On the dissipative spin-orbit problem

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Abstract

The spin-orbit model studies the movement of a satellite orbiting about a central body on a Keplerian elliptical orbit with eccentricity $e \in [0, 1)$. The main difference with the classical two-body problem is, that one takes care of the complex structure of the satellite. In particular we imagine the central planet as a point of mass and the satellite, having an ellipsoidal form, moving about it. Thus rotations of the satellite on its own axis are considered. We assume that the satellite has fixed spin axis, coinciding with the shortest physical axis of the ellipsoid, perpendicular to the orbital plane (no obliquity). This (apparently small) distinction makes the study of this problem very interesting.

One of the most natural question is about possible resonances of the system, i.e., under which conditions do $(p, q)$−periodic orbits exist, $p, q \in \mathbb{Z}$. We say that the system has a $(p, q)$−resonance (or equivalently that the two bodies move on a $(p, q)$−periodic orbit) if the satellite makes $p$−revolutions on itself when it completes $q$−revolutions about the central planet. The most famous example is the Earth-Moon system, which is a $(1, 1)$−periodic orbit, indeed from the Earth we see always the same half of the Moon. The curious fact is that the $(1, 1)$−resonance seems to be the most stable one. Indeed, with the important exception of Mercury-Sun system, which is in $(3, 2)$−resonance, we observe only $(1, 1)$−resonances in the solar system. But why does this happen? In *Low-order resonances in weakly dissipative spin-orbit models* L. Biasco and L. Chierchia give conditions for the existence of periodic orbits for $q = 1, 2$ and $4$ and estimates on the measure of the basins of attraction of stable periodic orbits for $q = 1, 2$ are discussed.

Since the main motivation of this dissertation is the better understanding of the dissipative spin-orbit problem, the principal goals of this thesis are:

- Verifying explicitly that the conditions for the coexistence of periodic orbits for $q = 1, 2$ and $4$, given by L. Biasco and L. Chierchia in *Low-order resonances in weakly dissipative spin-orbit models*, are satisfied if the eccentricity is small enough, providing also concrete applications to all observed systems in the Solar System admitting a $(p, q)$−periodic orbit.
- Finding explicit conditions for the existence of $(p, q)$-periodic orbits for all $q \geq 3, q \neq 4$, and discussing lower bounds on the basins of attraction for any $q \geq 3$;
- Developing a numerical method to compute approximations of $(p, q)$−periodic orbits.
Riassunto

Il problema spin-orbita studia il movimento di un satellite, che orbita su una traiettoria ellittica con eccentricità $e \in [0, 1]$ attorno ad un pianeta principale. La principale differenza con il famoso problema dei due corpi, è che in questo problema tiene conto della forma del satellite. Nello specifico si assume che il satellite abbia una forma ellissoidale, mentre il pianeta principale si assume essere puntiforme. Di conseguenza vanno considerate le rotazioni del satellite attorno al proprio asse di rotazione. Inoltre si assume che l’asse di rotazione coincida con il semiasse polare dell’ellissoide e che esso sia perpendicolare al piano orbitale. Questa differenza (apparentemente piccola) rende lo studio del problema spin-orbita molto interessante.

Una delle domande più naturali che ci si può porre studiando questo problema riguarda le condizioni di esistenza di orbite $(p,q)$-periodiche con $p, q \in \mathbb{Z}$. Con orbita o risonanza $(p,q)$-periodica si intende una traiettoria in cui il satellite compie $p$ rotazioni su se stesso nel tempo in cui compie $q$ rivoluzioni attorno al pianeta principale. L’esempio più famoso di questo fenomeno è il sistema Terra-Luna. La Luna descrive un’orbita $(1,1)$-periodica attorno alla Terra, infatti noi (dalla Terra) vediamo sempre la stessa faccia della Luna. È curioso il fatto che la risonanza $(1,1)$ sembra essere la più stabile. Infatti, eccezion fatta per Mercurio che descrive un orbita $(3,2)$-periodica con il Sole, nel nostro sistema solare la risonanza $(1,1)$ è l’unica astronomicamente osservabile. Come si spiega questo comportamento? Nell’articolo *Low-order resonances in weakly dissipative spin-orbit models* L. Biasco e L. Chierchia danno condizioni per l’esistenza di orbite periodiche nei casi in cui $q = 1, 2$ e $4$ e stimano i bacini d’attrazione di orbite periodiche nei casi in cui $q = 1, 2$.

Siccome la motivazione principale di questa tesi di dottorato è una migliore comprensione del problema spin-orbita dissipativo ne elenchiamo qui gli obiettivi principali:

- Verificare quantitativamente che le condizioni per la coesistenza di orbite periodiche per $q = 1, 2$ e $4$, date da L. Biasco e L. Chierchia in *Low-order resonances in weakly dissipative spin-orbit models*, siano soddisfatte per valori dell’eccentricità sufficientemente piccoli, dando anche applicazioni concrete a tutti i sistemi pianeta-satellite nel sistema solare, che ammettono un’orbita $(p,q)$-periodica;

- Trovare condizioni per l’esistenza di orbite $(p,q)$-periodiche per $q \geq 3$, $q \neq 4$ e stimare inferiormente il bacino d’attrazione di orbite $(p,q)$-periodiche per $q \geq 3$;

- Sviluppare un metodo numerico per calcolare approssimativamente orbite $(p,q)$-periodiche.
Introduction

The Solar System is fascinating in several different ways. To name only one of which, synchronous moving patterns of moons around their planet has to be mentioned. Earthlings, for example, are facing always the same side of their satellite.
In the Solar System we observe 18 moons\(^1\) in a so-called 1:1 spin–orbit resonance: the moons show always the same face to their hosting planets, i.e. during one orbital revolution around their host, they rotate one time around their proper spin axis. This spin axis is approximately perpendicular to the plane of revolution (see Figure 1).

\[ \text{Figure 1: Ellipsoidal satellite in a 1 : 1 spin-orbit resonance.} \]

The 3:2 resonance of Mercury around the Sun is the only exception of 1:1 spin-orbit resonance observable in the Solar System (see Figure 2).

\[ \text{Figure 2: (3, 2)–periodic orbit.} \]

In general we say that the system has a \((p, q)\)–resonance (or equivalently that the two bodies move on a \((p, q)\)–periodic orbit) if the satellite makes \(p\)–revolutions on itself

\(^1\)The complete list is: Moon (Earth); Io, Europa, Ganymede, Callisto (Jupiter); Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Iapetus, (Saturn); Ariel, Umbriel, Titania, Oberon, Miranda (Uranus); Charon (Pluto); minor bodies with mean radius smaller than 100 km are not considered.
when it completes $q$–revolutions about the central planet.

Figure 3: Spin–orbit problem (equatorial section).

The dissipative spin–orbit problem\(^2\) studies the motion of a satellite $S$ orbiting around a principal planet $P$ (modelled by a point of mass), under the following assumptions (see Figure 3):

- the center of mass of $S$ moves on a Keplerian ellipse with eccentricity $e \in [0, 1)$ around $P$;
- $S$ is a triaxial ellipsoid with axes $a_S \geq b_S \geq c_S > 0$, where $a_S$ and $b_S$ are the “equatorial radii” and $c_S$ the polar radius;
- the polar radius of $S$ coincides with the spin-axis of $S$ and it is perpendicular to the orbital plane;
- we consider the problem with dissipation.

The equation of motion is

$$\ddot{x} + \tilde{\eta}(\dot{x} - \tilde{\nu}) + \tilde{\varepsilon}f(x,t) = 0,$$

where:

- $t$ is the mean anomaly;
- $x$ is the angle (see Figure 3) formed by the direction of the major equatorial axis of the satellite with the direction of the semi–major axis of the Keplerian ellipse plane; ‘dot’ represents derivative with respect to $t$ and $e$ is the eccentricity of the orbital ellipse;
- the dissipation parameters $\tilde{\eta} = \tilde{\eta}_e$ and $\tilde{\nu} = \tilde{\nu}_e$ are real-analytic functions of the eccentricity $e$.
- the constant $\tilde{\varepsilon}$ measures the oblateness (or “equatorial ellipticity”) of the satellite;
- the function $f$ is the (“dimensionless”) Newtonian potential given by

$$f(x,t) := -\frac{1}{2(1 - e \cos(u_e(t)))^3} \cos(2x - 2f_e(t)),$$

where $f_e(t)$ is the true anomaly and $u = u_e(t)$ is the eccentric anomaly.

\(^2\)a more detailed description of the problem will be given in Chapter 1.
In the conservative case $\bar{\eta} = 0$ the spin-orbit problem (1) exhibits very interesting dynamics (periodic orbits of any period, KAM tori, AubryMather sets, etc.). This problem is very well studied in [11], [12]. A summary of the results about this problem can be found in [4].

In the dissipative case $\bar{\eta} > 0$ we have to distinguish two different cases. If $\bar{\varepsilon} = 0$ and $\bar{\eta} \neq 0$ hold, then the solution of (1) to the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$ is given by

$$x(t) = x_0 + \bar{\nu} t + \frac{1 - e^{-\eta t}}{\bar{\eta}} (v_0 - \bar{\nu}),$$

showing that $\{\dot{x} \equiv \bar{\nu}\}$ is a global attractor. In the second case $\bar{\varepsilon} \neq 0$ and $\bar{\eta} > 0$ the dynamic is much more varied. In [23] probabilities of capturing a planet (or satellite) into one of the commensurate rotation states as it is being despun by tidal friction (see [33]) are calculated. In particular in [19] is motivated why Mercury is in a $(3,2)$—periodic resonance with the Sun. In [15], [8] it is proved that for the dissipative spin-orbit problem the coexistence of low-order spin-orbit resonances is typical. Furthermore in [16] it is shown that the dissipative spin-orbit problem may also admit quasi-periodic solutions (attractors) $x(t) = \omega t + U(\omega t, t)$, with $U(\theta, t)$ analytic on $\mathbb{T}^2$ and $\omega$ Diophantine, provided $\bar{\varepsilon}$ is small enough and $\bar{\nu}$ satisfies the compatibility condition $\bar{\nu} = \omega(1 + \langle U_0^2 \rangle)$. Furthermore it is proved that, if it exits, the quasi-periodic attractor is unique.

In [15] the measure of the basins of attraction associated to periodic and quasi-periodic attractors is numerically investigated.

In this thesis we particularly focus our attention on the results of [8]. In that paper the authors, Luigi Chierchia and Luca Biasco, give explicit conditions for the coexistence of $(p,1)$—, $(p,2)$— and $(p,4)$—periodic orbits and provide analytical estimates on the measure of the basins of attraction of stable $(p,1)$— and $(p,2)$—periodic orbits. In Section 1 of [8] they suggest the following investigations:

1) Provide explicit conditions for the existence of $(p,q)$—periodic orbits for any $q$;

2) Discuss lower bounds on the basins of attraction for any $q$;

3) Prove (or disprove) that for $q = 1, 2$, the basin of attraction of a $(p,q)$-periodic orbit is actually “large”;

4) Discuss more general models (nonrestricted, nonplanar, obliquity, more general dissipations,...);

5) Extending the approach presented here, give lower bounds on the basins of attraction of the quasi-periodic attractors found in [16]. At this respect, let us remark that, numerically, the basins of attraction of quasi-periodic solutions appear to be much larger than those of periodic solutions (compare [15]).

The topics treated in this thesis cover the points 1) – 3) and partially 4). 5) is still an open problem.

The manuscript is organised as follows: In Chapter 1 we motivate the equation of motion (1) for the dissipative spin-orbit problem using the relation between this problem and the two body problem. Chapter 2 is devoted to the study of the Fourier coefficients $\alpha_j = \alpha_j(e)$ of the Newtonian potential $f$ given in equation (2), i.e.

$$f(x,t,e) = \sum_{j \in \mathbb{Z}} \alpha_j(e) \cos(2x - jt).$$
We will give integral formulas to compute these coefficients and to understand their asymptotical behaviour.

In Chapter 3 we study a more general problem than the dissipative spin-orbit problem. In particular we let the potential function

$$f \in C^\infty(\mathbb{R}^2, \mathbb{R})$$

be any analytic $2\pi$–periodic function in both variables $x, t$ and having zero average. For this problem we find conditions for the existence of a $(p, q)$–periodic orbit.

In Chapter 4 we prove that the condition, for the existence of $(p, q)$–periodic orbits of the dissipative spin-orbit problem for $q = 1, 2$ and $4$ given by L. Biasco and L. Chierchia in [8], is satisfied provided that the eccentricity and, in the case $q = 4$, also the dissipation, is smaller than an explicitly given bound.

In Chapter 5 we develop Matlab programs in order to compute numerically the solution of the dissipative spin-orbit problem (in a constructive way).

In Chapter 6 we prove the existence of periodic orbits for astronomical parameter values corresponding to all satellites of the Solar System observed in spin-orbit resonance.

In Chapter 7 we study the basin of attraction of $(p, q)$–periodic orbits for the dissipative spin-orbit problem giving two improvements of Theorem 1.3 in [8]: in the first result we allow $q$ to be any number greater than 2 (Theorem 1.3 in [8] covered only the cases $q = 1, 2$); the second result is a quantitative reformulation of Theorem 7.1, which makes concrete applications possible.

Finally, in Chapter 8 we aim to improve the numerical results on the basin of attraction of $(p, q)$–periodic orbits for the dissipative spin-orbit problem obtained by Celletti and Chierchia in [15]. Although our purposes are not yet all achieved, we present some partial results.
1 The physical model

The spin-orbit problem studies the movement of a satellite $S$ about a principal body $P$. Under the gravitational influence of $P$ the satellite follows a closed elliptical orbit. Furthermore rotations of the satellite are considered. In particular the spin-orbit problem assumes that the satellite has an ellipsoidal shape. This fact makes the spin-orbit problem a much more difficult problem to solve than the two-body problem. However this two problems are correlated and in order to well understand the physical origins of the equations, which govern the spin-orbit problem, one has to know some important relations arising from the two-body problem.

Chapter 1 is organised as follows: In Subsection 1.1 we analyse the two-body problem in details. In particular we prove the relation between the eccentric anomaly and the true anomaly (see equation (1.20)) and the Kepler equation (see equation (1.23)). These equations hold also in the spin-orbit problem. In Subsection 1.2 we consider first the conservative spin-orbit problem and then the dissipative spin-orbit problem, giving the equation of motion (see (1.24)), which will be used in the following chapters of this thesis. Finally in Subsection 1.3 we give a first definition of $\frac{p}{q}$ periodic resonances of the dissipative spin-orbit problem, which are the main object of interest in this thesis.

1.1 Two-body problem

1.1.1 Historical introduction (based on [17])

Ever since mankind existed on earth, men had great interest in studying stars, which was then in times solely based on observation with the naked eye. People in Mesopotamia, Central America, China, India and Egypt differentiated astronomers from their priests. Besides many other duties, astronomers were responsible to decide on the calendar, based on moon phases, which was fundamentally important for these agricultural people. The Greeks gave astronomy very important impulses. Following, some especially important thinkers will be mentioned: Aristarchus of Samos (approximately 310 BC - 230 BC) was the first postulating a heliocentric theory of the universe. According to his theory the earth was moving on a circular trajectory around the sun. Claudius Ptolemy (approximately 90 AD - 168 AD) crowned the geocentric theory of the universe to be the absolute truth, in his treatise *Almagesto*, written in AD 150. This paper served as basis for astronomic knowledge in all of Europe and Asia till the end of the medieval era. The science of astronomy had to wait until the Renaissance to finally receive new impulses. We will look at them in chronological order: Nicolaus Copernicus (1473-1543) formulated again, in his book *De revolutionibus orbium coelesticum*, published in the year of his death, the hypothesis of a heliocentric universe. This hypothesis was supported by Galileo Galilei (1564-1642), who could, for the first time, give scientific proof to the heliocentric theory through observing moon phases of Venus with his own invention, the telescope. Johannes Kepler (1571-1630) was able to push the theory forward, after getting access to precise astronomic data observed by Tycho Brahe. He formulated the (nowadays very famous) three Kepler’s laws.
Kepler’s Laws

**First Law:** The orbit of every planet is an ellipse with the Sun at one of the two foci.

**Second Law:** A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

**Third Law:** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

The credit for the mathematical demonstration of these laws has to be given to the genius of Isaac Newton (1642-1727), who elaborated the principles of celestial mechanics and the law of universal gravity in this book *Philosophiae Naturalis Principia Mathematica*, written in 1687. He summarized the principles and postulated three laws of motion and the law of universal gravitation (see below), through which he was able to derive Kepler’s laws.
Newton's laws of motion

**First law:** Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

**Second law:** The alternation of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

**Third law:** To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

**Newton’s law of universal gravitation**

Every point mass attracts every single other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between them.

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*Quote of I. Newton from *The Mathematical Principles of Natural Philosophy, Book I*, 1729.*


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The laws of Kepler and of Newton will allow us, in the course of this section, to understand the equation for the motion of two punctiform bodies. This is a fundamental step forward in the understanding of the spin-orbit problem, which will be tackled in the following subsection.

### 1.1.2 The model (based on [6] and [7])

We consider a principal body \( P \) (with mass \( M \)) and a satellite \( S \) (with mass \( m \)) interacting with each other under the gravitational force.
Newton’s second law implies

\begin{align*}
F_{PS}(x_1, x_2) &= M\ddot{x}_1, \\
F_{SP}(x_1, x_2) &= m\ddot{x}_2,
\end{align*}

(1.1)

(1.2)

where $F_{PS}$ is the force on $P$ due to $S$ and $F_{SP}$ is the force on $S$ due to $P$, respectively. By Newton’s third law and by Newton’s law of universal gravitation

\[ F_{SP} = -F_{PS} = \frac{-GMm}{r^3} \]

holds, where $G \approx 6.67 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ is the gravitational constant, and where

\[ x = x_2 - x_1 \quad \text{and} \quad r = |r|. \]

(1.3)

The center of mass

Adding up equations (1.1) and (1.2) we get

\[ 0 = F_{PS} + F_{SP} = M\ddot{x}_1 + m\ddot{x}_2 = (M + m)\ddot{R}, \]

(1.4)

where

\[ R = \frac{Mx_1 + mx_2}{M + m}, \]

(1.5)

gives the position of the center of mass of the two-body system. Notice that the initial two-body problem has been simplified to a one-body problem (the motion of center of mass). Furthermore the condition $\ddot{R} = 0$ shows that the velocity $v = \frac{dR}{dt}$ of the center of mass is constant.

Displacement vector $r$

Another possibility to reduce the problem to a one-body problem, arises subtracting equation (1.2) to the equation (1.1):

\[ \ddot{r} = \ddot{x}_2 - \ddot{x}_1 = \frac{F_{SP}(x_1, x_2)}{m} - \frac{F_{PS}(x_1, x_2)}{M} = F_{SP}(x_1, x_2) \left( \frac{1}{M} + \frac{1}{m} \right). \]

This is equivalent to

\[ \ddot{r} = -\frac{\mu}{r^3} r \]

(1.6)

with $\mu = G(M + m)$. 

Figure 1.3: Sketch of the two-body problem with origin in $O$. 

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4 Chapter 1. The physical model
1.1. Two-body problem

Conservation of Mechanical Energy

Scalar multiplication of (1.6) with \( \ddot{\mathbf{r}} \) implies

\[
\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}} + \frac{\mu}{r^3} \dot{\mathbf{r}} \cdot \mathbf{r} = 0.
\]

Let \( \mathbf{v} = \dot{\mathbf{r}} \) and recall that \( \mathbf{w} \cdot \dot{\mathbf{w}} = \dot{\mathbf{w}} \mathbf{w} \) for any vector \( \mathbf{w} \), then the previous equation is equivalent to

\[
v\ddot{\mathbf{v}} + \frac{\mu}{r^2} \dot{\mathbf{r}} = 0.
\]

Since \( \frac{d}{dt} \left( \frac{v^2}{2} \right) = v\dot{v} \) and \( \frac{d}{dt} \left( -\frac{1}{r} \right) = \frac{\dot{r}}{r^2} \) we get

\[
\frac{d}{dt} \left( \frac{v^2}{2} - \frac{\mu}{r} \right) = 0.
\]

Therefore the total specific mechanical energy of a satellite must be constant along its orbit, i.e.

\[
E = T + U = \frac{v^2}{2} - \frac{\mu}{r} = \text{const}, \quad (1.7)
\]

where \( T = \frac{v^2}{2} \) is the specific kinetic energy and \( U = -\frac{\mu}{r} \) is the specific potential energy of the satellite, respectively.

Conservation of Angular Momentum

Recall the specific angular momentum \( \mathbf{h} \) of a satellite is defined by

\[
\mathbf{h} = \mathbf{r} \times \mathbf{v}. \quad (1.8)
\]

In order to show that \( \mathbf{h} \) is constant, we differentiate equation (1.8) and use (1.6). So we have

\[
\frac{d}{dt} \mathbf{h} = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dddot{\mathbf{r}} \overset{(1.6)}{=} -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{r} = 0,
\]

which proves the conservation of the angular momentum.

The trajectory of the satellite

Cross multiplication of (1.6) by \( \mathbf{h} \) leads to

\[
\dot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{h}. \quad (1.9)
\]

For the left hand side of (1.9) the conservation of the angular momentum implies that

\[
\dot{\mathbf{r}} \times \mathbf{h} = \dot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \frac{\mathbf{h}}{0} = \frac{d}{dt} (\dot{\mathbf{r}} \times \mathbf{h})
\]

holds. Recalling the identities \( \mathbf{w}_1 \times (\mathbf{w}_2 \times \mathbf{w}_3) = \mathbf{w}_2 (\mathbf{w}_1 \cdot \mathbf{w}_3) - \mathbf{w}_3 (\mathbf{w}_1 \cdot \mathbf{w}_2) \) and \( \mathbf{w} \cdot \dot{\mathbf{w}} = \dot{\mathbf{w}} \mathbf{w} \), the right hand side of (1.9) becomes

\[
-\frac{\mu}{r^3} \mathbf{r} \times \mathbf{h} = \frac{\mu}{r^3} \mathbf{h} \times \mathbf{r} = \frac{\mu}{r^3} \left( \mathbf{v} (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} (\mathbf{r} \cdot \dot{\mathbf{r}}) \right)
\]

\[
= \frac{\mu}{r^3} \left( r^2 \mathbf{v} - r r \dot{\mathbf{r}} \right) = \frac{\mu}{r} \mathbf{v} - \frac{\mu \dot{r}}{r^2} \mathbf{r}
\]
\[ \frac{d}{dt} \left( \frac{\mathbf{r}}{r} \right) = \mu \frac{d}{dt} \left( \frac{\mathbf{r}}{r} \right). \]

So (1.9) is equivalent to
\[ \frac{d}{dt} (\dot{\mathbf{r}} \times \mathbf{h}) = \mu \frac{d}{dt} \frac{\mathbf{r}}{r}. \]

Integrating the last equation leads to
\[ \dot{\mathbf{r}} \times \mathbf{h} = \mu \frac{\mathbf{r}}{r} + \mathbf{B}, \quad (1.10) \]

where the vector \( \mathbf{B} \) is a constant. Scalar multiplication of (1.10) by \( \mathbf{r} \) implies
\[ \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \mu \frac{\mathbf{r} \cdot \mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{B}. \]

Since \( \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{h} \cdot \mathbf{h} = h^2 \) it follows
\[ h^2 = \mu r + rB \cos(\omega), \]

where \( \omega := \angle(\mathbf{B}, \mathbf{r}) \). In conclusion we find for \( r \) the following relation:
\[ r = \frac{h^2}{\mu} \left( \frac{1}{1 + \frac{B}{\mu} \cos(\omega)} \right). \quad (1.11) \]

Defining \( \sigma^2 := \frac{h^2}{\mu} \) and \( e := \frac{B}{\mu} \) equation (1.11) can also be rewritten as
\[ r = \frac{\sigma^2}{1 + e \cos(\omega)}, \quad (1.12) \]

which is the polar equation (with respect to the angle \( \omega \)) of a conic section (see Figure 1.4). Hence, the possible orbits in a two-body problem are: ellipses, if \( 0 \leq e < 1 \); parabolas, if \( e = 1 \); hyperbolas, if \( e > 1 \).

![Figure 1.4: Polar representation of a conic section.](image)

Since we are interested in the case of a satellite orbiting on a closed orbit around a principal body, we only consider the case of a closed elliptical curve.
1.1. Two-body problem

The case $M \gg m$ and the proof of the 1st and the 2nd Kepler-law.

Consider the case where the mass $m$ of the satellite $S$ is much smaller than the mass $M$ of the principal body $P$. By (1.3) and (1.5), the original trajectories $x_1$ and $x_2$ in Figure 1.3 are given by

\[
x_1(t) = R(t) - \frac{m}{M + m}r(t) \approx R(t), \quad (1.13)
\]
\[
x_2(t) = R(t) + \frac{M}{M + m}r(t) \approx R(t) + r(t). \quad (1.14)
\]

Therefore the center of mass of the system is approximately in $P$, i.e. the planet lies in one focus of the elliptical orbit. This proves the first Kepler-law. Furthermore by the conservation of angular momentum and by equation (1.8) and the definition of $\sigma$ we have

\[
A(t) = \int_0^t \frac{1}{2} |r \times v| dt = \int_0^t \frac{h}{2} dt = \frac{\sigma \sqrt{\mu}}{2} t. \quad (1.15)
\]

This proves the second Kepler-law.

---

Figure 1.5: Second Kepler-law: the sectors $A_1$, $A_2$ and $A_3$ have the same area. The ray from the central body to the satellite sweeps out equal areas in equal times.

Elliptic motion

Since we established that in the two body problem the orbit is an ellipse (see equation (1.12)), it is useful to recall some properties of this curve. It is obvious, comparing Figure 1.4 and 1.6 that $\omega = f_e$, which is called true anomaly. By (1.12) we have that the apoapsis and the periapsis\(^2\) are given by

\[
r_{\text{apoapsis}} = \frac{\sigma^2}{1 - e} \quad \text{and} \quad r_{\text{periapsis}} = \frac{\sigma^2}{1 + e}.
\]

The semi-major axis $a$ of the elliptical orbit fulfils the relation

\[
2a = r_{\text{apoapsis}} + r_{\text{periapsis}},
\]

hence we get

\[
\sigma^2 = a(1 - e^2). \quad (1.16)
\]

\(^2\)apogee/aphelion and perigee/perihelion for orbits around the Earth and the Sun, respectively.
Inserting (1.16) in (1.12) we get
\[ r = \frac{a(1 - e^2)}{1 + e \cos(f_e)}. \] (1.17)

Further interesting facts follow by the elementary geometry of the ellipse. By (1.17) the coordinates of the foci are \((\pm c, 0) = (ae, 0)\). The semi-minor axis \(b\) is then given by
\[ b^2 = a^2 - c^2 = a^2(1 - e^2). \] (1.18)

Choosing the directions of \(a\) and \(b\) as the reference axes of the system with origin \(O\), and introducing the eccentric anomaly \(u_e\) as in Figure 1.6, the satellite \(S\) has coordinates \((a \cos(u_e), b \sin(u_e))\). The theorem of Pythagoras implies
\[ r^2 = (a \cos(u_e) - ae)^2 + b^2 \sin^2(u_e) \]
\[ = a^2 \left[ (\cos(u_e) - e)^2 + (1 - e^2) \sin^2(u_e) \right] \]
\[ = a^2 (1 - e \cos(u_e))^2, \]

hence
\[ r = a \left(1 - e \cos(u_e)\right). \] (1.19)

**The Kepler equation**

Notice that by (1.19) and (1.17) we get
\[ \cos(u_e) = \frac{1}{e} - \frac{r}{ae} = \frac{1}{e} \left(1 - \frac{(1 - e^2)}{1 + e \cos(f_e)}\right) = \frac{e + \cos(f_e)}{1 + e \cos(f_e)} \]

The true anomaly \(f_e\) and the eccentric anomaly \(u_e\) are related as follows:
\[ \tan^2 \left(\frac{u_e}{2}\right) = \frac{1 - \cos(u_e)}{1 + \cos(u_e)} = \frac{(1 - e)(1 - \cos(f_e))}{(1 + e)(1 + \cos(f_e))} = \frac{1 - e}{1 + e} \tan^2 \left(\frac{f_e}{2}\right). \] (1.20)
Therefore \( f_e = 2 \arctan \left( \sqrt{\frac{1+e}{1-e}} \tan \left( \frac{u_e}{2} \right) \right) \) holds. Taking the derivative with respect to \( u_e \) of this formula we obtain

\[
\frac{df_e}{du_e} = \sqrt{\frac{1+e}{1-e}} \frac{1+\tan^2 \left( \frac{u_e}{2} \right)}{1+\frac{e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)}.
\]

We are now able to compute the integral of the second Kepler-law (see (1.15)), recalling that for the angular velocity \( v = r \frac{df_e}{dt} \). Hence we have:

\[
\sigma \sqrt{\mu t} = \int_0^\tau |\mathbf{r} \times \mathbf{v}| dt = \int_0^\tau r^2 df_e = \int_0^{u_e} a^2 (1-e^2)^2 \frac{a^2 (1-e^2)^2}{(1+e \cos (f_e))^2} df_e
\]

\[
= a^2 (1-e^2)^2 \sqrt{\frac{1+e}{1-e}} \int_0^{u_e} \frac{1}{1+e \frac{1+e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)} \frac{1+\tan^2 \left( \frac{u_e}{2} \right)}{1+\frac{e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)} du_e.
\]

Using the elementary trigonometric identity

\[
\cos(f_e) = \frac{1-\tan^2 \left( \frac{u_e}{2} \right)}{1+\tan^2 \left( \frac{u_e}{2} \right)} = \frac{1-\frac{1+e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)}{1+\frac{1+e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)}
\]

we have (according to [34])

\[
\sigma \sqrt{\mu t} = a^2 (1-e^2)^2 \sqrt{\frac{1+e}{1-e}} \int_0^{u_e} \frac{1}{1+e \frac{1+e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)} \frac{1+\tan^2 \left( \frac{u_e}{2} \right)}{1+\frac{e}{1-e} \tan^2 \left( \frac{u_e}{2} \right)} du_e
\]

\[
= a^2 (1-e^2)^2 \sqrt{\frac{1+e}{1-e}} \int_0^{u_e} \frac{1+\tan^2 \left( \frac{u_e}{2} \right) - e + e \tan^2 \left( \frac{u_e}{2} \right)}{(1-e^2)^2 (1+\tan^2 \left( \frac{u_e}{2} \right))} du_e
\]

\[
= a^2 \sqrt{1-e^2} \int_0^{u_e} \left( 1 - \frac{1-\tan^2 \left( \frac{u_e}{2} \right)}{1+\tan^2 \left( \frac{u_e}{2} \right)} \right) du_e
\]

\[
= a^2 \sqrt{1-e^2} \int_0^{u_e} (1-e \cos(u_e)) du_e = a^2 \sqrt{1-e^2} (u_e - e \sin(u_e)).
\]

So \( \sigma \sqrt{\mu t} = a^2 \sqrt{1-e^2} (u_e - e \sin(u_e)) \) holds, which by (1.16) is equivalent to

\[
nt = u_e - e \sin(u_e),
\]

where \( n := \sqrt{\frac{\mu}{a^3}} = 2 \pi \) (where \( T \) is the orbital period\(^3\)) is called \textit{mean motion}. Defining \( \tau := nt \) as the \textit{mean anomaly} (1.22) is equivalent to

\[
\tau = u_e - e \sin(u_e),
\]

which we will refer to as the \textit{Kepler equation}.\(^3\)

\(^3\)By (1.15) integrating over one complete orbital period \( T \) it follows

\[
\pi ab = A(T) = \frac{\sigma \sqrt{\mu t}}{2} T.
\]

Since (1.16) and (1.18) this is equivalent to

\[
4\pi^2 = \frac{\mu T^2}{a^3} = n^2 T^2.
\]

This proves the third Kepler-law.
Remark 1.1. Consider the Banach space

\[ \chi := \{ v(\tau) \in C(\mathbb{R}, \mathbb{R}) \mid v \text{ is } 2\pi - \text{periodic and odd} \} \]

endowed with the sup-norm \( \| v(\tau) \|_\infty := \sup_{\tau \in \mathbb{R}} |v(\tau)| \). For \( e \in (0,1) \) the map

\[ \Phi : \chi \rightarrow \chi \quad v \mapsto \Phi(v(\tau)) := \sin(\tau + ev(\tau)) \]

is a contraction with Lipschitz constant \( e \). Hence for \( u, v \in \chi \)

\[ \| \Phi(v) - \Phi(u) \|_\infty = \sup_{\tau \in \mathbb{R}} |\sin(\tau + ev(\tau)) - \sin(t + eu(\tau))| \]

\[ = \sup_{\tau \in \mathbb{R}} \left| 2 \cos \left( \tau + \frac{v(\tau) + u(\tau)}{2} \right) \sin \left( e \frac{v(\tau) - u(\tau)}{2} \right) \right| \]

\[ \leq 2 \sup_{\tau \in \mathbb{R}} \left| \sin \left( e \frac{v(\tau) - u(\tau)}{2} \right) \right| \]

\[ \leq e \sup_{\tau \in \mathbb{R}} |v(\tau) - u(\tau)| = e \| v - u \|_\infty \]

holds. From the fixed point theorem there exists a unique function \( U_e(\tau) \in \chi \) such that

\[ U_e(\tau) = \sin(\tau + eU_e(\tau)) \].

Notice that the function \( \tau + eU_e(\tau) \) satisfies the Kepler equation (1.23) and therefore

\[ u_e(\tau) = \tau + eU_e(\tau) \]

holds. It follows that \( u_e \) is continuous and odd \( 2\pi \)-periodic function of the mean anomaly \( \tau \). Defining \( f_e(\pi) = \pi \) the true anomaly \( f_e \) in (1.20) is also continuous and odd \( 2\pi \)-periodic function of the mean anomaly \( \tau \).

1.2 The spin-orbit problem

1.2.1 The model

As the two-body problem, the spin-orbit problem studies the motion of a satellite revolving around a principal planet. The difference, which makes the study of this problem much more interesting, is that the shape of the satellite is considered, i.e. rotations of the satellite on its own spin axis can occur. In particular we imagine the central planet as a point of mass and the satellite as triaxial ellipsoid with axes \( a_S \geq b_S \geq c_S > 0 \), where \( a_S \) and \( b_S \) are the “equatorial radii” and \( c_S \) the polar radius. We assume that the satellite moves about the central planet on a Keplarian elliptical orbit with semi-major axis \( a \). Furthermore the spin-axis is assumed to be perpendicular to the Keplarian orbit plane\(^4\).

Under the above hypotheses, the differential equation governing the motion of the satellite is then given by

\[ \ddot{x} + \eta(\dot{x} - \bar{v}) + \varepsilon f_\alpha(x,t) = 0, \quad (1.24) \]

where:

\(^4\)For the motivation of this assumption see Subsection 1.2.3
1.2. The spin-orbit problem

\( \rho_e \) is the orbital radius divided by the total area times 2\( \pi \). Notice that in (1.23) we used \( \tau \) for the mean anomaly;

(b) \( x \) is the angle (mod 2\( \pi \)) formed by the direction of (say) the major equatorial axis of the satellite with the direction of the semi-major axis of the Keplerian ellipse plane; ‘dot’ represents derivative with respect to \( t \) and \( e \) is the eccentricity of the ellipse;

(c) the dissipation parameters \( \bar{\eta} = K \Omega_e \) and \( \bar{\nu} = \bar{\nu}_e \) are real-analytic functions of the eccentricity \( e \): \( K \geq 0 \) is a physical constant depending on the internal (non-rigid) structure of the satellite and

\[
\begin{align*}
\Omega_e &:= \left(1 + 3e^2 + \frac{3}{8}e^4 \right) \frac{1}{(1 - e^2)^{9/2}}, \quad (1.25) \\
N_e &:= \left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6 \right) \frac{1}{(1 - e^2)^{5}}, \quad (1.26) \\
\bar{\nu}_e &:= \frac{N_e}{\Omega_e}; \quad (1.27)
\end{align*}
\]

(d) the constant \( \bar{\varepsilon} \) measures the oblateness (or “equatorial ellipticity”) of the satellite and it is defined as

\[
\varepsilon = \frac{3}{2} \frac{B - A}{C}, \quad (1.28)
\]

where 0 < \( A \leq B \leq C \) are the principal moments of inertia of the satellite (\( C \) being referred to the polar axis);

(e) the function \( f \) is the (“dimensionless”) Newtonian potential given by

\[
f(x, t) := -\frac{1}{2\rho_e(t)^3} \cos(2x - 2f_e(t)), \quad (1.29)
\]

where \( \rho_e(t) \) and \( f_e(t) \) are, respectively, the (normalized) orbital radius

\[
\rho_e(t) := 1 - e \cos(u_e(t)) \quad (1.30)
\]

and the true anomaly

\[
f_e(t) := 2 \arctan \left( \sqrt{\frac{1 + e}{1 - e}} \tan \left( \frac{u_e(t)}{2} \right) \right), \quad (1.31)
\]
while the eccentric anomaly $u = u_e(t)$ is defined implicitly by the Kepler equation\footnote{It is well known (see [47]) that for every $t \in \mathbb{R}$ the function $C \ni e \mapsto u_e(t)$ is holomorphic for $|e| < r_*$, with $r_* := \max_{y \in \mathbb{R}} \frac{y}{\cosh(y)} = \frac{y_+}{\cosh(y_+)} = 0.6627434 \cdots$ and $y_* = 1.1996786 \cdots$.}

$$t = u - e \sin(u).$$  \hspace{1cm} (1.32)

Notice that the Newtonian potential $f(x, t)$ is a doubly–periodic function of $x$ and $t$, with periods $2\pi$.

We should remark that the relations (1.31) and (1.32) for the spin-orbit problem come from the two-body problem, in particular from the equations (1.20) and (1.23).

For further references on this model, see [20], [23], [48], [12] and [13]; for a different approach to this problem, see [5].

1.2.2 The Moment of inertia tensor

Let $(x, y, z)$ be a given reference frame. For any 3–dimensional body with volume $V$ and mass $m$ we define the quantities

$$I_{11} = \iiint_V (y^2 + z^2) \, dm =: A$$

$$I_{22} = \iiint_V (x^2 + z^2) \, dm =: B$$

$$I_{33} = \iiint_V (x^2 + y^2) \, dm =: C$$

as the moments of inertia and

$$I_{23} = I_{32} = -\iiint_V yz \, dm =: -F$$

$$I_{12} = I_{21} = -\iiint_V xz \, dm =: -G$$

$$I_{13} = I_{31} = -\iiint_V xy \, dm =: -H$$

as the products of inertia.

The matrix

$$I = I(x, y, z) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

is called the inertia tensor. Notice that $I$ depends on the choice of a reference frame $(x, y, z)$.

We are interested in computing the moment of inertia of a satellite $S$ with an ellipsoidal shape with axes $a_S \geq b_S \geq c_S > 0$, choosing the principal axes of inertia as reference frame. Hence the ellipsoid is given by the relation

$$\frac{x^2}{a_S^2} + \frac{y^2}{b_S^2} + \frac{z^2}{c_S^2} = 1.$$ 

Assuming the density $\rho = \frac{m}{\frac{4}{3} \pi a_S b_S c_S}$ of the satellite to be constant, we get

$$A = \iiint_S (y^2 + z^2) \, dm = \rho \iiint_S (y^2 + z^2) \, dx dy dz.$$
1.2. The spin-orbit problem

Substituting \( x' = \frac{x}{\rho_S}, \ y' = \frac{y}{\rho_S}, \ z' = \frac{z}{\rho_S} \), we integrate now on \( S^3 \) (the sphere of radius 1). The Jacobi determinant implies \( dx' dy' dz' = \rho_S dx dy dz \). Thus we have

\[
A = \rho_S b_S c_S \int_{S^3} \left( b_S^2 (y')^2 + c_S^2 (z')^2 \right) dx' dy' dz'.
\]

Transforming to spherical coordinates

\[
x' = s \sin(\theta) \cos(\phi), \quad y' = s \sin(\theta) \sin(\phi), \quad z' = s \cos(\theta)
\]

we obtain

\[
A = \rho_S b_S c_S \int_0^{2\pi} \int_0^\pi \int_0^1 \left[ b_S^2 s^2 \sin^2(\theta) \sin^2(\phi) + c_S^2 s^2 \cos^2(\theta) \right] s^2 \sin(\theta) ds d\theta d\phi
\]

\[
= \frac{4 \rho_S b_S c_S (b_S^2 + c_S^2) \pi}{15} = m (b_S^2 + c_S^2) / 5.
\]

Analogously we get

\[
B = \frac{1}{5} m (a_S^2 + c_S^2), \quad C = \frac{1}{5} m (a_S^2 + b_S^2).
\]

Furthermore, choosing the principal axes of inertia as reference frame, the products of inertia are all zero. For example, making the same substitutions as above, let compute the first product of inertia \( F \). By definition we have

\[
F = \int \int \int_S yz dm = \rho \int \int \int_V yz dx dy dz
\]

\[
= a_S b_S c_S \int_0^{2\pi} \int_0^\pi \int_0^1 s^4 \sin(\theta)^2 \cos(\theta) \sin(\phi) ds d\theta d\phi = 0.
\]

The same holds for the other two products of inertia: \( G = H = 0 \).

We conclude that choosing the principal axes of inertia as reference frame corresponds to diagonalise the inertia tensor \( I \), i.e.

\[
I = I_{\alpha_S \beta_S \gamma_S} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.
\]

(1.33)

1.2.3 Motivation of the rotational assumption about the \( z \)-axes

Let \( \Omega \) be the center of mass of the satellite and let \((\Omega, x, y, z)\) be the reference frame pointing in the direction of the satellite’s principal axes of inertia. The moment of inertia tensor is then given by (1.33). In the following lines we explain why, the assumption that “the spin polar axis is perpendicular to the Keplerian orbit plane”, makes sense.

By definition the angular momentum is

\[
L = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = I \cdot \omega = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} A \omega_x \\ B \omega_y \\ C \omega_z \end{pmatrix}.
\]

(1.34)

Let \( L^2 := L_x^2 + L_y^2 + L_z^2 \). Furthermore using (1.34) the rotational energy is given by

\[
E = \frac{1}{2} \omega \cdot I \cdot \omega
\]

\[
= \frac{1}{2} (A \omega_x^2 + B \omega_y^2 + C \omega_z^2)
\]
\[
\begin{align*}
\frac{1}{2} (A\omega_x)^2 + \frac{1}{2} (B\omega_y)^2 + \frac{1}{2} (C\omega_z)^2 \\
= \frac{1}{2} \left( \frac{L_x^2}{A} + \frac{L_y^2}{B} + \frac{L_z^2}{C} \right).
\end{align*}
\]

The angular momentum \( \mathbf{L} \) lies therefore in the intersection of the sphere \( x^2 + y^2 + z^2 = L^2 \) with the ellipsoid \( E : \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = E \). The size of \( \mathbf{L} \) is conserved, if we neglect external torques. The tidal term dissipates the energy \( E \). So the new system has rotational energy \( E_\ast < E \). The ellipsoid \( E \) shrinks into \( E_\ast \) (see Figure 1.8) and therefore the direction of \( \mathbf{L} \) tends to the axis of the ellipsoid, with biggest moment of inertia. Since, in our case \( A \leq B \leq C \), this is the \( z \)-axis. So the assumption is plausible.

### 1.2.4 MacCullagh’s Potential (based on [20])

Consider a triaxial ellipsoidal body with center of mass in \( \Omega \). Assume the density of the body to be constant. In the following \( V \) denotes the volume of the body. We define \((\Omega, x, y, z)\) to be the reference frame, where the \( x \)-, \( y \)-, \( z \)-axes point in the direction of the principal axes of inertia of the ellipsoidal body. Let \( \mathcal{O} \) be a point with coordinates \((\lambda, \mu, \nu)\). The vector \( \mathbf{r} \) represents \( \mathcal{O}\Omega \). The length of the vector \( \mathcal{O}\Omega \) is equal to \( r = \sqrt{\lambda^2 + \mu^2 + \nu^2} \). Let \( \mathbf{P}(x_P, y_P, z_P) \) be any point of the triaxial ellipsoidal body with mass \( dm \). The vector \( \mathbf{s} \) represents \( \Omega\mathbf{P} \).

![Figure 1.8](image.png)

**Figure 1.8:** While the energy is dissipated the angular momentum \( \mathbf{L} \) tends to align itself to the longest axis of the ellipsoid.

![Figure 1.9](image.png)

**Figure 1.9:** Gravitational potential due to a triaxial ellipsoidal body on distant unit mass in \( \mathcal{O} \).

With these hypotheses the gravitational potential \( f \), the so-called MacCullagh potential, on distant unit mass, lying in \( \mathcal{O} \), can be given depending on the coordinates
1.2. The spin-orbit problem

$(\lambda, \mu, \nu)$ of the point $O$ (see Figure 1.9).

$$f(\lambda, \mu, \nu) := -G \iiint_V \frac{dm}{|r + s|} = -\frac{G}{r} \iiint_V \frac{dm}{\sqrt{1 - 2s^2 \cos(\theta) + \frac{s^2}{r^2}}}$$

$$= -\frac{G}{r} \iiint_V dm \left( 1 + \frac{s}{r} \cos(\theta) - \frac{1}{2} \frac{s^2}{r^2} \right. + \frac{3}{2} \frac{s^2}{r^2} \cos^2(\theta) + O \left( \frac{s}{r}^3 \right) \right)$$

$$= -\frac{G}{r} \iiint_V dm \left( 1 + \frac{s}{r} \cos(\theta) + \frac{1}{2} \frac{s^2}{r^2} \left[ 2 - 3 \sin^2(\theta) \right] + O \left( \frac{s}{r}^3 \right) \right), \quad (1.35)$$

where the law of cosines has been used. By symmetry the integral $\iiint_V \frac{s}{r} \cos(\theta) dm = 0$. Furthermore since $s << r$ we can say that

$$f(\lambda, \mu, \nu) = -\frac{G m}{r} - \frac{G}{2r^3} (A + B + C - 3I_{O\Omega}), \quad (1.36)$$

where $A, B, C$ are the principal moments of inertia of the satellite and

$$I_{O\Omega} := \iiint_S s^2 \sin^2(\theta) dm = \frac{A\lambda^2 + B\mu^2 + C\nu^2}{r^2} \quad (1.37)$$

is the moment of inertia of the satellite along the axes $O\Omega$. Equation (1.36) follows directly from (1.35), from the definition of the principal moment of inertia (see Subsection 1.2.2) and the observation that

$$A + B + C = 2 \iiint_S s^2 dm.$$

Equation (1.37) can be explained with the ellipsoidal of inertia $Ax^2 + By^2 + Cz^2 = 1$. This surface has the interesting property (see [24]) that the moment of inertia along any axis $\delta$ through the center of mass $\Omega$ of the body is equal to $1/d^2$, where $d$ is the distance between $D$, the intersection point of the axis $\delta$ with the ellipsoidal of inertia, and $\Omega$ (see Figure 1.10).

$$Ax^2 + By^2 + Cz^2 = 1$$

\[\delta\]

Figure 1.10: Two-dimensional section of the ellipsoid of inertia.

With other words the point $D$ lies on the circle with radius

$$d = \frac{1}{\sqrt{I_\delta}}, \quad (1.38)$$

where $I_\delta$ is the moment of inertia of the body along the axis $\delta$. 
Combining Figure 1.9 and Figure 1.10 one obtains the following figure:

![Diagram](image)

Figure 1.11: Ellipsoidal body with its ellipsoid of inertia.

The point $O(\lambda, \mu, \nu)$ lies on the circle with radius $r$, since $\sqrt{\lambda^2 + \mu^2 + \nu^2} = r$. Let $O'(\lambda', \mu', \nu')$ be the intersection point between the line $\Omega O$ and the ellipsoidal of inertia $Ax^2 + By^2 + Cz^2 = 1$. By (1.38) $O'$ lies also on the circle with radius $1/\sqrt{I_{O\Omega}}$, where $I_{O\Omega}$ is the moment of inertia along the axis $O\Omega$. Furthermore choose $\tilde{r}$ such that the ellipsoid $Ax^2 + By^2 + Cz^2 = \tilde{r}^2$ contains $O$ and is similar to the ellipsoid of inertia (see Figure 1.11). By similarity it follows

$$\frac{r^2}{I_{O\Omega}} = \frac{A\lambda^2 + B\nu^2 + C\mu^2}{A(\lambda')^2 + B(\nu')^2 + C(\mu')^2} = \frac{A\lambda^2 + B\nu^2 + C\mu^2}{1}.$$

This fact explains why (1.37) holds.

### 1.2.5 The Euler equations and the equation of motion (based on [20])

Let $\Omega$ be the center of mass of the triaxial ellipsoidal satellite and let $(\Omega, \hat{x}, \hat{y}, \hat{z})$ be the reference frame pointing in the direction of the principal moment of inertia of satellite.

![Diagram](image)

Figure 1.12: Spin-orbit problem with reference frame $(\Omega, \hat{x}, \hat{y}, \hat{z})$ (equatorial section).
1.2. The spin-orbit problem

By assumption the spin axis of the satellite coincides with its shortest physical axis and is perpendicular to the orbital plane, i.e. \((\Omega, \hat{x}, \hat{y}, \hat{z})\) is a rotating reference frame with angular velocity

\[
\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}.
\]

It is well known that the time derivative in a rotating frame follows the rule (see [18])

\[
\frac{d}{dt}w = \frac{\partial w}{\partial t} + \omega \times w
\]

for any vector \(w\). Therefore the derivative of the angular momentum \(L = I(\hat{x}, \hat{y}, \hat{z})\omega\) of the satellite with respect to \(t\) is equal to

\[
\frac{d}{dt}L = \frac{d}{dt}(I(\hat{x}, \hat{y}, \hat{z}) \cdot \omega) = \frac{\partial}{\partial t}(I(\hat{x}, \hat{y}, \hat{z}) \cdot \omega) + \omega \times (I(\hat{x}, \hat{y}, \hat{z}) \cdot \omega)
\]

\[
= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\omega}_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ C \omega_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ C \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ C \dot{\omega}_z \end{pmatrix}.
\]

Moreover, the angular momentum also satisfies the relation

\[
L = m \mathbf{h} = \mathbf{r} \times m \mathbf{v}.
\]

Again taking the derivative with respect to \(t\), one obtains

\[
\frac{d}{dt}(\mathbf{r} \times m \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times m \mathbf{v} + \mathbf{r} \times m \frac{d\mathbf{v}}{dt} = \mathbf{r} \times ma = \mathbf{r} \times \mathbf{F}_{PS} = \mathbf{\Gamma},
\]

where \(\mathbf{F}_{PS}\) is the gravitational force exerted from the principal planet \(P\) on the satellite \(S\). \(\mathbf{\Gamma}\) is called torque and indicates the tendency of a force to rotate an object. By the third Newton-law the satellite exerts an equal and opposite force on the planet \(P\), i.e. \(\mathbf{F}_{PS} = -\mathbf{F}_{SP}\). By definition of the MacCullagh potential \(f\) (see equation (1.36)) it follows

\[
\mathbf{F}_{SP} = -M \nabla f(\lambda, \mu, \nu),
\]

where \((\lambda, \mu, \nu)\) are the coordinates of the principal planet \(P\) in the \((\Omega, \hat{x}, \hat{y}, \hat{z})\) rotating reference frame, i.e. (see Figure 1.12)

\[
\lambda = r \cos(\phi), \quad \mu = r \sin(\phi), \quad \nu = 0,
\]

holds. So we have

\[
\mathbf{F}_{PS} = M \nabla f(\lambda, \mu, \nu) = \frac{3MG}{2r^3} \nabla I_{\Omega P} = \frac{3MG}{r^5} \begin{pmatrix} A \lambda \\ B \mu \\ C \nu \end{pmatrix}.
\]

The torque \(\mathbf{\Gamma}\) in (1.40) can therefore be rewritten as follows:

\[
\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} = \frac{3MG}{r^5} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \times \begin{pmatrix} A \lambda \\ B \mu \\ C \nu \end{pmatrix} = \frac{3MG}{r^5} \begin{pmatrix} (C - B) \mu \nu \\ (A - C) \lambda \nu \\ (B - A) \lambda \mu \end{pmatrix}.
\]
The *Euler equations* arise by comparing (1.39), (1.40) and (1.42). The first two components are trivially satisfied since $\nu = 0$ by (1.41). The equation in the third component is

$$C \dot{\omega}_z = \frac{3MG}{r^3}(B - A)\lambda \mu \quad (1.41)$$

$$= \frac{3MG}{r^3}(B - A)\cos(\phi)\sin(\phi). \quad (1.43)$$

Let $x$ be the angle (mod $2\pi$) formed by the direction of the major equatorial axis of the satellite with the major axis of the Keplerian ellipse plane (as in the Figure 1.12). Since the sum of the angles in a triangle is equal to $\pi$, we get the relation

$$\phi + x + f_e = \pi \quad \Rightarrow \quad \phi = -(x - f_e).$$

The equation (1.43) is taken to

$$\dot{\omega}_z = -MG \frac{3(B - A)}{2C} \sin(2x - 2f_e). \quad (1.44)$$

Notice that by equation (1.28) we know $\bar{\varepsilon} = \frac{3}{2} \frac{B-A}{C}$, where $0 < A \leq B \leq C$ are the principal moments of inertia of the satellite. Then (1.44) is equivalent to

$$\dot{\omega}_z = -\frac{\bar{\varepsilon}MG}{a^3} \sin(2x - 2f_e) \left(\frac{r}{a}\right)^3, \quad (1.45)$$

where $a$ is the length of the major semiaxis of the elliptical orbit. Since $M >> m$ the following approximation is possible:

$$\frac{MG}{a^3} \approx \frac{(M + m)G}{a^3} = \frac{\mu}{a^3} = n^2,$$

where $\mu := G(M + m)$ (as in (1.6)) and $n$ is the mean motion defined in (1.22). By (1.23) the mean anomaly satisfies $\tau = nt$ and therefore

$$\dot{\omega}_z = \frac{d\omega_z}{dt} = \frac{1}{n^2} \frac{d\omega_z}{d\tau}.$$ 

So we get (according to [12])

$$\omega_z = \frac{dx}{d\tau} = -\frac{d}{d\tau}(\phi - f_e)$$

and from (1.45) one gets the differential equation

$$\frac{d^2x}{d\tau^2} + \bar{\varepsilon} \sin(2x - 2f_e) \left(\frac{r}{a}\right)^3 = 0.$$ 

Recalling the definition of the Newtonian potential $f$ in (1.29), of the (normalized) orbital radius $\rho_e$ in (1.30), and of the function $u_e = u_e(\tau)$ in the Kepler equation (1.23), respectively, the differential equation governing the motion of the satellite (also according to [13]) is then given by

$$\frac{d^2x}{d\tau^2} + \bar{\eta}f_e(x, \tau) = 0, \quad (1.46)$$

which is equivalent to (1.24) (where dot means the derivative with respect to $\tau$) for $\bar{\eta} = 0$. The case $\bar{\eta} > 0$ will be treated in the next subsection.
1.2.6 The dissipative case

To the spin-orbit problem (1.46) we include also small dissipative effects due to the possible internal non-rigid structure of the satellite. According to the “viscous–tidal model, with a linear dependence on the tidal frequency” (see [19]), the dissipative term is given by the average over one revolution period (i.e., $2\pi$ with our normalization) of the so-called MacDonald torque [33]. Hence the tidal torque is

$$T(\dot{x}) = -\bar{\eta} (\dot{x} - \bar{\nu}).$$  \hspace{1cm} (1.47)

Compare also [23], [35], and more recently [13].

The dissipation parameters

$$\bar{\eta} = \eta_e = K \Omega_e \quad \text{and} \quad \bar{\nu} = \bar{\nu}_e$$  \hspace{1cm} (1.48)

are real-analytic functions of the eccentricity $e$, where $\Omega_e, N_e$ and $\bar{\nu}_e$ are defined in (1.25), (1.26) and (1.27). For the Moon and Mercury an accepted value for $\bar{\eta}$ is $\sim 10^{-8}$. The dissipation constant $K \geq 0$ in (1.48) has the form

$$K = 3n \frac{k_2 M}{Q m} \left( \frac{R_{\text{eq}}}{a} \right)^3,$$  \hspace{1cm} (1.49)

where:

- $M$ and $m$ are the masses of the principal body and of the satellite, respectively;
- $R_{\text{eq}}$ is the satellite’s mean equatorial radius;
- $a$ is the semi-major axis of the Keplerian elliptical orbit;
- $n = \sqrt{GM/a^3}$ is the orbital mean motion, where $G$ is the universal gravitational constant;
- $Q$ is called quality factor. $1/Q$ is defined as $\sin \epsilon$, where $\epsilon$ is a phase lag at this frequency;
- $k_2$ is called Love number and is defined by

$$k_2 = \frac{1.5}{1 + \frac{1}{2gR_{\text{eq}}}},$$  \hspace{1cm} (1.50)

where

• $\varsigma$ is the rigidity of the satellite,
• $\rho$ is the mean density of the satellite,
• $g$ is the surface gravity of the satellite.

There is no universally accepted determination of the tidal ratio $k_2/Q$ in (1.49) for most satellites of the Solar System. Significant amount of work on $k_2/Q$ of large icy satellites (Europa, Ganymede, Callisto, Titan) has been made in [26]. However, the only outer Solar System body, for which $k_2/Q$ has been measured, is Titan in [28]. Of course, you can make detailed models in individual cases. Some examples are: Io and Jupiter see [30], Saturn see [31] and Iapetus see [9].

In any case one notice, that $k_2$ in (1.50) is smaller than 1.5. Furthermore, by definition of $Q$, it is impossible to get $Q < 1$. In conclusion we can be sure that $\frac{k_2}{Q} \leq 1.5$. We will use this estimate in further chapters.

Equation (1.24) follows now from (1.46) and (1.47).

---

6In [19] (see Eqns (3) and (4)) $\Omega_e$ and $N_e$ are denoted, respectively, $\Omega(e)$ and $N(e)$, while, in [35], they are denoted, respectively, by $f_1(e)$ and $f_2(e)$.
1.3 Definition of spin-orbit resonances

As briefly mentioned in the introduction we say (according to [39]) that two bodies are in a $p : q$ spin-orbit resonance (with $p$ and $q$ co–prime non–vanishing integers) if

$$\frac{T_{\text{rev}}}{T_{\text{rot}}} = \frac{p}{q},$$

with $p, q \in \mathbb{Z}_+$, where $T_{\text{rev}}$ are the period of revolution of the satellite around the central planet, and $T_{\text{rot}}$ the period of rotation of the satellite around its spin-axis, respectively (see Figure 1.7).

**Definition 1.1.** A $(p,q)$–periodic orbit (with $p$ and $q$ co–prime non–vanishing integers) for the dissipative spin-orbit problem is a solution $x_{pq}(t) : \mathbb{R} \rightarrow \mathbb{R}$ of (1.24) satisfying

$$x_{pq}(t + 2\pi q) = x_{pq}(t) + 2\pi p, \quad \forall t. \tag{1.51}$$

**Remark 1.2.** (i) Indeed, for orbits as in (1.51), after $q$ revolutions of the orbital radius, $x$ has made $p$ complete rotations\(^7\). This explains why $p : q$ spin-orbit resonances are $(p,q)$–periodic orbits and vice versa.

(ii) The double periodicity of the Newtonian potential in (1.29) implies that the (extended) phase space for equation (1.24) is

$$\mathcal{M} := \{(x,t), \dot{x}) \in \mathbb{T}^2 \times \mathbb{R}\},$$

where $\mathbb{T}^2 = \mathbb{R}^2 / (2\pi \mathbb{Z}^2)$.

(iii) Notice that the winding (or rotation) number of a $(p,q)$–periodic orbit $x_{pq}(t)$ is equal to

$$\omega = \lim_{s \rightarrow \infty} \frac{1}{s} x_{pq}(t + s) = \frac{p}{q}.$$

\(^7\)Of course, in physical space, $x$ and $t$ being angles, are defined modulus $2\pi$, but to keep track of the topology (windings and rotations) one needs to consider them in the universal cover $\mathbb{R}$ of $\mathbb{R} / (2\pi \mathbb{Z})$. 
2 On the Fourier coefficients of the Newtonian Potential

In this chapter we analyse the Fourier coefficients \( \alpha_j = \alpha_j(e) \) of the Newtonian potential \( f \) given in equation (1.29), i.e.

\[
f(x, t; e) = \sum_{j \in \mathbb{Z}} \alpha_j(e) \cos(2x - jt).
\] (2.1)

In particular in Subsection 2.1 we give two equivalent integral formulas to compute \( \alpha_j \) for all \( j \in \mathbb{Z} \). In Subsection 2.2 we study the asymptotic behaviour in the eccentricity of \( \alpha_j(e) \) for all \( j \in \mathbb{Z} \). Both results will be very useful in the following chapters of this thesis.

2.1 Two Integral Formulas for \( \alpha_j \)

The Fourier coefficients \( \alpha_j \) coincide with the Fourier coefficients \( G_j(e) = G_j \) of

\[
G_e(t) := -\frac{e^{2i\tilde{f}_e(t)}}{2\tilde{\rho}_e(t)^3} = \sum_{j \in \mathbb{Z}} G_j \exp(ijt),
\] (2.2)

where \( \rho_e(t) \) and \( f_e(t) \) are defined in (1.30) and (1.31), respectively. The following lemma states this result and gives a first integral formula for the coefficients \( \alpha_j \) with \( j \in \mathbb{Z} \).

**Lemma 2.1.** The coefficients \( \alpha_j = \alpha_j(e) \) defined in (2.1) satisfy

\[
\alpha_j = \alpha_j(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(2\tilde{f}_e(u) - ju + e\sin(u))}{\tilde{\rho}_e^2} du = G_j,
\] (2.3)

where \( G_j \) for \( j \in \mathbb{Z} \) are defined in (2.2) and

\[
\tilde{f}_e = \tilde{f}_e(u) := 2\arctan\left(\sqrt{1 + \frac{e}{1 - e}} \tan\left(\frac{u}{2}\right)\right),
\]

\[
\tilde{\rho}_e = \tilde{\rho}_e(u) := 1 - e\cos(u).
\]

holds. In particular, \( \alpha_0 = \alpha_0(e) = 0 \) for \( e \in [0, 1) \) holds.

**Remark 2.1.** Since \( \rho_e(t) \) in (1.30) is even and \( f_e(t) \) in (1.31) is odd, by (2.2) we find that

\[
\sum_{j \in \mathbb{Z}} G_je^{-ijt} = G_e(-t) = \overline{G_e(t)} = \sum_{j \in \mathbb{Z}} G_je^{ijt},
\]

holds, proving that the coefficients \( G_j \) are real for all \( j \in \mathbb{Z} \).
Proof. (We follow the proof of Lemma 2.1 given in Appendix A of [8])

By definition of \( f \) in (1.29) it follows

\[
f(x, t; e) = -\frac{1}{2\rho_e(t)^3} \cos(2x - 2f_e(t)) = \text{Re} \left( -\frac{1}{2} e^{i2x} \cdot \frac{e^{-2f_e(t)}}{\rho_e(t)^3} \right).
\]

By Remark 2.1 one gets

\[
f(x, t; e) = \text{Re} \left( e^{i2x} G_e(-t) \right) = \text{Re} \left( \sum_{j \in \mathbb{Z}} G_j e^{-ijt} \right) = \sum_{j \in \mathbb{Z}} G_j \cos(2x - jt),
\]

proving that \( \alpha_j = G_j \) for all \( j \in \mathbb{Z} \). Furthermore, by definition of the Fourier coefficients of a \( 2\pi \)-periodic function we have

\[
\alpha_j = G_j := \frac{1}{2\pi} \int_0^{2\pi} G_e(t) e^{-ijt} dt \\
\overset{(2.2)}{=} -\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i2f_e(t) - ij} - e^{-i2f_e(t) - ij}}{\rho_e(t)^3} dt \\
= -\frac{1}{4\pi} \int_0^{2\pi} \left[ \frac{\cos(2f_e(t) - jt)}{\rho_e(t)^3} + i \frac{\sin(2f_e(t) - jt)}{\rho_e(t)^3} \right] dt.
\]

Since \( H_e(t) \) is a \( 2\pi \)-periodic and odd function, \( \int_0^{2\pi} H_e(t) dt = 0 \) holds. Therefore one gets

\[
\alpha_j = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(2f_e(t) - jt)}{\rho_e(t)^3} dt.
\]

From the Kepler equation (1.32) and (1.30), it follows that

\[
u_e(t)' = \frac{1}{\rho_e(t)}.
\]

Changing the variable of integration from \( t \) to \( u = u_e \) we obtain

\[
\alpha_j = \alpha_j(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(2\tilde{r}_e(u) - ju + ej \sin(u))}{\tilde{\rho}_e(u)^2} du, \tag{2.4}
\]

for \( j \in \mathbb{Z} \). In order to finish the proof of Lemma 2.1 we need to check the hypothesis \( \alpha_0 = \alpha_0(e) = 0 \) for \( e \in [0, 1) \) for the special case \( j = 0 \). First we prove the following claim.

Claim:

\[
\cos(2\tilde{r}_e) = \frac{3e^2 - 4e \cos(u) + (2 - e^2) \cos(2u)}{2\tilde{\rho}_e^2}. \tag{2.5}
\]

For every \( c \in \mathbb{R} \) the equality

\[
c^2 = \tan^2(\arctan(c)) = \frac{1 - \cos^2(\arctan(c))}{\cos^2(\arctan(c))}
\]
Then we have:

\[ \cos^2(\arctan(c)) = \frac{1}{1 + c^2}. \quad (2.6) \]

In particular choosing \( c = \sqrt{\frac{1 + e}{1 - e}} \tan\left(\frac{\pi}{2}\right) \) we get

\[ \cos^2\left(\arctan\left(\sqrt{\frac{1 + e}{1 - e}} \tan\left(\frac{u}{2}\right)\right)\right) = \frac{1}{1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)}. \quad (2.7) \]

Since for every angle \( \beta \) the relation \( \cos(2\beta) = 2\cos^2(\beta) - 1 \) holds, it follows

\[ \cos(4\beta) = 2\cos^2(2\beta) - 1 = 2\left[2\cos^2(\beta) - 1\right]^2 - 1 = 8\cos^4(\beta) - 8\cos^2(\beta) + 1. \quad (2.8) \]

From (1.31), (2.7) and (2.8) choosing \( \beta = \arctan\left(\sqrt{\frac{1 + e}{1 - e}} \tan\left(\frac{\pi}{2}\right)\right) \) we get

\begin{align*}
\cos(2\tilde{\varphi}) & = \cos\left(4 \arctan\left(\sqrt{\frac{1 + e}{1 - e}} \tan\left(\frac{u}{2}\right)\right)\right) \\
& = 8 \left[1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)\right]^2 - 8 \left[1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)\right] + 1 \\
& = \frac{8 - 8 \left[1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)\right] + \left[1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)\right]^2}{\left[1 + \frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right)\right]^2} \\
& = \frac{1 - 6\frac{1 + e}{1 - e} \tan^2\left(\frac{u}{2}\right) + \left(\frac{1 + e}{1 - e}\right)^2 \tan^4\left(\frac{u}{2}\right)}{(1 - e)^2 \left[(1 - e) + (1 + e) \tan^2\left(\frac{u}{2}\right)\right]^2} \\
& = \frac{(1 - e)^2 - 6(1 - e)(1 + e) \tan^2\left(\frac{u}{2}\right) + (1 + e)^2 \tan^4\left(\frac{u}{2}\right)}{[(1 - e) + (1 + e) \tan^2\left(\frac{u}{2}\right)]^2}.
\end{align*}

Let define

\[ \iota := \tan\left(\frac{u}{2}\right). \quad (2.9) \]

Then we have:

\[ \cos(2\tilde{\varphi}) = \frac{(1 - e)^2 - 6(1 - e^2)\iota^2 + (1 + e)^2 \iota^4}{[(1 - e) + (1 + e)\iota^2]^2}. \]

Since \( \cos(u) = \frac{1 - \iota^2}{1 + \iota^2} \) and

\[ \left[(1 - e) + (1 + e) \tan^2\left(\frac{u}{2}\right)\right]^2 = (1 + \iota^2)^2 \left[1 - \cos(u)\right]^2 = (1 + \iota^2)^2 \rho_e^2, \]

hold, we have to prove that

\[ \frac{(1 - e)^2 - 6(1 - e^2)\iota^2 + (1 + e)^2 \iota^4}{(1 + \iota^2)^2} = \frac{3\iota^2 - 4\cos(u) + (2 - e^2)\cos(2u)}{2}, \]

which is equivalent to

\[ 2(1 - e)^2 - 12(1 - e^2)\iota^2 + 2(1 + e)^2 \iota^4 = (1 + \iota^2)^2 \left(3\iota^2 - 4\cos(u) + (2 - e^2)\cos(2u)\right). \quad (2.10) \]
The left hand side of (2.10) can be computed directly
\[
2(1 - e)^2 - 12(1 - e^2)\epsilon^2 + 2(1 + e)^2 \epsilon^4 = \\
= 2 - 4e + 2e^2 - 12\epsilon^2 + 12e^2\epsilon^2 + 2\epsilon^4 + 4e\epsilon^4 + 2e^2\epsilon^4. \tag{2.11}
\]
The substitution (2.9) implies \( \cos(u) = \frac{1 - \epsilon^2}{1 + \epsilon^2} \), \( \cos(2u) = 2\left[\frac{1 - \epsilon^2}{1 + \epsilon^2}\right]^2 - 1 \). Therefore the right hand side of (2.10) is given by:
\[
(1 + \epsilon^2)^2 \left( 3e^2 - 4e \cos(u) + (2 - e^2) \cos(2u) \right) = \\
= (1 + \epsilon^2)^2 \left( 3e^2 - 4e \frac{1 - \epsilon^2}{1 + \epsilon^2} + (2 - e^2) \left( 2\left[\frac{1 - \epsilon^2}{1 + \epsilon^2}\right]^2 - 1 \right) \right) \\
= 3e^2(1 + \epsilon^2)^2 - 4e(1 - \epsilon^2)(1 + \epsilon^2) + 2(2 - \epsilon^2)(1 - \epsilon^2)^2 - (2 - \epsilon^2)(1 + \epsilon^2)^2 \\
= 3e^2 + 6e^2\epsilon^2 + 3e^2\epsilon^4 - 4e + 4e\epsilon^4 + 4 - 8\epsilon^2 + 4\epsilon^4 - 2e^2 + 4e^2\epsilon^2 - 2e^2\epsilon^4 + \\
+ e^2 + 2e^2\epsilon^2 + e^4\epsilon^4 - 2 - 4\epsilon^2 - 2\epsilon^4 \\
= 2 - 4e + 2e^2 - 12\epsilon^2 + 12e^2\epsilon^2 + 2\epsilon^4 + 4e\epsilon^4 + 2e^2\epsilon^4. \tag{2.12}
\]
(2.11) and (2.12) are equal. This concludes the proof of Claim (2.5).

We are now able to prove that:
\[
\alpha_0 = \alpha_0(e) = 0, \quad \text{for} \quad e \in [0, 1). \tag{2.13}
\]

By (2.4) and (2.5) it follows:
\[
\alpha_0(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(\frac{2\pi}{\rho_\epsilon(u)})}{\rho_\epsilon(u)^2} du = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\cos(2\pi u)}{\rho_\epsilon(u)^2} du \\
= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{3e^2 - 4e \cos(u) + (2 - e^2) \cos(2u)}{2\rho_\epsilon(u)^4} du.
\]
The substitution (2.9), \( \cos(u) = \frac{1 - \epsilon^2}{1 + \epsilon^2} \), \( \sin(u) = \frac{2\epsilon}{1 + \epsilon^2} \) and \( du = \frac{2d\epsilon}{1 + \epsilon^2} \) implies
\[
\alpha_0(e) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{3e^2 - 4e \frac{1 - \epsilon^2}{1 + \epsilon^2} + (2 - e^2) \left[\left(\frac{1 - \epsilon^2}{1 + \epsilon^2}\right)^2 - \left(\frac{2\epsilon}{1 + \epsilon^2}\right)^2\right]}{2 \left(\frac{1 - \epsilon^2}{1 + \epsilon^2}\right)^4} \frac{2d\epsilon}{1 + \epsilon^2} \\
= -\frac{1}{4\pi} \int_{-\infty}^{\infty} (1 + \epsilon^2) \frac{3e^2(1 + \epsilon^2)^2 - 4e(1 - \epsilon^4) + (2 - e^2) \left[ (1 - \epsilon^2)^2 - (2\epsilon)^2 \right]}{(1 + \epsilon^2)^2 - e(1 - \epsilon^2)^4} d\epsilon \\
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + \epsilon^2) \frac{1 - 6\epsilon^2 + 4\epsilon^4 - 2e(1 - \epsilon^4) + e^2(1 + 6\epsilon^2 + \epsilon^4)}{(1 + \epsilon^2)^2 - e(1 - \epsilon^2)^4} d\epsilon \\
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + \epsilon^2) \frac{(1 - \epsilon)^2 - 6(1 + e)(1 - \epsilon)\epsilon^2 + (1 + e)^2\epsilon^4}{(1 - e + (1 + e)\epsilon^2)^4} d\epsilon.
\]
Let us define \( \frac{1}{b_\epsilon} := \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \in (0, 1] \). Then:
\[
\alpha_0(e) = -\frac{1 + (b_\epsilon^{-1})^2)^2}{8\pi} \int_{-\infty}^{\infty} (1 + \epsilon^2) \frac{(b_\epsilon^{-1})^4 - 6(b_\epsilon^{-1})^2\epsilon^2 + \epsilon^4}{((b_\epsilon^{-1})^2 + \epsilon^2)^4} d\epsilon.
\]
Define the functions \( h(z) \) and \( \tilde{h}(z) \) as follow:
\[
h(z) := (1 + z^2)\frac{(b_\epsilon^{-1})^4 - 6(b_\epsilon^{-1})^2z^2 + z^4}{((b_\epsilon^{-1})^2 + z^2)^4}
\]
\[
\tilde{h}(z) := (1 + z^2)\frac{(b_\epsilon^{-1})^4 - 6(b_\epsilon^{-1})^2z^2 + z^4}{((b_\epsilon^{-1})^2 + z^2)^4}
\]
2.1. Two Integral Formulas for $\alpha_j$

\[ = (1 + z^2) \frac{(b_e^{-1})^4 - 6(b_e^{-1})^2 z^2 + z^4}{(z - ib_e^{-1})^4(z + ib_e^{-1})^4} = \frac{\tilde{h}(z)}{(z - ib_e^{-1})^4}. \]

$h(z)$ has a pole of order 4 (3 if $b_e^{-1} = 1$) in $z_e = ib_e^{-1}$. If $b_e^{-1} \neq 1$ we have

\[ \text{Res}_{z_e} h(z) = \frac{1}{6} \lim_{z \to z_e} \frac{\partial^3}{\partial z^3} [(z - z_e)^4 h(z)] = \frac{1}{6} \lim_{z \to z_e} \frac{\partial^3}{\partial z^3} \tilde{h}(z). \]

Define functions $p(z) := (1 + z^2)((b_e^{-1})^4 - 6(b_e^{-1})^2 z^2 + z^4)$ and $q(z) := (z + i(b_e^{-1}))^4$. Then it follows:

\[ \frac{\partial^3}{\partial z^3} \left( \frac{p(z)}{q(z)} \right) = \frac{\partial^2}{\partial z^2} \left( \frac{p'q - pq'}{q^2} \right) = \frac{\partial}{\partial z} \left( \frac{q^2(q''p' - q'p'') - 2q'(q'p'' - pq'')}{q^4} \right) \]

\[ = \frac{1}{q^3} \left[ (2q'q'' - pq''') + q^2(q'p'' + qp''' - p'q'' - pq''') \right. \]

\[ \left. - 2(q'q' - pq') - 2q(q''(qp' - pq') + q'(qp'' - pq'')) \right] q^4 \]

\[ - 4q^3q'(q''pq - pq'' - 2q'(qp' - pq')). \]

Inserting $z = z_e$ we get $\text{Res}_{z_e} h(z) = 0$ (we performed the computation with Mathematica). If $b_e^{-1} = 1$ we have

\[ \text{Res}_i h(z) = \frac{1}{2} \lim_{z \to i} \frac{\partial^2}{\partial z^2} [(z - i)^3 h(z)] = \lim_{z \to i} \frac{\partial^2}{\partial z^2} \frac{1 - 6z^2 + z^4}{(z + i)^3}. \]

Define functions $p(z) := 1 - 6z^2 + z^4$ and $q(z) := (z + i)^3$. Then it follows:

\[ \frac{\partial^2}{\partial z^2} \left( \frac{p(z)}{q(z)} \right) = \frac{\partial}{\partial z} \left( \frac{p'q - pq'}{q^2} \right) = \frac{q^2(q''p' - q'p'') - 2q'(q'p'' - pq'')}{q^4}. \]

Inserting $z = i$ we get $\text{Res}_i h(z) = 0$ (we performed the computation with Mathematica).

![Integration’s path](image)

Figure 2.1: Integration’s path $\gamma = \gamma_1 \circ \gamma_2$.

Let the path $\gamma := \gamma_1 \circ \gamma_2$, with $\gamma_1$ and $\gamma_2$ as in Figure 2.1. Then we have

\[ 0 = \text{Res}_i h(z) = \text{Res}_{z_e} h(z) \]

\[ = \int_{\gamma} h(z)dz = \int_{\gamma_1} h(z)dz + \int_{\gamma_2} h(z)dz \xrightarrow{R \to \infty} \int_{-\infty}^{\infty} h(z)dz, \]

since

\[ \left| \int_{\gamma_2} h(z)dz \right| = \left| \int_{0}^{\pi} h(Re^{i\phi})Re^{i\phi} d\phi \right| \leq O \left( \frac{1}{R} \right) R \xrightarrow{R \to \infty} 0. \]

This proves (2.13) and terminates the proof of Lemma 2.1. \qed
In the next lemma we provide a second equivalent integral formula for the Fourier coefficients \( \alpha_j \) of the Newtonian potential \( f \) in (1.29).

**Lemma 2.2.** For \( j \in \mathbb{Z} \setminus \{0\} \) the coefficients \( \alpha_j = \alpha_j(e) \) defined in (2.1) satisfy

\[
\alpha_j = -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{1}{\rho_e^2(w_e^2 + 1)^2} \left[ (w_e^4 - 6w_e^2 + 1)c_j(u) - 4w_e(w_e^2 - 1)s_j(u) \right] du,
\]

where \( w_e = w(u; e) := \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \), \( \rho_e := \rho_e(u) = 1 - e \cos u \) and

\[
c_j(u) := \cos(ju - je \sin u), \quad s_j(u) := \sin(ju - je \sin u).
\]

**Proof.** If \( z = \arctan w \), then

\[
e^{2iz} = \frac{i - w}{w + i} = \frac{(w - i)^2}{w^2 + 1},
\]

so that if \( w_e(t) := w(u_e(t), e) \) one has \( f_e(t) = 2 \arctan(w_e(t)) \) and

\[
G_e(t) \overset{(2.2)}{=} -\frac{1}{2\rho_e(t)^3} \left( e^{2i\frac{w_e(t)}{2}} \right)^2 \overset{(2.15)}{=} -\frac{1}{2\rho_e(t)^3} \left( \frac{w_e(t) - i}{w_e(t) + i} \right)^4
\]

\[
= -\frac{1}{2\rho_e(t)^3} \left( \frac{w_e(t)^4 - 6w_e(t)^2 + 1}{w_e(t)^2 + 1} \right)^2
\]

\[
= -\frac{1}{2\rho_e(t)^3} \left( \frac{w_e(t)^4 - 6w_e(t)^2 + 1 - 4iw_e(t)(w_e(t)^2 - 1)}{w_e(t)^2 + 1} \right). \quad (2.16)
\]

By Remark 2.1 we know that \( G_j \in \mathbb{R} \) for every \( j \in \mathbb{Z} \), i.e. \( G_j = \text{Re}(G_j) \). Thus, the following holds for \( j \in \mathbb{Z} \):

\[
\alpha_j = G_j = \text{Re} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} G_e(t) e^{-jt} dt \right]
\]

\[
(2.16) \overset{(2.16)}{=} -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{2\pi (w_e(t)^4 - 6w_e(t)^2 + 1) \cos(jt) - 4w_e(t)(w_e(t)^2 - 1) \sin(jt))}{\rho_e(t)^3(w_e(t)^2 + 1)^2} dt.
\]

Making again the change of variable given by the Kepler equation (1.32), i.e. integrating with respect to \( u \) instead of \( t \) and using \( u_e(t)' = \frac{1}{\rho_e(t)} \) we find

\[
\alpha_j = -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{(w_e^4 - 6w_e^2 + 1) \cos(ju - je \sin u) - 4w_e(w_e^2 - 1) \sin(ju - je \sin u)}{\rho_e^2(w_e^2 + 1)^2} du.
\]

Hence equation (2.14) holds. This terminates the proof of Lemma 2.2. \( \square \)

### 2.2 Asymptotic behaviour

The following theorem studies the expansion of the Fourier coefficients \( \alpha_j = \alpha_j(e) \), \( 0 \neq j \in \mathbb{Z} \) defined in (2.1) with respect to the eccentricity \( e \).

**Theorem 2.1.** Let \( \alpha_j(e) \) be as in (2.1). Then for all \( 0 \neq j \in \mathbb{Z} \) we have

\[
\alpha_j(e) = \bar{\alpha}_j e^{j\bar{e}} + O(e^{j\bar{e}+1}),
\]

where

\[
\bar{\alpha}_j = \begin{cases} 
P_{2j}(j), & \text{for } j \leq 2, \\
Q_{j-2}(j), & \text{for } j > 2.
\end{cases}
\]

(2.17)
2.2. Asymptotic behaviour

$(P_k)_{k \geq 0}$ and $(Q_k)_{k \geq 0}$ are two families of polynomial functions given by the relations

$$
P_k(x) := \left(-\frac{1}{2}\right)^{k+1} \frac{x^k}{k!}, \quad (2.19)$$

$$
Q_k(x) := -\frac{1}{2^{k+1}} \sum_{l=0}^{k} \frac{1}{l!} \left(\frac{k-l+3}{3}\right)^l x^l. \quad (2.20)
$$

Theorem 2.1 can be interpreted as a Corollary of A. Cayley [10], F. Tisserand [43] and W. M. Kaula [29], where the main question was about the two-body problem and not the spin-orbit problem. Many recent papers use Theorem 2.1 as a well known results, referring to [10], [43] and [29] for the proof.

However A. Cayley in [10] presents the problem using an old mathematical approach, which makes the proof difficult to be followed at the current days. On the other hand in Chapter 3.3 of [29] (see equations (3.67)-(3.71)) Kaula gives an explicit formula for $\bar{\alpha}_j$, but there is no proof for it. Anyway one can find those formulas in Remark 2.3. Finally F. Tisserand in [43] gives an elegant proof, due to Hansen, which is postponed to the following subsection. I found equations (2.19) and (2.20) independently and for completeness I will give my alternative proof of Theorem 2.1 in Subsection 2.2.2.

2.2.1 Hansen’s Proof

In the proof of Theorem 2.1 one uses the following result, which is due to Hansen (see Chapter XV of [43]).

**Lemma 2.3. (Hansen)** Let $u, t, f_e(t)$ and $\rho_e(t)$ be as in Chapter 1, $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and

$$
\beta := \frac{e}{1 + \sqrt{1 - e^2}} = \frac{1 - \sqrt{1 - e^2}}{e}. \quad (2.21)
$$

Furthermore define $P_{0,m}^{n,m} = Q_{0,m}^{n,m} := 1$ and for $k \geq 1$ define

$$
P_{k,m}^{n,m}(\nu) := \frac{\nu^k}{k!} + \sum_{l=0}^{k-1} \frac{\nu^l \hat{n}(\hat{n}-1)\ldots(\hat{n}-(k-l)+1)}{(k-l)!}, \quad (2.22)
$$

$$
Q_{k,m}^{n,m}(\nu) := (-1)^k \frac{\nu^k}{k!} + \sum_{l=0}^{k-1} \frac{\nu^l}{l!} (-1)^{\hat{n}} \frac{\hat{n}(\hat{n}-1)\ldots(\hat{n}-(k-l)+1)}{(k-l)!}, \quad (2.23)
$$

where

$$
\nu := \frac{je}{2\beta}, \quad \hat{n} = n + m + 1 \quad \text{and} \quad \hat{n} = n - m + 1. \quad (2.24)
$$

Then the coefficients $X_l^{n,m}$ in the expansion

$$(\rho_e(t))^n e^{im\epsilon(t)} =: \sum_{j \in \mathbb{Z}} X_j^{n,m} e^{ijt}, \quad (2.25)$$

are given, for $j \neq 0$, by the following formula:

$$
X_j^{n,m} = \begin{cases} 
(1 + \beta^2)^{-n-1}(\beta)^{j-m} \sum_{k=0}^{\infty} Q_{j-m+k}^{n,m}(\nu)P_{k}^{n,m}(\nu)\beta^{2k}, & \text{if } j > m, \\
(1 + \beta^2)^{-n-1}(\beta)^{m-j} \sum_{k=0}^{\infty} P_{m-j+k}^{n,m}(\nu)Q_{k}^{n,m}(\nu)\beta^{2k}, & \text{if } j \leq m.
\end{cases} \quad (2.26)
$$
Remark 2.2. Also the formula for $X_0^{n,m}$ is known, (see again Chapter XV of [43]), namely $X_0^{n,m} = 0$ if $n \geq -m - 1$ and

$$X_0^{n,m} = \frac{(-\beta)^m}{(1-\beta^2)^n+1} \frac{(n+2)(n+3) \cdots (n+m+1)}{m!} F(m-n-1,-n-1,m+1,\beta)$$

(2.27)

if $n < -m - 1$, where $F$ is the hypergeometric function.

Proof. Set

$$x = e^{i\varphi(t)}, \quad y = e^{iu} \quad \text{and} \quad z = e^{it}. \quad (2.28)$$

From the Kepler equation (1.32) it follows that

$$z = e^{it} = e^{i(\varphi(t)+\pi)} = ye^{-\frac{\pi}{2}(y-rac{1}{y})}$$

(2.29)

holds. Since $\tan(\alpha/2) = \frac{e^{\alpha/2}-1}{e^{\alpha/2}+1}$ holds for every angle $\alpha$, we can find the relation between $x$ and $y$ in (2.28). By (2.21) and (2.28) the equation (1.31) is then equivalent to

$$y - \frac{1}{y} + 1 = 1 - \beta \frac{x - 1}{x + 1},$$

since

$$\frac{1-\beta}{1+\beta} = \frac{1+\sqrt{1-e^2} - e}{1+\sqrt{1-e^2} + e} = \sqrt{1-e}.$$

Solving this equation with respect to $x$ we get

$$(1-x)(1+y - \beta - \beta y) = (1+x)(1-y + \beta - \beta y)$$

$$x = \frac{y - \beta}{1-\beta y} = y \frac{1 - \frac{\beta}{y}}{1 - \frac{\beta}{y}}. \quad (2.30)$$

Now we express the normalized radius $\rho_e(t)$ in (1.30) as function of $y$, i.e.

$$\rho_e(t) = 1 - \frac{e}{2} \left( y + \frac{1}{y} \right) = -\frac{ey^2 - 2y + e}{2y}$$

$$= -\frac{e}{2y} \left( y - \frac{1 + \sqrt{1-e^2}}{e} \right) \left( y - \frac{1 - \sqrt{1-e^2}}{e} \right)$$

$$= -\frac{e}{2y} (y - \beta) \left( y - \frac{1}{\beta} \right) = \frac{e}{2\beta} (1 - \beta y) \left( 1 - \frac{\beta}{y} \right)$$

(2.31)

Furthermore, from the Kepler equation (1.32) we get

$$dt = (1 - e \cos(u)) du = \frac{r}{a} du. \quad (2.32)$$

Now multiplying (2.25) by $z^{-j} = e^{-ijt}$ and integrating from 0 to $2\pi$ with respect to $t$, we get

$$X_j^{n,m} = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r}{a} \right)^n x^m z^{-j} dt = \frac{1}{2\pi} \int_0^{2\pi} \rho_e(t)^n x^m z^{-j} dt. \quad (2.33)$$

Using (2.29), (2.30), (2.31) and (2.32) the equation (2.33) becomes

$$2\pi X_j^{n,m} = \int_0^{2\pi} \left( \frac{1}{1+\beta^2} (1-\beta y) \left( 1 - \frac{\beta}{y} \right) \right)^n \left( y - \frac{1 - \frac{\beta}{y}}{1 - \beta y} \right)^m \left( ye^{-\frac{\pi}{2}(y-rac{1}{y})} \right)^{-j} du$$
\[ = (1 + \beta^2)^{-n-1} \int_0^{2\pi} y^{m-j}(1 - \beta y)^{n-m+1} \left( 1 - \frac{\beta}{y} \right)^{n+m+1} e^{j\frac{\pi}{2}(y-\frac{1}{y})} du. \] (2.34)

The terms \( y^{m-j}, (1 - \beta y)^{n-m+1}, \left( 1 - \frac{\beta}{y} \right)^{n+m+1} \) and \( e^{j\frac{\pi}{2}(y-\frac{1}{y})} \) are functions, which may be expanded in a power series of \( y \). We define

\[ \Phi = (1 - \beta y)^{n-m+1} \left( 1 - \frac{\beta}{y} \right)^{n+m+1} e^{j\frac{\pi}{2}(y-\frac{1}{y})}. \] (2.35)

For \( q \in \mathbb{Z} \) we have

\[ \int_0^{2\pi} y^{q} du = \begin{cases} 2\pi, & \text{if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases} \] (2.36)

Let expand \( \Phi \) in (2.35) in a power series of \( y \) and denote the coefficient \( y^{j-m} \) by \( A \).

Now equation (2.35) can be equivalently rewritten as

\[ \Phi = \Theta \Theta_1, \] (2.38)

where

\[ \Theta(y, n, m, \nu) = (1 - \beta y)^{n-m+1} e^{\nu \beta y}, \quad (2.39) \]

\[ \Theta_1(y, n, m, \nu) = \left( 1 - \frac{\beta}{y} \right)^{n+m+1} e^{-\frac{\nu \beta}{y}} = \Theta \left( \frac{1}{y}, n, -m, -\nu \right), \quad (2.40) \]

and \( \nu \) is defined in (2.24). Notice that, if we find the expansion of \( \Theta \), we have automatically also the expansion of \( \Theta_1 \). By definition of the exponential function we know that

\[ e^{\nu \beta y} = \sum_{k=0}^{\infty} \frac{(\nu \beta y)^k}{k!} = 1 + \frac{\nu \beta}{1} y + \frac{\nu^2 \beta^2}{2!} y^2 + \ldots. \]

Since

\[ \frac{d^k}{dy^k}(1 - \beta y)^r \bigg|_{y=0} = (1 - \beta y)^{r-k}(-\beta)^k r(r-1) \ldots (r-k+1) \bigg|_{y=0} = (-\beta)^k r(r-1) \ldots (r-k+1) \]

holds, we get by the Taylor Theorem

\[ (1 - \beta y)^r = \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} (\beta y)^k. \]

Choosing \( r = n - m + 1 \) and multiplying the two series together we find

\[ \Theta \overset{(2.39)}{=} e^{\nu \beta y}(1 - \beta y)^{n-m+1} = \sum_{k=0}^{\infty} \frac{(\nu \beta y)^k}{k!} \sum_{l=0}^{k} (-1)^l \binom{r}{l} (\beta y)^l. \]
The Cauchy product of two series tells us that
\[
\Theta = \sum_{k=0}^{\infty} (-1)^{k-l} \frac{r^k}{k!} \left( \frac{\nu}{k-l} \right) (\beta y)^k = \sum_{k=0}^{\infty} (-1)^k Q_{n,m}^k (\beta y)^k
\]
holds by (2.23). The same for \( \Theta_1 \), i.e. defining \( \tilde{r} = n + m + 1 \) we have that
\[
\Theta_1 = \Theta \left( \frac{1}{y}, n, -m, -\nu \right) = \sum_{k=0}^{\infty} (-1)^k \sum_{l=0}^{\nu} \frac{\nu^l}{l!} \left( \frac{\tilde{r}}{k-l} \right) (\beta y)^k =: \sum_{k=0}^{\infty} (-1)^k P_{n,m}^k (\beta y)^k
\]
holds by (2.22). Since \( A \) in (2.37) is the coefficient of \( y^{j-m} \) of \( \Phi \), and \( \Phi \) is given by
\[
\Phi = \Theta \Theta_1
\]
we have to take \( k + l = j - m \) in order to get \( A \). We distinguish two cases:

(i) If \( j - m > 0 \) then we have
\[
A = (-1)^{j-m} \beta^{j-m} \sum_{k=0}^{\infty} Q_{j-m+k}^{n,m} P_k^{n,m} \beta^{2k}.
\]

(ii) If \( j - m < 0 \) then we have
\[
A = (-\beta)^{m-j} \sum_{k=0}^{\infty} P_{m-j+k}^{n,m} Q_k^{n,m} \beta^{2k}.
\]

Equation (2.26) follows directly by (2.37), (2.42) and (2.43). This terminates the proof of Lemma 2.3.

**Proof.** (Proof of Theorem 2.1) By (2.1), (2.2), (2.3) and Remark 2.1 we have
\[
-\frac{1}{2\rho_\alpha(t)} e^{2\alpha(t)} = \sum_{j\neq 0} \alpha_j(e) e^{ijt}
\]
and, therefore, by (2.25) with \( n = -3, \ m = 2 \) we get
\[
\alpha_j = -\frac{X_{j,0}^{-3,2}}{2}, \quad \text{for all } 0 \neq j \in \mathbb{Z}.
\]

In order to find \( \bar{\alpha}_j \) in (2.17), we need to find the leading coefficient in the expansion of \( -\frac{X_{j,0}^{-3,2}}{2} \) with respect to \( e \), i.e. we must compute \( X_{j,0}^{-3,2} \) in
\[
X_j^{-3,2} = X_{j,0}^{-3,2} e^{\left|j-2\right|} + O(e^{\left|j-2\right|+1}).
\]
Let us recall that we have to prove (2.17) and (2.18) only for \( j \neq 0 \). Since
\[
\alpha = (1 + \beta^2)^{-n-1}(2.21) = 1 + O(e), \quad \forall n \in \mathbb{Z},
\]
\[
\beta_{|j-2|} = \frac{e^{|j-2|}}{2|j-2|} + O(e^{|j-2|+2}),
\]
we get from (2.26) for \( n = -3, m = 2 \) that
\[
X_{j,0}^{-3} = \begin{cases} \left( -\frac{1}{2} \right)^{j-2}Q_{j-2}^0P_{0}^0, & \text{if } j > 2, \\ \left( -\frac{1}{2} \right)^{2-j}P_{2-j}^0Q_{0}^0, & \text{if } 0 \neq j \leq 2, \end{cases}
\]
holds, where \( Q_{j-2}^0 := Q_{j-2}^{-3}(j), \ P_{2-j}^0 := P_{2-j}^{-3}(j) \). Namely \( P_{2-j}^0 \) and \( Q_{j-2}^0 \) are the zero coefficients in the \( e \)-expansion of \( P_{2-j}^{-3}, Q_{j-2}^{-3} \) defined in (2.22) and (2.23), respectively.

Let us consider first the case \( 0 \neq j \leq 2 \). Since \( \bar{n} = n + m + 1 = -3 + 2 + 1 = 0 \) by (2.22) we get
\[
P_{2-j}^0 = P_{2-j}^{-3}(j) = \frac{j^{2-j}}{(2-j)!}.
\]
Since \( Q_0^0 = 1 \) (recall that \( Q_0^0 \equiv 1 \)) it follows
\[
\alpha_j = -\frac{X_{j,0}^{-3}}{2} = \left( -\frac{1}{2} \right)^{3-j}P_{2-j}^0 = \left( -\frac{1}{2} \right)^{3-j} \frac{j^{2-j}}{(2-j)!} = P_{2-j}(j),
\]
which proves Theorem 2.1 for \( 0 \neq j \leq 2 \).

In the case \( j > 2 \), we have \( \bar{n} = n + m + 1 = -3 - 2 + 1 = -4 \) and from (2.23) we get
\[
Q_{j-2}^0 = Q_{j-2}^{-3}(j) = (-1)^j \sum_{l=0}^{j-2} \frac{j!}{l!(j-l-2)!} = \sum_{l=0}^{j-2} \frac{j!}{l!} \left( j-l+1 \right) \left( j-l+2 \right) \ldots \left( j-l+3 \right).
\]

Since \( P_0^0 = 1 \) (recall that \( P_0 \equiv 1 \)) we have
\[
\alpha_j = -\frac{X_{j,0}^{-3}}{2} = \left( -\frac{1}{2} \right)^{j-1}Q_{j-2}^0 = \left( -\frac{1}{2} \right)^{j-1} \frac{1}{2j-1} \sum_{l=0}^{j-2} \left( j+1-l \right) \frac{j!}{l!}
\]
\[
= Q_{j-2}(j).
\]
This concludes the proof of Theorem 2.1 also in the case \( j > 2 \). \( \square \)

**Remark 2.3.** In [29] Kaula uses coefficients \( G_{l,p,q} \) with \( l, p, q \in \mathbb{Z} \) to describe the Newtonian potential. The relation between \( G_{l,p,q} \) and \( \alpha_j \) in (2.1) is
\[
\alpha_j(e) = \frac{-G_{2,0,j-2}(e)}{2},
\]
i.e. we have to choose \( l = 2, p = 0 \) and \( j = q + 2 \) to get the spin-orbit case. Here we write (without proof) the formula for the coefficients \( G_{l,p,q} \).

If \( l - 2p + q = 0 \) define

\[
G_{lpq}(e) = \frac{1}{(1 - e^2)^{l/2}} \sum_{d=0}^{p'+1} \left( \frac{l - 1}{2d + l - 2p'} \right) \left( \frac{d}{2} \right)^{2d + l - 2p'},
\]

where \( p' = \begin{cases} p, & \text{for } p \leq l/2, \\ l - p, & \text{for } p \geq l/2. \end{cases} \)

If \( l - 2p + q \neq 0 \) define

\[
G_{lpq}(e) = (-1)^{|q|} (1 + \beta^2)^{|q|} \sum_{k=0}^{\infty} P_{lpqk} Q_{lpqk} \beta^{2k},
\]

where

\[
\beta = \frac{e}{1 + \sqrt{1 - e^2}},
\]

\[
P_{lpqk} = \sum_{r=0}^{h} \left( \frac{2p - 2l}{h - r} \right) \frac{(-1)^r}{r!} \left( \frac{(l - 2p' + q')e}{2\beta} \right)^r,
\]

\[
h = \begin{cases} k + q', & \text{for } q' > 0, \\ k, & \text{for } q' < 0, \end{cases}
\]

\[
Q_{lpqk} = \sum_{r=0}^{h} \left( \frac{-2p'}{h - r} \right) \frac{1}{r!} \left( \frac{(l - 2p' + q')e}{2\beta} \right)^r,
\]

\[
h = \begin{cases} k, & \text{for } q' > 0, \\ k - q', & \text{for } q' < 0, \end{cases}
\]

and

\[
(p', q') = \begin{cases} (p, q), & \text{for } p \leq l/2, \\ (l - p, -q), & \text{for } p > l/2. \end{cases}
\]

### 2.2.2 Alternative Proof of Theorem 2.1

Before starting with an alternative proof of Theorem 2.1, which was found independently of Hansen (see [43]), we need some preparatory lemmata.

**Lemma 2.4.** Let \((Q_k)_{k \geq 0}\) be defined as in (2.20). Then

\[
2(k + 1)Q_{k+1}(x) = (x + k + 4)Q_k(x) - \frac{x}{2} Q_{k-1}(x)
\]

holds for all \( k \geq 1 \).

**Proof.** With some straightforward computations we get

\[
(x + k + 4)Q_k(x) - \frac{x}{2} Q_{k-1}(x) =
\]

\[
= \frac{(x + k + 4)Q_k(x)}{2k+1} \sum_{l=0}^{k} \frac{1}{l!} \binom{k - l + 3}{3} x^l + \frac{x}{2k+1} \sum_{l=0}^{k-1} \frac{1}{l!} \binom{k - l + 2}{3} x^l
\]

\[
= -\frac{1}{2k+1} \sum_{l=0}^{k} \frac{1}{(l+1)!} \binom{k - l + 4}{3} x^l - \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l
\]

\[
= \frac{-1}{2k+1} x^{k+1} \left[ \left( \frac{k + 4}{3} \right) + \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l \right]
\]

\[
= \frac{1}{2k+1} x^{k+1} \left[ \left( \frac{k + 4}{3} \right) - \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l \right]
\]

\[
= \frac{1}{2k+1} x^{k+1} \left[ \left( \frac{k + 4}{3} \right) + \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l \right]
\]

\[
= \frac{1}{2k+1} x^{k+1} \left[ \left( \frac{k + 4}{3} \right) + \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l \right]
\]

\[
= \frac{1}{2k+1} x^{k+1} \left[ \left( \frac{k + 4}{3} \right) + \frac{1}{2k+1} \sum_{l=1}^{k} \frac{1}{(l-1)!} \binom{k - l + 4}{3} x^l \right]
\]
2.2. Asymptotic behaviour

\[ + \frac{1}{l!} \left( \begin{array}{c} k - l + 3 \\ 3 \end{array} \right) (k + 4) - \frac{1}{(l-1)!} \left( \begin{array}{c} k - l + 3 \\ 3 \end{array} \right) x^l \]

\[ = - \frac{(k+1)}{2^{k+1}} \sum_{l=0}^{k+1} \frac{1}{l!} \left( \begin{array}{c} k - l + 4 \\ 3 \end{array} \right) x^l = 2(k+1)Q_{k+1}(x). \]

Lemma 2.5. Let \( \tilde{\tau}_e = \tilde{\tau}_e(u) \) be as in Lemma 2.1. In order to simplify the notation we write \( D := \frac{\partial}{\partial e}, \, \iota := \tan \left( \frac{u}{2} \right), \, b_e := \sqrt{\frac{1+b_e}{1-b_e}}. \) Then for all \( k \geq 1 \)

\[ D^k \tilde{\tau}_e = \sum_{l=1}^{k} A_k^l(e)g_l, \] (2.48)

holds, where

\[ g_l := \frac{\iota^{2l-1}}{(1 + (b_e \iota)^2)^l} \]

and the coefficients \( A_k^l(e) \) are recursively defined as follows:

\[
\begin{align*}
A_1^1(e) &= 2b_e', \\
A_{k+1}^l(e) &= D A_k^l(e), \\
&= \begin{cases} D A_k^l(e) - 2(l-1)b_e b_e' A_{k-1}^l(e), & \text{for } l = 2 \leq l \leq k, \\
-2kb_e b_e' A_k^l(e), & \text{for } l = k + 1. \end{cases}
\end{align*}
\] (2.49)

Remark 2.4. We observe that

\[ Dg_l = -2lb_e b_e' g_{l+1} \]

holds for all \( l \geq 1. \)

Proof. (Induction with respect to \( k \))

\( k = 1: \) By definition of \( \tilde{\tau}_e \) in Lemma 2.1 we have \( D \tilde{\tau}_e = \frac{\partial}{\partial e} \tilde{\tau}_e(u) = \frac{2b_e'}{1+(b_e \iota)^2}. \) On the other hand by (2.48) and (2.49) we get \( D \tilde{\tau}_e = A_1^1(e)g_1 = 2b_e' \frac{1}{1+(b_e \iota)^2}, \) which proves the assertion for \( k = 1. \)

\( k \to k + 1: \) Induction step:

\[
D^{k+1} \tilde{\tau}_e = D \left[ D^k \tilde{\tau}_e \right] \]

\[ = D \left[ \sum_{l=1}^{k} A_k^l(e)g_l \right] \]

\[ = \sum_{l=1}^{k} \left[ (DA_k^l(e))g_l + A_k^l(e)Dg_l \right] \]

\[ \text{Rmk. 2.4:} \sum_{l=1}^{k} \left[ (DA_k^l(e))g_l - 2lb_e b_e' A_k^l(e)g_{l+1} \right] \]

\[ = (DA_1^1(e))g_1 + \sum_{l=2}^{k} \left[ (DA_k^l(e)) - 2(l-1)b_e b_e' A_{k-1}^l(e) \right] g_l \\
-2kb_e b_e' A_k^k(e)g_{k+1} \]
This concludes the proof of Lemma 2.5. \(\blacksquare\)

**Remark 2.5.** \(\tilde{I}_e\) defined in Lemma 2.1 is an analytic function of \(e\). Therefore it can be expanded about \(e = 0\), i.e.

\[
\tilde{I}_e(u) = u + e g(u; e),
\]

where

\[
g(u; e) = \sum_{k=1}^{\infty} g_k(u) e^{k-1}
\]

holds for some odd functions \(g_k(u)\), which will be analysed in details in the next lemma.

**Lemma 2.6.** Let \(g(u; e)\) be as in Remark 2.5. Then for all \(k \geq 1\)

\[
g_k(u) = \sum_{l=1}^{k} g_{k,l} \sin(lu),
\]

(2.50)

holds with \(g_{k,l} \in \mathbb{R}\) and \(g_{k,k} = \frac{1}{k^2 - 1}\).

**Proof.** From the Taylor expansion of \(\tilde{I}_e\) around \(e = 0\) we know that

\[
g_k(u) = \frac{D^k \tilde{I}_e}{k!} \bigg|_{e=0}
\]

holds for all \(k \geq 1\). Using the same notation as in Lemma 2.5, we define

\[
A_{k,l} := A_l^k(0).
\]

From (2.48) and (2.49) it follows

\[
g_k(u) = \frac{1}{k!} \sum_{l=1}^{k} A_l^k(0) g_l = \frac{1}{k!} \sum_{l=1}^{k} A_{k,l} \frac{u^{2l-1}}{(1 + u^2)^l}.
\]

Recall that \(\iota = \tan \left( \frac{u}{2} \right)\). So for all \(l \geq 1\) we have

\[
\frac{u^{2l-1}}{(1 + u^2)^l} = \frac{\sin^{2l-1} \left( \frac{u}{2} \right) \cos^{2l-1} \left( \frac{u}{2} \right)}{\cos^{2l-1} \left( \frac{u}{2} \right)} = \cos \left( \frac{u}{2} \right) \sin^{2l-1} \left( \frac{u}{2} \right)
\]

\[
= \cos \left( \frac{u}{2} \right) \sin \left( \frac{u}{2} \right) \left[ \sin^2 \left( \frac{u}{2} \right) \right]^{l-1} = \frac{1}{2} \sin(u) \left[ 1 - \cos(u) \right]^{l-1}
\]

\[
= \frac{1}{2^l} \sin(u) \left[ 1 - \cos(u) \right]^{l-1}.
\]

Using the binomial formula we obtain

\[
\frac{u^{2l-1}}{(1 + u^2)^l} = \frac{1}{2^l} \sin(u) \sum_{i=0}^{l-1} (-1)^i \binom{l-1}{i} \cos^i(u), \quad \text{for all } l \geq 1,
\]

and therefore

\[
g_k(u) = \frac{1}{k!} \sum_{l=1}^{k} A_{k,l} \frac{1}{2^l} \sin(u) \sum_{i=0}^{l-1} (-1)^i \binom{l-1}{i} \cos^i(u)
\]

\[
\text{for all } k \geq 1.
\]
2.2. Asymptotic behaviour

\[ k - \sum_{i=0}^{k-1} \left( \sum_{l=i+1}^{k} \frac{A_{k,l}(-1)^i \binom{l-1}{i}}{k! 2^l} \right) \sin(u) \cos^i(u). \]

Using the trigonometric identity \( \cos(\beta_1) \sin(\beta_2) = \frac{1}{2} [\sin(\beta_1 + \beta_2) - \sin(\beta_1 - \beta_2)] \) we get (2.50). Furthermore using equation (2.49) recursively, it is easy to verify that

\[ g_{k,k} = \frac{(-1)^{k-1}}{k! 2^{2k-1}} A_{k,k} (-2(k-1)) A_{k-1,k-1} \]

\[ = \ldots \]

\[ = \frac{(-1)^{k-1}}{k! 2^{2k-1}} (-2)^{k-1}(k-1)! A_{1,1} = \frac{1}{k 2^{k-1}}, \]

since \( A_{1,1} = 2 \).

**Remark 2.6.** Let \( g(u; e) \) as in Remark 2.5. Then we define

\[ \tilde{g}(u; e, j) := 2g(u; e) + j \sin(u) = \sum_{k=1}^{\infty} \tilde{g}_k^j(u) e^{k-1}, \]

where \( \tilde{g}_k(u) \) are given by

\[ \tilde{g}_k^j(u) := \begin{cases} 2g_1(u) + j \sin(u), & \text{for } k = 1, \\ 2g_k(u), & \text{for } k \geq 2. \end{cases} \]

From Lemma 2.6 we know that

\[ \tilde{g}_k^j(u) = \sum_{l=1}^{k} \tilde{g}_{k,l}^j \sin(lu), \]

holds with

\[ \tilde{g}_{k,k}^j := \begin{cases} 2 + j, & \text{for } k = 1, \\ \frac{1}{k 2^{k-2}}, & \text{for } k \geq 2. \end{cases} \]

**Lemma 2.7.** Let \( \tilde{g}(u; e, j) \) be defined as in equation (2.51). Then we have

\[ \frac{\cos(e \tilde{g}(u; e, j))}{(1 - e \cos(u))^2} = \sum_{k=0}^{\infty} c_k^j(u) e^k, \]

\[ \frac{\sin(e \tilde{g}(u; e, j))}{(1 - e \cos(u))^2} = \sum_{k=0}^{\infty} s_k^j(u) e^k, \]

where \( c_k^j(u), s_k^j(u) \) have the Fourier expansions

\[ c_k^j(u) := \sum_{l=0}^{k} c_{k,l}^j \cos(lu), \]

\[ s_k^j(u) := \sum_{l=1}^{k} s_{k,l}^j \sin(lu), \]

for some real coefficients \( c_{k,l}^j, s_{k,l}^j \).
Proof. Let us define

\[ C(u; e, j) := \cos \left( \sum_{k \geq 1} \tilde{g}_k^j(u) e^k \right) \quad \text{and} \quad S(u; e, j) := \sin \left( \sum_{k \geq 1} \tilde{g}_k^j(u) e^k \right), \]

where \( \tilde{g}_k^j(u) \) are defined in (2.52). Since \( g \) and therefore also \( \tilde{g} \) are odd with respect to \( u \), it is easy to verify that \( C(u; e, j) \) and \( S(u; e, j) \) are even and odd in the variable \( u \), respectively. Therefore \( C \) will have only cosine-terms in is expansion and \( S \) only sine-terms. Furthermore \( C(u; e, j) \) and \( S(u; e, j) \) are analytic in \( e \). We make the following formal Ansatz for their power series

\[ C(u; e, j) = \sum_{k \geq 0} C_k^j(u) e^k, \quad S(u; e, j) = \sum_{k \geq 0} S_k^j(u) e^k. \]

In [45] we find recursive formula to compute the coefficients \( C_k^j(u), S_k^j(u) \):

\[
\begin{cases}
    C_0^j(u) = 1, & C_k^j(u) = -\frac{1}{k} \sum_{l=1}^{k} l \tilde{g}_l^j(u) S_{k-l}^j(u), \\
    S_0^j(u) = 0, & S_k^j(u) = \frac{1}{k} \sum_{l=1}^{k} l \tilde{g}_l^j(u) C_{k-l}^j(u). 
\end{cases} \tag{2.58}
\]

Using induction with respect to \( k \) and the fact that \( \tilde{g}_k^j(u) \) are trigonometric polynomials of degree \( k \) (due to (2.53)), from (2.58) it is easy to prove that for some constant \( C_{k,l}, S_{k,l} \in \mathbb{R} \) the functions \( C_k^j(u), S_k^j(u) \) have the form

\[
C_k^j(u) := \sum_{l=0}^{k} C_{k,l}^j \cos(lu), \quad \tag{2.59}
\]

\[
S_k^j(u) := \sum_{l=0}^{k} S_{k,l}^j \sin(lu), \quad \tag{2.60}
\]

Furthermore using the geometric series expansion of the function \( \frac{1}{(1 - e \cos(u))^2} \) with respect to \( e \), i.e.

\[
\frac{1}{(1 - e \cos(u))^2} = \sum_{k=1}^{\infty} k \cos^{k-1}(u) e^{k-1},
\]

and the trigonometric identities \( \cos(\beta_1) \cos(\beta_2) = \frac{1}{2} (\cos(\beta_1 + \beta_2) + \cos(\beta_1 - \beta_2)) \) and \( \cos(\beta_1) \sin(\beta_2) = \frac{1}{2} (\sin(\beta_1 + \beta_2) - \sin(\beta_1 - \beta_2)) \) we get (2.55), (2.56) and (2.57).

Remark 2.7. Recall that: \( C_k^j(u) \) and \( S_k^j(u) \) are even and odd trigonometric polynomials of degree \( k \), respectively (see (2.59) and (2.60)); the trigonometric identities

\[
\sin(\beta_1) \sin(\beta_2) = \frac{1}{2} \left[ -\cos(\beta_1 + \beta_2) + \cos(\beta_1 - \beta_2) \right],
\]

\[
\sin(\beta_1) \cos(\beta_2) = \frac{1}{2} \left[ \sin(\beta_1 + \beta_2) - \sin(\beta_1 - \beta_2) \right]
\]

hold; \( \tilde{g}_k^j(u) \) has the form (2.53). Then for the coefficients \( C_{k,k}^j \) and \( S_{k,k}^j \) in (2.59) and (2.60) it follows that

\[
\begin{cases}
    C_{0,0}^j = 1, & C_{k,k}^j = \frac{1}{k} \sum_{l=1}^{k-1} l \tilde{g}_l^j S_{k-l-1}^j, \\
    S_{0,0}^j = 0, & S_{k,k}^j = \frac{1}{k} \sum_{l=1}^{k-1} l \tilde{g}_l^j C_{k-l-1}^j + \tilde{g}_{k,k}^j,
\end{cases} \tag{2.61}
\]

hold, where \( \tilde{g}_{k,k}^j \) is given in (2.54).
Remark 2.8. Using (2.61) we compute $C^j_{1,1} = 0$ and $S^j_{1,1} = 2 + j$.

Remark 2.9. From the proof of Lemma 2.7 we know that

$$
\sum_{k \geq 0} C^j_k(u) e^k = \left( \sum_{k \geq 0} C^j_k(u) e^k \right) \left( \sum_{k=0}^{\infty} (k+1) \cos^k(u) e^k \right)
$$

holds. Therefore we get

$$
c^j_k(u) = \sum_{l=0}^{k} (k+1-l)C^j_l(u) \cos^{k-l}(u), \quad \text{for } k \geq 0. \quad \text{(2.62)}
$$

Using (2.56), (2.59) and the trigonometric identities

$$
\cos(\beta_1) \cos(\beta_2) = \frac{1}{2} \left[ \cos(\beta_1 + \beta_2) + \cos(\beta_1 - \beta_2) \right] \quad \text{and} \quad \cos^{k-l}(u) = \frac{1}{2^{k-l}} \cos((k-l)u) + TP_{k-l-1}(u), \quad \text{where } TP_{k-l-1}(u) \text{ is a trigonometric polynomial in } u \text{ of degree } k-l-1
$$

from (2.62) it follows:

$$
\begin{align*}
\left\{ \begin{array}{l}
c^j_{k,k}(u) = \frac{(k+1)}{2^{k+1}} + \sum_{l=1}^{k} \frac{C^j_l(k+1-l)}{2^{k+1}} , \\
c^j_{0,0}(u) = 1.
\end{array} \right. \\
\text{for } k \geq 1,
\end{align*}
$$

(2.63)

Analogously we can prove that

$$
s^j_k(u) = \sum_{l=0}^{k} (k+1-l)S^j_l(u) \cos^{k-l}(u), \quad \text{for } k \geq 0. \quad \text{(2.64)}
$$

Using (2.57), (2.60) and the trigonometric identities

$$
\sin(\beta_1) \cos(\beta_2) = \frac{1}{2} \left[ \sin(\beta_1 + \beta_2) + \sin(\beta_1 - \beta_2) \right] \quad \text{and} \quad \cos^{k-l}(u) = \frac{1}{2^{k-l}} \cos((k-l)u) + TP_{k-l-1}(u), \quad \text{where } TP_{k-l-1}(u) \text{ is a trigonometric polynomial in } u \text{ of degree } k-l-1
$$

from (2.64) it follows:

$$
\begin{align*}
\left\{ \begin{array}{l}
s^j_{k,k}(u) = \sum_{l=1}^{k} \frac{S^j_l(k+1-l)}{2^{k+1}} , \\
s^j_{0,0}(u) = 0.
\end{array} \right. \\
\text{for } k \geq 1,
\end{align*}
$$

(2.65)

Definition 2.1. Let $C^j_{k,k}$ and $S^j_{k,k}$ be as in (2.61). Then for $k \geq 0$ we define

$$
\Delta^j_{k+1} := C^j_{k+1,k+1} \pm S^j_{k+1,k+1},
$$

$$
\Omega^j_{k+1} := \frac{1}{2^{k+2}} + \sum_{l=1}^{k+1} \frac{C^j_l + S^j_l}{2^{k+1-l}}.
$$

Lemma 2.8. Let $\Delta^j_{k+1}$ and $\Omega^j_{k+1}$ be as in Definition 2.1. Let $(Q_k)_{k \geq 0}$ be defined as in (2.20) and furthermore define $Q_{-1}(x) := 0$. Then for $k \geq 0$ we have

$$
\left\{ \begin{array}{l}
\Delta^j_{k+1} = -4Q_{k+1}(j) + 4Q_k(j) - Q_{k-1}(j), \\
\Omega^j_{k+1} = 2Q_k(j) - 4Q_{k+1}(j),
\end{array} \right. \quad \text{(2.66)}
$$

and

$$
\left\{ \begin{array}{l}
\Delta^j_{k+1} = \frac{(-1)^{k+1} j^k}{k!2^k} \left[ \frac{(j+k+1)(j+1)}{k+1} \right], \\
\Omega^j_{k+1} = \frac{(-1)^{k+1} j^k}{k!2^k} \left( \frac{j}{k+1} + 1 \right).
\end{array} \right. \quad \text{(2.67)}
$$
By (2.20), (2.61) in Remark 2.7, Remark 2.8 and Definition 2.1 we have
\[
\Delta_1^{j,+} = C_{1,1}^j + S_{1,1}^j = 2 + j = 4 + j - 2 = -4Q_1(j) + 4Q_0(j) - Q_{-1}(j),
\]
\[
\Omega_1^{j,+} = C_{0,0}^j + S_{0,0}^j + C_{1,1}^j + S_{1,1}^j = 3 + j = -1 + 4 + j = 2Q_0(j) - 4Q_1(j),
\]
\[
\Delta_1^{j,-} = C_{1,1}^j - S_{1,1}^j = -2 - j,
\]
\[
\Omega_1^{j,-} = C_{0,0}^j - S_{0,0}^j + C_{1,1}^j - S_{1,1}^j = 1 - (2 + j) = -1 - j.
\]

This proves the assertion for \( k = 0 \).

\( k + 1 \rightarrow k + 2 \). By Definition 2.1 we observe that
\[
\Omega_{k+2}^{j,\pm} = \frac{1}{2k+1} + \sum_{l=1}^{k+2} \frac{C_{l,l}^j \pm S_{l,l}^j}{2^{k+2-l}} = \frac{1}{2} \Omega_{k+1}^{j,\pm} + \Delta_{k+2}^{j,\pm}.
\]

On the other side
\[
\Delta_{k+2}^{j,\pm} = C_{k+2,k+2}^j + S_{k+2,k+2}^j = \left\{ \begin{array}{l}
\frac{1}{k+2} \sum_{l=1}^{k+2} \left[ \frac{S_{k+2-l,k+2-l}^j \pm C_{k+2-l,k+2-l}^j}{2} \right] \\
\pm \frac{1}{2k+1} \end{array} \right.
\]

\[
\Delta_{k+2}^{j,\pm} = \left\{ \begin{array}{l}
\frac{k}{2(k+2)} \frac{(-1)^{k+1} j^k}{k!} \frac{1}{k+1} \left( \frac{j}{k+1} + 1 \right) \\
\frac{j}{2(k+2)} \frac{(-1)^{k+1} j^{k-1}}{k!} \left[ \frac{(j+k+1)k + j(j+k+2)}{k+1} \right] \\
\frac{(-1)^{k+1} j^k}{(k+2)!2^{k+1}} [k(j + k + 1) - (j + k + 1)k - j(j + k + 2)] \\
\frac{(-1)^{k+2} j^{k+1}}{(k+2)!2^{k+1}} [j + k + 2] \\
\frac{(-1)^{k+2} j^{k+1}}{(k+1)!2^{k+1}} \left[ \frac{j}{k+2} + 1 \right]\end{array} \right.
\]

As a consequence we get
\[
\Omega_{k+2}^{j,\pm} = \left\{ \begin{array}{l}
\frac{1}{2} \Omega_{k+1}^{j,\pm} + \frac{1}{2} \Delta_{k+1}^{j,\pm} \\
\frac{k}{2(k+2)} \Delta_{k+1}^{j,\pm} + \frac{1}{2} \Omega_{k+1}^{j,\pm} \end{array} \right.
\]

Using the induction step assumption in the previous equation we have
\[
\Omega_{k+2}^{j,-} = \frac{k}{2(k+2)} \frac{(-1)^{k+1} j^k}{k!} \frac{1}{k+1} \left( \frac{j}{k+1} + 1 \right)
\]

\[
- \frac{j}{2(k+2)} \frac{(-1)^{k+1} j^{k-1}}{k!} \left[ \frac{(j+k+1)k + j(j+k+2)}{k+1} \right]
\]

\[
\Delta_{k+2}^{j,-} = \frac{k}{2(k+2)} \frac{(-1)^{k+1} j^k}{k!} \frac{1}{k+1} \left( \frac{j}{k+1} + 1 \right)
\]

\[
- \frac{j}{2(k+2)} \frac{(-1)^{k+1} j^{k-1}}{k!} \left[ \frac{(j+k+1)k + j(j+k+2)}{k+1} \right]
\]
\[
\Omega_{k+2}^{\pm} = \frac{k + 4}{2(k + 2)} (2Q_k(j) - 4Q_{k+1}(j)) + \frac{j}{2(k + 2)} (-4Q_{k+1}(j) + 4Q_k(j) - Q_{k-1}(j)) = -\frac{1}{k+2} \left( 2(2 + j)Q_{k+1}(j) - (2 + 2j)Q_k(j) + \frac{j}{2} Q_{k-1}(j) \right)
\]

and finally
\[
\Delta_{k+2}^{\pm} = \frac{j}{2(k + 2)} (-4Q_{k+1}(j) + 4Q_k(j) - Q_{k-1}(j)) + \frac{1}{k+2} (2Q_k(j) - 4Q_{k+1}(j)) = \frac{1}{(k + 2)} \left( -2(2 + j)Q_{k+1}(j) + (2 + 2j)Q_k(j) - \frac{j}{2} Q_{k-1}(j) \right)
\]

\[\text{Lemma 2.4} \hspace{1cm} -4Q_{k+2}(j) + 4Q_{k+1}(j) - Q_k(j).\]

\[\text{Remark 2.10. For } k \geq 1 \text{ using (2.63), (2.65) and Definition 2.1 it follows}
\]
\[
c_{k+1,k+1}^{j} \pm s_{k+1,k+1}^{j} = \frac{k + 2}{2^k} + \sum_{l=1}^{k+1} \frac{(C_{l,l}^{j} \pm S_{l,l}^{j})(k + 2 - l)}{2^{k+1-l}}
\]
\[
= \frac{k + 1}{2^k} + \sum_{l=1}^{k+1} \frac{(C_{l,l}^{j} \pm S_{l,l}^{j})(k + 1 - l)}{2^{k+1-l}} + \frac{1}{2^k} \sum_{l=1}^{k+1} C_{l,l}^{j} \pm S_{l,l}^{j}
\]
\[
= \frac{1}{2} \left( c_{k,k}^{j} \pm s_{k,k}^{j} \right) + \Omega_{k+1}^{\pm}.
\]
and furthermore
\[
c_{1,1}^{j} \pm s_{1,1}^{j} = c_{0,0}^{j} \pm s_{0,0}^{j} + \Omega_{1}^{\pm}
\]
holds.
Now we are able to prove Theorem 2.1.

**Proof.** (Proof of Theorem 2.1) From equation (2.3) and Remark 2.5 we have

\[
\alpha_j(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(2u - ju + e(2g(u) + j \sin(u)))}{(1 - e \cos(u))^2} \, du
\]

\[
(2.51)
\]

\[
= -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos((j - 2)u - e\tilde{g}(u; e, j))}{(1 - e \cos(u))^2} \, du.
\]

Using the trigonometric identity \(\cos(\beta_1 - \beta_2) = \cos(\beta_1) \cos(\beta_2) + \sin(\beta_1) \sin(\beta_2)\) we get

\[
\alpha_j(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos((j - 2)u) \cos(e\tilde{g}(u; e)) + \sin((j - 2)u) \sin(e\tilde{g}(u; e))}{(1 - e \cos(u))^2} \, du.
\]

By Lemma 2.7 we have

\[
\alpha_j(e) = I + II,
\]

where

\[
I := -\frac{1}{4\pi} \int_0^{2\pi} \cos((j - 2)u) \sum_{k \geq 0} e^k c^j_k(u) \, du,
\]

and

\[
II := -\frac{1}{4\pi} \int_0^{2\pi} \sin((j - 2)u) \sum_{k \geq 0} e^k s^j_k(u) \, du.
\]

We evaluate \(I\) and \(II\) separately. Using equation (2.56) and the obvious fact that for \(n, m \in \mathbb{Z}\)

\[
\frac{1}{2\pi} \int_0^{2\pi} \cos(nx) \cos(mx) \, dx = \begin{cases} 1, & \text{for } n = m = 0, \\ \frac{1}{2}, & \text{for } n = m \neq 0, \\ 0, & \text{otherwise,} \end{cases}
\]

we obtain:

\[
I \overset{(2.56)}{=} -\frac{1}{4\pi} \sum_{k \geq 0} e^k \int_0^{2\pi} \cos((j - 2)u) \sum_{l=0}^{[k/2]} c^j_{k,k-2l} \cos((k - 2l)u) \, du
\]

\[
= -\frac{1}{4\pi} \sum_{k \geq |j-2|} e^k \int_0^{2\pi} \cos(|j - 2|u) \sum_{l=0}^{[k/2]} c^j_{k,k-2l} \cos((k - 2l)u) \, du
\]

\[
= -\frac{1}{4\pi} e^{|j-2|} \int_0^{2\pi} \cos(|j - 2|u) \sum_{l=0}^{[|j-2|/2]} c^j_{|j-2|,|j-2|-2l} \cos((|j - 2| - 2l)u) \, du
\]

\[
+ O(e^{|j-2|+1})
\]

\[
(2.56)
\]

\[
= \frac{1}{2} e^{|j-2|} c^j_{|j-2|,|j-2|} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(|j - 2|u) \, du + O(e^{|j-2|+1})
\]

\[
= \frac{1}{2} e^{|j-2|} c^j_{|j-2|,|j-2|} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(|j - 2|u) \, du + O(e^{|j-2|+1})
\]
It remains to prove the formula (2.18). For j by definition

\[ \alpha_j := -\frac{1}{2} \left( c^j_{|j-2|,|j-2|} + \text{sign}(j-2)s^j_{|j-2|,|j-2|} \right), \quad \text{for all } \mathbb{Z} \ni j \neq 2. \]  

(2.71)

where the \( c^j_{k,k} \) are defined in equation (2.63).

Analogously using equation (2.57) and the obvious fact that for \( n, m \in \mathbb{Z} \)

\[ \frac{1}{2\pi} \int_0^{2\pi} \sin(nx) \sin(mx) dx = \begin{cases} 1, & \text{for } n = m = 0, \\ \frac{1}{\pi}, & \text{for } n = m \neq 0, \\ 0, & \text{otherwise,} \end{cases} \]

we obtain:

\[ II = -\frac{1}{4\pi} \sum_{k \geq 0} e^{k} \int_0^{2\pi} \sin((j-2)u) s^j_k(u) du \]

(2.57)

\[ = \begin{cases} O(e), & \text{for } j = 2, \\ -\frac{1}{4} \text{sign}(j-2) s^j_{|j-2|,|j-2|} e^{j-2|} + O(e^{j-2|+1}), & \text{for } j \neq 2, \end{cases} \]

where the \( s^j_{k,k} \) are defined in equation (2.65).

Till now we have proved:

\[ \alpha_j = \begin{cases} -\frac{1}{2} + O(e), & \text{for } j = 2, \\ \frac{1}{2} e^{j-2|} + O(e^{j-2|+1}), & \text{for } j \neq 2, \end{cases} \]  

(2.70)

with

\[ \bar{\alpha}_j := -\frac{1}{4} \left( c^j_{|j-2|,|j-2|} + \text{sign}(j-2)s^j_{|j-2|,|j-2|} \right), \quad \text{for all } \mathbb{Z} \ni j \neq 2. \]

It remains to prove the formula (2.18). For \( j = 2 \) by (2.70) it is trivially true, because by definition \( R_0(j) = Q_0(j) = -\frac{1}{2} \) (see (2.19) and (2.20)). For \( j \neq 2 \), we distinguish the cases \( j < 2 \) and \( j > 2 \).

- **Case 1:** \( j < 2 \). It is obvious that in this case we have \( \text{sign}(j-2) = -1 \), therefore (2.71) takes the form

\[ \bar{\alpha}_j := -\frac{1}{4} \left( c^j_{|j-2|,|j-2|} - s^j_{|j-2|,|j-2|} \right), \quad \text{for all } \mathbb{Z} \ni j < 2. \]

From equations (2.68) and (2.69) we get

\[ c^j_{k,k} - s^j_{k,k} = \frac{1}{2} \left( c^j_{k-1,k-1} - s^j_{k-1,k-1} \right) + \Omega^{-j}_{k} \]

\[ = \frac{1}{4} \left( c^j_{k-2,k-2} - s^j_{k-2,k-2} \right) + \frac{1}{2} \Omega^{j}_{k-1} + \Omega^{j}_{k} \]

\[ = \ldots \]

\[ = \frac{1}{2k-1} \left( c^j_{0,0} - s^j_{0,0} \right) + \sum_{i=1}^{k} \frac{1}{2k-i} \Omega^{j-i}_{i}. \]

Using Lemma 2.8 and the fact that \( c^j_{0,0}(u) - s^j_{0,0}(u) = 1 \) we get

\[ c^j_{k,k} - s^j_{k,k} = \frac{1}{2k-1} + \sum_{i=1}^{k} \frac{1}{2k-i} \frac{(-1)^i}{2^{i-1}} \frac{j-i-1}{(i-1)!} \left( \frac{j}{i} + 1 \right) \]
\[
\begin{align*}
\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} \sum_{i=1}^{k} (-1)^i j^{i-1} \left( \frac{j}{i} + 1 \right) \\
\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} \sum_{i=1}^{k} (-1)^i \frac{j^i}{i!} + \frac{1}{2^{k-1}} \sum_{i=1}^{k} (-1)^i \frac{j^{i-1}}{(i-1)!} \\
\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} \sum_{i=1}^{k-1} (-1)^i \frac{j^i}{i!} + (-1)^k j^k \\
\frac{1}{2^{k-1}} - \frac{1}{2^{k-1}} \sum_{i=2}^{k} (-1)^i \frac{j^{i-1}}{(i-1)!} \\
\frac{(-1)^k j^k}{2^{k-1} k!}.
\end{align*}
\]

Thus we get \( \frac{1}{4} (c^j_{k,k} - s^j_{k,k}) = P_k(j) \), for \( k \geq 1 \). Choosing \( k = 2 - j \) proves formula (2.18) for the case \( j < 2 \).

**Case 2:** \( j > 2 \). It is obvious that in this case we have \( \text{sign}(j - 2) = 1 \), therefore (2.71) has the form

\[
\tilde{\alpha}_j := -\frac{1}{4} \left( c^j_{|j-2|,|j-2|} + s^j_{|j-2|,|j-2|} \right), \quad \text{for all } Z \ni j < 2.
\]

From equations (2.68) and (2.69) we get

\[
\begin{align*}
c^j_{k,k} + s^j_{k,k} &= \frac{1}{2} \left( c^j_{k-1,k-1} + s^j_{k-1,k-1} \right) + \Omega^j_{k+} \\
&= \frac{1}{4} \left( c^j_{k-2,k-2} - s^j_{k-2,k-2} \right) + \frac{1}{2} \Omega^j_{k-1} + \Omega^j_k \\
&= \ldots \\
&= \frac{1}{2^{k-1}} \left( c^j_{0,0} + s^j_{0,0} \right) + \sum_{i=1}^{k} \frac{1}{2^{k-1}} \Omega^j_i.
\end{align*}
\]

Using Lemma 2.8 and the fact that \( c^j_{0,0}(u) - s^j_{0,0}(u) = 1 \) we get

\[
\begin{align*}
c^j_{k,k} + s^j_{k,k} &= \frac{1}{2^{k-1}} + \sum_{i=1}^{k} \frac{1}{2^{k-1}} \left( 2Q_i(j) - 4Q_i(j) \right) \\
&= \frac{1}{2^{k-1}} + \sum_{i=1}^{k} \frac{1}{2^{k-i}} \sum_{l=0}^{i} \frac{1}{i!} \left( \binom{i}{3} j^l \right) \\
&= \frac{1}{2^{k-1}} - \sum_{i=0}^{k-1} \frac{1}{2^{k-1}} \sum_{l=0}^{i} \frac{1}{i!} \left( \binom{i}{3} j^l \right) \\
&+ \sum_{i=1}^{k} \frac{1}{2^{k-1}} \sum_{l=0}^{i} \frac{1}{i!} \left( \binom{i}{3} j^l \right) \\
&= \frac{1}{2^{k-1}} - \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} \sum_{l=0}^{k} \frac{1}{l!} \left( k-l+3 \right) j^l
\end{align*}
\]
Thus we get \( \frac{1}{4} \left( c_{k,k}^j + s_{k,k}^j \right) = Q_k(j) \), for \( k \geq 1 \). Choosing \( k = 2 - j \) proves formula (2.18) for the case \( j > 2 \).

This concludes the proof of Theorem 2.1.
3 Generalized Newtonian Potential

In this chapter we study a more general problem than the dissipative spin-orbit problem presented in Subsection 1.3.1. In the equation of motion (1.24) we let the potential function

$$ f \in C^\infty(\mathbb{R}^2, \mathbb{R}) $$

be any $2\pi$–periodic function in both variables $x, t$ and having zero average. More precisely, we consider the differential equation

$$ 0 = \ddot{x} + \bar{\eta}(\dot{x} - \bar{\nu}) + \bar{\varepsilon} f_x(x, t), \quad (3.1) $$

where we assume that $f$ is a real analytic function with Fourier expansion

$$ f(x, t) = \sum_{\mathbb{Z}^2 \ni (m, n) \neq (0,0)} f_{m,n} e^{imx} e^{int}. \quad (3.2) $$

We assume $f_{m,n} = f_{-m,-n}$ for all $\mathbb{Z}^2 \ni (m, n) \neq (0,0)$, since we want $f$ to be real for real arguments. We call this problem the general dissipative spin-orbit problem.

Remark 3.1. Of course the dissipative spin-orbit problem (see equation (1.24)) is a special case of the general dissipative spin-orbit problem. Indeed rewriting the Newtonian potential defined in (1.29) we have

$$ f(x, t) = \sum_{n \neq 0, n \in \mathbb{Z}} \alpha_n \cos(2x - nt) = \sum_{n \neq 0, n \in \mathbb{Z}} \alpha_n \frac{e^{i2x} e^{-int} + e^{-i2x} e^{int}}{2} $$

Therefore the dissipative spin-orbit problem is a special case of the general dissipative spin-orbit problem, where $m$ only takes the values $\pm 2$. For the Fourier-coefficients of the functions $f_2(t)$ and $f_{-2}(t)$ we get the relations:

$$ f_{2,n} = \frac{\alpha_{-n}}{2} \quad \text{and} \quad f_{-2,n} = \frac{\alpha_n}{2}. \quad (3.3) $$

The aim of this chapter is to find conditions for the existence of a $(p,q)$–periodic orbit for the general dissipative spin-orbit problem, which are analogously defined as for the classical spin-orbit case (cf. Definition 1.1. in Chapter 1). Hence:

Definition 3.1. A $(p,q)$–periodic orbit (with $p$ and $q$ co–prime non–vanishing integers) for the general dissipative spin-orbit problem is a solution $x_{pq}(t) : \mathbb{R} \rightarrow \mathbb{R}$ of (3.1) satisfying

$$ x(t + 2\pi q) = x(t) + 2\pi p, \quad \forall t, \quad (3.4) $$

which corresponds to the spin-orbit resonance of the system.
3.1 Introduction

In this section we rewrite some important results, which can be found in Sections 2.1 – 2.4 of [8].

Since the potential $f$ in (3.1) is $2\pi$–periodic in $x$ and $t$, a $(p, q)$–periodic solution can be written as

$$x_{pq}(t) = \xi + \frac{p}{q}t + u \left(t \frac{q}{p}\right), \quad \langle u \rangle := \frac{1}{2\pi} \int_0^{2\pi} u(s)ds = 0, \quad (3.5)$$

where $\xi \in \mathbb{R}$ and $u$ is $2\pi$–periodic. Without loss of generality let $p$ and $q$ be positive coprime integers, then $x_{pq}(t)$ in (3.5) is a solution of (3.1) if and only if $u$ satisfies the functional equation

$$L_\eta u = \Phi_\xi(u), \quad (3.6)$$

where

$$
\begin{cases}
L_\eta u & := u'' + \eta u', \\
[\Phi_\xi(u)](t) & := [\Phi_\xi(u; \eta \nu, \varepsilon, p, q)](t) = \eta \nu - \varepsilon f_x(\xi + pt + u(t), qt),
\end{cases}
$$

and

$$\eta := \frac{q\bar{\eta}}{p}, \quad \nu := \frac{q\bar{\nu}}{p}, \quad \varepsilon := q^2 \bar{\varepsilon}.$$ 

Note that for convenience we again denote the new time $t/q$ by $t$ and $u', u''$ denote the derivatives with respect to the new time.

**Remark 3.2.** The nonlinear operator $\Phi_\xi$ is $2\pi$–periodic in $\xi$. Thus we can assume $\xi \in [0, 2\pi]$.

**Definition 3.2.** Let $C_{per}^k$ be the Banach space of $2\pi$–periodic $C^k(\mathbb{R})$ functions endowed with the $C^k$-norm; let $C_{per,0}^k$ be the closed subspace of $C_{per}^k$ formed by functions with vanishing average over $[0, 2\pi]$; finally, denote by $\mathbb{B} := C_{per,0}^0$ the Banach space of $2\pi$–periodic continuous functions with zero average (endowed with the sup–norm).

**Lemma 3.1.** For all $0 < \eta$ the linear operator $L_\eta : C_{per,0}^2 \to \mathbb{B}$ in (3.6) maps injectively $C_{per,0}^2$ onto $\mathbb{B}$. Furthermore, there exist constants $\eta_0 > 0$ and $\kappa_0 = \kappa_0(\eta_0) > 0$ such that for all $\eta \in (0, \eta_0]$ the inverse operator (the “Green operator”) $G_\eta = L_\eta^{-1}$ is a bounded linear isomorphism given by

$$G_\eta[g(t)] = G_\eta \left[ \sum_{n \neq 0} g_n e^{int} \right] := \sum_{n \neq 0} \frac{g_n}{i\eta n - n^2} e^{int} \quad (3.7)$$

with

$$\|G_\eta\| := \|G_\eta\|_{L(\mathbb{B}, C_{per,0}^2)} \leq \kappa_0.$$ 

**Proof.** The operator $L_\eta$ is injective since for $\eta > 0$ its kernel $\ker(L_\eta)$ is given by

$$\ker(L_\eta) = \{ g \in C_{per,0}^2 \mid g'' + \eta g' = 0 \} = \{ g = 0 \}.$$ 

Equation (3.7) follows since the expression in Fourier series of the operator $L$ is

$$L_\eta[g(t)] = L_\eta \left[ \sum_{n \neq 0} g_n e^{int} \right] = g'' + \eta g' = \sum_{n \neq 0} (i\eta n - n^2) g_n e^{int}.$$ 

\hspace{1cm} \text{where} \hspace{1cm} \|v\|_{C^k} := \sup_{0 \leq j \leq k} \sup_{t \in \mathbb{R}} |D^j v(t)|.
for every function $g \in C^2_{\text{per},0}$.

The existence of $\kappa_0(\eta_0) > 0$ such that $\|G_\eta\|_{L(B,C^2_{\text{per},0})} \leq \kappa(\eta_0)$ follows easily from the definition of the operator $G_\eta$. For $0 < \eta \leq \eta_0$ and $g \in B$ we have

$$\begin{align*}
\|G_\eta\|_{L(B,C^2_{\text{per},0})} & \leq \sup_{\|g\|_{C^0}=1} \|G_\eta(g)\|_{L(B,C^2_B)} \leq \sup_{\|g\|_{C^0}=1} \left| \sum_{n \neq 0} \frac{g_n}{i\eta n - n^2} e^{int} \right| \\
& \leq \frac{1}{1 + \eta_0^2} \sup_{\|g\|_{C^0}=1} \|g\|_{C^0} = \frac{1}{1 + \eta_0} =: \kappa_0(\eta_0).
\end{align*}$$

This proves the boundedness\(^2\) of the operator $G_\eta$.

### 3.1.1 Lyapunov-Schmidt decomposition

Performing a Lyapunov-Schmidt decomposition, equation (3.6) can be splitted in a bifurcation equation

$$\langle \Phi_\xi(u) \rangle = 0 \quad \iff \quad \phi_u(\xi) := \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt + u(t), qt) dt = \frac{\eta \nu}{\varepsilon}, \quad (3.8)$$

and a range equation

$$u = G_\eta (\Phi_\xi(u) - \langle \Phi_\xi(u) \rangle) = -\varepsilon G_\eta (f_x(\xi + pt + u(t), qt) - \phi_u(\xi)). \quad (3.9)$$

The unknowns in (3.8), (3.9) are $u \in B$ and $\xi \in [0, 2\pi]$. The idea to find a solution of the system is: first solve the range equation for arbitrary $\xi$ to get a function $u = u(\cdot; \xi, \varepsilon)$; then put this function $u$ into the bifurcation equation in order to determine $\xi$.

### 3.1.2 The range equation

The range equation (3.9) is solved using a contraction argument.

**Lemma 3.2.** Let $0 < \kappa_1 \leq 1$ and fix $\xi \in [0, 2\pi]$. For all $\eta \in (0, \eta_0]$ let $\kappa_0 = \kappa(\eta_0)$ be as in Lemma 3.1. Let $\kappa_2 := 2\kappa_0 \|f\|_{C^2}$ and $B_{\kappa_1} := \{v \in B \|v\|_{C^0} \leq \kappa_1\}$. For $\Phi_\xi(v; \varepsilon)$ as in (3.6) define

$$\hat{\Phi}_\xi(v; \varepsilon) := \frac{\Phi_\xi(v; \varepsilon) - \langle \Phi_\xi(v; \varepsilon) \rangle}{\varepsilon} = -f_x(\xi + pt + v(t), qt) + \phi_v(\xi),$$

where $\phi_v(\xi)$ is as in (3.8). Then for $|\varepsilon| \leq \varepsilon_0 := \frac{2}{2\kappa^2}$ the function

$$\varphi : v \in B_{\kappa_1} \to \varphi(v) := \varepsilon G_\eta \left( \hat{\Phi}_\xi(v; \varepsilon) \right)$$

is a contraction with Lipshitz constant $|\varepsilon|\kappa_2 \leq \varepsilon_0 \kappa_2 \leq \frac{1}{2}$. Moreover, for every $\xi \in [0, 2\pi]$ there exists a unique fixed point $u := u(\cdot; \xi) \in B_{\kappa_1}$ of the function $\varphi$ and defining $\varphi^k(0) = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{k \text{ times}}(0)$ the following holds:

$$u = \sum_{k=1}^{\infty} \left( \varphi^k(0) - \varphi^{k-1}(0) \right), \quad (3.10)$$

with

$$\|u\|_{C^0} \leq \frac{2|\varepsilon|\kappa_0 \|f\|_{C^2}}{1 - \frac{2|\varepsilon|\kappa_0 \|f\|_{C^2}}{4|\varepsilon|\kappa_0 \|f\|_{C^2}} \leq 4|\varepsilon|\kappa_0 \|f\|_{C^2}. \quad (3.11)$$

\(^2\)The constant $\kappa_0(\eta_0)$ is not optimal. In Chapter 6 for a particular value of $\eta_0$ we will find sharp estimates for this bound.
Proof. Notice that $\hat{\Phi}_\xi(\cdot;\varepsilon) \in C^1(\mathbb{B},\mathbb{B})$ and for every $u \in \mathbb{B}$ and $\xi \in [0,2\pi]$ we have

$$\|\hat{\Phi}_\xi(v;\varepsilon)\|_{C^0} \leq 2 \sup_{T^2} |f_x| \leq 2 \|f\|_{C^2},$$

$$\|D_v \hat{\Phi}_\xi(v;\varepsilon)\|_{L(\mathbb{B},\mathbb{B})} \leq 2 \sup_{T^2} |f_{xx}| \leq 2 \|f\|_{C^2}.$$

The function $\phi$ maps $\mathbb{B}_{\kappa_1}$ into itself, since for every $v \in \mathbb{B}$ with $\|v\|_{C^0} < \kappa_1$ we get

$$\|\phi(v)\|_{C^0} = |\varepsilon| \|G_0(\hat{\Phi}_\xi(v;\varepsilon))\|_{C^0} \leq 2 \varepsilon_0 \kappa_0 \|f\|_{C^2} = \frac{\kappa_1}{2} < \kappa_1.$$

Furthermore the Lipshitz-continuity follows from the linearity of the operator $G_0$ and from the mean value theorem, hence for every $v_1, v_2 \in \mathbb{B}_{\kappa_1}$

$$\|\phi(v_1) - \phi(v_2)\|_{C^0} \leq 2 |\varepsilon| \kappa_0 \|f\|_{C^2} \|v_1 - v_2\|_{C^0}$$

holds. This proves that $\phi$ is a contraction with Lipshitz constant $|\varepsilon| \kappa_2 \leq \varepsilon_0 \kappa_2 = \frac{\kappa_1}{2} \leq \frac{\kappa_1}{4}$.

The existence and the uniqueness of $u$ follow directly from the Banach fixed-point theorem. The sequence $(\phi^k(0))_{k \in \mathbb{N}}$ converges uniformly to $u$, since

$$\|\phi^k(0) - \phi^{k-1}(0)\|_{C^0} \leq (2 |\varepsilon| \kappa_0 \|f\|_{C^2})^k \to 0, \quad \text{for } k \to \infty,$$

(3.12) holds. This proves equation (3.10). Using (3.10), (3.12) and a geometric series argument we find the upper bound for $\|u\|_{C^0}$ in (3.11). This terminates the proof of Lemma 3.2. \qed

Remark 3.3. (i) Since $f$ in (1.29) is of class $C^\infty$ in the variables $t, \varepsilon, \xi$ and $\eta$, also the fixed point $u$ is of class $C^\infty$ in all these variables.

(ii) Since $f$ in (3.2) is real-analytic in $\varepsilon$ the fixed point $u(t;\xi, \varepsilon)$ is also real-analytic in $\varepsilon$. From Lemma 3.2 and from the Cauchy estimates we get

$$u(t;\xi, \varepsilon) = \sum_{k=1}^{\infty} u_k(t;\xi) \varepsilon^k,$$

and $\|u_k\| \leq 4 |\varepsilon| \kappa_0 \|f\|_{C^2} \varepsilon_0^{-k}$ on suitable complex domains. To simplify the notation we will often write $u_k$ instead of $u_k(t;\xi)$.

(iii) From (3.13) we conclude that also the function $\phi_u(\xi)$ in (3.8) can be expressed in a $\varepsilon$ power series, i.e.

$$\phi_u(\xi) = \sum_{k=0}^{\infty} \phi^{(k)}(\xi) \varepsilon^k.$$  

(3.14)

3.1.3 The bifurcation equation

For later use we formulate Propositon 2.7 of [8]:

Theorem 3.1. (Propositon 2.7 of [8]) For $|\varepsilon| \leq \varepsilon_0$ let $u$ be a solution of (3.9) as in Lemma 3.2. Assume $\phi$ in (3.8) can be written as

$$\phi_u(\xi; \varepsilon) = \phi^{(0)}(\xi) + \varepsilon \phi^{(1)}(\xi; \varepsilon) = \phi^{(0)}(\xi) + \varepsilon^2 \phi^{(1)}(\xi) + \varepsilon^2 \phi^{(2)}(\xi; \varepsilon)$$

with

$$\sup_{|\varepsilon| \leq \varepsilon_0, \xi \in [0,2\pi]} \left| \phi^{(i)}(\xi, \varepsilon) \right| \leq M_i$$

for suitable $M_i > 0$ and $i = 1, 2$. 


(i) Assume
\[ \phi_0 := \min_{\xi \in [0,2\pi]} \phi(0)(\xi) < \max_{\xi \in [0,2\pi]} \phi(0)(\xi) =: \phi_+. \]

Let
\[ |\varepsilon| \leq \varepsilon_1 := \min \left( \varepsilon_0, \frac{\delta}{M_1} \right), \]

where
\[ 0 < \delta < \frac{\phi_+ - \phi_0}{2}. \]

Then there exists \( \xi \in [0,2\pi] \) solving the bifurcation equation (3.8) provided
\[ \frac{\eta \nu}{\varepsilon} \in \left[ \phi_0^+ + \delta, \phi_+ - \delta \right] \]
holds.

(ii) Assume \( \xi \to \phi(0)(\xi) \) is identically zero and
\[ \phi_1 := \min_{\xi \in [0,2\pi]} \phi(1)(\xi) < \max_{\xi \in [0,2\pi]} \phi(1)(\xi) =: \phi_+. \]

Let
\[ |\varepsilon| \leq \varepsilon_2 := \min \left( \varepsilon_0, \frac{\delta}{M_2} \right), \]

where
\[ 0 < \delta < \frac{\phi_1 - \phi_0}{2}. \]

Then there exists \( \xi \in [0,2\pi] \) solving the bifurcation equation (3.8) provided
\[ \frac{\eta \nu}{\varepsilon} \in \left[ \varepsilon \left( \phi_0^+ + \delta \right), \varepsilon \left( \phi_+ - \delta \right) \right] \]
holds.

In either case, for the above \( \varepsilon, \xi, \eta \) and \( \nu \), the couple \( (u, \xi) \) solves (3.8)-(3.9), so that \( x_{pq}(t) \) defined in (3.5) solves (3.1).

This procedure can also be extended to higher order of non-degeneration. Hence we have:

1. **Non-degenerate case (at zeroth degree),** i.e. \( \exists \xi \) such that \( \partial_\xi \phi(0)(\xi) \neq 0 \).
   In this case, we can guarantee the existence of a solution for the bifurcation equation (3.8) under suitable conditions on \( \varepsilon \) (depending on the maximum and minimum of the function \( \phi(0)(\xi) \)). From Subsection 3.1.1 and Lemma 3.2 there exists a solution \( u(t; \xi, \varepsilon) \) of the functional equation (3.6), which by (3.5) implies the existence of a \( (p,q) \)-periodic solution \( x_{pq}(t) \) of (3.1).

2. **Degenerate case (at zeroth degree),** i.e. \( \partial_\xi \phi(0)(\xi) \equiv 0 \). If that happens, we analyse the next term of \( \phi \), i.e. \( \phi(1)(\xi) \). Again we need to distinguish two different cases:

   2.1 **Non-degenerate case (at first degree),** i.e. \( \exists \xi \) such that \( \partial_\xi \phi(1)(\xi) \neq 0 \).
   In this case, we can guarantee the existence of a solution for the bifurcation equation (3.8) under suitable conditions on \( \varepsilon \) (depending on the maximum and minimum of the function \( \phi(1)(\xi) \)). From Subsection 3.1.1 and Lemma 3.2 there exists a solution \( u(t; \xi, \varepsilon) \) of the functional equation (3.6), which by (3.5) implies the existence of a \( (p,q) \)-periodic solution \( x_{pq}(t) \) of (3.1).
2.2 Degenerate case (of first degree) if $\partial_{\xi} \phi^{(1)}(\xi) \equiv 0$. In this case, we analyse the next term of $\phi$, i.e. $\phi^{(2)}$ and so on.

To study the non-degeneration at higher degree we need the following definition.

**Definition 3.3.** We say that a $(p,q)$-periodic orbit $x_{pq}$ is

- degenerate at $k_{th}$-degree, if $\phi^{(l)}(\xi) : \xi \in [0, 2\pi] \to \mathbb{R}$ are constant functions for every $0 \leq l \leq k$.

- non-degenerate at $k_{th}$-degree, if it is degenerate at $(k-1)_{th}$-degree and $\phi^{(k)}(\xi)$ is non-constant.

The following theorem is the generalisation of Theorem 3.1 to an arbitrary degree.

**Theorem 3.2.** For $|\varepsilon| \leq \varepsilon_0$ let $u$ be as in Lemma 3.2. Assume $x_{pq}$ is non-degenerate at $k_{th}$-degree. Furthermore, assume that $\phi$ is (3.8) can be expanded with respect to $\varepsilon$ up to any order, i.e.

$$\phi_u(\xi; \varepsilon) = \sum_{l=1}^{k} \phi^{(l)}(\xi) \varepsilon^{l-1} + \varepsilon^k \tilde{\phi}^{(k+1)}(\xi; \varepsilon)$$

with

$$\sup_{|\varepsilon| \leq \varepsilon_0, \xi \in [0, 2\pi]} |\tilde{\phi}^{(k+1)}(\xi; \varepsilon)| \leq M_{k+1}$$

for some suitable $M_{k+1} > 0$.

Assume

$$\phi_-^{(k)} := \min_{\xi \in [0, 2\pi]} \phi^{(k)}(\xi) < \max_{\xi \in [0, 2\pi]} \phi^{(k)}(\xi) =: \phi_+^{(k)}.$$

Let

$$|\varepsilon| \leq \varepsilon_k := \min \left( \varepsilon_0, \frac{\delta}{M_{k+1}} \right),$$

where

$$0 < \delta < \frac{\phi_+^{(k)} - \phi_-^{(k)}}{2}.$$

Then there exists $\xi \in [0, 2\pi]$ solving the bifurcation equation (3.8) provided

$$\frac{\eta \mu}{\varepsilon} \in \left[ \varepsilon^k \left( \phi_-^{(k)} + \delta \right), \varepsilon^k \left( \phi_+^{(k)} - \delta \right) \right]$$

holds. For the above $\varepsilon, \xi, \eta$ and $\nu$, the pair $(u, \xi)$ solves (3.8)-(3.9), so that $x_{pq}(t)$ defined in (3.5) solves (3.1).

We need to study the functions $\phi^{(k)}(\xi)$ for $k \geq 0$ in order to find conditions for the existence of periodic orbits depending only on the general Newtonian potential $f$ in (3.1). In Section 3.2 we develop a method to compute recursively the orbits $\phi^{(k)}(\xi)$ for $k \geq 0$. Then in Section 3.3 we study the relation between $q$ and $k$. 

3.2 Computing $\phi^{(k)}(\xi)$ for $k \geq 0$

First method

We expand $f_x(\xi + pt + u, qt)$ with respect to $\varepsilon$ and get

$$f_x(\xi + pt + u, qt) = f_x(\xi + pt, qt) + \varepsilon f_{xx}(\xi + pt, qt)u_1 + \varepsilon^2 \left[ f_{xxx}(\xi + pt, qt)u_2 + f_{xxxx}(\xi + pt, qt)\frac{u_1^2}{2} \right] + \varepsilon^3 \left[ f_{xxxx}(\xi + pt, qt)u_3 + f_{xxxx}(\xi + pt, qt)\frac{2u_1u_2}{2} + f_{xxxx}(\xi + pt, qt)\frac{u_1^3}{6} \right] + \ldots$$

Taking the average with respect to $t$ and comparing coefficients of the same order in $\varepsilon$ with (3.14) leads to

$$\phi^{(0)} = \langle f^{(0)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt, qt)dt,$$

$$\phi^{(1)} = \langle f^{(1)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(\xi + pt, qt)u_1dt,$$

$$\phi^{(2)} = \langle f^{(2)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left[ f_{xxx}(\xi + pt, qt)u_2 + f_{xxxx}(\xi + pt, qt)\frac{u_1^2}{2} \right] dt,$$

$$\phi^{(3)} = \langle f^{(3)} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left[ f_{xxxx}(\xi + pt, qt)u_3 + f_{xxxx}(\xi + pt, qt)\frac{2u_1u_2}{2} + f_{xxxx}(\xi + pt, qt)\frac{u_1^3}{6} \right] dt,$$

and so on. The problem is that there is no simple formula for $\phi^{(k)}$ with $k \in \mathbb{N}$.

Second method

A similar method, as the one developed in this subsection, was already used in [14].

This approach allows us to compute $u_k$ and $\phi^{(k)}_u$ recursively for $k \in \mathbb{N}$.

Definition 3.4. Define $\beta^{m,n}_k(t; \xi)$ for $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$ to be the coefficients of the formal expansion of the function $e^{i(\nu p + \nu q) t + mu(\xi, \varepsilon)}$ with respect to $\varepsilon$, i.e.

$$e^{i(\nu p + \nu q) t + mu(\xi, \varepsilon)} = \sum_{k=0}^{\infty} \beta^{m,n}_k(t; \xi)\varepsilon^k,$$  \hspace{1cm} (3.19)

with

$$\beta^{m,n}_0(t; \xi) := e^{i(\nu p + \nu q) t}.$$  \hspace{1cm} (3.20)

To simplify the notation we will often write $\beta^{m,n}_k$ instead of $\beta^{m,n}_k(t; \xi)$.

Lemma 3.3. For $k \geq 0$ let $f^{(k)}$ be as in (3.15) and $\beta^{m,n}_k$ be as in (3.19)-(3.20). Then the relation between $f^{(k)}$ and $\beta^{m,n}_k$ is

$$f^{(k)} = i \sum_{(m,n) \neq (0,0)} m f_{m,n} e^{im\xi} \beta^{m,n}_k, \hspace{1cm} \forall k \geq 0.$$  \hspace{1cm} (3.21)
Moreover, for \( u_k \) in (3.13) we get
\[
 u_k = -G_\eta(f^{(k-1)} - \phi^{(k-1)}), \quad \forall k \geq 1, \tag{3.22}
\]
where \( \phi^{(k-1)} \) are given by
\[
 \phi^{(k-1)}(\xi) = \left\langle f^{(k-1)} \right\rangle = \left\langle i \sum_{(m,n) \neq (0,0)} m f_{m,n} e^{im\xi} \beta_{m,n}^{k-1} \right\rangle \tag{3.23}
\]
for \( k \geq 1 \).

**Proof.** Using (3.13) every function \( \frac{\partial f}{\partial x^l}(\xi + pt + u, t) \) for \( l \in \mathbb{N} \) can be expressed in a \( \varepsilon \) power series. In particular, for \( l = 1 \) we have from (3.2) and (3.19):
\[
 f_x(\xi + pt + u(t; \xi, \varepsilon), qt) = \left\langle f^{(k-1)} \right\rangle = \sum_{(m,n) \neq (0,0)} (im)^l f_{m,n} e^{im\xi} e^{i(mp+nq)t} e^{imu(t; \xi, \varepsilon)} \\
= \sum_{(m,n) \neq (0,0)} (im)^l f_{m,n} e^{im\xi} \beta_{m,n}^{k-1}. \tag{3.24}
\]
Equation (3.21) follows directly by comparing the coefficients in the \( \varepsilon \)–expansion. Putting (3.14) and (3.15) into the range equation (3.9) and using (3.13) and the linearity of \( G_\eta \) we get
\[
 \sum_{k=1}^{\infty} u_k \varepsilon^k = -\varepsilon G_\eta \left( \sum_{k=0}^{\infty} \left[ f^{(k)} - \phi^{(k)} \right] \varepsilon^k \right) \\
= -\sum_{k=0}^{\infty} G_\eta(f^{(k)} - \phi^{(k)}) \varepsilon^{k+1}.
\]
Again by comparing coefficients we obtain (3.22). To find \( \phi^{(k-1)}(\xi) \) in (3.23) we have to compute the mean over \([0, 2\pi]\) of the \((k-1)\)th coefficient of the \( \varepsilon \) expansion of \( f_x(\xi + pt + \sum_{k=1}^{\infty} u_k \varepsilon^k) \), i.e. by (3.15) we have \( \phi^{(k-1)}(\xi) = \left\langle f^{(k-1)} \right\rangle \). From formula (3.21) we have
\[
 \phi^{(k-1)}(\xi) = \left\langle i \sum_{(m,n) \neq (0,0)} m f_{m,n} e^{im\xi} \beta_{m,n}^{k-1} \right\rangle 
\]
for all \( k \geq 1 \). This proves equation (3.23). \( \square \)

**Remark 3.4.** Furthermore, notice that the derivatives of the Newtonian potential in (3.2) along the unperturbed periodic orbit \( t \to (\xi + pt, qt) \) are given by
\[
 \frac{\partial f}{\partial x^l}(\xi + pt, qt) = \sum_{(m,n) \neq (0,0)} (im)^l f_{m,n} e^{imqt} e^{i(mp+nq)t} \\
= \sum_{(m,n) \neq (0,0)} (im)^l f_{m,n} e^{im\xi} e^{i(mp+nq)t}. \tag{3.24}
\]
The hypothesis follows directly by the definition of $(3.25)$. Follows from $(3.25)$, $(3.26)$ and from the inductive step, since for $(3.19)$ $k$

Comparing coefficients in the $\epsilon$-terms for some function $(3.19)$ for all $k \geq 1$.

**Proof.** Taking the derivative of equation (3.19) with respect to $\epsilon$ and using (3.13) we obtain

\[
(\im) e^{i[(mp+nu)\xi]} \sum_{k=1}^{\infty} k u_k \xi^{k-1} = \sum_{k=0}^{\infty} k^2 \beta_k^{m,n} \xi^{k-1},
\]

\[
\Leftrightarrow (\im) \xi^{-1} \left[ \sum_{k=0}^{\infty} \beta_k^{m,n} \xi^{k} \right] \cdot \left[ \sum_{k=0}^{\infty} k u_k \xi^{k} \right] = \sum_{k=1}^{\infty} k \beta_k^{m,n} \xi^{k-1}.
\]

Using the Cauchy-formula for the product of two series we get the following equivalent statements

\[
\Leftrightarrow (\im) \xi^{-1} \sum_{k=1}^{\infty} \left[ \sum_{l=0}^{k} l u_l \beta_k^{m,n} \right] \xi^{k-1} = \sum_{k=1}^{\infty} k \beta_k^{m,n} \xi^{k-1},
\]

\[
\Leftrightarrow (\im) \sum_{k=1}^{\infty} \left[ \sum_{l=1}^{k} l u_l \beta_k^{m,n} \right] \xi^{k-1} = \sum_{k=1}^{\infty} k \beta_k^{m,n} \xi^{k-1}.
\]

Comparing coefficients in the $\xi$ expansions, we find the recursion formula (3.26) for the functions $\beta_k^{m,n}$. Finally equation (3.25) follows from (3.21), (3.22) and (3.23). This terminates the proof of Lemma 3.4.

**Lemma 3.5.** The function $\beta_k^{m,n}(t; \xi)$ can also be written in the following form:

\[
\beta_k^{m,n}(t; \xi) = \sum_{m_1, n_1} \Xi(\xi; m_1, \ldots, m_{k+1}) e^{i(m_1 + \ldots + m_{k+1}) \xi + (n_1 + \ldots + n_{k+1}) \eta]t},
\]

for some function $\Xi$.

**Proof.** (Induction on $k$)

$k = 0$: the hypothesis follows directly by the definition of $\beta_0^{m,n}(t; \xi)$ in (3.20).

$k \rightarrow k+1$: Follows from (3.25), (3.26) and from the inductive step, since for $G_\eta$ as in (3.7) for all $l \geq 0$ we have

\[
G_\eta \left( e^{i(m_1 + \ldots + m_l) \xi + (n_1 + \ldots + n_l) \eta]t} \right) = K(m_1, \ldots, m_l, n_1, \ldots, n_l) e^{i(m_1 + \ldots + m_l) \xi + (n_1 + \ldots + n_l) \eta]t},
\]

where $K(m_1, \ldots, m_l, n_1, \ldots, n_l) \in \mathbb{C}$. \hfill \Box
Remark 3.5. (i) Starting from (3.20) and using repeatedly (3.25) and (3.26) we can compute \( u_1, \phi_1^{m,n}, u_2, \phi_2^{m,n}, \ldots \). The solution \( u(t; \xi, \epsilon) \) of the range equation (3.9) and the function \( \phi_\epsilon(\xi) \) in the bifurcation equation (3.8) can be computed recursively with arbitrary accuracy. Their numerical computation for the dissipative spin-orbit problem (using Matlab) is presented in the next chapter and is based on this idea.

(ii) In order to compare the first and the second method, we explicitly compute \( \phi^{(0)}, \phi^{(1)} \) and \( \phi^{(2)} \) using equation (3.23).

\[
\phi^{(0)}(\xi) = \left( \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi \beta_0^{m,n}} \right) = \left( \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi e^{i(mp+nq)t}} \right) = \left( \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi e^{i(mp+nq)t}} \right) = \langle f_x(\xi + pt, qt) u_1 \rangle .
\]

Using equation (3.26) we have \( \beta_1^{m,n} = \im u_1 \beta_0^{m,n} \) and this implies

\[
\phi^{(1)}(\xi) = \left( \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi \beta_1^{m,n}} \right) = \left( \sum_{(m,n) \neq (0,0)} (im)^2 f_{m,n} e^{im\xi \beta_0^{m,n}} \right) = \langle f_{xx}(\xi + pt, qt) u_1 \rangle .
\]

Using two times equation (3.26) we get

\[
\beta_2^{m,n} = \frac{im}{2} (u_1 \phi_1^{m,n} + 2u_2 \beta_0^{m,n}) = \frac{im}{2} (u_1^2 \phi_1^{m,n} + 2u_2 \beta_0^{m,n})
\]

and this implies

\[
\phi^{(2)}(\xi) = \left( \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi \beta_2^{m,n}} \right) = \left( \sum_{(m,n) \neq (0,0)} \left( \frac{u_1^2}{2} f_{m,n} e^{im\xi \beta_0^{m,n}} + u_2 (im)^2 f_{m,n} e^{im\xi \beta_0^{m,n}} \right) \right) = \langle f_{xxx}(\xi + pt, qt) u_1^2 + f_{xx}(\xi + pt, qt) u_2 \rangle .
\]

Formulas (3.27), (3.28) and (3.30) (computed with the second method) are equal to formulas (3.16), (3.17) and (3.18) (computed with the first method), respectively. However, the second method is easier to handle because of the simple recursion formula.

### 3.3 Condition for non-degeneration of \( \phi_u(\xi) \) at any degree

#### 3.3.1 Condition for non-degeneration of \( \phi_u(\xi) \) at zero\(_n\)th degree

From (3.27) and Definition 3.3 the condition for the non-degeneration at zero\(_n\)th degree of \( \phi_u(\xi) \) is

\[
\phi^{(0)}(\xi) = \phi^{(0)}(\xi; p, q) := \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt, qt) dt \neq \text{const},
\]

for \( \xi \in [0, 2\pi] \). By (3.24) we have

\[
\phi^{(0)}(\xi; p, q) = \sum_{(m,n) \neq (0,0)} \im f_{m,n} e^{im\xi} \cdot \frac{1}{2\pi} \int_0^{2\pi} e^{i(mp+nq)t} dt.
\]

Notice that:
3.3. Condition for non-degeneration of $\phi_u(\xi)$ at any degree

(i) $\frac{1}{2\pi} \int_0^{2\pi} e^{i(mp+nq)t} dt = \begin{cases} 1, & \text{if } mp + nq = 0, \\ 0, & \text{otherwise}. \end{cases}$

(ii) Since $p$ and $q$ are coprime $mp + nq = 0$ is equivalent to $m = lq$ and $n = -lp$ with $l \in \mathbb{Z}$.

So, for $\phi^{(0)}(\xi)$ it follows:

$$\phi^{(0)}(\xi; p, q) = \sum_{l \neq 0} ilq f_{lq, -lp} e^{ilq\xi}$$

$$= \frac{1}{2} \left[ \sum_{l \neq 0} ilq f_{lq, -lp} e^{ilq\xi} + \sum_{l \neq 0} ilq f_{lq, -lp} e^{ilq\xi} \right]$$

$$= \frac{1}{2} \left[ \sum_{l \neq 0} ilq f_{lq, -lp} e^{ilq\xi} + \sum_{l \neq 0} (-ilq) f_{-lq, lp} e^{-ilq\xi} \right]$$

$$= \frac{1}{2} \sum_{l \neq 0} (ilq) \left[ f_{lq, -lp} e^{ilq\xi} - f_{-lq, lp} e^{-ilq\xi} \right].$$

Since by definition of $f$ (see equation (3.2)) $f_{lq, -lp} = \overline{f_{-lq, lp}}$ holds, we have

$$\phi^{(0)}(\xi; p, q) = \frac{1}{2} \sum_{l \neq 0} (ilq) 2i \text{Im} \left( f_{lq, -lp} e^{ilq\xi} \right)$$

$$= -\sum_{l \neq 0} lq \text{Im} \left( f_{lq, -lp} e^{ilq\xi} \right)$$

$$= -\sum_{l > 0} \left[ lq \text{Im} \left( f_{lq, -lp} e^{ilq\xi} \right) - lq \text{Im} \left( f_{-lq, lp} e^{-ilq\xi} \right) \right]$$

$$= -2q \sum_{l > 0} l \text{Im} \left( f_{lq, -lp} e^{ilq\xi} \right).$$ (3.31)

Remark 3.6. If $\phi^{(0)}(\xi) = \text{const for } \xi \in [0, 2\pi]$ then $\phi^{(0)}(\xi) \equiv 0$.

Remark 3.7. By Remark 3.1 (see in particular (3.3)) equation (3.31) for the special case of the dissipative spin-orbit problem is equivalent to

$$\phi^{(0)}(\xi) = \begin{cases} -2\alpha_2 p \sin(2\xi), & \text{for } q = 1, \\ -2\alpha_2 p \sin(2\xi), & \text{for } q = 2, \\ 0, & \text{otherwise}. \end{cases}$$ (3.32)

which was already found in [8] (see formula (56)). This is true, since for $m = \pm 2$

$$\left\{ l \in \mathbb{Z} \mid l > 0 \text{ and } l = \frac{m}{q} \right\} = \begin{cases} \{2\}, & \text{for } q = 1, \\ \{1\}, & \text{for } q = 2, \\ \emptyset, & \text{otherwise}. \end{cases}$$

holds.

Lemma 3.6. Let $p, q$ be positive coprime integers and assume that there exists $l \in \mathbb{Z} \setminus \{0\}$ such that

$f_{lq, -lp} \neq 0,$

where the coefficients $f_{m,n}$ are given in (3.2). Then the general spin-orbit problem in (3.1) is non-degenerate at degree zero, i.e. $\phi^{(0)}(\xi) \neq \text{const}$ according to Definition 3.3.
Proof. For $\xi \in [0, 2\pi]$ we define $f_{lq,-lp} = A_l + iB_l$, with $A_l, B_l \in \mathbb{R}$. Then we have
\[
f_{lq,-lp} e^{ilq\xi} = A_l \cos(lq\xi) - B_l \sin(lq\xi) + i(B_l \cos(lq\xi) + A_l \sin(lq\xi)).
\]
The degenerate case $\phi^{(0)} \equiv 0$ happens if and only if the imaginary part of the right hand side in the last equation disappears, i.e.
\[
B_l \cos(lq\xi) + A_l \sin(lq\xi) = 0, \text{ for all } \xi \in [0, 2\pi].
\]
Choosing $\xi = 0$ we get $B_l = 0$. This means $f_{lq,-lp} \in \mathbb{R}$ for all $l \neq 0$. So it follows:
\[
\phi^{(0)}(\xi; p, q) \overset{(3.31)}{=} -2q \sum_{l>0} l \text{ Im} \left( f_{lq,-lp} e^{ilq\xi} \right) = -2q \sum_{l>0} l f_{lq,-lp} \sin(lq\xi).
\]
The functions $\sin(lq\xi)$, $\sin(lq\xi)$ are linearly independent for $l \neq \tilde{l}$ and $l, \tilde{l} > 0$. This fact implies that
\[
\phi^{(0)}(\xi; p, q) = \text{const} \iff f_{lq,-lp} = 0, \forall l \neq 0.
\]
This terminates the proof of Lemma 3.6. \qed

3.3.2 Condition for the non-degeneration of $\phi_u(\xi)$ at first degree
We assume that $\phi^{(0)}(\xi) \equiv \text{const}$, which by Remark 3.6 implies $\phi^{(0)}(\xi) \equiv 0$. From (3.28) and Definition 3.3 the condition for the non-degeneration of $\phi_u(\xi)$ at first degree is
\[
\phi^{(1)}(\xi) = \frac{1}{2\pi} \int_{0}^{2\pi} f_{xx}(\xi + pt, qt) u_1(t) dt \neq \text{const},
\]
for $\xi \in [0, 2\pi]$. Using (3.25) with $k = 1$ it follows
\[
u_1 = -G_\eta \left( i \sum_{(m,n) \neq (0,0)} m_f e^{im\xi} \gamma^{m,n}_0 - \left( i \sum_{(m,n) \neq (0,0)} m_f e^{im\xi} \gamma^{m,n}_0 \right) \right)
\overset{(3.20)}{=} -G_\eta \left( \sum_{(m,n) \neq (0,0)} (im) f_m e^{im\xi} e^{ip\xi} + \left( \sum_{(m,n) \neq (0,0)} (im) f_m e^{im\xi} e^{ip\xi} \right) \right)
\overset{(3.24)}{=} -G_\eta \left( f_{xx}(\xi + pt, qt) - \left( f_{xx}(\xi + pt, qt) \right) \right).
\]
By (3.7), (3.28), (3.33) we get
\[
\phi^{(1)}(\xi) = -\left( f_{xx}(\xi + pt, qt) G_\eta \left( f_{xx}(\xi + pt, qt) - \frac{1}{2\pi} \int_{0}^{2\pi} f_{xx}(\xi + pt, qt) dt \right) \right).
\]
From the Fourier-expansion of the Newtonian potential $f(x, t)$ in (3.2) and from equations (3.7) and (3.24) we have
\[
\phi^{(1)}(\xi) = -\left( f_{xx}(\xi + pt, qt) G_\eta \left( f_{xx}(\xi + pt, qt) \right) \right)
= \left( \sum_{(m_1,n_1) \neq (0,0)} (im_1)^2 f_{m_1,n_1} e^{im_1\xi} e^{ip_{1}t+n_{1}q} \right).
\]
3.3. Condition for non-degeneration of $\phi_u(\xi)$ at any degree

We notice that $m_1 p + n_1 q + m_2 p + n_2 q = 0$ if and only if $n_1 = \frac{-(m_1 + m_2) p}{q} - n_2$. Since for $n \in \mathbb{Z}$ we have \[ \frac{1}{2\pi} \int_0^{2\pi} e^{int} dt = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases} \]

\[ \phi^{(1)}(\xi) = -im_1^2 m_2 \sum_{\substack{(m_1, m_2) \neq (0, 0) \\ n_1 \notin \{0, -\frac{(m_1 + m_2) p}{q}\}}} \frac{\tilde{f}_{m_1, n_1} e^{i m_1 \xi} \tilde{f}_{m_2, -\frac{(m_1 + m_2) p}{q} - n_1} e^{i m_2 \xi}}{(-m_1 p - n_1 q)^2 - i \eta(-m_1 p - n_1 q)} \]

\[ = -im_1^2 m_2 \sum_{\substack{(m_1, m_2) \neq (0, 0) \\ n_1 \notin \{0, -\frac{(m_1 + m_2) p}{q}\}}} \frac{\tilde{f}_{m_1, n_1} \tilde{f}_{m_2, -\frac{(m_1 + m_2) p}{q} - n_1} e^{i(m_1 + m_2) \xi}}{(m_1 p + n_1 q)^2 + i \eta(m_1 p + n_1 q)}. \] (3.34)

Furthermore notice that, if $m_1 + m_2 = 0$ then $\phi^{(1)}(\xi)$ is constant. So we have proved that:

**Lemma 3.7.** The existence of exponents $m_1 p + n_1 q$ for $i = 1, 2$ in the $t$-expansion of $f(x, t)$ with $f_{m_1, n_1}, f_{m_2, n_2} \neq 0$ for some $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$ such that $q$ divides $m_1 + m_2 \neq 0$ is a necessary condition for the non-degeneration of $\phi_u(\xi)$ at first degree, i.e.

\[ \phi^{(1)}(\xi) \neq \text{const} \quad \Rightarrow \quad \exists m_1, m_2 \quad \text{with } m_1 + m_2 \neq 0 \quad \text{and} \]

\[ f_{m_1, n_1}, f_{m_2, n_2} \neq 0 \quad \text{for some } n_1, n_2 \in \mathbb{Z} \setminus \{0\} \]

such that $q | (m_1 + m_2)$.

**Remark 3.8.** As we know from Remark 3.1 in the dissipative spin-orbit problem $m = \pm 2$. Assuming $\phi^{(0)}(\xi) \equiv 0$ (i.e. $q \neq 1, 2$ cf. (3.32)) and noticing that $\pm 2 \pm 2 = \{-4, 0, 4\}$ we have

\[ q \pm 4 \quad \Rightarrow \quad q = 4. \]

This implies that the only non-degenerate case at first degree for the dissipative spin-orbit problem is $q = 4$. From the proof of Lemma 3.7 and in particular from (3.34) (with $m_1 = m_2 = 2$ or $m_1 = m_2 = -2$) the function $\phi^{(1)}(\xi)$ is given by

\[ \phi^{(1)}(\xi) = \text{const} + \sum_{n \neq 0, -p} \frac{-i \tilde{f}_{2, n} \tilde{f}_{-2, p-n} e^{i4 \xi}}{(2p + nq)^2 + i \eta(2p + 4n)} - \sum_{n \neq 0, p} \frac{-i \tilde{f}_{-2, n} \tilde{f}_{2, p-n} e^{-i4 \xi}}{(-2p + 4n)^2 + i \eta(-2p + 4n)} \]

\[ = \text{const} + \frac{8}{i} \left[ \sum_{n \neq 0, p} \frac{\tilde{f}_{-2, n} \tilde{f}_{2, p-n} e^{i4 \xi}}{(2p - 4n)^2 + i \eta(2p - 4n)} - \sum_{n \neq 0, p} \frac{\tilde{f}_{-2, n} \tilde{f}_{2, p-n} e^{-i4 \xi}}{(2p - 4n)^2 - i \eta(2p - 4n)} \right]. \]

Moreover, by (3.2) and (3.3) we have

\[ f_{2, -n} = \overline{f_{-2, n}} = f_{-2, n} \quad \text{and} \quad f_{2, -p+n} = \overline{f_{-2, p-n}} = f_{-2, p-n}, \]

since for the dissipative spin-orbit problem the coefficients $f_{\pm 2, n}$ of the Newtonian potential are real. So it follows

\[ \phi^{(1)}(\xi) - \text{const} = 16 \text{Im} \left[ \sum_{n \neq 0, p} \frac{\tilde{f}_{2, n} \tilde{f}_{-2, p-n} e^{i4 \xi}}{(2p - 4n)^2 + i \eta(2p - 4n)} \right]. \]
The hypothesis follows from (3.25), (3.26), Lemma 3.5 and the fact that for all $\phi$ we know from (3.23) the term $\phi^{(1)}(\xi) = \text{const} - 4 \sin(4\xi) \sum_{j \in \mathbb{Z}, j \neq 0} \frac{\alpha_{p-j} \alpha_j}{4(p-2j)^2 + \eta^2}$.

where we used that the function $s(n) := \frac{1}{(2p-4n)^2 + \eta^2} (2p-4n) \sin(4\xi) - \cos(4\xi) s(n) f_{2,-p+n}$

for some $\eta$. By (3.3) we know that $f_{2,-n} = \frac{\alpha_n}{2}$ and $f_{2,-p+n} = \frac{\alpha_{p-n}}{2}$ hold, so we have

$$\phi^{(1)}(\xi) = \text{const} - 4 \sin(4\xi) \sum_{j \in \mathbb{Z}, j \neq 0} \frac{\alpha_{p-j} \alpha_j}{4(p-2j)^2 + \eta^2} \quad (3.35)$$

with const $\in \mathbb{R}$, which was already found in [8] (see formula (58)).

3.3.3 Condition for the non-degeneration of $\phi_n(\xi)$ at $k_{th}$ degree

In the following theorem a necessary condition for the non-degeneration of the general spin-orbit problem in (3.2) at any degree is given. This allows us to predict, at which level of degeneration some $(p, q)$-orbit may exists.

**Theorem 3.3.** Let $p, q$ be positive coprime integers. The existence of exponents $m_ip + n_iq$ for $i = 1, 2, \ldots, k+1$ in the $t$-expansion of $f(x, t)$ with $f_{m_1, n_1}, f_{m_2, n_2}, \ldots, f_{m_{k+1}, n_{k+1}} \neq 0$ for some $n_1, n_2, \ldots, n_{k+1} \in \mathbb{Z} \setminus \{0\}$ such that $q$ divides $m_1 + m_2 + \ldots + m_{k+1} \neq 0$ is a necessary condition for the non-degeneration at $k_{th}$-degree of the general spin-orbit problem, i.e.

$$\phi^{(k)}(\xi) \neq \text{const} \quad \Rightarrow \quad \exists m_1, m_2, \ldots, m_{k+1}, n_1, n_2, \ldots, n_{k+1} \in \mathbb{Z} \setminus \{0\}$$

with $m_1 + m_2 + \ldots + m_k \neq 0$,

$$f_{m_1, n_1}, f_{m_2, n_2}, \ldots, f_{m_k, n_k}, f_{m_{k+1}, n_{k+1}} \neq 0$$

and $q|(m_1 + m_2 + \ldots + m_{k+1})$.

**Proof.** As we know from (3.23) the term $\phi^{(k)}(\xi)$ has the form

$$\phi^{(k)}(\xi) = \left( i \sum_{(m,n) \neq (0,0)} m f_{m,n} e^{im\xi} \beta_k^{m,n} \right).$$

The hypothesis follows from (3.25), (3.26), Lemma 3.5 and the fact that for all $\mathbb{Z} \ni k \neq 0$ we have $\int_0^{2\pi} e^{ikt} dt = 0$. □

3.4 Application to the classical spin-orbit problem

The condition of Theorem 3.3 allows us to formulate a conjecture for the degeneration of $(p, q)$-periodic orbits for the dissipative spin-orbit problem. Since in this special case $m$ takes only the values $\pm 2$, we have that $q$ divides

$$\pm 2 \pm 2 \pm \ldots \pm 2$$

$k_{th}$ times, in order to get, that $\phi_n(\xi)$ is non-degenerate at $k_{th}$ degree.
Theorem 3.4. If the spin-orbit problem is non-degenerate at $k_{th}$-degree, then $q | 2(k+1)$ and if $k \neq 0$ also $q \nmid 2l$ for $0 < l \leq k$ hold.

Proof. The proof follows directly from Theorem 3.3.

Remark 3.9. (i) For example if $q = 7$ the system is non-degenerate at $6_{th}$-degree, since $7 | 14 = 2 \cdot (6 + 1)$ and $7 \nmid \{2, 4, 6, 8, 10, 12\}$.

(ii) For $q$ even the system is non-degenerate at $(\frac{q}{2} - 1)_{th}$-degree. For $q$ odd the system is non-degenerate at $(q - 1)_{th}$-degree.

(iii) Theorem 3.4 gives a necessary condition for the non-degeneration at $k_{th}$-degree, for $k \geq 0$. The question arises: Is this condition also sufficient? In Chapter 4 we answer this question for the cases $q = 1, 2$ and $4$. Hence in these special cases we give the proof of “$\Leftarrow$” of Theorem 3.4.

Conjecture 3.5

Let $k > 0$. If $q | 2(k+1)$ and $q \nmid 2l$ for $0 < l \leq k$ then the dissipative spin-orbit problem is non-degenerate at $k_{th}$-degree.

In Chapter 5 we will do a numerical study for $4 \neq q \geq 3$, in order to give numerical evidence for Conjecture 3.5.
4 Quantitative conditions for the existence of low-order spin–orbit resonances

This chapter will appear in Note di Matematica, (2014), (see [1]).

We consider \((p,q)\)-periodic orbits of a dissipative spin-orbit model. L. Biasco and L. Chierchia in [8] give a sufficient condition for the existence of such orbits for \(q = 1, 2\) and 4, imposing that the functions given in (3.32) and in (3.35) are not constant. In this chapter we prove that this condition is satisfied provided that the eccentricity and, in the case \(q = 4\), also the dissipation, is smaller than an explicitly given bound.

4.1 Results

Recall that we define \(\alpha_j = \alpha_j(e)\) in (2.1) to be the Fourier coefficients of the Newtonian potential \(f\) given in equation (1.29). Since by Lemma 2.1 we know that \(\alpha_0(e) = 0\) for \(e \in [0, 1)\) this implies

\[
f(x, t; e) = \sum_{j \neq 0, j \in \mathbb{Z}} \alpha_j(e) \cos(2x - j t).
\]

From Theorem 1.2 and Proposition 2.10 of [8] L. Biasco and L. Chierchia we can formulate the following theorem:

**Theorem 4.1.** Let \(p\) and \(q\) be positive coprime integers\(^1\), with \(q = 1, 2\) or 4. Let \(\alpha_j = \alpha_j(e)\) be as in (2.1). There exist \(\bar{\varepsilon}_0 > 0\) and \(\bar{\eta}_0 > 0\) such that if \(0 < \bar{\varepsilon} \leq \bar{\varepsilon}_0\), \(0 < \bar{\eta} \leq \bar{\eta}_0\) and

\[
\left| \bar{\nu} - \frac{p}{q} \right| < \begin{cases} 
\frac{\bar{\varepsilon}}{\bar{\eta}} |\alpha_{2p}|, & \text{if } q = 1, \\
\frac{\bar{\varepsilon}}{\bar{\eta}} |\alpha_p|, & \text{if } q = 2, \\
32 \frac{\bar{\varepsilon}}{\bar{\eta}} \left| \sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{\alpha_{p-j}(e)\alpha_j(e)}{4(p-j)^2 + q^2 \eta^2} \right|, & \text{if } q = 4,
\end{cases}
\]

hold, then the dissipative spin-orbit problem modelled by equations (1.24)-(1.32) admits a \((p,q)\)-periodic solution \(x_{pq}\).

**Remark 4.1.** Theorem 4.1 is applicable if we can prove that it exists \(\exists \mathbf{e}_* > 0\) such that

\[
0 \neq \begin{cases} 
\alpha_{2p}(e), & \text{for } q = 1, \\
\alpha_p(e), & \text{for } q = 2, \\
\sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{\alpha_{p-j}(e)\alpha_j(e)}{4(p-j)^2 + q^2 \eta^2}, & \text{for } q = 4,
\end{cases}
\]

(4.1)

for all \(0 < \mathbf{e} \leq \mathbf{e}_*\).

\(^1\)Equivalently \((p, q) = 1\).
For $0 < b < 1$ let
\[ r(b) := \frac{b}{\cosh b} \quad \text{and} \quad M(b) := \frac{2}{(1 - b)^2} \left( (1 + r(b))(1 + \cosh b) + 1 - b \right)^2. \] (4.2)

Moreover define
\[ c_{p,1} := \frac{|2p|^{2p-2}}{2^{2p-1}+1|2p-2|!}, \] (4.3)
\[ c_{p,2} := \frac{|p|^{p-2}}{2^{p-1}+1|p-2|!}, \] (4.4)
\[ c_{p,4} := \begin{cases} \frac{1}{2^{p-1}p(p-4)(p-2)!}, & \text{for } p \leq 1, \\ (2-p)^{1-p} & \text{for } p = 3, \\ \frac{(p-2)^{p-2}}{2^{p-1}(p-4)(p-2)!}, & \text{for } p \geq 5. \end{cases} \] (4.5)

Note that
\[ c_{p,1} \approx \frac{1}{\sqrt{p}} \left( \frac{e}{2} \right)^{|2p|} \to \infty, \quad c_{p,2} \approx \frac{1}{\sqrt{p}} \left( \frac{e}{2} \right)^{|p|} \to \infty \quad \text{and} \quad c_{p,4} \approx \frac{1}{p^2\sqrt{p}} \left( \frac{e}{2} \right)^{|p|} \to \infty \] (4.6)
as $p \to \pm \infty$.

The main result of this chapter is stated in the following theorem.

**Theorem 4.2.** Besides the assumptions of Theorem 4.1 on $p, q, \bar{\varepsilon}, \bar{\eta}, \bar{\nu}$, for $0 < b < 1$ let $r(b)$, $M(b)$, $c_{p,q}$ be as in equations (4.2)-(4.5).

- If $q = 1, 2$, assume
  \[ 0 < e \leq c_{p,q} \frac{1}{M(b)} \left( \frac{r(b)}{2} \right)^{\frac{|2q-2|+1}{q}}. \] (4.7)

- If $q = 4$, assume
  \[ 0 < e \leq \frac{r(b)}{2\pi M(b)} \left( \frac{r(b)}{\bar{\nu}(c_{p,q} r(b))^{p-4}+|p-4|} \right)^{1/2} \] (4.8)
\[ \bar{\eta} < \frac{3\sqrt{3}(c_{p,q})^{p-4/2}}{2\pi M(b)}. \] (4.9)

Then there exists a $(p,q)$–periodic orbit of the dissipative spin-orbit problem modelled by equations (1.24)-(1.32).

**Remark 4.2.** (i) Condition (4.1) depends only on $e$ when $q = 1, 2$ while it depends also on $\bar{\eta}$ when $q = 4$. This is why, for $q = 4$, we have to assume also the condition (4.9).

(ii) Conditions on the existence of $(p,q)$–periodic orbits are obtained combining Theorem 4.1 and Theorem 4.2. For the case $p/q = 1/1$ equation (4.7) is equivalent to $0 < e \leq 0.000412055$, choosing $b = 0.148642$. This condition on the eccentricity is satisfied by Tethys (a satellite of Saturn), which has an eccentricity approximately equal to 0.0001 and is known (from astronomical observation) to be locked in a $(1,1)$–periodic spin-orbit.
4.2 Proof of Theorem 4.2

For the case \( p/q = 3/2 \) equation (4.7) is equivalent to \( 0 < e \leq 0.0000609886 \), choosing \( b = 0.253122 \). Mercury (seen as satellite of the Sun), which represents the only observed example of the (3, 2)-periodic spin-orbit in our Solar system, doesn’t fulfill this condition, since its eccentricity is approximately equal to 0.2056.

In Chapter 6 one focus on the cases \( p/q = 1/1, 3/2 \) improving the estimate for the existence of \( (p, q) \)-periodic orbits in order to cover the values of the eccentricity of all satellites in our Solar system, which are observed to be in a \((1, 1)\)– or a \((3, 2)\)–periodic orbit.

(iii) We also note that the proof of the theorem above is quite simple in the cases \( q = 1, 2 \): one has to prove that some analytic function of \( e \) (namely \( \alpha_{2p} \) and \( \alpha_p \), respectively) are not identically zero. Since the behavior of such functions is well known (see Lemma 2.1), this goal is easily obtained. On the other hand the case \( q = 4 \) is more difficult (already in the case \( \bar{\eta} = 0 \)), since one has to prove that a series of functions with changing signs is not identically zero and compensations could occur.

To prove Theorem 4.2 we have to find a lower bound for the absolute value of the right hand side of (4.1). Since this quantity is a function of \( e \), we evaluate the leading term of the \( e \) expansion and we prove that, if \( e \) satisfies the conditions (4.7)-(4.9), the difference between the leading term and the rest do not change sign.

### 4.2 Proof of Theorem 4.2

In order to prepare and to understand the proof of Theorem 4.2 we need some intermediate results. The following lemma gives an upper bound on the Fourier coefficients \( \alpha_j \).

**Lemma 4.1.** Let \( 0 < b < 1 \) and let \( r(b) \) and \( M(b) \) be as in (4.2). The solution \( u_e(t) \) of the Kepler equation (1.32) is a holomorphic function in the ball

\[
|e| < r(b) = \frac{b}{\cosh b}
\]

satisfying

\[
\sup_{t \in \mathbb{R}} |u_e(t) - t| \leq b.
\]

The functions \( \rho_e(t) \) in (1.30) and \( G_e(t) \) in (2.2) satisfy

\[
|\rho_e(t)| \geq 1 - b, \quad \forall t \in \mathbb{R}, \; |e| < r(b),
\]

and

\[
|G_e(t)| \leq \frac{2}{(1 - b)^3} \left( |1 - e|(1 + \cosh b) + 1 - b \right)^2, \quad \forall t \in \mathbb{R}, \; |e| < r(b),
\]

respectively.

**Proof.** Using that

\[
\sup_{|\text{Im} \; z| \leq b} |\sin z| = \sup_{|\text{Im} \; z| \leq b} |\cos z| = \cosh b,
\]

it is simple to verify that for \( |e| < r(b) \) the map

\[
v \mapsto \chi(v; e)
\]
with
\[ (\chi(v; e))(t) := e \sin(v(t) + t) \]
is a contraction in the closed ball of radius \( b \) in
\[ C(\mathbb{R}, \mathbb{C})_b := \left\{ v(t) \in C(\mathbb{R}, \mathbb{C}) : \|v(t)\|_{\infty} = \sup_{t \in \mathbb{R}} |v(t)| \leq b \right\}, \]
the space of continuous functions endowed with the sup-norm. In fact, by \(|e| < r(b) = \frac{b}{\cosh b}\) and (4.13) the function \( \chi \) maps \( C(\mathbb{R}, \mathbb{C})_b \) into itself, indeed
\[ \|\chi(v; e)\|_{\infty} \leq r(b) \cosh(b) = b. \]
Furthermore for \( u, v \in C(\mathbb{R}, \mathbb{C})_b \) we have
\[ \|\chi(v; e) - \chi(u; e)\|_{\infty} = |e| \sup_{t \in \mathbb{R}} |\sin(v(t) + t) - \sin(u(t) + t)| \]
\[ = |e| \sup_{t \in \mathbb{R}} \left| \int_0^1 (v(t) - u(t)) \cos(u(t) + t + s(v(t) - u(t))) ds \right| \]
\[ \leq |e| \cosh(b) \|v - u\|_{\infty}, \]
\[ (4.13) \]
since
\[ |\text{Im} \ (u(t) + t + s(v(t) - u(t)))| = (1 - s) |\text{Im} \ (u(t))| + s |\text{Im} \ (v(t))| \leq b \]
holds for \( s \in [0, 1] \). By the Banach fixed point theorem there exists a unique \( \tilde{v}_e(t) \in C(\mathbb{R}, \mathbb{C})_b \) such that
\[ \tilde{v}_e(t) = e \sin(\tilde{v}_e(t) + t). \]
Notice that \( u_e(t) := \tilde{v}_e(t) + t \) satisfies the Kepler equation (1.32), indeed we have
\[ \tilde{v}_e(t) + t - e \sin(\tilde{v}_e(t) + t) = \tilde{v}_e(t) + t - \tilde{v}_e(t) = t. \]
Moreover, since \( \chi(v; e) \) is holomorphic in \( e \), the same holds for the fixed point \( \tilde{v}_e(t) \) of \( \chi \). Equation (4.10) follows by the fact that \( \tilde{v}_e(t) \in C(\mathbb{R}, \mathbb{C})_b \). Since by (4.10) we get
\[ |\text{Im} \ (u_e(t))| \leq b, \quad \forall t \in \mathbb{R}, \ |e| < r, \]
(4.14)
estimate (4.11) follows by
\[ |\rho_e(t)| \geq 1 - |e| \|\cos(u_e(t))\| \geq 1 - r \cosh b = 1 - b. \]
As in Lemma 2.2 define \( w_e = w(u; e) := \sqrt{\frac{1 + e}{1 - e}} \tan \frac{\theta}{2} \) and \( w_e(t) := w(u_e(t), e) \). Then by (2.15) we get
\[ |e^{2i u_e(t)}| = \left| \frac{[w_e(t) - i]^4}{[w_e(t)]^2 + 1} \right| = \left( \frac{|w_e(t) - i|^2}{[w_e(t)]^2 + 1} \right)^2 \]
\[ \leq \left( \frac{4}{[w_e(t)]^2 + 1} \right)^2 \quad \left( \frac{1 - |e| |1 + \cos u_e(t)|}{|1 - e \cos u_e(t)|} + 1 \right)^2. \]  
(4.15)
To prove the inequality (\( * \)) we notice that for \( x = |w - i| \) we have
\[ 4 + 2|w - i||w + i| - |w - i|^2 \geq 4 + 2x(x - 2) - x^2 = (x - 2)^2 \geq 0, \]
since \( |w + i| = |w - i + 2i| \geq |w - i| - 2 \). To prove (**) we used that \( \tan^2(\beta/2) = (1 - \cos \beta)/(1 + \cos \beta) \) holds for every angle \( \beta \). Then (4.12) follows by (4.11), (4.15), (4.13) and (4.14), i.e.

\[
|G_e(t)| = \frac{|e^{2it_e(t)}|}{2|\rho_e(t)|^3} \leq 4 \left( \frac{1 - |e| |1 + \cos u_e(t)|}{|1 - e \cos u_e(t)| + 1} \right)^2 \frac{1}{2(1 - b)^3} \leq \frac{2}{(1 - b)^5} \left( 1 - |e|(1 + \cosh b) + 1 - b \right)^2.
\]

This terminates the proof of Lemma 4.1.

**Lemma 4.2.** Let \( 0 < b < 1 \) and let \( r(b) \) and \( M(b) \) be as in (4.2). Then

\[
|\alpha_j(e)| \leq \frac{2}{(1 - b)^5} \left( 1 - |e|(1 + \cosh b) + 1 - b \right)^2 \leq M(b), \quad \forall j \in \mathbb{Z} \quad \forall |e| < r(b),
\]

holds.

**Proof.** From the proof of Lemma 2.1 we know that \( \alpha_j = \frac{1}{2\pi} \int_0^{2\pi} G_e(t)e^{-ijt}dt \) holds for all \( j \in \mathbb{Z} \), with \( G_e(t) \) defined in (2.2). This proves equation (4.16).

In the following lemma we determine the sign of the leading term in the \( e \) expansion of \( \alpha_j(e) \).

**Lemma 4.3.** Let \( \tilde{\alpha}_j \) as in (2.17). Then \( \tilde{\alpha}_j < 0 \) for all \( j \in \mathbb{Z} \setminus \{0, 1\} \), \( \tilde{\alpha}_0 = 0 \) and \( \tilde{\alpha}_1 = \frac{1}{4} > 0 \).

**Proof.** \( \tilde{\alpha}_1 = \frac{1}{4} > 0 \) follows directly from (2.18) and (2.19). Moreover, Lemma 2.1 implies \( \tilde{\alpha}_0 = 0 \). If \( j \geq 2 \) we see from (2.20) that all coefficients \( (q_i)_{0 \leq i \leq k} \) of \( Q_k(x) = q_0 + q_1x + \ldots + q_kx^k \) are strictly negative for every \( k \geq 0 \). By (2.18) we have \( \tilde{\alpha}_j = Q_{j-2}(j) < 0 \). Otherwise if \( j \leq -1 \) we have from (2.18), (2.19) that

\[
\tilde{\alpha}_j = \left( -\frac{1}{2} \right)^{3-j} \frac{j^{2-j}}{(2-j)!} \frac{(-1)^{3-j}(-1)^{2-j}}{2^{3-j}} \frac{|j|^{2-j}}{(2-j)!} = \frac{-1}{2^{3-j}} \frac{|j|^{2-j}}{(2-j)!} < 0.
\]

In the next lemma we estimate the absolute value of the difference between \( \alpha_j(e) \) and its leading term in the \( e \) expansion.

**Lemma 4.4.** Let \( r(b), M(b) \) be as in (4.2) and \( \alpha_j(e) \) be as in (2.1) and let

\[
\bar{r}_j(b) := \frac{|j|^{j-2}|}{2|j-1|^{j+1}|j - 2|! M(b) \left( \frac{r(b)}{2} \right)^{|j-2|+1}}, \quad \text{for all } j \in \mathbb{Z}.
\]

Then

\[
|\alpha_j(e) - \tilde{\alpha}_j e^{j-2}| \leq \frac{1}{2} |\tilde{\alpha}_j| |e||j-2| \quad \text{for all } |e| \leq \bar{r}_j(b).
\]

holds for all \( |e| \leq \bar{r}_j(b) \). Moreover, it follows

\[
\text{sgn}(\alpha_j(e)) = \text{sgn}(\tilde{\alpha}_j)
\]

for \( 0 \leq e \leq \bar{r}_j(b) \).
Remark 4.3. Note that $\tilde{r}_j(b) \approx \left( \frac{e^{r(b)}}{4} \right)^j \to 0$ as $j \to \infty$.

Proof. For $j = 0$ this lemma is trivial, since by Lemma 2.1 we know that $\alpha_0(e) = \tilde{\alpha}_0 = 0$ holds. Fix $b$ and $r(b)$ as in (4.2) and set $\rho_j(b) := r(b) - \tilde{r}_j(b)$. Then by the Cauchy estimate and (4.16) we obtain

$$\frac{d^n}{de^n}\alpha_j(e) \leq \frac{n!M(b)}{\rho_j(b)^n}, \quad \forall |e| \leq \tilde{r}_j(b), j \in \mathbb{Z}, n \in \mathbb{N}. \quad (4.19)$$

By (2.17) and the integral form for the remainder in the Taylor series we have

$$\alpha_j(e) = \bar{\alpha}_j(e)^{|j-2|} + R_{j-2}(e), \quad (4.20)$$

$$R_{j-2}(e) := \frac{e^{|j-2|+1}}{|j-2|!} \int_0^1 \frac{d^{j-2}|+1}\alpha_j(es) ds \left( 1 - s \right)^{|j-2|} ds. \quad (4.21)$$

Using (4.19) and (4.21) with $n = |j-2| + 1$ we have

$$|R_{j-2}(e)| \leq \frac{|e^{|j-2|+1}|}{|j-2|!} \frac{M(b)(|j-2|+1)!}{\rho_j(b)^{|j-2|+1}} \int_0^1 (1 - s)^{|j-2|} ds = \frac{M(b)|e^{|j-2|+1}|}{\rho_j(b)^{|j-2|+1}}, \quad (4.22)$$

Notice that by (2.19) and (2.20) we have:

$$|P_{j-2}(j)| = \frac{|j|^{|j-2|}}{2^{j-1}j!|j-2|!}, \quad \text{for } j \leq 2; \quad (4.23)$$

$$|Q_{j-2}(j)| = \frac{1}{2^{j-1}} \sum_{l=0}^{j-2} j! \binom{j-2}{l} 2^l \geq \frac{|j|^{|j-2|}}{2^{j-1}j!}, \quad \text{for } j > 2. \quad (4.24)$$

From (4.23) and (4.24) we get

$$|\bar{\alpha}_j| \geq \frac{|j|^{|j-2|}}{2^{j-1}j!}, \quad \text{for all } j \in \mathbb{Z}. \quad (4.25)$$

By (4.20), (4.22) and (4.25) we have that (4.18) follows if we prove that

$$|R_{j-2}(e)| \leq \frac{1}{2} |\bar{\alpha}_j| |e^{|j-2|}$$

or equivalently

$$\frac{M(b)\tilde{r}_j(b)}{\rho_j(b)^{|j-2|+1}} \leq \frac{|j|^{|j-2|}}{2^{j-1}j!}. \quad (4.26)$$

Since $\rho_j(b) \geq r(b)/2$ (with $r(b)$ defined in (4.2)), (4.26) follows by the definition of $\tilde{r}_j(b)$ in (4.17). \qed

Remark 4.4. The estimate given in (4.18) is not optimal\footnote{Better estimates in the cases of astronomical interest, as for example Earth-Moon or Mercury-Sun system, can be found in Chapter 6.} for $\tilde{r}_j(b)$.

Lemma 4.5. Let $p \in \mathbb{Z}$. It follows

$$|p - 4| \leq |p - j - 2| + |j - 2|, \quad \forall j \in \mathbb{Z}$$

and equality holds if one of the following two cases occurs:

- **Case 1:** $p \geq 4$ and $2 \leq j \leq p - 2$;
4.2. Proof of Theorem 4.2

- Case 2: \( p \leq 4 \) and \( p - 2 \leq j \leq 2 \).

**Proof.** Let us define

\[ v(p, j) := |p - j - 2| + |j - 2| = |j - (p - 2)| + |j - 2|. \]

In the first case, when \( p \geq 4 \), the function \( v(p, j) \) takes is minimum for \( 2 \leq j \leq p - 2 \) and for these values of \( j \) we have \( |p - 4| = |j - (p - 2)| + |j - 2| \) (see Figure 4.1).

\[ y = |j - 2| \quad y = |j - (p - 2)| \]

Figure 4.1: Sketch of the function \( v(p, j) \) in the case \( p \geq 4 \) (the case \( p \leq 4 \) is analogous).

Analogously, when \( p \leq 4 \), the function \( v(p, j) \) takes is minimum for \( p - 2 \leq j \leq 2 \) and, as in the first case, for these values of \( j \) we have \( |p - 4| = |j - (p - 2)| + |j - 2| \). □

**Lemma 4.6.** Let \( p \in \mathbb{Z} \) be odd. Then we have

\[
\begin{align*}
\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_{j}}{4(p-2)j^2} &> \frac{(2-p)(4-p)^{p-2}}{32(p-4)^2(p-2)^2} > 0, \quad \text{for } p \leq 1, \\
\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_{j}}{4(p-2)j^2} &< \frac{1}{16} < 0, \quad \text{for } p = 3, \quad \text{(4.27)} \\
\sum_{j=2}^{p-2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_{j}}{4(p-2)j^2} &> \frac{(p-2)^{p-2}}{2^{p-1}(p-4)^2(p-2)^2} > 0, \quad \text{for } p \geq 5.
\end{align*}
\]

**Proof.** If \( p \leq 1 \) using Lemma 4.3 we have

\[
\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_{j}}{4(p-2)j^2} = 2 \sum_{j=1}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_{j}}{4(p-2)j^2} > \frac{1}{2} \left( \frac{\bar{\alpha}_{p-1}\bar{\alpha}_{1}}{(p-2)^2} + \frac{\bar{\alpha}_{p-2}\bar{\alpha}_{2}}{(p-4)^2} \right) > \frac{1}{32} \frac{\bar{\alpha}_{p-2}\bar{\alpha}_{2}}{(p-4)^2}.
\]

The last estimate is obtained as follows. Since \( \bar{\alpha}_2 = -\frac{1}{2}, \bar{\alpha}_1 = \frac{1}{4} \) it is equivalent to

\[
\left( \frac{1}{2} - \frac{1}{32} \right) \frac{\bar{\alpha}_{p-2}\bar{\alpha}_{2}}{(p-4)^2} > -\frac{1}{2} \frac{\bar{\alpha}_{p-1}\bar{\alpha}_{1}}{(p-2)^2} \\
\frac{15}{64} \frac{\bar{\alpha}_{p-2}(p-2)^2}{(p-2)^2} > \frac{1}{8} \frac{\bar{\alpha}_{p-1}(p-4)^2}{(p-4)^2} \\
15 \frac{\bar{\alpha}_{p-2}(2-p)^2}{(2-p)^2} < 8 \frac{\bar{\alpha}_{p-1}(4-p)^2}{(4-p)^2}.
\]

(4.28)

Using (2.18) and (2.19) we know that

\[ \bar{\alpha}_p = \left( -\frac{1}{2} \right)^{3-p} \frac{p^{2-p}}{(2-p)!}, \quad \text{for all } p \leq 1 \]

holds. Since \( p \leq 1 \) and \( p \) is odd, from (4.29) the equation (4.28) is equivalent to

\[
\frac{15(p-2)^{4-p}}{2^{p-3}(p-3)!} (2-p)^2 < \frac{8(p-1)^{3-p}}{2^{4-p}(3-p)!} (4-p)^2.
\]
With Mathematica it is easy to verify that the last inequality is true for all $p \leq 1$, $p$ odd. So we proved that

$$
\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} > \frac{1}{32(p-4)^2} \frac{(2-p)^{4-p}}{2^{11-p}(4-p)^2(4-p)!}, \quad \text{for } p \leq 1, p \text{ odd}.
$$

By Theorem 2.1 we have $\bar{\alpha}_2 = -\frac{1}{2}$ and $\bar{\alpha}_1 = \frac{1}{4}$. Therefore the case $p = 3$ reduces to a direct computation, i.e.

$$
\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} = \frac{\bar{\alpha}_1\bar{\alpha}_2}{2} = -\frac{1}{16}.
$$

By (2.18) and (2.20) we know that $\bar{\alpha}_p = -\frac{1}{2}$ holds. So in the case $p \geq 5$ and $p$ odd, by (4.30) we get

$$
\sum_{j=2}^{p-2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} \geq \frac{\bar{\alpha}_{p-2}\bar{\alpha}_2}{2(p-4)^2} > \frac{(p-2)^{p-2}}{2^{p-1}(p-4)^2(p-2)!}.
$$

This terminates the proof of Lemma 4.6. \hfill \Box

Now we are able to prove Theorem 4.2.

**Proof. (Theorem 4.2)** If $q = 1, 2$ condition (4.1), which guarantees the existence of a $(p,q)$–periodic orbit of the dissipative spin-orbit problem modelled by equations (1.24)-(1.32), follows from Lemma 4.3 and 4.4 taking $|\mathbf{e}| \leq \bar{r}_{2p}(b)$. By (4.17) this is equivalent to

$$
|\mathbf{e}| \leq \bar{r}_{2p} = c_{p,1} \frac{1}{M(b)} \left( \frac{r(b)}{2} \right)^{|2p-2|+1}, \quad \text{for } q = 1,
$$

$$
|\mathbf{e}| \leq \bar{r}_p = c_{p,2} \frac{1}{M(b)} \left( \frac{r(b)}{2} \right)^{|p-2|+1}, \quad \text{for } q = 2,
$$

where $c_{p,1}$ and $c_{p,2}$ are defined in (4.3) and (4.4), respectively. This implies (4.7) and so for $q = 1, 2$ the proof of Theorem 4.2 is finished.
It remains to prove the case $q = 4$. Since $(p, q) = 1$ it follows that $p$ is odd, therefore also $p - 2$ is odd. Let us define

$$s(e, \zeta) := \sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{\alpha_{p-j}(e)\alpha_j(e)}{4(p - 2j)^2} + \zeta.$$  \hspace{1cm} (4.31)

Recalling Theorem 4.1 and condition (4.1), we have to prove that $s(e, 16\eta^2) \neq 0$ for $e$ and $\eta$ satisfying (4.8) and (4.9), respectively.

Let us start with the case $\eta = 0$. For $0 < b < 1$ and $|e| < r(b) = b / \cosh b$ by (4.16) we get

$$|s(e, 0)| = \left| \sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{\alpha_{p-j}(e)\alpha_j(e)}{4(p - 2j)^2} \right| \leq \frac{\pi^2 M^2(b)}{4} \sum_{k \neq 0, k \in \mathbb{Z}} \frac{1}{k^2} = \frac{\pi^2 M^2(b)\pi^2}{12}. \hspace{1cm} (4.32)$$

Furthermore by (2.17)

$$\alpha_{p-j}(e)\alpha_j(e) = \left[ \tilde{\alpha}_{p-j}e^{p-j-2} + O(e^{p-j-2}+1) \right] \left[ \tilde{\alpha}_j e^{j-2} + O(e^{j-2}+1) \right]$$

$$= \tilde{\alpha}_{p-j} \tilde{\alpha}_j e^{p-j-2+j-2} + O(e^{p-j-2+j-2}). \hspace{1cm} (4.33)$$

Using Lemma 4.5 and (4.33) we have

$$s(e, 0) = \begin{cases} \sum_{j=0}^{p-2} \frac{\tilde{\alpha}_{p-j}\tilde{\alpha}_j}{4(p-2j)^2} e^{p-4} + R_p(e), & \text{for } p \leq 3, \\ \sum_{j=p-2}^{p-2} \frac{\tilde{\alpha}_{p-j}\tilde{\alpha}_j}{4(p-2j)^2} e^{p-4} + R_p(e), & \text{for } p > 4, \end{cases}\hspace{1cm} (4.34)$$

where $R_p(e) = O(e^{p-4}+1)$ is the Taylor remainder. Take

$$0 < r' < r(b).$$

By the Cauchy estimate and (4.32) we get

$$|R_p(e)| \leq \frac{\pi^2 M^2(b)}{12(r - r')^{p-4}+1} |e|^{p-4}+1, \quad \forall |e| \leq r'(b). \hspace{1cm} (4.35)$$

Let $c_{p, 4}$ be as in (4.5). Choosing $r' = r_p'(b)$ as the unique solution in $(0, r(b))$ of

$$\tilde{r} = \frac{6|c_{p, 4}|}{\pi^2 M^2(b)} (r(b) - \tilde{r})^{p-4}+1 = h(\tilde{r}) \hspace{1cm} (4.36)$$

(see Figure 4.2), we get by (4.35)

$$|R_p(e)| \leq \frac{|c_{p, 4}|}{2h(r_p'(b))} |e|^{p-4}+1 \leq \frac{|c_{p, 4}|}{2r_p'(b)} |e|^{p-4}+1 \leq \frac{2|c_{p, 4}|}{2} |e|^{p-4} \quad \forall |e| \leq r_p'(b). \hspace{1cm} (4.37)$$

Then for $p$ odd we have

$$|s(e, 0)| = \begin{cases} \sum_{j=p-2}^{p-2} \frac{\tilde{\alpha}_{p-j}\tilde{\alpha}_j}{4(p-2j)^2} e^{p-4} + R_p(e), & \text{if } p \leq 3, \\ \sum_{j=2}^{p-2} \frac{\tilde{\alpha}_{p-j}\tilde{\alpha}_j}{4(p-2j)^2} e^{p-4} + R_p(e), & \text{if } p > 4, \end{cases}\hspace{1cm} (4.34)$$
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(4.27) \[
\geq \begin{cases} 
\frac{|2^{(2-p)} - p|}{2^{(4-p)^2(4-p)}} |e|^{p-4} - |R_p(e)|, & \text{if } p \leq 1, \\
\frac{-1}{16} |e|^{p-4} - |R_p(e)|, & \text{if } p = 3, \\
\frac{2^{(p-2)^2(4-p)^2(4-p)}}{2^{(p-2)^2} |e|^{p-4} - |R_p(e)|, & \text{if } p \geq 5, 
\end{cases}
\]

(4.5) & (4.37) \[
\geq \frac{|c_{p,4}| |e|^{p-4} - \frac{|c_{p,4}|}{2} |e|^{p-4}}{2 |e|^{p-4}}, \quad \forall |e| \leq r'_p(b). 
\]

Figure 4.2: Geometric explanation of equation (4.41).

Let us now consider the case \( \bar{\eta} > 0 \). For \( |e| < r(b) \) and \( \zeta > 0 \) we have by (4.16) and (4.31)

\[
|\partial_\zeta s(e, \zeta)| \leq \sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{|\alpha_{p-j}(e)| \alpha_j(e)}{(4(p-2j)^2 + \zeta)^2} \leq \frac{M^2(b)}{16} \sum_{0 \leq k \in \mathbb{Z}} \frac{1}{k^4} \leq \frac{5}{720} M^2(b). 
\]

Then by (4.38) and (4.39), for \( 0 < |e| \leq r'_p(b) \), we obtain

\[
|s(e, 16\bar{\eta}^2)| \geq |s(e, 0)| - \frac{\pi^2 M^2(b)}{720} 16\bar{\eta}^2 \geq \frac{|c_{p,4}|}{2} |e|^{p-4} - \frac{\pi^2 M^2(b)}{45} \bar{\eta}^2 \geq \frac{|c_{p,4}|}{4} |e|^{p-4} > 0 
\]

if we assume that

\[
\bar{\eta} \leq \frac{3\sqrt{5}|c_{p,4}|}{2\pi M(b)} |e|^{p-4}/2, 
\]

which corresponds to (4.9). Notice that (4.8) is equivalent to

\[
0 < e \leq r'' := \frac{r(b)}{\frac{\pi^2 M^2(b)}{6|c_{p,4}|r(b)^{p-4} + |p - 4|}. 
\]

Condition (4.1), which guarantees the existence of a \((p, 4)\)-periodic orbit of the dissipative spin-orbit problem modelled by equations (1.24)-(1.32), follows if we prove that

\[
r'' \leq r'_p(b). 
\]
From (4.36)

\[ h(\tilde{r}) \geq h(0) + h'(0)\tilde{r} \]

\[ = \frac{6|\eta|}{\pi^2 M^2(b)} (r(b))^{(p-4)^+1} - \frac{6|p - 4|\eta}{\pi^2 M^2(b)} (r(b))^{(p-4)^-1} \tilde{r} =: g(\tilde{r}) \]

holds for \(0 \leq \tilde{r} \leq r(b)\). Since \(r''\) (defined in (4.40)) is the unique solution of the equation \(\tilde{r} = g(\tilde{r})\) and \(r'\) is the unique solution of the equation \(\tilde{r} = h(\tilde{r})\), the inequality (4.41) is true (see Figure 4.2). This terminates the proof of Theorem 4.2.
5 Numerical analysis of \((p, q)\)-periodic orbits of the DSOP

The aim of this chapter, as already anticipated in Remark 3.5, is to develop a Matlab program in order to compute recursively the functions \(\phi^{(k)}\) in (3.23) and \(u_k\) in (3.25). Chapter 5 is organized as follow: In Section 5.1 we explain the basic notation and functions, which we will use; In Section 5.2 we develop the programs to find numerically the periodic solutions of the dissipative spin-orbit problem (DSOP); Finally, in Section 5.3 we will use this program to verify numerically the Conjecture 3.5 for diverse values of \(p\) and \(q\).

5.1 Basic notation and functions

5.1.1 Functions of \(t\) in matrix form

For the programs in Matlab we will use the following notation. Every function \(u(t) : \mathbb{R} \supseteq [0, 2\pi] \rightarrow \mathbb{C}\) with \(N\)-truncated Fourier-expansion equal to

\[
u(t) = \sum_{n=-N}^{N} u_n e^{int},
\]

with \(N \in \mathbb{N}_{\geq 1}\) and \(u_n \in \mathbb{C}\), will be expressed with Matlab using the \(2 \times (N + 1)\) complex matrix

\[
U = \begin{pmatrix}
u_0 & u_0 \\
u_{-1} & u_1 \\
\vdots & \vdots \\
u_{-N} & u_N
\end{pmatrix} \in \mathbb{C}^{2 \times (N+1)}.
\]

(5.1)

In the rest of this subsection we implement basic programs (sum, product, ...) to work with functions as above.

**Sum**

Given \(U \in \mathbb{C}^{2 \times N}\) and \(V \in \mathbb{C}^{2 \times M}\) representing the truncated Fourier expansion of the functions \(u(t)\) and \(v(t)\), respectively (as in (5.1)), we compute \(W \in \mathbb{C}^{2 \times \max(N, M)}\) the matrix representing truncated Fourier expansion of the the sum \(u(t) + v(t)\).

_Idea:_ If \(N = M\) the matrices \(U\) and \(V\) can be summed up together, hence \(W = U + V\). If (without loss of generality) \(N < M\), then

\[
\left\{
\begin{array}{ll}
w_{i,j} = u_{i,j} + v_{i,j}, & \text{for } i = 1, 2 \text{ and } 0 \leq j \leq N, \\
w_{i,j} = v_{i,j}, & \text{for } i = 1, 2 \text{ and } N + 1 \leq j \leq M,
\end{array}
\right.
\]

hold, where \(u_{i,j}, v_{i,j}\) and \(w_{i,j}\) denote the element in the \(i\)th-row and in the \(j\)th-column of the matrices \(U\), \(V\) and \(W\), respectively.
Chapter 5. Numerical analysis of \((p,q)\)-periodic orbits of the DSOP

**SUM of** \(u(t)\) **and** \(v(t)\)

\[
%\begin{comment}
% INPUT: u is a 2xN matrix, v is a 2xM matrix
% OUTPUT: w is a 2xP matrix, where P = max(N,M)
% DESCRIPTION: this program sums two truncated Fourier-expansions
%\end{comment}

function w = sumfourier(u,v)
    [N, due_u] = size(u);
    [M, due_v] = size(v);
    if (M > N)
        w = v;
        w(1:N,:) = w(1:N,:) + u;
    elseif (N > M)
        w = u;
        w(1:M,:) = w(1:M,:) + v;
    else % M = N
        w = u + v;
    end
end

**Product**

Given \(U \in \mathbb{C}^{2 \times N}\) and \(V \in \mathbb{C}^{2 \times M}\) representing the truncated Fourier expansion of the functions \(u(t)\) and \(v(t)\), respectively (as in (5.1)), we compute \(W \in \mathbb{C}^{2 \times N+M-1}\) the matrix representing the truncated Fourier expansion of the product \(u(t) \cdot v(t)\).

**Idea:** Since by (5.1) the functions \(u(t), v(t)\) are given by

\[
    u(t) = \sum_{j=-N}^{N} u_j e^{ijt} \quad \text{and} \quad v(t) = \sum_{k=-M}^{M} v_k e^{ikt},
\]

for the product \(u(t) \cdot v(t)\) we have

\[
    u(t) \cdot v(t) = \left( \sum_{j=-N}^{N} u_j e^{ijt} \right) \cdot \left( \sum_{k=-M}^{M} v_k e^{ikt} \right) = \sum_{l=-N-M}^{N+M} c_l e^{ilt}.
\]

The coefficients \(c_l \in \mathbb{C}\), for \(-N - M \leq l \leq N + M\), can be computed using the Cauchy formula for the product of two series, i.e. in the case \(N \leq M\)

\[
    c_l = \sum_{j+k=l} u_j v_k = \sum_{j=-N}^{N} u_j v_{l-j}
\]

holds (the case \(M > N\) is similar).

**PRODFOURIER of** \(u(t)\) **and** \(v(t)\)

\[
%\begin{comment}
% INPUT: u is a 2xN matrix, v is a 2xM matrix
% OUTPUT: w is a 2xP matrix, where P = N + M + 1
% DESCRIPTION: this program multiplies two truncated Fourier-expansions
%\end{comment}

function w = prodfourier(u,v)
\[ [N, \text{due}_u] = \text{size}(u); \]
\[ [M, \text{due}_v] = \text{size}(v); \]
\[ w = \text{zeros}(N+M-1,2); \] % create the bi-vector that will be the output

\% convert bi-vectors in vectors
\[ u_{\text{vector}} = [u(\text{end}:-1:2,1) ; u(:,2)]; \]
\[ v_{\text{vector}} = [v(\text{end}:-1:2,1) ; v(:,2)]; \]
\[ n = \text{length}(u_{\text{vector}}); \]
\[ m = \text{length}(v_{\text{vector}}); \]
\[ w_{\text{vector}} = \text{zeros}(2*(N+M-2)+1,1); \]
\[ \text{for } k = 1:n \]
\[ w_{\text{vector}}(k:(m-1+k)) = w_{\text{vector}}(k:(m-1+k)) + u_{\text{vector}}(k) \times v_{\text{vector}}; \]
\[ \text{end} \]
\% convert the solution in a bi-vector
\[ w(:,1) = w_{\text{vector}}((N+M-1):-1:1); \]
\[ w(:,2) = w_{\text{vector}}((N+M-1):end); \]
\[ \]

\textbf{Green operator}

Given \( V \in \mathbb{C}^{2 \times N} \) as in (5.1) representing the truncated Fourier expansion of the functions \( v(t) \), we compute \( W \in \mathbb{C}^{2 \times N} \) the matrix representing truncated Fourier expansion of \(-G_\eta(v(t))\), where the Green operator \( G_\eta \) (see equation (3.7))

\[
G_\eta \left[ \sum_{n \neq 0} v_n e^{int} \right] := \sum_{n \neq 0} \frac{v_n}{imn - n^2} e^{int},
\]

is diagonal in Fourier space, being just the multiplication of the \( n \)-th Fourier coefficient by \( \frac{1}{imn-n^2} \). Therefore\(^1\) we have

\[
W = \begin{bmatrix}
    v_0 & v_0 \\
    \frac{v_1}{im+1^2} & -\frac{v_1}{im+1^2} \\
    \vdots & \vdots \\
    \frac{v_{N-1}}{im(N-1)^2} & -\frac{v_{N-1}}{im(N-1)^2} \\
\end{bmatrix} \in \mathbb{C}^{2 \times N}.
\]

\textbf{MINUS GREEN OPERATOR of} \( v(t) \)

\[ \]
\[ ^1\text{The Matlab code works also if } v_0 \neq 0. \]
Chapter 5. Numerical analysis of \((p,q)\)-periodic orbits of the DSOP

**Average**

Given \(V \in \mathbb{C}^{2 \times N}\) as in (5.1) representing the truncated Fourier expansion of the function \(v(t)\), we compute \(W \in \mathbb{C}^{2 \times N}\) the matrix representing truncated Fourier expansion of \(v(t) - \langle v(t) \rangle\), where

\[
\langle v(t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} v(t) dt = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n \in \mathbb{Z}} v_n e^{int} dt = v_0.
\]

Therefore it follows

\[
W = \begin{bmatrix}
0 & 0 \\
v_{-1} & v_1 \\
\vdots & \vdots \\
v_{-N} & v_N
\end{bmatrix} \in \mathbb{C}^{2 \times (N+1)}.
\]

**MINUS MEDIA**: compute \(<v(t)>\) and \(v(t) - <v(t)>\)

\[
%-------------------------------------------------------------------------------
% INPUT: v is a 2xN matrix 
% OUTPUTs: w is a 2xN matrix, phi is a 1x1 matrix 
% DESCRIPTION: phi=<v> can be read in the zero coefficient of v, 
% w=v-<v> can be computed subtracting phi to v. 
%-------------------------------------------------------------------------------
function [w,phi] = minus_media(v)

w = v;
phi = v(1,1);
w(1,:) = [0 0];
end
\]

**Signum function**

Define

\[
\text{signum}(x) := \begin{cases}
1, & \text{if } x \leq 0, \\
2, & \text{if } x > 0.
\end{cases}
\]

This function is helpful to move a pointer inside a matrix as in (5.1).

**SIGNUM**: help function

\[
%-------------------------------------------------------------------------------
% INPUT: x is a real number 
% OUTPUT: r is a real number 
% DESCRIPTION: if x>0 then r=2, otherwise r=1. 
%-------------------------------------------------------------------------------
function [r]=signum(x)

if x>0
    r=2;
else
    r=1;
end
end
\]

5.1.2 Function of $\xi$ and $t$ in cell array form

For the programs in Matlab we will use the following notation. Every function $u(\xi, t) : [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{C}$ with $(\mathcal{M}, N)$—truncated Fourier-expansion equal to

$$u(\xi, t) = \sum_{m \in \mathcal{M}} \left( e^{im\xi} \sum_{n=-N}^{N} u_{m,n} e^{int} \right),$$

with $N \in \mathbb{N}_{\geq 1}$, $u_{m,n} \in \mathbb{C}$ and set of $\xi$—exponents

$$\mathcal{M} = \{m_1, m_2, \ldots, m_{|\mathcal{M}|}\} \subset \mathbb{Z}$$

with $|\mathcal{M}| < \infty$, will be expressed with Matlab using the $2|\mathcal{M}|$—cell array

$$U = \begin{pmatrix} u_{m,0} & u_{m,0} \\ u_{m,-1} & u_{m,1} \\ \vdots & \vdots \\ u_{m,-N} & u_{m,N} \end{pmatrix} \in \mathbb{C}^{2 \times (N+1)},$$

where for $m \in \mathcal{M}$ we have

$$U_m = \begin{pmatrix} u_{m,0} & u_{m,0} \\ u_{m,-1} & u_{m,1} \\ \vdots & \vdots \\ u_{m,-N} & u_{m,N} \end{pmatrix} \in \mathbb{C}^{2 \times (N+1)}.$$

**Sum of functions of $\xi$ and $t$ in cell array form**

Let $N \in \mathbb{N}_{\geq 1}$. Let $U \in \text{Cell}^{2|\mathcal{M}_1|}$ and $V \in \text{Cell}^{2|\mathcal{M}_2|}$ as in (5.2) be $(\mathcal{M}_i, N)$—truncated Fourier expansion for $i = 1, 2$ of the functions $u(\xi, t)$ and $v(\xi, t)$, respectively. We compute $W \in \text{Cell}^{2|\mathcal{M}_1 \cup \mathcal{M}_2|}$ the cell array representing the $(\mathcal{M}_1 \cup \mathcal{M}_2, N)$—truncated Fourier expansion of the sum $u(\xi, t) + v(\xi, t)$.

**Idea:** For $i = 1, 2$ let

$$\mathcal{M}_i = \{m^i_1, m^i_2, \ldots, m^i_{|\mathcal{M}_i|}\} \subset \mathbb{Z}$$

with $|\mathcal{M}_i| < \infty$. Define

$$\mathcal{M}_1 \cup \mathcal{M}_2 = (\mathcal{M}_1 \setminus \mathcal{M}_2) \cup (\mathcal{M}_1 \cap \mathcal{M}_2) \cup (\mathcal{M}_2 \setminus \mathcal{M}_1)$$

and put $l = |\mathcal{M}_1 \cup \mathcal{M}_2| \geq 0$. Without loss of generality we assume

$$m^i_k = m^j_k, \quad \text{for } 1 \leq k \leq l.$$ 

Then $W \in \text{Cell}^{2|\mathcal{M}|}$ as the following form

$$W = \begin{pmatrix} W_{\mathcal{M}_1 \setminus \mathcal{M}_2} & W_{\mathcal{M}_1 \cap \mathcal{M}_2} & W_{\mathcal{M}_2 \setminus \mathcal{M}_1} \end{pmatrix},$$

where

$$W_{\mathcal{M}_1 \setminus \mathcal{M}_2} = \begin{pmatrix} U_m^{(1)} & m^{(1)}_{|\mathcal{M}_1|-l+1} & \cdots & U_m^{(1)}_{|\mathcal{M}_1|} \\ m^{(1)}_{|\mathcal{M}_1|} & \cdots & U_m^{(1)}_{|\mathcal{M}_1|-l+1} & \cdots & m^{(1)}_{|\mathcal{M}_1|} \end{pmatrix},$$

$$W_{\mathcal{M}_1 \cap \mathcal{M}_2} = \begin{pmatrix} U^{(1)} m^{(2)} & m^{(1)} & \cdots & U^{(1)} m^{(2)} & m^{(1)} \end{pmatrix},$$

$$W_{\mathcal{M}_2 \setminus \mathcal{M}_1} = \begin{pmatrix} V^{(2)} m^{(2)} & m^{(2)}_{|\mathcal{M}_2|-l+1} & \cdots & V^{(2)} m^{(2)} & m^{(2)}_{|\mathcal{M}_2|} \end{pmatrix}.$$

Notice that to compute the sums in $W_{\mathcal{M}_1 \cap \mathcal{M}_2}$ we have to use the program SUM-FOURIER of the previous subsection.

---

2A cell array is an array that can contain data of varying types and sizes.
**SUMCELLS**: computes the sum of two functions given as cell arrays

%------------------------------------------------------------------------
% INPUT:  u  function u(t,xi) given in cell form (see description)
%  v  function v(t,xi) given in cell form
% OUTPUT: w  function w(t,xi) = u(t,xi) + v(t,xi) given in cell form
% %
% % DESCRIPTION: this program computes the sum of two functions given in this
% form
% |
% | --------- | ----- | --------- | ----- |
% | u_1(t) | e_1 | ... | u_n(t) | e_n |
% %
% %------------------------------------------------------------------------

function w = sumcells(u,v)

[uno_u,length_u] = size(u);
[uno_v,length_v] = size(v);

%-------------to eliminate potential empty cells---------------------
index_u = length_u;
index_v = length_v;
while ( (index_u > 0) && (isempty(u{1,index_u})) )
index_u = index_u - 2;
end
while ( (index_v > 0) && (isempty(v{1,index_v})) )
index_v = index_v - 2;
end
if (index_u == 0)
%u = {};
w = v;
return;
else
u = u(1,1:index_u);
length_u = index_u;
end
if (index_v == 0)
%v = {};
w = u;
return;
else
v = v(1,1:index_v);
length_v = index_v;
end

%------------------------------------------------------------------------

index_u = 2;
index_v = 2;

w = cell(1,length_u + length_v);
index_w = 2;
while (index_u ~= length_u + 2) && (index_v ~= length_v + 2)
if (u{1,index_u} > v{1,index_v})
w{1 , index_w-1} = u{1 , index_u-1};
w{1 , index_w} = u{1 , index_u};
index_u = index_u + 2;
elseif (u{1,index_u} < v{1,index_v})
w{1 , index_w-1} = v{1 , index_v-1};
w{1 , index_w} = v{1 , index_v};
index_v = index_v + 2;
else % u{1,index_u} = v{1,index_v} i.e. the exponent is the same
w{1 , index_w-1} = sumfourier(u{1 , index_u-1},v{1 , index_v-1});
w{1 , index_w} = v{1 , index_v};
index_u = index_u + 2;
index_v = index_v + 2;
end
end
%-------------to eliminate potential empty cells---------------------
w = w(1,1:index_w-2);

if (index_u ~= length_u + 2)
w = w; 
    w = [w u(1,index_u-1:end)];
    %index_w = index_w + length_u - index_u + 2;
elseif (index_v ~= length_v + 2)
w = w; 
    w = [w v(1,index_v-1:end)];
    %index_w = index_w + length_v - index_v + 2;
end

end

Product of functions of $\xi$ and $t$ in cell array form

Let $N \in \mathbb{N}_{\geq 1}$. Let $U \in \text{Cell}^{2,|\mathcal{M}_1|}$ and $V \in \text{Cell}^{2,|\mathcal{M}_2|}$ as in (5.2) be $(\mathcal{M}_i,N)$—truncated Fourier expansion for $i = 1,2$ of the functions $u(\xi,t)$ and $v(\xi,t)$, respectively. We compute $W \in \text{Cell}^{2,|\mathcal{M}_1| \cup \mathcal{M}_2|}$ the cell array representing the $(\mathcal{M}_1 \cup \mathcal{M}_2,N)$—truncated Fourier expansion of the product $u(\xi,t) \cdot v(\xi,t)$.

**Idea:** For $i = 1,2$ let

\[
\mathcal{M}_i = \left\{ m^i_1, m^i_2, \ldots, m^i_{|\mathcal{M}_i|} \right\} \subset \mathbb{Z}
\]

with $|\mathcal{M}_i| < \infty$. By (5.2) it follows

\[
U = \begin{bmatrix} U_{m^1_1} & m^1_1 & \cdots & U_{m^1_{|\mathcal{M}_1|}} & m^1_{|\mathcal{M}_1|} \\ V_{m^2_1} & m^2_1 & \cdots & V_{m^2_{|\mathcal{M}_2|}} & m^2_{|\mathcal{M}_2|} \end{bmatrix},
\]

We define

\[
U_{m^1_l} := U_{m^1_l} \begin{bmatrix} m^1_l \\ \end{bmatrix}, \quad \text{for } 1 \leq l \leq |\mathcal{M}_1|,
\]

\[
V_{m^2_k} := V_{m^2_k} \begin{bmatrix} m^2_k \\ \end{bmatrix}, \quad \text{for } 1 \leq k \leq |\mathcal{M}_2|.
\]

Using the distributive property we get

\[
W = U \cdot W = \sum_{1 \leq l \leq |\mathcal{M}_1|} \sum_{1 \leq k \leq |\mathcal{M}_2|} U_{m^1_l} V_{m^2_k}.
\]

Notice that in the last formula the sums may be performed using the program SUM-CELLS, since we are adding cell arrays together. Therefore it is sufficient to know how to compute $U_{m^1_l} V_{m^2_k}$. By definition of $U_{m^1_l}$ and $V_{m^2_k}$, we know that

\[
U_{m^1_l} = e^{im^1_l \xi} \sum_{n=-N}^{N} u_{m^1_l,n} e^{int},
\]

\[
V_{m^2_k} = e^{im^2_k \xi} \sum_{n=-N}^{N} v_{m^2_k,n} e^{int},
\]
for $1 \leq l \leq |M_1|$ and for $1 \leq k \leq |M_2|$. Hence

$$U_{m_1(1)} V_{m_2(2)} = e^{i(m_1(1) + m_2(2))t} \left( \sum_{n=-N}^{N} u_{m_1(1),n} e^{int} \right) \cdot \left( \sum_{n=-N}^{N} v_{m_2(2),n} e^{int} \right).$$

In Matlab the right hand side is represented by the cell array

$$U_{m_1(1)} \cdot V_{m_2(2)} \quad m_1(1) + m_2(2).$$

To compute the term $U_{m_1(1)} \cdot V_{m_2(2)}$ the program PRODFOURIER of the previous section may be used.

**PRODCELLS:** computes the product of two functions given as cell arrays

```matlab
function w = prodcells(u,v)

[uno_u,length_u] = size(u);
[uno_v,length_v] = size(v);

%--------------to eliminate potential empty cells---------------------
index_u = length_u;
index_v = length_v;
while ( (isempty(u{1,index_u})) )
  index_u = index_u - 2;
end
while ( (isempty(v{1,index_v})) )
  index_v = index_v - 2;
end
if (index_u == 0)
  u = {};
else
  u = u(1,1:index_u);
  length_u = index_u;
end
if (index_v == 0)
  v = {};
else
  v = v(1,1:index_v);
  length_v = index_v;
end

%----------------------------------------------------------------------

w = {}; 
for index_u = 1:2:length_u-1
  for index_v = 1:2:length_v-1
    w = sumcells( w , ... 
                { prodfourier(u{1,index_u}, v{1,index_v}) , ... 
                  u{1,index_u+1} + v{1,index_v+1} } ... 
                )
  end
end
```
Plot functions given in cell array form

Let \( u = u(\xi, t) \) be a function with zero average with respect to \( t \). The following program plots the input function \( u \) as in (5.2) on the interval \( \xi \in [0, 2\pi] \)

**PLOTTARE:** plots a cell array function with zero mean

```plaintext
function [dati]=ploottare(f,color)
[a,b]=size(f);
dati=0;
for j=2:2:b
    ff=f{a,j-1};
    dati=ff*exp(1i.*f{a,j}*[0:2 *pi/100:2*pi])+dati;
end
plot([0:2 *pi/100:2*pi],dati,color)
```

5.2 The DSOP and its representation with Matlab

Recall from (2.1), Lemma 2.1 and Remark 3.1 that the Newtonian potential \( f(x, t; e) \) for the dissipative spin-orbit problem is given by

\[
f(x, t; e) = \sum_{j \in \mathbb{Z}, j \neq 0} \alpha_j \cos(2x - jt)
\]

\[
= f_{-2}(t) e^{i(-2)x} + f_2(t) e^{i2x},
\]

with

\[
f_{\pm 2}(t) = \sum_{j \in \mathbb{Z}} \frac{\alpha_{\pm j}}{2} e^{ijt}.
\]

Hence the dissipative spin-orbit problem (1.24) is a special case of the generalized spin-orbit problem in (3.1), where \( m \) only admits the values 2 and \(-2\).

By Lemma 2.1 the coefficients \( \alpha_j \) \((j \in \mathbb{Z})\) depend on the eccentricity \( e \in [0, 1) \) as follows:

\[
\alpha_j(e) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\cos(2\tilde{\rho}_e - ju + \epsilon j \sin(u))}{\tilde{\rho}_e^2} du,
\]

with

\[
\tilde{\rho}_e = \tilde{\rho}_e(u) := 2 \arctan \left( \sqrt{\frac{1+e}{1-e}} \tan \left( \frac{u}{2} \right) \right),
\]

\[
\tilde{\rho}_e = \tilde{\rho}_e(u) := 1 - e \cos(u).
\]

Given \( N \in \mathbb{N}_{\geq 1} \) the following Matlab program computes an approximation of the coefficients \( \alpha_j \) of the Newtonian potential \( f(x, t; e) \) in (2.1) for \(-N \leq j \leq N\).

**COEFF-ALPHA:** compute the coefficients \( \alpha_j \)

```plaintext
%-------------------------------------------------------------------------
% INPUT: N number of Fourier-coefficients we consider
% e the eccentricity
% OUTPUT: alpha vector of the coefficients of f(x,t;e)
%-------------------------------------------------------------------------
function [alpha]=coeff_alpha(e,N)
```

```plaintext
%-------------------------------------------------------------------------
% INPUT: N number of Fourier-coefficients we consider
% e the eccentricity
% OUTPUT: alpha vector of the coefficients of f(x,t;e)
%-------------------------------------------------------------------------
function [alpha]=coeff_alpha(e,N)
```
alpha = zeros(N,2);
for j=1:N
  g1 = @(x) cos(4.*atan(sqrt((1+e)/(1-e)).*tan(x/2))-j.*x+e.*j.*sin(x))./(1-e.*cos(x)).^2;
  g2 = @(x) cos(4.*atan(sqrt((1+e)/(1-e)).*tan(x/2))+j.*x-e.*j.*sin(x))./(1-e.*cos(x)).^2;
  alpha(j,2) = -1 / (4*pi) * quad(g1,0,2*pi);
  alpha(j,1) = -1 / (4*pi) * quad(g2,0,2*pi);
end

Recall that by (3.13), (3.14) and (3.15) we have
\[ u(t; \xi, \varepsilon) = \sum_{k=1}^{\infty} u_k \varepsilon^k, \]
\[ \phi(u) := \langle f_x(\xi + pt + u(t), qt) \rangle = \sum_{k=0}^{\infty} \phi^{(k)} \varepsilon^k, \]
\[ f_x(\xi + pt + u, qt) = \sum_{k=0}^{\infty} f^{(k)} \varepsilon^k. \]

By Lemma 3.4 we know that we can compute \( u_k \) and \( \phi^{(k)} \) recursively, using equations (3.20), (3.21),(3.23), (3.25) and (3.26). More precisely for the dissipative spin-orbit problem we have that:

- Equation (3.26) is equivalent to
  \[
  \beta_1^{2,n}(t) = \frac{2i}{l} \sum_{k=1}^{l} k u_k \beta_{l-k}^{2,n}, \\
  \beta_1^{-2,n}(t) = -\frac{2i}{l} \sum_{k=1}^{l} k u_k \beta_{l-k}^{-2,n},
  \]
  for \( 1 \leq k \leq l \), where \( \beta_0^{\pm 2,n} \) in (3.20) are given by:
  \[
  \beta_0^{2,n} = e^{i(2p-nq)t}, \\
  \beta_0^{-2,n} = e^{i(-2p+nq)t}.
  \]

- Equations (3.21) and (3.23) say
  \[
  f^{(k)} = 2ie^{2i\xi} \sum_{n \neq 0} f_{2,n} \beta_k^{2,n} - 2ie^{-2i\xi} \sum_{n \neq 0} f_{-2,n} \beta_k^{-2,n}, \\
  \phi^{(k)} = \langle f^{(k)} \rangle,
  \]
  for all \( k \geq 0 \).

- Equation (3.25) implies
  \[
  u_k = u_k(t; \xi) = -G_\eta \left( f^{(k-1)} - \phi^{(k-1)} \right)
  \]
  for all \( k \geq 1 \).
5.2. The DSOP and its representation with Matlab

The basic idea of the following program is to implement this recursion, i.e. starting with $\beta^{2,n}_0, \beta^{-2,n}_0$ (which are explicitly given) we first compute $f^{(0)}, \phi^{(0)}$ and $u_1$. Then we are able to compute $\beta^{2,n}_1, \beta^{-2,n}_1$ and automatically $f^{(1)}, \phi^{(1)}$ and $u_2$. And so on.

**Idea:** Let $N \in \mathbb{N}_{\geq 1}$ be the truncation number. Since from Lemma 2.1 we know that $\alpha_0 \equiv 0$, the function $f^{(0)}$ in (5.5), can be approximated as follows

$$f^{(0)} \approx e^{i2\xi} \cdot \frac{1}{2} \sum_{j=-N}^{N} (2i)\alpha_j e^{i(2p-jq)t} + e^{-i2\xi} \cdot \frac{1}{2} \sum_{j=-N}^{N} (-2i)\alpha_j e^{-i(2p-jq)t}$$

$$=: e^{i2\xi} \cdot f_2^{(0)}(t; N) + e^{-i2\xi} \cdot f_{-2}^{(0)}(t; N), \quad (5.8)$$

where

$$f_{\pm 2}^{(0)}(t; N) = \sum_{k=-2p-Nq}^{2p+Nq} a_k e^{ikt} \quad (5.9)$$

with $a_k \in \mathbb{C}$ defined by the relation (5.8). In Matlab we describe the approximating function (5.8) using cell arrays as in (5.2), i.e. for the right hand side of (5.8) we use the cell array

$$\begin{array}{cc}
F_2^{(0)} & 2 \\
F_{-2}^{(0)} & -2
\end{array}$$

where $F_2^{(0)}$ and $F_{-2}^{(0)}$ are matrices in $\mathbb{C}^{2 \times (2p+Nq+1)}$ of the form

$$F_{\pm 2}^{(0)} = \begin{bmatrix}
  a_{\pm 2,0} & a_{\pm 2,0} \\
  a_{\pm 2,-1} & a_{\pm 2,1} \\
  \vdots & \vdots \\
  a_{\pm 2,-2p-Nq} & a_{\pm 2,2p+Nq}
\end{bmatrix}$$

representing the truncated Fourier expansion of the functions $f_2^{(0)}(t; N)$ and $f_{-2}^{(0)}(t; N)$ in (5.9), respectively. Then by (5.6) and (5.8) $\phi^{(0)}$ is given by

$$\phi^{(0)} = \left\langle f^{(0)} \right\rangle = ie^{i2\xi} \alpha \frac{2p}{T} - ie^{-i2\xi} \alpha \frac{2p}{T}$$

and we save it in the cell array

$$\begin{array}{cc}
i\alpha \frac{2p}{T} & 2 \\
-2i\alpha \frac{2p}{T} & -2
\end{array}$$

From the definition of the Green operator (see program MINUS GREEN OPERATOR in Section 5.1) and from equation (5.7) the function

$$u_1 = -G_{\eta} \left( f^{(0)} - \phi^{(0)} \right),$$

is saved in Matlab using the following cell array:

$$\begin{array}{cc}
U_2^{(1)} & 2 \\
U_{-2}^{(1)} & -2
\end{array}$$

where

$$U_{\pm 2}^{(1)} = \begin{bmatrix}
  0 & 0 \\
  \frac{a_{\pm 2,-1}}{i\eta + 1} & \frac{a_{\pm 2,1}}{-i\eta + 1} \\
  \vdots & \vdots \\
  \frac{a_{\pm 2,-2p-Nq}}{(2p-Nq)(2p+Nq)t} & \frac{a_{\pm 2,2p+Nq}}{-(2p+Nq)(2p+Nq)t}
\end{bmatrix}.$$
We introduce help functions in cell array form

$$b_{\pm 2}^{(0)} = \begin{bmatrix} B_{\pm 2}^{(0)} & 0 \end{bmatrix},$$

where

$$B_{\pm 2}^{(0)} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

In (5.3) and (5.4) replace $\beta_l$ by $b^{(l)}$. So we have

$$b^{(l)}_{\pm 2} = \frac{\pm 2i}{l} \sum_{k=1}^{l} k u_k b^{(l-k)}_{\pm 2},$$

(5.10) for $1 \leq k \leq l$. (5.10) can be performed with the programs SUMCELLS and PRODCELLS of the previous subsection. Then we save the function $f^{(k)}$ in (5.5) with the cell array

$$b_2^{(k)} \cdot F_2^{(0)} + b_{-2}^{(k)} \cdot F_{-2}^{(0)}.$$

“+” , “·” are here the SUMCELLS and PRODCELLS function in Matlab developed above. Finally $\phi^{(k)}$ and $u_k$ are computed with (5.6) and (5.7), respectively. The idea behind this is that the recursion with the $b^{(l)}_{\pm 2}$ is easier than the recursion with the $\beta_l_{\pm 2,n}$ and by replacing at the end $b^{(0)}_{\pm 2}$ by $\beta_0^{\pm 2,n}$ we get the same result.

**SPIN-ORBIT:** given $k,p,q$ compute the function $\phi^{(k-1)}$ in cell format

```matlab
% INPUT: N number of Fourier-coefficients we consider
% k degree
% e the eccentricity
% eta dissipative constant
% p,q constants of the periodic solution
% OUTPUT: phi=(phi_l)_l, where phi_l are cell arrays for 1 <= l <= k.

function [phi]=spin_orbit(k,p,q,e,eta,N)

alpha=coeff_alpha(e,N);

U = cell(k,1);
beta1 = cell(1,2);
beta_1 = cell(1,2);
phi = cell(k,1);

% Create beta0:
N_max = 2*p - (-N)*q;
% the minimal exponent is N_min=-2*p-N*q, obviuosly N_min = -N_max
beta10 = zeros(N_max+1,2); % this part will be the bi-vector for e^i2xi
beta_10 = zeros(N_max+1,2); % this part will be the bi-vector for e^-i2xi
j = [(-N:-1) (1:N)];
exponent = 2*p - j*q;
beta10( abs(exponent) + 1 + (exponent >= 0)*(N_max+1) ) = alpha(abs(j) + (j>0)*N) * (1i);
beta_10( abs(exponent) + 1 + (exponent < 0)*(N_max+1) ) = alpha(abs(j) + (j>0)*N) * (-1i);

% here the values of the zero-elements are copied on the other
% component of F_2 and F_{-2}
beta1{1,1}= beta10; beta1{1,2}=2;
beta_1{1,1}=beta_10; beta_1{1,2}=-2;
```

% output in the cell form
beta1(1,1)= beta10; beta1(1,2)=2;
beta_1(1,1)=beta_10; beta_1(1,2)=-2;
5.3 Control and results

In the following program we compute \( \phi^{(0)} \) and \( \phi^{(1)} \) using equations (3.32) and (3.35), because we want to check if the program SPIN-ORBIT works correctly. We will do the control only at the zeroth degree and at the first degree of non-degeneration.

CONTROL: given \( k = 0,1 \) compute \( \phi^{(k)} \) with the formulas (3.32) and (3.35)
function [phik_test, alpha] = control(k, p, q, e, eta, N)

[F, alpha] = f_coeff(N, 1, e, p, q);
if k == 1
  \% ZERO DEGREE:
  if q == 1
    PHI_test{1,1} = -alpha(2*abs(p), signum(2*p))/i; \% 1+(p>0)*1
    PHI_test{1,2} = 2;
    PHI_test{1,3} = alpha(abs(2*p), signum(2*p))/i;
    PHI_test{1,4} = -2;
  elseif q == 2
    PHI_test{1,1} = -alpha(abs(p), signum(p))/i;
    PHI_test{1,2} = 2;
    PHI_test{1,3} = alpha(abs(p), signum(p))/i;
    PHI_test{1,4} = -2;
  else
    PHI_test = [0, 0];
  end
  phik_test = PHI_test(1,:);
else
  \% FIRST DEGREE:
  if q == 4
    alpha(N+1:p*N,:) = 0;
    uno = -2/p*alpha(abs(p), signum(p))^2/(4*p^2+eta^2);
    if p > 0
      dueA = 0;
      for j = -N:1:-1
        dueA = dueA + (alpha(abs(j), signum(j))^2 - alpha(abs(p-j), signum(p-j))^2)...
          /((p-2*j)*(4*(p-2*j)^2+eta^2)) + dueA;
      end
      dueB = 0;
      for j = 1:1:p-1
        dueB = dueB + (alpha(abs(j), signum(j))^2 - alpha(abs(p-j), signum(p-j))^2)...
          /((p-2*j)*(4*(p-2*j)^2+eta^2)) + dueB;
      end
      dueC = 0;
      for j = p+1:N
        dueC = dueC + (alpha(abs(j), signum(j))^2 - alpha(abs(p-j), signum(p-j))^2)...
          /((p-2*j)*(4*(p-2*j)^2+eta^2)) + dueC;
      end
      due = dueA + dueB + dueC;
    else
      dueA = 0;
      for j = -N:1:p-1
        dueA = dueA + (alpha(abs(j), signum(j))^2 - alpha(abs(p-j), signum(p-j))^2)...
          /((p-2*j)*(4*(p-2*j)^2+eta^2)) + dueA;
      end
    end
  end
end
5.3. Control and results

\[ \text{dueB} = 0; \]
\[ \text{for } j = p+1:1:-1 \]
\[ \text{dueB} = (\alpha(\text{abs}(j), \text{signum}(j))^2 - \alpha(\text{abs}(p-j), \text{signum}(p-j))^2) \]
\[ /((p-2*j) + 4*(p+2*j)^2*\text{eta}^2)) + \text{dueB}; \]
\[ \text{end} \]
\[ \text{dueC} = 0; \]
\[ \text{for } j = 1:1:N \]
\[ \text{dueC} = (\alpha(\text{abs}(j), \text{signum}(j))^2 - \alpha(\text{abs}(p-j), \text{signum}(p-j))^2) \]
\[ /((p-2*j) + 4*(p+2*j)^2*\text{eta}^2)) + \text{dueC}; \]
\[ \text{end} \]
\[ \text{due} = \text{dueA} + \text{dueB} + \text{dueC}; \]
\[ \text{PARTE1} = \eta*(\text{uno} + \text{due}); \]
\[ \text{quattroA} = 4/i*\sum(\alpha(N:-1:p+1,2).*\alpha(N-p:-1:1,1)./((4.*(p-2.*(p-N:1:-1)).^2+\text{eta}^2))') \]
\[ \text{quattroB} = 4/i*\sum(\alpha(p-1:-1:1,2).*\alpha(1:p-1,2)./((4.*(p-2.*(1:p-1)).^2+\text{eta}^2))') \]
\[ \text{quattroC} = 4/i*\sum(\alpha(1:N-p,1).*\alpha(p+1:1:N,2)./((4.*(p-2.*(p+1:N)).^2+\text{eta}^2))') \]
\[ \text{quattro} = \text{quattroA} + \text{quattroB} + \text{quattroC}; \]
\[ \text{PHI_test}(2,1)=\text{quattro}/2; \]
\[ \text{PHI_test}(2,2)=4; \]
\[ \text{PHI_test}(2,3)=\text{PARTE1}; \]
\[ \text{PHI_test}(2,4)=0; \]
\[ \text{PHI_test}(2,5)=\text{quattro}/2; \]
\[ \text{PHI_test}(2,6)=4; \]
\[ \text{phik_test} = \text{PHI_test}(2,:); \]
\[ \text{else} \]
\[ \text{uno} = 0; \]
\[ \text{for } j = -N:1:-1 \]
\[ \text{uno} = (\alpha(\text{abs}(j), \text{signum}(j))^2)/(2*p-j*q)^3 + \eta^2*(2*p-j*q) + \text{uno}; \]
\[ \text{end} \]
\[ \text{for } j = 1:1:N \]
\[ \text{uno} = (\alpha(\text{abs}(j), \text{signum}(j))^2)/(2*p-j*q)^3 + \eta^2*(2*p-j*q) + \text{uno}; \]
\[ \text{end} \]
\[ \text{PHI_test}(2,1)=4*\eta*\text{uno}; \]
\[ \text{PHI_test}(2,2)=0; \]
\[ \text{phik_test} = \text{PHI_test}(2,:); \]
\[ \text{else} \]
\[ \text{phik_test} = \{'\text{no informations'}\}; \]
\[ \text{end} \]

The next Matlab program compares the programs SPIN-ORBIT and CONTROL. The result is plotted with the function PLOTTARE.

**LANCIA: SPIN-ORBIT vs CONTROL.**

```matlab
%=============================================================================%
% INPUT:   N   number of Fourier-coefficients we consider   %
%          k   degree                                          %
%          e   the eccentricity                                %
%          eta  dissipative constant                          %
%          p,q  constants of the periodic solution              %
% OUTPUT:  phi=(phi_l)_l, where phi_l are cell arrays for 1 <= l <= k computed with SPIN-ORBIT%
%          phi_test=(phi_l)_l, where phi_l are cell arrays for 1 <= l <= k computed with CONTROL  %
%=============================================================================%
```

\(^3\)The output \(\phi^{(k)}\) can only be compared for \(k = 0, 1\). If \(k \geq 2\) we have only the value obtained with the program SPIN-ORBIT.
Finally with the following Matlab program we check the hypothesis of Conjecture 3.5 for all $p, q \in \mathbb{N}$, where $(p, q) = 1$, $1 \leq p \leq p_{\text{max}}$ and $1 \leq q \leq q_{\text{max}}$. $p_{\text{max}}, q_{\text{max}} \in \mathbb{N}$ are given as inputs.

Idea: If $q | 2(k + 1)$ we check that $\phi^{(l)} = \text{const}$ for all $0 \leq l \leq k - 1$ and $\phi^{(k)} \neq \text{const}$. The output $R_{p_{\text{max}}, q_{\text{max}}}$ is a $4 \times |\mathcal{I}_{p_{\text{max}}, q_{\text{max}}}|$ matrix, where

$$\mathcal{I}_{p_{\text{max}}, q_{\text{max}}} := \{(p, q) | p, q \in \mathbb{N} \text{ coprime}, 1 \leq p \leq p_{\text{max}}, 1 \leq q \leq q_{\text{max}}\}.$$ 

For example for $p_{\text{max}} = 3$ and $q_{\text{max}} = 4$ we obtain

$$R_{3,4} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 2 & 3 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 3 & 4 & 1 & 1 \end{bmatrix}.$$ 

Every row indicates a test. In the example the sixth row of $R_{3,4}$ says that for $p = 1$, $q = 3$ we have $k = 2$ and the hypothesis of Conjecture 3.5 is true, i.e. $\phi^{(0)}, \phi^{(1)}$ are constant and $\phi^{(2)} \neq \text{const}$. 

MULTI-SPIN: check the hypothesis.

```
% function [phi,phi_test]=lancia(k,p,q)
[phi]=spin_orbit(k,p,q,e,eta,N);
[phi_test]=controllo(k,p,q,e,eta,N);
hold on
plottare(phi{k,1},'r')
hold on
plottare(phi_test,'blue')
hold off
```

```
Finally with the following Matlab program we check the hypothesis of Conjecture 3.5 for all $p, q \in \mathbb{N}$, where $(p, q) = 1$, $1 \leq p \leq p_{\text{max}}$ and $1 \leq q \leq q_{\text{max}}$. $p_{\text{max}}, q_{\text{max}} \in \mathbb{N}$ are given as inputs.

Idea: If $q | 2(k + 1)$ we check that $\phi^{(l)} = \text{const}$ for all $0 \leq l \leq k - 1$ and $\phi^{(k)} \neq \text{const}$. The output $R_{p_{\text{max}}, q_{\text{max}}}$ is a $4 \times |\mathcal{I}_{p_{\text{max}}, q_{\text{max}}}|$ matrix, where

$$\mathcal{I}_{p_{\text{max}}, q_{\text{max}}} := \{(p, q) | p, q \in \mathbb{N} \text{ coprime}, 1 \leq p \leq p_{\text{max}}, 1 \leq q \leq q_{\text{max}}\}.$$ 

For example for $p_{\text{max}} = 3$ and $q_{\text{max}} = 4$ we obtain

$$R_{3,4} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 2 & 3 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 3 & 4 & 1 & 1 \end{bmatrix}.$$ 

Every row indicates a test. In the example the sixth row of $R_{3,4}$ says that for $p = 1$, $q = 3$ we have $k = 2$ and the hypothesis of Conjecture 3.5 is true, i.e. $\phi^{(0)}, \phi^{(1)}$ are constant and $\phi^{(2)} \neq \text{const}$. 

MULTI-SPIN: check the hypothesis.

```
% INPUT: N number of Fourier-coefficients we consider
% k degree
% e the eccentricity
% eta dissipative constant
% p,q constants of the periodic solution
% OUTPUT: prova =
```

```
function [R]=multi_spin(p_max,q_max,e,eta,N)
R=[];
for q=1:1:q_max
  k=1;
  q_test=[-2*k,2*k];
  while prod(mod(q_test,q)) ~= 0
    k=k+1;
    q_test=[-2*k,2*k];
  end
```

4the number in the fourth column can only takes the values 1, when the hypothesis is satisfied, and 0 when it is not satisfied.
for p=1:1:p_max
    if gcd(p,q)==1
        [phi]=spin_orbit(k,p,q,e,eta,N);
        result_tot=1;
        for j1=1:1:k-1
            [uno,b]=size(phi{j1,1});
            j2=1;
            while j2<b && ( phi{j1,1}{1,j2}==0 || phi{j1,1}{1,j2+1} ==0 )
                j2=j2+2;
            end
            if j2==b+1
                result=1;
            else
                result=0;
            end
            result_tot=result_tot*result;
            % if the hypothesis is satisfied result_tot = 1, otherwise zero
        end
        [uno,b]=size(phi{k,1});
        j2=2;
        while j2<b && ( phi{k,1}{1,j2}==0 || phi{k,1}{1,j2+1} ==0 )
            j2=j2+2;
        end
        if j2<=b
            result=1;
        else
            result=0;
        end
        result_tot=result_tot*result;
        prova=[prova;[p,q,k-1,result_tot]];
        [k,p,q,result_tot]
    else
        % nothing should be done in this case
    end
end
end
Chapter 5. Numerical analysis of \((p, q)\)-periodic orbits of the DSOP

Remark 5.1. (i) The truncation parameter \(N\) depends on the choice of \(p\) and \(q\). For example from equations (3.35), (4.31), (4.33) and (4.34) we see that the leading term in the \(e\)-expansion of \(\phi^{(1)}\) depends on the coefficients \(\alpha_j\) with

\[
\begin{cases}
2 \leq j \leq p - 2, & \text{if } p \leq 3, \\
p - 2 \leq j \leq 2, & \text{if } p > 4.
\end{cases}
\]

Only a finite number of coefficients \(\alpha_j\) play a decisive role to determine \(\phi^{(1)}\), in particular we have \(N(p) \geq p - 2\).

(ii) Till today we have verified the hypothesis of Conjecture 3.5 for all pairs \((p, q)\) of coprime positive integers with \(1 \leq p \leq 50\) and \(1 \leq q \leq 8\). To do that we have chosen \(N = 100\). Notice that \(N\) is larger than \(p - 2 \leq 48\) in order to get a better approximation of \(\phi^{(k)}\), not only considering the leading term in the \(e\)-expansion but also some further terms (although from Theorem 2.1 we know that the coefficients in the \(e\)-expansion of \(\alpha_j(e)\) decay exponentially fast).

(iii) Numerical experiments (see Figures 5.1 and 5.2) suggest that the following conjecture holds:

\textbf{Conjecture 5.1}

Fix \(k \geq 0\) and \(q \geq 1\). Then \(\phi^{(k)}\) corresponding to \((q + 1, q)\)-periodic orbit has the largest amplitude among all \(\phi^{(k)}\) corresponding to \((p, q)\)-periodic orbit for any other \(p\).
Figure 5.1: Let $e = 0.0549$ be the moon eccentricity, $\eta = 10^{-4}$ and $N = 100$. For $1 \leq q \leq 6$ we plot the $\phi^{(k)}(\xi)$ on the interval $[0, 2\pi]$, corresponding to the $(p, q)$--periodic orbit, with the largest amplitude among all $p$. 
Figure 5.2: Let $e = 0.2056$ be the mercury eccentricity, $\eta = 10^{-4}$ and $N = 100$. For $1 \leq q \leq 6$ we plot the $\phi^{(k)}(\xi)$ on the interval $[0, 2\pi]$, corresponding to the $(p,q)$-periodic orbit, for some values of $p$. 
6 The spin-orbit resonances of the Solar System: a simple mathematical theory

Sections 6.1-6.4 of this chapter are joint work with L. Biasco and L. Chierchia and will appear in Journal of Nonlinear Science, (2014), (see [3]).

Consider the dissipative spin-orbit problem introduced in Subsection 1.2.1. Recall that in this framework a \( pq \) spin–orbit resonance (with \( p \) and \( q \) co–prime non–vanishing integers) is, by (1.51), a solution \( t \in \mathbb{R} \rightarrow x(t) \in \mathbb{R} \) of (1.24) such that

\[
x(t + 2\pi q) = x(t) + 2\pi p ;
\]

indeed, for such orbits, after \( q \) revolutions of the orbital radius, \( x \) has made \( p \) complete rotations.

In Sections 6.1-6.4 of this Chapter we discuss a mathematical theory, which is consistent with the existence of all spin–orbit resonances of the Solar System; in other words, we prove a theorem, establishing the existence of periodic orbits for parameter values corresponding to all the satellites (or Mercury) in our Solar system observed in spin–orbit resonance. Furthermore in Section 6.5 we discuss the existence of spin-orbit resonances considering the orbital eccentricity \( e \) to be variable.

6.1 Spin–orbit resonances in the Solar System

As already mentioned in the introduction of this thesis there are eighteen moons of the Solar System which are observed in a \((1,1)\)–periodic orbit. The list is the following: Moon (Earth); Io, Europa, Ganymede, Callisto (Jupiter); Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Iapetus, (Saturn); Ariel, Umbriel, Titania, Oberon, Miranda (Uranus); Charon (Pluto); minor bodies with mean radius smaller than 100 Km are not considered (see, however, Section 6.4).
The known relevant physical parameter values of these 18 moons are reported in Table 6.1.

<table>
<thead>
<tr>
<th>Principal body</th>
<th>Satellite</th>
<th>Eccentricity</th>
<th>( a ) \text{(km)}</th>
<th>( b ) \text{(km)}</th>
<th>Oblateness ( \varepsilon = \frac{3a^2 - b^2}{2(a^2 + b^2)} )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>Moon [37]</td>
<td>0.0549</td>
<td>1740.19</td>
<td>1737.31</td>
<td>0.00248454179</td>
<td>1.018088056</td>
</tr>
<tr>
<td>Jupiter</td>
<td>Io [40]</td>
<td>0.0041</td>
<td>1829.7</td>
<td>1819.2</td>
<td>0.000863266715</td>
<td>1.00010086</td>
</tr>
<tr>
<td></td>
<td>Europa</td>
<td>0.0094</td>
<td>1561.3</td>
<td>1560.3</td>
<td>0.00096104552</td>
<td>1.000530163</td>
</tr>
<tr>
<td></td>
<td>Ganymede</td>
<td>0.0011</td>
<td>2632.9</td>
<td>2629.5</td>
<td>0.0019382783</td>
<td>1.00000726</td>
</tr>
<tr>
<td></td>
<td>Callisto</td>
<td>0.0074</td>
<td>2411.8</td>
<td>2408.8</td>
<td>0.00186698679</td>
<td>1.000328561</td>
</tr>
<tr>
<td>Saturn</td>
<td>Mimas [21]</td>
<td>0.0193</td>
<td>208.3</td>
<td>196.2</td>
<td>0.08966019091</td>
<td>1.002234993</td>
</tr>
<tr>
<td></td>
<td>Enceladus [21]</td>
<td>0.0047</td>
<td>257.2</td>
<td>251.2</td>
<td>0.03540026218</td>
<td>1.00013254</td>
</tr>
<tr>
<td></td>
<td>Tethys [21]</td>
<td>0.0001</td>
<td>538.7</td>
<td>527.0</td>
<td>0.03293212897</td>
<td>1.00000006</td>
</tr>
<tr>
<td></td>
<td>Dione [21]</td>
<td>0.0022</td>
<td>564.0</td>
<td>560.8</td>
<td>0.0053478156</td>
<td>1.00002904</td>
</tr>
<tr>
<td></td>
<td>Rhea [21]</td>
<td>0.001</td>
<td>766.8</td>
<td>762.8</td>
<td>0.0008127957</td>
<td>1.000006</td>
</tr>
<tr>
<td></td>
<td>Iapetus [21]</td>
<td>0.0288</td>
<td>2575.239</td>
<td>2574.932</td>
<td>0.0017882901</td>
<td>1.00497691</td>
</tr>
<tr>
<td></td>
<td>Ariel [41]</td>
<td>0.0012</td>
<td>582.0</td>
<td>577.3</td>
<td>0.012162311957</td>
<td>1.0000864</td>
</tr>
<tr>
<td></td>
<td>Umbriel [41]</td>
<td>0.0039</td>
<td>587.5</td>
<td>581.9</td>
<td>0.01436601227</td>
<td>1.00009126</td>
</tr>
<tr>
<td></td>
<td>Titania [41]</td>
<td>0.0011</td>
<td>790.7</td>
<td>787.1</td>
<td>0.00684493838</td>
<td>1.00000726</td>
</tr>
<tr>
<td></td>
<td>Oberon [41]</td>
<td>0.0014</td>
<td>764.0</td>
<td>758.8</td>
<td>0.01024416739</td>
<td>1.00001176</td>
</tr>
<tr>
<td></td>
<td>Miranda [41]</td>
<td>0.0013</td>
<td>241.0</td>
<td>233.3</td>
<td>0.04869051956</td>
<td>1.00001014</td>
</tr>
<tr>
<td></td>
<td>Charon [38]</td>
<td>0.0022</td>
<td>655.0</td>
<td>602.2</td>
<td>0.006955821306</td>
<td>1.00002904</td>
</tr>
</tbody>
</table>

Table 6.1: Physical data of the moons in 1:1 spin–orbit resonance
(see also http://ssd.jpl.nasa.gov/?sat_phys_par and http://ssd.jpl.nasa.gov/?sat_elem)

Mercury is apparently trapped in a 3:2 spin–orbit resonance around the Sun. Its corresponding data are given in Table 6.2.

<table>
<thead>
<tr>
<th>Principal body</th>
<th>Satellite</th>
<th>Eccentricity</th>
<th>( a ) \text{(km)}</th>
<th>( b ) \text{(km)}</th>
<th>Oblateness ( \varepsilon = \frac{3a^2 - b^2}{2(a^2 + b^2)} )</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>Mercury</td>
<td>0.2056</td>
<td>2440.7</td>
<td>2439.7</td>
<td>0.00061470369</td>
<td>1.255835458</td>
</tr>
</tbody>
</table>


No further pairs of planet-satellite (where the mean radius of the minor body is bigger than 100 Km) are observed in other spin-orbit resonances in the Solar System.
6.2 Theorem for Solar System spin–orbit resonances

Our main result can now be stated as follows:

**Theorem 6.1.** [Moons] The differential equation (1.24) admits spin–orbit resonances (6.1) with \( p = q = 1 \) provided \( e, \bar{\nu} \) and \( \bar{\varepsilon} \) are as in Table 6.1 and \( 0 \leq \bar{\eta} \leq 0.008 \).

[Mercury] The differential equation (1.24) admits spin–orbit resonances (6.1) with \( p = 3 \) and \( q = 2 \) provided \( e, \bar{\nu} \) and \( \bar{\varepsilon} \) are as in Table 6.2 and \( 0 \leq \bar{\eta} \leq 0.002 \).

The proof of this theorem follows closely the proof of (the more general) Theorem 1.2 in [8], where, however, no explicit computations of constants (size of admissible \( \bar{\varepsilon} \), size of admissible \( \bar{\eta} \), ...) have been carried out. Thus, the proof presented here, although complete and self-contained, is somewhat complementary to that presented in [8].

The main idea (as in Chapter 3) is to reduce the existence of spin–orbit resonances to a fixed point problem containing parameters: then such a problem is solved by a Lyapunov–Schmidt or “range–bifurcation equation” decomposition (see Chapter 3). The “range equation” is solved by standard contraction mapping method and the “bifurcation equation” is solved by a topological argument.

Clearly, the main point for the application to the Solar System is to compute everything explicitly with attention to optimal estimates so as to include all cases of physical interest.

### 6.3 Proof of Theorem 6.1

**Step 1. Reformulation of the problem of finding spin–orbit resonances**

Let \( x(t) \) be a \( p:q \) spin–orbit resonance and let \( u(t) := x(qt) - pt - \xi \). Then, by (6.1) and choosing \( \xi \) suitably one sees immediately that \( u \) is \( 2\pi \)-periodic and satisfies the differential equation

\[
\begin{align*}
  u''(t) + \eta (u'(t) - \nu) + \varepsilon f_x(x + pt + u(t), qt) &= 0, \\
  \langle u \rangle &= 0,
\end{align*}
\]  

(6.2)

where \( \langle \cdot \rangle \) denotes the average over the period\(^1\) and

\[
\eta := q\bar{\eta}, \quad \nu := q\bar{\nu} - p, \quad \varepsilon := q^2\bar{\varepsilon}.
\]  

(6.3)

Separating the linear part from the non–linear one, we can rewrite (6.2) as follows: let

\[
\begin{cases}
  L_\eta u := u'' + \eta u', \\
  [\Phi_\xi(u)](t) := \eta \nu - \varepsilon f_x(x + pt + u(t), qt),
\end{cases}
\]  

(6.4)

then, the differential equation in (6.2) is equivalent to the functional equation

\[
L_\eta u = \Phi_\xi(u),
\]  

(6.5)

(see also (3.6)).

\(^1\)The parameter \( \xi \) represents the average of the periodic part of \( x(t) \) and will be our “bifurcation parameter”.

---

*Page 95*
Step 2. The Green operator $G_\eta = L_\eta^{-1}$

As in Definition 3.2 let $B := C^0_{\text{per}, 0}$ be the Banach space of $2\pi$–periodic continuous functions with zero average (endowed with the sup–norm). The following lemma gives sharp estimates, which will be used later.

**Lemma 6.1.**

$$v \in B \cap C^1 \implies \|v\|_{C^0} \leq \frac{\pi}{2}\|v'\|_{C^0} \quad (6.6)$$

$$v \in B \cap C^2 \implies \|v\|_{C^0} \leq \frac{\pi}{8}\|v''\|_{C^0} \quad (6.7)$$

**Proof.** We first prove (6.6). By a rescaling it is sufficient to prove (6.6) with $\|v'\|_{C^0} = 1$. Assume by contradiction that $\|v\|_{C^0} =: c > \pi/2$.

Note that it is obvious that $c \leq \pi$ since $v$ has zero average and, therefore, it must vanish at some point. Up to a translation we can assume that $|v|$ attains its maximum in $-c$. Furthermore we can also assume that $v$ attains its minimum in $-c$ (if not multiply $v$ by $-1$), namely

$$\|v\|_{C^0} = c = -v(-c).$$

![Figure 6.1: The integral over $[-2c, 2\pi - 2c]$ of a function with values in the filled area is less than 0 (see [44]).](image)

Since $\|v'\|_{C^0} = 1$ we get

$$v(t) \leq -c + |t + c|, \quad \forall t \in [-2c, 0],$$

and, therefore (see Figure 6.1) we have

$$v(0) \leq 0, \quad v(-2c) \leq 0, \quad \int_{-2c}^{0} v \leq -c^2. \quad (6.8)$$

Analogously for $t \geq 0$, since $\|v'\|_{C^0} = 1$ we get

$$v(t) \leq \pi - c - |t - \pi + c|, \quad \forall t \in [0, 2\pi - 2c].$$
Then (see again Figure 6.1) we have
\[ \int_0^{2\pi - 2c} v \leq (\pi - c)^2. \] (6.9)

From (6.8) and (6.9) we get
\[ \int_{-2c}^{2\pi - 2c} v \leq (\pi - c)^2 - c^2 = \pi(\pi - 2c) < 0, \]
which contradicts the fact that \( v \) has zero average, proving (6.6).

We now prove (6.7). Up to a rescaling we can prove (6.7) assuming \( \|v''\|_{C^0} = 1 \).
Assume by contradiction that
\[ \|v\|_{C^0} =: c > \frac{\pi^2}{8}. \] (6.10)

Up to a translation we can assume that \( |v| \) attains its maximum at 0. Furthermore we can also assume that \( v \) attains its minimum in 0 (if not multiply \( v \) by \(-1\)), namely
\[ \|v\|_{C^0} = c = -v(0). \]

![Figure 6.2: The integral over \([\sqrt{2c}, 2\pi - \sqrt{2c}]\) of a function with values in the filled area is less than 0.](image)

Since \( \|v''\|_{C^0} = 1 \) we get
\[ v(t) \leq -c + \frac{t^2}{2} \quad \forall t \in \left[-\sqrt{2c}, \sqrt{2c}\right]. \]

Since \( v \) has zero average there must exist (see Figure 6.2)
\[ t_1 \leq -\sqrt{2c}, \ t_2 \geq \sqrt{2c}, \ t_2 - t_1 < 2\pi, \ \text{s.t.} \ v(t_1) = v(t_2) = 0, \ v(t) < 0 \ \forall t \in (t_1, t_2). \]

Moreover
\[ \int_{t_1}^{t_2} v \leq \int_{-\sqrt{2c}}^{\sqrt{2c}} -c + \frac{1}{2} t^2 = \frac{2}{3} (2c)^{3/2}. \] (6.11)

Analogously for \( t \geq \sqrt{2c} \), since \( \|v''\|_{C^0} = 1 \) we get
\[ v(t) \leq \frac{(t - \sqrt{2c})(2\pi - \sqrt{2c} - t)}{2}, \quad \forall t \in [\sqrt{2c}, 2\pi - \sqrt{2c}]. \]
Then (see again Figure 6.2) we have
\[
\int_{\sqrt{2c}}^{2\pi - \sqrt{2c}} v \leq \frac{2(3\sqrt{2c} - 2c\sqrt{2c} - 6c\pi - \pi^3)}{3}. \tag{6.12}
\]
From (6.11) and (6.12) we get
\[
\int_{-\sqrt{2c}}^{2\pi - \sqrt{2c}} v \leq -\frac{2}{3}(2c)^{3/2} + \frac{2}{3}(3\sqrt{2c} - 2c\sqrt{2c} - 6c\pi - \pi^3)
\]
\[
= -\frac{c^{3/2}8\sqrt{2}}{3} + 4\sqrt{2c}^2\sqrt{c} + \frac{2}{3}\pi^3
\]
\[
< - \left(\frac{\pi^2}{8}\right)^{3/2} \frac{8\sqrt{2}}{3} + 4\sqrt{2}\pi^2 - \frac{2\sqrt{2}\pi^2}{3} \frac{2}{3}\pi^3
\]
\[
= 0,
\]
which contradicts the fact that \(v\) has zero average, proving (6.7).
\[\square\]

Recall that the linear operator \(L_\eta : C_{\text{per},0}^2 \to \mathbb{B}\) defined in (6.4) is the same as in (3.6). From Definition 3.2 and Lemma 3.1 we know that the inverse operator \(G_\eta = L_\eta^{-1}\) is a bounded linear isomorphism. The following elementary Lemma finds the optimal bound for the norm of the operator \(G_\eta\). In its proof we will use Lemma 6.1.

**Lemma 6.2.** Let \(\eta < 2/\pi\). Then it follows:
\[
\|G_\eta\|_{L(\mathcal{B},\mathcal{B})} = \sup_{\eta : \|g\|_{C^0} = 1} \|G_\eta(g)\|_{C^0} \leq \left(1 + \eta \frac{\pi}{2}(1 - \frac{\pi}{2}\eta)^{-1}\right) \frac{\pi^2}{8}.\tag{6.13}
\]
In particular, assuming
\[
\eta \leq \frac{\pi}{5}\left(\frac{10}{\pi^2} - 1\right), \tag{6.14}
\]
which by (6.3) is equivalent to
\[
\tilde{\eta} \leq \left\{\begin{array}{ll}
\frac{\pi}{5} \left(\frac{10}{\pi^2} - 1\right) = 0.0083\cdots, & \text{if } (p,q) = (1,1), \\
\frac{\pi}{10} \left(\frac{10}{\pi^2} - 1\right) = 0.0041\cdots, & \text{if } (p,q) = (3,2),
\end{array}\right. \tag{6.15}
\]
one gets
\[
\|G_\eta\|_{L(\mathcal{B},\mathcal{B})} \leq \frac{5}{4}. \tag{6.16}
\]
**Proof.** Given \(g \in \mathcal{B}\) with \(\|g\|_{C^0} = 1\) we have to prove that if \(u \in \mathcal{B}\) is the unique solution of \(u'' + \eta u' = g\) with \(\langle u \rangle = 0\), then (6.13) is equivalent to
\[
\|u\|_{C^0} = \|G_\eta(g)\|_{L(\mathcal{B},\mathcal{B})} \leq \left(1 + \eta \frac{\pi}{2}(1 - \frac{\pi}{2}\eta)^{-1}\right) \frac{\pi^2}{8}. \tag{6.17}
\]
We note that, setting \(v := u'\), we have that \(v \in \mathcal{B}\) and \(v' = -\eta v + g\). Then we get
\[
\|v\|_{C^0} \leq \frac{\pi}{2}\|v\|_{C^0} - \eta v + g\|_{C^0} \leq \frac{\pi^2}{2}(\eta\|v\|_{C^0} + 1),
\]
which implies
\[
\|u'\|_{C^0} = \|v\|_{C^0} \leq \left(1 - \frac{\pi}{2}\eta\right)^{-1}\frac{\pi}{2}. \tag{6.18}
\]
Since \(u'' = -\eta u' + g\), we have
\[
\|u\|_{C^0} \leq \frac{\pi^2}{8}\| - \eta u' + g\|_{C^0} \leq \frac{\pi^2}{8}(1 + \eta\|u'\|_{C^0})
\]
and (6.17) follows by (6.18). Furthermore (6.16) follows directly by (6.13) and (6.14). \[\square\]
Step 3. Lyapunov–Schmidt decomposition

Here we use the main ideas already developed in Section 3.1 for the general spin-orbit problem. Solutions of (6.5) are recognized as fixed points of the operator $\mathcal{G}_\eta \circ \Phi_\xi$:

$$u = \mathcal{G}_\eta \circ \Phi_\xi(u),$$  \hspace{1cm} (6.19)

where $\xi$ appears as a parameter. To solve equation (6.19), we shall perform a Lyapunov–Schmidt decomposition. Let us denote by $\hat{\Phi}_\xi : C^0_{\text{per}} \to \mathbb{B} = C^0_{\text{per},0}$ (as in Lemma 3.2) the operator

$$\hat{\Phi}_\xi(u) := \frac{1}{\varepsilon} [\Phi_\xi(u) - \langle \Phi_\xi(u) \rangle] = -f_x(\xi + pt + u(t;\xi),qt) + \phi_u(\xi),$$  \hspace{1cm} (6.20)

Then, equation (6.19) can be splitted into a “range equation” (recall (3.9))

$$u = \varepsilon \mathcal{G}_\eta \circ \hat{\Phi}_\xi(u),$$  \hspace{1cm} (6.21)

(where $u = u(\cdot;\xi)$) and a “bifurcation (or kernel) equation” (recall (3.8))

$$\phi_u(\xi) = \frac{\eta \nu}{\varepsilon} \iff \langle \Phi_\xi(u(\cdot;\xi)) \rangle = 0.$$  \hspace{1cm} (6.22)

Remark 6.1. (i) If $(u,\xi) \in \mathbb{B} \times [0,2\pi]$ solves (6.21) and (6.22), then $x(t) = \xi + \frac{\eta}{q} t + u(t/q;\xi)$ solves (1.24).

(ii) Recall from the proof of Lemma 3.2 that $\hat{\Phi}_\xi \in C^1(\mathbb{B}, \mathbb{B})$ holds $\forall \xi \in [0,2\pi]$ and $\forall (u,\xi) \in \mathbb{B} \times [0,2\pi]$ we have

$$\|\hat{\Phi}_\xi(u)\|_{C^0} \leq 2 \sup_{T^2} |f_x|, \quad \|D_u \hat{\Phi}_\xi\|_{L(\mathbb{B}, \mathbb{B})} \leq 2 \sup_{T^2} |f_{xx}|.$$  \hspace{1cm} (6.23)

Analogously as in Subsection 3.1.1, the usual way to proceed to solve (6.21) and (6.22) is the following:

1. for any $\xi \in [0,2\pi]$, find $u = u(\cdot;\xi)$ solving (6.21);

2. insert $u = u(\cdot;\xi)$ into the kernel equation (6.22) and determine $\xi \in [0,2\pi]$ so that (6.22) holds.

Step 4. Solving the range equation (contracting map method)

For $\varepsilon$ small the range equation is easily solved by a standard contraction argument. Let

$$R := \frac{5}{2} \varepsilon \sup_{T^2} |f_x|,$$  \hspace{1cm} (6.24)

and let

$$\mathbb{B}_R := \{ v \in \mathbb{B} : \|v\|_{C^0} \leq R \},$$  \hspace{1cm} (6.25)

$$\varphi : \mathbb{B}_R \to \varphi(v) := \varepsilon \mathcal{G}_\eta \circ \hat{\Phi}_\xi(v).$$
Lemma 6.3. Assume that \( \eta \) satisfies (6.14) and that
\[
\frac{5}{2} \varepsilon \sup_{T^2} |f_{xx}| < 1. \tag{6.26}
\]
Then, for every \( \xi \in [0, 2\pi] \), there exists a unique \( u := u(\cdot; \xi) \in \mathbb{B}_R \) such that \( \varphi(u) = u \).

Proof. By (6.14) the map \( \varphi \) in (6.25) maps \( \mathbb{B}_R \) into itself, since
\[
\|\varphi(v)\|_{C^0} \overset{(6.25)}{=} \varepsilon \|G_{\eta}(\hat{\Phi}_\xi(v))\|_{C^0} \leq \varepsilon \|G_{\eta}\|_{L(B,B)} \|\hat{\Phi}_\xi(v)\|_{C^0}
\]
\[
\overset{(6.16) & (6.23)}{\leq} \frac{5}{4} \sup_{T^2} |f_{xx}| \overset{(6.24)}{=} R. \tag{6.27}
\]
From the mean-value theorem we have
\[
\|\varphi(u) - \varphi(v)\|_{C^0} \leq \varepsilon \|\hat{\Phi}_\xi(u) - \hat{\Phi}_\xi(v)\|_{C^0}
\]
\[
\leq \varepsilon \frac{5}{4} \|D_u \hat{\Phi}_\xi\|_{L(B,B)} \|u - v\|_{C^0}
\]
\[
\overset{(6.23)}{\leq} \frac{5}{2} \sup_{T^2} |f_{xx}| \|u - v\|_{C^0} \tag{6.28}
\]
and so we proved that \( \varphi \) is a contraction with Lipschitz constant smaller than 1 by (6.26). The existence and uniqueness of \( u := u(\cdot; \xi) \in \mathbb{B}_R \) follows by the standard fixed point theorem. This terminates the proof of Lemma 6.3. \( \square \)

Remark 6.2. By (1.29), (1.30) we notice that
\[
\sup_{T^2} |f_x| \leq \frac{1}{|1 - e|^3}, \tag{6.29}
\]
\[
\sup_{T^2} |f_{xx}| \leq \frac{2}{|1 - e|^3}, \tag{6.30}
\]
hold. By (6.29) and (6.30) the “range condition” (6.26) follows from
\[
\bar{\varepsilon} < \begin{cases} \frac{(1-e)^3}{3}, & \text{if } (p,q) = (1,1), \\ \frac{(1-e)^3}{20}, & \text{if } (p,q) = (3,2). \end{cases} \tag{6.31}
\]

Step 5. Solving the bifurcation equation (6.22)
From Taylor’s theorem the function \( \phi_u(\xi) \) in (6.20) can be written as
\[
\phi_u(\xi) = \phi^{(0)}(\xi) + \varepsilon \phi_u^{(1)}(\xi; \varepsilon), \tag{6.32}
\]
with
\[
\phi^{(0)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} f_x(\xi + pt, qt) dt,
\]
\[
\phi_u^{(1)}(\xi; \varepsilon) = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} u(t; \xi) \left( \int_0^1 f_{xx}(\xi + pt + su(t; \xi), qt) ds \right) dt.
\]
Since for \( \varepsilon \) satisfying (6.26) by Lemma 6.3 we have \( u(t; \xi) = \varphi(u(t; \xi)) \in \mathbb{B}_R \), then
\[
\sup_{\xi \in [0, 2\pi]} |\phi_u^{(1)}| \leq \sup_{T^2} |f_{xx}| \frac{R}{\varepsilon} \leq \frac{5}{2} \left( \sup_{T^2} |f_x| \right) \left( \sup_{T^2} |f_{xx}| \right) \tag{6.33}
\]
holds. From Remark 6.2 the condition (6.26) follows from (6.31). Hence, by (6.29), (6.30) and (6.33) for \( \varepsilon \) satisfying (6.31) one finds immediately
\[
\sup_{\xi \in [0, 2\pi]} |\tilde{\varphi}(1)| \leq \frac{5}{(1 - e)^6} =: M_1. \tag{6.34}
\]

From (3.32) we know that \( \phi(0)(\xi) \) can also be written as follows:
\[
\phi(0)(\xi) = \begin{cases} 
-2\alpha_2(e) \sin(2\xi), & \text{if } (p,q) = (1,1), \\
-2\alpha_3(e) \sin(2\xi), & \text{if } (p,q) = (3,2),
\end{cases} \tag{6.35}
\]
where \( \alpha_j(e) \) for \( j \in \mathbb{Z} \) are defined in (2.1). Define
\[
a_{pq} := \begin{cases} 
2|\alpha_2(e)| - \varepsilon M_1, & \text{if } (p,q) = (1,1), \\
2|\alpha_3(e)| - \varepsilon M_1, & \text{if } (p,q) = (3,2).
\end{cases} \tag{6.36}
\]

Then, from (6.32), (6.34), (6.35) and (6.36), it follows that \( \phi_u([0,2\pi]) \) contains the interval \([-a_{pq}, a_{pq}]\), which is non empty provided (recall (6.3), (6.34) and (6.36))
\[
\bar{\varepsilon} < \begin{cases} 
\frac{2(1-e)^6}{5}|\alpha_2(e)|, & \text{if } (p,q) = (1,1), \\
\frac{(1-e)^6}{10}|\alpha_3(e)|, & \text{if } (p,q) = (3,2). \tag{6.37}
\end{cases}
\]

Therefore, we can conclude that the bifurcation equation (6.22) admits a solution if one assumes \( \frac{2\varepsilon}{|\varepsilon|} \leq a_{pq} \), i.e. (recall again (6.3), (6.34) and (6.36)), if
\[
\bar{\eta} < \begin{cases} 
\frac{\varepsilon}{|\varepsilon| - 1} \left( 2|\alpha_2(e)| - \frac{5\varepsilon}{(1-e)^6} \right), & \text{if } (p,q) = (1,1), \\
\frac{2\varepsilon}{|2\varepsilon - 1|} \left( 2|\alpha_3(e)| - \frac{20\varepsilon}{(1-e)^6} \right), & \text{if } (p,q) = (3,2). \tag{6.38}
\end{cases}
\]

We have proven the following:

**Proposition 6.1.** Let \( (p,q) = (1,1) \) or \( (p,q) = (3,2) \) and assume (6.15), (6.31), (6.37) and (6.38). Then (1.24) admits a \( p:q \) spin-orbit resonance.

**Step 6. Lower bounds on \( |\alpha_2(e)| \) and \( |\alpha_3(e)| \)**

In order to complete the proof of the Theorem 6.1, checking the conditions of Proposition 6.1 for the resonant satellites of the Solar System, we need to give lower bounds on the absolute values of the Fourier coefficients \( \alpha_2(e) \) and \( \alpha_3(e) \). To do this we will simply use the Taylor formula to develop \( \alpha_j(e) \) in powers of \( e \) up to a suitably large order\(^2\)
\[
\alpha_j(e) = \sum_{k=0}^{h} \alpha_j^{(k)} e^k + R_j^{(k)}(e) \tag{6.39}
\]
and use the analyticity property of the function
\[
G_e(t) := -\frac{e^{2i\rho e(t)}}{2\rho e(t)^2} = \sum_{j \in \mathbb{Z}} \alpha_j(e) e^{ijt}
\]
\(^2\)We shall choose \( h = 4 \) for the 1:1 resonances and \( h = 21 \) for the 3:2 case of Mercury.
(already defined in Definition 2.2), to get an upper bound on $R_j^{(h)}$ by means of standard Cauchy estimates for holomorphic functions. To use Cauchy estimates, we need an upper bound of $G_e$ in a complex eccentricity region. The following simple result will be enough.

**Lemma 6.4.** Let $R_j^{(h)}(e)$ be as in (6.39), $0 < b < 1$ and $0 < e < r(b) := b / \cosh b$. Then,

$$|R_j^{(h)}(e)| \leq R_j^{(h)}(e; b)$$

with

$$R_j^{(h)}(e; b) := \frac{2}{(1 - b)^3} \left( (1 + \frac{b}{\cosh b} - e)(1 + \cosh b + 1 - b) \right)^2 \frac{e^{b+1}}{(\cosh b - e)^{b+1}}.$$

**Proof.** For $e, \varrho > 0$ we set

$$[0, e]_\varrho := \{ z \in \mathbb{C} | z = z_1 + z_2, \ z_1 \in [0, e], \ |z_2| < \varrho \}.$$

![Figure 6.3: The set $[0, e]_\varrho$ with $\varrho = r - e$ (see [44]).](image)

Since $0 < b < 1$ and $r(b) := \frac{b}{\cosh b}$ are as in (4.2), by Lemma 4.1 and standard (complex) Cauchy estimates imply, for $0 \leq s \leq 1$,

$$|D^{h+1} \alpha_j(se)| \leq \frac{(h + 1)!}{(r - e)^{h+1}} \sup_{[0, e], r-e} |\alpha_j(e)|$$

and, therefore,

$$|R_j^{(h)}(e)| \leq \frac{e^{b+1}}{(r - e)^{h+1}} \sup_{[0, e]_\varrho} |\alpha_j(e)|.$$

By (4.16) we obtain

$$\sup_{[0, e]_\varrho} |\alpha_j(e)| \leq \frac{2}{(1 - b)^3} \left( (1 + r - e)(1 + \cosh b + 1 - b) \right)^2$$

from which Lemma 6.4 follows.  

Now, in order to check the conditions of Proposition 6.1 we will expand $\alpha_2(e)$ in powers of $e$ up to order $h = 4$ and $\alpha_3(e)$ up to order $h = 21$. Using the representation formula (2.14) and Mathematica we find:

$$\alpha_2(e) = -\frac{1}{2} + \frac{5}{4} e^2 - \frac{13}{32} e^4 + R_2^{(4)}(e), \quad (6.40)$$
\[ \alpha_3(e) = -\frac{7}{4} e + \frac{123}{32} e^3 - \frac{489}{256} e^5 + \frac{1763}{4096} e^7 - \frac{13527}{32768} e^9 + \frac{180369}{13107200} e^{11} \\
+ \frac{5986093}{734003200} e^{13} + \frac{24606987}{3355443200} e^{15} + \frac{33790034193}{526134937600} e^{17} \\
+ \frac{1193558821627}{21045397504000} e^{19} + \frac{467145991400853}{92599494901760000} e^{21} + R_3^{(21)}(e). \] (6.41)

From (6.40) and (6.41) in view of Lemma 6.4, we choose, respectively, \( b = 0.462678 \) and \( b = 0.768368 \) to get lower bounds:

\[ |\alpha_2(e)| \geq \frac{1}{2} - \frac{5}{4} e^2 + \frac{13}{32} e^4 - |R^{(4)}(e; 0.462678)| \] (6.42)

\[ |\alpha_3(e)| \geq \sum_{k=0}^{21} \alpha_3^{(k)} e^k - |R^{(21)}(e; 0.768368)|. \] (6.43)

**Step 7. Checking the conditions and conclusion of the proof**

We are now ready to check all conditions of Proposition 6.1 with the parameters of the satellite in spin–orbit resonance given in Table 6.1 and 6.2.

In Table 6.3 we report:

- in column 2: the lower bounds on \( |\alpha_2(e)| \) as obtained in Step 6 using (6.42) and (6.43) (with the eccentricities listed in Table 6.1 and 6.2);
- in column 3: the difference between the right hand side and the left hand side of the inequality\(^4\) (6.31);
- in column 4: the difference between the right hand side and the left hand side of the inequality (6.37);
- in column 5: the right hand side and the left hand side of the inequality (6.38), which is an upper bound for the admissible values of the dissipative parameter \( \bar{\eta} \).

\(^3\)The values for \( b \) are rather arbitrary (as long as \( 0 < b < 1 \)); our choice is made for optimizing the estimates.

\(^4\)Thus, the inequality is satisfied if the numerical value in the column is positive; the same applies to the fourth and to the fifth column.
Table 6.3: Check of the hypotheses of Proposition 6.1 for the satellites in spin–orbit resonance

| Satellite | lower bound on $|\alpha_{2p/q}|$ | r.h.s. - l.h.s. of Eq. (6.31) | r.h.s. - l.h.s. of Eq. (6.37) | r.h.s. - l.h.s. of Eq. (6.38) |
|-----------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| Moon      | 0.45475265                   | 0.1663508                     | 0.127144                      | 0.1225335                    |
| Io        | 0.49997893                   | 0.1889174                     | 0.186489                      | 81.800325                    |
| Europa    | 0.49988598                   | 0.1934518                     | 0.187978                      | 1.8031043                    |
| Ganymede  | 0.49999849                   | 0.1974024                     | 0.196745                      | 264.3751                     |
| Callisto  | 0.49993049                   | 0.1937258                     | 0.189389                      | 5.6260606                    |
| Mimas     | 0.49938883                   | 0.0989819                     | 0.088051                      | 19.852395                    |
| Enceladus | 0.49997228                   | 0.161793                      | 0.159015                      | 218.44519                    |
| Tethys    | 0.49999999                   | 0.1670079                     | 0.166948                      | 458.43746                    |
| Dione     | 0.49999395                   | 0.1901481                     | 0.188837                      | 281.18521                    |
| Rhea      | 0.49999875                   | 0.1895878                     | 0.18899                       | 1554.7362                    |
| Titan     | 0.49776167                   | 0.1830341                     | 0.166905                      | 0.0357326                    |
| Iapetus   | 0.49790449                   | 0.171834                      | 0.155986                      | 2.2484865                    |
| Ariel     | 0.4999982                    | 0.1871186                     | 0.186401                      | 1321.448                     |
| Umbriel   | 0.499998095                  | 0.1833031                     | 0.180992                      | 145.83674                    |
| Titania   | 0.49999849                   | 0.1924958                     | 0.191838                      | 910.34423                    |
| Oberon    | 0.49997555                   | 0.188917                      | 0.188081                      | 826.10305                    |
| Miranda   | 0.49997989                   | 0.1503505                     | 0.149754                      | 3623.6286                    |
| Charon    | 0.49999395                   | 0.1917247                     | 0.190414                      | 231.15781                    |
| Mercury   | 0.27                         | 0.0244515                     | 0.006171                      | 0.0012363                    |

The positive values reported in the fourth and fifth column means that the range condition (6.31) and the “topological condition (6.37) are satisfied for all the moons in 1:1 resonance and for Mercury; the bifurcation condition (6.38) yields an upper bound on the admissible value for $\bar{\eta}$. Thus, $\bar{\eta}$ has to be smaller than the minimum between the value in the fifth column of Table 6.3 and the value in the right hand side of Eq. (6.15) (needed to give a bound on the Green operator): such minimum values is seen to be $0.008 \cdots$ for the moons in 1:1 resonance and $0.002 \cdots$ for Mercury.

The proof of the Theorem 6.1 is complete.
6.4 Small bodies

In the Solar System besides the eighteen moons listed in Table 6.2 and Mercury there are other five minor bodies with mean radius smaller than 100 km observed in 1:1 spin–orbit resonance around their planets: Phobos and Deimos (Mars), Amalthea (Jupiter), Janus and Epimetheus (Saturn).

<table>
<thead>
<tr>
<th>Principal body</th>
<th>Satellite</th>
<th>Eccentricity</th>
<th>a (km)</th>
<th>b (km)</th>
<th>Oblateness $\varepsilon = \frac{3}{2} \frac{a^2 - b^2}{a^2 + b^2}$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mars</td>
<td>Phobos [40],[42]</td>
<td>0.0151</td>
<td>13.4</td>
<td>11.2</td>
<td>0.26616393443443</td>
<td>1.00136808</td>
</tr>
<tr>
<td></td>
<td>Deimos [40],[42]</td>
<td>0.0002</td>
<td>7.5</td>
<td>6.1</td>
<td>0.30558527712</td>
<td>1.00000024</td>
</tr>
<tr>
<td>Jupiter</td>
<td>Amalthea [40]</td>
<td>0.0031</td>
<td>125</td>
<td>73</td>
<td>0.73704304667</td>
<td>1.00005766</td>
</tr>
<tr>
<td>Saturn</td>
<td>Janus [36]</td>
<td>0.0073</td>
<td>97.4</td>
<td>96.9</td>
<td>0.007719969946</td>
<td>1.000319741</td>
</tr>
<tr>
<td></td>
<td>Epimetheus [36]</td>
<td>0.0205</td>
<td>58.7</td>
<td>58.0</td>
<td>0.017994211119</td>
<td>1.002521568</td>
</tr>
</tbody>
</table>

Table 6.4: Physical data of minor bodies in 1:1 spin–orbit resonance (see also http://solarsystem.nasa.gov/planets/profile.cfm?Object=Mars&Display=Sats).

Besides being small, such bodies have also a quite irregular shape and only Janus and Epimetheus have a good equatorial symmetry. Indeed, for these two small moons, Theorem 6.1 holds as shown by the data reported in Table 6.5:

| Satellite | lower bound on $|\alpha_{20/4}|$ | r.h.s. - l.h.s. of Eq. (6.31) | r.h.s. - l.h.s. of Eq. (6.37) | r.h.s. - l.h.s. of Eq. (6.38) |
|-----------|----------------------------------|-------------------------------|-------------------------------|-------------------------------|
| Phobos    | 0.49996743                       | -0.075088                     | -0.08373                      | -89.23777                    |
| Deimos    | 0.49999995                       | -0.105705                     | -0.10583                      | -674530.2                   |
| Amalthea  | 0.49998797                       | -0.538897                     | -0.54074                      | -35210.01                   |
| Janus     | 0.4999324                        | 0.1879319                     | 0.183652                      | 23.167321                   |
| Epimetheus| 0.49927518                       | 0.1699562                     | 0.158377                      | 6.3987689                   |

Table 6.5: Check of the hypotheses of Proposition 6.1 for the small satellites in spin–orbit resonance.
6.5 Eccentricity Intervals

In this section we let the eccentricity \( e \) of the orbit of the satellite be a variable. This assumption is plausible since we know that the eccentricity \( e \) is a function of the time. For example, in [32] it is explained how the eccentricity of Mercury oscillates during the last 3 Gyr.

Some natural questions now come into mind: for all of the above satellites what are the intervals of eccentricity for which the \((1, 1)\)−periodic orbit exists? And what are the intervals of eccentricity for which the \((3, 2)\)−periodic orbit exists? Are there satellites for which both resonances simultaneously exists?

Condition (6.31) is weaker than condition (6.37). Therefore, using a truncated version of \( \alpha_2(e) \) and of \( \alpha_3(e) \) at 6-th and 21-th order in the \( e \)-expansion, respectively, we can plot in a \( e - \varepsilon \) plane an approximation of the right hand side of (6.37). These lines correspond to the conditions for the existence of the \((1, 1)\)− and of the \((3, 2)\)−periodic orbit, respectively.

At the same time for every satellite \( S \) in Tables 6.1, Table 6.5 and for Mercury we draw the point \((e_S, \varepsilon_S)\) in the \( e - \varepsilon \) plane corresponding to its values of eccentricity \( e_S \) and oblateness \( \varepsilon_S \). If the point \((e_S, \varepsilon_S)\) stays below the line corresponding to the condition for the existence of the \((1, 1)\)− or/and the \((3, 2)\)−periodic orbit, then the \((1, 1)\)− or/and the \((3, 2)\)−periodic orbits exist, respectively. Otherwise, if the point \((e_S, \varepsilon_S)\) stays above the line corresponding to the condition for the existence of the \((1, 1)\)− or/and the \((3, 2)\)−periodic orbit, then our model says nothing on the existence of the \((1, 1)\)− or/and the \((3, 2)\)−periodic orbits, respectively. The results are presented in Figures 6.4 and 6.5.

![Figure 6.4: e - \varepsilon plane. Existence of the (1, 1)− and the (3, 2)−periodic orbit.](image)

In the Figure 6.4 one can see that only three points are above the (blue) line, corresponding to the condition for the existence of a \((1, 1)\)−periodic orbit. These are the small satellites Phobos, Deimos and Amalthea studied in Section 6.4 (recall Table 6.5). For all the other satellites we are considering the \((1, 1)\)−periodic orbit exists. Moreover the Moon, Titan and Mercury have a large eccentricity, which makes them recognizable in Figure 6.4. They lie clearly below the (red) line, corresponding to the condition for the existence of \((3, 2)\)−periodic orbits, i.e. for them both resonances are possible.
The region near the origin in Figure 6.4 is densely filled with points and Figure 6.5 is a zoom of this area.

Figure 6.5: Zoom in near to the origin of Figure 6.4.

Besides the objects observed in Figure 6.5 one can see: a (yellow) horizontal line representing the maximum of the condition for the existence of $(3, 2)$—periodic orbits; a (green) horizontal corresponding to $\varepsilon = \varepsilon_{\text{Rhea}} = 0.0098127957$; a (blue) horizontal corresponding to $\varepsilon = \varepsilon_{\text{Callisto}} = 0.00186698679$. Notice that:

- Rhea is the satellite with the lowest value for the oblateness for which the condition for the existence of $(3, 2)$—periodic orbits is not satisfied. Every satellite $S$ with $\varepsilon_S \geq \varepsilon_{\text{Rhea}}$ does not fulfil the condition for the existence of $(3, 2)$—periodic orbits.

- The point of the satellite Europa lies below the condition for the existence of $(3, 2)$—periodic orbits. So there are four satellites (Europa, Moon, Titan and Mercury), for which both (the 1 : 1 and the 3 : 2) periodic resonances exist at the same time.

- Seven satellites (Io, Ganymede, Callisto, Dione, Janus, Titania and Charon) lie below the yellow horizontal line. Their eccentricity is however too small to satisfy the condition for the existence of $(3, 2)$—periodic orbits. Since from [32] we know that the eccentricity varies as a function of time, it is possible that long time ago some of those seven satellites were in a $(3, 2)$—periodic orbits. For example the point of Callisto is really near to the line of the condition for the existence of $(3, 2)$—periodic orbits, suggesting that he is the latest (among this seven satellites) to have lefted the region below this curve. Unfortunately we can not verify this assertion, since the astronomical observations do not go that back in the past.

- To satisfy both conditions a satellite should have a really small value of oblateness (i.e. it should be almost spherical) and a relatively large orbital eccentricity.
In Table 6.6, for all the satellites in Tables 6.1, Table 6.5 and for Mercury we give the eccentricity intervals $I_1$ and $I_2$ for the existence of the $(1,1)$– and of the $(3,2)$–periodic orbits, respectively.

<table>
<thead>
<tr>
<th>Satellite</th>
<th>$I_1$</th>
<th>$I_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moon</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.02, 0.38]$</td>
</tr>
<tr>
<td>Phobos</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Deimos</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Io</td>
<td>$[0, 0.36]$</td>
<td>$[0.09, 0.19]$</td>
</tr>
<tr>
<td>Europa</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.01, 0.37]$</td>
</tr>
<tr>
<td>Ganymede</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.02, 0.41]$</td>
</tr>
<tr>
<td>Callisto</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.02, 0.41]$</td>
</tr>
<tr>
<td>Amalthea</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Mimas</td>
<td>$[0, 0.11]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Enceladus</td>
<td>$[0, 0.23]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Tethys</td>
<td>$[0, 0.24]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Dione</td>
<td>$[0, 0.36]$</td>
<td>$[0.09, 0.19]$</td>
</tr>
<tr>
<td>Rhea</td>
<td>$[0, 0.35]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Titan</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.01, 0.58]$</td>
</tr>
<tr>
<td>Iapetus</td>
<td>$[0, 0.34]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Janus</td>
<td>$[0, 0.37]$</td>
<td>$[0.07, 0.22]$</td>
</tr>
<tr>
<td>Epimetheus</td>
<td>$[0, 0.3]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Ariel</td>
<td>$[0, 0.33]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Umbriel</td>
<td>$[0, 0.32]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Titania</td>
<td>$[0, 0.38]$</td>
<td>$[0.06, 0.24]$</td>
</tr>
<tr>
<td>Oberon</td>
<td>$[0, 0.35]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Miranda</td>
<td>$[0, 0.19]$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Charon</td>
<td>$[0, 0.38]$</td>
<td>$[0.06, 0.24]$</td>
</tr>
<tr>
<td>Mercury</td>
<td>$[0, 0.41721847]$</td>
<td>$[0.01, 0.41721847]$</td>
</tr>
</tbody>
</table>

Table 6.6: $I_1$ and $I_2$ are the intervals of the eccentricity for the existence of the $(1,1)$– and of the $(3,2)$–periodic orbits, respectively. Notice that $0.41721847 = r(0.462678) = \frac{b}{\cosh(b)}$ as in (6.42) and (6.43).
7 Estimates on the basin of attraction

This chapter is joint work with L. Biasco and L. Chierchia, Preprint (2014) (see [2]).

In Theorem 1.3 of [8] Biasco and Chierchia give estimates of the basin of attraction of stable $(p, 1)$– and $(p, 2)$–periodic orbits for $p \in \mathbb{Z}$. In this chapter we present two results, which concern this particular theorem, which we write here for completeness:

**Theorem 7.1.** Let $x_{pq}$ be a $(p, q)$–periodic orbit with $q = 1, 2$. Assume that

$$\theta_0 := \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(\xi + pt, qt)dt > 0.$$ 

Then for $\bar{\varepsilon}_0$ and $\bar{\eta}_0$ as in Theorem 4.1 there exist $\bar{\varepsilon}^*$ and $\bar{\eta}^*$ with $0 < \bar{\varepsilon}^* \leq \bar{\varepsilon}_0$ and $0 \leq \bar{\eta} \leq \bar{\eta}^*$ and

$$\bar{\eta}^2 < \bar{\varepsilon} \min\{\theta_0, 1\},$$

every solution $x(t)$ with initial conditions sufficiently close to the initial conditions of $x_{pq}$ tends exponentially to $x_{pq}(t)$; more precisely, if $x(t)$ is a solution of (1.24) with

$$\sqrt{\bar{\varepsilon}}|x(0) - x_{pq}(0)| + |\dot{x}(0) - \dot{x}_{pq}(0)| \leq \bar{c}_1 \bar{\eta},$$

then

$$\sqrt{\bar{\varepsilon}}|x(t) - x_{pq}(t)| + |\dot{x}(t) - \dot{x}_{pq}(t)| \leq \bar{c}_2 \bar{\eta} e^{-\bar{\eta} t/2}.$$ 

The first result of this chapter, Theorem 7.2, is the generalisation of Theorem 7.1 to $(p, q)$–periodic orbits for arbitrary $q \geq 3$. The second result, Theorem 7.3 is a quantitative reformulation of Theorem 7.1. This makes concrete applications possible. We will present the case of Titan (as satellite of Saturn).

### 7.1 Main results and applications

As already mentioned before, for $q = 1, 2$ estimates on the basin of attraction of periodic orbits are given in Theorem 7.1. In the following theorem we give estimates on the basin of attraction of $x_{pq}$ for $q \geq 3$.

**Theorem 7.2.** Let $q \geq 3$ and $(p, q) = 1$. Let $x_{pq}$ as in (3.5) be a $(p, q)$–periodic solution of the spin-orbit problem. Assume that $x_{pq}$ is non-degenerate at $k_{th}$-degree for some $k \geq 1$ (according to Definition 3.3) and

$$\theta_k := \partial^k \Theta^k(\xi) > 0.$$ 

Then there exist positive $\bar{\varepsilon}_*$ and $\bar{\eta}_*$ and constants $0 < \bar{c}_1 \leq \bar{c}_2$ such that for all $\bar{\varepsilon}, \bar{\eta}$ with $0 < \bar{\varepsilon} \leq \bar{\varepsilon}_*$, $0 \leq \bar{\eta} \leq \bar{\eta}_*$ and

$$\bar{\eta}^2 < q^2 \bar{\varepsilon}^2 \min\left[\left[q^2 \bar{\varepsilon}\right]^{k-1} \theta_k, 4\right],$$ 

(7.2)
every solution \( x(t) \) with initial conditions sufficiently close to the initial conditions of \( x_{pq} \) tends exponentially to \( x_{pq}(t) \); more precisely, if \( x(t) \) is a solution of (1.24) with

\[
|x(0) - x_{pq}(0)| + \frac{|\dot{x}(0) - \dot{x}_{pq}(0)|}{q\bar{\varepsilon}} \leq \bar{c}_1 \bar{\eta},
\]

then

\[
|x(t) - x_{pq}(t)| + \frac{|\dot{x}(t) - \dot{x}_{pq}(t)|}{q\bar{\varepsilon}} \leq \bar{c}_2 \bar{\eta}e^{-\bar{\eta}t/2}, \quad \forall t \geq 0.
\]

The following theorem is a quantitative reformulation of Theorem 7.1. Besides the assumptions (7.3) - (7.5), which were already taken in Theorem 7.1, we have an extra condition (equation (7.6)) on \( \bar{\varepsilon} \). Under these assumptions we give explicit formulas for the constants \( \bar{c}_1 \) and \( \bar{c}_2 \) in equation (7.7), which give a lower estimate of the basin of attraction of a \((p,1)\)— or a \((p,2)\)—periodic orbit.

**Theorem 7.3.** Let \( x_{pq} \) be a \((p,q)\)—periodic orbit with \( q = 1, 2 \) as in (3.5). Assume that

\[
\theta_0 := \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(\xi + pt, qt) dt > 0. \quad (7.3)
\]

Furthermore define:

\[
K := 2 \sup_{t \in [0,2\pi]} |f_{xx}(\bar{x}(t), qt)|,
\]

\[
\bar{K} := 1 + \frac{4K}{\pi^2 \theta_0},
\]

\[
\lambda_1 := \sqrt{\frac{1}{8\pi^2 \bar{K}}},
\]

\[
M_1 := 8\pi^2 \bar{K} \lambda_1^2 + \lambda_1^2 64\pi^3 \bar{K} \left( \frac{1}{2\pi} \left| \frac{2i\lambda_1}{1 - e^{-2\pi\lambda_1}} \right| + 1 \right) \left( 1 + \pi \bar{K} \lambda_1 \right),
\]

\[
M_2 := \lambda_1 \left[ 2\pi \bar{K} + 2\lambda_1 M_1 (1 + \pi \bar{K} \lambda_1) \right],
\]

\[
C' := (2M_2 + \lambda_1 (1 + M_1)) \left( 1 + \frac{\lambda_1 + M_2}{|\lambda_1 - M_2|} \right),
\]

\[
S := \frac{1 + M_1}{|\lambda_1 - M_2|},
\]

\[
S' := \frac{2M_2 + \lambda_1 (1 + M_1)}{|\lambda_1 - M_2|},
\]

\[
c_1 := \sup_{t \in [0,2\pi]} \sup_{\zeta \in [\bar{x}(t), \bar{x}(t) + \bar{w}(t)]} \frac{|f_{xxx}(\zeta, qt)|}{2},
\]

\[
c_2 := \max \left( C, \frac{\sqrt{2}}{\pi \sqrt{\theta_0}} S, \sqrt{\frac{576\pi^2 \theta_0}{47}} C', S' \right).
\]

Then for \( \varepsilon, \bar{\varepsilon} := q^2 \bar{\varepsilon} \) and \( \bar{\eta} \) so small such, that

\[
\eta^2 < \bar{\varepsilon} \min \{ \theta_0, 1 \}, \quad (7.4)
\]

\[
\frac{1}{2} \theta_0 \leq \theta := \theta(\varepsilon) := \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(x_{pq}(qt), qt) dt \leq 2\theta_0 \quad (7.5)
\]
and

\[ \varepsilon \leq \min \left( \frac{1}{8\theta_0 \pi^2 \left(1 + \frac{8K}{\theta_0}\right) \left[4\pi \left(1 + \frac{8K}{\theta_0}\right) + 1\right]} \right) \tag{7.6} \]

hold, every solution \( x(t) \) with initial conditions sufficiently close to the initial conditions of \( x_{pq}(t) \); more precisely, if \( x(t) \) is a solution of (1.24) with

\[ \sqrt{\varepsilon} |x(0) - x_{pq}(0)| + |\dot{x}(0) - \dot{x}_{pq}(0)| \leq \bar{c}_1 \bar{\eta}, \]

then

\[ \sqrt{\varepsilon} |x(t) - x_{pq}(t)| + |\dot{x}(t) - \dot{x}_{pq}(t)| \leq \bar{c}_2 \bar{\eta} e^{-\bar{\eta}t/2}, \]

where

\[ \bar{c}_1 = \frac{1}{64c_1 c_2^3} \quad \text{and} \quad \bar{c}_2 = \frac{3}{8c_1 c_2^2}. \tag{7.7} \]

**Remark 7.1.** Theorem 7.3 can be applied to Titan (a satellite of Saturn). For Titan we have \( e_{\text{Titan}} = 0.0288, \bar{\varepsilon}_{\text{Titan}} = 1.7882901 \cdot 10^{-4} \). Numerical approximations show that

\[ \theta_0 = 1.995068948321280 > 0 \]

and

\[ \theta(\bar{\varepsilon}_{\text{Titan}}) = 1.997250241599330, \]

therefore conditions (7.3) and (7.5) are satisfied. By (1.25) we have

\[ \bar{\eta}_{\text{Titan}} \approx 10^{-3} \cdot \Omega_{\text{Titan}} \]

\[ = \left(1 + 3e_{\text{Titan}}^2 + \frac{3}{8}e_{\text{Titan}}^4\right) \frac{1}{(1 - e_{\text{Titan}}^2)^{9/2}} = 0.00100638906743. \]

**Condition (7.4) holds since**

\[ 1.012516717318596 \cdot 10^{-6} \approx \bar{\eta}_{\text{Titan}}^2 < \bar{\varepsilon}_{\text{Titan}} \min \{\theta_0, 1\} \approx 3.567762049100357 \cdot 10^{-4}. \]

**Computing the right hand side of (7.6) we get that**

\[ \min \left( \frac{1}{8\theta_0 \pi^2 \left(1 + \frac{8K}{\theta_0}\right) \left[4\pi \left(1 + \frac{8K}{\theta_0}\right) + 1\right]} \right) \approx 4.679380623399785 \cdot 10^{-4}, \]

which is larger than \( \bar{\varepsilon}_{\text{Titan}} \). Therefore also (7.6) is satisfied. Then by (7.7) we get \( \bar{c}_1 = 0.02631663091091581 \) and \( \bar{c}_2 = 93.314130269171429 \).

### 7.2 Proof of Theorem 7.2

#### 7.2.1 Introduction

We consider orbits \( x(t) \) starting near the \((p, q)\)-periodic solution \( x_{pq}(t) \), cf. (3.5), i.e.

\[ x(t) = x_{pq}(t) + \tilde{w}(t). \tag{7.8} \]

Inserting \( x(t) \) in equation (1.24) we get:

\[ (x_{pq}(t) + \tilde{w}(t))^\prime + \eta ((x_{pq}(t) + \tilde{w}(t))^\prime - \nu) + \varepsilon f_x(x_{pq}(t) + \tilde{w}(t), t) = 0 \]
\[ x''_{pq}(t) + \tilde{\eta}_t((x_{pq}(t) - \bar{v}) + \tilde{w}' + \tilde{\eta} \tilde{w}' + \varepsilon f_x(x_{pq}(t) + \tilde{w}(t), t) = 0. \]

Setting \( w(t) := \tilde{w}(qt) \) and replacing \( t \) by \( qt \), the condition that \( x(t) \) is a solution of (1.24) implies

\[ w''(t) + \eta w'(t) + \varepsilon f_x(\bar{x}(t) + w(t), qt) - \varepsilon f_x(\bar{x}(t), qt) = 0, \]

where

\[ \bar{x}(t) = x_{pq}(qt) = \xi + pt + u(t), \quad (7.9) \]
\[ \eta = q\tilde{\eta}, \quad (7.10) \]
\[ \varepsilon = q^2\tilde{\varepsilon}. \quad (7.11) \]

Defining

\[
\begin{aligned}
(Q(w))(t) &:= f_x(\bar{x}(t) + w(t), qt) - f_x(\bar{x}(t), qt) - f_{xx}(\bar{x}(t), qt)w(t), \\
\gamma(t) &:= f_{xx}(\bar{x}(t), qt) - \theta, \\
\alpha &:= \frac{\gamma}{2},
\end{aligned}
\]

for \( \theta \) as in (7.5) we get

\[
\begin{aligned}
w''(t) + \eta w'(t) + \varepsilon f_x(\bar{x}(t) + w(t), qt) - \varepsilon f_x(\bar{x}(t), qt) & = 0 \quad \Leftrightarrow \\
w''(t) + \eta w'(t) + \varepsilon (Q(w))(t) + \varepsilon f_{xx}(\bar{x}(t), qt)w(t) & = 0 \quad \Leftrightarrow \\
w''(t) + \eta w'(t) + \varepsilon(\theta + \gamma(t))w(t) & = -\varepsilon (Q(w))(t).
\end{aligned}
\]

Setting \( w(t) = z(t)e^{-\alpha t} \) the last equation is equivalent to

\[
Lz \triangleq z'' + ((\varepsilon \theta - \alpha^2) + \varepsilon \gamma(t))z = -\varepsilon e^{\alpha t} (Q(e^{-\alpha t}z))(t). \quad (7.13)
\]

**Remark 7.2.** \( Q \) is a quadratic operator, i.e. there exists a constant \( c_1 \) (depending only on \( f \)) such that

\[
|Q(w)(t)| \leq c_1|w(t)|^2, \quad \forall t \in \mathbb{R}.
\]

*From the Taylor theorem we get*

\[
c_1 := \sup_{t \in [0,2\pi]} \sup_{\zeta \in [\bar{x}(t), \bar{x}(t) + w(t)]} \frac{|f_{xxx}(\zeta, qt)|}{2}.
\]

**Remark 7.3.** From (3.14) also the function \( u(t; \xi, \varepsilon) \) can be written in the form

\[
u(t; \xi, \varepsilon) = \sum_{k=1}^{\infty} u_k \varepsilon^k.
\]

*Using Taylor expansion we define the following formal series in \( \varepsilon \):

\[
f_{xx}(\bar{x}(t), qt) =: f_2^{(0)} + \varepsilon f_2^{(1)} + \varepsilon^2 f_2^{(2)} + \ldots, \]

\[
\theta = \theta(\varepsilon) \quad (7.5) \quad := \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(\bar{x}(t), qt)dt =: \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots,
\]

\[
\gamma = \gamma(t) \quad (7.12) \quad := f_{xx}(\bar{x}(t), qt) - \theta =: \gamma_0 + \varepsilon \gamma_1 + \varepsilon^2 \gamma_2 + \ldots.
\]

**Remark 7.4.** For \( \phi_u(\xi) \) in (3.8) notice that

\[
\partial_\xi \phi_u(\xi) = \partial_\xi \frac{1}{2\pi} \int_0^{2\pi} f_x(\bar{x}(t), qt)dt = \frac{1}{2\pi} \int_0^{2\pi} f_{xx}(\bar{x}(t), qt)dt
\]

implies \( \partial_\xi \phi^{(l)}(\xi) = \theta_l \) for every \( \mathbb{Z} \ni l \geq 1 \).
7.2.2 The homogeneous equation $Lz = 0$

The substitution $u_1 = z, u_2 = z'$ takes the homogeneous equation $Lz = 0$ to

$$
\begin{pmatrix}
  u_1' \\
  u_2'
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  -\left[(\varepsilon^{k+1}\tilde{\theta} - \alpha^2) + \varepsilon \gamma(t)\right]
\end{pmatrix}
\begin{pmatrix}
  u_2 \\
  u_1
\end{pmatrix},
$$

(7.14)

where $\tilde{\theta} := \theta_k + \varepsilon\theta_{k+1} + \varepsilon^2\theta_{k+2} + \ldots$ for $\theta_l$ for $l \in \mathbb{N}$ as in Remark 7.3. Since $\varepsilon = q^2 \bar{\varepsilon}, \eta = q \bar{\eta}$ and $\alpha = \frac{\eta}{2}$ hold, equation (7.2) implies

$$
\frac{\eta^2}{4} = \alpha^2 < \frac{\varepsilon^{k+1}\theta_k}{4}.
$$

(7.15)

The fundamental solution of (7.14), i.e. the two-by-two matrix $U$ satisfying

$$
U' = \begin{pmatrix}
  0 \\
  -\left[(\varepsilon^{k+1}\tilde{\theta} - \alpha^2) + \varepsilon \gamma(t)\right]
\end{pmatrix}U,
$$

with $U(0) = Id,$

(7.16)

is given by

$$
U(t) := U(t; \varepsilon) := \begin{pmatrix}
  c(t) & s(t) \\
  c'(t) & s'(t)
\end{pmatrix}.
$$

(7.17)

We rewrite (7.16) as follows:

$$
U' = AU, \quad \text{with } U(0) = Id,
$$

(7.18)

where

$$
\begin{align*}
A & := T + \varepsilon D_0(t) + \varepsilon^2 D_1(t) + \ldots + \varepsilon^k D_{k-1}(t) + \varepsilon^{k+1} E + O(\varepsilon^{k+2}), \\
T & := \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}, \\
E(t) & := \begin{pmatrix}
  0 \\
  \alpha^2
\end{pmatrix} - \begin{pmatrix}
  f_1(t) \\
  0
\end{pmatrix}, \\
D_l(t) & := \begin{pmatrix}
  0 & 0 \\
  -\gamma(t) & 0
\end{pmatrix}, \quad \text{for } 0 \leq l \leq k - 1.
\end{align*}
$$

Lemma 7.1. Let

$$
\Delta := -2\pi \int_0^{2\pi} (\Gamma_0(s))^2 \, ds + \left(\int_0^{2\pi} \Gamma_0(s) \, ds\right)^2,
$$

(7.19)

where $\Gamma_0$ is defined by

$$
\Gamma_0(t) := \int_0^t \gamma_0(s) \, ds.
$$

(7.20)

Then we have $\Delta < 0$.

**Proof.** The proof follows from the Cauchy-Schwarz inequality

$$
\left(\int_a^b f(s)g(s) \, ds\right)^2 \leq \int_a^b f(s)^2 \, ds \int_a^b g(s)^2 \, ds,
$$

taking $f(s) = \Gamma_0(s),\ g(s) \equiv 1,\ a = 0$ and $b = 2\pi$.

In Subsection 7.2.3 we show that the trace $\text{tr}(U(2\pi))$ of $U(2\pi)$ plays a crucial role for the determination of the fundamental solutions of the homogeneous equation $Lz = 0$. In the following lemma we find a formula for $\text{tr}(U(2\pi))$ depending on the degree of non-degeneration $k$ of the periodic orbit (according to Definition 3.3).
Lemma 7.2. Let \((\theta_i)_{i \geq 0}\) be as in Remark 7.3. Let \(U(t; \varepsilon)\) be the solution of (7.18). Assume (7.15), \(\theta_0 = 0, \theta_1 = 0, \ldots, \theta_{k-1} = 0\) and \(\theta_k > 0\) with \(k \geq 1\). Then
\[
\text{tr}(U(2\pi)) = \begin{cases} 
2 + \varepsilon^2 \left( -4\pi^2 \left( \theta_1 - \frac{\Delta}{\varepsilon^2} \right) + \Delta \right) + O(\varepsilon^3), & \text{if } k = 1, \\
2 + \varepsilon^2 \Delta + O(\varepsilon^3), & \text{if } k \geq 2,
\end{cases}
\]
holds, where \(\Delta\) is defined in (7.19).

Proof. Introducing the formal expansion \(U(t, \varepsilon) := U_0(t) + \varepsilon U_1(t) + \varepsilon^2 U_2(t) + \ldots\) and comparing terms in equation (7.18) of the same order in \(\varepsilon\) we have
\[
U'_0 = TU_0, \\
U'_1 = TU_1 + D_0 U_0, \\
U'_2 = TU_2 + D_0 U_1 + D_1 U_0, \\
\vdots \\
U'_{k} = TU_k + D_0 U_{k-1} + D_1 U_{k-2} + \ldots + D_{k-1} U_0, \\
U'_{k+1} = TU_{k+1} + D_0 U_k + D_1 U_{k-1} + \ldots + D_{k-1} U_1 + EU_0.
\]

- The system of ordinary differential equations of zero degree in \(\varepsilon\) can be easily solved:
\[
U'_0 = TU_0, \quad U_0(0) = Id \quad \implies \quad U_0(t) = e^{Tt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\]

- For the term of order one in \(\varepsilon\) we have
\[
U'_1 = TU_1 + D_0 U_0, \quad \text{with initial condition } U_1(0) = 0.
\]

Using the variation of constant formula\(^1\) we get
\[
U_1(t) = e^{Tt} \int_0^t e^{-Ts} D_0(s) e^{Ts} ds
= e^{Tt} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\gamma_0(s) & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} ds
= e^{Tt} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\gamma_0(s) & -s\gamma_0(s) \end{pmatrix} ds.
\]

Since by assumption \(\langle \gamma_0 \rangle = \theta_0 = 0\), it follows that
\[
U_1(2\pi) = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \int_0^{2\pi} \begin{pmatrix} s\gamma_0(s) & s^2\gamma_0(s) \\ -\gamma_0(s) & -s\gamma_0(s) \end{pmatrix} ds
= \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = \begin{pmatrix} a & b - 2\pi a \\ 0 & -a \end{pmatrix},
\]
where \(a := \int_0^{2\pi} s\gamma_0(s) ds\) and \(b := \int_0^{2\pi} s^2\gamma_0(s) ds\). Therefore \(U_1(2\pi)\) has zero trace, i.e.
\[
\text{tr} \left( U_1(2\pi) \right) = a - a = 0.
\]

\(^1\)Let \(y'(t) + a(t)y(t) = f(t)\) be a first order inhomogeneous linear ordinary differential equation. The homogeneous solution is given by \(y_{\text{hom}}(t) = ce^{-A(t)}\), where \(A(t)\) is the primitive of \(a(t)\). By the variation of constants formula one gets the solution \(y(t) = e^{-A(t)} \int_0^t f(s)e^{A(s)} ds\).
Now we have to distinguish two cases:

**Case 1:** $k = 1$, i.e. $q = 4$.

- For the term of order two in $\varepsilon$ (because $k = 1$), we have

$$U_2' = TU_2 + D_0U_1 + EU_0,$$

with initial condition $U_2(0) = 0$.

Again using the variation of constant formula we get

$$U_2(t) = e^{Tt} \int_0^t e^{-Ts} [D_0(s)U_1(s) + E(s)U_0(s)] \, ds$$

$$= e^{Tt} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \cdot \left[ \begin{pmatrix} 0 & 0 \\ -\gamma_0(s) & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \int_0^s \begin{pmatrix} \tau\gamma_0(\tau) & \tau^2\gamma_0(\tau) \\ -\gamma_0(\tau) & -\tau\gamma_0(\tau) \end{pmatrix} d\tau \\ \begin{pmatrix} 0 & 0 \\ -(f_2^{(1)}(s) - \frac{\alpha^2}{\varepsilon}) & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \right] ds$$

$$= e^{Tt} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \left[ \begin{pmatrix} 0 & 0 \\ -\gamma_0(s) & -s\gamma_0(s) \end{pmatrix} \int_0^s \begin{pmatrix} \tau\gamma_0(\tau) & \tau^2\gamma_0(\tau) \\ -\gamma_0(\tau) & -\tau\gamma_0(\tau) \end{pmatrix} d\tau \\ \begin{pmatrix} 0 & 0 \\ -c(s) & -c(s)s \end{pmatrix} \right] ds$$

where $c(t) := f_2^{(1)}(t) - \frac{\alpha^2}{\varepsilon}$. We compute now the trace of $U_2$ for $t = 2\pi$:

$$\text{tr}(U_2(2\pi)) = \text{tr}(I) + \text{tr}(II),$$

where

$$I := \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \int_0^{2\pi} \begin{pmatrix} s\gamma_0(s) & s^2\gamma_0(s) \\ -\gamma_0(s) & -s\gamma_0(s) \end{pmatrix} \int_0^s \begin{pmatrix} \tau\gamma_0(\tau) & \tau^2\gamma_0(\tau) \\ -\gamma_0(\tau) & -\tau\gamma_0(\tau) \end{pmatrix} d\tau ds,$$

$$II := \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \int_0^{2\pi} \begin{pmatrix} c(s)s & c(s)s^2 \\ -c(s) & -c(s)s \end{pmatrix} ds.$$

For the term $II$ we have

$$II = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \int_0^{2\pi} c(s)s ds & \int_0^{2\pi} c(s)s^2 ds \\ \int_0^{2\pi} -c(s) ds & \int_0^{2\pi} -c(s)s ds \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^{2\pi} c(s)s ds - 2\pi \int_0^{2\pi} c(s) ds & \int_0^{2\pi} c(s)s ds \\ \int_0^{2\pi} -c(s) ds & \int_0^{2\pi} -c(s)s ds \end{pmatrix} \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c(s)s & c(s)s^2 \\ -c(s) & -c(s)s \end{pmatrix} ds.$$

Therefore

$$\text{tr}(II) = -2\pi \int_0^{2\pi} c(s) ds = -2\pi \int_0^{2\pi} \left[ f_2^{(1)}(s) - \frac{\alpha^2}{\varepsilon} \right] ds.$$
\[
= -2\pi \left( 2\pi \theta_1 - 2\pi \frac{\alpha^2}{\varepsilon^2} \right) = -4\pi^2 \left( \theta_1 - \frac{\alpha^2}{\varepsilon^2} \right),
\]
where we used the fact that
\[
\int_0^{2\pi} f_2^{(1)}(t) dt = \int_0^{2\pi} f_{xxx}(\xi + pt, qt) dt = \partial_\xi \int_0^{2\pi} f_{xx}(\xi + pt, qt) dt = 2\pi \theta_1.
\]
The term I is more complicated to compute:
\[
I = \left( \begin{array}{cc} 1 & 2\pi \\ 0 & 1 \end{array} \right) \int_0^{2\pi} \left( \begin{array}{cc} s\gamma_0(s) & s^2\gamma_0(s) \\ -s\gamma_0(s) & -s\gamma_0(s) \end{array} \right) \int_0^s \left( \begin{array}{cc} \tau \gamma_0(\tau) & \tau^2\gamma_0(\tau) \\ -\gamma_0(\tau) & -\tau\gamma_0(\tau) \end{array} \right) d\tau \, ds
\]
\[
= \int_0^{2\pi} \int_0^s \gamma_0(s)\gamma_0(\tau) \left( \tau s - 2\pi \tau - s^2 + 2\pi s - \tau^2 + s \right) d\tau \, ds.
\]
Therefore we have
\[
\text{tr}(I) = \int_0^{2\pi} \int_0^s \gamma_0(s)\gamma_0(\tau) \left[ \tau s - 2\pi \tau - s^2 + 2\pi s - \tau^2 + s \right] d\tau \, ds
\]
\[
= \int_0^{2\pi} \int_0^s \gamma_0(s)\gamma_0(\tau) \left[ 2\pi(s - \tau) - (s - \tau)^2 \right] d\tau \, ds
\]
\[
= 2\pi \int_0^{2\pi} \gamma_0(s) \int_0^s \gamma_0(\tau)(s - \tau) d\tau \, ds - \int_0^{2\pi} \gamma_0(s) \int_0^s \gamma_0(\tau)(s - \tau)^2 d\tau \, ds. \tag{IV}
\]
The integrals III and IV can be computed by partial integration. Defining
\[
\Lambda_0(t) := \int_0^t \Gamma_0(s) ds
\]
with \( \Gamma_0 \) in (7.20) and recalling the obvious \( \Gamma_0(2\pi) - \Gamma_0(0) = \Gamma_0(2\pi) = 0 \) we get
\[
III = \int_0^s \gamma_0(\tau)(s - \tau) d\tau = \int_0^s \gamma_0(\tau)(s - \tau)^2 \bigg|_0^s + \int_0^s \Gamma_0(\tau) d\tau = \int_0^s \Gamma_0(\tau) d\tau
\]
and
\[
IV = \int_0^s \gamma_0(\tau)(s - \tau)^2 d\tau = \Gamma_0(\tau)(s - \tau)^2 \bigg|_0^s + 2 \int_0^s \Gamma_0(\tau)(s - \tau) d\tau
\]
\[
= 2 \left[ \Lambda_0(\tau)(s - \tau) \bigg|_0^s + \int_0^s \Lambda_0(\tau) d\tau \right] = 2 \int_0^s \Lambda_0(\tau) d\tau.
\]
Again by partial integration we have
\[
\text{tr}(I) = 2\pi \left[ \int_0^{2\pi} \gamma_0(s) \int_0^s \Gamma_0(\tau) d\tau \, ds - 2 \int_0^{2\pi} \gamma_0(s) \int_0^s \Lambda_0(\tau) d\tau \, ds \right]
\]
\[
= 2\pi \left[ \int_0^{2\pi} \Gamma_0(s) \int_0^s \Gamma_0(\tau) d\tau \bigg|_0^{2\pi} - \int_0^{2\pi} \Gamma_0(s)^2 d\tau \right]
\]
\[
- 2 \left[ \int_0^{2\pi} \Lambda_0(\tau) d\tau \bigg|_0^{2\pi} - \int_0^{2\pi} \Gamma_0(s) \Lambda_0(\tau) d\tau \bigg|_0^{2\pi} \right].
\]
7.2. Proof of Theorem 7.2

This terminates the proof of Lemma 7.2.

**Case 2: k ≥ 2, i.e. q ≥ 3 and q ≠ 4.**

- For the second degree in \( ε \) (because \( k ≥ 2 \)) we have

\[
U_2' = TU_2 + D_0 U_1 + D_1 U_0, \quad \text{with initial condition } \quad U_2(0) = 0.
\]

Using the variation of constant formula we get

\[
U_2(t) = e^{Tt} \int_0^t e^{-Ts} [D_0(s)U_1(s) + D_1(s)U_0(s)] \, ds
\]

\[
= e^{Tt} \int_0^t e^{-Ts} D_0(s)U_1(s)ds + e^{Tt} \int_0^t e^{-Ts} D_1(s)U_0(s)ds
\]

\[
= e^{Tt} \int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\gamma_0(s) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} \, ds + e^{Tt} \int_0^t e^{-Ts} \gamma_0(t) \begin{pmatrix} s \gamma_1(s) & s^2 \gamma_1(s) \\ -s \gamma_0(s) & s \gamma_1(s) \end{pmatrix} \, ds
\]

\[
= e^{Tt} \int_0^t \begin{pmatrix} s \gamma_0(s) & s^2 \gamma_0(s) \\ -s \gamma_0(s) & s \gamma_0(s) \end{pmatrix} \int_0^s \begin{pmatrix} \tau \gamma_0(\tau) & \tau^2 \gamma_0(\tau) \\ -\tau \gamma_0(\tau) & -\tau^2 \gamma_0(\tau) \end{pmatrix} \, d\tau ds + e^{Tt} \int_0^t \begin{pmatrix} s \gamma_1(s) & s^2 \gamma_1(s) \\ -s \gamma_1(s) & -s \gamma_1(s) \end{pmatrix} \, ds.
\]

We compute now the trace of \( U_2 \) for \( t = 2\pi \). So we have

\[
\text{tr}(U_2(2\pi)) = \text{tr}(\tilde I) + \text{tr}(\tilde II),
\]

where

\[
\begin{align*}
\tilde I & := \left( \begin{array}{cc} 1 & 2\pi \\ 0 & 1 \end{array} \right) \int_0^{2\pi} \left( \begin{array}{cc} s \gamma_0(s) & s^2 \gamma_0(s) \\ -s \gamma_0(s) & s \gamma_0(s) \end{array} \right) \left( \begin{array}{cc} \tau \gamma_0(\tau) & \tau^2 \gamma_0(\tau) \\ -\tau \gamma_0(\tau) & -\tau^2 \gamma_0(\tau) \end{array} \right) \, d\tau ds, \\
\tilde II & := \left( \begin{array}{cc} 1 & 2\pi \\ 0 & 1 \end{array} \right) \int_0^{2\pi} \left( \begin{array}{cc} s \gamma_1(s) & s^2 \gamma_1(s) \\ -s \gamma_1(s) & -s \gamma_1(s) \end{array} \right) \, ds.
\end{align*}
\]

Since by assumption \( \langle \gamma_1 \rangle = \theta_1 = 0 \), it follows that

\[
\tilde II = \left( \begin{array}{cc} 1 & 2\pi \\ 0 & 1 \end{array} \right) \int_0^{2\pi} \left( \begin{array}{cc} s \gamma_1(s) & s^2 \gamma_1(s) \\ -s \gamma_1(s) & s^2 \gamma_1(s) \end{array} \right) \, ds
\]

\[
= \left( \begin{array}{cc} 1 & 2\pi \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \tilde a & \tilde b \\ 0 & -\tilde a \end{array} \right) = \left( \begin{array}{cc} \tilde a & \tilde b - 2\pi \tilde a \\ 0 & -\tilde a \end{array} \right),
\]

where \( \tilde a := \int_0^{2\pi} s \gamma_1(s) \, ds \) and \( \tilde b := \int_0^{2\pi} s^2 \gamma_1(s) \, ds \).

This implies that the trace of \( \tilde II \) is zero. Notice that \( \tilde I \) is equal to \( I \), defined in the previous subsection. It follows that

\[
\text{tr}(U_2(2\pi)) = \text{tr}(\tilde I) = -2\pi \int_0^{2\pi} (\gamma_0(s))^2 \, ds + \left( \int_0^{2\pi} \gamma_0(s) \, ds \right)^2 = (7.19) \Delta.
\]

This terminates the proof of Lemma 7.2. □
Chapter 7. Estimates on the basin of attraction

7.2.3 Fundamental solutions of the homogeneous equation \( Lz = 0 \)

We define \( 2\delta := \text{tr}(U(2\pi)) \).

**Remark 7.5.** From Lemma 7.1, 7.2 and condition (7.15) we get \( 0 < \delta < 1 \) for \( \varepsilon \) small enough.

We interpret the second order differential equation \( Lz = 0 \) as a Hill equation \( z'' + A(t)z = 0 \) with \( A(t+2\pi) = A(t) := (\varepsilon\theta - \alpha^2) + \varepsilon \gamma(t) \) for all \( t \in \mathbb{R} \). From the Floquet Theory (see [22], [46]) we know that

\[
\text{tr}(U(2\pi)) = c(2\pi) + s'(2\pi),
\]

where \( c(t), s(t) \) are the fundamental solutions of \( Lz = 0 \) with initial data

\[
c(0) = 1 = s'(0) \quad \text{and} \quad c'(0) = 0 = s(0).
\]

The characteristic equation has the form

\[
\begin{align*}
\rho^2 - \left[c(2\pi) + s'(2\pi)\right] \rho + 1 &= 0 \\
\rho^2 - 2\delta \rho + 1 &= 0
\end{align*}
\]

and we find

\[
\rho_\pm = \delta \pm \sqrt{\delta^2 - 1}.
\]

Since \( 0 < \delta < 1 \) the two solutions \( \rho_\pm \) are complex conjugate. By [46] (see Theorem at p. 4) the equation \( Lz = 0 \) has two independent solutions of the form

\[
z_\pm(t) = e^{i\lambda t}P_\pm(t),
\]

where \( P_\pm(t) \) are \( 2\pi \)-periodic functions and \( \lambda \) satisfies

\[
\cos(2\pi \lambda) \pm i \sin(2\pi \lambda) = e^{\pm i 2\pi \lambda} = \rho_\pm = \delta \pm \sqrt{\delta^2 - 1}.
\]

**Remark 7.6.** Comparing the imaginary parts it is easy to see that \( \lambda \sim \varepsilon \) for both cases \( (k = 1 \text{ and } k \geq 2) \), i.e. there exist constants \( c, \tilde{c} \in \mathbb{R}_+ \) such that \( |\varepsilon| \leq c|\lambda| \) and \( |\lambda| \leq \tilde{c}|\varepsilon| \) for all \( k \geq 1 \).

We define

\[
\omega^2(\varepsilon) := \varepsilon\theta - \alpha^2 = \varepsilon^{k+1}\theta_k - \alpha^2 + O(\varepsilon^{k+2}).
\]

(7.23)

**Remark 7.7.** Of course \( \omega^2 \sim \varepsilon^{k+1} \).

We can resume the results of this section in the following lemma:

**Lemma 7.3.** Let \( \omega(\varepsilon) \) be as in (7.23). For \( \omega(\varepsilon) > 0 \) small enough there exists \( 0 < \delta < 1 \) such that the solutions of the characteristic equation (7.21) associated to \( Lz = 0 \) are given by \( \rho_\pm = \delta \pm \sqrt{\delta^2 - 1} \) and, hence, are distinct. Thus \( \lambda \) in (7.22) is real, \( \lambda \sim \varepsilon \) and \( \omega^2 \sim \varepsilon^{k+1} \) and all solutions of \( Lz = 0 \) are bounded together with their derivatives. Finally \( \delta \) and \( \lambda \) smoothly depend on \( \varepsilon \).
7.2.4 Fixed point argument

\( \zeta_+(t) = e^{i\lambda t} \mathcal{P}_+(t) \) is a solution\(^2\) of \( \mathcal{L}z = 0 \) with Dirichlet boundary conditions

\[
\mathcal{P}_+(0) = \mathcal{P}_+(2\pi) = 1
\]

if and only if \( \mathcal{P}_+ \) satisfies

\[
\ddot{\mathcal{P}}_+ + 2i\lambda \dot{\mathcal{P}}_+ + \left[(\varepsilon \theta - \alpha^2) + \varepsilon \gamma(t) - \lambda^2\right] \mathcal{P}_+ = 0.
\]

Defining \( \bar{\gamma}(t) := \gamma(t) - \gamma_0(t) - \varepsilon \gamma_1(t) - \ldots - \varepsilon^{k-1} \gamma_{k-1}(t) \) (see Remark 7.3) we get

\[
0 = \ddot{\mathcal{P}}_+ + 2i\lambda \dot{\mathcal{P}}_+ + \left[\omega^2(1 + \frac{\varepsilon^{k+1}}{\omega^2} \bar{\gamma}(t)) - \lambda^2\right] \mathcal{P}_+ + \varepsilon \gamma_0(t) \mathcal{P}_+ + \ldots + \varepsilon^{k} \gamma_{k-1}(t) \mathcal{P}_+ + \varepsilon \gamma_0(t) \mathcal{P}_+ + \ldots + \varepsilon^{k} \gamma_{k-1}(t) \mathcal{P}_+ \tag{7.24}
\]

where \( \mathcal{G}(\varepsilon; t) := \frac{\omega^2}{\lambda^2}(1 + \frac{\varepsilon^{k+1}}{\omega^2} \bar{\gamma}(t)) - 1 \). We notice that \( \mathcal{G}(\varepsilon; t) = -1 + O(\varepsilon^{k-1}) \) smoothly depends on \( \varepsilon \). Defining \( \zeta_-(t) = e^{-i\lambda t} \overline{\mathcal{P}_+(t)} \) it follows that \( \zeta_-(t) \) is a solution of \( \mathcal{L}z = 0 \). The function \( \mathcal{P}_-(t) := \overline{\mathcal{P}_+(t)} \) satisfies the equation

\[
0 = \ddot{\mathcal{P}}_- - 2i\lambda \dot{\mathcal{P}}_- + \lambda^2 \mathcal{G}(\varepsilon; t) \mathcal{P}_-(t) + \varepsilon \gamma_0(t) \mathcal{P}_-(t) + \ldots + \varepsilon^{k} \gamma_{k-1}(t) \mathcal{P}_-(t) \tag{7.25}
\]

Lemmata 7.4 - 7.8 are devoted to prove that:

- \( \mathcal{P}_+(t) \) exists and can be extended to a \( 2\pi \)-periodic function over \( \mathbb{R} \);
- For \( \mathcal{P}_\pm(t) \) as in (7.22) there are constants \( c_\pm \in \mathbb{C} \) such that \( \mathcal{P}_\pm(t) = c_\pm \mathcal{P}_\pm(t) \) holds;
- \( \mathcal{P}_-(t) = \overline{\mathcal{P}_+(t)} \).

These results will be very useful in Lemma 7.10 in order to understand the behaviour of \( \mathcal{P}_\pm(t) = \mathcal{P}_\pm(t; \lambda) \) in terms of the parameter \( \lambda \) (cf. equation (7.29)).

**Lemma 7.4.** Let \( 0 < |\lambda| < \frac{1}{2} \). Then the two solutions \( \zeta_\pm(t) = e^{i\lambda t} \mathcal{P}_\pm(t) \) are linearly independent.

**Proof.** By contradiction assume that there exists \( c \in \mathbb{C} \) such that \( \zeta_-(t) = c \zeta_+(t) \).

Then for \( t \in [0, 2\pi] \) we have

\[
e^{-i\lambda t} \overline{\mathcal{P}_+(t)} = ce^{i\lambda t} \mathcal{P}_+(t)
\]

Choosing \( t = 0 \) we get \( c = 1 \). Analogously for \( t = 2\pi \) it follows

\[
1 = \mathcal{P}_+(2\pi) = e^{i4\lambda \pi} \mathcal{P}_+(2\pi) = e^{i4\lambda \pi},
\]

which is a contradiction, since \( 0 < |\lambda| < \frac{1}{2} \). □

**Lemma 7.5.** Let \( 0 < |\lambda| < \frac{1}{2} \). \( \mathcal{P}_+(t) \) and \( \mathcal{P}_-(t) := \overline{\mathcal{P}_+(t)} \) can be extended on \( \mathbb{R} \) to two \( 2\pi \)-periodic functions, such that equations (7.24) and (7.25) still hold.

\(^2\)The existence of \( \mathcal{P}_+(t) \) will be proved in Lemma 7.7.
**Proof.** Assume that \( z_\pm(t) = e^{\pm i \lambda t} P_\pm(t) \) are independent solutions of \( L z = 0 \) on \( \mathbb{R} \). By hypothesis \( \zeta_\pm(t) = e^{i \lambda t} P_\pm(t) \) is a solution of \( L z = 0 \) on the interval \([0, 2\pi]\). From Lemma 7.4 we know that \( \zeta_\pm(t) = e^{\pm i \lambda t} P_\pm(t) \) with \( P_-(t) := \overline{P_+(t)} \) are independent. Thus on \( t \in [0, 2\pi] \) there exists \( c_\pm, d_\pm \in \mathbb{C} \) such that
\[
e^{i \lambda t} P_+(t) = c_+ e^{i \lambda t} P_+(t) + c_- e^{-i \lambda t} P_-(t),
\]
\[
e^{-i \lambda t} P_-(t) = d_+ e^{i \lambda t} P_+(t) + d_- e^{-i \lambda t} P_-(t).
\]
For \( t \in [0, 2\pi] \) we can therefore write
\[
P_+(t) = c_+ P_+(t) + c_- e^{-i 2 \lambda t} P_-(t),
P_-(t) = d_+ e^{i 2 \lambda t} P_+(t) + d_- P_-(t).
\]
Setting \( t = 0, 2\pi \) and defining \( C_\pm := c_\pm P_\pm(0), D_\pm := d_\pm P_\pm(0) \) we get
\[
C_+ + C_- = 1, \quad D_+ + D_- = 1, \quad C_+ + C_- e^{-i 4 \pi \lambda} = 1, \quad D_+ e^{i 4 \pi \lambda} + D_- = 1.
\]
Subtracting the first equation from the third and the second from the fourth yields
\[
\begin{cases}
C_-(e^{i 4 \pi \lambda} - 1) = 0, \\
D_+(e^{i 4 \pi \lambda} - 1) = 0.
\end{cases}
\]
Since by hypothesis \( 0 < |\lambda| < \frac{1}{2} \) it follows that \( e^{\pm i 4 \pi \lambda} - 1 \neq 0 \). This implies that \( C_- = D_+ = 0 \). So we have
\[
c_- P_-(0) = 0 \quad \text{and} \quad d_+ P_+(0) = 0.
\]
Since \( z_\pm \) are independent solution, \( P_+(0) \) and \( P_-(0) \) cannot be both zero, therefore \( c_- d_+ = 0 \). If \( c_- = 0 \) we have \( P_+(t) = c_+ P_+(t) \). This implies that \( P_+(t) \) and \( P_-(t) := \overline{P_+(t)} \) can be extended to two \( 2\pi \)-periodic functions defined on \( \mathbb{R} \), which solves (7.24) and (7.25). The same arguments also hold for \( d_+ = 0 \). \( \square \)

The following lemma will be used in the proof of Lemma 7.7.

**Lemma 7.6.** Let \( f : [0, 2\pi] \to \mathbb{C}, t \mapsto f(t) \) be a continuous function. Then the following holds:
\[
\left\| \int_0^t f(s) ds \right\|_{C^0([0,2\pi],[\mathbb{C}])} \leq 2\pi \left\| f \right\|_{C^0([0,2\pi],[\mathbb{C}])},
\]
\[
\left\| \int_0^t \int_0^s f(\tau) d\tau ds \right\|_{C^0([0,2\pi],[\mathbb{C}])} \leq 4\pi^2 \left\| f \right\|_{C^0([0,2\pi],[\mathbb{C}])}.
\]

**Proof.** Recall that for any \( f \in C^0([0,2\pi],[\mathbb{C}]) \) we define \( \left\| f \right\|_{C^0([0,2\pi],[\mathbb{C}])} := \sup_{t \in [0,2\pi]} |f(t)| \). Therefore we have
\[
\left\| \int_0^t f(s) ds \right\|_{C^0([0,2\pi],[\mathbb{C}])} = \sup_{t \in [0,2\pi]} \left| \int_0^t f(s) ds \right| \leq \sup_{t \in [0,2\pi]} \int_0^t |f(s)| ds \leq \int_0^{2\pi} |f(s)| ds \leq 2\pi \left\| f \right\|_{C^0([0,2\pi],[\mathbb{C}])}
\]
and
\[
\left\| \int_0^t \int_0^s f(\tau) d\tau ds \right\|_{C^0([0,2\pi],[\mathbb{C}])} = \sup_{t \in [0,2\pi]} \left| \int_0^t \int_0^s f(\tau) d\tau ds \right|.
\]
Since the function $P$ implies
Integrating again we obtain
\[
|f(t)| = 4\pi^2 \| f \|_{C^0([0,2\pi], \mathbb{C})}.
\]

Finally we can state the following fixed point lemma.

**Lemma 7.7.** The function $P_+(t) : [0, 2\pi] \to \mathbb{C}$ is a solution of the equation $P = \psi(P)$, where $\psi : C^0([0,2\pi], \mathbb{C}) \to C^0([0,2\pi], \mathbb{C})$ is defined by
\[
[\psi(P; \lambda)](t) := 1 + \lambda v_0(t) - 2i\lambda \int_0^t [P(s) - 1] ds - \lambda^2 \int_0^t \int_0^s G(\xi, \xi) P(\xi) d\xi ds +
- \varepsilon \int_0^t \int_0^s \gamma_0(\xi) P(\xi) d\xi ds - \ldots - \varepsilon^k \int_0^t \int_0^s \gamma_{k-1}(\xi) P(\xi) d\xi ds,
\]
where
\[
v_0(P; \lambda) := \frac{i}{\pi} \int_0^{2\pi} [P(s) - 1] ds + \frac{\lambda}{2\pi} \int_0^{2\pi} \int_0^s G(\xi, \xi) P(\xi) d\xi ds +
+ \frac{\varepsilon}{2\pi \lambda} \int_0^{2\pi} \gamma_0(\xi) P(\xi) d\xi ds + \ldots + \frac{\varepsilon^k}{2\pi \lambda} \int_0^{2\pi} \gamma_{k-1}(\xi) P(\xi) d\xi ds.
\]

**Proof.** Integrating (7.24) and using the fact that $P_+(0) = 1$ we get
\[
\dot{P}_+(t) = -2i\lambda \dot{P}_+(t) - \lambda^2 G(\xi; t) P_+(t) +
- \varepsilon \gamma_0(t) P_+(t) - \ldots - \varepsilon^k \gamma_{k-1}(t) P_+(t)
\]
\[
\dot{P}_+(s) - \dot{P}_+(0) = -2i\lambda [P_+(s) - P_+(0)] - \lambda^2 \int_0^s G(\xi, \xi) P_+(\xi) d\xi +
- \varepsilon \int_0^s \gamma_0(\xi) P_+(\xi) d\xi - \ldots - \varepsilon^k \int_0^s \gamma_{k-1}(\xi) P_+(\xi) d\xi
\]
\[
\dot{P}_+(s) = \dot{P}_+(0) - 2i\lambda [P_+(s) - 1] - \lambda^2 \int_0^s G(\xi, \xi) P_+(\xi) d\xi +
- \varepsilon \int_0^s \gamma_0(\xi) P_+(\xi) d\xi - \ldots - \varepsilon^k \int_0^s \gamma_{k-1}(\xi) P_+(\xi) d\xi.
\]

Integrating again we obtain
\[
P_+(t) = 1 + \dot{P}_+(0) t - 2i\lambda \int_0^t [P_+(s) - 1] ds - \lambda^2 \int_0^t \int_0^s G(\xi, \xi) P_+(\xi) d\xi ds +
- \varepsilon \int_0^t \int_0^s \gamma_0(\xi) P_+(\xi) d\xi ds - \ldots - \varepsilon^k \int_0^t \int_0^s \gamma_{k-1}(\xi) P_+(\xi) d\xi ds.
\]
Since the function $P_+(t)$ is $2\pi$-periodic, we have $P_+(2\pi) = P_+(0) = 1$. This condition implies
\[
1 = 1 + \dot{P}_+(0) 2\pi - 2i\lambda \int_0^{2\pi} [P_+(s) - 1] ds - \lambda^2 \int_0^{2\pi} \int_0^s G(\xi, \xi) P_+(\xi) d\xi ds
\]
We define the constants \( M \) and so equations (7.26) and (7.27) are satisfied.

Lemma 7.8. For \( \lambda \) small enough \( \psi(\mathcal{P}_+; \lambda) \) defined in Lemma 7.7 is a contraction in the space \( C^0([0, 2\pi], \mathbb{C}) \).

Proof. Furthermore the function \( \psi(\mathcal{P}_+, \lambda) \) smoothly depends on \( \varepsilon \) (because \( \lambda(\varepsilon) \sim \varepsilon \)). We need to show that \( \psi \) is a contraction. For \( \mathcal{P}_+, \tilde{\mathcal{P}}_+ \in C^0([0, 2\pi], \mathbb{C}) \)

\[
\| \psi(\mathcal{P}_+; \lambda)(t) - \psi(\tilde{\mathcal{P}}_+; \lambda)(t) \|_{C^0([0, 2\pi])} \leq 2\lambda \int_0^t (\mathcal{P}_+ - \tilde{\mathcal{P}}_+) ds \|_{C^0([0, 2\pi])} + \lambda^2 \int_0^t \int_0^s G(\mathcal{P}_+ - \tilde{\mathcal{P}}_+) \, d\xi ds \|_{C^0([0, 2\pi])} + \varepsilon \int_0^t \int_0^s \delta \mathcal{P}_+ \, d\xi ds \|_{C^0([0, 2\pi])} + \ldots + \varepsilon^k \int_0^t \int_0^s \delta \mathcal{P}_+ \, d\xi ds \|_{C^0([0, 2\pi])}.
\]

Using Lemma 7.6 we have

\[
\| \psi(\mathcal{P}_+; \lambda)(t) - \psi(\tilde{\mathcal{P}}_+; \lambda)(t) \|_{C^0([0, 2\pi])} \leq 4\pi \lambda \left( 1 + \lambda \pi \max_{[0, 2\pi]} \frac{\omega^2}{\lambda^2} \left( 1 + \frac{4\gamma(t)}{\theta_k} \right) + 1 \right)
\]

\[
+ 4\pi^2 \varepsilon \left( \gamma_0 \left[ \mathcal{P}_+ - \tilde{\mathcal{P}}_+ \right] \right)_{C^0([0, 2\pi])} + \ldots + 4\pi^2 \varepsilon^k \left( \gamma_{k-1} \left[ \mathcal{P}_+ - \tilde{\mathcal{P}}_+ \right] \right)_{C^0([0, 2\pi])}
\]

For \( \lambda \) small enough there exist \( D_1, D_2 \in \mathbb{R}_+ \) such that

\[
\frac{\varepsilon}{\lambda} \leq D_1, \quad \frac{\omega^2}{\lambda^2} \leq D_2.
\]

We define the constants \( M_0, M_1, \ldots, M_{k-1}, \tilde{M} \in \mathbb{R}_+ \) as follow

\[
M_i := \max_{t \in [0, 2\pi]} |\gamma_i(t)| \quad \text{for } i = 0, 1, \ldots, k-1,
\]

\[
\tilde{M} := \max_{[0, 2\pi]} D_2 (1 + 4\pi \varepsilon^k) + 1 \geq \max_{t \in [0, 2\pi]} \left( \frac{\omega^2}{\lambda^2} (1 + \gamma_{k-1}(t) \right) + 1.
\]
\( \psi \) is a contraction if the the following condition holds:

\[
\mathcal{H}_k(\lambda) := 4\pi \lambda \left[ 1 + \lambda \pi \bar{M} + D_1 \pi M_0 + D_2 \pi \lambda M_1 + \ldots + D_k \pi \lambda^{k-1} M_{k-1} \right] < 1.
\]

\( \mathcal{H}_k(\lambda) \) is a polynomial of degree \( k \) in \( \lambda \). Since \( \lambda = 0 \) is a zero of this polynom (i.e. \( \mathcal{H}_k(0) = 0 \)), there exists an interval \([0, \lambda_*] \) such that

\[ |\mathcal{H}_k(\lambda)| < 1, \quad \text{for} \ \lambda \in [0, \lambda_*). \]

Therefore \( \psi \) is a contraction for \( \lambda \) small enough. This terminates the proof of Lemma 7.7. \( \square \)

### 7.2.5 Estimates on the fundamental solutions \( c(t) \) and \( s(t) \)

**Lemma 7.9.** The fixed point \( \mathcal{P}_+ = \psi(\mathcal{P}_+) \) of Lemma 7.7 is in \( C^\infty([0,2\pi], \mathbb{C}) \).

**Proof.** (Bootstrap argument) We know that \( \mathcal{P}_+ \in C^0([0,2\pi], \mathbb{C}) \). Furthermore \( \mathcal{P}_+ \) satisfies the equation

\[ \mathcal{P}_+(t) = \psi(\mathcal{P}_+; \lambda)(t), \]

where \( \psi \) is defined in Lemma 7.7. Every term on the right-hand side is differentiable with respect to \( t \), because \( \mathcal{P}_+ \in C^0([0,2\pi], \mathbb{C}) \). Therefore \( \mathcal{P}_+ \in C^1([0,2\pi], \mathbb{C}) \). The same argument can be used to prove recursively that

\[ \mathcal{P}_+ \in C^2([0,2\pi], \mathbb{C}), \quad \mathcal{P}_+ \in C^3([0,2\pi], \mathbb{C}), \quad \mathcal{P}_+ \in C^4([0,2\pi], \mathbb{C}), \]

and so on. \( \square \)

**Lemma 7.10.** For \( \lambda \) small enough there exists \( c_2 \) such that the fundamental solutions \( c(t) \) and \( s(t) \) of \( \mathcal{L}z = 0 \) satisfy

\[ |c(t)|, |s(t)|, \frac{|c'(t)|}{\varepsilon}, |s'(t)| \leq c_2, \quad \forall t \geq 0. \quad (7.28) \]

**Proof.** Since \( \psi \) smoothly depends on \( \lambda \), the fixed point \( \mathcal{P}_+ \) also depends smoothly on \( \lambda \). We assume

\[ \mathcal{P}_+(t; \lambda) = 1 + \sum_{l=1}^{\infty} A_l(t) \lambda^l, \]

with \( A_l := A_l(t) \in \mathbb{C} \) and \( A_l(0) = A_l(2\pi) = 0 \) for all \( l \geq 1 \). Putting this Ansatz in the equations (7.26) and (7.27) of Lemma 7.7 we get:

- \( v_0 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^s \gamma_0 + O(\lambda); \)
- \( \lambda \gamma_0 - 2\lambda \int_0^t (\mathcal{P}_+ - 1) - \lambda^2 \int_0^t \int_0^s GP_+ = \frac{\varepsilon}{2\pi} \int_0^{2\pi} \int_0^s \gamma_0 + O(\lambda^2); \)
- \( -\varepsilon \int_0^t \int_0^s \gamma_0 \mathcal{P}_+ = O(\lambda) = B_1 \lambda + O(\lambda^2) \), where the term \( B_1 \) is given by

\[ B_1 = -\frac{\varepsilon}{\lambda} \int_0^t \int_0^s \gamma_0 d\xi ds. \]

Defining \( B(t) = (B_1(t) - \langle B_1 \rangle) \in \mathbb{R} \) and recalling that \( \mathcal{P}_-(t) = \overline{\mathcal{P}_+(t)} \) we have

\[
\begin{align*}
\mathcal{P}_+(t; \lambda) &= 1 + B(t) \lambda + O(\lambda^2), \\
\mathcal{P}_-(t; \lambda) &= B(t) \lambda + O(\lambda^2), \\
\hat{\mathcal{P}}_+(t; \lambda) &= 1 + \overline{B(t)} \lambda + O(\lambda^2), \\
\hat{\mathcal{P}}_-(t; \lambda) &= \overline{B(t)} \lambda + O(\lambda^2).
\end{align*}
\]

(7.29)
The fundamental solutions of the equation $Lz = 0$, i.e. the solutions $c(t)$ and $s(t)$ with initial conditions $c(0) = s'(0) = 1$ and $c'(0) = s(0) = 0$, can be expressed as linear combination of the two known solutions $\zeta_\pm(t) = e^{\pm i\lambda t}P_\pm(t)$. So we have

$$
\begin{align*}
\begin{cases}
  c(t) &= c_+ \zeta_+(t) + c_- \zeta_-(t), \\
  s(t) &= s_+ \zeta_+(t) + s_- \zeta_-(t),
\end{cases}
\end{align*}
$$

(7.30)

where

$$
\begin{align*}
c_+ &= 1 - c_- = \frac{i\lambda - \mathcal{P}_-(0)}{2i\lambda + \mathcal{P}_+(0) - \mathcal{P}_-(0)} = \frac{1}{2} + O(\lambda), \\
s_+ &= -s_- = \frac{1}{2i\lambda + \mathcal{P}_+(0) - \mathcal{P}_-(0)} = \frac{1}{2i\lambda} + O(1).
\end{align*}
$$

(7.31) (7.32)

c_+, s_+ \in \mathbb{C}$. Equation (7.31) and (7.32) can be proved as follows: Using the initial conditions for the function $c(t)$ we obtain the following system of equations:

$$
\begin{align*}
\begin{cases}
  1 &= c(0) = c_+ \zeta_+(0) + c_- \zeta_-(0), \\
  0 &= c'(0) = c_+ \zeta_+(0) + c_- \zeta_-(0).
\end{cases}
\end{align*}
$$

From the first equation it follows $c_- = 1 - c_+$. This implies that

$$
\begin{align*}
0 &= i\lambda c_+ + c_+ \mathcal{P}_+(0) + c_- \mathcal{P}_-(0) - i\lambda c_- \mathcal{P}_-(0) + c_- \mathcal{P}_-(0) \\
0 &= 2i\lambda c_+ - i\lambda + \mathcal{P}_-(0) + c_+ \left[ \mathcal{P}_+(0) - \mathcal{P}_-(0) \right].
\end{align*}
$$

Solving this equation for $c_+$ and observing that $\mathcal{B}(0) = 0$ we get (7.31). The proof for (7.32) is analogous. Then by equation (7.30) we get:

$$
\begin{align*}
\begin{cases}
  c(t) &= \cos(\lambda t) + O(\lambda), \\
  s(t) &= \frac{\sin(\lambda t)}{\lambda} + O(1).
\end{cases}
\end{align*}
$$

(7.33)

From Remark 7.6 we know that $\lambda \sim \varepsilon$. We define the constant $c \in \mathbb{R}_+$ such that $|\varepsilon| \leq c$. By (7.33) for $0 < |\lambda| < \frac{1}{2}$ small enough there exist constants $M, \tilde{M}, \bar{M}, \bar{\tilde{M}}$ such that

$$
\begin{align*}
|c(t)| &= |\cos(\lambda t) + O(\lambda)| \leq 1 + M =: c_{2,1}, \\
|\varepsilon c(t)| &= |\frac{\sin(\lambda t)}{\lambda} + O(1)| \leq c + \tilde{M} =: c_{2,2}, \\
\left| \frac{c'(t)}{\varepsilon} \right| &= \frac{1}{\varepsilon} \left| \lambda \sin(\lambda t) + O(\lambda) \right| \leq c + \bar{M} =: c_{2,3}, \\
|s'(t)| &= |\cos(\lambda t) + O(\lambda)| \leq 1 + \bar{M} =: c_{2,4}.
\end{align*}
$$

Lemma 7.10 follows defining $c_2 := \max_{i=1,2,3,4} c_{2,i}$. □
7.2.6 Technical estimates

Recall that \( z(t) \) is the solution of the inhomogeneous equation (7.13) with initial condition

\[
a := z(0), \quad b := z'(0).
\]

We define

\[
z_{\text{hom}}(t) := a \cdot c(t) + b \cdot s(t), \quad v(t) := z(t) - z_{\text{hom}}(t).
\]

**Remark 7.8.** Since for the fundamental solutions \( c(t), s(t) \) of \( \mathcal{L}z = 0 \) we have \( c(0) = s'(0) = 1 \) and \( c'(0) = s(0) = 0 \), then for \( v \)

\[
\begin{align*}
v(0) &= z(0) - z_{\text{hom}}(0) = a - (a \cdot c(0) + b \cdot s(0)) = a - a = 0, \\
v'(0) &= z'(0) - z'_{\text{hom}}(0) = b - (a \cdot c'(0) + b \cdot s'(0)) = b - b = 0,
\end{align*}
\]

hold.

**Remark 7.9.** By linearity of the operator \( \mathcal{L} \) in (7.13) we obtain \( \mathcal{L}z = \mathcal{L}z_{\text{hom}} + \mathcal{L}v = \mathcal{L}v \).

Define

\[
(\mathcal{G}[h])(t) := s(t) \int_0^t c(\tau)h(\tau)d\tau - c(t) \int_0^t s(\tau)h(\tau)d\tau
\]

and notice that

\[
\begin{align*}
\mathcal{L}(\mathcal{G}[h])(t) &= \mathcal{L} \left( s(t) \int_0^t c(\tau)h(\tau)d\tau - c(t) \int_0^t s(\tau)h(\tau)d\tau \right) \\
&= s''(t) \int_0^t c(\tau)h(\tau)d\tau + 2s'(t)c(t)h(t) + s(t) \left[ c'(t)h(t) + c(t)h'(t) \right] \\
&\quad - c''(t) \int_0^t s(\tau)h(\tau)d\tau - 2c'(t)s(t)h(t) - c(t) \left[ s'(t)h(t) + s(t)h'(t) \right] \\
&\quad + \left[ (\varepsilon \theta - \alpha^2) + \varepsilon \gamma(t) \right] \left( s(t) \int_0^t c(\tau)h(\tau)d\tau - c(t) \int_0^t s(\tau)h(\tau)d\tau \right) \\
&= [s'(t)c(t) - c'(t)s(t)] h(t),
\end{align*}
\]

where we have used that \( c(t), s(t) \) solve \( \mathcal{L}z = 0 \). Notice that

\[
s'(t)c(t) - c'(t)s(t) \quad (7.17) \quad \det(U(t)) = \det(U_0)e^{\int_0^t \text{tr}(A(s))ds} = 1, \forall t,
\]

where we have used that for \( A \) in (7.18) \( \text{tr}(A) = 0 \) holds. Furthermore notice that for every function \( h \) we have

\[
\begin{align*}
(\mathcal{G}[h])(0) &= 0, \\
(\mathcal{G}[h])'(0) &= s(0)c(0)h(0) - c(0)s(0)h(0) = 0.
\end{align*}
\]

Then (7.13) is equivalent to

\[
\begin{align*}
\mathcal{L}z &= \mathcal{L}v = -\varepsilon e^{\alpha t}Q(e^{-\alpha t}z) \\
\iff v &= \mathcal{G} \left[ -\varepsilon e^{\alpha t}Q(e^{-\alpha t}z_{\text{hom}}(t) + e^{-\alpha t}v(t)) \right].
\end{align*}
\]

The next lemma contains the most important technical estimate of this section. The idea is that if the initial data of \( z(t) \) are small enough, then \( v(t) \) and its derivative \( v'(t) \) remain small for every \( t \).
Lemma 7.11. Let $c_1$ be as in Remark 7.2 and $c_2$ be as in Lemma 7.10. Define $c_3 := \frac{1}{32c_1^2}$ and $c_4 := \frac{1}{16c_1^2}$. If $|a| \frac{\varepsilon}{\varepsilon} \leq c_3 \alpha$ then

$$|v(t)| + \frac{|v'(t)|}{\varepsilon} < c_4 \alpha, \quad \forall t \geq 0.$$  

Proof. By contradiction we assume that there exists $\bar{t} > 0$ such that

$$|v(t)| + \frac{|v'(t)|}{\varepsilon} < c_4 \alpha, \quad \text{for } 0 \leq t < \bar{t},$$

$$|v(\bar{t})| + \frac{|v'(\bar{t})|}{\varepsilon} = c_4 \alpha.$$  

Using the result of Lemma 7.10 we have

$$|z_{\text{hom}}(t)| = |a \cdot c(t) + b \cdot s(t)| \leq |a| \cdot |c(t)| + |b| \cdot |s(t)| \leq c_3 \alpha c_2 + \varepsilon c_3 \alpha \frac{c_2}{\varepsilon} = 2c_2 c_3 \alpha,$$  

for all $t \geq 0$. By Remark 7.2 we easily get that

$$|Q(e^{-\alpha t}z_{\text{hom}}(t) + e^{-\alpha t}v(t))| \leq c_1 e^{-2\alpha t} (|z_{\text{hom}}(t)| + |v(t)|)^2 \leq c_1 e^{-2\alpha t} (2c_2 c_3 \alpha + c_4 \alpha)^2$$

$$= c_1 \alpha^2 e^{-2\alpha t} \left( \frac{2c_2}{32c_1^2} + c_4 \right)^2$$

$$= c_1 \alpha^2 e^{-2\alpha t} \left( \frac{1}{16c_1^2} + c_4 \right)^2$$

$$= 4c_1^2 \alpha^2 e^{-2\alpha t}. \quad (7.38)$$  

From Lemma 7.10 it follows that for the operator $G$ in (7.36) the following two inequalities hold for all $t \geq 0$ and for any function $h$:

$$(\star) \quad \varepsilon |G[h](t)| \leq \varepsilon \left| s(t) \int_0^t c(\tau) h(\tau) d\tau - c(t) \int_0^t s(\tau) h(\tau) d\tau \right| \leq 2c_2^2 \int_0^t |h(\tau)| d\tau; \quad (7.39)$$

$$(\star\star) \quad \left| \frac{d}{dt} (G[h](t)) \right| = \left| s'(t) \int_0^t c(\tau) h(\tau) d\tau + s(t)c(t)h(t) - c'(t) \int_0^t s(\tau) h(\tau) d\tau - s(t)c(t)h(t) \right| \leq 2c_2^2 \int_0^t |h(\tau)| d\tau. \quad (7.40)$$  

So it follows that

$$c_4 \alpha = \left| v(\bar{t}) \right| + \frac{|v'(\bar{t})|}{\varepsilon} \leq \varepsilon G \left[ e^{\alpha t} Q \left( e^{-\alpha t} z_{\text{hom}}(\bar{t}) + e^{-\alpha t} v(\bar{t}) \right) \right]$$

$$+ \frac{1}{\varepsilon} \left| \frac{d}{dt} \left( G \left[ e^{\alpha t} Q \left( e^{-\alpha t} z_{\text{hom}}(t) + e^{-\alpha t} v(t) \right) \right](\bar{t}) \right) \right| \leq 4c_2^2 \int_0^{\bar{t}} e^{\alpha t} |Q(e^{-\alpha t}z_{\text{hom}}(t) + e^{-\alpha t}v(t))| dt \quad (7.39)\&(7.40).$$
7.2. Proof of Theorem 7.2

\[ 7.2.7 \text{ Proof of Theorem 7.2} \]

**Proof.** The assumptions (7.1) and (7.2) imply (7.15). From Lemma 7.1, 7.2 and condition (7.15) we get \( 0 < \delta < 1 \) for \( \varepsilon \) small enough (see Remark 7.5). From Lemma 7.3 we know that \( \lambda \sim \varepsilon \), i.e. “for \( \varepsilon \) small enough” is the same as “for \( \lambda \) small enough”. Therefore Lemmata 7.4 - 7.10 hold.

Since (7.8), (7.10), (7.11) hold the assumption

\[ |x(0) - x_{pq}(0)| + \left| \frac{\dot{x}(0) - \dot{x}_{pq}(0)}{q\varepsilon} \right| \leq \tilde{c}_1 \tilde{\eta} \]

of Theorem 7.2 is equivalent to

\[ |\tilde{w}(0)| + \left| \frac{\dot{\tilde{w}}(0)}{\varepsilon} \right| \leq \frac{\tilde{c}_1 \eta}{q} \].

Since \( w(t) := \tilde{w}(qt) \) we have \( \tilde{w}(0) = w(0) \) and \( q\dot{\tilde{w}}(0) = \dot{w}(0) \). Therefore we get

\[ |w(0)| + \left| \frac{\dot{w}(0)}{\varepsilon} \right| \leq \frac{\tilde{c}_1 \eta}{q} \].

This implies

\[ |w(0)| \leq \frac{\tilde{c}_1 \eta}{q} \quad \text{and} \quad \left| \frac{\dot{w}(0)}{\varepsilon} \right| \leq \frac{\tilde{c}_1 \eta}{q} \] \hspace{1cm} (7.41)

Let \( c_1 \) be as in Remark 7.2 and \( c_2 \) be as in Lemma 7.10. Define \( c_3 := \frac{1}{32\alpha c_2} \). Choosing \( \tilde{c}_1 := \frac{q\alpha}{4} \) equation (7.41) is equivalent to

\[ |w(0)| \leq \frac{c_3 \alpha}{2} \quad \text{and} \quad \left| \frac{\dot{w}(0)}{\varepsilon} \right| \leq \frac{c_3 \alpha}{2} \] \hspace{1cm} (7.42)

where we used \( \eta = 2\alpha \), cf. (7.15). Recalling the relation between \( z(t) \) and \( w(t) \), i.e. \( z(t) := e^{\alpha t}w(t) \), it is easy to prove that \( z(t) \) has the following initial conditions:

\[ z(0) = w(0), \]
\[ z'(0) = \left[ \alpha e^{\alpha t}w(t) + e^{\alpha t}w'(t) \right]_{t=0} = \alpha \cdot w(0) + w'(0). \] \hspace{1cm} (7.43) \hspace{1cm} (7.44)

From (7.42), (7.43) and (7.44) we get

\[ |z(0)| = |w(0)| \leq \frac{c_3 \alpha}{2} < c_3 \alpha, \] \hspace{1cm} (7.45)
\[ \left| \frac{z'(0)}{\varepsilon} \right| = \left| \frac{\alpha \cdot w(0) + w'(0)}{\varepsilon} \right| \leq \alpha \cdot \left| \frac{w(0)}{\varepsilon} \right| + \left| \frac{w'(0)}{\varepsilon} \right| \leq \frac{\alpha}{\varepsilon} \left( \frac{c_3 \alpha}{2} + \frac{c_3 \alpha}{2} \right) \leq c_3 \alpha, \] \hspace{1cm} (7.46)

where we used that by (7.2) \( \frac{\alpha}{\varepsilon} \leq 1 \) holds\(^3\). From (7.34), (7.45), (7.46) we see that the conditions of Lemma 7.11 are satisfied, i.e.

\[ |a| = |z(0)| \leq c_3 \alpha, \] \hspace{1cm} (7.47)

\(^3\)Recall that by definition \( \alpha := \frac{\eta}{2} \).
\[
\frac{|b|}{\varepsilon} = \frac{|z'(0)|}{\varepsilon} \leq c_3 \alpha. \quad (7.48)
\]

Then from Lemma 7.11 for \( c_4 := \frac{1}{16c_1c_2^2} \) and \( v = z - z_{\text{hom}} \) we get
\[
|v(t)| + \frac{|v'(t)|}{\varepsilon} < c_4 \alpha, \quad \text{for all } t \geq 0. \quad (7.49)
\]

Furthermore, for \( z_{\text{hom}}(t) \) defined in (7.35), using (7.47), (7.48) and the estimates in Lemma 7.10 we have
\[
|z_{\text{hom}}(t)| + \frac{|z'_{\text{hom}}(t)|}{\varepsilon} \leq |a| \cdot |c(t)| + |b| \cdot |s(t)| + \left| \frac{|a|}{\varepsilon} \cdot |c'(t)| + \frac{|b|}{\varepsilon} \cdot |s'(t)| \right|
\leq c_3 \alpha \cdot c_2 + \varepsilon c_3 \alpha \cdot \frac{c_2}{\varepsilon} + \frac{c_3 \alpha}{\varepsilon} \cdot \varepsilon c_2 + c_3 \alpha \cdot c_2 = 4c_2 c_3 \alpha = \frac{4c_2}{32c_1c_2^2} \alpha = \alpha \frac{1}{8c_1c_2} = 2c_4 \alpha. \quad (7.50)
\]

For \( z(t) \) in (7.13) we have
\[
|z(t)| + \frac{|z'(t)|}{\varepsilon} = |z_{\text{hom}}(t) + v(t)| + \frac{|z'_{\text{hom}}(t) + v'(t)|}{\varepsilon}
\leq |z_{\text{hom}}(t)| + \frac{|z'_{\text{hom}}(t)|}{\varepsilon} + |v(t)| + \frac{|v'(t)|}{\varepsilon}
\leq 2c_4 \alpha + c_4 \alpha = 3c_4 \alpha, \quad (7.51)
\]

for all \( t \geq 0. \) From (7.51) and the fact that \( w(t) = e^{-at}z(t) \) we have
\[
|w(t)| + \frac{|w'(t)|}{\varepsilon} = |e^{-at}z(t)| + \frac{|ae^{-at}z(t) + e^{-at}z'(t)|}{\varepsilon}
\leq e^{-at} \cdot \left( |z(t)| + \frac{\alpha}{\varepsilon} |z(t)| + \frac{|z'(t)|}{\varepsilon} \right)
\leq e^{-at} \cdot \left( 2|z(t)| + \frac{|z'(t)|}{\varepsilon} \right) \leq 2e^{-at} \cdot \left( |z(t)| + \frac{|z'(t)|}{\varepsilon} \right)
\leq 2e^{-at} \cdot 3c_4 \alpha = 6c_4 \alpha e^{-at}. \quad (7.52)
\]

Since \( w(t) = \tilde{w}(qt) \) and \( w'(t) = q \tilde{w}(qt) \) from (7.52) we get
\[
|\tilde{w}(qt)| + q \frac{|\tilde{w}'(qt)|}{\varepsilon} \leq 6c_4 \alpha e^{-at}.
\]

Choosing \( \tilde{c}_2 := 3qc_4 \) and replacing \( t \) by \( t/q \) we obtain
\[
|x(t) - x_{pq}(t)| + \frac{|\dot{x}(t) - \dot{x}_{pq}(t)|}{q\varepsilon} \leq \tilde{c}_2 \tilde{y} \tilde{e}^{-\bar{\eta}t/2}, \quad \forall t \geq 0.
\]

This terminates the proof of Theorem 7.2. \( \square \)

### 7.3 Proof of Theorem 7.3

The general idea to prove Theorem 7.3 is the same as for Theorem 7.1. The main difference is that every estimate is given explicitly.
Lemma 7.12. Let $K$ be as in Theorem 7.3. Let $\bar{x}(t)$ be as in (7.9) and

$$\gamma(t) = f_{xx}(\bar{x}(t), qt) - \theta$$

be as in (7.12) (see equation (64) of [8]). Let $\omega^2 := \varepsilon \theta - \frac{\pi^2}{4}$ as in (7.23) (as in equation (74) of [8]) and let $g(t) := \frac{1}{2\omega^2}\gamma(t)$ (as in (75) of [8]). Then the following statements hold:

(i) $\sup_{t \in [0, 2\pi]} |\gamma(t)| \leq 2 \sup_{t \in [0, 2\pi]} |f_{xx}(\bar{x}(t), qt)| = K$;

(ii) $\frac{1}{4} \sup_{\theta \in \theta_0} \omega^2 \leq 2 \varepsilon \theta_0$;

(iii) $\sup_{t \in [0, 2\pi]} [1 + g(t)] \leq 1 + \frac{8K}{\theta_0}$.

Proof. (i) By definition of $\gamma(t)$ and using the triangle inequality we get

$$\sup_{t \in [0, 2\pi]} |\gamma(t)| = \sup_{t \in [0, 2\pi]} |f_{xx}(\bar{x}(t), qt) - \langle f_{xx}(\bar{x}(t), qt) \rangle| \leq 2 \sup_{t \in [0, 2\pi]} |f_{xx}(\bar{x}(t), qt)|.$$

(ii) The hypothesis follows directly from (7.4) and (7.5).

(iii) By definition of $g(t)$ we get

$$\sup_{t \in [0, 2\pi]} [1 + g(t)] \leq 1 + \sup_{t \in [0, 2\pi]} |g(t)| \leq 1 + \frac{4}{\theta_0} \sup_{t \in [0, 2\pi]} |\gamma(t)|$$

$$\leq 1 + \frac{8}{\theta_0} \sup_{t \in [0, 2\pi]} |f_{xx}(\bar{x}(t), qt)|.$$

\[\square\]

Lemma 7.13. Let $U$ be the solution of (7.16) or (equivalently of (7.18)) with $k = 0$. For $\omega^2$ as in (7.23) this is equivalent to

$$U' = (T + \omega^2 \tilde{E}(t))U, \quad \text{with } U(0) = Id,$$

where

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{E}(t) = \begin{pmatrix} 0 & 0 \\ -1 - g(t) & 0 \end{pmatrix}.$$

Expanding $U = U(t; \omega^2) = \sum_{l \geq 0} U_l(t; \omega^2)\omega^{2l}$ in powers of $\omega^2$ we obtain

$$U''_0 = TU_0, \quad U_0(0) = Id, \quad \quad \quad \text{(7.53)}$$

$$U'_l = TU_l + \tilde{E}(t)U_{l-1}, \quad U_l(0) = 0, \quad \text{for } l \geq 1. \quad \text{(7.54)}$$

Then, for $K$ as in Theorem 7.3, the following holds:

$$|\text{tr}(U_l(2\pi))| \leq \left[ 4\pi^2 \left( 1 + \frac{8K}{\theta_0} \right) \right]^l, \quad \text{for all } l \geq 1.$$

Proof. From (7.53) we know that

$$U_0(t) = e^{Tt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

By (7.54) and the of variation of constants formula we have

$$U_l(t) = e^{Tt} \int_0^t e^{-Ts} \tilde{E}(s)U_{l-1}(s) ds.$$
Define
\[ U_{l-1}(s) = \begin{pmatrix} U_{1,1}^{l-1}(s) \\ U_{1,2}^{l-1}(s) \\ U_{2,1}^{l-1}(s) \\ U_{2,2}^{l-1}(s) \end{pmatrix}. \]

Notice that
\[ e^{-Ts} \tilde{E}(s) U_{l-1}(s) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_{1,1}^{l-1} \\ U_{1,2}^{l-1} \\ U_{2,1}^{l-1} \\ U_{2,2}^{l-1} \end{pmatrix} = \begin{pmatrix} s(1 + g)U_{1,1}^{l-1} \\ -(1 + g)U_{1,1}^{l-1} \end{pmatrix}. \]

Therefore we have
\[ U_i(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \int_0^t s(1 + g)U_{1,1}^{l-1}ds \\ \int_0^t (1 + g)U_{1,1}^{l-1}ds \end{pmatrix} - \begin{pmatrix} \int_0^t s(1 + g)U_{1,2}^{l-1}ds \\ \int_0^t (1 + g)U_{1,2}^{l-1}ds \end{pmatrix} \] (7.55)
and for \( t = 2\pi \) we get
\[ U_i(2\pi) = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \int_0^{2\pi} s(1 + g)U_{1,1}^{l-1}ds \\ \int_0^{2\pi} (1 + g)U_{1,1}^{l-1}ds \end{pmatrix} - \begin{pmatrix} \int_0^{2\pi} s(1 + g)U_{1,2}^{l-1}ds \\ \int_0^{2\pi} (1 + g)U_{1,2}^{l-1}ds \end{pmatrix}. \]

So it follows
\[ \text{tr}(U_i(2\pi)) = \int_0^{2\pi} s(1 + g)U_{1,1}^{l-1} - 2\pi \int_0^{2\pi} (1 + g)U_{1,1}^{l-1} - \int_0^{2\pi} (1 + g)U_{1,2}^{l-1}ds \]
\[ = -\int_0^{2\pi} (1 + g) \left[U_{1,1}^{l-1}(2\pi - s) + U_{1,2}^{l-1}\right]. \] (7.56)

By applying (7.55) recursively we obtain
\[ U_{1,1}^{l-1}(t) = \int_0^t (s - t)(1 + g)U_{1,1}^{l-2}(s)ds \]
\[ = \int_0^t (s_1 - t)(1 + g(s_1)) \int_0^{s_1} (s_2 - s_1)(1 + g(s_2))U_{1,1}^{l-3}(s_2)ds_2ds_1 \]
\[ = \int_0^t (s_1 - t)(1 + g(s_1)) \int_0^{s_1} (s_2 - s_1)(1 + g(s_2))\ldots \]
\[ \ldots \int_0^{s_{l-2}} (s_{l-1} - s_{l-2})(1 + g(s_{l-1}))U_{1,1}^{0}ds_{l-1}\ldots ds_2ds_1. \]

This implies
\[ |U_{1,1}^{l-1}(t)| \leq \sup_{t \in [0,2\pi]} |1 + g(t)|^{l-1}(4\pi^2)^{l-1}, \quad \text{for all } t \in [0,2\pi]. \]

Analogously for \( U_{1,2}^{l-1}(t) \) we get
\[ |U_{1,2}^{l-1}(t)| \leq \sup_{t \in [0,2\pi]} |1 + g(t)|^{l-1}(4\pi^2)^{l-1}2\pi, \quad \text{for all } t \in [0,2\pi], \]
since \( U_{1,2}^0(t) = t \). Therefore using (7.56) we have an upper bound for the absolute value of \( \text{tr}(U_i(2\pi)) \):
\[ |\text{tr}(U_i(2\pi))| \leq \left| \int_0^{2\pi} (1 + g) \left[U_{1,1}^{l-1}(2\pi - s) + U_{1,2}^{l-1}\right]ds \right|. \]
Lemma 7.12. (iii) \[ \sup_{t \in [0, 2\pi]} |1 + g(t)| \leq \left(1 + \frac{8K}{\theta_0}\right)^l (4\pi^2)^l, \quad \text{for all } l \geq 1. \]

This terminates the proof of Lemma 7.13. \qed

**Lemma 7.14.** Assuming
\[ 2\varepsilon\theta_0 < \frac{1}{4\pi^2 \left(1 + \frac{8K}{\theta_0}\right) \left[4\pi \left(1 + \frac{8K}{\theta_0}\right)^2 + 1\right]} \] (7.57)
we get
\[ \pi \omega^2 \leq |\delta - 1| := \left| \frac{\text{tr}(U(2\pi)) - 2}{2} \right| \leq 3\pi^2 \omega^2. \]

**Proof.** From Lemma 7.13 we have
\[
\left| \sum_{l \geq 2} \text{tr}(U_l(2\pi)) \omega^{2l} \right| \leq \sum_{l \geq 2} |\text{tr}(U_l(2\pi))| \omega^{2l} \\
\leq \sum_{l \geq 2} \left[ 4\pi^2 \omega^2 \left(1 + \frac{8K}{\theta_0}\right)^l \right] \\
\overset{\text{Geom. Series}}{=} \frac{\left[4\pi^2 \omega^2 \left(1 + \frac{8K}{\theta_0}\right)^2\right]}{1 - 4\pi^2 \omega^2 \left(1 + \frac{8K}{\theta_0}\right)},
\]
if \( \omega \) is so small such that
\[ 4\pi^2 \omega^2 \left(1 + \frac{8K}{\theta_0}\right) < 1 \quad \text{or, equivalently} \quad \omega^2 < \frac{1}{4\pi^2 \left(1 + \frac{8K}{\theta_0}\right)}. \] (7.58)
Since by point (ii) of Lemma 7.12 we have \( \omega^2 \leq 2\varepsilon\theta_0 \) and since \( 4\pi \left(1 + \frac{8K}{\theta_0}\right) + 1 > 1 \) holds, it is clear that condition (7.57) is stronger and implies (7.58). So we get
\[ |\delta - 1| = \left| \frac{\text{tr}(U(2\pi)) - 2}{2} \right| = -2\pi^2 \omega^2 + \sum_{l \geq 2} \text{tr}(U_l(2\pi)) \omega^{2l} \]
implying
\[ |\delta - 1| \leq 2\pi^2 \omega^2 + \pi^2 \omega^2 = 3\pi^2 \omega^2 \quad \text{and} \quad |\delta - 1| \geq 2\pi^2 \omega^2 - \pi^2 \omega^2 = \pi^2 \omega^2, \]
where we used the triangle inequality
\[ ||x| - |y|| \leq |x + y| \leq |x| + |y|. \]

**Lemma 7.15.** (i) Let condition (7.57) be satisfied. Then the following statements hold: (A) \( \frac{\lambda_2^2}{\lambda_1^2} \leq \frac{1}{2\pi^2} \), (B) \( \frac{\lambda_1^4}{\lambda_2^4} \leq 2\frac{8\pi^2}{47} \), (C) \( \frac{\lambda_2}{\lambda_1} \leq \frac{\sqrt{3}}{\pi\sqrt{\theta_0}} \) and (D) \( \frac{\lambda_1}{\sqrt{\lambda_2}} \leq \sqrt{\frac{376\pi^2 \theta_0}{47}}. \)
(ii) Define $G(t; \lambda) := \frac{\omega^2}{\lambda^2}(1 + g(t)) - 1$ (as in equation (Eq.) in [8]). Then we have:

$$\sup_{t \in [0, 2\pi]} |G(t; \lambda)| \leq 1 + \frac{4K}{\pi^2 \theta_0} = \tilde{K}$$

for $\tilde{K}$ defined as in Theorem 7.3.

**Proof.** (i) By [8] we know the relation

$$\cos(\lambda) = \frac{\text{tr}(U(2\pi))}{2} = 1 - 2\pi^2 \omega^2 + \sum_{l \geq 2} \text{tr}(U_l(2\pi)) \omega^{2l}.$$ 

Since $\delta > 0$ using Lemma 7.14 and the expansion of $\cos(\lambda)$ we get:

(A) $1 - \frac{\lambda^2}{2} \leq 1 - \pi^2 \omega^2 \implies \frac{\omega^2}{\lambda^2} \leq \frac{1}{2\pi^2};$

(B) $1 - 3\pi^2 \omega^2 \leq 1 - \frac{\lambda^2}{2} + \frac{\lambda^4}{24} \implies \frac{\lambda^2}{\omega^2} \leq \frac{3\pi^2}{\frac{1}{2} - \frac{(1/2)^2}{24}} = \frac{288\pi^2}{47}.$

(C) and (D) follows directly from (A), (B) and point (ii) of Lemma 7.12. (ii) Follows from Lemma 7.12 (i) and (ii). \(\Box\)

**Lemma 7.16.** Let $\tilde{K}, \lambda_1$ and $M_1$ be as in Theorem 7.3. Furthermore assume that:

(a) condition (7.57) holds;

(b) $\lambda$ is so small that $0 \leq |\lambda| \leq \lambda_1 = \sqrt{\frac{1}{8\pi^2 \tilde{K}}}$.

Then we have

$$\sup_{t \in [0, 2\pi]} |\mathcal{P}_+(t)| \leq 1 + M_1.$$ 

**Proof.** From [8] we know that $\mathcal{P}_+(t)$ solves the differential equation

$$\dot{\mathcal{P}}_+(t) + 2i\lambda \mathcal{P}_+(t) + \lambda^2 \mathcal{P}_+(t) = 0,$$

with boundary conditions $\mathcal{P}_+(0) = \mathcal{P}_+(2\pi) = 1$. $\mathcal{P}_+(t)$ has the from$^5$

$$\mathcal{P}_+(t) = 1 + A_2(t)\lambda^2 + A_3(t)\lambda^3,$$

with $A_2(t) := \int_0^t \int_0^s G - \frac{1}{2\pi} \int_0^{2\pi} \int_0^s G$. Notice that from the definition of $\tilde{K}$ in Theorem 7.3 for $A_2(t)$ and $A_3(t)$ the following statements hold:

$$\sup_{t \in [0, 2\pi]} |A_2(t)| \leq 8\pi^2 \tilde{K}; \quad (7.59)$$

$$\sup_{t \in [0, 2\pi]} |A_3(t)| = \sup_{t \in [0, 2\pi]} \left| \int_0^s G - \frac{1}{2\pi} \int_0^{2\pi} \int_0^s G \right| \leq 4\pi \tilde{K}. \quad (7.60)$$

The equation describing $A_3$ is

$$\dot{A}_3(t) + 2i\lambda \dot{A}_3(t) + \lambda^2 A_3(t) = -2i \dot{A}_2(t) - \lambda GA_2(t).$$

Defining $v(t) := \dot{A}_3(t)$ and $f(t) := f(t, A_3(t)) = -2i \dot{A}_2(t) - \lambda GA_2(t) - \lambda^2 A_3(t)$ we get

$$\dot{v}(t) + 2i\lambda v(t) = f(t).$$

$^4$This condition implies $4\pi^2 \lambda^2 \tilde{K} \leq \frac{1}{2}$.

$^5$This is different from (7.29) (where $k \geq 1$) since here $k = 0$ holds.
With the variation of constant formula we get
\[ v(t) = e^{-2i\lambda t} \left( c + \int_0^t e^{i2\lambda s} f(s) ds \right), \quad (7.61) \]
where
\[ c = \frac{2i\lambda \int_0^{2\pi} \int_0^s e^{i2\lambda (\tau - s)} f(\tau) d\tau ds}{1 - e^{-14\pi\lambda}} \quad (7.62) \]
is given by the condition \( A_3(2\pi) = 0 \). Estimating the norm of \( A_3 \) (remember that \( \lambda \) is real) we have:
\[
\sup_{t \in [0, 2\pi]} |A_3(t)| = \sup_{t \in [0, 2\pi]} \left| \int_0^t v(s) ds \right|
\leq 2 \pi \left| c \right| \left| \frac{2i\lambda}{1 - e^{-14\pi\lambda}} \right| \left| 4 \pi^2 \sup_{t \in [0, 2\pi]} |f(t)| + 4 \pi^2 \sup_{t \in [0, 2\pi]} |f(t)| \right|.
\]
For real values of \( \lambda \) with \( 0 \leq \lambda < \frac{1}{2} \) the function \( n(\lambda) := \left| \frac{2i\lambda}{1 - e^{-14\pi\lambda}} \right| \) is strictly increasing. Since \( \lambda_1 < \frac{1}{2} \) we get
\[
\sup_{t \in [0, 2\pi]} |A_3(t)| \leq 4\pi^2 \left( 2\pi n(\lambda_1) + 1 \right) \sup_{t \in [0, 2\pi]} |f(t)|
\leq 4\pi^2 \left( 2\pi n(\lambda_1) + 1 \right) \left( \sup_{t \in [0, 2\pi]} |h(t)| + \lambda^2 \tilde{K} \sup_{t \in [0, 2\pi]} |A_3(t)| \right)
\leq 4\pi^2 \left( 2\pi n(\lambda_1) + 1 \right) \left( 8\pi \tilde{K} + 8\pi^2 \tilde{K}^2 \lambda + \lambda^2 \tilde{K} \sup_{t \in [0, 2\pi]} |A_3(t)| \right),
\] (7.63)
where we have used the following calculation:
\[
\sup_{t \in [0, 2\pi]} |h(t)| = \sup_{t \in [0, 2\pi]} \left| -2i\lambda \dot{A}_2(t) - \lambda G \dot{A}_2(t) \right|
\leq 2 \sup_{t \in [0, 2\pi]} |\dot{A}_2(t)| + \lambda \tilde{K} \sup_{t \in [0, 2\pi]} |\dot{P}_2(t)|
\leq 8\pi \tilde{K} + 8\pi^2 \tilde{K}^2 \lambda.
\] (7.59) & (7.60)
Equation (7.63) implies
\[
\sup_{t \in [0, 2\pi]} |A_3(t)| \leq \frac{4\pi^2 \left( 2\pi n(\lambda_1) + 1 \right) \left( 8\pi \tilde{K} + 8\pi^2 \tilde{K}^2 \lambda \right)}{1 - 4\pi^2 \tilde{K}^2 \lambda^2}
\leq 64\pi^3 \tilde{K} \left( 2\pi n(\lambda_1) + 1 \right) \left( 1 + \pi \tilde{K} \lambda_1 \right).
\] (ii)
Therefore now we are able to conclude that
\[
\sup_{t \in [0, 2\pi]} |\dot{P}_+(t)| = \sup_{t \in [0, 2\pi]} \left| 1 + \lambda^2 \dot{A}_2(t) + \lambda^3 A_3(t) \right|
\leq 1 + \lambda^2 \sup_{t \in [0, 2\pi]} |\dot{A}_2(t)| + \lambda^3 \sup_{t \in [0, 2\pi]} |A_3(t)|
\leq 1 + 8\pi^2 \tilde{K} \lambda^2 + |\lambda|^3 64\pi^3 \tilde{K} \left( 2\pi n(\lambda_1) + 1 \right) \left( 1 + \pi \tilde{K} \lambda_1 \right).
\]
Since \( 0 \leq |\lambda| \leq \lambda_1 \) this terminates the proof of Lemma 7.16. \( \square \)
Lemma 7.17. As in [8] let \( v_0 := \frac{1}{i} \int_0^{2\pi} (P_+ - 1) + \frac{\lambda}{2\pi} \int_0^{2\pi} s \int_0^s G \mathcal{P}_+ \) and let \( K, \lambda_1, M_1, M_2 \) be defined as in Theorem 7.3. Then under the assumptions of Lemma 7.16 the following statements hold:

(i) \( \dot{P}_+(0) = \lambda v_0 \);

(ii) \( |\dot{P}_+(0)| \leq M_2 \);

(iii) \( |\dot{P}_+(0) - \dot{P}_-(0)| \leq 2M_2 \);

(iv) \( |\dot{P}_+(t)| \leq 2M_2 \), for all \( t \in [0, 2\pi] \).

Proof. (i) By the fixed point equation we know that \( \dot{P}_+ \) satisfies

\[
\dot{P}_+ = \lambda v_0 - 2i\lambda (P_+ - 1) - \lambda^2 \int_0^t G \mathcal{P}_+.
\]

(7.64)

For \( t = 0 \) we get

\[
\dot{P}_+(0) = \lambda v_0
\]

since by definition \( P_+(0) = 1 \).

(ii) Since

\[
|v_0| = \left| \frac{i}{\pi} \int_0^{2\pi} (P_+ - 1) + \frac{\lambda}{2\pi} \int_0^{2\pi} \int_0^s G \mathcal{P}_+ \right|
\]

\[
\leq 2|P_+ - 1| + 2\pi K|\lambda||P_+|,
\]

we get by Lemma 7.16 that

\[
|v_0| \leq 2 \sup_{t \in [0, 2\pi]} \left| \lambda^2 P_2(t) + \lambda^3 P_3(t) \right| + 2\pi K|\lambda| \sup_{t \in [0, 2\pi]} \left| 1 + \lambda^2 P_2(t) + \lambda^3 P_3(t) \right|
\]

\[
\leq 2M_1 + 2\pi K_1 \lambda_1 [1 + M_1] = \frac{M_2}{\lambda_1}
\]

holds. Therefore we have

\[
|\dot{P}_+(0)| = |\lambda||v_0| \leq \lambda_1 \frac{M_2}{\lambda_1} = M_2.
\]

(iii) We notice that

\[
\dot{P}_-(0) = \bar{\dot{P}_+(0)} = \bar{\dot{P}_+(0)} = \bar{\lambda} v_0.
\]

By the triangle inequality we have

\[
|\dot{P}_+(0) - \dot{P}_-(0)| \leq |\dot{P}_+(0)| + |\dot{P}_-(0)| = 2|\dot{P}_+(0)| \leq 2M_2.
\]

(iv) From (7.64) we get

\[
|\dot{P}_+| \leq |\lambda||v_0| + 2|\lambda||P_+ - 1| + 2\pi K\lambda^2 |P_+|
\]

\[
\leq M_2 + 2\lambda_1 M_1 + 2\pi K\lambda_1^2 (1 + M_1)
\]

\[
\leq M_2 + M_2 = 2M_2.
\]

\[
\square
\]

Lemma 7.18. Let \( c(t) = c_+ \zeta_+(t) + c_- \zeta_-(t) \) and \( s(t) = s_+ \zeta_+(t) + s_- \zeta_-(t) \) as in (7.30) (see also equation (91) of [8]). Let \( \lambda_1, M_1, M_2 \) be as in Theorem 7.3. Under the assumptions of Lemma 7.16 we have:

\[
|c_+| = |1 - c_-| \leq \frac{\lambda_1 + M_2}{|2\lambda_1 - 2M_2|}, \quad (7.65)
\]

\[
|s_+| = |s_-| \leq \frac{1}{|2\lambda_1 - 2M_2|}. \quad (7.66)
\]
Moreover for all \( t \in [0, 2\pi] \) we have:

\[
|c(t)| \leq C := (1 + M_1) \left( 1 + \frac{\lambda_1 + M_2}{|\lambda_1 - M_2|} \right), \tag{7.67}
\]

\[
|\dot{c}(t)| \leq C' := (2M_2 + \lambda_1(1 + M_1)) \left( 1 + \frac{\lambda_1 + M_2}{|\lambda_1 - M_2|} \right), \tag{7.68}
\]

\[
|s(t)| \leq S := \frac{1 + M_1}{|\lambda_1 - M_2|}, \tag{7.69}
\]

\[
|\dot{s}(t)| \leq S' := \frac{2M_2 + \lambda_1(1 + M_1)}{|\lambda_1 - M_2|}. \tag{7.70}
\]

**Proof.** (7.65) and (7.66) follow from (7.31), (7.32), the triangle inequality and Lemma 7.17. (7.67) - (7.70) follow from (7.30) and the facts that

\[
\sqrt{\frac{1}{\varepsilon}} \ \text{with} \ \varepsilon \ \text{then}
\]

So from Theorem 7.1 we have that for \( \bar{\varepsilon} \)

\[
\text{Lemma 7.16}
\]

there holds, since \( P_- = \overline{P_+} \).

**Remark 7.10.** By Lemma 7.18 and the point (i) of Lemma 7.15, choosing

\[
c_2 := \max \left( C, \frac{\sqrt{2}}{\pi \sqrt{\theta_0}} S, \sqrt{\frac{576\pi^2 \theta_0}{47}} C', S' \right)
\]

we have:

\[
|c(t)|, \sqrt{\bar{\varepsilon}}|s(t)|, \frac{|c'(t)|}{\sqrt{\bar{\varepsilon}}}, |s'(t)| \leq c_2.
\]

**Proof. (Theorem 7.3)** From (7.3), (7.4) and (7.5) Theorem 7.1 holds. In the proof Theorem 7.1 (see [8]) the constants \( c_1 \) and \( c_2 \) have to satisfy the following conditions:

- For \( \varepsilon \) small enough the constant \( c_2 \) satisfies \( |c(t)|, \sqrt{\varepsilon}|s(t)|, \frac{|c'(0)|}{\sqrt{\varepsilon}}, |s'(0)| \leq c_2 \),

where \( c(t) \) and \( s(t) \) are the fundamental solutions of \( \mathcal{L}z = 0 \) with initial conditions \( c(0) = 1 = s'(0), c'(0) = 0 = s(0) \) (see Lemma 3.4 of [8]).

As in the proof of Theorem 7.1 (see (7.7)) we take

\[
\bar{c}_1 = \frac{1}{64c_1 c_2^2} \quad \text{and} \quad \bar{c}_2 = \frac{3}{8c_1 c_2^2}.
\]

So from Theorem 7.1 we have that for \( \bar{\varepsilon} \) and \( \bar{\eta} \) small enough, if \( x(t) \) is a solution of (1.24) with

\[
\sqrt{\bar{\varepsilon}}|x(0) - x_{pq}(0)| + |\dot{x}(0) - \dot{x}_{pq}(0)| \leq \bar{c}_1 \bar{\eta},
\]

then

\[
\sqrt{\bar{\varepsilon}}|x(t) - x_{pq}(t)| + |\dot{x}(t) - \dot{x}_{pq}(t)| \leq \bar{c}_2 \bar{\eta} e^{-\bar{\eta}t/2}.
\]

In order to finish the proof of Theorem 7.3 we have give quantitative estimates of the constants \( c_1 \) and \( c_2 \) using the extra assumption (7.6). These estimates are done in Lemmata 7.12 - 7.18 and the final results are summarised in Remarks 7.2 and 7.10. 

\( \square \)
8 Numerical estimates on the basin of attraction

In this last chapter we aim to improve the numerical results obtained by Celletti and Chierchia in [15]. Although our purposes are not yet all achieved, we present some partial results.

8.1 General setting

The problem (same as in [15]):
For all coprime integers \( p,q \) determine the intersection of the basin of attraction of the \((p,q)\)−periodic orbit with some fixed domain \( \Omega = \{0 \leq x \leq 2\pi, 0 \leq \dot{x} \leq 5\} \).

Numerical approach:
By a Monte-Carlo method choose 1000 initial conditions \((x_0,v_0) \in \Omega\). For every initial condition integrate the equation of motion (1.24) numerically for a fixed time \( T_K = 10^3/K \), where \( K \) is given in (1.49). Determine whether the solution has approached some \((p,q)\)−periodic orbit or the quasi-periodic orbit with frequency \( \omega \). For every observed \((p,q)\)−pair determine the measure of its basin of attraction by counting the percentage of the initial data which have approached them.

In a similar approach Celletti and Chierchia [15] analyse for the eccentricities of the Moon and of Mercury \(^1\) quasi-periodic attractors and \((p,q)\)−periodic orbits with \( p, q \leq 5 \) and \( \frac{p}{q} \leq 3 \). Using Yoshidas algorithm [49] they integrate the equation of motion (1.24) for a long time \( T_K = 10^3/K \), where \( K \) was defined in (1.49) and belongs to the interval \([5 \cdot 10^{-6}, 10^{-3}]\). As results they give tables with measures of the basin of attraction for every numerically observed frequency \( \omega \) of the attractor.

Achivements and possible developments\(^2\):

(i) Develop an explicit integrator of order two being symplectic for \( \eta = 0 \);

(ii) Give phase space plots for the dissipative spin-orbit problem for Mercury\(^3\);

(iii) For fixed \( \varepsilon \) we give numerical measure of the basin of attraction for Moon and Mercury (computed with the method developed in (i)) and we compared these results with those found in [15];

(iv) Give numerical measure of the basin of attraction for all the other satellites in spin-orbit resonance (see list of 18 satellites in Chapter 6);

(v) Compare the numerical result in (ii) with Theorem 7.2;

(vi) Develop explicit integrators of higher order being symplectic for \( \eta = 0 \).

In this chapter we report results on the points (i) and (ii).

\(^1\) \( e = 0.0549 \) and \( e = 0.2056 \), corresponding to driving frequencies \( \nu = 1.01809 \) and \( \nu = 1.25584 \), respectively.

\(^2\) Currently I am working on these topics with Prof. Corrado Falcolini.

\(^3\) The phase space of other satellites will look similar to those of Mercury.
8.2 An explicit integrator of order two

Consider the Hamiltonian differential equation

\[ \ddot{x} = g(t, x) \]  

(8.1)

where \( g \) is \( 2\pi \)-periodic with respect to \( t \). For given initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = v_0 \) we want to find an approximation to \( x(2\pi) \).

The Störmer-Verlet-Method is a second order symplectic integrator for (8.1) (see [25]). Choosing uniform steps of length \( h := \frac{2\pi}{n} \) and denoting with \( x_k \) the approximation of \( x(t_k) = x(\frac{2\pi k}{n}) \) for \( 0 \leq k \leq n \) we have:

\[ x_{k+1} = -x_{k-1} + 2x_k + h^2 g(t_k, x_k), \]

with starting values

\[ x_0 := x(0), \quad x_{-1} := x_0 - hv_0 + \frac{h^2}{2} g(t_0, x_0). \]

Notice that with this choice

\[ v_0 = \frac{x_1 - x_{-1}}{2h} \]

holds.

However, the dissipative spin-orbit problem \( \ddot{x} + \eta(\dot{x} - \nu) + \varepsilon f_x(x, t) = 0 \) is not Hamiltonian (since \( \eta > 0 \)). The right-hand side \( g(t, x, \dot{x}) = -\eta(\dot{x} - \nu) - \varepsilon f_x(t, x) \) of the differential equation depends not only on \( t \) and \( x \) but also on \( \dot{x} \). Thus, we modify the method in the following way: Choosing uniform steps of length \( h := \frac{2\pi}{n} \) and denoting with \( x_k, v_k \) the approximations of \( x(t_k) = x(\frac{2\pi k}{n}) \) and of \( \dot{x}(t_k) = \dot{x}(\frac{2\pi k}{n}) \) for \( 0 \leq k \leq n \), respectively, we have:

\[
\begin{align*}
  x_{k+1} &= -x_{k-1} + 2x_k + h^2 g(t_k, x_k, v_k), \\
  v_k &= \frac{x_{k+1} - x_{k-1}}{2h},
\end{align*}
\]

(8.2)

with starting values

\[ x_0 := x(0), \quad x_{-1} := x_0 - hv_0 + \frac{h^2}{2} g(t_0, x_0, v_0). \]

The scheme (8.2) is an implicit method. For the dissipative spin-orbit problem it is equivalent to

\[
\begin{align*}
  x_{k+1} &= -x_{k-1} + 2x_k + h^2 g(t_k, x_k, v_k) \\
  &= -x_{k-1} + 2x_k + h^2 \left[ -\eta(v_k - \nu) - \varepsilon f_x(t_k, x_k) \right] \\
  &= -x_{k-1} + 2x_k + h^2 \left[ -\eta \left( \frac{x_{k+1} - x_{k-1}}{2h} - \nu \right) - \varepsilon f_x(t_k, x_k) \right].
\end{align*}
\]

Solving with respect to \( x_{k+1} \) leads to

\[
\begin{align*}
  x_{k+1} &= \frac{-x_{k-1} + 2x_k + h^2 \left[ -\eta \left( \frac{x_{k+1} - x_{k-1}}{2h} - \nu \right) - \varepsilon f_x(t_k, x_k) \right]}{1 + \frac{h^2}{2} \eta} \\
  v_k &= \frac{x_{k+1} - x_{k-1}}{2h},
\end{align*}
\]

(8.3)

(8.3) is an explicit integrator of order two being symplectic for \( \eta = 0 \). Given initial conditions \( (x(0), \dot{x}(0)) \) we are now able to compute a numerical approximation of \( (x(2\pi), \dot{x}(2\pi)) \) with the method (8.3). Iterating this process we construct the approximate Poincaré section for the dissipative spin-orbit problem.
8.3 Phase portrait of Mercury

Using (8.3) with $h = \frac{2\pi}{200}$ we plot the Poincaré section of every (of the 1000) initial condition in the same compact region \( \{ 0 \leq x \leq 2\pi, 0 \leq \dot{x} \leq 5 \} \). We integrate the equation of motion up to a fixed time $T_K$ and therefore \((x_k, v_k)\) is the sequence of points corresponding to an initial condition \((x_0, v_0)\). In order to make the plots more understandable we paint groups on initial conditions tending to the same attractor with the same color: the dark blue points correspond to the \((1,1)\)–periodic resonance; the light blue points correspond to the \((2,1)\)–periodic resonance; the light green points correspond to the \((3,2)\)–periodic resonance; the fluorescent green points correspond to the \((5,4)\)–periodic resonance; the red points converge to the yellow quasi periodic attractor.

This procedure leads to the plot (a) in Figure 8.1. To understand what is happening in this plot we have to enlarge the picture in the regions corresponding to the frequency $\omega$ of attractors, which we observe numerically. The plot (b) in Figure 8.1 shows the region, where $\omega \approx \nu$. The yellow object is the unique quasi-periodic attractor of the system, whose existence was proved in [16].

The plot (c)-(f) in Figure 8.1 show the regions, where $\omega$ is near 1, 1.5, 2, 1.25 and they correspond to the \((1,1)\), \((3,2)\), \((2,1)\)– and \((5,4)\)–resonances of the system, respectively.

The theory on \((p,q)\)–periodic orbits developed in this thesis (in particular see Chapters 3 and Chapter 7) tells us, that \((p,q)\)–periodic orbits exist also for other values of $q$. Thus, similar plots to the plots (c)-(f) in Figure 8.1 should be observable for every frequency $\frac{p}{q}$. Since the basin of attraction of a \((p,q)\)–periodic orbit decreases as the order of non-degeneration increases, we are not able to observe these attractors numerically. This explains, why with this numerical experiment we only find a few periodic orbits (the most probable).
Figure 8.1: Phase space of Mercury for the values $e = 0.2056, K = 5 \cdot 10^{-5}, T_K = 10^5$ and $n = 200$. 
Bibliography


