \geq c \cdot n^2 \log(n)^2$.
$\square$

Chapter 7

The Largest Laplacian Eigenvalue and Partiteness

As we have seen there is a strong and well-studied relation between the second Laplacian eigenvalue of a graph and its edge expansion. It is a rather simple observation that each of these two graph parameters is zero if and only if the other one is. The Cheeger inequality extends this observation by demonstrating that each of these two values basically determines the behaviour of the other. Very recently, similar results have been achieved for the eigenvalue at the other end of the spectrum.

It is again a simple observation that, for a d-regular graph, say, the largest eigenvalue of the Laplacian is $2d$ if and only if G has a bipartite connected component. In [109] Trevisan presents a strengthening of this observation, analogous to the Cheeger inequality: The distance of the largest eigenvalue from its extremal value is a measure of how close a graph is to having a bipartite component.

More precisely, Trevisan introduces the following parameter, the *bipartiteness ratio* of a d-regular graph G:

$$\beta(G) := \min_{S \subset V, L \sqcup R = S} \frac{2|E(R)| + 2|E(L)| + |E(S, V \setminus S)|}{d|S|},$$

where for a vertex set $S \subset V$ we have $E(S) = \{e \in E : e \subset S\}$. Thus, a bipartiteness ratio of 0 means that there is a connected component S that is bipartite. His result is then phrased as follows:

$$1/2(1 - |\lambda_n|) \leq \beta(G) \leq \sqrt{2(1 - |\lambda_n|)}.$$

Here λ_n is the smallest eigenvalue of the matrix $M = 1/dA$. Hence, $1 - \lambda_n$ is the largest eigenvalue of the normalized Laplacian of G and

$|\lambda_n|$ can be at most 1. Moreover, G has a bipartite connected component if and only if $\lambda_n = -1$. Trevisan's result is the promised more qualified version of this statement: The distance of $|\lambda_n|$ to 1 captures how close G is to having such a component.

In this chapter, we study possible generalizations for bipartiteness for 2-dimensional simplicial complexes and find an analogue of the upper bound of Trevisan's result. While the actual aim of Trevisan's paper [109] is an approximation algorithm for the Maxcut problem that is based on the result described here, our results so far are of purely theoretical interest.

Generalization to 2-Complexes Phrased directly in terms of the largest eigenvalue of the normalized Laplacian, Trevisan's results can also be applied to general graphs. The outcome is the following theorem:

Theorem 7.1 ([109]). *Let G be a graph without isolated vertices. Let $\lambda_n(\Delta)$ be the largest eigenvalue of the normalized Laplacian $\Delta = \Delta(G)$. Define*

$$\beta(G) := \min_{S \subset V, L \sqcup R = S} \frac{2|E(R)| + 2|E(L)| + |E(S, V \setminus S)|}{\sum_{v \in S} \deg(v)}.$$

Then $1/2(2 - \lambda_n(\Delta)) \leq \beta(G) \leq \sqrt{2(2 - \lambda_n(\Delta))}$. *Alternatively:*

$$2(1 - \beta(G)) \leq \lambda_n(\Delta) \leq 2 - \frac{1}{2}\beta(G)^2.$$

In the following, we want to formulate a 2-dimensional analogue of $\beta(G)$. To make this notationally simpler, note that for a graph G we have $\beta(G) = 1 - \beta'(G)$ where

$$\beta'(G) := \max_{S \subset V, L \sqcup R = S} \frac{2|E(L, R)|}{\sum_{v \in S} \deg(v)}.$$

Our definition of $\beta'(X)$ for a 2-complex $X = (V, E, T)$ is inspired by the notion of expansion defined by Parzanchevski, Rosenthal and Tessler in [97] (see also Section 5.2). For $A_0, A_1, A_2 \subset V$ we let $T(A_0, A_1, A_2) = \{t \in T : |t \cap A_i| = 1 \text{ for all } i\}$ and define $\beta(X) := 1 - \beta'(X)$, where

$$\beta'(X) := \max_{\substack{S \subset V, \\ A_0 \sqcup A_1 \sqcup A_2 = S}} \frac{3|T(A_0, A_1, A_2)|}{\sum_{e \in E(S)} \deg(e)}.$$

We have $\beta(X) = 0$ if and only if there exists a vertex set S and a partition $A_0 \sqcup A_1 \sqcup A_2 = S$ such that $T(A_0, A_1, A_2) = T(S)$, where $T(S)$ is the set of triangles with at least one edge in $E(S)$. Thus, any triangle with an edge in $E(S)$ has all vertices in S, i.e., $E(S)$ contains a connected component of X with respect to hypergraph connectivity. Furthermore, every triangle in $T(S)$ is spanned by all three of the A_i.

Therefore, anticipating a definition from the next-section, $\beta(X)$ is 0 if and only if X has a *vertex-3-partite component* (w.r.t. hypergraph connectivity). We can show:

Theorem 7.2. *Let X be a pure 2-dimensional complex. Then*

$$3(1 - \beta(X)) \leq \lambda_{\max}(\Delta_1^{\mathrm{up}}),$$

where $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ is the largest eigenvalue of the normalized Laplacian $\Delta_1^{\mathrm{up}} = \Delta_1^{\mathrm{up}}(X)$.

It is known that $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ is at most 3. Thus, Theorem 7.2 tells us that $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ reaches this extremal value if $\beta(X) = 0$ and that $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ approaches 3 if X is close to having a vertex-3-partite connected component.

Next to vertex-3-partiteness, we suggest several other possible generalizations of bipartiteness and study how these relate to each other. This will shed some light on possible analogues of the upper bound in Theorem 7.1.

7.1 3-Partiteness for 2-Complexes

While higher-dimensional generalizations of the graph Laplacian are well studied, it is not clear how bipartiteness should be adapted to higher dimensions.

As we already observed, the definition of $\beta(X)$ suggests one notion, *vertex 3-partiteness*, for 2-complexes and Theorem 7.2 tells us that vertex 3-partiteness implies $\lambda_{\max}(\Delta_1^{\mathrm{up}}) = 3$ for a hypergraph-connected complex. For graphs, the converse also holds: A connected graph is bipartite if and only if the largest eigenvalue of its normalized Laplacian is 2. In this section, we will see that the corresponding statement is not true for 2-complexes. In response we will therefore introduce several possible notions of 3-partiteness and study their relations.

In Section 2.6, we already cited the combinatorial criterion of Horak and Jost [63] for $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ achieving the extremal value. Here, we recall their result for 2-complexes:

Lemma 7.3 ([63, Theorem 7.1])**.** *Let $X = (V, E, T)$ be a 2-complex. The largest eigenvalue of the normalized Laplacian Δ_1^{up} is 3 if and only if there is a connected component C of X (w.r.t. hypergraph connectivity) and an orientation of the triangles of X such that*

$$[t : e] = [t' : e] \text{ for all } e \subset t, t' \in C.$$

This is equivalent to the existence of orientations of the edges and triangles of X such that $[t : e] = 1$ for all $e \subset t \in C$.

Furthermore, this condition is equivalent to C not containing any orientable 2-circuits of odd length or non-orientable 2-circuits of even length.

The last condition can be seen as the analogue of the statement that a graph is bipartite if and only if it has no odd cycles. Recall that a pure 2-dimensional simplicial complex Y is a 2-*circuit of length* $(m-1)$ if there is an ordering of its triangles $t_1, t_2, \ldots, t_m = t_1$, such that any t_i and t_j with $|j - i| = 1$ share a common edge. It is *orientable* if it is possible to assign an orientation to all triangles of Y in a way such that any two simplices sharing a common edge induce a different orientation on this face.

Horak's and Jost's result inspires the following definition of *spectral partiteness*, based on the first condition on orientations. We moreover suggest the following additional possible generalizations of bipartiteness in graphs:

Definition 7.4. Let $X = (V, E, T)$ be a 2-complex.

1. X is *vertex-3-partite* if there are vertex sets $A_0, A_1, A_2 \subset V$ that partition V such that every $t \in T$ has exactly one vertex from each A_i.

2. X is *spectrally partite* if there are orientations of the edges and triangles of X such that $[t : e] = 1$ for all $e \subset t \in T$.

3. X is *vertex-link partite* if for every vertex $v \in V$ the link $\mathrm{lk}(v)$ is bipartite.

4. X is *edge-3-partite* if there is a partition of the edges of X into three pairwise-disjoint sets $B_0 \cup B_1 \cup B_2 = E$ such that every $t \in T$ has exactly one edge from each B_i, i.e., every triangle is a *rainbow triangle*.

The definitions of vertex-3-partiteness and edge-3-partiteness both seem natural generalizations of bipartiteness. As already explained, the first is motivated by the notion of expansion defined by Parzanchevski, Rosenthal and Tessler in [97]. Gower's notions of quasirandomness for 3-uniform hypergraphs [55] inspired the latter (see Section 6.2). The notion of vertex-link partiteness is more technical than the other notions but is linked to them as we see in the forthcoming lemma.

Remark 7.5. One might consider a complex X as $\mathbb{Z}_2$*-partite* if there is a set of edges $B \subset E$ such that every $t \in T$ has either exactly one or three edges from B. Unfortunately, this definition is satisfied by every 2-complex X, as this always holds for the set $B = E$, and also for $B = E \setminus E(S, V \setminus S)$ for any $S \subset V$.

Now, we consider the relations between the different partiteness properties presented above. We see that in contrast to the situation for graphs, the different notions are not equivalent for 2-complexes. Since the definition of the partiteness ratio $\beta(X)$ is based on the notion of vertex-3-partiteness, the first statement of the following lemma, which is illustrated in Figure 7.1, actually follows from Theorem 7.2, which gives a more refined account of the connection of the value of $\lambda_{\max}(\Delta_1^{\mathrm{up}})$ and vertex-3-partiteness.

As already mentioned in Section 2.6, Horak and Jost show that a 2-complex is spectrally partite if the chromatic number of its 1-skeleton is 3 [63, Theorem 7.3]. As a 3-colorable 1-skeleton implies vertex-3-partiteness, we strengthen their result here.

Lemma 7.6. *Let $X = (V, E, T)$ be a 2-complex.*

(a) If X is vertex-3-partite then X is spectrally partite, the converse does not hold.

(b) If X is spectrally partite then X is vertex-link partite, the converse does not hold.

(c) If X is vertex-3-partite then X is edge-3-partite. Edge-3-partiteness does not imply vertex-link partiteness. Furthermore, not all complexes are edge-3-partite.

Proof. (a) Let X be vertex-3-partite and fix a partition $A_0 \cup A_1 \cup A_2 = V$ such that every $t \in T$ has exactly one vertex from each A_i. To every edge $\{v_i, v_j\} \in E(A_i, A_j)$ with $i < j$ (or $i = 2$ and vertex 3-partite $\underset{\not\Leftarrow}{\Rightarrow}$ spectrally partite $\underset{\not\Leftarrow}{\Rightarrow}$ vertex-link partite

$\searrow$ edge 3-partite $\not\nearrow$

Figure 7.1: Lemma 7.6 illustrated

$j = 0$) and $v_i \in A_i$ give the orientation $[v_i, v_j]$. To every triangle $\{v_0, v_1, v_2\} \in T$ with $v_i \in A_i$ give the orientation $[v_0, v_1, v_2]$. These orientations show that X is spectrally partite. See Figure 7.2 for an illustration. To see that the converse does not hold consider the 2-complex depicted in Figure 7.3. It exhibits the right kind of orientation but it is impossible to partition the vertices in the desired way.

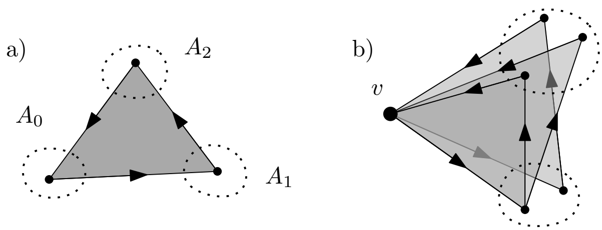

Figure 7.2: Illustrations for Lemma 7.6, parts a) and b)

(b) Let X be spectrally partite and consider a vertex $v \in V$. We get a bipartite partition of the vertices in $\mathrm{lk}(v)$ by forming two sets, according to the orientation of the edge they form with v (see Figure 7.2). An example of a 2-complex that is vertex-link partite but not spectrally partite is depicted in Figure 7.3: A triangulation of a cylindrical strip (the two edges labelled e should be identified) with an odd number of triangles.

(c) Let X be vertex-3-partite and fix a partition $A_0 \cup A_1 \cup A_2 = V$ such that every $t \in T$ has exactly one vertex from each A_i. Then X is edge-3-partite with respect to the edge partition $E(A_0, A_1)$, $E(A_1, A_2)$, $E(A_0, A_2)$. To see that edge-3-partiteness does not imply vertex-link partiteness consider again the boundary of a tetrahedron. A complex X that is not edge-3-partite is depicted in Figure 7.3. It consists of the boundaries of two tetrahedra

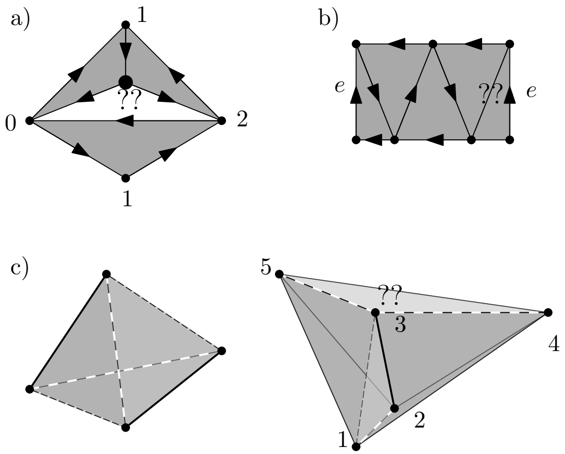

Figure 7.3: Counterexamples from Lemma 7.6

on the vertex sets $\{1,2,3,4\}$ and $\{1,2,3,5\}$ and of an additional triangle $\{3,4,5\}$. It is impossible to partition the edges into three sets such that every triangle is a rainbow triangle.

□

7.2 Proof of the Inequality

We now come to the proof of Theorem 7.2. It follows the idea of the proof of the corresponding inequality from Theorem 7.1 in [109] and also derives inspiration from the proof of the higher-dimensional discrete Cheeger inequality in [97].

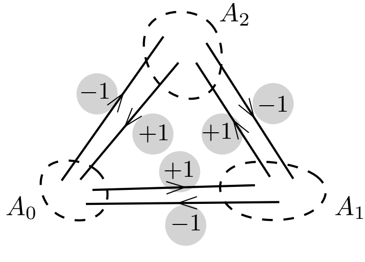

Figure 7.4: f illustrated

Proof of Theorem 7.2. Fix a set $S \subset V$ and a partition $A_0 \sqcup A_1 \sqcup A_2 = S$ such that $\beta'(X) = 1 - \beta(X) = \frac{3|T(A_1,A_2,A_3)|}{\sum_{e\in E(S)} \deg(e)}$. We define a function $f \in C^1(X)$ as depicted in Figure 7.4. Formally,

$$f(\{v,w\}) = \begin{cases} +1 & \text{if } v \in A_i, w \in A_j, j \equiv i+1 \pmod 3, \\ & \text{ and } e \text{ is oriented from } v \text{ to } w, \\ -1 & \text{if } v \in A_i, w \in A_j, j \equiv i+1 \pmod 3, \\ & \text{ and } e \text{ is oriented from } w \text{ to } v, \\ 0 & \text{otherwise.} \end{cases}$$

The important feature of f is that for $t \in T(A_0, A_1, A_2)$, $[t : e]f(e)$ is the same for any $e \subset t$. For $t \in T$ we have

$$\delta f(t) = \sum_{e \subset t} [t : e] f(e) = \begin{cases} \pm 3 & \text{if } t \in T(A_0, A_1, A_2), \\ \pm 1 & \text{if } t \in T(A_i, A_j, V \setminus S), i \neq j, \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\langle \Delta_1^{\mathrm{up}} f, f \rangle = \langle \delta f, \delta f \rangle = \sum_{t \in T} (\delta f(t))^2 \geq 9|T(A_0, A_1, A_2)|.$$

As furthermore $\langle f, f \rangle = \sum_{e\in E} \deg(e) f(e)^2 \leq \sum_{e \in E(S)} \deg(e)$, we get

$$\begin{aligned} \lambda_{\max}(\Delta_1^{\mathrm{up}}) &= \max_{g \neq 0} \frac{\langle \Delta_1^{\mathrm{up}} g, g \rangle}{\langle g, g \rangle} \\ &\geq \frac{\langle \Delta_1^{\mathrm{up}} f, f \rangle}{\langle f, f \rangle} \\ &\geq \frac{9|T(A_1, A_2, A_3)|}{\sum_{e \in E(S)} \deg(e)} = 3(1 - \beta(X)). \end{aligned}$$

□

In conclusion, we have seen a higher-dimensional analogue of the lower bound in Trevisan's result, Theorem 7.1 [109]. In light of the result of Lemma 7.6, we cannot expect an analogue of the upper bound in Theorem 7.1 involving the 3-partiteness ratio $\beta(X)$ as defined here. However, an analogue involving a partiteness ratio based on a different notion of partiteness than vertex-3-partiteness such as the ones we mentioned in Defintion 7.4 seems plausible.

Chapter 8

Mapping Simplicial Complexes

In this chapter we will consider maps of simplicial complexes into Euclidean spaces and study questions concerning the intersections of images of simplices under such maps.

In the first section we focus on affine maps. We apply a result of Parzanchevski, Rosenthal and Tessler from [97], a higher-dimensional analogue of the Expander Mixing Lemma, to establish a connection between the non-trivial eigenvalues of the Laplacian of a simplicial complex and the minimal number of crossings of image simplices.

In the second section, we consider a question concerning general continuous maps of simplicial complexes into Euclidean spaces. We are interested in the *overlap number* of a simplicial complex X, the largest constant c such that for any map there exists a point belonging to at least $c \cdot f_k(X)$ images of k-simplices from X. We will see that this can be understood as yet another measure of higher-dimensional expansion. Our result, however, only concerns the overlap number of the complete 3-complex. We study a structure, *pagodas*, that was introduced by Matoušek and Wagner in [91], in order to improve the known bounds for this overlap number.

The results in this chapter are joint work with Uli Wagner.

8.1 Laplacian Eigenvalues and the Crossing Number

For a k-dimensional simplicial complex X and a map $f : V \to \mathbb{R}^d$, we consider the affine extension $\|f\| : \|X\| \to \mathbb{R}^d$ of f, i.e., we extend f affinely on the relative interiors of the simplices of X. A pair of k-simplices (F_1, F_2), $F_i \in X_k$ is said to be *crossing* under f if F_1 and F_2 are vertex-disjoint and the relative interiors of their images under $\|f\|$ intersect. We define the d-dimensional *crossing number* $\mathrm{cr}_d(X)$ of X to be the minimal number of pairs of crossing k-simplices under any such affine map of X into $\mathbb{R}^d$. For any map we assume that the images of vertices are in general position to avoid degeneracies.

With parameters $k = 1$ and $d = 2$ this describes crossing numbers of graphs. The well-known Crossing Lemma of Ajtai, Chvátal, Newborn, and Szemerédi [2] states that for a graph $G = (V, E)$ on n vertices either $|E| = O(n)$, or $\mathrm{cr}_2(G) = \Omega(|E|^3/n^2)$. A connection between the Laplacian spectrum of a bipartite graph and its bipartite crossing number was established in [103]. For $k = 2$ and $d = 3$ the question has been studied by Dey and Edelsbrunner [33]. They show that $\mathrm{cr}_3(X) = \Omega((f_2(X))^4/n^6)$ for any 2-complex X with n vertices and $f_2(X) \geq 2n^2$. Note that $\mathrm{cr}_d(X) = 0$ for $d \geq 2k+1$ because any k-dimensional simplicial complex has a geometric realization in $\mathbb{R}^{2k+1}$. We will focus on the case $d = 2k$.

Our goal in this section will be to establish the following relation between the Laplacian spectrum of a k-complex and its $2k$-dimensional crossing number:

Proposition 8.1. *For any k and $0 < \varepsilon < \frac{1}{2(2k+3)!(k+1)!^2}$, there is $K = K(\varepsilon, k)$ such that the following holds:*

Let X be a k-dimensional simplicial complex with n vertices. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest, respectively, largest non-trivial eigenvalues of the Laplacian $L^{\mathrm{up}}_{k-1}(X)$. Denote furthermore by λ_{avg} the average of all non-trivial eigenvalues of $L^{\mathrm{up}}_{k-1}(X)$. Suppose that

$$|\lambda_{\min}/\lambda_{avg} - 1| \leq 1/K \text{ and } |\lambda_{\max}/\lambda_{avg} - 1| \leq 1/K$$

and let $\delta = 1 - K \cdot \max\big(|\lambda_{\min}/\lambda_{avg} - 1|, |\lambda_{\max}/\lambda_{avg} - 1|\big) > 0$. Then there are at least

$$\varepsilon\delta^2\lambda_{avg}^2 \cdot n^{2k}$$

pairs of crossing k-simplices under any affine map from X into $\mathbb{R}^{2k}$, i.e.,

$$\mathrm{cr}_{2k}(X) = \Omega(\delta^2 \lambda_{avg}^2 \cdot n^{2k}).$$

Our result heavily relies on a theorem by Fox, Gromov, Lafforgue, Naor and Pach on semi-algebraic relations [45] and further uses a higher-dimensional analogue of the Expander Mixing Lemma by Parzanchevski, Rosenthal and Tessler [97] for their notion of expansion, see also Sections 2.3 and 5.2.

A *semi-algebraic set* in $S \subseteq \mathbb{R}^d$ is the locus of all points that satisfy a given finite Boolean combination of polynomial equations and inequalities in the d coordinates. The *description complexity* of such a set S is the smallest number $s \geq d$ such that there is some representation of S that involves at most s equations and inequalities, each of degree at most s. Semi-algebraic sets are classically studied in real algebraic geometry, for more information see, e.g., [17]. Fox, Gromov, Lafforgue, Naor and Pach show:

Theorem 8.2 ([45, Theorem 1.10])**.** *For any $h, s \in \mathbb{N}$ and for any $\varepsilon > 0$, there exists $K = K(\varepsilon, h, s)$ satisfying the following condition. For any $\ell \geq K$ and for any semi-algebraic relation R on h-tuples of points in a Euclidean space $\mathbb{R}^d$ with description complexity at most s, every finite set $P \subset \mathbb{R}^d$ has an equipartition $P = P_1 \cup \ldots \cup P_\ell$ such that all but at most an ε-fraction of the h-tuples $(P_{i_1}, \ldots, P_{i_h})$ have the property that either all h-tuples of points with one element in each P_{i_j} are related with respect to R or none of them are.*

Here, an *equipartition* of a finite set is a partition of the set into subsets whose sizes differ by at most one. We want to apply this result to a semi-algebraic relation that is based on the intersection behaviour of the simplices spanned by tuples of points: For $2k+2$ points $x_1, x_2, \ldots, x_{k+1}$ and $y_1, y_2, \ldots, y_{k+1}$ in general position in $\mathbb{R}^{2k}$, the two k-simplices $\mathrm{conv}(x_1, x_2, \ldots, x_{k+1})$ and $\mathrm{conv}(y_1, y_2, \ldots, y_{k+1})$ either intersect in a common point or not at all. This can be decided by evaluating polynomial inequalities, see Lemma 8.7. The following proposition is hence a straight-forward consequence of Theorem 8.2:

Proposition 8.3. *For any k and $\varepsilon > 0$ there is a $K = K(\varepsilon, k)$ such that for all $\ell \geq K$ the following holds: For all finite point sets $P \subset \mathbb{R}^{2k}$ there is an equipartition $P_1 \cup P_2 \cup \ldots \cup P_\ell = P$ and a set $M \subset [\ell]^{2k+2}$,*

$|M| \geq (1-\epsilon)\ell^{2k+2}$ *such that for all* $(i_1, i_2, \ldots, i_{2k+2}) \in M$*:*

$$\begin{gathered}\operatorname{conv}(x_1, \ldots, x_{k+1}) \cap \operatorname{conv}(x_{k+2}, \ldots, x_{2k+2}) \neq \emptyset \\ \Leftrightarrow \\ \operatorname{conv}(y_1, \ldots, y_{k+1}) \cap \operatorname{conv}(y_{k+2}, \ldots, y_{2k+2})) \neq \emptyset\end{gathered}$$

for all $(x_1, x_2, \ldots, x_{2k+2}), (y_1, y_2, \ldots, y_{2k+2}) \in P_{i_1} \times P_{i_2} \times \ldots \times P_{i_{2k+2}}$.

We combine this with the following result by Parzanchevski, Rosenthal and Tessler [97], which is a higher-dimensional analogue of the Expander Mixing Lemma (see Theorem 2.3).

Theorem 8.4 ([97, Theorem 1.4])**.** *Let* X *be a* k*-dimensional simplicial complex with a complete* $(k-1)$*-skeleton. Let* $\lambda_{\min}$ *and* $\lambda_{\max}$ *be the smallest, respectively, largest non-trivial eigenvalues of the Laplacian* $L^{\mathrm{up}}_{k-1}(X)$*. Denote furthermore by* λ_{avg} *the average of all non-trivial eigenvalues of the Laplacian* $L^{\mathrm{up}}_{k-1}(X)$*. For any disjoint sets of vertices* $A_1, \ldots, A_{k+1}$ *(not necessarily a partition),*

$$\left| |F(A_1, \ldots, A_{k+1})| - \frac{\lambda_{avg} \cdot |A_1| \cdot \ldots \cdot |A_{k+1}|}{n} \right| \leq \rho \cdot (|A_1| \cdot \ldots \cdot |A_{k+1}|)^{\frac{k}{k+1}},$$

where $\rho = \max(|\lambda_{\max} - \lambda_{avg}|, |\lambda_{\min} - \lambda_{avg}|)$*.*

The Expander Mixing Lemma for graphs gives a bound for the deviation of the size of the edge set $E(S,T)$ from the expected size in a random graph of corresponding density. The same is true for the result above, even though the result is phrased in terms of λ_{avg}, the average of non-trivial eigenvalues of $L^{\mathrm{up}}_{k-1}(X)$. For a k-complex X with complete $(k-1)$-skeleton it is a simple observation that the density is identical with $\frac{\lambda_{\mathrm{avg}}}{n}$:

$$\frac{\lambda_{\mathrm{avg}}}{n} = \frac{\sum_{\lambda \text{ non-trivial eigenvalue}} \lambda}{n\binom{n-1}{k}} = \frac{\sum_{F \in X_k} \deg(F)}{(k+1)\binom{n}{k+1}} = \frac{f_k(X)}{\binom{n}{k+1}}.$$

Note that there are $\binom{n-1}{k} = \binom{n}{k} - \binom{n-1}{k-1}$ non-trivial eigenvalues of $L^{\mathrm{up}}_{k-1}(X)$, since for any k-complex X with complete $(k-1)$-skeleton, the space $B^{(k-1)}(X) = \operatorname{im} \delta_{k-2}$ has dimension $\binom{n-1}{k-1}$, see Lemma 2.5.

Proof of Proposition 8.1

We now come to the proof of Proposition 8.1. The basic idea is as follows: Given a partition $P_1 \cup P_2 \cup \ldots \cup P_K$ of the image points of the vertices of X into K sets as in Proposition 8.3, we choose one representative of each partition set and consider the induced map for the complete complex K_2^K on these vertices. We know that for most choices of $2k+2$ indices the intersection behaviour of the corresponding simplices is determined by the behaviour on the representatives. We will then use a lower bound for the number of crossing pairs in the complete complex to show that many of such choices of indices yield crossings. To achieve the bound for X we connect the number of simplices among any $k+1$ partition sets with the Laplacian eigenvalues of X, using Theorem 8.4.

To establish a lower bound on the crossing number $\mathrm{cr}_{2k}(K_n^k)$ of the complete complex, we use the following well-known theorem of van Kampen and Flores [44, 43, 114, 113]. A modern treatment can be found in [90].

Theorem 8.5 (Van Kampen-Flores Theorem). *For $d \geq 1$ the complex K_{2k+3}^k does not embed into $\mathbb{R}^{2k}$. More precisely, for every continuous map $f : ||K_{2k+3}^k|| \to \mathbb{R}^{2k}$ there exist two disjoint simplices $F_1, F_2 \in K_{2k+3}^k$ such that $f(F_1) \cap f(F_2) \neq \emptyset$.*

This yields the following simple bound:

Lemma 8.6. *Let $f : \|K_n^k\| \to \mathbb{R}^{2k}$. Then the number of pairs of crossing k-simplices under f is at least $\frac{1}{2k+3}\binom{n}{2k+2}$.*

Proof. By the Van Kampen-Flores Theorem among any $2k+3$ vertices of K_n^2 there are two disjoint crossing k-simplices. If we sum this up over all choices of $2k+3$ vertices, any such crossing pair can be counted at most $n-2k-2$ times. Hence, total number of crossing pairs is at least $\frac{1}{n-2k-2} \cdot \binom{n}{2k+3} = \frac{1}{2k+3}\binom{n}{2k+2}$. $\square$

We now have all ingredients to turn our attention to the main proof:

Proof of Proposition 8.1. Let X be a k-dimensional simplicial complex and fix some $0 < \varepsilon < \frac{1}{2(2k+3)!(k+1)!^2}$. For a map $f : V(X) \to \mathbb{R}^{2k}$ consider the set of image points $P = \{f(v) : v \in V\} \subset \mathbb{R}^{2k}$.

Choose a constant $K = K(\varepsilon, k)$ as in Proposition 8.3 and define $\varepsilon' = \frac{1}{2\cdot(2k+3)!} - (k+1)!^2 \cdot \varepsilon$. Choose an equipartition $P_1 \cup P_2 \cup \ldots \cup P_K = P$ and a set $M \subset [K]^{2k+2}$, $|M| \geq (1-\varepsilon')K^{2k+2}$ as in Proposition 8.3. Furthermore, let $\delta = 1 - K \cdot \max\big(|\lambda_{\min}/\lambda_{\text{avg}} - 1|, |\lambda_{\max}/\lambda_{\text{avg}} - 1|\big)$.

For every $i \in \{1, \ldots, K\}$ fix a point $p_i \in P_i$ and consider the map $g : [K] \to \mathbb{R}^{2k+2}$ defined by $g(i) = p_i$ and the straight-line mapping of

K_K^k to $\mathbb{R}^{2k}$ induced by g. By Lemma 8.6 the number of pairs of crossing k-simplices under g is at least $\frac{1}{2k+3}\binom{K}{2k+2}$. So the set

$$N := \{(i_1, \ldots, i_{2k+2}) : \operatorname{conv}(p_1, \ldots, p_{k+1}) \cap \operatorname{conv}(p_{k+2}, \ldots, p_{2k+2}) \neq \emptyset\}$$

has size at least

$$|N| \geq \frac{K^{2k+2}}{2 \cdot (2k+3)!}.$$

We are interested in the set $M \cap N$, as for $(i_1, i_2, \ldots, i_{2k+2}) \in M \cap N$ we know that

$$\operatorname{conv}(x_1, \ldots, x_{k+1}) \cap \operatorname{conv}(x_{k+2}, \ldots, x_{2k+2}) \neq \emptyset$$

for all $(x_1, x_2, \ldots, x_{2k+2}) \in P_{i_1} \times P_{i_2} \times \ldots \times P_{i_{2k+2}}$. By choice of M and the bound on $|N|$ given above, we have that

$$|M \cap N| \geq |M| + |N| - K^{2k+2} \geq \left(\tfrac{1}{2\cdot(2k+3)!} - \varepsilon'\right) K^{2k+2} \geq (k+1)!^2 \varepsilon K^{2k+2}.$$

Now, for $i_1, \ldots, i_{k+1} \in [K]$ pairwise distinct, let $F(P_{i_1}, \ldots, P_{i_{k+1}}) = \{\{v_1, \ldots, v_{k+1}\} \in X : f(v_j) \in P_{i_j}\}$. Since we can w.l.o.g. assume that X has a complete $(k-1)$-skeleton, by Theorem 8.4 we have

$$\begin{aligned}
|F(P_{i_1}, \ldots, P_{i_{k+1}})| &\geq \frac{\lambda_{\text{avg}} \cdot |P_{i_1}| \cdot \ldots \cdot |P_{i_{k+1}}|}{n} - \rho(|P_{i_1}| \cdot \ldots \cdot |P_{i_{k+1}}|)^{\frac{k}{k+1}} \\
&= (\lambda_{\text{avg}} - K \cdot \rho) \frac{n^k}{K^{k+1}},
\end{aligned}$$

where $\rho = \max(|\lambda_{\max} - \lambda_{\text{avg}}|, |\lambda_{\min} - \lambda_{\text{avg}}|)$. Observe that $\lambda_{\text{avg}} - K \cdot \rho = \lambda_{\text{avg}}\delta > 0$.

For $(i_1, \ldots, i_{2k+2}) \in M \cap N$ every $(2k+2)$-tuple $(x_1, \ldots, x_{2k+2}) \in P_{i_1} \times P_{i_2} \times \ldots \times P_{i_{2k+2}}$ defines a pair of crossing k-simplices. The number of such pairs is at least

$$|F(P_{i_1}, \ldots, P_{i_{k+1}})| \cdot |F(P_{i_{k+2}}, \ldots, P_{i_{2k+2}})| \geq \delta^2 \lambda_{\text{avg}}^2 \cdot \frac{n^{2k}}{K^{2k+2}}.$$

Accounting for multiple counting – each pair might have been counted $(k+1)!^2$ times – the total number of crossings is at least:

$$\frac{1}{(k+1)!^2} |M \cap N| \cdot \delta^2 \lambda_{\text{avg}}^2 \cdot \frac{n^2 k}{K^{2k+2}} \geq \varepsilon \delta^2 \lambda_{\text{avg}}^2 \cdot n^{2k}.$$

□

Lemma 8.7. *For* $x_1, \ldots, x_{k+1} \in \mathbb{R}^{2k}$ *let* $\mathrm{conv}(\bar{x}) = \mathrm{conv}(x_1, \ldots, x_{k+1})$. *Consider the following relation on* $(k+1)$*-tuples of points in* $\mathbb{R}^{2k}$*:*

$$R := \{(x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}) \in (\mathbb{R}^{2k})^{2k+2} : \mathrm{conv}(\bar{x}) \cap \mathrm{conv}(\bar{y}) \neq \emptyset\}.$$

Then R *is a semi-algebraic relation of constant description complexity.*

Proof. For $(\bar{x}, \bar{y}) = (x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1})$ with $x_i, y_i \in \mathbb{R}^{2k}$ define the following $(2k+2) \times (2k+2)$-matrix:

$$M = M(\bar{x}, \bar{y}) := \begin{pmatrix} | & & | & | & & | \\ x_1 & \ldots & x_{k+1} & -y_1 & \ldots & -y_{k+1} \\ | & & | & | & & | \\ 1 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{pmatrix}.$$

Then $(\bar{x}, \bar{y}) \in R$ if and only if there is a vector $z \in \mathbb{R}^{2k+2}$ that satisfies $Mz = (0, \ldots, 0, 1, 1)^T$ and such that $z_i \in [0, 1]$ for $0 \leq i \leq 2k+2$. A solution of $Mz = (0, \ldots, 0, 1, 1)^T$ exists if and only if $\det M \neq 0$. If it does, we have by Cramer's Rule

$$z_i = \frac{\det M_i}{\det M},$$

where M_i is the matrix formed by replacing the i-th column of M by $(0, \ldots, 0, 1, 1)^T$.

Thus,

$$R = \{(\bar{x}, \bar{y}) : 0 \leq \det M_i(\bar{x}, \bar{y}) \leq \det M(\bar{x}, \bar{y}) \text{ for } i = 0 \leq i \leq 2k+2\}.$$

As $\det M_i(\bar{x}, \bar{y})$ as well as $\det M(\bar{x}, \bar{y})$ are polynomials of constant degree, this is a semi-algebraic description of R with constant description complexity. □

8.2 Pagodas

In this section, instead of the number of crossings under a map, we are interested in the number of image simplices sharing a common point. For a k-dimensional simplicial complex X, its *overlap number* $c(X)$ is the largest $c \in (0, 1]$ such that for any map $f : V \to \mathbb{R}^k$ there is a point in $\mathbb{R}^k$ that is contained in the convex hulls of the images of the vertices of at least $c \cdot f_k(X)$ many k-simplices of X.

For a graph G it is easy to see that large expansion implies a large overlap number: For any map $f : V(G) \to \mathbb{R}$ the median of image points is contained in at least $\varepsilon(G)/4 \cdot |E(G)|$ many edges. The overlap number can hence be interpreted as yet another measure of higher-dimensional expansion. See, e.g., [45], where Fox, Gromov, Lafforgue, Naor and Pach show the existence of infinite families $\{X_n : n \in \mathbb{N}\}$ of higher dimensional complexes with bounded degree that are highly overlapping, i.e., there exists a $c > 0$ such that $c(X_n) > c$ for all n. In [97], Parzanchevski, Rosenthal and Tessler connect the notion of spectral expansion with the overlap number. Similar to our result from the previous section, they use their higher-dimensional analogue of the Expander Mixing Lemma and establish a lower bound for the overlap number under the condition that the Laplacian spectrum is sufficiently concentrated.

For complete complexes $X = K_n^k$ this question, phrased in terms of point sets in $\mathbb{R}^k$ and the simplices spanned by them, has received a lot of attention. It is known that $c(K_n^k)$ asymptotically does not depend on n: Boros and Füredi [20] showed that $c(K_n^2) = 2/9 - o(1)$. More precisely, they show that for every set P of n points in the plane there is a point in $\mathbb{R}^2$ that is contained in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ triangles spanned by P. This was extended to complete complexes in arbitrary dimension by Bárány [12].

Theorem 8.8 ([12])**.** *There is a constant $c_k > 0$ only depending on k such that for any set P of n points in $\mathbb{R}^k$ there is a point in $\mathbb{R}^k$ that is contained in at least $c_k\binom{n}{k+1} - O(n^k)$ k-simplices of P.*

While for $k = 2$ the factor $2/9$ is known to be asymptotically tight, the determination of the best value of c_k for larger k is the subject of ongoing research, see e.g. [22, 56, 70, 74, 91, 115]. Previously known bounds were recently improved by Gromov [56], who employed a topological proof method that also applies to continuous maps. We follow Matoušek's and Wagner's presentation of his result in [91].

The *topological overlap number* $c_{\text{top}}(X)$ of a k-dimensional simplicial complex X is the largest $c \in (0, 1]$ such that for any continuous map $f : \|X\| \to \mathbb{R}^k$ there is a point in $\mathbb{R}^k$ that is contained in the f-image of at least $c \cdot f_k(X)$ many k-simplices of X. Gromov shows that also $c_{\text{top}}(K_k^n)$ asymptotically does not depend on n:

Theorem 8.9 ([56])**.** *There exists a constant $c_k^{top} > 0$ only depending on k such that for any continuous map $f : \|K_k^n\| \to \mathbb{R}^k$ there is a point* *in $\mathbb{R}^k$ that is contained in the images of at least*

$$c_k^{top}\binom{n}{k+1} - O(n^k)$$

k-simplices of K_k^n. Furthermore,

$$c_k^{top} \geq \varphi_k(1/2 \cdot \varphi_{k-1}(1/3 \cdot \varphi_{k-2}(\ldots 1/k \cdot \varphi_1(1/(k+1)) \ldots))),$$

where the cofilling profile *φ_d for $d = 1, \ldots, k$ is defined as follows:*

$$\varphi_d(\alpha) := \liminf_{n\to\infty} \min_{\substack{f \in C^{d-1}(K_n^k;\mathbb{Z}_2) \\ \|[f]\|=\alpha}} \|\delta f\|.$$

Here, $\|\cdot\|$ denotes the weighted Hamming norm, as defined in Section 2.3.

In the remainder of this section, we denote by c_k and c_k^{top} the largest constants satisfying the statement of Theorem 8.8 and Theorem 8.9, respectively. The basic bound for combinatorial expansion, Proposition 2.4, that was observed by Gromov and independently also by Linial, Meshulam and Wallach [79, 92] and by Newman and Rabinovich [94], can in this context be phrased as the following basic bound on the coffiling profile:

$$\varphi_d(\alpha) \geq \alpha.$$

Combined with the simple observation that $\varphi_1(\alpha) = 2\alpha(1-\alpha)$, this yields

$$c_k \geq c_k^{\mathrm{top}} \geq \frac{2k}{(k+1)!(k+1)}.$$

This improves the best previously known lower bound of $c_k \geq \frac{k^2+1}{(k+1)^{k+1}}$ by Wagner [115], but is still far from the best known upper bound for arbitrary k by Bukh, Matoušek and Nivasch [22], which is

$$c_k \leq \frac{(k+1)!}{(k+1)^{k+1}} = e^{-\Theta(k)},$$

while Gromov's bound is of the order $e^{-\Theta(k\log(k))}$.

The method of Gromov can also be applied to arbitrary, non-complete, complexes X and yields yet another connection between the different notions of expansion: Suppose a k-dimensional complex X is combinatorially ε_i-expanding in dimension i with $\varepsilon_i > 0$ for every $1 \leq i \leq k$.

Then there exists $\varepsilon > 0$, depending on k and on the ε_i, such that the topological overlap number of X satisfies $c_{\text{top}}(X) > \varepsilon$.

Gromov's proof consist of a topological and of a combinatorial argument. Matoušek and Wagner in [91] describe both and suggest an additional structure, called *pagoda*, that can be used to improve the combinatorial component of the proof for the case of complete complexes. Our aim in this section is to study 3-dimensional pagodas, i.e., we restrict our attention to the case $k = 3$.

Pagodas - What and Why?

In the remainder of this section, we identify functions $f \in C^1(K_n^3, \mathbb{Z}_2)$ with their support as we have done before and write, e.g., δA instead of δf, where $A = \text{supp}(f)$. Recall that a set $A \subseteq \binom{V}{i}$ is *minimal* if $\|A\| = \|[A]\|$, i.e. if A contains at most half of the i-tuples from each set of the form δB for $B \subseteq \binom{V}{i-1}$.

Definition 8.10. Let $V = [n]$. A 3-dimensional *pagoda*

$$\mathcal{P} = (\{V_i\}, \{E_{ij}\}, \{F_{ijk}\}, G)$$

over V consists of:

- *minimal* sets $V_1, V_2, V_3, V_4 \subseteq V$ with $V_1 + V_2 + V_3 + V_4 \approx V$,
- *minimal* sets $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34} \subseteq \binom{V}{2}$ such that

$$\delta V_i \approx \sum_{j \neq i} E_{ij} \text{ for } i = 1, \ldots, 4,$$

- *minimal* sets $F_{123}, F_{124}, F_{134}, F_{234} \subseteq \binom{V}{3}$ with

$$\delta E_{ij} \approx \sum_{k \neq i,j} F_{ijk} \text{ for all } i \neq j,$$

- and a set $G \subseteq \binom{V}{4}$ such that $\delta F_{ijk} \approx G$ for all i, j, k.

Here, $X \approx Y$ for two sets of i-tuples means that $\|X + Y\| = o(1)$. The set G is called the *top* of the pagoda $\mathcal{P}$.

One part of Gromov's proof as phrased by Matoušek and Wagner in [91] is the following deep statement, which we state here without proof.

Lemma 8.11 ([56, 91]).

$$c_3 \geq c_3^{top} \geq \liminf_{n\to\infty} \min_{\substack{\mathcal{P} \text{ pagoda over } [n], \\ G \text{ top of } \mathcal{P}}} \|G\|.$$

The second, more combinatorial part, is the following lemma, completing Lemma 8.11 to a proof of the bound given in Theorem 8.9.

Lemma 8.12. *The top G of every 3-dimensional pagoda satisfies*

$$\|G\| \geq \varphi_3(1/2 \cdot \varphi_2(1/3 \cdot \varphi_1(1/4))) - o(1).$$

In the following sketch of proof for Lemma 8.12 we ignore terms of order $o(1)$ and hence assume that all relations in a pagoda are satisfied with equality. Let $\mathcal{P} = (\{V_i\}, \{E_{ij}\}, \{F_{ijk}\}, G)$ be such a pagoda. As $V_1 + V_2 + V_3 + V_4 = V$, one of the four vertex sets has size at least $n/4$, say w.l.o.g. $\|V_1\| \geq 1/4$. Thus, because v_1 is minimal, we get that

$$\|E_{12}\| + \|E_{13}\| + \|E_{14}\| \geq \|E_{12} + E_{13} + E_{14}\| = \|\delta V_1\| \geq \varphi_1(1/4).$$

Hence, one of the three summands, w.l.o.g. $\|E_{12}\|$, is at least $\varphi_1(1/4)/3$. We can then argue analogously and employ the minimality of E_{12} in order to see that, say, $\|F_{123}\| \geq 1/2 \cdot \varphi_2(1/3 \cdot \varphi_1(1/4))$ and hence

$$\|G\| = \|\delta F_{123}\| \geq \varphi_3(1/2 \cdot \varphi_2(1/3 \cdot \varphi_1(1/4))).$$

With the basic cofilling bounds and the exact value of $\varphi_1(1/4) = 3/8$, this yields a lower bound of $1/16 - o(1)$ for the top of every 3-dimensional pagoda. Matoušek and Wagner employ a more involved argument and show that the top G of every pagoda with sufficiently large vertex set satisfies $\|G\| \geq \frac{1}{16} + \varepsilon_0$ for $\varepsilon_0 > 0.00082$, which yields $c_3 \geq c_3^{\text{top}} \geq 0.06332$. Their argument uses the basic bounds for the cofilling profiles φ_2 and φ_3. Combined with an improved lower bound on φ_2 from [74], which gives $\varepsilon_0 > 0.012589$, we get the best currently known lower bound of

$$c_3 \geq c_3^{\text{top}} \geq 0.07509.$$

Here we will not present a result on the size of pagodas, but a reformulation of the problem. We show that one can without loss of generality assume that the V_i are pairwise disjoint and that $E_{ij} = V_i \times V_j$, if one allows some of the E_{ij} and F_{ijk} not to be minimal.

While our result falls short of giving an improved bound for c_3 and it is actually not even clear whether too much of the structure of the pagoda is lost to gain any, we think that the method employed to attain it could be of independent interest and might be strengthened to achieve a more satisfying result.

The Reformulation

Our result is the following proposition:

Proposition 8.13. *Let $\mathcal{P}$ be a pagoda. For every i_1, i_2, i_3 the top G of $\mathcal{P}$ is also the top of a pagoda-like structure $(\{V_i'\}, \{E_{ij}'\}, \{F_{ijk}'\}, G)$ of the following form:*

- $V_1', V_2', V_3', V_4' \subseteq V$ *are* minimal *and* pairwise disjoint *sets such that* $V_1' + V_2' + V_3' + V_4' \approx V$*;*
- $E_{ij}' = V_i' \times V_j' \subseteq \binom{V}{2}$ *for all* $i \neq j$*;*
- $F_{123}', F_{124}', F_{134}', F_{234}' \subseteq \binom{V}{3}$ *satisfy* $\delta E_{ij}' \approx \sum_{k \neq i,j} F_{ijk}'$ *and we have that* $F_{i_1 i_2 i_3}' = F_{i_1 i_2 i_3}$ *and is hence* minimal*;*
- $\delta F_{ijk}' \approx G$ *for all* i, j, k*.*

It is not hard to see that disjointness properties as the ones in Proposition 8.13 can be very helpful in studying the structure of a pagoda. The problem with this result is that it can only guarantee minimality for one of the F_{ijk}'s. This makes it hard to connect the size of the top of the pagoda with the size of the sets E_{ij}'.

Basic Idea. The proof of Proposition 8.13 uses concepts from homology with $\mathbb{Z}_2$-coefficients, but only for the 3-simplex Δ_3, which we also know as the complex K_4^3. We now give a very short introduction. Readers not familiar with these notions are, e.g., referred to [93].

The space $C_i(\Delta_3) = \mathbb{Z}_2^4$ corresponds to the set of i-element subsets of $[4]$. The $\mathbb{Z}_2$-boundary of a set $A \subseteq \binom{[4]}{i+1}$ is the set $\partial_i A$ containing all $F \in \binom{[4]}{i}$ that are contained in an odd number of elements from A. Just as for the coboundary, we have $\partial_{i-1}\partial_i A = \emptyset$, and can consider the sets

$$B_i(\Delta_3) = \operatorname{im} \partial_{i+1} \subseteq Z_i(\Delta_3) = \ker \partial_i \subseteq \binom{4}{i+1}$$

and the homology group $H_i(\Delta_3; \mathbb{Z}_2) = Z_i(\Delta_3)/B_i(\Delta_3)$.

We now illustrate the basic idea of the proof by showing how, in a pagoda where all relations are satisfied with equality, the vertex sets V_i can be made disjoint: Since $V_1 + V_2 + V_3 + V_4 = V$, every vertex $v \in V$ has to be contained in an odd number of V_i. If we let

$$S_v = \{i : v \in V_i\},$$

we can phrase this fact in a very elaborate way by saying that the set $S_v + \{\min(S_v)\}$ is contained in $Z_0(\Delta_3)$, the space of 0-dimensional cycles of Δ_3. We then let $V_i' = V_i + \{v \in V : i \in Z_v\}$, so that every $v \in V$ is now only contained in $V'_{\min(S_v)}$.

In order to keep the structure of a pagoda, we need to change the sets E_{ij} along with the $V_i's$. This is were our elaborate way of formalizing becomes useful. We have

$$\delta V_i' = \sum_{j \neq i} E_{ij} + \delta\{v : i \in Z_v\}.$$

Since the 3-simplex has trivial homology, there is $C_v \in C_1(\Delta_3)$ with $Z_v = \partial C_v$. This allows us to write the second summand also as a sum of sets over all $j \neq i$:

$$\delta\{v : i \in Z_v\} = \sum_{i \in Z_v} \delta v = \sum_{j \neq i} \sum_{\{i,j\} \in C_v} \delta v.$$

The summand δv appears an odd number of times in the sum on the right-hand side if and only if $i \in Z_v$. Hence, we can set $E'_{ij} = E_{ij} + \sum_{\{i,j\} \in C_v} \delta v$ and maintain the structure of the pagoda, while we loose minimality of the E_{ij}.

This was already most of the first part of the following proof of Proposition 8.13. It presents a similar argument for actual pagodas, taking into account the terms of order $o(1)$, and also for the edge sets.

Proof of Proposition 8.13. Let $\mathcal{P} = (\{V_i\}, \{E_{ij}\}, \{F_{ijk}\}, G)$ be a 3-dimensional pagoda. We show that we can change $\mathcal{P}$ into a structure as above while keeping F_{123} and G unchanged. We consider the index set $\{1, 2, 3, 4\}$ as the vertex set of the 3-simplex Δ_3.

Making the V_i's disjoint. For a vertex $v \in V$, define the set $S_v = \{i : v \in V_i\} \in C_0(\Delta_3)$. As we have $V_1 + V_2 + V_3 + V_4 \approx V$, except for $o(n)$ vertices, every vertex is contained in an odd number of the sets V_i. So for all but $o(n)$ vertices $|S_v|$ is odd, and hence $S_v = \{\min(S_v)\} + Z_v$, for a $Z_v \in Z_0(\Delta_3)$. For the remaining vertices there is $Z_v \in Z_0(\Delta_3)$ such that $S_v = Z_v$.

We define

$$V_i^{\text{new}} = V_i + \{v \in V : i \in Z_v\}.$$

Then $\sum_i V_i^{\text{new}} + \sum_i V_i = \sum_i \{v \in V : i \in Z_v\} = 0$, as every v appears an odd number of times in this sum. So $\sum_i V_i^{\text{new}} \approx V$. The sets V_i^{new} are minimal, as subsets of the V_i.

As $Z_0(\Delta_3) = B_0(\Delta_3)$, there is $C_v \in C_1(\Delta_3)$ with $Z_v = \partial C_v$. We define

$$E_{ij}^{\text{new}} = E_{ij} + \sum_{\{i,j\} \in C_v} \delta v.$$

For any i we have $\sum_{j \neq i} \{v : \{i,j\} \in C_v\} = \{v : i \in Z_v\}$, and hence $\sum_{j \neq i} E_{ij}^{\text{new}} + \delta V_i^{\text{new}} = \sum_{j \neq i} E_{ij} + \delta V_i$. Thus,

$$\sum_{j \neq i} E_{ij}^{\text{new}} \approx \delta V_i^{\text{new}}.$$

As $\delta E_{ij}^{\text{new}} = \delta E_{ij}$, the relations defining a pagoda are still satisfied.

Making the E_{ij}'s disjoint. Let now $V_i = V_i^{\text{new}}$ and $E_{ij} = E_{ij}^{\text{new}}$, so that we have $(\{V_i\}, \{E_{ij}\}, \{F_{ijk}\}, G)$, satisfying the pagoda relations, such that the V_i are pairwise disjoint and minimal, the E_{ij} might not be minimal and the F_{ijk} and G are unchanged.

For an edge $e \in \binom{V}{2}$ let $S_e = \{\{i,j\} : e \in E_{ij}\}$. Consider an edge $e \in V_{i_1} \times V_{i_2}$ with $i_1 \neq i_2$. Then $e \in \delta V_{i_1}, \delta V_{i_2}$ and $e \notin \delta V_{i_3}, \delta V_{i_4}$. We have $\delta V_i \approx \sum_{j \neq i} E_{ij}$. Hence, for all but $o(n^2)$ such edges e, we have $e \in \sum_{j \neq i_l} E_{i_l j}$ for $l = 1, 2$ and $e \notin \sum_{j \neq i_l} E_{i_l j}$ for $l = 3, 4$. For these edges, S_e contains an odd number of the pairs $\{i_1, i_2\}, \{i_1, i_3\}, \{i_1, i_4\}$ and of the pairs $\{i_1, i_2\}, \{i_2, i_3\}, \{i_2, i_4\}$ and an even number of the pairs $\{i_1, i_3\}, \{i_2, i_3\}, \{i_3, i_4\}$ and $\{i_1, i_4\}, \{i_2, i_4\}, \{i_3, i_4\}$. Thus, $\{i_1, i_2\} + S_e$ contains an even number of edges at every vertex of Δ_3 and hence $\{i_1, i_2\} + S_e = Z_e \in Z_1(\Delta_3)$.

For edges $e \in V_i \times V_i$ by a similar argument we get that for all but $o(n^2)$ such edges $S_e = Z_e \in Z_1(\Delta_3)$. The number of edges that are neither in some $V_{i_1} \times V_{i_2}$ or some $V_i \times V_i$ is $o(n^2)$, because the number of vertices not contained in any V_i is $o(n)$.

We let

$$E_{ij}^{\text{new}} = V_i \times V_j.$$

Then $\delta V_i \approx \sum_{j \neq i} E_{ij}$, because there are at most $o(n^2)$ edges in δV_i that are not in any set $V_i \times V_j$.

In order to correct the F_{ijk}'s and verify their relation with the E_{ij}^{new}, we first compare E_{ij}^{new} to $E'_{ij} = E_{ij} + \{e : \{i,j\} \in Z_e\}$. By the considerations above, $E_{ij}^{\text{new}} \approx E'_{ij}$, and hence $\delta E_{ij}^{\text{new}} \approx \delta E'_{ij}$.

We now establish the relation between $\delta E'_{ij}$ and the F_{ijk}'s. Let $C_e \in C_2(\Delta_3)$ with $Z_e = \partial C_e$. Because $\Delta_3 \setminus \{1, 2, 3\}$ also has trivial homology and contains all edges of Δ_3, we can choose C_e not containing $\{1,2,3\}$. Define

$$F_{ijk}^{\text{new}} = F_{ijk} + \sum_{\{i,j,k\}\in C_e} \delta e.$$

For any i,j we have $\sum_{k\neq i,j}\{e : \{i,j,k\} \in C_e\} = \{e : \{i,j\} \in Z_e\}$, and hence $\sum_{k\neq i,j} F_{ijk}^{\text{new}} + \delta E'_{ij} = \sum_{k\neq i,j} F_{ijk} + \delta E_{ij}$. Thus,

$$\sum_{k\neq i,j} F_{ijk}^{\text{new}} \approx \delta E'_{ij}.$$

As $\delta F_{ijk}^{\text{new}} = \delta F_{ijk}$, the relations defining a pagoda are still satisfied. $\square$

Bibliography

[1] R. M. Adin. Counting colorful multi-dimensional trees. *Combinatorica*, 12(3):247–260, 1992.

[2] M. Ajtai, V. Chvátal, M. M. Newborn, and E. Szemerédi. Crossing-free subgraphs. In *Theory and practice of combinatorics*, volume 60 of *North-Holland Math. Stud.*, pages 9–12. North-Holland, Amsterdam, 1982.

[3] M. Ajtai, J. Komlós, and E. Szemerédi. Topological complete subgraphs in random graphs. *Studia Sci. Math. Hungar.*, 14:293–297, 1979.

[4] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[5] N. Alon and F. R. K. Chung. Explicit construction of linear sized tolerant networks. In *Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986)*, volume 72, pages 15–19, 1988.

[6] N. Alon and V. D. Milman. λ_1, Isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[7] N. Alon, O. Schwartz, and A. Shapira. An elementary construction of constant-degree expanders. *Combin. Probab. Comput.*, 17(3):319–327, 2008.

[8] M. F. Anjos and J. B. Lasserre. *Introduction to semidefinite, conic and polynomial optimization*, volume 166 of *Internat. Ser. Oper. Res. Management Sci.* Springer, New York, 2012.

[9] L. Aronshtam, N. Linial, T. Łuczak, and R. Meshulam. Collapsibility and vanishing of top homology in random simplicial complexes. *Discrete Comput. Geom.*, 49:317–334, 2013.

[10] S. Arora, S. Rao, and U. Vazirani. Expander flows, geometric embeddings and graph partitioning. *J. ACM*, 56(2):Art. 5, 37, 2009.

[11] E. Babson, C. Hoffman, and M. Kahle. The fundamental group of random 2-complexes. *J. Amer. Math. Soc.*, 24(1):1–28, 2011.

[12] I. Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982.

[13] E. A. Bender and E. R. Canfield. The asymptotic number of rooted maps on a surface. *J. Combin. Theory Ser. A*, 43(2):244–257, 1986.

[14] E. A. Bender, E. R. Canfield, and R. W. Robinson. The enumeration of maps on the torus and the projective plane. *Canad. Math. Bull.*, 31(3):257–271, 1988.

[15] E. A. Bender and L. B. Richmond. A survey of the asymptotic behaviour of maps. *J. Combin. Theory Ser. B*, 40(3):297–329, 1986.

[16] M. Blum, R. M. Karp, O. Vornberger, C. H. Papadimitriou, and M. Yannakakis. The complexity of testing whether a graph is a superconcentrator. *Inform. Process. Lett.*, 13(4-5):164–167, 1981.

[17] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors.

[18] B. Bollobás. Random graphs. In *Combinatorics*, number 52 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1981.

[19] A. Borel. Cohomologie des certains groupes discrets et Laplacien p-adique. In *Séminaire Bourbaki, Exposé 437*, volume 431 of *Lecture Notes in Math.*, pages 12–35. Springer, 1973.

[20] E. Boros and Z. Füredi. The number of triangles covering the center of an n-set. *Geom. Dedicata*, 17(1):69–77, 1984.

[21] W. G. Brown, P. Erdős, and V. T. Sós. Some extremal problems on r-graphs. In *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., 1971)*, pages 53–63. Academic Press, New York, 1973.

[22] B. Bukh, J. Matoušek, and G. Nivasch. Stabbing simplices by points and flats. *Discrete Comput. Geom.*, 43(2):321–338, 2010.

[23] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

[24] F. Chung. *The Laplacian of a hypergraph*, volume 10 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 21–36. Amer. Math. Soc., 1993.

[25] F. Chung. *Spectral graph theory*, volume 92 of *CBMS Reg. Conf. Ser. Math.* Amer. Math. Soc., 1997.

[26] F. Chung and R. Graham. Sparse quasi-random graphs. *Combinatorica*, 22(2):217–244, 2002.

[27] F. Chung, R. Graham, and R. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[28] F. Chung, L. Lu, and V. Vu. The spectra of random graphs with given expected degrees. *Internet Mathematics*, 1(3):257–275, 2004.

[29] D. Cohen, A. Costa, M. Farber, and T. Kappeler. Topology of random 2-complexes. *Discrete Comput. Geom.*, 47:117–149, 2012.

[30] A. Coja-Oghlan. Spectral techniques, semidefinite programs, and random graphs. Habilitationsschrift, 2005.

[31] A. Coja-Oghlan. On the Laplacian eigenvalues of $G_{n,p}$. *Combin. Probab. Comput.*, 16(6):923–946, 2007.

[32] A. Coja-Oghlan and A. Lanka. The spectral gap of random graphs with given expected degrees. *Electron. J. Combin.*, 16(1):R138, 2009.

[33] T. Dey and H. Edelsbrunner. Counting triangle crossings and halving planes. *Discrete Comput. Geom.*, 12:281–289, 1994.

[34] J. Dodziuk. Finite-difference approach to the Hodge theory of harmonic forms. *Amer. J. Math.*, 98(1):79–104, 1976.

[35] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[36] J. Dodziuk and V. K. Patodi. Riemannian structures and triangulations of manifolds. *J. Indian Math. Soc. (N.S.)*, 40(1-4):1–52 (1977), 1976.

[37] X. Dong and M. L. Wachs. Combinatorial Laplacian of the matching complex. *Electron. J. Combin.*, 9(1):Research Paper 17, 11, 2002.

[38] D. Dotterrer and M. Kahle. Coboundary expanders. *Journal of Topology and Analysis*, 2012.

[39] A. M. Duval, C. J. Klivans, and J. L. Martin. Simplicial matrix-tree theorems. *Trans. Amer. Math. Soc.*, 361(11):6073–6114, 2009.

[40] B. Eckmann. Harmonische Funktionen und Randwertaufgaben in einem Komplex. *Comment. Math. Helv.*, 17:240–255, 1945.

[41] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.*, 5:17–61, 1960.

[42] U. Feige and E. Ofek. Spectral techniques applied to sparse random graphs. *Random Structures Algorithms*, 27(2):251–275, 2005.

[43] A. Flores. Über n-dimensionale Komplexe, die nicht in den $\mathbb{R}^{2n}$ topologisch einbettbar sind. *Ergebnisse eines mathematischen Kolloquiums*, 5:17–24, 1932/1933.

[44] A. Flores. Über n-dimensionale Komplexe, die im $\mathbb{R}^{2n+1}$ absolut selbstverschlungen sind. *Ergebnisse eines mathematischen Kolloquiums*, 6:4–7, 1933/1934.

[45] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach. Overlap properties of geometric expanders. Preprint, `arXiv:1005.1392`, 2010.

[46] J. Friedman. Some geometric aspects of graphs and their eigenfunctions. *Duke Math. J.*, 69(3):487–525, 1993.

[47] J. Friedman. Computing Betti numbers via combinatorial Laplacians. *Algorithmica*, 21(4):331–346, 1998.

[48] J. Friedman and P. Hanlon. On the Betti numbers of chessboard complexes. *J. Algebraic Combin.*, 8(2):193–203, 1998.

[49] J. Friedman, J. Kahn, and E. Szemerédi. On the second eigenvalue of random regular graphs. In *Proc. 21st Ann. ACM Symp. Theor. Comput.*, STOC '89, pages 587–598, New York, 1989. ACM.

[50] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[51] O. Gabber and Z. Galil. Explicit constructions of linear-sized superconcentrators. *J. Comput. System Sci.*, 22(3):407–420, 1981.

[52] Z.-C. Gao. The number of rooted triangular maps on a surface. *J. Combin. Theory Ser. B*, 52(2):236 – 249, 1991.

[53] H. Garland. p-adic curvature and the cohomology of discrete subgroups of p-adic groups. *Ann. of Math. (2)*, 97(3):375–423, 1973.

[54] B. Gärtner and J. Matoušek. *Approximation algorithms and semidefinite programming.* Springer, Heidelberg, 2012.

[55] W. T. Gowers. Quasirandomness, counting and regularity for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):143–184, 2006.

[56] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010.

[57] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics: Study and Research Texts.* Springer-Verlag, Berlin, 1988.

[58] A. Gundert and U. Wagner. On Laplacians of random complexes. In *Proc. 28th Ann. Symp. Comput. Geom. (SoCG)*, pages 151–160, 2012.

[59] A. Gundert and U. Wagner. On the subdivision containment problem for random 2-complexes. In *Young Researchers Forum - Symposium on Computational Geometry*, 2013.

[60] C. Hoffman, M. Kahle, and E. Paquette. A sharp threshold for Kazhdan's property (T). Preprint, `arXiv:1201.0425`, 2012.

[61] C. Hoffman, M. Kahle, and E. Paquette. The threshold for integer homology in random d-complexes. Preprint, `arXiv:1308.6232`, 2013.

[62] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[63] D. Horak and J. Jost. Spectra of combinatorial Laplace operators on simplicial complexes. *Adv. Math.*, 244:303 – 336, 2013.

[64] R. A. Horn and C. R. Johnson. *Matrix analysis.* Cambridge University Press, Cambridge, second edition, 2013.

[65] J. P. Huneke. A minimum-vertex triangulation. *J. Combin. Theory Ser. B*, 24(3):258–266, 1978.

[66] S. Janson. On concentration of probability. In B. Bollobás, editor, *Contemporary Combinatorics*, volume 10 of *Boly. Soc. Math. Stud.*, pages 289–301. Springer, 2002.

[67] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* Wiley-Intersci. Ser. in Discrete Math. Optim. Wiley, 2000.

[68] G. Kalai. Enumeration of $\mathbb{Q}$-acyclic simplicial complexes. *Israel J. Math.*, 45:337–351, 1983.

[69] G. Kalai. Many triangulated spheres. *Discrete Comput. Geom.*, 3(1):1–14, 1988.

[70] R. N. Karasev. A simpler proof of the Boros-Füredi-Bárány-Pach-Gromov Theorem. *Discrete Comput. Geom.*, 47(3):492–495, 2012.

[71] Y. Kohayakawa, V. Rödl, M. Schacht, P. Sissokho, and J. Skokan. Turán's theorem for pseudo-random graphs. *J. Combin. Theory Ser. A*, 114(4):631–657, 2007.

[72] W. Kook, V. Reiner, and D. Stanton. Combinatorial Laplacians of matroid complexes. *J. Amer. Math. Soc.*, 13(1):129–148, 2000.

[73] D. Kozlov. The threshold function for vanishing of the top homology group of random d-complexes. *Proc. Amer. Math. Soc.*, 138(12):4517–4527, 2010.

[74] D. Kráľ, L. Mach, and J.-S. Sereni. A new lower bound based on Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 48(2):487–498, 2012.

[75] M. Krivelevich and B. Sudakov. Pseudo-random graphs. In *More sets, graphs and numbers*, volume 15 of *Bolyai Soc. Math. Stud.*, pages 199–262. Springer, 2006.

[76] M. Krivelevich and B. Sudakov. The phase transition in random graphs: A simple proof. *Random Structures Algorithms*, 43(2):131–138, 2013.

[77] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11(3):796–817, 2000/01.

[78] M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In *Emerging applications of algebraic geometry*, volume 149 of *IMA Vol. Math. Appl.*, pages 157–270. Springer, New York, 2009.

[79] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[80] L. Lu and X. Peng. Loose Laplacian spectra of random hypergraphs. Preprint, `arXiv:1109.3433`, 2011.

[81] A. Lubotzky. Expander graphs in pure and applied mathematics. Preprint, `arXiv:1105.2389`, 2011.

[82] A. Lubotzky. Ramanujan complexes and high dimensional expanders. Preprint, `arXiv:1301.1028`, 2013.

[83] A. Lubotzky and R. Meshulam. Random latin squares and 2-dimensional expanders. Preprint, `arXiv:1307.3582v1`, 2013.

[84] A. Lubotzky, R. S. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[85] A. Lubotzky, B. Samuels, and U. Vishne. Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$. *European J. Combin.*, 26(6):965–993, 2005.

[86] A. Lubotzky, B. Samuels, and U. Vishne. Ramanujan complexes of type $\tilde{A}_d$. *Israel J. Math.*, 149:267–299, 2005. Probability in mathematics.

[87] A. Marcus, D. A. Spielman, and N. Srivastava. Interlacing families i: Bipartite ramanujan graphs of all degrees. Preprint, `arXiv:1304.4132`, 2013.

[88] G. A. Margulis. Explicit constructions of expanders. *Problemy Peredači Informacii*, 9(4):71–80, 1973.

[89] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988.

[90] J. Matoušek. *Using the Borsuk-Ulam theorem.* Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

[91] J. Matoušek and U. Wagner. On Gromov's method of selecting heavily covered points. Preprint, `arXiv:1102.3515v1`, 2011.

[92] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures Algorithms*, 34(3):408–417, 2009.

[93] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[94] I. Newman and Y. Rabinovich. On multiplicative λ-approximations and some geometric applications. In *Proc. 23rd Ann. ACM-SIAM Symp. Discrete Algorithms.* SIAM, 2012.

[95] A. Nilli. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991.

[96] O. Parzanchevski and R. Rosenthal. Simplicial complexes: spectrum, homology and random walks. 2012. Preprint, `arXiv:1211.6775`.

[97] O. Parzanchevski, R. Rosenthal, and R. J. Tessler. Isoperimetric inequalities in simplicial complexes. Preprint, `arXiv:1207.0638`, 2012.

[98] J. Pfeifle and G. M. Ziegler. Many triangulated 3-spheres. *Math. Ann.*, 330(4):829–837, 2004.

[99] P. Raghavendra. Optimal algorithms and inapproximability results for every CSP? [extended abstract]. In *STOC'08*, pages 245–254. ACM, New York, 2008.

[100] P. Raghavendra and D. Steurer. How to round any CSP. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, pages 586–594. IEEE Computer Soc., Los Alamitos, CA, 2009.

[101] P. Raghavendra and D. Steurer. Integrality gaps for strong SDP relaxations of Unique Games. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, pages 575–585. IEEE Computer Soc., Los Alamitos, CA, 2009.

[102] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Ann. of Math. (2)*, 155(1):157–187, 2002.

[103] F. Shahrokhi, O. Sýkora, L. A. Székely, and I. Vrťo. Bipartite crossing numbers of meshes and hypercubes. In *Graph drawing (Rome, 1997)*, volume 1353 of *Lecture Notes in Comput. Sci.*, pages 37–46. Springer, Berlin, 1997.

[104] J. Steenbergen, C. Klivans, and S. Mukherjee. A Cheeger-type inequality on simplicial complexes. Preprint, `arXiv:1209.5091v3`, 2012.

[105] M. Szedlák. On the Cheeger inequality for simplicial complexes. Master's Thesis, ETH Zürich, 2013.

[106] A. Thomason. Pseudo-random graphs. In M. Karoński and Z. Palka, editors, *Random Graphs '85*, volume 33 of *Annals of Discrete Mathematics*, pages 307 – 331. North-Holland, 1987.

[107] L. Trevisan. Relaxations of edge expansion and their duals. `http://lucatrevisan.wordpress.com/2008/05/03/relaxations-of-edge-expansion-and-their-duals/`, May 2008. Blog: in theory.

[108] L. Trevisan. The spectral lower bound to edge expansion. `http://lucatrevisan.wordpress.com/2008/04/30/the-spectral-lower-bound-to-edge-expansion/`, April 2008. Blog: in theory.

[109] L. Trevisan. Max cut and the smallest eigenvalue. *SIAM J. Comput.*, 41(6):1769–1786, 2012.

[110] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[111] W. T. Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968.

[112] W. T. Tutte. The enumerative theory of planar maps. In *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, pages 437–448. North-Holland, Amsterdam, 1973.

[113] E. R. van Kampen. Berichtigung zu Komplexe in euklidischen Räumen. *Abh. Math. Sem. Hamburg*, 9:152–153, 1932.

[114] E. R. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Hamburg*, 9:72–78, 1932.

[115] U. Wagner. *On k-sets and applications.* PhD thesis, ETH Zürich, 2003.

[116] U. Wagner. Minors in random and expanding hypergraphs. In *Proc. 27th Ann. Symp. Comput. Geom. (SoCG)*, pages 351–360, 2011.