Master Thesis

Spatio-Temporal Linear Stability Analysis for Heated Coaxial Jet Flows

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Spatio-Temporal Linear Stability Analysis for
Heated Coaxial Jet Flows

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Master Program in Mechanical Engineering

Master’s Thesis, HS 2013
Institute of Fluid Dynamics
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Supervisor: MSc ETH CSE Michael Gloor
Professor: Prof. Dr. Leonhard Kleiser
Abstract

The main purpose of this thesis is the investigation of the instability behaviour of heated coaxial jet flows with the linear spatio-temporal stability theory. Theoretical and experimental studies in the last decades have provided strong evidence that the generation of self-sustained oscillations is due to a particular spatio-temporal instability: the absolute instability. These global modes may be related to the large-scale turbulent structures which act as efficient noise sources in the exhaust flow of a turbofan jet engine. The spatio-temporal stability characteristics can be defined by looking at the impulse response of the flow. The latter is numerically computed by solving the eigenvalue problem obtained from the linearised compressible inviscid equations of motion with a wave ansatz for the perturbations. The stability of the mode with zero group velocity determines whether the flow is convectively or absolutely unstable. In coaxial jets, the ratio between the jet-core- and the bypass-velocity can be referred to as bypass-velocity ratio. Previous studies of heated single round jets are extended to non-zero bypass-velocity ratios. Furthermore, the influence of jet heating, shear-layer thickness and Mach number on the spatio-temporal stability of axisymmetric modes is analysed. First azimuthal modes are also considered. Due to the presence of two separated shear layers, two spatio-temporal modes exist. When increasing the velocity of the bypass-flow, the inner mode is stabilized, whereas the outer mode, exclusively present in coaxial jets, shows a larger positive absolute temporal growth rate for bypass-velocities around one half. For thicker shear layers, the inner mode is stabilized or destabilized depending on the bypass-velocity ratio. The outer mode can become absolutely unstable also for an isothermal coaxial jet for certain shear-layer thicknesses. Inner modes are never absolutely unstable without density differences between the core-stream and the bypass-stream. Increasing the Mach number has a stabilizing effect on both modes. First azimuthal modes display absolute instability only in a very restricted parameter range. The local linear spatio-temporal theory can be generalized to weakly non-parallel flows. Linear global modes are obtained by a superposition of linear instability waves computed for different streamwise locations and assuming a locally parallel flow. The base flow profiles are taken from a temporally averaged Large Eddy Simulation. Both intrinsic and forced global modes are investigated. Self-sustained oscillations typically have low frequencies and weak temporal growth rates. Their maximum amplification is reached at the downstream location where the underlying mode turns from convectively unstable to stable on a local basis. In the cross-stream direction, the fluctuations are large in the jet-core region as well, indicating a similarity to a jet-column mode. For high forcing frequencies, global modes are of the shear-layer type and are very similar to Kelvin-Helmholtz instabilities.
Acknowledgements

My gratitude goes to several people, without whom this thesis wouldn't have been possible.

First, I would like to thank Prof. Dr. Leonhard Kleiser for giving me the opportunity of doing a project in the fascinating field of hydrodynamic stability, which I discovered during his very interesting and excellent classes of Turbulent Flows and Hydrodynamic Stability and Transition. His commitment to teaching was very appreciated.

Many thanks go to Michael Gloor for supervising my work and for the stimulating discussions we had, which always helped me to overcome the difficulties. Having the chance to work in his office and exchange ideas about the project was very enriching.

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A big thanks goes also to my family and friends who always supported me during this thesis and in general all along my studies.

Zurich, March 2014

Gioele Balestra
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Nomenclature

The symbols used throughout this work are listed and briefly described. The equation in which they first appear is also indicated.

**Greek Letters**

- $\alpha$ downstream spatial growth rate p. 89
- $\alpha$ axial wavenumber (2.12), p. 9
- $\alpha(\omega)^\pm$ spatial branches p. 22
- $\alpha^*$ stationary point of $\rho$ (2.45), p. 18
- $\alpha_{1,2}^\pm$ initial discrete spatial eigenvalues for the calculation of the pinching point (3.21), p. 31
- $\alpha_{1,2}^{*}\pm$ new initial spatial eigenvalues corresponding to $\omega_{1,2}^*$ p. 32
- $\alpha_{\text{max}}$ wavenumber with the maximal temporal growth rate for the impulse response (2.50), p. 20
- $\chi$ constant for the calculation of new frequencies for fixed parameters (3.22), p. 32
- $\delta$ Dirac impulse (2.27), p. 13
- $\epsilon$ criticality, departure from global instability threshold (5.34), p. 117
- $\eta$ eigenvalue of the shifted and inverted eigenvalue problem (3.17), p. 30
- $\gamma$ ratio of specific heats (2.1), p. 6
- $\hat{\rho}$ density for the disturbance energy norm (3.26), p. 35
- $\lambda_{\text{max}}$ largest local wavelength in the global mode (5.26), p. 103
- $\mu_{1,2}, i = 1, 2$ inverse width of refinement of the mapping (3.12), p. 28
- $\mu_{2,1}, i = 1, 2$ location of refinement in the mapping (3.12), p. 28
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<td>ν</td>
<td>shift value, i.e. guess of eigenvalue for the implicitly restarted Arnoldi algorithm</td>
<td>(3.17), p. 30</td>
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<td>ω</td>
<td>angular frequency</td>
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<td>ω(α)</td>
<td>temporal branch</td>
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<td>ω₁₂</td>
<td>frequencies corresponding to $\alpha_{1,2}^\pm$</td>
<td>(3.21), p. 31</td>
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<tr>
<td>ω₁₂⁺</td>
<td>new initial value of the frequencies for the calculation of the pinching point</td>
<td>(3.22), p. 32</td>
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<td>φ</td>
<td>azimuthal coordinate</td>
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<td>π</td>
<td>ratio of a circle’s circumference to its diameter</td>
<td>(2.35), p. 15</td>
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<td>Π</td>
<td>generic parameter (e.g. h, S, θ, Ma)</td>
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<td>ρ</td>
<td>mass density</td>
<td>(2.1), p. 6</td>
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<td>σ</td>
<td>temporal growth rate perceived by an observer moving along the ray with velocity V</td>
<td>p. 20</td>
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<td>τ, ξ₀, λ, κ₁, κ₂</td>
<td>parameters of the mapping in physical space with two shear-layer regions</td>
<td>(3.12), p. 28</td>
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<tr>
<td>Θ</td>
<td>momentum thickness</td>
<td>(5.25), p. 103</td>
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<tr>
<td>θ₁</td>
<td>thickness of inner shear layer</td>
<td>(2.4), p. 7</td>
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<td>θ₂</td>
<td>thickness of outer shear layer</td>
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<td>ε</td>
<td>small parameter for linear global mode theory in weakly non-parallel flows</td>
<td>(5.2), p. 79</td>
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<td>ϱ</td>
<td>exponential term in the computation of the impulse response</td>
<td>(2.43), p. 18</td>
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<td>ϑ</td>
<td>Ritz value of the Ritz eigenvector $\mathbf{y}$</td>
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<td>$\tilde{\alpha}_0$</td>
<td>absolute wavenumber in the moving frame with velocity V</td>
<td>(4.5), p. 77</td>
</tr>
<tr>
<td>$\tilde{\omega}_0$</td>
<td>absolute frequency in the moving frame with velocity V</td>
<td>(4.5), p. 77</td>
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<tr>
<td>$\xi_j$</td>
<td>Chebyshev Gauss-Lobatto collocation point</td>
<td>(3.5), p. 25</td>
</tr>
<tr>
<td>ζ</td>
<td>spatial integration variable for the linear global mode shape</td>
<td>(5.7), p. 81</td>
</tr>
</tbody>
</table>
Nomenclature

Latin Letters

\( \hat{c}_j \) coefficient for the calculation of the expansion coefficient of the Chebyshev collocation approximation

(3.3), p. 25

\( \Delta S \) stabilization due to Mach number

(4.4), p. 68

\( \Delta x \) healing length for the non-linear global modes

(5.34), p. 117

\( \exp = e \approx 2.71828 \ldots \) Euler’s number

(2.12), p. 9

\( \hat{r} \) radial coordinate between the two shear layers

(5.25), p. 103

\( \hat{u}_k \) expansion coefficient of Chebyshev series

(3.1), p. 24

\( R \) specific gas constant

(2.2), p. 6

\( S \) forcing term in the PDE

(2.29), p. 14

\( V \) group velocity, equal to the velocity of a moving observer along a ray in the \((x,t)\)-plane

p. 20

\( V^+ \) leading edge velocity of impulse response

(4.5), p. 77

\( V^- \) trailing edge velocity of impulse response

(4.5), p. 77

\( V_{\text{max}} \) group velocity giving rise to the maximal temporal growth rate of the impulse response

(2.50), p. 20

\( A \) multiplicative factor of the eigenfunction for the linear global mode shape (5.7), p. 81

\( \tilde{U} \) complex amplitude of the linear global mode

(5.5), p. 80

\( q_i \) orthonormal basis vector \( i \) of the Krylov subspace

p. 30

\( x_n \) residual vector after an Arnoldi iteration

(3.18), p. 30

\( s \) eigenvector of Hessenberg matrix \( H_n \) with unity norm

(3.19), p. 30

\( U \) vector of discretised cross-stream disturbance eigenfunctions

(3.15), p. 29

\( y \) Ritz eigenvector associated with the unity eigenvector \( s \) of the Hessenberg matrix \( H_n \)

p. 30

\( A, B, C \) linear operators for the generalized eigenvalue problem

(2.25), p. 12
Nomenclature

<table>
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<td>$A, B, C$</td>
<td>matrices corresponding to discretised linear operators for the generalized eigenvalue problem (3.15), p. 29</td>
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<td>$H_n$</td>
<td>Hessenberg matrix, projection of $S_n$ in the Krylov subspace (3.18), p. 30</td>
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<td>$L, R$</td>
<td>left and right hand side matrices corresponding to the discretised generalized eigenvalue problem (3.16), p. 29</td>
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<td>$Q_n$</td>
<td>matrix whose columns are the orthonormal basis vectors of the Krylov subspace (3.18), p. 30</td>
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<td>$S$</td>
<td>shifted and inverted matrix for the implicitly restarted Arnoldi algorithm (3.17), p. 30</td>
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<td>$D_{\xi\xi}$</td>
<td>second order derivative operator in computational space for the Chebyshev collocation (3.10), p. 26</td>
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<tr>
<td>$D_{\xi}$</td>
<td>first order derivative operator in computational space for the Chebyshev collocation (3.9), p. 26</td>
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<tr>
<td>$D_r$</td>
<td>first order derivative operator in physical space for the Chebyshev collocation (3.13), p. 28</td>
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<tr>
<td>$I$</td>
<td>identity matrix (3.14), p. 29</td>
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<tr>
<td>$M$</td>
<td>matrix of metric factors for the mapping from computational space $\xi$ to physical space $r$ (3.13), p. 28</td>
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<tr>
<td>$e$</td>
<td>basis vector in $e$-direction</td>
</tr>
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<td>$U$</td>
<td>vector of cross-stream disturbance eigenfunctions (2.13), p. 9</td>
</tr>
<tr>
<td>$u$</td>
<td>velocity vector (2.1), p. 6</td>
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<td>$U^+$</td>
<td>vector of cross-stream disturbance eigenfunctions of the $\alpha^+$-branch (5.23), p. 90</td>
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<td>$u_N$</td>
<td>vector of approximations of the solution with a discrete Chebyshev series of order $N$ for all the collocation points (3.9), p. 26</td>
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<tr>
<td>$u_N^{(p)}$</td>
<td>vector of $p$th derivatives of the Chebyshev collocation approximation for all the collocation points (3.9), p. 26</td>
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<tr>
<td>$x$</td>
<td>vector of spatial coordinates (2.12), p. 9</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
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<tr>
<td>$a_c$</td>
<td>speed of sound at the centerline</td>
</tr>
<tr>
<td>$a_{\infty}$</td>
<td>free-stream speed of sound</td>
</tr>
<tr>
<td>$b, z_1, z_2$</td>
<td>parameters of the mapping in physical space with two shear-layer regions</td>
</tr>
<tr>
<td>c.c.</td>
<td>complex conjugate</td>
</tr>
<tr>
<td>$c_1$</td>
<td>constant for the calculation of new frequency $\omega_1^*$ when the parameter are changed</td>
</tr>
<tr>
<td>$c_2$</td>
<td>constant for the calculation of new frequency $\omega_2^*$ when the parameter are changed</td>
</tr>
<tr>
<td>$c_{\text{ph}}$</td>
<td>phase velocity</td>
</tr>
<tr>
<td>$D_\varepsilon$</td>
<td>non-parallel flow correction of the dispersion relation</td>
</tr>
<tr>
<td>$d_{j,k}^{(p)}$</td>
<td>coefficient for the approximation of the $p$th derivative for the Chebyshev collocation method</td>
</tr>
<tr>
<td>$E$</td>
<td>kinetic disturbance energy norm</td>
</tr>
<tr>
<td>$f$</td>
<td>function used for the computation of the impulse response</td>
</tr>
<tr>
<td>$F_\alpha$</td>
<td>integration contour in the complex $\alpha$-plane</td>
</tr>
<tr>
<td>$G$</td>
<td>impulse response, i.e. the Green's function</td>
</tr>
<tr>
<td>$h$</td>
<td>bypass-velocity ratio</td>
</tr>
<tr>
<td>$i := \sqrt{-1}$</td>
<td>imaginary unit</td>
</tr>
<tr>
<td>$k$</td>
<td>number of sought eigenvalues with the implicitly restarted Arnoldi method</td>
</tr>
<tr>
<td>$L$</td>
<td>evolution length scale</td>
</tr>
<tr>
<td>$l$</td>
<td>constant for the branch-fitting</td>
</tr>
<tr>
<td>$L_\omega$</td>
<td>integration contour in the complex $\omega$-plane</td>
</tr>
<tr>
<td>$m$</td>
<td>azimuthal wavenumber</td>
</tr>
<tr>
<td>$N$</td>
<td>number of Chebyshev collocation points in $r$-direction</td>
</tr>
</tbody>
</table>
Nomenclature

\begin{itemize}
  \item \( n \) number of Arnoldi’s vectors forming the orthogonal basis (2.30, p. 30)
  \item \( P \) thermodynamic pressure cross-stream disturbance eigenfunction (2.13, p. 9)
  \item \( p \) thermodynamic pressure (2.1, p. 6)
  \item \( q \) mapping to physical space (3.11, p. 27)
  \item \( R \) mass density cross-stream disturbance eigenfunction (2.13, p. 9)
  \item \( r \) radial coordinate (2.3, p. 6)
  \item \( r_1 \) location of inner shear layer (2.4, p. 7)
  \item \( r_2 \) location of outer shear layer (2.4, p. 7)
  \item \( r_j \) Chebyshev collocation point mapped in physical space (3.11, p. 27)
  \item \( r_{\text{max}} \) size of physical domain in radial direction (3.11, p. 27)
  \item \( S \) ambient-to-core temperature ratio (2.6, p. 7)
  \item \( s \) constant for the branch-fitting (3.21, p. 31)
  \item \( T \) temperature (2.3, p. 6)
  \item \( t \) time (2.1, p. 6)
  \item \( T_k \) Chebyshev polynomial (3.1, p. 24)
  \item \( U \) axial velocity cross-stream disturbance eigenfunction (2.13, p. 9)
  \item \( u \) axial velocity (2.3, p. 6)
  \item \( u_1 \) jet-core velocity profile (2.4, p. 7)
  \item \( u_2 \) bypass-stream velocity profile (2.4, p. 7)
  \item \( u_N \) approximation of the solution with a discrete Chebyshev series of order \( N \) (3.1), p. 24
  \item \( u_{\text{N}}^{(p)} \) \( p \)th derivative of the Chebyshev collocation approximation (3.6, p. 25)
  \item \( u_x' \) amplitude of the global mode (5.24, p. 91)
  \item \( u_{\text{bypass}} \) axial velocity of the bypass-flow (2.5, p. 7)
  \item \( V \) radial velocity cross-stream disturbance eigenfunction (2.13, p. 9)
\end{itemize}
### Nomenclature

- $v$: radial velocity \( (2.3) \), p. 6
- $W$: azimuthal velocity cross-stream disturbance eigenfunction \( (2.13) \), p. 9
- $w$: azimuthal velocity \( (2.3) \), p. 6
- $X$: slow space scale \( (5.4) \), p. 80
- $x$: streamwise coordinate \( (2.3) \), p. 6
- $X_S$: source location \( (5.11) \), p. 83
- $y$: cross-stream direction in a Cartesian coordinate system \( (5.7) \), p. 81

### Dimensionless Numbers

- $Ma$: Mach number \( p. 6 \)
- $Ma_\infty$: Mach number (defined with free-stream speed of sound) \( (4.2) \), p. 58
- $Ma_{ph}$: phase Mach number \( (4.3) \), p. 62

### Indices

- $'$: disturbance quantity
- $^*$: normalized quantity
- $^\circ$: dimensional quantity
- $^\star$: quantity in the new iteration
- $^0$: absolute quantity
- $^1$: inner shear-layer quantity
- $^2$: outer shear-layer quantity
- $^b$: base flow quantity
- $^c$: centerline quantity
- $^f$: forced quantity
- $^G$: global mode quantity

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- $\infty$ free-stream quantity
- $\mathbf{v}$ vector quantity
- $\mathbf{M}$ matrix quantity
- $\tilde{\mathbf{v}}$ quantity $\mathbf{v}$ in the moving reference frame with velocity $\mathbf{V}$

Operators

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$ scalar product, defined using Einstein summation notation
- $\mathbf{v}^T$ transpose of quantity $\mathbf{v}$
- $\partial^{(n)} v$ $n$th derivative of quantity $v$
- $\mathfrak{I}\{\mathbf{v}\}$ imaginary part of $\mathbf{v}$
- $\mathfrak{F}\{\mathbf{v}\}$ real part of $\mathbf{v}$
- $\mathbf{v}^\ast$ complex conjugate of the quantity $\mathbf{v}$
- $\partial_v \mathbf{v} := \partial \mathbf{v}/\partial v$ partial derivative of $\mathbf{v}$ with respect to $v$
- $\nabla := \hat{e}_l \partial_l$ gradient, defined using Einstein summation notation
- $|\mathbf{v}|$ absolute value of quantity $\mathbf{v}$
- $D(\mathbf{v}, \phi) = 0$ dispersion relation between $\mathbf{v}$ and $\phi$
- $\text{Res}(\mathbf{v}, \phi)$ residue of $\mathbf{v}$ at the pole $\phi$

Abbreviations

AI Absolute Instability
CI Convective Instability
DMD Dynamic Mode Decomposition
DNS Direct Numerical Simulation
LES Large Eddy Simulation
### Nomenclature

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>LST</td>
<td>Linear Stability Theory</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>WKBJ</td>
<td>Wentzel-Kramers-Brillouin-Jeffreys approximation</td>
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### Sets

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>set of complex numbers</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>set of real numbers</td>
</tr>
</tbody>
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Chapter 1. Introduction

1. Introduction

1.1. Motivation

Nowadays one of the major issue of air transportation is the noise emitted by jet engines. Cities around the world are increasing in size, getting closer and closer to the airports, transforming the rising air traffic into a complex socio-economical problem.

A large amount of aircraft noise is due to the large- and fine-scale eddies generated by the turbulent shear-flow mixing at the turbine exit (Williams & Kempton 1978; Tam 1998; Papanoschou 2004). The engine design has been improved by adding a cold bypass-flow, which is only slightly accelerated by the fan, to the heated core-stream, flowing through the compressor, the combustion chamber and the turbine stages (see Fig. 1.1). This allows to increase the fuel efficiency and to reduce the aerodynamic noise (Gloor et al. 2013). In fact the jet velocity needed for the same thrust, which is directly proportional to the radiated noise (Lighthill 1952), is lowered.

The generation of aerodynamic noise from turbofan jet engines is however not the only motivation for the present study. There are indeed many other applications where understanding the transition mechanisms in this kind of jet flows is essential. Another example are dual-stream nozzles, e.g. for mixing devices, where an increase of the turbulence of the flow is desirable (Schumaker & Driscoll 2012).

The large-scale turbulent structures may be related to the self-sustained oscillations, which occur for a particular spatio-temporal instability (Leshaf& Huerre 2007). Gaining insight into the instability characteristics of a heated coaxial jet with a spatio-temporal analysis, in order to reduce or enhance the generation of turbulence, is therefore of significant interest.

![Figure 1.1: Sketch of a turbofan engine](http://deargradher.deviantart.com/art/Turbofan-Engine-271669034)
1.2. Background

Classically, local linear stability theory (LST) has been applied to study either the spatial or the temporal development of flow disturbances. Using a modal approach, the perturbations are decomposed into eigenmodes, which can be classified either as stable, marginally stable or unstable.

Several authors suggest that even in the turbulent regime, the linear stability theory is able to model the growth and decay of the large-scale structures (Monkewitz & Sohn 1988). This holds however only under the fundamental assumption on the magnitude of the instability wave amplitude. Non-linear terms can be neglected only if this amplitude remains small.

Huerre & Monkewitz (1985), following the terminology used by Briggs (1964) and Bers (1984) in the study of plasma instabilities, introduced the notion of absolutely unstable and convectively unstable flows in the context of spatio-temporal stability theory, where the impulse response of a flow disturbance is studied. This methodology allows to determine if either spatial or temporal linear stability theory is applicable. In fact, in convectively unstable flows disturbances are convected away from their initial location making a spatial analysis meaningful, while in absolutely unstable flows the long-time response of an impulse disturbance exhibits growth in the complete flow domain so that a temporal analysis is more suitable. The first category of flows act as amplifiers whereas the latter act as oscillators (Huerre & Monkewitz 1990). Furthermore, a distinction between the two instability characters of the impulse response is essential because self-sustained global instabilities require a region of absolute instability to develop (Huerre & Monkewitz 1990).

A short review of studies on open shear flows like mixing layers, wakes and rotating-disk boundary layers with the spatio-temporal stability theory can be found in Huerre (chap. 3.5, pp. 189-197 2000, and references therein). Absolute instabilities in rotating-disk boundary layers are thoroughly investigated by Healey (2006, 2007).

Despite jet flows are a classical subject of research in fluid mechanics, many studies have been done on single jets only and the majority considers uniquely a linear spatial stability analysis. One of the first works on heated single round jet, prior to the introduction of the notions of absolute and convective instabilities, is the review article by Michalke (1984). Spatio-temporal linear stability analysis of axisymmetric single jets with density variations, which could also be due to fluids with different densities (e.g. helium in air), have been done by Monkewitz & Sohn (1988); Sreenivasan et al. (1989); Monkewitz et al. (1990); Jendoubi & Strykowski (1994) and more recently by Juniper (2006, 2007); Lesshaft & Huerre (2007). Absolute instability was found for jets without counterflow at a critical temperature ratio between the surrounding and the jet-core-stream fluids around 0.7. Swirling flows were instead considered by Lim & Redekopp (1998); Gallaire & Chomaz (2003).

A comprehensive study of spatial linear stability characteristics of compressible, viscous, subsonic coaxial jet flows is found in Gloor et al. (2013). The work of Talamelli & Gavarini (2006) considers a local spatio-temporal linear stability analysis of incompressible and inviscid isother-
Pioneer works in the field of global stability analysis for non-parallel base flows were done by Crighton & Gaster (1976) for circular jets, by Gaster et al. (1985) for mixing layers and by Chomaz et al. (1988) for the Ginzburg-Landau model equation with varying coefficients. Pierrehumbert (1984) first studied the possible appearance of an intrinsic global oscillation when a region of local absolute instability is present for a baroclinic instability. Monkewitz et al. (1990) compared linear local spatio-temporal and global theories with experiments for a single heated jet.

The generalisation of the concept of absolute instability to weakly non-parallel flows resulting in a linear global mode is well described in Huerre & Monkewitz (1990); Chomaz et al. (1991); Monkewitz et al. (1993); Huerre (2000).

Juniper et al. (2011) use this theory based on the local linear stability theory to study the effect of confinement on planar wakes and show very good agreement with the results obtained by means of the classical, and much more computationally expensive, global linear stability theory. The latter is employed by Garnaud et al. (2011, 2013) for the study of the subsonic jet profile proposed in Monkewitz & Sohn (1988).

To overcome the limitations of the linear theory, the spatio-temporal stability analysis can be extended to the non-linear regime by studying the impulse response using the full non-linear disturbance equations. A first glimpse on absolute and convective instabilities for non-linear systems can be found in the letter of Chomaz (1992) which considers the real Ginzburg-Landau equation. Non-linear convective and absolute instabilities for parallel two-dimensional wakes were studied by Delbende & Chomaz (1998).

Non-linear global modes for slowly varying flows are nicely described for the real Ginzburg-Landau amplitude equation with varying coefficients by Couairon & Chomaz (1999a, 1996, 2001). For the non-linear and non-parallel configuration, the majority of the theoretical works considers for simplicity this model equation instead of the full Navier-Stokes equations. Non-linear global modes are however also studied by numerically solving the full non-linear governing equations and are discussed in Pier et al. (2001b); Chomaz (2003, 2004) for parallel and non-parallel wakes and in Lesshaft et al. (2006, 2007) for hot jets. These numerical investigations confirm very nicely the results of the theory based on the Ginzburg-Landau model equations.

The comprehensive review of Chomaz (2005) allows to critically compare the developed approaches used to study instabilities in open flows.

1.3. Objectives and outline

The first goal of the present project is the study of the local linear spatio-temporal instability characteristics of heated coaxial jet flows. A parametric study is performed in order to investigate the parameter influence on the transition between convective and absolute instabilities. Several parameters like the bypass-velocity ratio, the temperature ratio, the Mach number and the shear-layer thickness are varied so as to represent many possible configurations.

In addition, a global stability analysis based on this local linear spatio-temporal theory is com-
1.3. Objectives and outline

pleted for a weakly non-parallel base flow obtained from Large Eddy Simulation mean flow profiles of a heated coaxial jet. Both, forced and self-sustained global modes are investigated. This step is enlightening if one desires to study the large-scale structures which act as sources of the aerodynamic noise and the mechanism underlying the transition to turbulence.

The text is subdivided as follows. Chapter 2 presents the theoretical basis of the local linear spatio-temporal stability analysis. The equations governing the physics of the problem and the ones describing the considered base flow are presented in part 2.2. In section 2.3 the disturbance equations which are the basis of any linear stability analysis are derived with their respective boundary conditions. For reasons of completeness and to help the reader with the following part, the section is closed by a short introduction to temporal and spatial stability analysis. Section 2.4 presents the spatio-temporal linear stability theory. An intuitive criterion to distinguish between stable, convectively unstable and absolutely unstable flows is first presented (Sec. 2.4.2). Secondly, the mathematical solution for the impulse response is derived in part 2.4.3. The criterion used to find the character of the instability is also described.

The used numerical methods are presented in chapter 3. The discretisation method is introduced in section 3.2 whereas the used algorithms are listed in sections 3.3 and 3.4. In chapter 4, the results of the parametric study are presented. Section 4.2 presents the influence of the bypass-velocity ratio on the instability characteristic of the heated coaxial flow. The sensitivity analysis for the temperature ratio, the shear-layer thickness and the Mach number are found in sections 4.3, 4.4 and 4.5. Section 4.6 analyses the stability of the first azimuthal modes. The spatio-temporal instability characteristic of travelling modes is also briefly addressed in the last part of this chapter, section 4.7.

The global stability theory and the subsequent results are found in chapter 5. This second part begins with an introduction in section 5.2 on the theory of linear global modes for weak non-parallel flows, including the definition of global stability and instability. Section 5.3 presents the results obtained for the signalling problem (Sec. 5.3.2) and for the intrinsic global mode (Sec. 5.3.3). The limitations on the linearity of the theory are discussed in part 5.3.4. Section 5.4 treats the non-linear extension of the absolute and convective instabilities, while section 5.5 offers an overview on non-linear global modes in parallel, weakly non-parallel semi- or doubly-infinite domains.

Chapter 6 summarizes the findings of this project and provides some ideas for possible future works.
Chapter 2. Local linear stability theory

2. Local linear stability theory

2.1. Introduction

Linear stability theory is a relatively mature subject that studies the evolution of a flow when perturbed with small disturbances. The (laminar) base flow is considered stable when all the perturbations decay and unstable when at least some disturbance grows in time or space.

Usually, the disturbances are decomposed into eigenmodes using a Fourier transformation. In incompressible viscous flows, the evolution of such normal modes is given by the Orr-Sommerfeld equation. The inviscid counterpart is the Rayleigh equation.

The goal of this chapter is firstly to present the equations governing the dynamics of the disturbances for the case of a compressible inviscid flow and secondly to give a short introduction on spatio-temporal stability theory, necessary for the understanding of the following chapters. The base flow used for the local study is also presented.

In the first part of this project (chapters 2 to 4), a local linear stability theory is used. The base flow is assumed to be invariant under a streamwise translation, i.e. the velocity profile is strictly parallel. In the second part, chapter 5 the assumptions are successively relaxed. Firstly, the base flow is allowed to vary slowly and secondly also non-linear effects are taken into account.

For a comprehensive survey on Hydrodynamic Stability the reader is referred to the excellent books of Schmid & Henningson (2001); Criminale et al. (2003); Drazin & Reid (2004).

2.2. Governing equations

2.2.1. Equations of motion

This and the next sections are inspired by the paper of Lesshafft & Huerre (2007) and therefore a similar notation is used and only the most important steps are presented.

In the study, a cylindrical coordinate system \( x = (x, r, \phi) \) is considered: \( x \) is the \textit{axial} direction, \( r \) the \textit{radial} direction and \( \phi \) the \textit{azimuthal} direction. The velocity vector is decomposed along these coordinate directions: \( \mathbf{u} = (u, v, w) \). For a compressible inviscid flow with density \( \rho = \rho (x, t) \), pressure \( p = p (x, t) \) and constant heat capacity ratio \( \gamma \), the governing equations can be written as follow.
2.2. Governing equations

Continuity equation: \[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0
\] (2.1a)

Momentum equation: \[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p
\] (2.1b)

Energy equation: \[
\frac{\partial}{\partial t} \left( \frac{p}{\rho} \right) + u \cdot \nabla \left( \frac{p}{\rho} \right) = -\left( \gamma - 1 \right) \frac{p}{\rho} \nabla \cdot u
\] (2.1c)

Pressure and density are related by the equation of state for an ideal gas, i.e.

\[
p = R \rho T,
\] (2.2)

where \(T\) is the temperature and \(R\) the specific gas constant.

2.2.2. Nondimensionalisation

Throughout this work, all the quantities are dimensionless and are defined by:

\[
x := \frac{x^0}{r_1^0}, \quad r := \frac{r^0}{r_1^0}, \quad u := \frac{u^0}{u_c^0}, \quad v := \frac{v^0}{u_c^0}, \quad w := \frac{w^0}{u_c^0}, \quad t := \frac{t^0}{r_1^0/u_c^0}, \quad \rho := \frac{\rho^0}{\rho_c^0}, \quad T := \frac{T^0}{T_c^0}, \quad p := \frac{p^0}{\rho_c^0 u_c^2}, \quad \mathcal{R} := \frac{R^0}{u_c^2/T_c^0},
\] (2.3)

where the characteristic length \(r_1^0\) is the inner dimensional jet radius, the reference velocity \(u_c^0\), density \(\rho_c^0\) and temperature \(T_c^0\) are defined with the dimensional flow values at the centerline. The characteristic time is defined by \(r_1^0/u_c^0\), the pressure by the dynamic pressure \(\rho_c^0 u_c^2\) and the specific gas constant by \(u_c^2/T_c^0\). The dimensionless pressure and specific gas constant scale therefore as \(1/(\gamma M a^2)\).

2.2.3. Base flow

The base flow is assumed to be parallel in the axial direction and swirl free, with neither radial or azimuthal velocity components: \(u_b = (u_b, 0, 0)\). Extending the hyperbolic-tangent base flow
2.2. Governing equations

velocity profile for a single jet proposed by [Michalke 1984], typical of the potential core region, to a coaxial jet one obtains:

\[
u_b(r) = (1 - h)u_1(r) + hu_2(r) \quad \Leftrightarrow \quad u_b(r) = \frac{1 - h}{2}\left\{1 + \tanh\left[\frac{r_1}{4h_1}\left(\frac{r_1}{r} - \frac{r}{r_1}\right)\right]\right\} + \frac{h}{2}\left\{1 + \tanh\left[\frac{r_2}{4h_2}\left(\frac{r_2}{r} - \frac{r}{r_2}\right)\right]\right\}
\] (2.4)

where \( r_1 \) is the location of the inner shear layer which lies between the heated core-flow and the bypass-stream and \( r_2 \) the location of the outer shear layer, situated between the bypass-stream and the surrounding flow, which is assumed stationary. These shear layers have a characteristic momentum thickness of \( \theta_1 \) and \( \theta_2 \), respectively. The remaining parameter \( h \) is called bypass-velocity ratio and defines the velocity ratio between the bypass- and core-flows

\[
h = \frac{u_{bypass}}{u_c}.
\] (2.5)

Hyperbolic-tangent velocity profiles are commonly used for the study of free shear layers (for more examples see Huerre 2000).

Since the core-flow is heated, the radial temperature profile can be linked to the jet-core velocity profile via the Crocco-Busemann relation (see Crocco 1932, Busemann 1935, Michalke 1984, Lesshafft & Huerre 2007):

\[
T_b(r) = S + (1 - S)u_1(r) + \frac{\gamma - 1}{2}Ma^2 [1 - u_1(r)] u_1(r),
\] (2.6)

where the jet-core velocity profile is:

\[
u_1(r) = \frac{1}{2}\left\{1 + \tanh\left[\frac{r_1}{4h_1}\left(\frac{r_1}{r} - \frac{r}{r_1}\right)\right]\right\}
\] (2.7)

and \( S \) the ambient-to-core temperature ratio:

\[
S = \frac{T_\infty}{T_c}.
\] (2.8)

The Mach number depends on the velocity of the flow and the speed of sound at the centerline, i.e.

\[
Ma = \frac{u_c}{a_c}.
\] (2.9)
2.2. Governing equations

Using the ideal gas law (2.2) and the scaling for the specific gas constant, as well as the observation that at the centerline $\rho_b(0)$ and $T_b(0)$ are unity because of nondimensionalisation, the value of the constant pressure $p_b$ of an unperturbed jet is:

$$p_b(0) = \frac{1}{\gamma M_a^2} \rho_b(0) T_b(0) \Leftrightarrow p_b(r) = \frac{1}{\gamma M_a^2} \forall r.$$  

(2.10)

Furthermore, again with equation (2.2), one obtains a very easy relation for the density: it is given by the inverse of the temperature.

$$\rho_b(r) = \frac{1}{T_b(r)}$$  

(2.11)

**Remark** Throughout this study the shear layers are located at $r_1 = 1$, $r_2 = 2$ and the specific heat capacity ratio will be $\gamma = 1.4$. The value of the other parameters is further specified.

Figure 2.1 shows typical velocity and temperature profiles for the base flow with different bypass-velocity ratios. It can be observed that the outer base flow temperature is $T_b(r > r_1) = 0.3$, which is the chosen temperature ratio. By changing $S$, the temperature in the bypass-stream and the in the ambient flow can be modified.

**Figure 2.1.** Base flow profiles for the heated coaxial jet flow with different bypass-velocity ratios: $S = 0.3$, $\theta_1 = \theta_2 = 0.03$: $-$, $h = 0$; $- - -$, $h = 0.4$; $- - - -$, $h = 0.6$. 
2.3. Disturbance equations

Generally, in order to derive the disturbance equations, a perturbed flow field \((u', p', \rho')\) is added to the base flow \((u_b, p_b, \rho_b)\). The resulting total flow \((u, p, \rho) = (u_b + u', p_b + p', \rho_b + \rho')\) has to satisfy the governing equations. Since also the base flow has to fulfill the Navier-Stokes equations, these equations can be subtracted from the governing equations for the total flow.

Under the assumption of small perturbation amplitudes, the resulting equations can be linearized neglecting higher order terms. This is the fundamental reason of the major limitation of linear stability theory, i.e. its ability to predict only the onset of transition, when disturbance amplitudes are small and the mechanisms linear.

The base flow is defined by equations (2.4), (2.10) and (2.11). For the perturbations a normal mode ansatz is used. This procedure allows to decompose the perturbations into their Fourier modes and is justifiable by the observation of wave like solutions in experiments. Their general structure reads:

\[
u'(x, t) = U(r, \alpha, m) \cdot \exp\left[i(\alpha x + m\phi - \omega t)\right] + c.c., \quad (2.12)
\]

where \(\alpha\) is the complex axial wavenumber, the integer \(m\) the azimuthal wavenumber and \(\omega\) the complex angular frequency. The complex amplitude of the perturbation \(U(r, \alpha, m)\) depends on the radial coordinate, the complex axial wavenumber and the azimuthal wavenumber. Following Lesshaft & Huerre (2007), the normal mode ansatz for all the perturbations, omitting the dependence of the eigenfunctions on the wavenumbers and angular frequency, reads

\[
\begin{bmatrix}
\rho' \\
u' \\
v' \\
\omega' \\
p'
\end{bmatrix}(x, t) = 
\begin{bmatrix}
R(r) \\
U(r) \\
iV(r) \\
W(r) \\
P(r)
\end{bmatrix} \cdot 
\exp\left[i(\alpha x + m\phi - \omega t)\right] + c.c. \quad (2.13)
\]

Introducing the normal mode ansatz into the linearised governing equations leads to a major simplification, transforming the partial differential equations to ordinary differential equations in \(r\). Derivatives in time, axial and azimuthal directions are replaced by algebraic terms, i.e. the normal modes multiplied by \(-i\omega, i\alpha\) and \(im\), respectively.

By manipulating the resulting equations, the following system of ordinary differential equations is obtained. These equations are nothing else than the inviscid version of the one used by Lesshaft & Huerre (2007). We would like to highlight that there is a sign error in their continuity equation (A1).
2.3. Disturbance equations

Continuity:
\[

r \omega R(r) - \left[ \omega \rho_0(r) + r^2 \frac{dp_0(r)}{dr} \right] V(r) - r \rho_0(r) \frac{dV(r)}{dr} - m \rho_0(r) W(r) = ru_0(r) \alpha R(r) + r \rho_0(r) \alpha U(r) \quad (2.14a)

\]

\(x\)-momentum:

\[

\rho_0(r) \omega U(r) - \rho_0(r) \frac{du_0(r)}{dr} V(r) = \rho_0(r) u_0(r) \alpha U(r) + \alpha P(r) \quad (2.14b)
\]

\(r\)-momentum:

\[

\rho_0(r) \omega V(r) + \frac{dP(r)}{dr} = \rho_0(r) u_0(r) \alpha V(r) \quad (2.14c)
\]

\(\phi\)-momentum:

\[

-r^2 \rho_0(r) \omega W(r) + mrP(r) = -r^2 \rho_0(r) u_0(r) \alpha W(r) \quad (2.14d)
\]

Energy:

\[

\frac{\omega^2}{(\gamma - 1) \rho_0(r)} R(r) + \left[ - \frac{r^2 \frac{dp_0(r)}{dr}}{(\gamma - 1) \rho_0(r)} \right] V(r) + r^2 \frac{dV(r)}{dr} + mrW(r) - \frac{\gamma Ma^2 r^2 \omega}{\gamma - 1} P(r) =

\frac{r^2 u_0(r)}{(\gamma - 1) \rho_0(r)} \alpha R(r) - r^2 \alpha U(r) - \frac{\gamma Ma^2}{\gamma - 1} r^2 u_0(r) \alpha P(r) \quad (2.14e)
\]

**Remark** The incompressible limit of equations (2.14) is derived in the appendix A.1. These five equations are sufficient to solve for the five unknown perturbations. Because of the singular nature of the coordinate system, the boundary conditions have to be defined carefully to ensure bounded solutions for all perturbations. Khorrami et al. (1989) and Ash & Khorrami (1995, pp. 339 - 342) propose as compatibility conditions for incompressible flows, that the azimuthal derivatives of velocity and pressure perturbations vanish as \( r \to 0 \). In the present compressible study, the same requirement for the additional variable \( \rho \) is assumed. This yields to

\[

\lim_{r \to 0} \frac{\partial u'}{\partial \phi} = imU(0) \xi_x + [-mV(0) - W(0)] \xi_x + [iV(0) + imW(0)] \xi_\phi = 0 , \quad (2.15)
\]

\[

\lim_{r \to 0} \frac{\partial p'}{\partial \phi} = imP(0) = 0 , \quad (2.16)
\]

\[

\lim_{r \to 0} \frac{\partial p'}{\partial \phi} = imR(0) = 0 . \quad (2.17)
\]
2.3. Disturbance equations

On the centerline one has therefore to ensure:

\[ mR(0) = 0, \quad mU(0) = 0, \quad V(0) + mW(0) = 0, \quad mV(0) + W(0) = 0, \quad mP(0) = 0. \tag{2.18} \]

For \( m = \pm 1 \), the third and fourth conditions are identical, thus an additional boundary condition is necessary. This is provided by an examination of the radial or azimuthal momentum equations and reads

\[ \frac{dV(0)}{dr} = 0 \quad \text{or} \quad \frac{dW(0)}{dr} = 0. \tag{2.19} \]

The boundary conditions at the centerline for all azimuthal wavenumbers are obtained.

| \( m = 0 \) | \( V(0) = W(0) = 0 \) | \( R(0), U(0) \) and \( P(0) \) finite \tag{2.20a} |
| \( m = \pm 1 \) | \( V(0) \pm W(0) = 0, \frac{dV(0)}{dr} = 0 \) | \( R(0) = U(0) = P(0) = 0 \) \tag{2.20b} |
| \( |m| > 1 \) | \( R(0) = U(0) = V(0) = W(0) = P(0) = 0 \) \tag{2.20c} |

For the case of axisymmetric disturbances \( m = 0 \), the conditions for the density, axial velocity and pressure are obtained from Taylor expansions of the continuity, axial momentum and energy equations around the centerline. For \( r \to 0 \) this yields

\[ \frac{dU(0)}{dr} = 0, \]

\[ \frac{dP(0)}{dr} = 0, \]

\[ \frac{dR(0)}{dr} = \gamma M a^2 \frac{dP(0)}{dr} = 0. \]
2.3. Disturbance equations

Keeping in mind that the eigenfunctions should decay exponentially as $r \to \infty$, the far-field boundary conditions are simply zero Dirichlet boundary conditions.

\[ R(\infty) = U(\infty) = V(\infty) = W(\infty) = P(\infty) = 0 \quad (2.24) \]

The disturbance equations (2.14) with the corresponding boundary conditions (2.20) and (2.24) constitute a generalized eigenvalue problem of the form

\[ A U(r) = \omega B U(r) + \alpha C U(r), \quad (2.25) \]

where the disturbance amplitudes $U(r)$ are the eigenfunctions and $\alpha \in \mathbb{C}$ or $\omega \in \mathbb{C}$ the corresponding eigenvalue for a given $\omega$ or $\alpha$, respectively. The linear operators $A$, $B$ and $C$, given by the disturbance equations and the boundary conditions, are reported in the appendix A.2.

A non-trivial solution for the eigenfunctions is admitted if and only if the complex wavenumber $\alpha$ and angular frequency $\omega$ satisfy the dispersion relation

\[ D(\alpha, \omega) = 0. \quad (2.26) \]

2.3.1. Temporal stability analysis

In a temporal linear stability analysis $\alpha$ is constrained to be a real number $\alpha \in \mathbb{R}$ and temporal modes $\omega = \omega(\alpha)$ are sought as zeros of the dispersion relation. Only waves growing ($\Im \{\omega\} > 0$) or decaying ($\Im \{\omega\} < 0$) in time, but not in space, are considered.

2.3.2. Spatial stability analysis

In a spatial linear stability analysis $\omega$ is constrained to be a real number $\omega \in \mathbb{R}$. Zeros of the dispersion relation correspond to spatial modes $\alpha = \alpha(\omega)$, called also branches. By a sinusoidal forcing with frequency $\omega$ at a particular location, perturbations may decay or grow in space. Great care has to be used when analysing the spatial stability characteristics of a flow. In fact, the eigenvalues corresponding to a given parameter $\omega$ are different branches in the complex $\alpha$-plane which consist of down- and upstream travelling waves (see Ashpis & Reshotko 1990). A mode with $\Im \{\alpha\} < 0$ is unstable downstream of the point of excitation, but stable upstream.
2.4. Spatio-temporal linear stability theory

Remark A temporal neutral state corresponds to a spatial neutral state. For a spatio-temporal mode however, temporal neutral stability does not imply spatial neutral stability and vice versa since both frequency and wavenumber are complex quantities.

2.4. Spatio-temporal linear stability theory

2.4.1. Introduction

The theory presented in this section mainly follows the excellent reviews of Huerre & Monkewitz (1990); Huerre (2000). In spatio-temporal linear stability analyses, both wavenumber $\alpha$ and angular frequency $\omega$ are complex numbers. Waves are allowed to grow or decay in both space and time. Instead of a periodic forcing in space or time, the base flow field is excited with a Dirac impulse in either one of the five flow variables at the spatio-temporal origin

$$\delta(x,t) = \delta(x_0, \phi_0, t) = \delta(x)\delta(t), \quad \forall r_0, \forall \phi_0$$

(2.27)

and the impulse response is considered. All the wavenumbers and frequencies are equally excited. For large times, only the wave-modes satisfying the dispersion relation will remain in the flow. The response to any type of forcing can be obtained by convolution of the impulse response with the forcing signal (Huerre, 2000). However, the impulse response is sufficient to extract the spatio-temporal stability characteristic of the flow. Indeed, if the localized disturbance generated by the impulse grows both upstream and downstream of its original location, the flow is said to be absolutely unstable, whereas if it is swept upstream or downstream only, then it is called convectively unstable (Huerre & Monkewitz, 1990). In absolutely unstable flows, a disturbance at any location leads to exponential growth everywhere, independently of an external forcing. In convectively unstable flows in contrary, disturbances are convected away, so that spatially growing waves at the excitation frequency can be observed (Huerre & Monkewitz, 1990). Absolute or convective instabilities are the result of a competition between dispersion induced by the instability and the basic flow advection (Chomaz, 2005). Furthermore, the complex $\alpha$- and $\omega$-planes provide an useful criterion to distinguish between these two character of instabilities.

2.4.2. Linear impulse response: absolute/convective instability

As discussed previously in section 2.3, the dispersion relation encloses the properties of normal modes sustained by a parallel base flow.

$$D(\alpha, \omega) = 0$$

(2.28)
2.4. Spatio-temporal linear stability theory

Consider this dispersion relation being associated with a forced partial differential equation involving solely time and the streamwise coordinate, ignoring variations in the cross-stream and azimuthal directions:

\[
D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t} \right) u'_f(x,t) = S(x,t),
\]

where the normal mode decomposition for \( u'(x,t) \) leads to non-trivial solutions of the unforced equation if and only if \( \alpha \) and \( \omega \) satisfy the dispersion relation (2.28). The sought impulse response, given by the Green’s function \( G(x,t) \), has to fulfill this equation as well, i.e.

\[
D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t} \right) G(x,t) = \delta(x)\delta(t),
\]

where \( \delta \) denotes the Dirac delta function. The Green’s function \( G \) encapsulates all the information about the spatio-temporal characteristic of the perturbation (Huerre, 2000). Using the general property of a linear system, once \( G(x,t) \) is known, the solution to any other forcing \( S(x,t) \) is obtained by convolution. Nevertheless, as already announced, this is not necessary for the classification of the stability character.

A base flow is linearly stable if

\[
\lim_{t \to \infty} G(x,t) = 0 \quad \text{along all rays } \frac{x}{t} = \text{constant}
\]

and the impulse response is a decaying wavepacket.

A base flow is linearly unstable if

\[
\lim_{t \to \infty} G(x,t) \to \infty \quad \text{along at least one ray } \frac{x}{t} = \text{constant}
\]

and the impulse response consists of an unstable wavepacket confined within a wedge in the \((x,t)\)-plane. Convectively or absolutely unstable flows can be further distinguished by looking at the impulse response along the ray \( x/t = 0 \).

A base flow is convectively unstable if

\[
\lim_{t \to \infty} G(x,t) = 0 \quad \text{along at the ray } \frac{x}{t} = 0
\]
2.4. Spatio-temporal linear stability theory

and *absolutely unstable* if

\[ \lim_{t \to \infty} G(x, t) \to \infty \text{ along the ray } \frac{x}{t} = 0. \quad (2.34) \]

In the convectively unstable case, the wavepacket is advected away from the origin of the source (see Fig. 2.2a) whereas in the absolutely unstable case (see Fig. 2.2c) the wavepacket remains at the source location and contaminates the entire spatial domain. This different behaviour of the wavepacket permits to decide whether a spatial or a temporal stability analysis is more suitable. 

As presented in figure 2.2b, the absolute marginal instability is obtained when the trailing-edge of the wavepacket has a zero front velocity.

![Figure 2.2: Sketch of spatio-temporal instability states in the (x,t)-diagram (inspired from Huerre, 2000).](image)

2.4.3. Briggs-Bers criterion

In this section, a very useful mathematical criterion, which allows to analytically determine the nature of the instability, is presented. This is done by deriving the solution of equation (2.29). The Briggs-Bers criterion is based on the properties of the dispersion relation in the complex \( \omega \)- and \( \alpha \)-planes. The procedure follows the one proposed by Huerre (2000).

Assuming the solution \( u_f'(x, t) \) of equation (2.29) being known in the \( (\alpha, \omega) \)-space, its corresponding in the physical space is obtained by the double Fourier integral

\[ u_f'(x, t) = \frac{1}{(2\pi)^2} \int_{L_\omega} \int_{F_\alpha} u_f'(\alpha, \omega) e^{i(\alpha x - \omega t)} d\alpha d\omega, \quad (2.35) \]
where $L_\omega$ and $F_\alpha$ are the integration contours plotted in figure 2.3. Both integration paths are composed of a line and a semicircle at infinity to close them (see Fokas 2003 chap. 2-4). $L_\omega$ needs to be chosen in a way such that causality is satisfied. The line has to lie above all the singularities of $u'_f(\alpha, \omega)$ in the complex $\omega$-plane when $\alpha$ travels along the $\alpha_r$-axis. Furthermore, the closing semicircle at infinity is in the upper or lower part of the complex-plane where the integrand decays exponentially to zero, avoiding any unwanted contribution to the solution from these closing paths, i.e. above (below) the line for $t < 0$ ($t > 0$).

$F_\alpha$ is chosen along the $\alpha_r$-axis and the closing semicircle at infinity lies below (above) the line for $x < 0$ ($x > 0$), in the region where the integrand decays exponentially as well.

Invoking Cauchy’s residue theorem, the integral along $L_\omega$ and $F_\alpha$ can be then calculated. The causality requirement, stating that $u'_f(x, t) = u'(x, t) = 0$ if there is no forcing for $t < 0$, implies that no singularity of $u'(\alpha, \omega)$ can be located in the upper half-$\omega$-plane above $L_\omega$ (Huerre 2000).

In order to obtain the solution in the $(\alpha, \omega)$-space, it is useful to apply the double Fourier transform

$$u'_f(\alpha, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u'_f(x, t)e^{-i(\alpha x - \omega t)}dxdt \quad (2.36)$$

to $(2.29)$. The dispersion relation in spectral space reads

$$D(\alpha, \omega)u'_f(\alpha, \omega) = S(\alpha, \omega). \quad (2.37)$$
2.4. Spatio-temporal linear stability theory

Remark without forcing, i.e. \( S(\alpha, \omega) = 0 \), non-trivial solutions \( u'(\alpha, \omega) \neq 0 \) are obtained only if the dispersion relation \( D(\alpha, \omega) = 0 \) is satisfied.

The previous expression in spectral space leads to

\[
u_f'(\alpha, \omega) = \frac{S(\alpha, \omega)}{D(\alpha, \omega)}.
\] (2.38)

For sake of simplicity, assume both forcing and dispersion relation to be analytic expressions in \( \alpha \) and \( \omega \) so that the first inverse Fourier transform reads

\[
u_f'(\alpha, t) = \frac{1}{2\pi} \int_{L_\omega} S(\alpha, \omega) D(\alpha, \omega) e^{-i\omega t} d\omega.
\] (2.39)

The only singularities of the integrand are the poles associated with zeros of the dispersion relation. These are temporal modes \( \omega = \omega(\alpha) \) because \( \alpha \) plays the role of a parameter confined to the \( F_\alpha \) contour. While \( \alpha \) travels along this line, the pole describes a locus in the complex \( \omega \)-plane, which must lie below \( L_\omega \) for causality. For \( t > 0 \), the \( L_\omega \) contour is closed in the lower half-\( \omega \)-plane and application of Cauchy’s residue theorem for first order poles\(^1\) leads to

\[
u_f'(\alpha, t) = -i \frac{S[\alpha, \omega(\alpha)]}{(\partial D/\partial \omega)[\alpha, \omega(\alpha)]} e^{-i\omega(\alpha)t}.
\] (2.40)

Before proceeding further, it has to be said that while \( \omega(\alpha) \) travels along \( L_\omega \), the integrand of (2.35) with \( u_f'(\alpha, \omega) \) given by (2.38), displays pole singularities in the complex \( \alpha \)-plane at zeros of the dispersion relation, which are nothing else but the spatial branches \( \alpha^+(\omega) \) and \( \alpha^-(\omega) \).

These images of \( L_\omega \) by the dispersion relation \( \alpha = \alpha(\omega) \) must lie in the upper (resp. lower) half \( \alpha \)-planes. Any crossing of the contour \( F_\alpha \) by the spatial branches would correspond to a crossing of the locus of \( \omega(\alpha) \) with \( L_\omega \), which is excluded. Resuming, the causality principle requires that \( \omega(\alpha) \) lie below \( L_\omega \) in the complex \( \omega \)-plane and that the \{\( \alpha^+(\omega), \alpha^-(\omega) \} \) branches are confined within their respective half \( \alpha \)-planes (Huere 2000, chap. 3.3). To satisfy this condition, it is enough to define \( L_\omega \) sufficiently high in the complex \( \omega \)-plane, strictly above all singularities.

The remaining step is the second inverse Fourier transform, which is

\[
u_f'(x, t) = -i \int_{F_\alpha} S[\alpha, \omega(\alpha)] (\partial D/\partial \omega)[\alpha, \omega(\alpha)] e^{i(\alpha x - \omega(\alpha)t)} d\alpha.
\] (2.41)

Since only the impulse response \( G(x, t) \) along different spatio-temporal rays is looked for, this integral can be simplified considering \( S(\alpha, \omega) = 1 \), which is the Dirac delta at the origin of

\[
\text{Res}(f, c) = \frac{1}{2\pi i} \int_{\Gamma} f(z)dz = \frac{1}{2\pi i} \int_{\Gamma} g(z) h^*(z)dz = \frac{g(c)}{h'(c)}
\]

\(1\)For first order poles the residue of the analytic function \( f = g/h \) at \( c \) is given by:
2.4. Spatio-temporal linear stability theory

the \((x,t)\)-diagram in spectral space. As Huerre (2000) points out, the resulting wavepacket integral

\[
G(x,t) = -\frac{i}{2\pi} \int_{F_{\alpha}} \frac{1}{(\partial D/\partial \omega) [\alpha, \omega(\alpha)]} e^{i(\alpha x - \omega(\alpha)t)} d\alpha,
\]

belongs to the general class of integrals having the following structure

\[
G(x,t) = -\frac{i}{2\pi} \int_{F_{\alpha}} f(\alpha) e^{\varphi(\alpha; \frac{x}{t}) t} d\alpha,
\]

where here the functions correspond to

\[
f(\alpha) = \frac{1}{(\partial D/\partial \omega) [\alpha, \omega(\alpha)]},
\]

\[
\varphi(\alpha; \frac{x}{t}) = i \left[ \alpha \frac{x}{t} - \omega(\alpha) \right].
\]

Keeping in mind that the long-time impulse response is sought, \(t\) is a large parameter so that the integrand is composed by a fast exponential. For such integrals, the method of steepest descent is well suited to obtain an asymptotic approximation as \(t \to \infty\) (Bender & Orszag 1999, chap. 6.6) and (Bleistein & Handelsman 1975, chap. 7). In this limit, the order of magnitude of the integrand is dictated by the real part of the exponent, which corresponds to the height of the surface \(\varphi_r(\alpha; \frac{x}{t})\) in the \((\alpha_r, \alpha, \varphi_r)\)-space.

It is found that the complex exponent often admits a stationary point \(\alpha_s:\)

\[
\frac{\partial \varphi}{\partial \alpha} (\alpha_s; \frac{x}{t}) = i \left[ \frac{x}{t} - \frac{\partial \omega(\alpha_s)}{\partial \alpha} \right] = 0,
\]

so that in its vicinity, the complex function \(\varphi(\alpha; \frac{x}{t})\) is approximated by the Taylor expansion

\[
\varphi(\alpha; \frac{x}{t}) \sim \varphi(\alpha_s; \frac{x}{t}) + \frac{1}{2} \frac{\partial^2 \varphi}{\partial \alpha^2} (\alpha_s; \frac{x}{t}) (\alpha - \alpha_s)^2.
\]

Being a harmonic function, this stationary location is a saddle point (see Fig. 2.4). No local minima or maxima can exist (Kreyszig 2007).

For the next, it is assumed that the mode \(\omega(\alpha)\) gives rise to a single discrete stationary point \(\alpha_s\).

In order to apply the method of steepest descent, the original \(F_{\alpha}\) contour is deformed into the steepest descent path \(F_p\) through the saddle point without crossing any extrema of the surface.
2.4. Spatio-temporal linear stability theory

Figure 2.4.: Local topology of surface $\varrho_r(\alpha_r, \alpha_i)$ around the saddle point $\alpha_s$ (inspired from Huerre, 2000).

at infinity, so that the global maximum of $\varrho_r(\alpha; \bar{z})$ along $F_p$ is reached at $\alpha_s$ and the main contribution to the integral will be due to this point. This allows to restrict the steepest descent path to a small segment of length $2\epsilon$ around the saddle point so that the function $\varrho(\alpha; \bar{z})$ may be well approximated by its Taylor expansion and $f(\alpha)$ by $f(\alpha_s)$. Only at the saddle point the two distinct steepest curves are allowed to intersect. Assuming subdominant contributions from infinity, the final result for the leading order term of the impulse response is obtained by extending the integration path to infinity $\epsilon \to \infty$ in both directions and applying classical steepest descent theory (Bender & Orszag, 1999, chap. 6.6):

$$G(x, t) \sim \frac{f(\alpha_s)}{\left[2\pi \frac{\partial^2 \varrho}{\partial \alpha^2}(\alpha_s; \frac{\bar{z}}{t}) \right]^{1/2}} e^{i\varrho(\alpha_s; \bar{z})t}$$  \hspace{1cm} (2.47)

(we would like to highlight that there is a missing $t$ in the equation (3.37) of Huerre, 2000). In the original variables it reads, along each spatio-temporal ray $x/t = $ constant:

$$G(x, t) \sim \frac{\exp \left(i \left[\alpha_s x - \omega(\alpha_s)t\right]\right)}{\partial_D \left[\alpha_s, \omega(\alpha_s)\right] \left[-2\pi i \frac{\partial^2 \varrho}{\partial \alpha^2}(\alpha_s) t\right]^{1/2}} = \frac{\exp \left(i \left[\pi/4 + \alpha_s x - \omega(\alpha_s)t\right]\right)}{\partial_D \left[\alpha_s, \omega(\alpha_s)\right] \left[2\pi \frac{\partial^2 \omega}{\partial \alpha^2}(\alpha_s) t\right]^{1/2}},$$  \hspace{1cm} (2.48)

where the complex wavenumber $\alpha_s$ is obtained by the saddle point relation (2.45), namely

$$\frac{\partial \omega}{\partial \alpha}(\alpha_s) = \frac{x}{t},$$  \hspace{1cm} (2.49)
2.4. Spatio-temporal linear stability theory

Knowing the asymptotic estimate of the impulse response, the spatio-temporal dynamics of instability waves satisfying the dispersion relation with a single discrete temporal mode \( \omega(\alpha) \) can be studied.

As already anticipated in the figure 2.2, the Green’s function \( G(x, t) \) takes the form of a wavepacket in the \((x,t)\)-plane.

Consider an observer moving at the velocity \( \mathcal{V} \) along the spatio-temporal ray \( x/t = \mathcal{V} \in \mathbb{R} \) in the \((x,t)\)-space. Because of (2.49), the group velocity along this ray is precisely equal to \( \mathcal{V} \) and the observer perceives a complex wavenumber \( \alpha_* \), an angular frequency \( \omega_* = \omega(\alpha_*) \) and a temporal growth rate of \( \sigma(\mathcal{V}) = \omega_* - \mathcal{V}\alpha_* \). It is important to note that even if the group velocity may be complex, it assumes physical significance only when it is real (Huerre, 2000). To have an unstable flow, the perturbations along the considered ray have to increase exponentially, i.e. \( \sigma(\mathcal{V}) > 0 \). This condition defines the leading- and trailing-edge of the domain occupied by the unstable wavepackets (see Fig. 2.2).

In most problems of interest, the temporal growth rate \( \omega_i \) reaches a maximum \( \omega_{i,max} = \omega_i(\alpha_{max}) = \sigma(\mathcal{V}_{max}) \) for a wavenumber \( \alpha_{max} \), i.e.

\[
\frac{\partial \omega_i}{\partial \alpha}(\alpha_{max}) = 0
\]  

(2.50)

and a real group velocity

\[
\frac{\partial \omega}{\partial \alpha}(\alpha_{max}) = \frac{\partial \omega_r}{\partial \alpha}(\alpha_{max}) = \mathcal{V}_{max} \in \mathbb{R}
\]  

(2.51)

along the ray \( x/t = \mathcal{V}_{max} \). It can be shown that this growth rate is a global maximum over all \( \mathcal{V} \)-rays and is observed following the peak of the wavepacket (Huerre & Monkewitz, 1990). This leads to the following criterion for linear instability:

- \( \omega_{i,max} > 0 \): linearly unstable flow: the temporal growth rate is positive in a finite range of \( x/t = \mathcal{V} \),
- \( \omega_{i,max} < 0 \): linearly stable flow: the temporal growth rate is negative for all rays.

In order to further distinguish between convective and absolute instabilities, it is necessary to study the long-time behaviour of the wavenumber \( \alpha_0 \) observed along the ray \( x/t = \mathcal{V} = 0 \) at the source location. This so called absolute wavenumber \( \alpha_0 \) corresponds indeed to a wave with zero group velocity

\[
\frac{\partial \omega}{\partial \alpha}(\alpha_0) = \frac{x}{t} = \mathcal{V} = 0.
\]  

(2.52)
2.4. Spatio-temporal linear stability theory

The corresponding angular frequency \( \omega_0 = \omega(\alpha_0) \) is usually called \textit{absolute frequency}. The absolute growth rate \( \sigma(0) = \omega_{0,i} \) measures the growth or decay of a wavepacket in the laboratory frame. Said differently, it characterizes the temporal evolution of the wavenumber \( \alpha_0 \) observed at a fixed station in the limit \( t \to \infty \). The absolute frequency \( \omega_0 \) is an algebraic branch point of the function \( \omega(\alpha) \) in the complex \( \omega \)-plane and \( \alpha_0 \) is a saddle point of \( \alpha(\omega) \) in the complex \( \alpha \)-plane [Huerre & Monkewitz 1990]. Just as the sign of \( \omega_{i,\text{max}} \) indicates the unstable or stable nature of the flow, the sign of \( \omega_{0,i} \) determines its absolute or convective nature.

Eventually, the following Briggs-Bers criterion states:

- \( \omega_{0,i} > 0 \) : \textit{absolutely unstable flow}: disturbances grow exponentially in the laboratory frame, contaminating the entire medium,
- \( \omega_{0,i} < 0 \) : \textit{convectively unstable flow}: disturbances decay in the laboratory frame, leaving the source undisturbed. Another wavepacket travelling up- or downstream may be unstable, but not the one with zero group velocity.

\textbf{Remark} Because of the upper bound of the temporal growth rate, one has \( \omega_{0,i} \leq \omega_{i,\text{max}} \).

The maximal temporal growth rate \( \omega_{i,\text{max}} \) can be obtained with a linear temporal stability analysis.

Geometrically, the criterion can be seen from the \((x,t)\)-diagram of figure 2.2. When the group velocities of the leading- and trailing-edges of the domain of instability have opposite sign, the flow will be absolutely unstable and the perturbation will contaminate the whole domain. Group velocities of leading- and trailing edge having same sign correspond to convectively unstable flows, the wavepacket perturbation will be convected downstream with a non-zero group velocity, leaving every point unperturbed for \( t \to \infty \).

It is important to stress that the \((\alpha_0,\omega_0)\)-mode corresponds to a mode with zero group velocity. The absolute frequency is solely useful to distinguish the spatio-temporal stability character, but for the full impulse response it is necessary to consider all the modes with different group velocities. Indeed, travelling waves may experience larger growths. The maximal one is for the wave travelling at the velocity \( V_{\text{max}} \).

The absolute frequency will also play an important role in the global stability analysis (see Sec. 5.2.5). In the next chapter, only the spatio-temporal modes with vanishing group velocity are taken into account since the main interest of the work is the parameter influence on absolute instabilities and not the full impulse response.

The above criterion for absolute or convective instabilities is however not precise enough [Briggs 1964, Bers 1984]. An additional specification, explained in the next paragraph, is required.
2.4. Spatio-temporal linear stability theory

**Bers’ procedure to find a pinching point**

To determine whether an instability is absolute or convective, one has to find the absolute wavenumber and the absolute angular frequency defined by the zero group velocity condition (2.52).

Bers (1984), taking inspiration from the fact that the dispersion relation \( \omega(\alpha_0) = \omega_0 \) has a double root \( \alpha_0 \) at \( \omega_0 \) (zero and first derivative vanish), implying that \( \alpha_0 \) is a point of contact between the two spatial branches \( \alpha^+ \) and \( \alpha^- \), designed a very useful geometrical method to determine the absolute wavenumber and frequency.

Initially, the contour \( L_\omega \) lies far above the singularities in the complex \( \omega \)-plane to satisfy causality and \( F_\alpha \) correspond to the real \( \alpha_r \)-axis. The temporal branch \( \omega(\alpha) \) is below \( L_\omega \) (see Fig. 2.5a) and the spatial branches \( \{\alpha^+(\omega), \alpha^-(-\omega)\} \) lie above (resp. below) \( F_\alpha \) (see Fig. 2.5b).

When the integration contour \( L_\omega \) is gradually lowered (see Fig. 2.5c), its images through the dispersion relation, i.e. the spatial branches \( \{\alpha^+(\omega), \alpha^-(-\omega)\} \), are not confined to the upper (resp. lower) half \( \alpha \)-planes, but they migrate one towards another, crossing the real axis. In order to preserve causality, i.e. the images of \( F_\alpha \) in the complex \( \omega \)-plane must lie below \( L_\omega \), the contour \( F_\alpha \) must be deformed to maintain the two spatial branches on the same original side (see Fig. 2.5d).

By continuing lowering \( L_\omega \), a point where \( F_\alpha \) becomes pinched simultaneously by both \( \alpha^+ \)- and \( \alpha^- \)-branches occurs (see Fig. 2.5f). This point, called *pinching point*, corresponds to the saddle point \( \alpha_0 \) with the double root \( \omega(\alpha_0) = \omega_0 \). In the complex \( \omega \)-plane, the temporal mode shows a cusp at \( \omega_0 \), which is the result of the zero group velocity condition (see Fig. 2.5e). As a side remark, the imaginary part of \( \omega_0 \) for which pinching occurs in figure 2.5e and 2.5f is negative, indicating a convectively unstable mode.

If one tries to further lower \( L_\omega \), no contour \( F_\alpha \) that avoid intersecting the branches can be found because after having pinched they changed their nature (Huerre, 2000). This observation yields the additional necessary condition for the Briggs-Bers criterion: the spatial branches \( \{\alpha^+(\omega), \alpha^-(-\omega)\} \) must lie in the upper (resp. lower) half \( \alpha \)-planes for high enough \( L_\omega \). A saddle point between two \( \alpha^+ \)- or \( \alpha^- \)-branches is not physically relevant because they do not pinch \( F_\alpha \) (Huerre, 2000). Said differently, the saddle point must be formed by the coalescence of a \( \alpha^+ \)- and a \( \alpha^- \)-branch corresponding to waves travelling in the positive and negative \( x \)-direction, respectively. Furthermore, only the first pinching point obtained while lowering \( L_\omega \) gives a pertinent zero group velocity wavenumber which can be analysed with the criterion to see whether the flow is absolutely or convectively unstable.

**Other procedures to find a pinching point**

It has to be pointed out that other techniques to find the pinching point exist. Among them the Briggs’ method (see Schmid & Henningson, 2001, pp. 273 - 278) and the Cusp Map Procedure, which is based on the geometrical characteristic of \( \omega(\alpha) \) shown in figure 2.5e, see Kupfer et al. (1987) and Schmid & Henningson (2001, pp. 278 - 281). However, since the previous explained method is used in the implementation, the other strategies are not discussed further.
2.4. Spatio-temporal linear stability theory

\[ \omega(\alpha) \]

(a)

\[ L_\omega \]

(b)

\[ \alpha_i \]

(c)

\[ F_\alpha \]

(d)

\[ \omega_i \]

(e)

\[ \alpha_i \]

(f)

\[ \alpha_i \]

\[ \alpha^+(\omega) \]

\[ \alpha^-(\omega) \]

\[ x > 0 \]

\[ x < 0 \]

\[ t < 0 \]

\[ t > 0 \]

Figure 2.5.: Integration contours in the complex \( \omega \)- and \( \alpha \)-planes for successively lowered \( \omega_i \) values of \( L_\omega \). The sketched temporal and spatial branches are typical for the Ginzburg-Landau equation (Huërre 2000).
3. Numerical methods and algorithms

3.1. Introduction

This chapter deals with the implementation of the previously explained theory on a computer. The numerical method adopted to solve the generalised eigenvalue problem is described in section 3.2, whereas the algorithms to numerically find and track the pinching point are presented in sections 3.3 and 3.4. The Matlab’s codes for the eigenvalue problem resolution as well as the ones for the spatio-temporal analysis were written by Gloor et al. (2013) and Casagrande (2013). To conclude the chapter, the disturbance energy norm used to normalize the eigenfunctions is described in section 3.5.

3.2. Numerical resolution of the generalised eigenvalue problem

In order to numerically solve the generalized eigenvalue problem, Monkewitz & Sohn (1988) as well as Jendoubi & Strykowski (1994) employed a shooting method, while Lesshafft & Huerre (2007) made instead use of a Chebyshev collocation method close to the one described by Ash & Khorrami (1995). This method may exhibit spurious oscillations when functions with regions of rapid variation have to be approximated. Coordinate transformations have to be used to improve accuracy.

In the present study, a Chebyshev collocation technique with a mapping refining the two separated shear-layer regions is employed to discretise the governing equations in the radial direction.

3.2.1. Chebyshev collocation

In the computational space $\xi \in [-1, 1]$, the solution $u(\xi)$ is approximated with a truncated discrete Chebyshev series of order $N$:

$$u_N(\xi_j) = \sum_{k=0}^{N} \hat{u}_k T_k(\xi_j), \quad j = 0, \ldots, N$$

(3.1)

where $\hat{u}_k$ is the expansion coefficient and $T_k$ the Chebyshev polynomial. Chebyshev polynomials are orthogonal and constitute a good alternative to the Fourier basis for non-periodic problems.
3.2. Numerical resolution of the generalised eigenvalue problem

(Peyret, 2002). Their first kind are the eigenfunctions of the singular Sturm-Liouville problem and read:

\[ T_k(\xi) = \cos(kz), \quad z = \arccos(\xi). \quad (3.2) \]

It can be noted that they are nothing else than cosine functions after change of independent variable and posses therefore many valuable properties of the cosine Fourier series (Canuto et al., 2007, chap. 2.4).

The expansion coefficients \( \hat{u}_k \) are instead determined by imposing a vanishing residual \( R_N = u - u_N \) at the collocation points \( \xi_j \). Using the orthogonality condition to solve the obtained algebraic system of equations one obtains:

\[ \hat{u}_k = \frac{2}{\hat{c}_k N} \sum_{l=0}^{N} \frac{1}{\hat{c}_l} u(\xi_l) T_k(\xi_l), \quad k = 0, \ldots, N, \quad (3.3) \]

where

\[ \hat{c}_l := \begin{cases} 2, & l = 0, N, \\ 1, & 1 \leq l \leq N - 1 \end{cases} \quad (3.4) \]

The \( N + 1 \) Chebyshev Gauss-Lobatto points are defined by

\[ \xi_j = \cos \left( \frac{j\pi}{N} \right), \quad j = 0, \ldots, N. \quad (3.5) \]

The density of such points is higher at the boundaries ±1. Other families of collocation points exist (Canuto et al., 2007, chap. 2.4).

Since in a collocation method the solution is uniquely known at the grid points, the derivatives at any collocation point have to be expressed as functions of the grid values (Peyret, 2002). The \( p \)th derivative reads

\[ u_N^{(p)}(\xi_j) = \sum_{k=0}^{N} d_{j,k}^{(p)} u_N(\xi_j), \quad j = 0, \ldots, N. \quad (3.6) \]
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For the first order derivative, the coefficients are given by

\[
d_{j,k}^{(1)} := \begin{cases} 
\frac{\xi_j (-1)^{j+k}}{\xi_k (\xi_j - \xi_k)}, & 0 \leq j, k \leq N, j \neq k, j, \\
-\frac{\xi_j}{2(1-\xi_j^2)}, & 1 \leq k \leq j \leq N - 1, \\
\frac{2N^2+1}{6}, & j = k = 0, \\
-\frac{2N^2+1}{6}, & j = k = N.
\end{cases}
\]  

(3.7)

The first order derivative in a vector notation can be written as

\[
u_N^{(1)} = D_\xi u_N
\]

with

\[
u_N = \begin{bmatrix} u_N(\xi_0) \\ u_N(\xi_1) \\ \vdots \\ u_N(\xi_N) \end{bmatrix}, \quad \nu_N^{(1)} = \begin{bmatrix} u_N^{(1)}(\xi_0) \\ u_N^{(1)}(\xi_1) \\ \vdots \\ u_N^{(1)}(\xi_N) \end{bmatrix} \quad \text{and} \quad D_\xi = \begin{bmatrix} d_{0,0}^{(1)} & d_{0,1}^{(1)} & \cdots & d_{0,N}^{(1)} \\ d_{1,0}^{(1)} & d_{1,1}^{(1)} & \cdots & d_{1,N}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N,0}^{(1)} & d_{N,1}^{(1)} & \cdots & d_{N,N}^{(1)} \end{bmatrix}.
\]

(3.9)

It has to be pointed out that round-off errors increase when differentiating \cite{Peyret2002}. A first reason is the magnitude disparity in the differential operator matrix components. Secondly, almost equal numbers are subtracted in their denominators, i.e. \((\xi_j - \xi_k)\) and \((1 - \xi_j^2)\). To avoid the second inconvenient, trigonometric identities are used to express these quantities and the second order derivative matrix is numerically computed as the square of the first order matrix obtained using the trigonometrical relations. Thus,

\[
D_{\xi \xi} = D_\xi D_\xi.
\]  

(3.10)

For other correction techniques see \cite{Peyret2002} and references therein. Compared to finite difference methods, pseudo-spectral methods can exhibit enhanced accuracy for a fixed number of computational degrees of freedom \cite{Bayliss1995}. If the rapid variation regions are well resolved, the convergence is exponential and not algebraic as for finite difference schemes.
3.2. Numerical resolution of the generalised eigenvalue problem

3.2.2. Mapping

The Chebyshev collocation method explained in the previous section [3.2.1] can be directly applied for domains of size \([-1, 1]\) if the functions are smooth. In the present case however, a mapping is needed to convert this domain to the physical one going from the centerline to the largest radial location \(r_{\text{max}}\). Furthermore, a mapping \(r = q(\xi, \mu)\) transforming the function \(u(r)\) with rapid variation regions to a more gradually varying \(u[q(\xi, \mu)]\) improves the accuracy (Bayliss & Turkel, 1992; Bayliss et al., 1995). Please note that the \(\xi\)-space is called computational space just to distinguish it from the physical space, which is however employed for all the calculations.

The used mapping is:

\[
q(\xi, \mu) : [1, -1] \rightarrow [0, r_{\text{max}}]
\]

\[
\xi_j \mapsto r_j = \frac{2 - b \pm \sqrt{z_1 + z_2}}{4} r_{\text{max}}
\]

where

\[
b = (\mu_{12} + \mu_{22}) - \tau \left(1 \frac{1}{\mu_{21}} + \frac{1}{\mu_{11}}\right),
\]

\[
\tau = \cot[\lambda(\xi_j - \xi_0)],
\]

\[
\lambda = \frac{\kappa_1 + \kappa_2}{2},
\]

\[
\xi_0 = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2},
\]

\[
\kappa_1 = \arctan[\mu_{11}(1 + \mu_{12})] + \arctan[\mu_{21}(1 + \mu_{22})],
\]

\[
\kappa_2 = \arctan[\mu_{11}(1 - \mu_{12})] + \arctan[\mu_{21}(1 - \mu_{22})],
\]

\[
z_1 = \left[(\mu_{12} - \mu_{22}) + \tau \left(1 \frac{1}{\mu_{21}} - \frac{1}{\mu_{11}}\right)\right]^2,
\]

\[
z_2 = 4 \frac{1 + \tau^2}{\mu_{11}\mu_{21}},
\]

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3.2. Numerical resolution of the generalised eigenvalue problem

with the sign of (3.11) depending on the one of $\tau$. $\mu_{i2}, i = 1, 2$ defines the location of the refinement and $\mu_{i1}, i = 1, 2$ corresponds to its inverse width. As discussed later (see Sec. 4.5), the location and width of the refinement areas have to be tuned to have good representation of the functions.

Figure 3.1 shows the discretised velocity profile for $N = 128$ with the refinements exactly at the shear-layer locations and an inverse width of 15 for both shear layers. It can be seen that the collocation points are not concentrated at the boundaries as the Chebyshev Gauss-Lobatto points, but are dense where the function varies rapidly.

For sake of completeness, it has to be pointed out that the differentiation operator in physical space is obtained by multiplying the one in the computational space by the matrix of the metric factors $M$, thus

$$D_r := M D_\xi$$

(3.13)

where, using the mapping function and the identity matrix $I$:

$$M = \left[ \begin{array}{c} d\eta(\xi, \mu) \\ d\xi \end{array} \right]^{-1} I.$$

(3.14)
3.2. Numerical resolution of the generalised eigenvalue problem

3.2.3. Discretised problem

Using this technique, equation (2.25) can be discretised into

\[ A U = \omega B U + \alpha C U. \]  

(3.15)

The sparse matrices \( A, B \) and \( C \) of size \( 5(N + 1) \times 5(N + 1) \) are the linear operators containing the coefficients corresponding to the discretised disturbance equations (2.14), whereas the vector \( U \in \mathbb{C}^{5(N+1)} \) contains the discretised amplitudes of the perturbations at the collocation points. This eigenvalue problem is of first order since inviscid compressible equations are considered, see the disturbance equations (2.14).

As explained in section 2.4.3, all the eigenvalues in the complex \( \alpha \)-plane fulfilling the dispersion relation for a given complex angular frequency \( \omega \) are sought. The eigenvalue problem can therefore be written as a general eigenvalue problem of the form:

\[ L U = \alpha R U. \]  

(3.16)

with \( L := A - \omega B \) known for a given \( \omega \in \mathbb{C} \) and \( R := C \). When the full \( \alpha(\omega) \)-spectrum is sought, the Matlab’s \texttt{eig} function is employed. This uses the \texttt{ZGGEV} routine of the LAPACK library. A QZ algorithm, also known as the generalized Schur decomposition, yields the generalized right eigenvectors and the corresponding eigenvalues. The routine, which needs as input only the left and right hand side matrices, uses a preliminary balancing step improving the conditioning of \( L \) to produce more accurate results. However, the full spectrum is only necessary for the first iteration of the algorithm explained in the next section 3.3. In fact, for the following iterations it is sufficient to resolve the eigenspectrum only in a subregion to find the closest eigenvalue to a guessed value. For this purpose, the Matlab’s \texttt{eigs} function is employed. It uses the implicitly restarted Arnoldi algorithm (or implicitly restarted Lanczos algorithm if the matrix \( L \) is symmetric or Hermitian, which is however not the case here) implemented in the ARPACK library. This algorithm is very time efficient, especially for large sparse matrices, since it only solves a shifted and inverted eigenvalue problem to investigate the eigenspectrum around the guessed eigenvalue \( \nu \). As a input, beside the left and right hand side matrices, it also needs the shift value \( \nu \) and the number \( k \) of eigenvalues that have to be computed. Arnoldi’s method is an efficient iterative procedure for approximating a subset of the eigensystem of large sparse matrices (Lehoucq & Sorensen, 1996). One obtains an orthonormal basis \( Q_n \) of the Krylov subspace and the projection of the matrix in this subspace, \( H_n \), which is much smaller than \( L \) and whose eigenvalues are also eigenvalues of the original matrix. However, the number of iterations before the residual vector is small can be very high. In a nutshell, the idea of the implicitly restarted Arnoldi algorithm is therefore to interrupt the iterations after a certain amount of steps, to
3.2. Numerical resolution of the generalised eigenvalue problem

reduce the search space and finally to resume the Arnoldi iteration (Arbenz & Kressner, 2012, p. 177). Following Arbenz & Kressner (2012), the main steps for the resolution of a generalized eigenvalue problem are outlined hereafter. First, a shift-and-invert spectral transformation to (3.16) is performed:

$$S U = \eta U$$

where

$$S := (L - \nu R)^{-1} R , \quad \eta := \frac{1}{\alpha - \nu}.$$  (3.17)

After executing an Arnoldi iteration with $S$, composed by $n$ steps which build the orthogonal basis, one obtains:

$$S_n Q_n = Q_n H_n + \xi_n e_n^T,$$  (3.18)

where $Q_n$ is the $n \times n$ matrix whose columns are the orthonormal basis vectors $q_i$, $i = 1, ..., n$ of the Krylov subspace $K_n$ (also called Arnoldi’s or Lanczos’ vectors), $H_n$ is the projection of $S_n$ in the subspace, $\xi_n$ is the residual vector and $e_n^T = q_n^T Q_n$.

Subsequently, assume $s$ to be an eigenvector of the small upper Hessenberg matrix $H_n$ with unity norm and Ritz value $\vartheta$. Furthermore, let $y = Q_n s$ be the associated Ritz vector. Thus,

$$S_n Q_n s = Q_n H_n s + \xi_n e_n^T s \iff S_n y = \vartheta y + \xi_n e_n^T s.$$  (3.19)

Therefore, $\vartheta y + \xi_n e_n^T s$ can be considered as an improved approximation to the desired eigenvector and is obtained by a one step inverse iteration only (Arbenz & Kressner, 2012, p. 181ff.). In fact, one can write

$$S_n y - \xi_n e_n^T s = (S_n - \xi_n e_n^T) y = \vartheta y,$$  (3.20)

so that in the limiting case when the norm of the residual vector vanishes, the Ritz vector $y$ and a sought eigenvector $U$ of the large $S_n$ coincide. All the $n$ eigenvalues of the small matrix $H_n$ are also eigenvalues of the large sparse matrix $S_n$. These eigenvalues are calculated with a QR algorithm, affordable since the dense matrix $H_n$ is small. It is found that the Ritz eigenvalues converge to the extreme eigenvalues of $S_n$, the closest to the shift $\nu$. The quality of the approximation increases with the number of Arnoldi’s vectors, which can be defined as input parameter in the Matlab’s function. Please note that the amount of necessary Arnoldi vectors is linked with the number of eigenvalues one is looking for by the relation $n > k + 1$. In every restarted Arnoldi’s iteration, the needed initial vector may be constructed in a way such to have a fast convergence to the desired eigenvector (Wintergerst, 2002). In the `eigs` function, the vector are per default
3.3. Algorithm to find the pinching point

Monkewitz & Sohn (1988) proposed a very efficient method to find the pinching point for a given flow configuration. However, since a parametric study has to be performed, an automatic procedure to track this point while the parameters are varied has to be designed. Gloor et al. (2013) and successively Casagrande (2013) implemented a very powerful algorithm in Matlab for the first and second purpose, which are described in this and the next section, respectively.

Following the idea of Bers (1984), presented in section 2.4.3 in order to find the first pinching point, the image in the complex $\alpha$-plane through the dispersion relation $D(\alpha, \omega) = 0$ of a line $L_\omega = \{ \omega = \omega_r + i\omega_i \in \mathbb{C} | \omega_i = \text{const.} \}$ lying above all the singularities is tracked while lowering the value of $\omega_i$. At one point, the spatial branches $\alpha^+$ and $\alpha^-$ will be close enough and pinching occurs.

The line in the $\omega$-space is discretised in a way that allows the spatial continuous branches to be well represented by the discretised spatial eigenvalues. Typically $\omega_r \in [0, 2]$ with a uniform step of 0.05. Monkewitz & Sohn (1988) designed a reliable and very time efficient iterative algorithm which finds the absolute wavenumber $\alpha_0$ without the need of having continuous $\omega$- and $\alpha$-branches around $\omega_0$ and $\alpha_0$, respectively (Lesshafft & Huerre, 2007). The steps of this procedure are now listed:

1. The user has to compute the discretised $\alpha(\omega)$-spectrum of $L_\omega$ for successively lowered $\omega_i$-values until the first pinching between two branches satisfying the pinching requirement is identifiable.

2. The four discrete eigenvalues $\alpha_{1,2}^+($\omega_{1,2}$)$ lying on the $\alpha^+$ and $\alpha^-$ branches emanating from the pinching point $\alpha_0$ for the two frequencies $\omega_1$ and $\omega_2$ have to be selected.

3. The non-linear function proposed by Monkewitz & Sohn (1988) is then fitted to these four eigenvalues

$$\alpha^\pm - \alpha_0 = \pm s\sqrt{\omega - \omega_0} + l(\omega - \omega_0),$$

so that the constants $s$ and $l$, as well as $\alpha_0$ and $\omega_0$, approximations of the pinching point, are determined.
3.3. Algorithm to find the pinching point

4. New initial values for the angular frequencies, closer to the guessed absolute frequency, are defined by

\[ \omega_{1,2}^* = (1 - \chi)\omega_{1,2} + \chi \omega_0, \]  

(3.22)

where \( \chi = 0.2 \) for example. Approximations of the corresponding four eigenvalues in the complex \( \alpha \)-plane are computed via relation (3.21) with the previously obtained \( s, l, \omega_0 \) and \( \alpha_0 \). For reasons of robustness, the parameter \( \chi \in [0,1] \) should be small.

5. The actual corresponding eigenvalues \( \alpha_{1,2}^\pm \) are obtained by solving the eigenvalue problem (3.16) and taking the solutions closest to the previously computed approximations.

6. Equation (3.21) is fitted again with the new values \( \omega_{1,2} = \omega_{1,2}^* \) and \( \alpha_{1,2}^\pm = \alpha_{1,2}^* \). The procedure from step 3 is repeated until both \( \alpha_{1,2}^\pm \) and \( \omega_{1,2} \) converge to the fixed points \( \alpha_0 \) and \( \omega_0 \), i.e. the absolute values.

As suggested by Casagrande (2013), in order to ensure that the algorithm converges effectively to a pinching point, it is necessary to require that both wavenumber and angular frequency become stationary and close to the critical point values. In fact, if one selects a starting eigenvalue which does not belong to a \( \alpha^+ \)- or \( \alpha^- \)-branch, it is possible that the calculated value \( \omega_0 \) and \( \alpha_0 \) become stationary because the chosen eigenvalue does not move when \( \omega \) is changed, so that a criterion based only on \( \omega_0 \) or \( \alpha_0 \) is not sufficient. The used criterion reads

\[ |\alpha_1^+ - \alpha_0| + |\alpha_2^+ - \alpha_0| + |\alpha_1^- - \alpha_0| + |\alpha_2^- - \alpha_0| + |\omega_1 - \omega_0| + |\omega_2 - \omega_0| < 10^{-2}. \]  

(3.23)

Remark Relation (3.21) is a local approximation of the \( \alpha \)-branches near the pinching point and avoids the computation of the continuous Riemann sheets. It is founded on the fact that \( \omega_0 \) is a square-root singularity, i.e. \( |\alpha_1^+ (\omega) - \alpha_1^- (\omega)|^2 \) is locally a linear function of the distance from the branch point (Huerre & Monkewitz 1985).

Figure 3.2 presents the results of the algorithm for \( h = 1, S = 0.2, \theta_1 = \theta_2 = 0.03, Ma = 0 \) and \( m = 0 \). The successive approximations of the spatial branches as well as the chosen spatial eigenvalues are visible on figure 3.2a. The convergence toward the pinching point shown in figure 3.2b is clearly exponential.
3.3. Algorithm to find the pinching point

(a) Iterative results of the algorithm in the complex $\alpha$-plane

(b) Convergence of absolute wavenumber and angular frequency

Figure 3.2: Results of the iteratively fitted curves and convergence of absolute quantities for $h = 1$, $S = 0.2$, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 0$: $\times$, initial eigenvalues; $\bigcirc$, new initial eigenvalues at each iteration; $-----$, real part; $-\cdot\cdot\cdot$, imaginary part.
3.4. Algorithm to track the pinching point in the parametric study

The algorithm presented in the previous section 3.3 is very useful to find the saddle point in the complex $\alpha$-plane for a given base flow configuration. However, since a parametric study of the linear spatio-temporal stability characteristic has to be performed, it would become cumbersome for the user to manually identify the four starting eigenvalues for every set of parameters. The algorithm to track the critical point for varying parameters proposed by Casagrande (2013) is explained next.

To present the algorithm in a very general fashion, assume to be interested in the variation of the spatio-temporal stability when the generic parameter $\Pi$ changes. The location of the pinching point, under a little variation of this parameter, shouldn’t vary too much. The steps of the algorithm are:

1. Find $\omega_0, \alpha_0, s$ and $l$ by means of the algorithm proposed by Monkewitz & Sohn (1988), explained in section 3.3, for the starting base flow configuration.

2. Construct new values for the angular frequencies

   \begin{align}
   \omega_1^\star &= \omega_0 + c_1 + ic_2, \\
   \omega_2^\star &= \omega_0 - c_1 + ic_2,
   \end{align}

   where $c_1, c_2 \in \mathbb{R}$ (for instance $c_1 = 0.001$ and $c_2 = 0.002$ seem to be good choices) and compute the approximations of the corresponding four eigenvalues in the complex $\alpha$-plane with relation (3.21).

3. Find the actual corresponding eigenvalues $\alpha_{1,2}^\star(\omega_{1,2}^\star)$ by solving the eigenvalue problem (3.16) and taking the closest eigenvalues to the previously computed approximations (still using the old value for the parameter $\Pi$).

4. Update the parameter $\Pi$ with its new value and solve again the eigenvalue problem (3.16). The four eigenvalues which are the closest to the ones obtained in the previous iteration are the starting eigenvalues for the algorithm of Monkewitz & Sohn (1988) with the new value of the parameter $\Pi$.

5. Use the algorithm explained in section 3.3 to calculate $\omega_0, \alpha_0$ as well as $s$ and $l$ for the new base flow configuration. Repeat from step 2 until the complete parameter range is studied.

This algorithm works quite well for small variations in most of the parameters. A more detailed discussion about its limitations will follow in chapter 4.

Remark Step 2 effectively means to space out the four initial eigenvalues (increase $\omega_i$: slightly raise the line $L_{\omega_i}$, i.e. move apart the spatial branches; change $\omega_i^\star$: take more distant eigenvalues) so that a movement of the pinching point while varying the parameter $\Pi$ can still be tracked. This is exactly the contrary to what is done in every iteration of the algorithm of Monkewitz & Sohn (1988), where the new angular frequency goes toward the absolute one. Apart of this
3.5. Disturbance energy norm

and the recalculation of the eigenvalues for the new parameter (step 4) both algorithms are very similar.

3.5. Disturbance energy norm

In the following sections, eigenfunctions of different modes are reported. To allow a better comparison, they are normalized with the kinetic disturbance energy, i.e.

\[ E := \frac{1}{2} \hat{\rho} \int_{0}^{r_{\text{max}}} \left[ u' u' + v' v' + w' w' \right] r \, dr , \tag{3.26} \]

where \( \hat{\rho} \) are complex conjugate quantities. Since this energy is solely used to normalize the eigenfunctions, the uniform density \( \hat{\rho} \) is assumed to be unity. All the modes will therefore have the same kinetic disturbance energy and it will be possible to investigate how it is distributed between the different components depending on the base flow parameters.
4. Parametric study

4.1. Introduction

The algorithms presented in sections 3.3 and 3.4 are used to perform a sensitivity analysis of some flow parameters on the linear spatio-temporal instability characteristic of the heated coaxial jet flow presented in section 2.2.3. The varied parameters are the bypass-velocity ratio \( h \) (eq. (2.5)), the temperature ratio between ambient- and core-flows \( S \) (eq. (2.8)), the shear-layer thickness \( \theta \) and the Mach number \( Ma \) (eq. (2.9)). This allows to compute with a local linear stability analysis the behaviour of several heated coaxial jet flow configurations.

Two and three dimensional perturbations are considered.

The adopted procedure consists firstly on a loop over all bypass-velocity ratio \( h = [0, 1] \) for a given temperature ratio \( S \), shear-layer thickness \( \theta \) and \( Ma \), and secondly on a loop over \( S \) in order to be able to draw the absolute marginal instability map in the \((h, S)\)-plane for a given \( \theta \) and \( Ma \). The majority of the computations have been run on the high-performance cluster of ETH Zurich, Brutus.

**Remark** Except in Sec. 4.6 the perturbations are axisymmetric \((m = 0)\).

Since the base flow of a coaxial jet exhibits two distinct shear layers, satisfying both the necessary conditions of Rayleigh (1880) and Fjørtoft (1950), one expects to have two unstable modes. Therefore, two pinching points have to be sought and followed during the parameter variation. The linear spatio-temporal theory presented in section 2, which considers only one discrete mode, is assumed to be applicable to two discrete modes separately. However, only the mode with the highest imaginary absolute frequency \( \omega_{0,i} \) for a given parameter set will be physically relevant. Furthermore, it is found that the two pinchings result from the coalescence of the two \( \alpha^\pm \)- with two different \( \alpha^- \)-branches. This is consistent with the theory stating that, after a pinching, no integration contour \( F_\alpha \) can be drawn without crossing the original \( \alpha^\pm \)-branches, whose nature has changed. Nevertheless, such a path \( F_\alpha \) can still be found between two others \( \alpha^\pm \)-branches, which will give rise to the second pinching, when \( L_\omega \) is further lowered.

In the following, the two modes are referred as inner and outer.

Figure 4.1 shows the discretized \( \alpha^\pm \)-branches in the complex \( \alpha \)-plane for different values of \( \omega_i \). The pinching point \( \alpha_0 \), corresponding in this case to the inner mode, is clearly visible. Since the corresponding imaginary angular frequency is positive, the inner mode is absolutely unstable.
4.1. Introduction

In the literature (see for example Monkewitz et al. 1990; Jendoubi & Strykowski 1994; Lesshaft & Huerre 2007) absolute and global modes are often classified into jet-column or shear-layer modes. The first type of modes features a strong pressure peak at the centerline which decays monotonically in the radial direction. For the second type of modes, the pressure perturbation is instead solely located at the shear-layer regions (Jendoubi & Strykowski 1994). As it will be explained in section 4.7 Lesshaft & Huerre (2007) have found that the first type of modes have a temporal growth rate scaling with the jet-radius, whereas for the seconds it scales with the shear-layer thickness. Throughout this work, a clear distinction between these two types of modes cannot be done, but the general resemblance of a mode to a specific type is pointed out when possible.

It is found that the eigenvalues forming the \( \alpha^+ \)-branch have often eigenfunctions whose maximum amplitude locations vary from the centerline to the shear-layers regions for increasing frequencies. The \( \alpha^- \)-branch is instead frequently composed by eigenfunctions of the jet-column type, but this can change if merging with other \( \alpha^- \)-branches occurs before pinching. The typology of the eigenfunctions of the \( \alpha^- \)-branch depends on the base flow parameters. For this reason, a careful study of the eigenfunctions of the spatio-temporal mode has to be performed. Please note that absolute modes have zero group velocities. Travelling waves may be of a different type.

Figure 4.1.: Pinching point in the complex \( \alpha \)-plane for the inner mode: \( h = 0, S = 0.3, \theta_1 = \theta_2 = 0.03, Ma = 0, m = 0; \times, \omega_i = 0.35858; \circ, \omega_i = 0.30858; \times, \omega_i = 0.25858; \)
4.2. Influence of the bypass-velocity ratio $h$

As discussed in section 1.2, the majority of the studies on spatio-temporal instability done until now considers a single jet (see for example Monkewitz & Sohn 1988, Monkewitz et al. 1990, Jendoubi & Strykowski 1994, Lesshaft & Huerre 2007). The variation of the bypass-velocity ratio for $h > 0$ is therefore of great interest. The velocity of the heated-core jet is assumed to be larger than the one of the bypass-stream and both flows are in the same direction (no counterflow), so that $h \in [0, 1]$. This is the case for typical turbofan jet engines. The amount of bypass-flow depends on the mean velocity in the bypass-stream and the inner and outer nozzles size.

Figure 2.1 presented already the variation in the base flow velocity profile for different bypass-velocity ratios and $\theta_1 = \theta_2 = 0.03$ and $S = 0.3$. The temperature is not affected by the bypass-velocity ratio since it depends solely on the core-flow velocity profile.

For the extreme cases $h = 0$ and $h = 1$ only one shear layer exists. At most one unstable mode exists. $h = 0$ corresponds to a single heated jet, whereas for $h = 1$ the single jet is two times larger and only its core is heated. Assuming that the inner mode is more linked with the inner shear layer, i.e. the one between the heated jet-core and the bypass-stream, and the outer mode more with the outer shear layer, i.e. the one between the bypass-stream and the environment, the two modes can be found separately by looking at the unique pinching point for these extreme values of $h$. Once the saddle point being found, it can be followed for bypass-velocity ratios going from zero to unity for the inner mode and opposite for the outer one. The parameter $h$ is increased/decreased by a constant of $10^{-3}$ at every iteration in order not to lose the saddle point. The same variation is used for all the parameters in the next sections.

**Convergence Study** A convergence study is performed to validate the numerical results. The domain $r \in [0, 25]$ is discretised with $N = 96, 128$ and $160$ plus one collocation points. The obtained absolute wavenumber when varying the bypass-velocity ratio in the range $h \in [0, 1]$ is plotted for both modes in figure 4.2. Markers correspond to the pinching point locations for $h = 0, 0.1, ..., 1$. Discrepancies are only found for the inner mode with a bypass-velocity ratio approaching unity, when the absolute wavenumber is close to the origin of the complex $\alpha$-plane. However, since the results for $N = 128$ and $160$ are very similar for the whole $h$-parameter range and both modes, $N = 128$ is chosen for the following study to avoid unnecessary extra computational costs.
4.2. Influence of the bypass-velocity ratio $h$

**Figure 4.2:** Convergence study on the absolute wavenumber (pinching point location) for both modes, different bypass-velocity ratios $h \in [0, 1]$ and $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 0$: ○, $N = 96$; ×, $N = 128$; □, $N = 160$. 
4.2. Influence of the bypass-velocity ratio \(h\)

By running the algorithm of section 3.4, the stability properties of the inner and outer modes are obtained as the bypass-velocity ratio is varied for a fixed temperature ratio \(S = 0.3\), shear-layer thickness \(\theta_1 = \theta_2 = 0.03\), Mach number \(Ma = 0\) and azimuthal wave number \(m = 0\). The pinching point location and the corresponding absolute angular frequency \(\omega_0\) are plotted in figure 4.3.

![Figure 4.3](image)

**Figure 4.3.** Absolute/convective instability analysis for different bypass-velocity ratios: \(S = 0.3\), \(\theta_1 = \theta_2 = 0.03\), \(Ma = 0\), \(m = 0\): --, inner mode; ---, outer mode. \(\circ\), \(h = 0\); \(\bullet\), \(h = 1\); \(\triangle\) correspond to an increase in \(h\) of \(\Delta h = 0.05\).

From figure 4.3a it can be seen that both inner and outer modes may display an absolute instability \((\omega_{0,i} > 0)\) for certain bypass-velocity ratios. The fact that an absolute instability can be found over a large range of \(h\) is not totally surprising since the temperature ratio of \(S = 0.3\) is already quite strong. Furthermore, the absolute temporal growth rate of the inner mode for low
bypass-velocity ratios is much larger than the maximal one of the outer mode. In fact, for low $h$, both temperature and velocity profiles exhibit strong gradients around $r_1 = 1$. The outer mode reaches its maximal instability slightly above $h \approx 0.5$, i.e. when the velocity gradients are large on both inner and outer shear layers. For $h = 1$, only a temperature gradient remains at $r_1 = 1$, the velocity gradients being at $r_2 = 2$, yielding a stabilisation of the outer mode.

The saddle point locations plotted in figure 4.3c show that the higher the bypass-velocity ratio, the less the absolute wavenumber varies. Furthermore, the absolute wavenumber location of the inner mode changes more than the one of the outer mode and as $h \rightarrow 1$, $\alpha_0$ tends to the origin of the the complex $\alpha$-plane. A mode with a real wavenumber tending to zero corresponds to a mode of infinite axial wavelength. The frequency tends to zero as well (see Fig. 4.3b). The interpretation of such eigenmodes is still not completely understood. Jendoubi & Strykowski (1994) and Juniper (2006) first argued that such modes are non-physical since they violate in any physical situation the fundamental assumption of a uniform base flow over a distance of the order of the wavelength. However, Healey (2006) pointed out that these eigenmodes are permissible and represent growth and propagation in the cross-stream direction (Juniper, 2007).

For $h \rightarrow 0$, the outer mode tends to a mode of infinite period. Both modes vanish when the corresponding shear layer becomes very weak and disappears. Furthermore, from figure 4.3b it can be concluded that the inner mode has a real absolute frequency varying almost linearly from 0.94 to 0, whereas the one of the outer mode varies from 0 to 0.57, with a fast increase for low bypass-velocity ratios and little variations for large $h$.

Typical eigenfunctions of the inner mode are shown in figure 4.4. Density perturbations are compact around the inner shear layer, where the temperature gradients are high. Please note that since $Ma = 0$ the density fluctuations are solely due to the mixing of fluid with different densities, but not to compressibility. Similarly, the axial velocity perturbations are located at the base flow shear-layer regions but are non-zero also in the jet core. Also the disturbances of the radial velocity have their peaks near $r_1 = 1$ and $r_2 = 2$, but their supports are not compact. Since axisymmetric modes are sought, no disturbances are present in the azimuthal velocity. The pressure perturbation affects a larger range of the domain and its maximum is located at the centerline. The fact that the perturbations at the centerline are greater than zero indicates that disturbances can communicate across the jet column and may therefore enhance instability. This wouldn’t be the case in the solution of a plane mixing layer, where perturbations are non-zero only at the shear-layer regions (Jendoubi & Strykowski, 1994).
4.2. Influence of the bypass-velocity ratio $h$

Figure 4.4.: Disturbance eigenfunctions of the inner mode: $h = 0.5$, $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 0$: ——, magnitude; - - -, real part; ..., imaginary part.
4.2. Influence of the bypass-velocity ratio $h$

It is interesting to have a look at the disturbance eigenfunctions for both modes and different $h$-values.

The amplitude of density perturbations exhibits a compact peak at the inner shear-layer location for both modes since this is the only region with high temperature gradients. Its intensity increases for higher bypass-velocity ratios (see Fig. 4.5a and Fig. 4.6a). From the inner mode axial velocity perturbation eigenfunctions it can be seen that the amplitudes are higher around the shear layer with larger gradients, i.e. the inner (outer) one for low (high) $h$, and they decrease at the centerline with increasing bypass-velocity ratio (see Fig. 4.5b). For the outer mode in contrary, the largest axial velocity perturbations are always around $r_2$ (see Fig. 4.6b) and at the centerline they increase with $h$. Concerning the radial velocity fluctuations, their intensity at
4.2. Influence of the bypass-velocity ratio $h$

$r_1$ strongly decreases for the inner mode with increasing bypass-velocity ratios (see Fig. 4.5c). For the outer mode, the maximum amplitude of these fluctuations is instead shifted from a shear layer to the other (see Fig. 4.6c). Regarding the pressure perturbation amplitude at the centerline, it decreases drastically for the inner mode when $h$ is increased (see Fig. 4.5d), but it is only slightly lowered for the outer mode (see Fig. 4.6d). This observation allows the conclusion that the inner mode for high bypass-velocity ratios tends to become more of a shear-layer type, with disturbances only around the shear layers, whereas the outer mode remains a jet-column mode with a strong perturbation in the pressure eigenfunction at the centerline as well.

![Figure 4.6.](image)

**Figure 4.6.** Magnitude of disturbance eigenfunctions of the outer mode for different bypass-velocity ratios: $S = 0.3, \theta_1 = \theta_2 = 0.03, Ma = 0, m = 0$: $\cdots, h = 0.25; \cdots, h = 0.5; \cdots, h = 0.75; \cdots, h = 1$. 

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4.3. Influence of the temperature ratio $S$

**Remark** The spatio-temporal stability character of each mode, i.e. absolute or convective instability, is uniquely given by the sign of the imaginary part of the complex absolute angular frequency (see Fig. 4.3a). Eigenfunctions only help in the classification of the modes.

### 4.3. Influence of the temperature ratio $S$

In non-isothermal jet flows, the temperature ratio between the environment and the heated jet-core is given by the factor $S$. The results presented in the previous section 4.2 were for $S = 0.3$. Regions of absolute and convective instability were found. The target of this section is to investigate the spatio-temporal stability character when $S$ is varied. Keeping in mind that one major application of this study are turbofan jet engines, the considered range for the temperature ratios is $S \in [0.1, 1]$, which implies that the core-flow is always hotter than the bypass-stream. The value of $S$ depends on both the heating due to the combustion and the temperature of the surrounding fluid.

The influence of the temperature ratio on the temperature profile of the base flow is shown in figure 4.7.

![Figure 4.7](image)

**Figure 4.7.** Temperature profile for different temperature ratios: $h = 0.6$, $\theta_1 = \theta_2 = 0.03$, $r_1 = 1$, $r_2 = 2$: $S = 0.1$; $S = 0.4$; $S = 0.6$; $S = 1$.

Regarding the procedure used for this parameter variation, the saddle point for different $S$ is tracked for the inner and outer modes with the same algorithms as before, but using as initial conditions the results obtained in the $h$-loop. For a fixed $h$, the temperature ratio is varied until the character of the instability changes ($\omega_{0,i} = 0$). If the flow is initially convectively unstable, $S$ is decreased, whereas if it is absolutely unstable for $S = 0.3$, the temperature ratio is increased. Lowering the temperature ratio means increasing the temperature gradients such that the flow may become more unstable. The locus of all points in the $(h, S)$-plane for which $\omega_{0,i} = 0$ defines the absolute marginal instability curve, i.e. the frontier between absolute and convective instabilities. An absolute marginal unstable mode features no temporal growth at the location of the Dirac impulse (see Sec. 2.4.2).
4.3. Influence of the temperature ratio $S$

Regions of absolute (AI) and convective (CI) instability in the $(h, S)$-plane are shown in figure 4.8a. In the CI region without any pattern, inner and outer modes are convectively unstable. Disturbances are convected downstream and a spatial linear stability analysis is useful to study the most amplified angular frequency. If the parameter couple $(h, S)$ belongs to an AI region, the disturbance will grow in the whole domain, so that a temporal linear stability analysis becomes useful to identify the most amplified wavenumber. The AI region can be subdivided into three domains to classify the origin of the absolutely unstable mode. The dotted domain corresponds to $(h, S)$-values for which the inner mode is absolutely unstable. For the outer mode, absolute instability is found in the lined area. In the third subregion, both inner and outer modes present a positive imaginary absolute angular frequency. However, only the mode with the highest absolute temporal growth rate is physically meaningful. Figure 4.8b shows the mode with the higher $\omega_{0,i}$: in the grey area the inner mode will contaminate the entire base flow, while in the white area the absolute instability will be due to an outer mode.

A first validation of the results is found by looking at the absolute marginal instability for $h = 0$. The temperature ratio $S$ for which the spatio-temporal instability characteristic changes from convective to absolute is about 0.71. Keeping in mind that the used shear-layer thicknesses are $\theta_1 = \theta_2 = 0.03$, corresponding to $r_1/\theta_1 \approx 33$, this critical value for $S$ fits very well with $S = 0.713$ obtained by Lesshaft & Huerre (2007) for $r/\theta = 26$. Monkewitz & Sohn (1988) obtained very similar results as well ($S$ is there the density-instead of the temperature-ratio). Furthermore, the highest temperature ratio yielding an absolute marginal unstable outer mode is reached for $h = 0.51$ and is $S = 0.73$. An isothermal jet ($S = 1$) is always convectively unstable. The lowest temperature ratio for which the flow is not absolutely unstable is found at $h = 0.19$ and $S = 0.23$. If the temperature ratio is further lowered, then a mode will become absolutely unstable. Generally, the inner mode is more unstable for low or high bypass-velocity ratios, whereas the outer mode for bypass-velocity ratios around 0.5, as discussed for figure 4.3.
4.4. Influence of the shear-layer thickness $\theta$

The goal of this section is to study the variation in the absolute marginal instability curve of figure 4.8a when the shear-layer thickness $\theta$ is varied. In fact, as a jet flow develops downstream
4.4. Influence of the shear-layer thickness $\theta$

of a nozzle exit, the mixing layer will spread in the radial direction, while the jet half width will grow almost linearly.

For sake of simplicity, the same value for the inner and outer shear-layer thicknesses is chosen $\theta_1 = \theta_2 = \theta$.

Increasing (decreasing) the shear-layer thickness will decrease (increase) the gradients at the shear flow interfaces (see Fig. 4.9). For very large $\theta$, the base flow will have a flattened profile.

![Figure 4.9: Base flow profiles for heated coaxial jet flows with different shear-layer thicknesses: $h = 0.6, S = 0.3$: --- $\theta = 0.02$; - - - $\theta = 0.04$; - - - - $\theta = 0.1$; - - - - - $\theta = 0.2$.](image)

4.4.1. $\theta$-loop

In order to be able to use the same procedure as in sections 4.2 and 4.3, the initial conditions for the whole range of $\theta$-values have to be found. This is again obtained by using the algorithm explained in section 3.4 with $\theta$ being the parameter which is varied, $\theta \in [0.02, 0.2]$. The loop for the inner mode is performed with $h = 0$ and for the outer mode with $h = 1$. Figure 4.10 shows the obtained results.

With increasing shear-layer thicknesses, the inner mode of a single jet turns more stable, becoming even convectively unstable already for $h = 0$. The absolute temporal growth rate for the outer mode increases with $\theta$, but still lying below zero. The location of the pinching point in the complex $\alpha$-plane varies much more for the inner mode than for the outer one. In contrary to what has been obtained when the bypass-velocity ratio was varied (see Fig. 4.3c), the variation in the $\alpha_0$-location is almost constant for a given variation in $\theta$. 
4.4. Influence of the shear-layer thickness $\theta$

Figure 4.10.: Absolute/convective instability analysis for different shear-layer thicknesses: $h = 0$ for inner mode, $h = 1$ for outer mode, $S = 0.3$, $Ma = 0$, $m = 0$: - - - , inner mode; ---, outer mode. •, $\theta = 0.02$; o, $\theta = 0.2$.

4.4.2. h-loop

Once the pinching point is found for every $\theta$-value for the inner ($h = 0$) and outer ($h = 1$) modes, the loops over all bypass-velocity ratios can be performed. Inner and outer modes are treated separately for an easier representation.

Inner mode

Figure 4.11 presents the results for the spatio-temporal instability characteristic over all bypass-velocity ratios for different shear-layer thicknesses. The values for $h = 0$ are equivalent to the one obtained in figure 4.10. It is interesting to note that the most unstable mode does not correspond to the same $\theta$ when $h$ increases. For low bypass-velocity ratios, it is obtained from flows with thin shear layers, whereas for large $h$ it is generated from flows with large $\theta$. Furthermore, as the bypass-velocity ratio increases, the absolute angular frequencies and wavenumbers for the different shear-layer thicknesses become closer and closer indicating a certain insensitivity to this parameter for high $h$-values.

The magnitude of the eigenfunctions for different shear-layer thicknesses at $h = 0.1$ are plotted in figure 4.12. Generally speaking it can be said that the thicker the shear layers, the flatter and weaker the disturbance amplitudes. The amplitude of the axial velocity disturbance increases on the centerline. Large supports might indicate the formation of structures with scales having the size of the jet (see Jendoubi & Strykowski [1994]. The character of the modes becomes closer to the jet-column type for higher $\theta$, logical conclusion of the weakening of the shear-layer gradients. In figure 4.12b it can be seen that the peak around $r_1 = 1$ decreases with increasing $\theta$. 
4.4. Influence of the shear-layer thickness $\theta$

Figure 4.11: Absolute/convective instability analysis for the inner mode for different bypass-velocity ratios and shear-layer thicknesses: $S = 0.3$, $Ma = 0$, $m = 0$: $\theta = 0.02$; $\theta = 0.03$; $\theta = 0.06$; $\theta = 0.1$; $\theta = 0.16$; $\theta = 0.2$. $h = 0$; $h = 1$.

Figure 4.12: Magnitude of disturbance eigenfunctions of the inner mode for different shear-layer thicknesses: $h = 0.1$, $S = 0.3$, $Ma = 0$, $m = 0$: $\theta = 0.02$; $\theta = 0.04$; $\theta = 0.1$; $\theta = 0.2$. $\theta = 0.2$. 

(a) Absolute temporal growth rate
(b) Absolute wavenumber (pinching point location)

(a) Absolute/convective instability analysis for the inner mode for different bypass-velocity ratios and shear-layer thicknesses: $S = 0.3$, $Ma = 0$, $m = 0$: $\theta = 0.02$; $\theta = 0.03$; $\theta = 0.06$; $\theta = 0.1$; $\theta = 0.16$; $\theta = 0.2$. $h = 0$; $h = 1$.

(b) Absolute wavenumber (pinching point location)

(a) Density $|\rho|$
(b) Axial velocity $|U|$
(c) Radial velocity $|V|$
(d) Pressure $|P|$

Figure 4.12: Magnitude of disturbance eigenfunctions of the inner mode for different shear-layer thicknesses: $h = 0.1$, $S = 0.3$, $Ma = 0$, $m = 0$: $\theta = 0.02$; $\theta = 0.04$; $\theta = 0.1$; $\theta = 0.2$. $\theta = 0.2$. 

(a) Density $|\rho|$
(b) Axial velocity $|U|$
(c) Radial velocity $|V|$
(d) Pressure $|P|$
4.4. Influence of the shear-layer thickness $\theta$

**Outer mode**

The results for the outer mode are shown in figure 4.13. For $h$ close to unity, the most unstable mode arises in flows with thick shear layers, whereas for $h \approx 0.5$ it is generated in flows with thin shear layers.

![Graph showing absolute temporal growth rate and absolute wavenumber](image)

**Figure 4.13.** Absolute/convective instability analysis for the outer mode for different bypass-velocity ratios and shear-layer thicknesses: $S = 0.3$, $Ma = 0$, $m = 0$: \(-\ldots\), $\theta = 0.03$; \(-\ldots\), $\theta = 0.06$; \(-\ldots\), $\theta = 0.07$; \(-\ldots\), $\theta = 0.08$; \(-\ldots\), $\theta = 0.09$; \(-\ldots\), $\theta = 0.1$. •, $h = 0.07$; ×, $h = 1$.

Figure 4.14 shows the magnitude of the disturbance eigenfunctions for different $\theta$-values for $h = 0.5$. Also for the outer mode, increasing the shear-layer thickness results in flatter disturbance density amplitudes, smaller peaks of the axial velocity perturbations, whose amplitude increases at the centerline. However, the radial velocity disturbance and the pressure disturbance amplitudes increase.
4.4. Influence of the shear-layer thickness $\theta$

![Graphs](image)

Figure 4.14.: Magnitude of disturbance eigenfunctions of the outer mode for different shear-layer thicknesses: $h = 0.5, S = 0.3, Ma = 0, m = 0$: ---, $\theta = 0.03$; -- - - , $\theta = 0.06$; - - - , $\theta = 0.08$; ---, $\theta = 0.1$.

**Phase velocity**

Modes can also be classified depending on their phase velocity, which is defined in the axial direction by

$$c_{ph} = \frac{\omega_r}{\alpha_r}.$$  \hspace{1cm} (4.1)
4.4. Influence of the shear-layer thickness $\theta$

For the mode with zero group velocity, the phase velocity for different shear-layer thicknesses and all investigated bypass-velocity ratios is shown in figure 4.15. For $h \to 1$ the inner modes have a phase velocity corresponding to the velocity of the base flow at $r_1 = 1$, whereas for $h \to 0$ the phase velocity of the outer modes tends to $u_b(r = r_2) = 0$. This reinforce the idea that the inner, resp. outer, mode is linked with the inner, resp. outer, shear layer. The peak in the phase velocity for the inner mode is due to the rapid decrease of $\alpha_r$ (see Fig. 4.11b) when the value of $\omega_r$ is still large (see Fig. 4.3b). Concerning the dependence on the shear-layer thickness, both modes feature higher phase velocities for thin shear layers.

![Figure 4.15: Phase velocities of modes with zero group velocity. S = 0.3, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 0$: outer mode; $\cdots$, velocity at $r_1 = 1$; $\cdots$, velocity at $r_2 = 2$. The thicker the line, the thicker the shear layer.](image)

4.4.3. $S$-loop

Exactly in the same fashion as for the case presented in section 4.3 for $\theta = 0.03$, the temperature ratio is varied until the absolute marginal instability is found. This is done for all bypass-velocity ratios $h \in [0, 1]$ and different shear-layer thicknesses.
4.4. Influence of the shear-layer thickness $\theta$

**Inner mode**

The spatio-temporal instability characteristic in the $(h, S)$-plane for the inner mode and different $\theta$-values is shown in figure 4.16. It is interesting to note that while the temperature ratio needed for the absolute marginal instability for low bypass-velocity ratios is decreased for thick shear layers, which indicates a stabilisation of the flow, the contrary is observed for bypass-velocity ratios tending to unity.

A possible explanation for this can be found by observing the absolute wavenumber for different bypass-velocity ratios (see Fig. 4.3c). The real part of $\alpha_0$ becomes smaller for larger $h$ such that the wavelength in the axial direction $\lambda_{0,x} = 2\pi/\alpha_{0,r}$ increases in size. Assuming that the characteristic dimensions of the mode (wavelength) in the radial direction increases as well, it can be presumed that this large perturbation is more unstable in flows with thicker shear layers.

In general, the size of the domain of an absolute instability decreases for increasing $\theta$-values.

![Figure 4.16: Absolute marginal instability in the $(h, S)$-plane for the inner mode and various shear-layer thickness: $Ma = 0$, $m = 0$: ---, $\theta = 0.03$; -- , $\theta = 0.06$; - - - , $\theta = 0.1$. Above lines, convective instability CI; below lines, absolute instability AI.](image-url)
4.4. Influence of the shear-layer thickness $\theta$

**Outer mode**

For the outer mode an interesting phenomenon occurs. The variations in the absolute marginal instability curve are larger for a certain range of parameters. The results plotted in figure 4.17 indicate that the changes in the curve are not monotone when increasing $\theta$. The bell-shape of the absolutely unstable region reaches higher $S$-values for a shear-layer thickness up to $\theta \approx 0.07$ and decreases then for thicker shear layers, reaching almost the same shape as for $\theta = 0.03$. An isothermal jet with $\theta = 0.07$ may become absolutely unstable for a bypass-velocity ratio $h \in [0.2, 0.55]$.

Furthermore, the transition between absolutely and convectively unstable flows for bypass-velocity ratios above 0.65 is almost independent on the thickness of the shear layers. When $h$ is large, the inner shear layer where temperature gradients vary with $S$ becomes less relevant and a change in its thickness has only a little influence. For low bypass-velocity ratios, the marginal instability curve is very sensitive to the shear-layer thickness and temperature ratio because both velocity and temperature gradients are high at the inner shear region.

![Figure 4.17: Absolute marginal instability in the $(h, S)$-plane for the outer mode and various shear-layer thickness: $Ma = 0, m = 0$: \(\triangle\), $\theta = 0.03$; \(-\cdot\cdot\cdot\), $\theta = 0.04$; \(-\cdot\cdot\cdot\triangle\cdot\cdot\cdot\), $\theta = 0.05$; \(-\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\
4.4. Influence of the shear-layer thickness $\theta$

However, as it can be seen from figure 4.18 the absolute temporal growths near the absolute marginal unstable curve vary very slowly for $h \lesssim 0.6$. This explains the high sensitivity of the absolute marginal curve on the shear-layer thickness for low bypass-velocity ratios. Lesshaft & Huere (2007) and Jendoubi & Strykowski (1994) also found larger instability for a certain range of shear-layer thickness in a single jet. In the viscous study of the former, this effect is even more pronounced.

Also for the outer mode, the shape of the absolute marginal instability curve does not vary with $\theta$. The most unstable outer modes are always found for bypass-velocity ratios around $h \simeq 0.5$.

**Figure 4.18:** Temporal growth rates in the $(h, S)$-plane for the outer mode: $\theta = 0.05$, $Ma = 0$, $m = 0$. Colorbar indicates $\omega_{0,1}$ value. Above line, convective instability CI; below line, absolute instability AI.

In figure 4.19 the magnitude of the disturbance eigenfunctions for different temperature ratios is plotted. From figure 4.19a it can be seen that the density perturbation strongly decreases for $S \to 1$. The axial velocity disturbances exhibit larger amplitudes in the jet-core region for $S \to 0$ and larger ones in the bypass-stream and the ambient flow for temperature ratios closer to unity. When the temperature of the jet-core is increased or the one of the surrounding decreased, the
4.5. Influence of the Mach number $Ma$

Radial velocity fluctuation amplitudes become stronger in the region where temperature gradients are present. The pressure disturbances are also much larger for small $S$.

Figure 4.19: Magnitude of disturbance eigenfunctions of the outer mode for different temperature ratios: $h = 0.5$, $\theta = 0.05$, $Ma = 0$, $m = 0$: $S = 0.1$; $S = 0.4$; $S = 0.6$; $S = 1$.

4.5. Influence of the Mach number $Ma$

4.5.1. Introduction

In this section the spatio-temporal linear instability characteristic of a heated coaxial jet flow is studied for increasing Mach numbers. This is of fundamental importance if one wants to investigate this flow in the context of turbofan jet engines.
4.5. Influence of the Mach number $Ma$

Unfortunately, results were obtainable only for low $Ma$, in the subsonic regime $Ma \in [0, 0.6]$. However, it has to be pointed out that the considered Mach number is defined using centerline values (see eq. (2.9)), where the temperature reaches its maximum. The speed of sound in the surroundings is linked to the one at the centerline by the temperature ratio (assuming the same gas in both streams): $a_\infty = a_c \sqrt{S} \leq a_c$, so that the Mach number defined with the ambient speed of sound reads

$$Ma_\infty = \frac{u_c}{a_\infty} \geq Ma = \frac{u_c}{a_c},$$

(4.2)

which could also be supersonic. In the following, when referring to Mach number, the one with respect to the speed of sound at the centerline is considered.

For the range of chosen Mach numbers, the temperature profile is only slightly modified (see eq. (2.6)).

The followed procedure is the same as the one used for the study of the influence of the shear-layer thickness. First, a loop over all $Ma$ for some bypass-velocity ratios is performed. For certain $Ma$-values, $h$ is then varied and finally the temperature ratio $S$ is changed until the absolute marginal instability curve for different Mach numbers is obtained.

Nevertheless, difficulties are encountered while increasing $Ma$. In fact, the grid with $r_{max} = 25$ and $N = 128$, discussed in section 4.2, is not big and fine enough to allow a correct tracking of the saddle point. It is found that the algorithm of section 3.4 either does not find an eigenvalue any more or it may continue following an eigenvalue, which is though the wrong one.

By looking at the disturbance eigenfunctions for high Mach numbers it can be seen that the perturbations are large also far away from the centerline. The reason for this phenomenon is the fact that, for higher Mach numbers, energy is radiated away from the centerline. As it can be seen in figure 4.20 for the real part of the temporal pressure disturbance mode, this eigenfunction exhibits oscillations, whose peaks move towards higher $r$ as time evolves. This effect is however not surprising for high Mach values.
4.5. Influence of the Mach number $Ma$

Using zero Dirichlet boundary conditions (see eq. (2.24)) at $r_{max} = 25$ is therefore not a valid approximation and could generate spurious oscillations (in addition to the physical oscillations stemming from radiation) in the result over the whole domain. Such spurious global oscillations are due to the discretisation with a Chebyshev collocation method. Using a compact finite difference discretisation scheme would reduce spurious numerical oscillation over the whole domain, but it is found that the local oscillations are larger since the shear layers are not well resolved with the tested finite-difference discretization.

Possible solutions are either an increase of the domain size or changing the boundary conditions. Both strategies are investigated.

When the domain is enlarged, the number of collocation points has to be increased correspondingly. Additionally, if the domain is very large, the refinement regions, which are originally located at the shear-layer locations, have to be shifted in order to have a good resolution also far from the centerline. A possible solution would be setting a refinement area between the two shear layers, e.g. at $\hat{r} = (r_1 + r_2)/2$ and the second, with a much smaller inverse refinement width, far away from the centerline, e.g. at $r_{max}/2$.

The second kind of boundary conditions are non-reflecting boundary conditions which do not
impose a zero value of the eigenfunctions at the domain boundary, so that it is not necessary to increase the domain size to fulfill relation \ref{eq:2.24}. These boundary conditions were already implemented by Michael Gloor in the solver. Because of their dependence on the solution itself (wavenumber and frequency), an additional inner loop has to be computed at every iteration of the algorithm in order to ensure their correctness. For further detail the reader is referred to the Matlab’s code InviscidSolverComp.m.

For further detail the reader is referred to the Matlab’s code InviscidSolverComp.m.

**Figure 4.21.** Convergence study on \( \omega_{0,i} \) for different boundary conditions: outer mode, \( h = 1 \), \( S = 0.3, \theta_1 = \theta_2 = 0.03, m = 0 \): D, Dirichlet boundary condition; NR, non-reflecting boundary conditions; \( r_{max} \), mesh size; \( N + 1 \), number of collocation points.

Figures 4.21 and 4.22 show the convergence study of the outer mode at unity bypass-velocity ratio (large single jet) for different domain sizes, boundary conditions and number of collocation points. It can be seen that with Dirichlet boundary conditions and a relatively small domain, the functions are not monotone. This is suspicious because for an increasing Mach number one can expect the disturbances to be stabilized (see \cite{Monkewitz1988, Jendoubi1994, Lesshaft2007}). Increasing drastically the mesh size and the number of collocation points prevents the algorithm to follow an incorrect eigenvalue obtained from other \( \alpha \)-branches than the correct \( \alpha^+ \) and \( \alpha^- \) ones. However, this procedure is prohibitive due to the very high computational cost. It has to be stressed, that even if some grids seem to be coarse because of the large domain compared to the number of collocation points, only the shear-layer
4.5. Influence of the Mach number $Ma$

regions have to be finely resolved and this is always the case because of the mapping (see Sec. 3.2).

![Figure 4.22: Convergence study on $\alpha_0$ for different boundary conditions: outer mode, $h = 1$, $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $m = 0$; D, Dirichlet boundary condition; NR, non-reflecting boundary conditions; $r_{max}$, mesh size; $N+1$, number of collocation points.](image)

Using non-reflecting boundary conditions is found to be a good solution. The results for $r_{max} = 100$ and $N = 192$ or $N = 256$ are very close until $Ma \cong 0.4$. The mesh with $r_{max} = 100$, $N = 192$ and non-reflecting boundary conditions are used for the following calculations in this section.

For higher Mach values however, the algorithm loses again the correct saddle point. The eigenvalues spectrum when this occurs is plotted in figure 4.23.
4.5. Influence of the Mach number $Ma$

The location of the computed absolute mode is indicated by *. Identifying the $\alpha^\pm$-branches involved in the pinching is not an easy task. Many other eigenvalues and spatial branches, whose character differs from the $\alpha^\pm$-branches for low Mach numbers, are present. The exactness of the computed spatio-temporal mode is therefore very questionable for high Mach numbers.

The pressure disturbance eigenfunction of the closest eigenvalue $\odot$ to the pinching point is shown in the same figure. The oscillations indicate a radiative mode. In fact, its phase Mach number

$$Ma_{ph} = \frac{c_{0,ph}}{a_c} = \frac{\omega_{0,r}/\alpha_{0,r}}{1/Ma} = 5.48 > 1,$$

which is a characteristic of these undesired eigenvalues. Their physical meaning is unfortunately still unclear. However it can be said that the region in the complex $\alpha$-plane where they appear becomes larger and coincides with the location of the pinching point, which moves towards the
imaginary axis for higher $Ma$. It is therefore very likely that the algorithm picks four eigenvalues which do not belong to the correct $\alpha^+$- and $\alpha^-$-branches. The sudden incorrect movement of the saddle point location to higher imaginary values (see Fig. 4.22) is probably due to such a wrong interpretation of the $\alpha^-$-branch with one of the spatial branches visible on figure 4.23 which move to higher $\alpha_i$ for increasing Mach numbers.

In order to ensure the following of a correct saddle point, a couple of strategies can be used. Firstly, there is no reason for a sudden change in the trajectory of the pinching point in the complex $\alpha$-plane. Secondly, for increasing Mach numbers the absolute temporal growth rate should decrease. Lastly, if $\omega_{0,i}$ decreases abruptly when other parameters such as $h$ or $S$ are varied, it is very likely that the algorithm converged to an unwanted eigenvalue which is too damped, i.e. is a radiative mode.

To conclude this discussion on the difficulty of finding the absolute wavenumber for high Mach numbers, it has to be remembered that the main objective of the study is finding regions of absolute instability. Since the stabilisation of the flow is significant already for moderate Mach numbers, not being able to reach supersonic Mach numbers is not a too big limitation, because the transition from absolute to convective instability has already occurred at lower Mach numbers (at least for moderate temperature ratios).

### 4.5.2. $Ma$-loop

Instead of only performing a loop over all Mach numbers at $h = 0$ for the inner mode and at $h = 1$ for the outer one, an additional loop for both modes is computed at $h = 0.5$. For this bypass-velocity ratio, higher Mach numbers can be attained.

The results are presented in figure 4.24. As expected, for increasing $Ma$ the flow is stabilized. The modes lose energy through radiation in the far-field.

The maximum attainable Mach number is $Ma = 0.47$ ($Ma_\infty = 0.86$) for the inner mode at $h = 0$, $Ma = 0.41$ ($Ma_\infty = 0.75$) for the outer mode at $h = 1$ and around $Ma \cong 0.65$ ($Ma_\infty \cong 1.2$) for both modes at $h = 0.5$. The study of Lesshafft & Huerre (2007) considers only Mach numbers below 0.5.
4.5. Influence of the Mach number $Ma$

![Graph](a) Absolute temporal growth rate $\omega_i$ vs. $Ma$

4.5.3. $h$-loop

The influence of the bypass-velocity ratio on the instability characteristics for several Mach numbers is computed by varying $h$ starting from $h = 0$ (inner mode), $h = 1$ (outer mode) and $h = 0.5$ for both modes. This procedure allows to follow the saddle point for a parameter range of bypass-velocity ratios as big as possible.

**Inner mode**

Figure 4.24 shows the absolute temporal growth rate and the location of the pinching point in the complex $\alpha$-plane for different bypass-velocity ratios. Because of the reasons discussed in the previous section 4.5.1, the results are unfortunately not complete over the complete $h$-range. However, the stabilizing effect for high Mach number is still visible. Furthermore, this stabilization decreases for the inner mode as the bypass-velocity ratio increases.

The location of the saddle point in the complex $\alpha$-plane before interruption of the algorithm shows the deviation from the expected direction. The latter can be constructed by looking at the results obtained for higher Mach numbers when starting from other initial $h$-values and assuming a continuous trend in the results for increasing $Ma$.

The magnitudes of the disturbance eigenfunctions of the inner mode for different Mach numbers at $h = 0.1$ are plotted in figure 4.26. For the density and the radial velocity it can be seen that the amplitudes are higher over a larger radial domain when the Mach number is increased. Furthermore the peak of $|V|$ at the inner shear layer becomes stronger.

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**Figure 4.24:** Absolute/convective instability analysis for different Ma number: $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $m = 0$: - - - - inner mode for $h = 0$; - - - - outer mode for $h = 1$; - - - - - inner mode for $h = 0.5$; - - - , outer mode for $h = 0.5$. $\circ$, $Ma = 0$; $\triangleright$, $Ma = 0.47$; $\blacktriangle$, $Ma = 0.41$; $\blacksquare$, $Ma = 0.65$. 

**Figure 4.25:** Inner mode for different bypass-velocity ratios.
4.5. Influence of the Mach number $Ma$

Figure 4.25: Absolute/convective instability analysis for the inner mode for different bypass-velocity ratios and Mach numbers: $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $m = 0$: $\cdots$, $Ma = 0$; $\cdots$, $Ma = 0.1$; $\cdots$, $Ma = 0.2$; $\cdots \cdots$, $Ma = 0.3$; $\cdots\cdots\cdots$, $Ma = 0.38$; $\cdots\cdots\cdots$, $Ma = 0.6$.

Figure 4.26: Magnitude of disturbance eigenfunctions of the inner mode for different Mach numbers: $h = 0.1$, $\theta_1 = \theta_2 = 0.03$, $S = 0.3$, $m = 0$: $\cdots$, $Ma = 0$; $\cdots$, $Ma = 0.1$; $\cdots$, $Ma = 0.2$; $\cdots$, $Ma = 0.38$. 
4.5. Influence of the Mach number $Ma$

**Outer mode**

The instability characteristic of the outer mode for bypass-velocity ratios varying from zero to unity and different Mach numbers is presented in figure 4.27. The stabilizing effect of the Mach number is nicely visible on figure 4.27a. Similarly to what has been observed for the inner mode, this effect is more pronounced near the single jet configuration corresponding to the studied mode, here the outer one, i.e. $h = 1$.

![Graphs showing absolute temporal growth rate and absolute wavenumber](image)

**Figure 4.27.** Absolute/convective instability analysis for the outer mode for different bypass-velocity ratios and Mach numbers: $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $m = 0$: $\rightarrow$, $Ma = 0$; $\cdots \cdots$, $Ma = 0.1$; $\cdot \cdot \cdot$, $Ma = 0.2$; $\cdot \times \cdot \cdot \cdot$, $Ma = 0.3$; $\cdot \cdot \cdot \nabla \cdot \cdot \cdot$, $Ma = 0.38$; $\cdot \cdot \cdot \cdot \cdot \cdot \cdot$, $Ma = 0.6$.

In figure 4.28 the disturbance eigenfunctions of the outer mode at $h = 0.5$ for increasing Mach numbers are displayed. The increasingly large support of non-zero disturbance amplitudes for higher $Ma$ can be seen here as well (see for example Fig. 4.28a and Fig. 4.28c).
4.5. Influence of the Mach number $Ma$

4.5.4. $S$-loop

Once the pinching point location is obtained for all bypass-velocity ratios, different $Ma$ and fixed $S = 0.3$, the temperature ratio is increased/decreased until the absolute marginal state is found.

Figure 4.28.: Magnitude of disturbance eigenfunctions of the outer mode for different Mach numbers: $h = 0.5$, $\theta_1 = \theta_2 = 0.03$, $S = 0.3$, $m = 0$: $- - -$, $Ma = 0$; $- - -$, $Ma = 0.2$; $- - -$, $Ma = 0.38$; $- - -$, $Ma = 0.6$. 

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4.5. Influence of the Mach number $Ma$

**Inner mode**

Absolute marginal instability curves in the $(h, S)$-plane for different Mach numbers are shown in figure 4.29. Because of the stabilization for higher Mach values, lower temperature ratios are needed to have an absolutely unstable mode.

![Figure 4.29: Absolute marginal instability in the $(h, S)$-plane for the inner mode and different Mach numbers: $\theta_1 = \theta_2 = 0.03$, $m = 0$: $\cdots$, $Ma = 0$; $\cdots \cdots$, $Ma = 0.1$; $\cdots \cdots$, $Ma = 0.2$; $\cdots \cdots \cdots$, $Ma = 0.3$; $\cdots \cdots \cdots$, $Ma = 0.38$; $\cdots \cdots \cdots \cdots$, $Ma = 0.6$. Above lines, convective instability CI; below lines, absolute instability AI.](image)

Lesshaft & Huerre (2007) proposed a very simple relation to evaluate the stabilizing effect by looking at the difference in the temperature ratio needed to have a marginal state for a given $Ma$ with respect to the case with $Ma = 0$. Their formula, proposed for single jets, is slightly modified for coaxial jet flows:

$$
\Delta S(Ma, h) = -1.4 Ma^2 (1 - h)^2.
$$

(4.4)
4.5. Influence of the Mach number \( Ma \)

The validation of this expression with the computed results is shown in figure 4.30. Good agreement between theoretical and numerical results reinforces the applicability of equation (4.4), making it useful to directly predict the absolute marginal instability curve for the inner mode with a given Mach number. However, the maximal Mach number for which the expression is correct is uncertain (e.g., it does not hold for \( Ma = 0.6 \)) and as it is the case for \( Ma = 0.2 \), some inconsistencies may occur for low bypass-velocity ratios. Relation (4.4) indicates that for high bypass-velocity ratios the instability characteristic of the inner mode becomes independent of the Mach number (see also Fig. 4.25a).

\[ \Delta S = f(h) \]

**Figure 4.30.** Variation in the temperature ratio corresponding to the absolute marginal instability for a given bypass-velocity ratio with respect to the case with \( Ma = 0 \): \( \circ \), \( Ma = 0.1 \); \( * \), \( Ma = 0.2 \); \( \nabla \), \( Ma = 0.3 \); ——, numerical result; ——, theoretical value.

In the literature, results are often presented for constant Mach numbers based on the free-stream speed of sound: \( Ma_\infty \). Absolute marginal instability curves for different free-stream Mach numbers are obtained by varying simultaneously the temperature ratio \( S \) and \( Ma = Ma_\infty \sqrt{S} \) as well. The results for a heated coaxial jet are plotted in figure 4.31. The same considerations as for the Mach number defined with centerline quantities can be drawn.
4.5. Influence of the Mach number $Ma$

**Figure 4.31.** Absolute marginal instability in the $(h, S)$-plane for the inner mode and different free-stream Mach numbers: $\theta_1 = \theta_2 = 0.03$, $m = 0$: $Ma_\infty = 0$; $\cdots \cdots \cdot$, $Ma_\infty = 0.37$; $\cdots \cdots \cdot$, $Ma_\infty = 0.55$. Above lines, convective instability CI; below lines, absolute instability AI.

**Outer mode**

The absolute and convective instability regions in the $(h, S)$-plane for the outer mode and different Mach numbers are presented in figure 4.32. Increasing the Mach number drastically reduces the domain of an absolute instability. For $Ma = 0.6$, only convectively unstable flows can be found. Furthermore, the higher $Ma$, the bigger the variation in the stabilization. The maximum attainable temperature ratio, still giving rise to an absolute instability, shifts towards lower $h$ as the Mach number is increased.

It is very interesting to note that all the absolute marginal curves pass through the point $(0.1, 0.1)$. As it was already seen in figure 4.27a, for low bypass-velocity ratios, the instability characteristic of the outer mode is less sensitive to the Mach number.

For the outer mode no easy relation for the prediction of the absolute marginal instability curve with different Mach numbers could be found.

As for the inner mode, the absolute marginal instability curve for constant free-stream Mach numbers is computed and is presented in figure 4.33.
4.5. Influence of the Mach number $Ma$

**Figure 4.32.** Absolute marginal instability in the $(h, S)$-plane for the outer mode and different Mach numbers: $\theta_1 = \theta_2 = 0.03$, $m = 0$: $Ma = 0$; $\ldots$, $Ma = 0.1$; $\ldots$, $Ma = 0.2$; $\ldots \times \ldots$, $Ma = 0.3$; $\ldots \nabla \ldots$, $Ma = 0.38$. Above lines, convective instability CI; below lines, absolute instability AI.

**Figure 4.33.** Absolute marginal instability in the $(h, S)$-plane for the outer mode and different free-stream Mach numbers: $\theta_1 = \theta_2 = 0.03$, $m = 0$: $Ma_\infty = 0$; $\ldots$, $Ma_\infty = 0.37$; $\ldots$, $Ma_\infty = 0.55$. Above lines, convective instability CI; below lines, absolute instability AI.
All the results presented until now were for two-dimensional axisymmetric disturbances, i.e. \( m = 0 \). In this section, the spatio-temporal instability characteristic of three-dimensional disturbances for an azimuthal wavenumber \( m = 1 \) is discussed, still while considering a base flow without swirl.

The initial conditions for the inner and outer modes at \( h = 0 \) and \( h = 1 \), respectively, are found by looking at the \( \alpha^\pm \)-branches when the imaginary part of the integration path \( L_\omega \) is decreased, in the same manner as for the axisymmetric modes. The pinching points are then followed while the parameters \( h \) and \( S \) are successively varied.

4.6.1. \( h \)-loop

In figure 4.34 the absolute temporal growth rate and the saddle point location \( \alpha_0 \) for both inner and outer, axisymmetric and first azimuthal modes (\( m = 1 \)) with bypass-velocity ratios \( h \in [0, 1] \) are presented.

As already pointed out by Monkiewitz & Sohn (1988); Lesshafft & Huerre (2007) for single jets, the most unstable mode with zero group velocity corresponds to an axisymmetric mode. This result can be generalized also for coaxial jet flows.

Concerning the first azimuthal modes, the inner one exhibits a strong stabilisation for increasing \( h \)-values. The outer mode is instead less stabilized but it never shows absolute instability.

The locus of absolute wavenumbers in the complex \( \alpha \)-plane has a different character than for axisymmetric modes, even if for the inner mode the real wavenumber \( \alpha_{0,r} \) still tends to zero for bypass-velocity ratios approaching unity.
4.6. Azimuthal mode $m = 1$

The disturbance eigenfunctions are plotted in figures 4.35 and 4.36. In contrast to figure 4.4, the azimuthal velocity disturbance eigenfunction is not zero any more. In addition to that, the perturbations of the azimuthal inner mode are more compact around the inner shear layer also if $h = 0.5$. Furthermore, the amplitude of the pressure disturbance at the centerline vanishes for both modes because of the boundary condition. This indicates that azimuthal modes are more of the shear-layer type, explaining so their higher stability compared to axisymmetric modes, where pressure perturbations can communicate through the centerline (Jendoubi & Strykowski, 1994; Lesshafft & Huerre, 2007).

Looking at the eigenfunctions also permits to verify the correct implementation of the boundary conditions (see eq. (2.20)).
4.6. Azimuthal mode $m = 1$

Figure 4.35.: Disturbance eigenfunctions of the first azimuthal inner mode: $h = 0.5$, $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 1$: —— magnitude; - - - real part; ··· imaginary part.
4.6. Azimuthal mode $m = 1$

Figure 4.36: Disturbance eigenfunctions of the first azimuthal outer mode: $h = 0.5$, $S = 0.3$, $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = 1$: --- magnitude, - - - real part, ..., imaginary part.
4.6. Azimuthal mode $m = 1$

4.6.2. $S$-loop

Already from the results for $S = 0.3$ it is clear that the absolutely unstable region for the first azimuthal modes will be very small. This is confirmed by performing a loop over the temperature ratios. As it can be seen from figure 4.37, the outer mode never displays an absolute instability in the range of chosen parameters, whereas the inner mode becomes absolutely unstable only for very low bypass-velocity and temperature ratios. The outer mode imaginary absolute frequency presented in figure 4.34a is negative, close to zero, over all bypass-velocity ratios for $S = 0.3$. Even by decreasing the temperature ratio, the instability characteristic of this mode does not change significantly. The reason for this is the weaker influence of temperature variations on this mode, which can also be seen from the smaller amplitude of the density fluctuations in the first azimuthal outer mode. Compared to outer axisymmetric modes, the relative importance of the density perturbation with respect to other disturbances is smaller. It has to be kept in mind that temperature gradients only occur at the inner shear-layer location. Outer azimuthal modes are more sensitive to velocity gradients, which are in contrary also present at the outer shear-layer location.

![Figure 4.37: Absolute marginal instability in the ($h, S$)-plane for the first azimuthal modes: $\theta_1 = \theta_2 = 0.03$, $Ma = 0$, $m = \pm 1$: - - - , inner mode absolute marginal instability. Above line, convective instability CI; below line, absolute instability AI.](image)

Considering that first azimuthal modes are much more stable than the axisymmetric ones, the influence of the shear-layer thickness and the Mach number on their absolute marginal instability is not investigated. The study of thick shear layers would become interesting for azimuthal modes with group velocity greater than zero, which are more unstable and are observed in temporal stability studies (see Lesshaft & Huerre [2007]). The stabilizing effect for an increasing Mach number is presumed to hold also for azimuthal modes.
4.7. Spatio-temporal characteristics of travelling waves

Until now, spatio-temporal modes with zero group velocity were considered. They allow to distinguish between absolute and convective instabilities. In reality, any perturbation is a superposition of different waves whose group velocity is not necessarily vanishing and the most amplified mode is probably not the stationary one. In this section, a very brief glimpse to the stability of travelling waves is given. Such study is of fundamental importance if one desires to compute the full impulse response.

Absolute wavenumbers and frequencies are computed for several group velocities again by using the algorithm described in section 3.4. The instability characteristic is found for every travelling wave in the moving reference frame with velocity \( V \). In this reference system, the axial base flow velocity reads therefore \( \tilde{u}_b = u_b - V \). Spatio-temporal modes in this reference frame correspond to modes in the stationary frame with external co-flow or counterflow (Lesshaft & Huerre, 2007).

The spatio-temporal characteristic obtained in the moving reference frame can be translated to the fixed reference frame by:

\[
\omega_0 = \tilde{\omega}_0 + \tilde{\alpha}_0 V, \quad \alpha_0 = \tilde{\alpha}_0.
\]  
(4.5)

To define the trailing- and leading-edges of the linear impulse response (see Sec. 2.4.2), the growth rate in the moving reference frame has to be considered:

\[
\sigma(V) = \tilde{\omega}_{0,i}.
\]  
(4.6)

The results for an incompressible heated single jet with \( S = 0.5 \) and \( \theta_1 = 0.05 \) are shown in figure 4.38. The quantities in the laboratory frame present discontinuities. In fact, depending on the group velocity, the pinching results from the coalescence of different \( \alpha^- \) with \( \alpha^+ \)-branches. As suggested by Lesshaft & Huerre (2007), low-group-velocity modes correspond to absolutely unstable modes in jets with zero or moderate counterflow. Modes travelling with higher \( V \) correspond instead to absolutely unstable modes in jets with strong counterflow. They also showed that the growth rate of the former scales with the jet radius, meaning that these mode are of the jet-column type, whereas the growth rate of the latter scales with the shear-layer thickness, indicating a shear-layer mode type (Lesshaft & Huerre, 2007). Around \( V \approx 0.17 \), the mode giving rise to the higher absolute temporal growth rate in the moving frame, and therefore the only one satisfying the pinching requirement, switches from a jet-column to a shear-layer mode. The parabola-shaped variation of the temporal growth rate \( \sigma \) for large group velocity is typical of the Kelvin-Helmholtz instability for a plane shear layer (Lesshaft & Huerre, 2007). It can also be seen that jet-column modes have much larger wavelengths than shear-layer modes (see Fig. 4.38c). Furthermore, the axial phase velocity of jet-column modes is larger than the jet-centerline velocity, whereas for the shear-layer modes it is lower. In their paper, Lesshaft & Huerre (2007) never present the eigenfunction of the modes. Despite these clear differences between the two types of modes, no clear distinction of the mode types by looking at the eigenfunctions, as done by Jendoubi & Strykowski (1994), could however be found in the
present work. For both modes, the amplitudes of the perturbations are large at the centerline and at the shear-layer location. This confirms the difficulty of classifying the modes just by considering their eigenfunctions.

From figure 4.38a it can be seen that the most amplified mode in the impulse response is a travelling shear-layer mode with $V = 0.45$. For this group velocity, the downstream spatial growth rate vanishes. Its frequency is $\omega_{0,r} = 2$ and its wavenumber $\alpha_{0,r} = 4.3$. Furthermore, the travelling modes not presenting any growth (i.e. $\sigma(V) = 0$), defining the trailing- and leading-edges of the impulse response, are given by $V^+ = 0.83$ and $V^- = -0.07$. The flow in the laboratory frame is therefore absolutely unstable and the absolute mode is of the jet-column type for this set of parameters.

The obtained results are in excellent agreement with the ones of Lesshafft & Huerre (2007, compare with their Fig. 1).
5. Global stability analysis

5.1. Introduction

All previous results dealt with strictly parallel steady flows, i.e. invariant under a translation in the axial $x$-direction. The heated coaxial jet flow was assumed to have constant parameters along all the streamwise $x$-coordinates, making a local linear stability study possible. However, jet flows and most of shear flows in general, are non-uniform in the streamwise $x$-direction. In this chapter, the relationship between the local stability characteristics at each streamwise location and the global stability characteristics over some wavelengths is presented. Using the parallel flow theory to perform a global analysis for a slowly spatially developing flow shows the strength, but also the limitations, of the former.

The chapter is outlined as following. First, the theoretical background of the theory is presented taking inspiration from the review articles of Huerre & Monkewitz (1990); Huerre (2000), including the definitions of global stability or instability (see Sec. 5.2). Results obtained for forced and self-sustained oscillations are presented in section 5.3. The limitations of this approach are explained at the end of that section, before introducing the non-linear extension of the theory in sections 5.4 and 5.5.

5.2. Global linear stability theory

5.2.1. Weakly non-parallel flow assumption

The fundamental assumption of the linear global theory based on the local stability is the existence of two separated length scales in the flow. On one hand, the evolution length scale $L$ characterizing the spatial development of the base flow and on the other hand, a typical instability wavelength $\lambda$. The length scale $L$ can be defined by means of the momentum or vorticity thickness $\Theta = \Theta(x)$ and reads

$$ L \sim \left( \frac{1}{\Theta} \frac{d\Theta}{dx} \right)^{-1}. \quad (5.1) $$

The small parameter of interest is then

$$ \varepsilon \sim \frac{\lambda}{L} \ll 1, \quad (5.2) $$

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5.2. Global linear stability theory

making an asymptotic analysis possible (Huerre & Monkewitz, 1990). Therefore, the slow space scale can be defined as

\[ X = \varepsilon x . \] (5.3)

The locally parallel stability results would appear directly as the leading order approximations. It has to be kept in mind that if \( \varepsilon \) is not small, no connection between local and global stability properties can be found. However, if it is, the global mode is obtained by a superposition of linear instability waves that behave at each streamwise location as if the flow were parallel.

### 5.2.2. Introduction to linear global modes

Similarly to what was done in section 2.4 (compare with eq. (2.29)), the one dimensional dispersion relation without forcing reads now:

\[
D\left(-i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X\right) u'(x,t) + \varepsilon D_x \left(-i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X\right) u'(x,t) + O(\varepsilon^2) = 0,
\] (5.4)

where the second term represents the non-parallel flow corrections. It can be seen that every term contains the dependence on the slow streamwise coordinate \( X \), making the equation only invariant in time, but not in space any more. As for the local linear spatio-temporal stability theory, only a single unstable mode is assumed. The normal mode ansatz used for the strictly parallel case, i.e. \( u'(x,t) = U \exp[i(\alpha x - \omega t)] + c.c. \), has to be replaced by the time-harmonic normal mode:

\[
u'(x = X/\varepsilon, t) \sim \tilde{U}(X) \exp\left[i \left(\frac{1}{\varepsilon} \int_0^X \alpha(\zeta; \omega) d\zeta - \omega t\right)\right] + c.c.,
\] (5.5)

where \( \tilde{U}(X) \) is an unknown complex amplitude function and \( \omega \) the complex angular frequency. The perturbation field \( u'(x,t) \) is therefore composed by a slowly varying spatial envelope and a fast varying complex phase (Huerre, 2000).

Interestingly, the complex wavenumber \( \alpha(\zeta; \omega) \) is the solution of the local dispersion relation at location \( \zeta \):

\[ D(\alpha, \omega; \zeta) = 0 , \] (5.6)

which is nothing else than equation (2.28) solved with the base flow parameters at the downstream location \( \zeta \).

The complex amplitude is composed by the cross-stream eigenfunctions associated with the
eigenvalues $\alpha$ and $\omega$ at the location $X$ and a multiplicative factor only depending on $X$. The resulting two-dimensional perturbation field reads therefore:

$$u'(x = X/\varepsilon, y, t) \sim A(X)U(y; X) \exp \left[ i \left( \frac{1}{\varepsilon} \int_0^X \alpha(\zeta; \omega) d\zeta - \omega t \right) \right] + c.c., \quad (5.7)$$

where $U(y; X)$ is the cross-stream disturbance eigenfunction obtained from the local dispersion relation at $X$ for a given $\alpha(X; \omega)$.

**Remark** The multiplicative factor $A(X)$ could be obtained from a perturbation calculation, as explained in Huerre & Rossi (1998). However, it is considered here uniform because the inaccuracies in the calculation of the wavenumber are larger than the error of assuming a constant value for $A(X)$, as pointed out by Juniper et al. (2011).

### 5.2.3. Classes of spatially developing flows

The entire theory explained for the parallel flow case in section 2.4 can be extended to weakly non-parallel base flows just by adding $X$ as a frozen parameter in the arguments and operators. Assume now that equation (5.6) admits only one temporal mode $\omega(\alpha; X)$ (Huerre, 2000). As in equation (2.50), the local maximum temporal growth rate at each $X$-location $\omega_{i,\text{max}}(X) = \omega(\alpha_{\text{max}}; X)$ is given by:

$$\frac{\partial \omega_i}{\partial \alpha}(\alpha_{\text{max}}; X) = 0 \quad (5.8)$$

and as in equation (2.52), the absolute angular frequency $\omega_0(X) = \omega(\alpha_0; X)$ corresponding to the absolute wavenumber $\alpha_0(X)$ is defined by:

$$\frac{\partial \omega}{\partial \alpha}(\alpha_0; X) = \frac{x}{t} = \nu = 0. \quad (5.9)$$

In figure 5.1, an example of both local maximum temporal growth rate and absolute temporal growth rate as a function of the slow space scale $X$ is plotted. Following Huerre (2000), four stability classes of spatially developing flows can be distinguished.

As it can be seen in figure 5.1a, both maximum and absolute temporal growth rates are negative for all streamwise locations ($\omega_{i,\text{max}}(X) < 0$ and $\omega_{0,i}(X) < 0, \forall X$), implying that this situation corresponds to a locally stable flow everywhere. The second stability class shows an unstable region with positive maximal temporal growth rate $\omega_{i,\text{max}} > 0$, but the imaginary part of the absolute frequency is still negative $\omega_{0,i} < 0$ everywhere (Fig. 5.1b). This configuration is called locally convectively unstable. In figure 5.1c, the situation for marginally absolutely unstable flows is displayed. The absolute temporal growth rate reaches almost zero at a certain $X$ location, while the maximal temporal growth rate is positive. The fourth and last class presents a positive
maximal temporal growth rate $\omega_{i,max} > 0$ and also a finite region of absolute instability, i.e. $\omega_{0,i}(X) > 0$ for some $X$-values. The flow is said to be \textit{locally absolutely unstable} (Fig. 5.1d).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/stability.png}
\caption{Stability classes of spatially developing flows (inspired from \cite{Huerre1990, Huerre2000}).}
\end{figure}
5.2. Global linear stability theory

5.2.4. Signalling problem in locally convectively unstable flows

The goal of this section is to describe the global mode shape for convectively unstable flows as illustrated in figure 5.1b. The study of such flows was already performed before the introduction of the notions of absolute and convective instabilities and still nowadays, the majority of the linear stability analyses, considers convectively unstable configurations. Crighton & Gaster (1976) and Gaster et al. (1985) studied the stability for slowly diverging jet flows and mixing layers, respectively. Convectively unstable flows act as noise amplifiers (Huerre & Monkewitz, 1990) so that the stability of the flow depends on the growth or decay of external perturbations. In order to find the global mode shape, it is assumed that every streamwise section of the flow is excited with the same real frequency \( \omega_f \in \mathbb{R} \) and that a local linear stability analysis can be performed at each \( X \)-location.

Similarly to the unforced dispersion relation for a slowly diverging base flow (5.4), the solution is governed by the forced dispersion relation:

\[
D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X \right) u'_f(x, t) + \varepsilon D_x \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X \right) u'_f(x, t) = \delta(x_S) e^{-i \omega_f t},
\]

so that the time-harmonic response reads, similarly to equation (5.7):

\[
u'_f(x = X/\varepsilon, y, t) \sim A(X) U^{\pm}(y; X) \exp \left[ i \left( \frac{1}{\varepsilon} \int_{x_S}^X \alpha^{\pm}(\zeta; \omega_f) d\zeta - \omega_f t \right) \right] + c.c. \] (5.11)

The only difference is the forcing frequency and the superscripts \( \pm \) which refer to the upper or lower spatial \( \alpha \)-branches depending on the domain in which the solution is sought (+ for \( X > X_S \) and – for \( X < X_S \), if the source is at \( X_S \)). Again, the spatial branches satisfy the local linear stability problem at every \( X \)-location:

\[
D(\alpha^{\pm}, \omega_f; X) = 0,
\]

from which also the cross-stream eigenfunctions are obtained (Huerre, 2000).

5.2.5. Global intrinsic oscillations in locally absolutely unstable flows

Consider now a flow presenting a finite region of absolute instability (Fig. 5.1d). In this case, the flow can support a self-excited global mode also without external forcing, through a purely hydrodynamic feedback loop (Chomaz et al., 1991). It acts as an oscillator (Huerre & Monkewitz, 1990). The global impulse response is governed by:

\[
D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X \right) G(x, t) + \varepsilon D_x \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial t}; X \right) G(x, t) = \delta(x_S) \delta(t),
\]

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where \( G(x, t) \) is the global Green function. Global stability and instability can be now defined in the same manner as for the local counterparts of section 2.4 (Huerre, 2000).

A spatially developing base ow is globally stable if

\[
\lim_{t \to \infty} G(x, t) = 0 \quad \text{for all} \quad x, \tag{5.14}
\]

and it is globally unstable if

\[
\lim_{t \to \infty} G(x, t) \to \infty \quad \text{for some} \quad x. \tag{5.15}
\]

If the ow is globally unstable, it is reasonable to assume that the response of the ow is dominated by a global mode solution, neglecting all the eects of external forcing which would be relevant in a convectively unstable medium.

\[
u'_G(x, t) = U_G(x) e^{-i\omega_G t} \tag{5.16}
\]

Unlike in the signalling problem, the complex frequency \( \omega_G \) is unknown. Once the global frequency and its associated global eigenfunction \( U_G(x) \) are known, the global mode is obtained. This is the classical procedure for a two-dimensional global stability analysis. In this work however, no global eigenfunctions are computed, but the global mode shape is constructed with the cross-stream eigenfunctions obtained from local analyses at successive downstream locations.

The above criterion for global stability and instability can also be formulated by using the global mode frequency in the following manner:

- \( \omega_{G,i} < 0 \): globally stable ow: the global mode will decay in time,
- \( \omega_{G,i} > 0 \): globally unstable ow: the global mode will grow in time.

**Remark** The fact that convectively unstable ows are always globally stable justify the study of the spatial response to external perturbations with real frequency as done in section 5.2.4. A global mode can be sustained only if constant forcing is applied.

The next step is the calculation the global mode frequency. This is not an easy task and is still a subject of controversy. The procedure differs depending on the considered ow domain.

**Doubly-infinite ow domain** Assume a doubly-infinite ow domain. The global mode frequency can be obtained by an asymptotic analysis. At the leading order, it is given by

\[
\omega_G \sim \omega_0(X_S), \tag{5.17}
\]
5.2. Global linear stability theory

where $X_S$ is the saddle point in the complex $X$-plane satisfying:

$$\frac{\partial \omega_0}{\partial X} (X_S) = 0.$$  \hspace{1cm} (5.18)

In this complex plane, similarly as for the $\alpha^{\pm}$-branches in the complex $\alpha$-plane, the $X$-integration contour is pinched between two $X^{\pm}$-branches on which the group velocity is zero.

The absolute frequency in the complex $X$-plane is found first by calculating $\omega_0$ as a function of the slow downstream scale $X$ and secondly by analytically prolonging it into the complex $X$-plane. The existence of such a saddle point is related to the existence of a maximum of the absolute temporal growth rate $\omega_{0,i}$ within the flow domain.

For more details, the interested reader is referred to Huerre & Monkewitz (1990); Monkewitz et al. (1993); Huerre & Rossi (1998); Huerre (2000); Pier et al. (2001a).

**Semi-infinite flow domain** Assume a semi-infinite flow domain (e.g. $X \in [0, \infty]$ as for jets) and an absolutely unstable region located at the upstream flow boundary $X = 0$ (e.g. the inlet).

In this case, the pole $\omega_0(X_S)$ is pinned because the $X$-integration contour has to emanate from the boundary (Monkewitz et al., 1993). The global mode frequency is therefore the absolute frequency at the inlet:

$$\omega_G = \omega_0(X_S = 0).$$  \hspace{1cm} (5.19)

Furthermore, it can be pointed out that an asymptotic analysis provides analytic expressions for the global mode frequencies in terms of the local stability properties at the points $X_S$ (see Monkewitz et al., 1993).

As it can be concluded from the previous explanations and figure 5.1, the global growth rate is not greater than the largest absolute local temporal growth rate over all physical $X$-locations:

$$\omega_{G,i} \leq \omega_{0,i}|_{\text{max}}.$$  \hspace{1cm} (5.20)

To understand this better, remember that the former depends on the complex location of the saddle point $X_S$. The $\omega_{0,i}$-curve of figure 5.1d is a cut through its complex prolongation. When the imaginary part of $X$ is varied, the parabolic shape of the curve is shifted to lower values until the saddle point is reached at $X_S$. This means that the global growth rate is lower, or at most equal, to the maximal absolute temporal growth rate. They coincide when $X_S \in \mathbb{R}$ and the saddle point lies directly on the real axis, as it is the case for a semi-infinite flow domain. Equation (5.20) has been rigorously proven by Chomaz et al. (1991); Le Dizès et al. (1996).

Concerning the size of the absolutely unstable region, it is important to stress that it has to be larger than a critical value to have a global instability. If this is not the case, the global mode can still be stable since the imaginary part of the frequency at the saddle point can be negative,
5.3. Linear global modes in heated coaxial jet flows

and this even if the maximal imaginary absolute frequency along the real $X$-axis is positive. For an unstable linear global mode to appear, a finite region of absolute instability is necessary.

Once the global mode frequency is defined, the dispersion relation at every streamwise $X$-location

$$D(\alpha^\pm, \omega_G; X) = 0$$

(5.21)

can be solved to find the spatial branches and the cross-stream eigenfunctions. The time-harmonic response is then given by equation (5.7), which takes now the form:

$$u'(x = X/\varepsilon, y, t) \sim A(X)U^\pm(y; X) \exp \left[ i \left( \frac{1}{\varepsilon} \int_{X_S}^{X} \alpha^\pm(\zeta; \omega) d\zeta - \omega_G t \right) \right] + c.c.,$$

(5.22)

where the superscripts are $+$ for $X > X_S$ and $-$ for $X < X_S$. In the case of a semi-infinite flow domain, only the downstream branch will play a role.

5.3. Linear global modes in heated coaxial jet flows

In this section, the results obtained from the previous explained theories for a heated coaxial jet flow are presented. A semi-infinite flow domain is therefore considered. As base flow profiles the mean flow profiles obtained from the Large Eddy Simulations of Michael Gloor are used. The mean axial velocity and the mean temperature for a globally unstable configuration are shown in figure 5.2. Note that the axial velocity in the surrounding fluid is lowered from the original value 0.1 of the LES to zero in order to change the global stability character from stable to unstable. This is not done for the signalling problem which needs a convectively unstable medium. The other parameters of the flow remain unchanged and their values at the inlet are presented in table 5.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.64</td>
</tr>
<tr>
<td>$S$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.1</td>
</tr>
<tr>
<td>$r_1$</td>
<td>1</td>
</tr>
<tr>
<td>$r_2$</td>
<td>2</td>
</tr>
<tr>
<td>$Ma$</td>
<td>0</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Table 5.1.: Parameters of the base flow at $x = 0$. 86
5.3. Linear global modes in heated coaxial jet flows

![Figure 5.2: Base flow of the heated coaxial jet obtained from Large Eddy Simulations.](image)

**Remark** In the following, the computations are performed using the physical downstream coordinate $x$. The results could be easily transformed to the slow space scale $X$ just with the rescaling $X = \varepsilon x$, only the abscissa will change.

### 5.3.1. Numerical methods and procedure

The procedure to calculate the linear global mode shape slightly differs depending on the considered problem.

**Signalling problem**

Firstly, the spatial spectrum for some given real frequencies at the inlet location is computed (the source of a semi-infinite domain is at $x_S = 0$). Secondly, the selected eigenvalue of the correct spatial branch has to be followed in the complex $\alpha$-plane when the base flow changes for successive downstream sections, which are all excited with a real frequency $\omega_f$. Pure spatial problems instead of spatio-temporal ones have to be solved. The eigenfunctions have also to be stored. Lastly, the spatial wave amplification over the whole domain can be calculated via expression (5.11).

To resume, a local spatial stability analysis has to be performed at every $x$-location using the solver described in sections 2.3 and 3.2. This has to be repeated for every forcing frequency and for all the modes separately.
Global intrinsic oscillations

In the case of an intrinsic global oscillation, the global mode frequency has to be first computed with a linear spatio-temporal stability analysis (algorithm of Sec. 3.3). Assuming a semi-infinite flow domain, this has to be performed for the base flow at the inlet location \( x = 0 \).

Subsequently, the evolution of the eigenvalue belonging to the \( \alpha^+ \)-branch corresponding to \( \omega_G \) and its eigenfunctions have to be tracked. This is done by solving the local forced dispersion relation with forcing frequency \( \omega_G \) at the successively downstream \( x \)-sections. Finally, the global mode shape is computed with equation (5.22).

To resume, a local spatio-temporal stability analysis is performed at \( x = 0 \), followed by the solution of a local eigenvalue problem at every \( x \)-location using the solver presented in sections 2.3 and 3.2.

Remark The step in the downstream direction has to be small because a local eigenvalue problem, using as a guess the eigenvalue obtained from the previous \( x \)-section, is solved. The downstream step could also be adapted depending on the evolution of the base flow. However, because of the low computational cost of the calculations, this is not necessary and a small step is affordable.

Convergence study

A convergence study has been performed and a grid with \( N = 128 + 1 \) collocation points in the radial direction for a domain \( r \in [0, 25] \) and zero Dirichlet boundary conditions at \( r = 25 \) (far-field) have been chosen.

5.3.2. Forced global modes

The spatial spectrum for real discrete frequencies \( \omega_f \in [0, 8] \) for the base flow profile at the inlet location \( x = 0 \) is shown in figure 5.3. The \( \alpha^+ \) and \( \alpha^- \)-branches are clearly identifiable. There are two \( \alpha^+ \)-branches corresponding to the inner and outer modes.

It is interesting to note that, since this flow configuration is convectively unstable (\( \omega_{0,i} < 0 \)), the spatial spectrum for frequencies with \( \omega_f,i = 0 \) presents separated branches, i.e. the pinching between \( \alpha^+ \) and \( \alpha^- \)-branches has not occurred yet.

At this location \( (x = 0) \), the outer mode exhibits a higher spatial growth rate than the inner mode for a wavenumber smaller than \( \alpha_c \simeq 4.1 \), which corresponds to a forcing frequency \( \omega_f \simeq 1.7 \). For higher wavenumbers and frequencies, the inner mode is more unstable.

The unstable outer mode frequencies span the range \([0, 2]\), whereas the inner mode features spatial growth for frequencies in the larger range \([0, 8]\). This means that when exciting with a frequency \( \omega_f \in [0, 2] \) both modes are amplified in the flow. However, the one with the larger growth rate will dominate (see 5.3.2).
Figure 5.3.: Spatial spectrum in the complex $\alpha$-plane at location $x = 0$.

**Remark** Since the used theory considers only one mode, the response to forcing has to be calculated for the modes separately.

The evolution of the spatial eigenvalues of the $\alpha^+$-branches for the outer and inner modes for downstream locations going from $x = 0$ to $x = 20$ are displayed in figure 5.4. The full bullets correspond to the eigenvalues of figure 5.3 i.e. at $x = 0$, while the empty circles are the eigenvalues for the section at $x = 20$. As a side remark, it can be observed that for the base flow at the section $x = 20$, the outer mode is not unstable any more, whereas the inner mode still exhibits a downstream positive spatial growth for low frequencies. Furthermore, the downstream spatial growth rate is not monotonically decreasing over all the wavenumbers for increasing $x$. It is found instead that for some downstream locations the spatial growth rate slightly increases. This phenomenon is however only present for the wavenumbers of the most unstable mode, i.e. the outer for $\alpha_r < 4.1$ and the inner for larger wavenumbers.
5.3. Linear global modes in heated coaxial jet flows

The time-harmonic steady-state response is only calculated for some frequencies reported in Table 5.2 and the two modes separately. The outer mode is presented first since it is more relevant for self-sustained global modes.

Table 5.2: Selected frequencies $\omega_f$ for the forced response calculation of the two modes.

<table>
<thead>
<tr>
<th>outer</th>
<th>inner</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.35</td>
</tr>
<tr>
<td>0.65</td>
<td>2.35</td>
</tr>
<tr>
<td>1.1</td>
<td>4.15</td>
</tr>
<tr>
<td>1.5</td>
<td>6.05</td>
</tr>
</tbody>
</table>

For sake of clarity, the used expression for the perturbation response (5.11) is rewritten here:

$$u'_f(x, r, t) \sim U^+(r; x) \exp \left[ i \left( \int_0^x \alpha^+(\zeta; \omega_f) d\zeta - \omega_f t \right) \right] + c.c. \quad (5.23)$$

where the multiplicative factor $A$ is omitted as argued in section 5.2.2 and the slow space scale $X$ is replaced by the physical streamwise coordinate $x = X/\varepsilon$. Since a semi-infinite domain with the source at the inlet is considered, only the eigenvalues of the $\alpha^+$-branch play a role, as explained in section 5.2.5. The cross-stream direction is the radial direction $r$. Since two-dimensional perturbations are considered, the azimuthal velocity disturbance is zero: $u'_f(x, r, t) = [\rho'_f; u'_f; v'_f; 0; p'_f]^T(x, r, t)$.

Given that the cross-stream eigenfunctions at every section are normalized to have the same
disturbance kinetic energy (see Sec. 3.3), the only \(x\)-dependence in the amplitude of the response comes from the complex exponential term, which is given by:

\[
 u'_x(x, t) = \exp \left[ i \left( \int_0^x \alpha^+(\zeta; \omega_f) d\zeta - \omega_f t \right) \right].
\]  

(5.24)

The phase modulation of the global mode also depends only on \(u'_x\). The term \(\exp(-i\omega_f t)\) dictates the time evolving phase.

For both modes, the eigenvalues are followed until they become neutrally stable. Afterwards, the global mode shape is not amplified any more. The corresponding streamwise coordinate \(x\) depends on the mode and on the excitation frequency.

**Remark** Unfortunately, the stable region couldn’t be investigated since the used solver presented some difficulties in finding stable modes. The real axis in the complex \(\alpha\)-plane is indeed covered by a continuous spectrum. When an unstable mode becomes stable, the solver cannot follows it and only finds marginally stable modes.

**Outer mode**

The paths of the spatial eigenvalues corresponding to the frequencies of table 5.2 for the outer mode are shown in figure 5.5a. Figure 5.5b displays the growth of the global mode at every downstream location until the outer mode becomes marginally stable. It can be nicely seen that for the forcing frequency \(\omega_f = 0.65\) the amplitude of the global mode is the largest.

The real and imaginary parts of the normalized amplifications (normalized by their maximum) for the four chosen frequencies are shown in figures 5.5c and 5.5d. The amplification is not exactly periodic in space since the local wavelength varies at every streamwise location. However, it can be generally said that for low frequencies the wavenumbers are small, so that the wavelengths are large. Small wavelengths are instead obtained for high frequencies.
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(a) Paths of outer mode eigenvalues for increasing downstream locations: ●, $x = 0$; ○, location of neutral stability

(b) Absolute value of the amplitude of the global mode

(c) Normalized real part of the amplitude of the global mode

(d) Normalized imaginary part of the amplitude of the global mode

Figure 5.5.: Signalling problem for the outer mode for $t = 0$: $\cdots$, $\omega_f = 0.3$; $\cdots$, $\omega_f = 0.65$; $\cdots$, $\omega_f = 1.5$.

The time-harmonic steady-state response is obtained by multiplying the complex axial modulation $u'_x(x,t)$ by the complex cross-stream eigenfunctions $U^+(r;x)$ of every downstream $x$-location and by taking the real part of the result. For consistency reasons, the phase of all the eigenfunctions is shifted such that at the centerline the imaginary part the axial velocity disturbance eigenfunctions vanishes for $t = 0$.

To facilitate the spatial structure comparison, the global modes are normalized to unity for all frequencies.

Figures 5.6, 5.7, 5.8 and 5.9 show the results for $t = 0$. For low frequencies (see for example Fig. 5.6a), the axial velocity perturbations are much larger in the jet-core region $r < r_1 = 1$ than
5.3. Linear global modes in heated coaxial jet flows

anywhere else. As the frequency is increased, the perturbations become more and more compact around the second shear layer, i.e. at \( r_2 = 2 \). This is not surprising since for low frequencies the eigenvalues of the \( \alpha^+ \)-branch are close to the jet-column \( \alpha^- \)-branch (see Fig. 5.3). The same observation concerning the larger support of the axial velocity eigenfunctions of a spatial mode for decreasing frequencies was also found by Gloor et al. (2013). Also for the radial velocity and pressure disturbance eigenfunctions, the radial domain where they are large changes from the whole coaxial jet region to the location of the outer shear layer for increasing frequencies. The global mode changes from a jet-column to a shear-layer type.

\[ \text{Figure 5.6: Normalized axial velocity perturbations } u_f^*(x, r, t = 0) \text{ for the signalling problem for the outer mode.} \]
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Figure 5.7: Normalized radial velocity perturbations $v_f^r(x, r, t = 0)$ for the signalling problem for the outer mode.
5.3. Linear global modes in heated coaxial jet flows

Figure 5.8: Normalized density perturbations $\rho_f'(x, r, t = 0)$ for the signalling problem for the outer mode.
5.3. Linear global modes in heated coaxial jet flows

The same procedure is repeated for the inner mode. In figure 5.10a, the spatial eigenvalues paths for the frequencies reported in table 5.2 and increasing downstream locations are presented. The magnitudes of the global modes as a function of the x-coordinate are plotted in figure 5.10b. The maximum amplitude is reached for $\omega_f = 2.35$. In the last two figures 5.10c and 5.10d, the real and imaginary parts of the normalized streamwise amplifications are displayed. As the frequency is increased, the local wavenumber increases, remaining almost constant along all the x-sections, so that the wavelength becomes smaller. Furthermore, it can be observed that for high frequencies the normalized perturbation assumes large values already near the inlet. The fact that for $\omega_f = 0.35$ the neutrally stable state is reached only at $x \approx 28$, which is after
than for any outer mode, confirms the findings of figure 5.4 where it was shown that for \( x = 20 \) the inner mode was still unstable.

\[ \text{(a) Paths of inner mode eigenvalues for increasing downstream locations: } \bullet, x = 0; \circ, \text{ location of neutral stability} \]

\[ \text{(b) Absolute value of the amplitude of the global mode} \]

\[ \text{(c) Normalized real part of the amplitude of the global mode} \]

\[ \text{(d) Normalized imaginary part of the amplitude of the global mode} \]

**Figure 5.10.:** Signalling problem for the inner mode for \( t = 0 \): \( \cdots, \omega_f = 0.35; - \cdots, \omega_f = 2.35; \cdots, \omega_f = 4.15; \cdots, \omega_f = 6.05. \)

The two-dimensional axial velocity perturbations for the inner mode are presented in figure 5.11. Also for the inner mode and the frequency \( \omega_f = 0.35 \), the influence of the jet-column mode type can be observed. For the other three chosen frequencies, which are much higher, the inner mode is clearly of a shear-layer type with perturbations concentrated around the inner shear layer at \( r_1 = 1 \). For a high forcing frequency, e.g. \( \omega_f = 6.05 \), when the outer mode is not unstable any more, a Kelvin-Helmholtz instability is found. The radial velocity perturbations and the pressure fluctuations are presented in figures 5.12 and 5.13. Density perturbations are
5.3. Linear global modes in heated coaxial jet flows

not reported since they do not exhibit any interesting difference from the ones obtained for the outer mode (Fig. 5.8).

Figure 5.11.: Normalized axial velocity perturbations \( u_f^x(x, r, t = 0) \) for the signalling problem for the inner mode.
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Figure 5.12: Normalized radial velocity perturbations $v_f'(x, r, t = 0)$ for the signalling problem for the inner mode.
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Figure 5.13.: Normalized pressure perturbations $p_f^*(x,r,t=0)$ for the signalling problem for the inner mode.

Comparison between signalling problem results and a spatial stability analysis at the inlet

In figures 5.3.b and 5.10.b the magnitude of the response for some frequencies and downstream $x$-locations was presented. In order to generalize the results and see the dependence of the maximal amplification of the disturbances on the frequency, a signalling problem has been performed for several frequencies in the range $\omega_f \in [0, 8]$. Figure 5.14.a shows the obtained maximal magnitude of the response for both modes as a function of the forcing frequency. A flow perturbed with a frequency below $1.35$ will show a global mode more similar to the most amplified outer mode, while for higher frequencies the structure of the forced mode will be closer (or equal if $\omega_f > 2$)
5.3. Linear global modes in heated coaxial jet flows

to the inner one.

\[
\max \left\{ |u_x'| \right\}
\]

(a) Maximal amplification for the signalling problem as a function of the forcing frequency

(b) Spatial stability analysis at the location \( x = 0 \)

**Figure 5.14.** Comparison between maximum amplification of the global mode obtained with the signalling problem solution and a spatial stability analysis at the inlet: —, outer mode; ---, inner mode.
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It is very interesting to note the close similarity to the results of a spatial stability analysis at the inlet location $x = 0$, which is plotted in figure 5.14. For both studies, the largest amplification is obtained for the outer mode. The frequencies giving rise to the most amplified waves are reported in table 5.3.

<table>
<thead>
<tr>
<th>mode</th>
<th>signalling problem</th>
<th>spatial stability at $x = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>outer</td>
<td>0.35</td>
<td>0.75</td>
</tr>
<tr>
<td>inner</td>
<td>2.35</td>
<td>2.9</td>
</tr>
</tbody>
</table>

Table 5.3.: Frequencies corresponding to the maximal spatial amplifications for the signalling problem solution and for a spatial stability analysis at the inlet.

It can be concluded that the frequency corresponding to the maximal spatial amplification of a global mode is slightly overestimated by the spatial stability analysis at the inlet. This is however not surprising since the base flow at $x = 0$ exhibits the thinnest shear layers. Zaman and Hussain (1980) have also found that the natural roll-up frequency in a circular jet vortex pairing is lower than the most unstable mode frequency obtained with acoustic forcing. As the flow evolves downstream, the shear-layer thickness increases as it is shown in figure 5.2. As Gloor et al. (2013) demonstrated in their paper, when the shear-layer thickness increases, the frequency giving rise to the most amplified spatial mode decreases. The maximal amplification of the global mode obtained with the signalling problem solution does not only take the initial section into account, but also all the downstream positions, for which the local most amplified frequency is lowered.

To conclude, a spatial stability analysis at the inlet location is of great interest to approximately estimate the forcing frequency giving rise to the most amplified spatial wave over the complete domain. Nevertheless, to obtain the shape of the two-dimensional perturbations, the eigenfunctions at every downstream coordinate have to be computed.

**Calculation of the small parameter $\varepsilon$**

The reader has to be aware that the results presented in this section rely on the fundamental assumption $\varepsilon \ll 1$ defined in Sec. 5.2.1. This condition has to be verified a posteriori and is the goal of this section. The momentum thickness is computed via a slightly modified version of the formula used by Michalke (1984) and Garnaud et al. (2011) for a single jet. Since a coaxial jet is considered, the
total momentum thickness is assumed to be the sum of both shear layers considered separately. Additionally, due to the zero Mach number, the incompressible expression is used:

\[ \Theta(x) = \theta_1(x) + \theta_2(x) \]

\[ = \int_0^{\hat{r}} \frac{u_b(x,r) - u_b(x,\hat{r})}{u_b(x,0) - u_b(x,\hat{r})} \left( 1 - \frac{u_b(x,r) - u_b(x,\hat{r})}{u_b(x,0) - u_b(x,\hat{r})} \right) dr + \int_{\hat{r}}^{\infty} \frac{u_b(x,r) - u_b(x,\infty)}{u_b(x,\hat{r}) - u_b(x,\infty)} \left( 1 - \frac{u_b(x,r) - u_b(x,\infty)}{u_b(x,\hat{r}) - u_b(x,\infty)} \right) dr, \]  

(5.25)

where \( \hat{r} \) is the radial coordinate between the two shear layers.

A plot of the momentum thickness for the coaxial jet is presented in the next section on figure 5.22.

The largest wavelength is given for every mode and frequency by:

\[ \lambda_{\text{max}} = \frac{2\pi}{\alpha_{r,\text{min}}}. \]  

(5.26)

The evolution length scale \( L \) is computed using an arithmetic mean of the local evaluation of expression (5.1) over the \( x = x(\omega_f) \)-range for each frequency:

\[ L = \left[ \frac{1}{\Theta(x)} \frac{d\Theta}{dx}(x) \right]^{-1}. \]  

(5.27)

The small parameter of interest is then calculated by \( \varepsilon \sim \lambda_{\text{max}} / L \).

The results for both modes and the chosen frequencies are reported in table 5.4. The evolution length scale differs from case to case since the \( x \)-domain, where the arithmetic average is performed, varies. For all the selected configurations, the fundamental assumption \( \varepsilon \ll 1 \) is satisfied.

<table>
<thead>
<tr>
<th>mode</th>
<th>( \omega_f )</th>
<th>( \lambda_{\text{max}} )</th>
<th>( L )</th>
<th>( \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>16.0</td>
<td>70.2</td>
<td>0.2</td>
</tr>
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<td>76.8</td>
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<td>80.7</td>
<td>0.03</td>
</tr>
<tr>
<td>outer</td>
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<td>2.0</td>
<td>102.4</td>
<td>0.02</td>
</tr>
<tr>
<td>inner</td>
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<td>14.6</td>
<td>54.9</td>
<td>0.3</td>
</tr>
<tr>
<td>inner</td>
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<td>2.0</td>
<td>71.2</td>
<td>0.03</td>
</tr>
<tr>
<td>inner</td>
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<td>84.8</td>
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</tr>
<tr>
<td>inner</td>
<td>6.05</td>
<td>0.8</td>
<td>142.7</td>
<td>0.006</td>
</tr>
</tbody>
</table>

*Table 5.4.* Calculation of the small parameter \( \varepsilon \).
5.3. Linear global modes in heated coaxial jet flows

### 5.3.3. Self-sustained global modes

In this section, the problem concerning intrinsic global oscillations is addressed. As explained in section 5.2.5, a global mode appears because of the presence of a finite region of absolute instability in the flow. The self-sustained oscillation overwhelms any other oscillation due to external forcing or noise. For a semi-infinite flow domain, the global mode frequency corresponds to the absolute frequency of the most unstable mode obtained by a linear spatio-temporal stability analysis at the inlet. For the parameters reported in Table 5.1 with zero coflow velocity \( u_{\infty} \), the obtained frequency is:

\[
\omega_G \equiv \omega_0(x = 0) = 0.380 + 0.038i. \tag{5.28}
\]

Some interesting conclusions can already be drawn just by inspection of this value. First, the base flow at the inlet effectively exhibits an absolute instability in the sense defined in section 2.4.2. Secondly, the absolute temporal growth rate is rather small. Recalling the results obtained in the parametric study, this is not surprising since the inlet parameters of the heated coaxial jet (see Tab. 5.1) give rise to an absolute instability with a larger temporal growth rate for the outer mode, always having rather small \( \omega_{0,i} \) compared to the inner mode for lower bypass-velocity ratios, see for example Fig. 4.8b. That figure is for \( \theta_1 = \theta_2 = 0.03 \), but it has to be remembered that for a thicker shear layer at \( r_1 \), the inner mode is stabilized (see Fig. 4.16), while a shear-layer thickness of 0.05 at \( r_2 \) destabilizes the outer mode even more (see Fig. 4.17). As discussed in section 2.4.3, only the pinching point with the largest absolute temporal growth rate has to be taken into account. At the inlet it is found that this is the case for the outer mode. Furthermore, since the adopted linear global theory assumes only one mode, the inner mode is not considered for the self-sustained global mode. In a heated coaxial jet flow the inner mode wouldn’t give rise to a global instability since it does not present a region of absolute instability. Lastly, the real global mode frequency is very close to the smallest one chosen for the signalling problem. Since the global temporal growth rate is small, a similarity between the shape of the intrinsic global mode and the global mode obtained with a forcing frequency \( \omega_f = 0.3 \) for the outer mode is expected, also for \( t > 0 \).

Figure 5.15a shows the \( \alpha^+ \)- and \( \alpha^- \)-branches for the base flow at the inlet, while the eigenvalues corresponding to \( \omega_G \) are better visible on figure 5.15b. These eigenvalues have to be followed when the base flow evolves downstream and is forced with the global frequency \( \omega_G \). For the computation of the global mode shape, only the one belonging to the \( \alpha^+ \)-branch is useful, but for sake of clarity and to ensure that the computation is correct, both are tracked.
5.3. Linear global modes in heated coaxial jet flows

The paths of the spatial eigenvalues corresponding to $\omega_G$ are presented in figure 5.16a. From figures 5.16b and 5.16c it can be noticed that the location of the eigenvalues changes drastically for $x \gtrsim 10$. As it can be observed from figure 5.17, the reason is the invariance of the base flow profiles in the first sections. The local absolute frequency would be close to $\omega_G$ and therefore the spatial $\alpha^+\text{-branches}$ do not differ much from those at $x = 0$.

The eigenvalue of the $\alpha^+\text{-branch}$ for $\omega_G$ can be tracked almost until $x \approx 20$. At this coordinate, the mean axial velocity features a thin inner shear layer, but a very thick outer one (see Fig. 5.17a). The downstream spatial growth rate of the outer mode becomes negative, i.e. it is stable (see Fig. 5.16c). This mode can not be followed further downstream since the eigenvalue is located in the region of the continuous spectrum, which for $\omega_G,i > 0$ lies in the upper half of the complex $\alpha$-plane (see the remark in Sec. 5.3.2). A continuous spectrum is neglected in the spatio-temporal theory (Huerre & Monkewitz 1990). However, to be really rigorous, the applicability of the linear spatio-temporal stability theory explained in section 2.4 is questionable for a stable mode since an unstable wavepacket is no longer present and the result (2.48), obtained from an asymptotic analysis for $t \to \infty$, no longer holds. Therefore, the computation of the global mode should be interrupted at the marginal stability location $x \approx 18$. 

Figure 5.15.: Spatial branches for frequencies close or equal to the global mode frequency and the base flow of location $x = 0$: \(\triangle\), $\alpha^+$-branch; \(\triangledown\), $\alpha^-$-branch; \(\circ\), exact pinching location.

(a) Eigenvalues corresponding to discretised frequencies $\omega \in [\omega_G - 5 \cdot 10^{-5}, \omega_G + 5 \cdot 10^{-5}]$

(b) Eigenvalues corresponding to $\omega_G$ (magnified)
5.3. Linear global modes in heated coaxial jet flows

As a side remark, it can be pointed out that reaching a downstream location of \( x \approx 18 \) might seem unsatisfactory but it is not. A global mode analysis like the one presented here has been performed for the heated single jet flow profile proposed by [Monkewitz & Sohn (1988)] and also used by [Garmaud et al. (2011)]. For such a flow without bypass-stream \((h = 0)\), only the inner mode is present and as seen in the parametric study, its absolute temporal growth rate is much larger (see for example Fig. 4.3). Furthermore, the used heated single jet base flow evolves much faster than the coaxial one \((\varepsilon \ll 1 \) no longer holds). The last two considerations yield the fact that when exciting every downstream section of the single jet flow with the absolute frequency of
5.3. Linear global modes in heated coaxial jet flows

the inlet, the $\alpha^+ (\omega_G)$-mode turns to be stable for a much smaller $x$-location. For this reason, the
global mode results for a heated single jet are not reported in this text. Another performed study
not reported here concerns an analytic jet profile, for which the user can specify the downstream
evolution of the parameters. Having a base flow profile evolving slowly and a spatio-temporal
mode not too sensitive to the variations in the parameters are therefore fundamental ingredients.
This is the case for the outer mode of a heated coaxial jet flow.

Before proceeding further, the used expression for the perturbation response (5.22) is restated
here.

$$u'(x,r,t) \sim U^+(r;x) \exp \left[ i \left( \int_0^x \alpha^+(\zeta;\omega_G) d\zeta - \omega_G t \right) \right] + c.c. \quad (5.29)$$

The same observations done for equation (5.23) hold here as well. The $x$-dependence in the
amplitude of the response reads now:

$$u'_x(x,t) = \exp \left[ i \left( \int_0^x \alpha^+(\zeta;\omega_G) d\zeta - \omega_G t \right) \right]. \quad (5.30)$$

The real part of the term $\exp(-i\omega_G t)$ is responsible for the temporal growth of the global mode.

The normalized streamwise dependence of the magnitude of the global mode is drawn in figure
5.18. It can be observed that the wavelength decreases downstream. In contrary to the signalling
problem where the local wavenumber was almost constant, here it is almost doubled from the inlet.
5.3. Linear global modes in heated coaxial jet flows

\((x = 0)\) to the most downstream location \((x \approx 20)\). It is enlightening to note that the location of the largest global mode amplitude is far away from the region where the local absolute temporal growth rate is maximum. As Huerre & Monkewitz (1990) point out, this is not an exceptional situation and dispels the notion that the region where the local absolute temporal growth rate is positive coincides with the maximum amplitude location of the global mode. Instead, it can be understood as a self-excited \textit{wavemaker} in the region of local absolute instability that acts as a source for the downstream instability waves, which develop in the streamwise direction similarly to spatially developing waves, reaching their maximum amplitude where they turn locally neutrally stable. The amplitude of the non-normalized \(u'_x\) reaches a value of the order of \(8 \cdot 10^3\) which is similar as for the outer mode in the signalling problem with \(\omega_f = 0.3\).

![Normalized downstream amplitude of the global mode for \(t = 0\): ---, absolute value; - - -, real part; , , imaginary part.](image)

By multiplying the amplitude of the global mode by the cross-stream eigenfunctions obtained at every \(x\)-section, the two-dimensional perturbations can be visualized (see Fig. 5.19). As already hypothesized at the beginning of this section, the intrinsic global mode shape is very similar to the spatial wave obtained in the signalling problem for the outer mode with \(\omega_f = 0.3\) (compare for example Fig. 5.6a and Fig. 5.19a). Concerning the axial velocity perturbation, the influence of the jet-column type on the mode is also relatively strong. The self-sustained mode exhibits therefore strong oscillations in the axial velocity at downstream locations close to the local neutral stability margin of the outer mode, mainly in the jet-core region \(r < r_1 = 1\). The radial velocity disturbances, showed in figure 5.19b, are on a wider \(r\)-domain, with peaks on both shear layers. Much more compact around the inner shear layer are instead the density fluctuations (Fig. 5.19c). Pressure perturbations are intense in both jet-core and shear-layers regions, but do not vanish at the centerline (Fig. 5.19d).
5.3. Linear global modes in heated coaxial jet flows

Figure 5.19: Normalized perturbations at $t = 0$ for the intrinsic global mode.

In figure 5.20, both local absolute temporal growth rate and the largest local temporal growth rate of a pure temporal stability analysis are plotted. The region of absolute instability is relatively large and goes until $x \approx 16.2$. From this location until $x \approx 18$ the outer mode is convectively unstable. Further downstream, it is stable and the positive maximal temporal growth rate $\omega_{i,max}$ between $x \approx 18$ and $x \approx 30$ is a result of the unstable inner mode, for which the corresponding shear layer is still thin. However, also the inner mode, which is linked with the inner shear layer, is drastically stabilized when the shear-layer thickness at $r_1 = 1$ increases in size (see Fig. 5.17 and Fig. 5.22). This picture is the corresponding of 5.1d calculated for a real heated coaxial jet flow.
5.3. Linear global modes in heated coaxial jet flows

**Figure 5.20.** Absolute temporal growth rate and largest temporal growth rate of a temporal stability analysis as a function of the downstream location: \(- - - \), $\omega_0,i$; \(\longrightarrow\), $\omega_{i,max}$.

From the spatial spectrum plotted in figures 5.21a to 5.21c it can be seen again how the most unstable mode changes from the outer to the inner (stable modes are not shown). This is similar to what was discussed for figure 5.4. Please also note from figure 5.21a that for real frequencies, the outer mode $\alpha^+$- and $\alpha^-$-branches have already pinched since the absolute temporal growth rate is positive.
5.3. Linear global modes in heated coaxial jet flows

Figure 5.21: Spatial spectrum in the complex $\alpha$-plane for different downstream locations.
5.3. Linear global modes in heated coaxial jet flows

**Calculation of the small parameter \( \varepsilon \)**

The small parameter \( \varepsilon \) is again calculated a posteriori. Since the wavelength is varying locally, the largest one is taken to be conservative:

\[
\lambda_{\text{max}} = \frac{2\pi}{\alpha r_{\text{min}}} \approx 13.4. \tag{5.31}
\]

The momentum thickness is computed via formula (5.25) and is plotted on figure 5.22. The shear-layers thicknesses at the inlet presented in table 5.1 are well predicted by expression (5.25).

![Figure 5.22: Momentum thickness for the flow of figure 5.2a](image)

For the evolution length scale \( L \), the arithmetic mean of the local evaluation of expression (5.1) over all \( x \) is used:

\[
L = \frac{1}{\Theta(x)} \frac{d\Theta}{dx}(x) \quad \approx 72.2. \tag{5.32}
\]

The small parameter of interest is then

\[
\varepsilon \sim \frac{\lambda}{L} \approx 0.19 \ll 1. \tag{5.33}
\]
The applicability of the theory is proved a posteriori. However, it has to be kept in mind that only the region until \( x \approx 20 \) is considered. Further downstream, the base flow would evolve much more, violating the weakly non-parallel assumption.

### 5.3.4. Assessment of the linear global mode analysis

The linear global mode analysis presented in section 5.2 has been performed since it is a very nice continuation and application of the local linear spatio-temporal theory used for the parametric study. It allows to extend the local parallel theory to a weakly non-parallel medium and to obtain a linear global mode. This step is fundamental if one wants to compare the results with numerical simulations or real experiments of spatially developing systems.

Juniper et al. (2011) showed for a confined planar wake (doubly-infinite domain) how well the results of the linear global theory based on the linear local spatio-temporal stability analysis fit with the ones of a classical linear global stability analysis. The latter solves a large eigenvalue problem to directly find a linear global mode of the form \( U_G(x,r) \exp(-i\omega_G t) \) and is therefore much more computationally expensive, but is also applicable to fully non-parallel flows. The local theory only slightly overestimates the growth rate, probably since its calculation considers only the most unstable section \( x = 0 \), pushing the maximum of the global mode slightly upstream, but as an advantage it gives more physical insight about the flow, e.g. the location of the wavemaker (Juniper et al., 2011). The good agreement in the shape of the global mode from the two approaches shows that it is adequate to assume that the slowly evolving amplitude function \( A(X) \) is uniform.

Other studies also confirmed the good prediction of the linear global analysis based on the local stability concepts for global modes of finite amplitude.

As Huerre & Monkewitz (1990) pointed out, open flows acting as noise amplifiers or as oscillators exhibit strong destabilizing non-linearities. Unfortunately, the extension of linear global modes to the non-linear regime is very questionable. Recent studies used the fully global theory, nowadays affordable because of the increased computational capabilities, to study open flows. It is found that the linear global modes are strongly non-orthogonal, experiencing transient growth. In fact, the linear evolution operator corresponding to the linearised disturbance equations is non-normal because of the base flow advection. Quoting Chomaz (2005, p. 357), "when the flow is weakly non-parallel, this limitation is so severe that the linear global mode theory is of little help". It would be then more appropriate to use a fully non-linear formulation. For a discussion about the non-normality of the linear operator, the interested reader is referred to Cossu & Chomaz (1997); Chomaz (2005); Schmid & Henningson (2001). Garnaud et al. (2011, 2013) performed a linear global study of a heated single jet considering transient growths as well.
5.4. Non-linear absolute/convective instability and front selection problem

The linear global mode theory is able to describe the physical mechanism responsible for the resonance in weakly non-parallel open flows, but it fails immediately beyond the threshold since transient growth yields large amplitudes.

5.4. Non-linear absolute/convective instability and front selection problem

In the light of the limitations of the linear stability theory for non-linear systems, explained in section 5.3.4, the generalisation of the linear spatio-temporal stability theory to the non-linear regime is now outlined. For now, the flow is assumed parallel.

As suggested by Chomaz (1992, p. 1931), the definitions of non-linear stability or absolute and convective instabilities follow naturally the ones for the linear case (see Sec. 2.4.2). The basic flow state is:

- **stable**: “if for all initial perturbations of finite extent and finite amplitude, the flow relaxes to the basic state everywhere in any moving frame”. If this is not the case, the flow is unstable.

- **non-linearly convectively unstable**: “if for all initial perturbations of finite extent and finite amplitude, the flow relaxes to the basic state everywhere in the laboratory frame”.

- **non-linearly absolutely unstable**: “if there exist an initial condition of finite extent and finite amplitude, and a location where the system does not relax to the basic state”.

Similarly to what was shown in figure 2.2 for the linear case, the non-linear response to a perturbation (droplet) of finite extent and amplitude is shown in figure 5.23. It is reminded that the linear wavepacket is delimited by the leading and trailing edges having velocities $V^+$ and $V^-$, respectively, such that $\sigma(V^\pm) = 0$. Linear absolute instability appears when the edges present opposite front velocity signs. The same holds for the non-linear regime.

The problem becomes therefore the computation of the front velocity at the trailing edge. The front velocity selection criteria is a research topic itself and goes beyond the scope of this work, only the main conclusions are reported. A small perturbation in a unstable medium grows first according to the linear instability theory and its ends propagate at the linearly selected velocities. When the non-linear terms compensate the exponential term, the wave packet saturates and the fronts become non-linear (Delbende & Chomaz 1998). However, it is found that the front velocity separating the basic state from the bifurcated state is often given by a linear selection criterion (Huerre 2000). Such a front moving at the linearly determined velocity is called *pulled* front since the dynamics is governed by the instability properties upstream of the front, where linear theory holds. This is not valid only if there exists a non-linear faster front with larger non-linear spatial decay rate (Huerre 2000). Such a front is called *pushed* front since the saturated region drives the dynamics and forces the linear region upstream of the front (Chomaz 2003). For further informations see Van Saarloos (1988, 1989) or the review of Van Saarloos (2003). The great conclusion is that when the front velocity is linearly selected, the absolute or convective nature of the non-linear instability is determined by the same condition as for the linear
5.4. Non-linear absolute/convective instability and front selection problem

![Figures](image1)

(a) Stable  
(b) Non-linearly convectively unstable  
(c) Non-linearly absolutely unstable  
(d) Linearly convectively unstable but non-linearly absolutely unstable

**Figure 5.23.** Sketch of impulse response in the \((x,t)\)-diagram for the non-linear velocity selection criterion.

The only effect of non-linear terms is on the amplitude, which is found to be limited at some saturations in the wavepacket core. These results were verified with direct numerical simulations (DNS) by Delbende & Chomaz (1998). Furthermore, it can be proven (Huerre, 2000) that linear absolute instability is a sufficient condition for non-linear absolute instability because the non-linear wave packet contains the linear one (see Fig. 5.23c). The contrary is however not true (see Fig. 5.23d).

To conclude, the absolute instability threshold predicted by the linear absolute instability theory is not affected by non-linear terms (Chomaz, 2003). The performed parametric study with linear theory also holds for the investigation of non-linear absolute instability regions. Concerning the frequency, the non-linearity causes the saturation of the mode so that the frequency corresponds to the linear absolute frequency at the threshold and to its real part when saturation is attained.
5.5. Non-linear global modes

The goal of this section is to briefly outline the theory on non-linear global modes, which have to be preferred to the linear ones as soon as their amplitude is not small any more (see Sec. 5.3.4). As for the linear case, the non-linear absolute and convective instability concepts might be extended to the study of non-linear global modes. Unfortunately, the theory becomes more complex so that studies consider either model equations or they perform direct numerical simulations. No easy analytical theory for the Navier-Stokes equations exists. To compare theoretical with numerical results, quantities like the healing length $\Delta x$ for the semi-infinite flow domain, i.e. the characteristic growth length defined as the distance at which the solution reaches 99% of its maximum amplitude, and like the amplitude itself, are employed.

In the following, non-linear global modes will also be classified in several categories.

5.5.1. Semi-infinite flow domain

Parallel medium

First attempts to generalize the linear global theory to the non-linear regime were done for the subcritical real Ginzburg-Landau equation with constant coefficients by Couairon & Chomaz (1996, 1997a). They discovered that the onset of non-linear absolute instability coincides with the appearance of a non-linear global mode for a pulled front. Furthermore, the selected frequency at the threshold is exactly the linear absolute frequency at $x = 0$. It is found that the global mode saturates closer to the boundary and could be described by a front blocked by the inlet condition. Downstream, the saturated amplitude remains constant.

The scaling law for the healing length $\Delta x$, found to vary as the inverse of the square root of the criticality, has been validated by direct numerical simulations (Couairon & Chomaz, 1997b) for the Rayleigh-Bénard problem. Chomaz (2003) showed with a numerical study on parallel wakes, to which he added a body force to compensate for the basic flow diffusion and keep the medium parallel, that the results obtained with the model equation do apply to such flows as well.

Weakly non-parallel medium

Consider now the non-linear counterpart of section 5.2. The paper of Couairon & Chomaz (1999a) offers an exhaustive description of non-linear global modes obtained by using the real Ginzburg-Landau amplitude equation with varying coefficients and a matched asymptotic expansion with a boundary condition at $x = 0$. It has to be pointed out that a fully non-linear problem is considered. The weakly non-linear problem is found to be ill posed in the case of weakly non-parallel flows (Le Dizès et al., 1993).
5.5. Non-linear global modes

For the model equation, the same scaling law for the healing length as in the case of a parallel medium holds:

\[ \Delta x \simeq \frac{1}{\sqrt{\epsilon}}, \tag{5.34} \]

where the criticality \( \epsilon \) is the departure from the global instability threshold. A scaling law is found also for the maximum amplitude of the global mode as a function of the criticality and the coefficients of the Ginzburg-Landau equation. These laws are validated with experimental data (Couairon & Chomaz, 1999a) and numerical studies, e.g. the ones of Lesshaft et al. (2006, 2007) among others.

The necessary condition for the existence of a non-linear self-sustained mode is that the saturation occurs within the locally absolutely unstable domain. Also for a non-linear self-sustained mode in a weakly non-parallel medium, the selected frequency at the threshold corresponds to the absolute frequency at the inlet. The dynamics of the non-linear mode of a parallel medium holds in the weakly non-parallel case as well if the healing length \( \Delta x \) is smaller than the base flow evolution length scale \( L \) (eq. (5.1)). In contrary, the saturated amplitude of the modes slowly decays far downstream.

It is interesting to point out that saturated non-linear global modes can sustain a secondary instability. Absolute and convective secondary instabilities can be defined for this saturated state as well. The secondary instability can be absolute either before or after the primary. In the latter case, a stable non-linear global mode appears first and it is destabilized by a secondary instability only when the control parameter is increased above the threshold of a secondary absolute instability. For the former case instead, it is found that a secondary absolute instability preceding the primary one could constitute a new generic scenario for a one-step bifurcation to turbulence (Couairon & Chomaz, 1999b; Chomaz, 2004). This so called AA route could maybe explain how hot jets experience transition to turbulence.

Remark  Pushed global modes exist as well. In that case, the medium does not need to be linearly absolutely unstable, it is sufficient to be non-linearly absolutely unstable (i.e. it might be linearly convectively unstable). The saturated wave downstream of the front sets the frequency of the mode. Since such modes are less frequent and seem not to appear for the class of flows under consideration, the reader is referred to Couairon & Chomaz (2001) for further information. To see what kind of mode exists, numerical simulation are performed and the linear and non-linear impulse responses are compared.

5.5.2. Doubly-infinite flow domain

For sake of completeness, non-linear global modes occurring in doubly-infinite domains are also briefly described. Such modes do not play a role in stability studies of flows such as jets, which
5.5. Non-linear global modes

remain the core of the investigations in this work.
As it was shortly presented in section 5.2.5, the frequency selection criterion for a domain extending to infinity in both directions differs to the one for a semi-infinite flow. The non-linear global frequency is however real since the solution is saturated and therefore the above mentioned criterion can be simplified.

Fully non-linear global modes in an unbounded domain can be classified in two different categories: steep global modes and soft global modes.
For steep global modes, the global frequency corresponds to the real absolute frequency at the location where the linear instability changes from convective to absolute (also called marginal stability criterion), making the sophisticated linear frequency selection theory in the complex $X$-plane meaningless. The name of this mode is due to the steep front which is found at that particular coordinate and acts as a wavemaker. It can be remarked that the frequency of such modes also relies only on the local linear dispersion relation. No absolutely unstable region of finite extent is needed for a steep mode to appear.
For soft global modes instead, the frequency selection criterion is formally identical to the saddle-point linear selection criterion except that the frequency is obtained from the non-linear theory and $X_S$ is real (Huerre 2000). As the name indicates, this mode does not feature a sharp front. The mode with the larger frequency will dominate (Pier et al. 2001a). Steep global modes appear as soon as a point of local linear absolute stability exists, while soft global modes are obtained only further above the onset of steep global modes, if the base flow advection is not too strong (Huerre 2000). This condition implies that steep global modes may be present in a linearly globally stable medium as well.
For additional information about steep and soft modes in doubly-infinite domains, the exhaustive review of Pier et al. (2001a) is suggested. Pier et al. (2001a) validated the theory of steep modes by performing a numerical study of a spatially developing synthetic wake (i.e. without recirculation zone).
Chapter 6. Summary, conclusions and outlooks

6. Summary, conclusions and outlooks

6.1. Summary and conclusions

First, a parametric study of the local spatio-temporal linear stability characteristic of a heated coaxial jet flow has been performed. Bypass-velocity and temperature ratios, shear-layer thicknesses and Mach numbers were varied for two instability modes: the inner and the outer. Axisymmetric and azimuthal disturbances were considered. Configurations yielding an absolute instability could be found.

Secondly, a linear global stability analysis based on a WKBJ approximation scheme has been carried out to extend the previous obtained local results to the real heated coaxial jet base flows obtained from the Large Eddy Simulations of Michael Gloor. Forced and self-sustained oscillations have been investigated.

Spatio-temporal linear stability theory was reviewed at the beginning of this work. In this context, the stability character of a mode is totally described by the absolute wavenumber and the absolute angular frequency. The former is located in the complex \( \alpha \)-plane where coalescence between an upstream and a downstream spatial branch occurs. The sign of the imaginary part of the latter reveals the instability nature of the mode.

The procedure from the compressible inviscid governing equations leading to the disturbance equations via the normal mode ansatz was described. With the corresponding boundary conditions in cylindrical coordinates, these form a generalized eigenvalue problem. For the discretization, a Chebyshev collocation method was used. Following Bers (1984) and the algorithm proposed by Monkevitz & Sohn (1988), the pinching point for a given parameter configuration could be found. A similar algorithm allowed to track the absolute wavenumber when the parameters were changed.

The inner mode was found to be more related to the shear layer between the heated core-flow and the bypass-stream, whereas the outer one was related to the shear layer between the bypass-stream and the ambient flow. Even if the support of the disturbance eigenfunctions is not restricted to one or the other shear layer, a direct influence of one shear layer to its corresponding mode could be verified. The bypass-velocity ratio is defined as the ratio between the velocity of the bypass-stream and the velocity at the centerline, i.e. \( h = u_{\text{bypass}} / u_c \). For the extreme single jet cases with bypass-velocity ratios unity or zero, only the mode of the remaining shear layer was identifiable. For \( h = 0 \) the inner mode was absolutely unstable, whereas for \( h = 1 \) the outer mode was convectively unstable, probably because in the latter case the temperature and velocity gradients are not at the same location.

The linear spatio-temporal instability characteristic for different parameters was presented in the bypass-velocity-temperature ratios plane. The temperature ratio is the ratio between the
temperature in the far-field and the temperature at the centerline, i.e. \( S = T_\infty / T_c \). The inner mode exhibited a region of absolute instability for low bypass-velocity ratios going up to a temperature ratio of \( S = 0.710 \) and extending to higher bypass-velocity ratios for lower temperature ratios. This value agrees very nicely to the one obtained by Monkewitz & Sohn (1988); Lesshaft \\& Huerre (2007) for single heated jets, namely \( S = 0.713 \). The shape of the absolute instability region for the outer mode featured instead a peak when the jet-core velocity is twice the bypass-stream velocity, i.e. when at the inner shear layer both temperature and velocity vary strongly.

The maximum attainable temperature ratio giving an absolute marginal state is \( S = 0.73 \). At this value the inner mode is convectively unstable. The minimum temperature ratio is instead \( S = 0.23 \). Larger temperature differences between the streams result in an absolute instability.

The absence of an absolute instability for isothermal flows obtained by the previous authors, for a specified shear-layer thickness, could be extended to all bypass-velocity ratios as well. It is enlightening to point out that the increase of the bypass-velocity ratio has a different effect on the modes. In fact, the velocity differences across the shear layers vary, but temperature gradients are uniquely present between the jet-core and the bypass-flow and do not vary with this parameter. For increasing bypass-velocity ratios, the inner mode was found to be stabilized, similarly as for increasing co-flow velocities in single jets. This inner mode for \( h = 0 \) is the one studied by Jendoubi & Strykowski (1994); Lesshaft \\& Huerre (2007). However, coaxial jets are destabilized for non-zero bypass-velocity ratios because of the presence of an absolutely unstable outer mode, whose largest absolute temporal growth rate is not for unity bypass-velocity ratio, i.e. when the base flow is a thicker single jet. Furthermore, the absolute instability of the outer mode was found to be sustained for smaller temperature differences. Nevertheless, its absolute growth rate is much smaller than the one of the most unstable inner mode.

Increasing the thickness of both shear layers could stabilize or destabilize the flow depending on the thickness, the mode and the bypass-velocity ratio. For the inner mode, weaker gradients in the shear layers stabilized the flow for low bypass-velocity ratios and destabilized it for high bypass-velocity values. Regions of absolute instability for the outer mode were much larger for shear-layers thicknesses around \( \theta = 0.07 \). For such values, an absolute instability could also arise from the outer mode in an isothermal coaxial jet.

Higher Mach numbers exhibited a monotonic stabilisation of the modes. For the inner one, an analytical relation which predicts the temperature ratio needed for the absolute marginal instability state depending on the bypass-velocity ratio, Mach number and critical temperature ratio for the incompressible case, could be formulated.

Regarding the first azimuthal modes, they were found to be much more stable. No region of absolute instability for the first azimuthal outer mode could be found. Absolute instability only arises from the inner mode in a very narrow parameter range.

From the spatio-temporal stability analysis of travelling waves, the results of Lesshaft \\& Huerre (2007) for a single jet could be nicely reproduced. The most amplified mode was found to be not the one with zero group velocity, but a travelling mode.

In the second part of this thesis, the theoretical bases of the used linear global stability analysis for weakly non-parallel flows were first elucidated. Global perturbations composed by a slowly varying envelope and a fast varying complex phase that obey the local dispersion relation are
assumed, making this study an intriguing extension of the previously employed local theory. Global modes can be either stable or unstable depending on the sign of the imaginary part of their frequency. Linearly unstable global modes require a finite region of absolute instability to occur. Such modes act as oscillators. Their intrinsic frequency corresponds, in the case of a semi-infinite domain, to the absolute frequency at the upstream boundary. Convectively unstable flows act as amplifiers and can sustain linear global modes when a continuous forcing is applied. This is known as the signalling problem. Both intrinsic and forced global modes were investigated for a weakly non-parallel base flow obtained by the mean flow profiles of Large Eddy Simulations. Depending on the external ambient axial velocity, convectively or absolutely unstable configurations could be generated.

For the signalling problem, the spatial wave amplification corresponding to an outer mode was found to be larger than the one of an inner mode. This is in good agreement with a pure spatial stability analysis performed at the inlet. Also the forcing frequency giving rise to the maximal growth could be compared. A spatial analysis at the upstream boundary overestimates the most amplified frequency since it considers only the base flow profile at \( x = 0 \), exhibiting the thinnest shear layers. When the flow evolves downstream, the shear-layer thickness increases, lowering the most amplified frequency. Furthermore, it was observed that the lower the frequency, the more the global mode features large perturbations in the jet-core region. This is due to the vicinity of the corresponding spatial branch to the jet-column-mode branch. For high frequencies instead, the disturbances are compact around the corresponding shear layer, resembling very nicely to Kelvin-Helmholtz instabilities.

The intrinsic linear global mode oscillation frequency \( \omega_G = 0.38 + 0.038i \), namely the absolute frequency at the inlet, agrees very well with the one obtained for the outer mode and the corresponding settings in the parametric study. It can be noted that the global growth rate is weak. The needed absolute instability region of finite extent located upstream acts as a wavemaker, triggering the global mode which reaches its maximum amplitude where the underlying mode turns from locally convectively unstable to stable on a local basis. This is a general characteristic of linear global modes for semi-infinite domains. Since the oscillation frequency is low, the self-sustained mode features strong perturbations in the jet-core region as well, resembling more a jet-column mode.

Linear global mode results should be considered only for the very first stages of the instability growth, when the amplitude is small. Indeed, the operator governing the physics is non-normal, leading to large transient growths. Non-linear global modes should be preferred in this case. However, it is found that even non-linear global modes are in part dictated by the linear theory. In fact, the front velocity is often linearly selected, making the threshold for a linear and a non-linear absolute instability identical. Furthermore, the beating frequency of non-linear global modes is still the linear absolute frequency at the upstream boundary.

To conclude, even if the linear theory fails in predicting the shape of the global mode, it is able to estimate the global frequency and the threshold for an unstable non-linear pulled global mode in a heated coaxial jet. The performed parametric study is therefore of great help if one wants to investigate non-linear global modes in this kind of jet flows.
6.2. Outlooks

The performed parametric study was vast but other parameter ranges could be further explored. Possible extensions concern bypass-velocity ratios going below zero (bypass-stream in other direction) and above unity (bypass-stream faster than core-flow). Also the temperature ratio could have values larger than one. Although such conditions do not correspond to exhaust flows of turbofan jets engines, they are likely to occur in mixing devices with dual-stream nozzles. Filtering the modes appearing near the pinching point for higher Mach numbers would permit to reach the supersonic regime.

Furthermore, it could be interesting to consider other distances between the shear layers in order to investigate the limit when the modes become only sustained by one shear layer. Considering different thicknesses for the inner and outer shear layers could help analysing the link between the modes and their origin.

The used velocity profile assumed infinitely thin walls between the streams. It could be instructive to consider a velocity deficit profile with a wake region behind the separation walls. This would however give rise to an additional mode, making the study more cumbersome. Since the absolute instability regions were at the center of the investigation, modes with zero group velocity were mainly considered. Nevertheless, taking into account modes with group velocity greater than zero would help distinguishing the type of modes (see Lesshafft & Huerre, 2007). Additionally, travelling modes may experience a larger temporal growth, obscuring the standing modes when a temporal stability analysis is performed. The brief study of the spatio-temporal characteristics of travelling waves addressed in section 4.7 may be extended to coaxial jets with different parameters.

Looking at the disturbance energy balance could explain the mechanisms yielding a (de)stabilisation of the flow while varying the parameters.

For completeness, performing a viscous analysis would probably confirm the stabilizing effect of viscosity for this kind of flows (see Lesshafft & Huerre, 2007). Concerning the base flow, the addition of swirl may be considered.

The performed linear global stability study reached the limits of the current theory. Theoretical extensions to non-linear global modes are mainly done with model equations like the Ginzburg-Landau equation. Because of the increasingly high computational capabilities, the problem can also be studied numerically. Concerning hot single jets, Lesshafft et al. (2006, 2007) numerically investigated non-linear global modes. A similar computational study on the heated coaxial jet flow might help understanding the non-linear global mode shapes, their link to acoustic noise generation and the transition mechanism. Furthermore, the threshold between convective and absolute instability would correspond to the instant when a perturbation is not swept downstream any more but contaminates the entire flow field. An attempt to numerically study global modes was done using the PARACONCYL code. Unfortunately, a numerical instability arose because of the low Mach number needed for an absolute instability and the treatment of the singular boundary condition at the centerline. An additional improvement of the code is suitable if one desires to use it for low Mach number regimes.

The global mode frequency could also be verified with a Dynamic Mode Decomposition (DMD) analysis of the LES data (see Schmid, 2010; Song et al., 2013).
A. Appendix

A.1. Derivation of the disturbance equations for an incompressible flow

A compressible solver is used (see Sec. 2.2). Its applicability to configurations with a Mach number tending to zero as well as for uniform density flows is proved in this section.

The continuity and the momentum equations in the three directions are not affected by $Ma$.

For vanishing Mach numbers, the energy equation becomes the divergence-free condition for an incompressible flow.

Starting from the disturbance energy equation (2.14e), restated here:

$$
\frac{1}{\gamma - 1} \frac{i \omega r^2}{\rho_b(r)} R(r) + \left[ i r - \frac{i r^2}{(\gamma - 1) \rho_b(r)} \frac{d \rho_b(r)}{dr} \right] V(r) + i r^2 \frac{dV(r)}{dr} + i m r W(r) - \frac{\gamma M a^2}{\gamma - 1} i r^2 \omega P(r) \\
= \frac{i r^2 u_b(r)}{(\gamma - 1) \rho_b(r)} \alpha R(r) - i r^2 \alpha U(r) - \frac{\gamma M a^2}{\gamma - 1} r^2 u_b(r) \alpha P(r)
$$

(A.1)

and taking the limit for $Ma \to 0$ one obtains:

$$
\frac{1}{\gamma - 1} \frac{\omega r^2}{\rho_b(r)} R(r) + \left[ r - \frac{r^2}{(\gamma - 1) \rho_b(r)} \frac{d \rho_b(r)}{dr} \right] V(r) + r^2 \frac{dV(r)}{dr} + m r W(r) \\
= \frac{r^2 u_b(r)}{(\gamma - 1) \rho_b(r)} \alpha R(r) - r^2 \alpha U(r),
$$

(A.2)

and multiplying by $(\gamma - 1) \rho_b(r)$ yields:

$$
\omega r^2 R(r) + \left[ r(\gamma - 1) \rho_b(r) - r^2 \frac{d \rho_b(r)}{dr} \right] V(r) + (\gamma - 1) \rho_b(r) r^2 \frac{dV(r)}{dr} + (\gamma - 1) \rho_b(r) m r W(r) \\
= r^2 u_b(r) \alpha R(r) - (\gamma - 1) \rho_b(r) r^2 \alpha U(r)
$$

(A.3)

which gives:
\[ \frac{\omega^2 R(r) + r \gamma \rho_b(r) V(r) - r \rho_b(r) V(r) - r^2 \frac{\rho_b(r)}{dr} V(r) + \gamma \rho_b(r) r^2 \frac{dV(r)}{dr} - \rho_b(r) r^2 \frac{dV(r)}{dr}}{\gamma \rho_b(r) m r W(r) - \rho_b(r) m r W(r)} = \frac{\gamma \rho_b(r) m r W(r) - \rho_b(r) m r W(r)}{- \gamma \rho_b(r) r^2 \alpha U(r)}. \]  

(A.4)

The underlined terms are nothing else than the continuity equation (2.14a) multiplied by \( r \). The only remaining terms are:

\[ r \gamma \rho_b(r) V(r) + \gamma \rho_b(r) r^2 \frac{dV(r)}{dr} + \gamma \rho_b(r) m r W(r) = - \gamma \rho_b(r) r^2 \alpha U(r), \]  

(A.5)

which can be further simplified.

\[ \frac{dV(r)}{dr} + \frac{V(r)}{r} + \frac{m W(r)}{r} + \alpha U(r) = 0 \]  

(A.6)

Equation (A.6) is the divergence-free condition for an incompressible flow in a cylindrical coordinate system [Ash & Khorrami 1995].

Furthermore, if the flow has a uniform density, which is however not the case in a heated coaxial jet, the density perturbations and the spatial variations of the base flow density vanish. In this case, the continuity equation (2.14a) reduces to (A.6) as well, energy and continuity equation are identical. The resulting equations are the ones reported by Ash & Khorrami (1995). The compressible solver can be therefore used without issues also for \( \text{Ma} = 0 \) and uniform densities. An incompressible solver, not reported in this text for sake of brevity, has been written to validate the results.

### A.2. Matrices of the generalized eigenvalue problem

The matrices appearing in the generalized eigenvalue problem are presented in detail in this section.

As seen in equation (2.25), the linearized disturbance equations can be written in the matrix formulation:

\[ \mathbf{A} \mathbf{U} = \omega \mathbf{B} \mathbf{U} + \alpha \mathbf{C} \mathbf{U}, \]  

(A.7)

where the eigenvector solution is:

\[ \mathbf{U} := \begin{bmatrix} R \\ U \\ iV \\ W \\ P \end{bmatrix}. \]  

(A.8)
A.2. Matrices of the generalized eigenvalue problem

The linear operators are the sparse matrices of the form:

\[ A := A^1 D + A^0, \] (A.9)

where

\[ A^0 := \begin{bmatrix}
0 & 0 & A_{13}^0 & A_{14}^0 & 0 \\
0 & 0 & A_{23}^0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{45}^0 & 0 \\
0 & 0 & A_{53}^0 & A_{54}^0 & 0
\end{bmatrix}, \] (A.10)

\[ A^1 := \begin{bmatrix}
0 & 0 & A_{13}^1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{35}^1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{53}^1 & 0 & 0
\end{bmatrix}, \] (A.11)

and the differential operator \( D \) is the diagonal matrix whose elements are the differentiation matrices \( D_r \) obtained with the Chebyshev collocation method (see Sec. 3.2):

\[ D := \begin{bmatrix}
D_r & 0 & 0 & 0 & 0 \\
0 & D_r & 0 & 0 & 0 \\
0 & 0 & D_r & 0 & 0 \\
0 & 0 & 0 & D_r & 0 \\
0 & 0 & 0 & 0 & D_r
\end{bmatrix}. \] (A.12)

The other two temporal and spatial linear operators have the form:

\[ B := \begin{bmatrix}
B_{11} & 0 & 0 & 0 & 0 \\
0 & B_{22} & 0 & 0 & 0 \\
0 & 0 & B_{33} & 0 & 0 \\
0 & 0 & 0 & B_{44} & 0 \\
B_{51} & 0 & 0 & 0 & B_{55}
\end{bmatrix}, \] (A.13)

\[ C := \begin{bmatrix}
C_{11} & C_{12} & 0 & 0 & 0 \\
0 & C_{22} & 0 & 0 & C_{25} \\
0 & 0 & C_{33} & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 \\
C_{51} & C_{52} & 0 & 0 & C_{55}
\end{bmatrix}. \] (A.14)
A.2. Matrices of the generalized eigenvalue problem

The components of the sparse matrices are given by the disturbance equations (2.14). Please note that the components are presented in their non-discretised form and therefore the single (double) underlying indicating a vector (matrix) quantity is not necessary.

*Continuity equation*

\[
A_{13}^0 := - \left( \rho_b + r \frac{d\rho_b}{dr} \right)
\]
\[
A_{14}^0 := - m\rho_b
\]
\[
A_{13}^1 := - r\rho_b
\]
\[
B_{11} := - r
\]
\[
C_{11} := ru_b
\]
\[
C_{12} := r\rho_b
\]

*x-momentum*

\[
A_{23}^0 := - \rho_b \frac{du_b}{dr}
\]
\[
B_{22} := - \rho_b
\]
\[
C_{22} := \rho_b u_b
\]
\[
C_{25} := 1
\]

*r-momentum*

\[
A_{35}^1 := 1
\]
\[
B_{33} := - \rho_b
\]
\[
C_{33} := \rho_b u_b
\]

*φ-momentum*

\[
A_{45}^0 := mr
\]
\[
B_{44} := r^2 \rho_b
\]
\[
C_{44} := - r^2 \rho_b u_b
\]
A.2. Matrices of the generalized eigenvalue problem

Energy

\[ A_{33}^0 := r - \frac{r^2}{(\gamma - 1)\rho_b} \frac{d\rho_b}{dr} \]
\[ A_{34}^0 := mr \]
\[ A_{33}^1 := r^2 \]
\[ B_{51} := -\frac{r^2}{(\gamma - 1)\rho_b} \]
\[ B_{55} := \frac{\gamma Ma^2 r^2}{\gamma - 1} \]
\[ C_{51} := \frac{r^2 u_b}{(\gamma - 1)\rho_b} \]
\[ C_{52} := -r^2 \]
\[ C_{55} := -\frac{\gamma Ma^2 r^2 u_b}{\gamma - 1} \]
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