Doctoral Thesis

Modeling and experimental study of a flexural vibration sensor for density measurements

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Modeling and experimental study of a flexural vibration sensor for density measurements

A thesis submitted to attain the degree of
DOCTOR OF SCIENCES of ETH ZURICH
(Dr. sc. ETH Zurich)

presented by

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2014
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Abstract

Measurement of fluid viscosity based on torsional oscillations of a cylindrical tube is well established. However, in order to determine absolute viscosity values, the fluid density must be known. Based on flexural oscillations of the same tube, the fluid density could be determined, and the existing viscosity sensor could be developed into a combined density and viscosity sensor. The accuracy and versatility of the existing viscosity sensor could greatly be enhanced. Current work is therefore focused on the development of a density sensor based on flexural oscillations of a cylindrical tube. The objective of this work is the study of different modeling approaches for the structure and the fluid-structure interaction of such a sensor, as well as the experimental characterization of the sensor, and the validation of the models.

A structure oscillating in a viscous fluid experiences inertial and damping forces, which in general depend on both fluid density and fluid viscosity. At the beginning of this work the possibility of separating the influences of fluid density and fluid viscosity, in order to measure one or the other property with sufficient accuracy, is discussed. The concept of combining the viscosity measurement and the density measurement is presented.

In the first part of this work the design and the mechanical model of the flexural oscillator is considered. Based on practical requirements and in order to have a manageable system complexity, that can be captured by modeling, the existing sensor is adapted. A suitable asymmetry is introduced to separate modes that would otherwise be too close. Two analytical models and a FEM (Finite Element Method) model for the flexural oscillator are introduced. Because higher flexural oscillation modes shall be used for the density measurement, the analytical models are based on Timoshenko beam theory. In the analytical models the flexural oscillator is represented by a cantilever beam and a rigid mass (with rotary inertia) attached at its free end. The first analytical model is based on a transfer matrix method, in which the beam and the rigid mass are described as separate parts, and an excitation moment acts at the rigid mass. In this model, the driving point mobility between the rotational velocity of the rigid mass and the excitation moment is calculated, which can numerically be evaluated in order to determine the resonance frequency and the bandwidth of the flexural modes. In the second analytical model the rigid mass is modelled as a concentrated mass/concentrated inertia (point mass/point inertia), which is directly introduced in the differential equations of the Timoshenko beam using distribution theory (via Dirac delta functions). In this second analytical model a frequency equation is derived, whose roots describe the natural frequencies of the flexural modes. Damping effects cannot be included in this model. With the FEM model, a ‘natural frequency extraction’ procedure is used which performs an
Abstract

eigenvalue extraction in order to determine the natural frequencies of the modes. Damping effects are not included in the FEM model. The models depend on various geometrical and material properties. An accurate determination of these parameters is crucial in order to obtain an accurate description of the flexural oscillator. A procedure is presented in order to determine the required parameters. Experiments are performed in order to validate the models, to characterize the influence of the clamping of the sensor, and to determine the influence of temperature variations on resonance frequency and damping of the flexural oscillations.

The second part of this work considers the modeling of the resonator in a fluid. The description of the fluid-structure interaction relies on fluid models and their description of the fluid forces which act on the resonator. Considering a dimensional analysis of the Navier-Stokes equation and the continuity equation of a compressible fluid, the importance of the various fluid effects can be assessed. Different analytical fluid models from the literature which can be used together with the analytical structural models of the flexural oscillator are introduced, which describe the fluid as 2D incompressible viscous unbounded/bounded fluid, as 2D compressible viscous bounded fluid, or as 3D compressible viscous unbounded fluid. In the existing mechanical FEM model of the flexural oscillator, the surrounding fluid is modeled as an acoustic medium.

All experiments presented in this work are performed with the third flexural oscillation mode. Experiments with varying oscillation amplitudes are performed as an experimental check that the measured resonance frequencies and damping values are amplitude independent and therefore a linear fluid behavior can be assumed. This assumption is also made in the introduced analytical fluid models. Measurements are conducted with different dimensions of the fluid container in order to investigate the influence of fluid boundaries, and to assess which dimension of the fluid container is adequate for further measurements. A series of measurements in various test fluids is performed for the later assessment of the sensor models and the density measurement capabilities of the sensor. The considered test fluids cover a density range of 700 kg/m³ to 1000 kg/m³, a viscosity range of 0.5 mPas to 70 mPas, and speeds of sound in the range of 1100 m/s to 1500 m/s.

Before combining the different analytical fluid models with the mechanical models of the resonator, the fluid forces predicted by the models are compared to each other and to the results from the FEM model. It allows to recognize the differences and similarities between the formulations. Against the background of measurements, e.g. the measurements with different sizes of the fluid container, the suitability of the fluid models for the current application can be assessed. Finally the model predictions for resonance frequency and bandwidth obtained from analytical models (together with different fluid models) and from the FEM model (considering an acoustic fluid) are compared with the results from the measurements in the test fluids. Excellent results for the modeling of the density influence on the resonance frequencies are obtained. An accurate prediction of the bandwidths seems to be more difficult though.
The third part of this work considers the inverse problem, i.e. the determination of fluid density (and viscosity) based on the measured resonance frequency and damping of the flexural oscillator. Based on considerations of a simple Euler Bernoulli beam and a description of the fluid forces in an incompressible viscous fluid for the case of small viscous penetration depths, a general form of the fluid density-resonance frequency relationship is found, that contains two calibration constants. The calibration constants capture the effects not accounted for by the considered simplified model. Difficulties are encountered in applying a similar procedure to a Timoshenko beam or a beam with an attached mass. The available fluid density-resonance frequency relationship is successfully applied to the measurements in the test fluids. Based on a calibration with two test fluids, the densities of the other test fluids are predicted with good accuracy.

The present thesis presents three different modeling approaches that are able to accurately predict the resonance frequencies of the density sensor in the considered test fluids. It has been shown that an accurate density measurement is possible with the current density sensor, such that together with a torsional mode both viscosity and density can be fully determined.
Zusammenfassung


Im ersten Teil dieser Arbeit wird das Design und das mechanische Modell des Biegeschwingers betrachtet. Basierend auf praktischen Anforderungen und um eine handhabbare Komplexität des betrachteten Systems zu erhalten, welche in einem Modell abgebildet werden kann, wird der bestehende Sensor angepasst. Eine geeignete Asymmetrie wird einge führt, um Moden zu separieren, welche andernfalls zu nahe beieinander wären. Zwei analytische Modelle und ein FEM (Finite Elemente Methode) Modell des Biegeschwingers werden präsentiert. Da für die Dichtemessung höhere Biegemoden verwendet werden sollen, basieren die analytischen Modelle auf der Timoshenko-Balkentheorie. In den analytischen Modellen wird der Biegeschwinger als ein einseitig eingespannter Balken mit einer starren Masse (mit Rotationsträgheit) am freien Ende abgebildet. Das erste analytische Modell basiert auf einer Übertragungsmatrix-Methode, in welchem der Balken und die starren Masse als getrennte Teile beschrieben werden, und in dem ein Anregungsmoment an der starren Masse angreift. In diesem Modell wird die Admittanz am Anregungspunkt zwischen der Verdrehungsgeschwindigkeit der starren Masse und dem Anregungsmoment berechnet, welche numerisch ausgewertet werden kann, um die Resonanzfrequenz und die Bandbreite der Biegeschwingungsmoden zu bestimmen. Im zweiten analytischen Modell wird die starre Masse als konzentrierte Masse/konzentrierte Trägheit (Punktmasse/"Punktträgheit") beschrieben, welche mittels Distributions-
Zusammenfassung


In allen Experimenten welche in dieser Arbeit präsentiert werden, wird mit dem dritten Biegemode gearbeitet. Experimente mit unterschiedlichen Schwingungsamplituden werden durchgeführt zwecks experimenteller Überprüfung, dass die gemessenen Resonanzfrequenzen und Dämpfungswerte amplitudenumabhängig sind, und folglich das Fluidverhalten als linear angenommen werden kann. Diese Annahme ist auch in den präsentierten analytischen Fluidmodellen gemacht worden. Messungen in Fluidbehältern unterschiedlicher Dimensionen werden durchgeführt, um den Einfluss der Fluidbegrenzungen zu untersuchen, und um zu beurteilen, welche Dimension des Fluidgefässes für weitere Messungen angemessen ist. Eine Messreihe in verschiedenen Testfluiden wird durchgeführt zwecks späterer Beurteilung der Sichtbarkeit und der Tauglichkeit des Sensors zur Dichtemessung. Die betrachteten Testfluiden umfassen einen Dichtebereich von 700 kg/m³ bis 1000 kg/m³, einen Viskositätsbereich von 0,5 mPas bis 70 mPas und Schallgeschwindigkeiten im Bereich von 1100 m/s bis 1500 m/s.

Bevor die verschiedenen analytischen Fluidmodelle mit den mechaniischen Modellen der schwingenden Struktur kombiniert werden, werden die von den Modellen vorausgesagten Fluidkräfte miteinander und mit den Resultaten aus dem FEM-Modell verglichen. Dies erlaubt, die Unterschiede und Ähnlichkeiten zwischen den ver-


Die vorliegende Doktorarbeit präsentiert drei verschiedene Modellierungsansätze, welche in der Lage sind, die Resonanzfrequenzen des Dichtesensors in den betrachteten Testfluiden genau vorauszusagen. Es konnte gezeigt werden, dass eine genaue Dichtemessung mit dem gegenwärtigen Dichtesensor möglich ist, so dass, zusammen mit einem Torsionsmode, sowohl Viskosität als auch Dichte vollständig bestimmt werden können.
# List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>EMAT</td>
<td>Electromagnetic Acoustic Transducer</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite Element Method</td>
</tr>
<tr>
<td>MEMS</td>
<td>Micro Electrical Mechanical System</td>
</tr>
<tr>
<td>&quot;1 dof oscillator&quot;</td>
<td>Vibrations of a single-degree-of-freedom system</td>
</tr>
</tbody>
</table>
# List of Symbols

**Roman alphabet**

- $a$ Non-dimensional parameter according to [23]; Length scale of solid object; Calibration constant
- $A$ Area of beam cross-section
- $a_1, a_2, a_3, a_4$ Constants
- $b$ Non-dimensional parameter according to [23]; Calibration constant
- $c$ Stiffness (1 dof oscillator); Constant
- $c_f$ Speed of sound in fluid; Viscous damping coefficient
- $c_s$ Shear wave speed of a circular cylindrical rod
- $c_0$ Longitudinal wave speed
- $c_2$ Shear wave speed of a rod with a general cross-section
- $C$ Complex constant
- $C()$ Operator in description of a Timoshenko beam describing fluid damping
- $C_n$ Mode dependent constant of mode $n$ (Euler Bernoulli beam)
- $d$ Viscous damping coefficient (1 dof oscillator); Radius of oscillating cylinder
- $d_f$ Bandwidth
- $d_{f_{\text{fluid}}}$ Bandwidth due to fluid influence alone
- $d_{f_n}$ Bandwidth of mode $n$
- $D$ Radius of cylindrical fluid boundary
- $E$ Young’s modulus
- $E_d$ Complex Young’s modulus if a hysteretic damping model is used
- $f$ Frequency; Resulting fluid force per unit length
- $f$ Scaled amplitude of excitation force (1 dof oscillator)
- $f_n$ Resonance frequency of $n$-th longitudinal mode
- $f_r, f_t$ parameters for the description of the resilient foundation via rotational/linear springs in the DeqD model
- $f_{\text{res}}$ Resonance frequency
- $f_{\text{res, fluid}}$ Resonance frequency in fluid
- $f_0$ Amplitude of harmonic excitation force
- $F$ Amplitude of harmonic excitation force (1 dof oscillator)
- $F_n$ Modal force of mode $n$
- $g$ Acceleration of gravity
- $G$ Shear modulus
- $G_d, G_v$ Transfer function relating deflection/velocity magnitude with magnitude of excitation force (1 dof oscillator)
- $GI^*$ Torsional rigidity
- $H$ Complex-valued function in description of fluid forces in [9]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Area moment of inertia</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Polar moment of area</td>
</tr>
<tr>
<td>$I_s$</td>
<td>Mass moment of inertia with respect to the center of mass of the rigid mass in the mobility model</td>
</tr>
<tr>
<td>$j_c$</td>
<td>Concentrated rotary inertia on an Euler Bernoulli beam (treatment as a 1 dof oscillator)</td>
</tr>
<tr>
<td>$j_y$</td>
<td>Radius of gyration</td>
</tr>
<tr>
<td>$J_c$</td>
<td>Concentrated rotary inertia in the DeqD model; Modal parameter for the description of a concentrated rotary inertia on an Euler Bernoulli beam</td>
</tr>
<tr>
<td>$k$</td>
<td>Wavenumber; Wavenumber of Euler Bernoulli beam</td>
</tr>
<tr>
<td>$k_f$</td>
<td>Acoustic wavenumber in fluid</td>
</tr>
<tr>
<td>$k_n$</td>
<td>Wavenumber of mode $n$ (Euler Bernoulli beam)</td>
</tr>
<tr>
<td>$k_{r, l}$</td>
<td>Rotational/Linear spring stiffness</td>
</tr>
<tr>
<td>$k_s$</td>
<td>Shear wave number in fluid</td>
</tr>
<tr>
<td>$k_x, k_y$</td>
<td>Wavenumber in x-/y-direction</td>
</tr>
<tr>
<td>$k_z$</td>
<td>Axial wavenumber of flexurally oscillating cylinder</td>
</tr>
<tr>
<td>$k_1, k_2, k_3, k_4$</td>
<td>Wavenumbers of Timoshenko beam in mobility model</td>
</tr>
<tr>
<td>$K$</td>
<td>Fluid compressibility</td>
</tr>
<tr>
<td>$K_f$</td>
<td>Bulk modulus of fluid (FEM model)</td>
</tr>
<tr>
<td>$K_n$</td>
<td>Modal stiffness of mode $n$</td>
</tr>
<tr>
<td>$l$</td>
<td>Vertical dimension of fluid flow</td>
</tr>
<tr>
<td>$l_{rm}$</td>
<td>Length of rigid mass (mobility model)</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of beam</td>
</tr>
<tr>
<td>$L(\cdot)$</td>
<td>Operator in the description of a Timoshenko beam</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass (1 dof oscillator); Rigid mass in mobility model; Non-dimensional parameter according to [23]</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>Mass per length (of the beam)</td>
</tr>
<tr>
<td>$m_b$</td>
<td>Mass of bushing (mobility model)</td>
</tr>
<tr>
<td>$m_c$</td>
<td>Concentrated mass in the DeqD model; Concentrated mass on an Euler Bernoulli beam (treatment as a 1 dof oscillator)</td>
</tr>
<tr>
<td>$m_f$</td>
<td>Added mass coefficient</td>
</tr>
<tr>
<td>$m_{rs}$</td>
<td>Parameter in description of fluid forces according to [30] $(r, s \in {1, 2, 3, 4})$</td>
</tr>
<tr>
<td>$m_m$</td>
<td>Mass of magnet (mobility model)</td>
</tr>
<tr>
<td>$m_{tab}$</td>
<td>Mass of tube around bushing (mobility model)</td>
</tr>
<tr>
<td>$M$</td>
<td>(Bending) moment; Non-dimensional parameter describing influence of fluid viscosity; Mass per unit length of oscillating rigid cylinder in [9]</td>
</tr>
<tr>
<td>$M$</td>
<td>Matrix in description of fluid forces according to [30]</td>
</tr>
<tr>
<td>$M(\cdot)$</td>
<td>Operator in the description of a Timoshenko beam</td>
</tr>
<tr>
<td>$M_n$</td>
<td>Modal mass of mode $n$</td>
</tr>
</tbody>
</table>
$M_{nf}$  Modal mass including added mass due to surrounding fluid

$M_p$  Non-dimensional parameter describing the fluid compressibility

$p$  Non-dimensional parameter for description of concentrated rotary inertia in the DeqD model; Non-dimensional variation of fluid pressure; Fluid pressure in an incompressible fluid

$p'$  Variation of fluid pressure in a compressible fluid

$p_0$  Pressure amplitude in fluid

$q$  External distributed load

$q_n(\omega)$  “Complex modal amplitude of the $n$-th natural mode of the beam” according to [14]

$Q$  Quality factor; Shear force

$Q_{\text{fluid}}$  Quality factor due to fluid influence alone

$Q_{\text{mat}}$  Quality factor due to material or structural damping

$Q_n$  Quality factor of mode $n$

$Q_{\text{total}}$  Overall quality factor

$r$  Non-dimensional parameter according to [23]; Radius of polar coordinates

$r_i$  Inner radius of tube

$R$  Outer radius of tube; Radius circular cylinder

$Re$  Reynolds number; Parameter in description of fluid forces according to [30]; Reynolds number in description of fluid forces according to [52]

$Re_s$  Streaming Reynolds number

$s$  Non-dimensional parameter in description of a Timoshenko beam

$s$  Non-dimensional parameter according to [23]

$s_p$  Sensitivity of resonance frequency with respect to fluid density

$S$  Parameter in the depiction of function $H$ in [9]

$t$  Time

$T_{w}, T_{\infty}$  Temperature at surface of obstacle/in the fluid stream

$u$  Velocity of harmonically oscillating cylinder

$\vec{u}$  Vector of fluid particle displacement (FEM model)

$u_0$  Velocity amplitude of harmonically oscillating cylinder

$U_\infty$  Velocity amplitude of oscillation

$v$  Velocity component in $y$-direction

$v$  Fluid velocity vector

$v_f$  Velocity component in $y$-direction in fluid

$v_r, v_\theta$  Radial/Tangential fluid velocity (in polar coordinates)

$v_0$  Amplitude of velocity component in $y$-direction

$\vec{w}(x, \omega)$  Complex displacement response of the Timoshenko beam

$\vec{W} = \begin{pmatrix} \frac{Y(x)}{\Psi(x)} \end{pmatrix}$  Vector of spatial functions describing deflection of centroidal axis $Y(x)$ and slope due to bending $\Psi(x)$ of a Timoshenko beam

$\vec{W}_n = \begin{pmatrix} \frac{Y_n(x)}{\Psi_n(x)} \end{pmatrix}$  Vector of real $n$-th eigenfunctions of Timoshenko beam

$x$  Spatial coordinate; Deflection (1 dof oscillator); Position along undeformed centroidal axis of beam

$x_c$  Position of concentrated mass and rotary inertia in the DeqD model; Position of concentrated mass/rotary inertia on an Euler Bernoulli
List of Symbols

beam (treatment as a 1 dof oscillator)

- $x_e$: Position of excitation force
- $x_s$: Spatial position of center of mass of rigid mass in the mobility model
- $y$: Spatial coordinate; Deflection of centroidal axis of beam
- $\hat{y}(x, \omega)$: “Complex displacement response of the beam at position $x$” according to [14]
- $Y$: “Normal function of $y$” according to [23]; Deflection of cylinder axis in [52]
- $Y_0$: Deflection amplitude of cylinder axis in [52]
- $Y_{drive}$: Driving point mobility function
- $z$: Spatial coordinate
- $z_p$: Complex deflection in the particular solution of the 1 dof oscillator

Greek alphabet

- $\alpha$: Dissipative part of the fluid force (per unit length)
- $\beta$: Inertial part of the fluid force (per unit length)
- $\hat{\beta}$: Parameter in the frequency equation of the DeqD model
- $\gamma$: Parameter in the frequency equation of the DeqD model
- $\omega_0$: Slope due to shearing effects (Timoshenko beam theory)
- $\Gamma$: Hydrodynamic function in description of fluid forces according to [52]
- $\delta$: Penetration depth, boundary layer thickness
- $\delta_1$: Damping parameter (1 dof oscillator)
- $\Delta \rho_e$: Absolute error of predicted fluid density
- $\varepsilon$: Non-dimensional parameter describing magnitude of inertial effects
- $\zeta$: Non-dimensional length according to [23]; Normalized wavenumber of fluid in description of fluid forces according to [52]
- $\eta$: Dynamic viscosity of fluid; Deflection of rigid mass (mobility model)
- $\eta_b$: Dimensionless bulk viscosity of fluid
- $\eta_B$: Bulk viscosity of fluid
- $\eta_{loss}$: Loss factor
- $\theta$: Polar angle of polar coordinates
- $\kappa$: Shear correction factor; Normalized wavenumber of cylinder in description of fluid forces according to [52]
- $\lambda_f$: Acoustic wavelength in fluid
- $\nu$: Kinematic viscosity of fluid; Poisson’s ratio
- $\rho$: Density of beam material; Non-dimensional variation of fluid density
- $\hat{\rho}$: Fluid density in a compressible fluid
- $\rho'$: Variation of fluid density in a compressible fluid
- $\rho_I$: Density of fluid (of fluid at rest)
- $\sigma$: Shear stress
- $\sigma_{rr}, \sigma_{r\theta}$: Normal/Tangential stress in polar coordinates
- $\varphi$: Phase; Rotation of axis of rigid mass (mobility model)
- $\phi_n(x)$: “Real modal function of the $n$-th natural mode of the beam” according to [14]
\( \chi_1, \chi_2 \) Parameters in description of fluid forces according to [30]

\( \psi \) Slope due to bending (Timoshenko beam theory)

\( \Psi \) "Normal function of \( \psi \)" according to [23]

\( \omega \) Angular frequency

\( \omega_0 \) Parameter in description of fluid forces according to [30]

\( \omega_f \) Angular resonance frequency in fluid

\( \omega_n \) Natural angular frequency of mode \( n \); Angular resonance frequency of mode \( n \) in fluid

\( \omega_{nc} \) Natural angular frequency of mode \( n \) of an Euler Bernoulli beam with concentrated mass/rotary inertia

\( \omega_{vac} \) Angular resonance frequency in vacuum

\( \omega_{0n} \) Natural angular frequency (1 dof oscillator)

\( \Omega \) Angular excitation frequency (1 dof oscillator)

**Miscellaneous**

\( i \) Imaginary unit \((i = \sqrt{-1})\)

\( \Re \{ \} \) Real part of a complex expression

\( \Im \{ \} \) Imaginary part of a complex expression

\( \arg \{ \} \) Argument of a complex expression

\( H_0^{(1)}, H_1^{(1)} \) Zeroth-/first-order Hankel function of the first kind

\( J_1, Y_1 \) Bessel function of the first/second kind and order 1

\( I_1, K_1 \) Modified Bessel function of the first/second kind and order 1

\( \delta (\cdot) \) Dirac delta function

\( (r, \theta) \) Polar coordinates (radius \( r \), polar angle \( \theta \))

\( \nabla \) Nabla operator

\( \frac{\partial \psi}{\partial t} \equiv \dot{y} \equiv y_t \) Partial differentiation with respect to time

\( \frac{\partial \psi}{\partial x} \equiv y_x \) Partial differentiation with respect to spatial coordinate \( x \)
1. Introduction

1.1. Background and motivation

An oscillating structure in a viscous fluid experiences changes in resonance frequency and damping dependent on the properties of the surrounding fluid. Conversely, these changes can be used to determine the properties of the surrounding fluid. In this thesis Newtonian fluids are considered. The fluid parameters of interest are therefore the density and the viscosity of the fluid. Various measurement principles and sensors have been developed to measure either one or both of these parameters. Generally devices for the measurement of density or viscosity can be subdivided into laboratory and process instruments (cp. [2, 3]). The former allow a very accurate determination of the absolute values of the density or viscosity of a fluid sample (e.g. Anton Paar oscillating U-tube density meter, capillary viscometer). Process instruments are intended for continuous measurements in order to monitor or control production processes. They may have lower accuracy than laboratory instruments and may provide only relative density or viscosity measurements (meaning that they detect changes from a reference state instead of determining absolute density or viscosity values). Examples are vibration sensors as resonating tubes or vibrating forks. Dual presented in his dissertation [12] a viscosity sensor based on torsional vibrations of a cylindrical tube. In this sensor, the fluid viscosity is calculated from the measured damping of the vibrating tube. The high precision, the simple geometry of the wetted parts and the quick evaluation make it a valuable tool for in-line measurements. In order to calculate absolute viscosity values, however, the fluid density must be supplied to the sensor. In processes where density variations are small, inserting an average density has little effect on the measurement accuracy. Further, the density value can be supplied by an external measurement. On the other hand it is clear that the combined measurement of density and viscosity with the same sensor would lead to a more accurate and more versatile instrument. Density measurement could be realized with the same sensor, if the cylindrical tube would be excited to flexural vibrations. The fluid density could then be determined from the change in resonance frequency of the flexurally oscillating tube due to the surrounding fluid. The motivation of the work presented here is to provide a basis that allows developing the existing sensor into a combined viscosity and density sensor. In this regard, the objective of this work is the study of different modeling approaches for the structure and the fluid-structure interaction of a flexural vibration sensor for density measurements, as well as the experimental characterization of the sensor, and the validation of the models.
1.2. Literature overview

A sensor for the combined in-line measurement of density and viscosity is a valuable tool to monitor and control industrial production processes in food, cosmetics, petrochemical and general chemical industries. An example from the food industry would be the production of fruit juices from a concentrate. In the petroleum industry the measurement of density and viscosity is required in order to assess the value of petroleum reservoirs and to determine the production strategy (Goodwin et al. [20], [21]). Such a sensor can also be used for the blending of fuels (Sparks et al. [49]). Ghatkesar et al. [17] has the intention to use the sensitivity of a micromechanical device on fluid density and viscosity to monitor chemical reactions, protein aggregation, and for blood plasma rheology for medical applications.

In the next paragraphs density or combined density/viscosity sensors based on vibrating structures which can be found in literature are presented. The focus is on their application areas, principles of operation, obtained accuracy, and also on the modeling approaches.

According to Majer and Padua [33], vibrating-wire densimeters and vibrating-tube densimeters are the most important vibration sensors for density measurements, and an extensive survey on the principles of operation, working equations, applications, and current developments is presented. A vibrating-wire densimeter consists of a vertical wire which is fixed on one end and carries a mass on the other end. The wire and the mass are immersed in fluid. Due to the buoyancy forces acting on the attached mass, the tension force on the wire and thereby the resonance frequency of the wire depend on the fluid density. Therefore the fluid density can be related to the resonance frequency of the wire. Additionally, the damping of the vibrations can be related to the fluid viscosity and the device can be used as a combined density and viscosity sensor. According to Majer and Padua [33] vibrating-wire sensors can give accuracies of about 0.1 % in density and 2 % in viscosity, and precisions of ±0.03 % in density and ±0.5 % in viscosity. Vibrating-tube densimeters are usually built by a U-shaped tube which performs bending oscillations with deflections normal to the plane containing the tube axis. The tube is filled with fluid and the resonance frequency of the oscillating tube can be related to the density of the fluid. The tube is usually oscillating in its fundamental mode. An exception is the design of Chang and Moldover [8] which uses the third bending mode (which has the benefit of better decoupling from the mounting and less susceptibility to disturbing external vibrations). The well-known Anton Paar densimeter is an example of a vibrating-tube densimeter. According to Majer and Padua [33] vibrating tube densimeters typically have a resolution and precision of about 0.01 kg/m³.

A detailed theory and design criteria for density and viscosity sensors based on flexural vibrations of a solid rod clamped on both ends and surrounded by a fluid are presented in Retsina et al. [39], [40]. The presented theory treats the description of the mechanical oscillator, the description of fluid forces and sensitivity analyses.
The sensitivity analysis in Retsina et al. [39] shows that for density measurements, the principal sources of error arise from errors in the density of the rod material and the measurement of the resonant frequency. The use of two different operation modes is proposed. A forced mode where the resonance frequency is measured in order to obtain the density of the liquid (viscosity known). Then the setup is run in free decay mode, where the logarithmic decrement is measured in order to obtain the viscosity of the fluid (density known). An application of the presented theory is given in Bett et al. [6].

In patent literature, Fitzgerald et al. [15], an in-line vibratory viscometer-densitometer is found which consists of a hollow tube clamped on both ends, and with the fluid flowing through the tube. The tube is excited to simultaneously oscillate in a flexural and torsional mode. The power required to maintain a constant amplitude of the torsional vibrations is related to the fluid viscosity. The fluid density can be inferred from the resonance frequency of the flexural vibrations.

In Goodwin et al. [20], [21] bending vibrations of a plate are used for the combined measurement of fluid density and viscosity. Their Micro Electrical Mechanical System (MEMS) sensor is intended to be used in the petroleum industry in order to assess the value of petroleum reservoirs and to determine the production strategy. It is noted that for such an application an accuracy of $\pm 1\%$ in density and $\pm 10\%$ in viscosity is required. The lowest bending mode is used for measurements, even though it is mentioned that higher bending modes could offer better decoupling from the support. For the measurement of density and viscosity, the resonance frequency and Q-factor of the flexural oscillations are evaluated using semi-empirical working equations. These equations contain empirical constants that account for uncertainties in geometry and material parameters, and in the effective area (regarding fluid-structure interaction), and must be determined by calibration. In Goodwin et al. [21] measurements are presented with fluid densities in the range 408 to 1834 kg/m$^3$ and fluid viscosities in the range 0.038 to 275 mPas at temperatures between 313 and 373 K and pressures up to 60 MPa, for which predictions of the densities and viscosities could be obtained with the sensor that showed deviations of $\pm 2\%$ in density (for viscosities between 0.1 and 100 mPas) and deviations less than $\pm 30\%$ in viscosity.

For the measurement of density and viscosity, Reichel et al. [38] uses two parallel rectangular diaphragms that perform flexural oscillations, and the fluid is located between the two diaphragms. It is noted in Reichel et al. [38] that a numerical treatment is not practical because the need to resolve the viscous penetration depth leads to a too large number of elements. Instead an analytical approach is pursued and an energy method is applied. The velocity field is calculated from which kinetic and dissipated energies can be derived. The energy expressions are dependent on fluid density and viscosity, and can be related to the measurable resonance frequency and Q-factor of the oscillating system. Only a measurement of the unloaded system is needed for calibration. Experiments are performed with mixtures of DI-water and glycerine that cover a density range of 998 to 1139 kg/m$^3$ and a viscosity range of 0.9
Chapter 1. Introduction

to 7.1 mPas. The densities and viscosities are predicted by the sensor with relative errors of maximally 5.4\% in density and up to 93.1\% in viscosity.

Ghatkesar et al. [17] considers an array of micro-cantilevers in water which are excited in the lowest 18 flexural resonance modes. It is found that the sensitivity on fluid density increases with mode order, and that the influence of viscosity on the added mass cannot be neglected even for higher modes (because of the small beam dimensions). Experimentally determined eigenfrequency shifts for the test fluids with respect to water are compared to model predictions (using a model from literature). For this comparison the thickness of the beam has been adapted in the model in order that the eigenfrequency in water agrees between model and experiment.

In Youssry et al. [56] a rectangular microcantilever oscillating in its lowest flexural mode is used to measure fluid density and viscosity. The beam is modeled as an Euler Bernoulli beam and a modal analysis approach is used to express the transfer function for the beam deflection due to an excitation force acting on the free end of the beam. Using a simplified expression for the fluid forces allows to formulate analytic expressions relating the fluid density and viscosity to the measured eigenfrequency and damping ratio. Experiments are presented that have been conducted in fluids with densities of 750 kg/m$^3$ and 1000 kg/m$^3$ and viscosities between 1.67 and 553 mPas. The calculated densities show errors up to 14.5\% (for viscosities up to 54.2 mPas) and the calculated viscosities show errors up to 54.6\% (for viscosities up to 54.2 mPas).

In the context of atomic force microscopy Sader [44] considers beams with circular and rectangular cross-sections performing small amplitude flexural oscillations in a viscous, incompressible fluid. The beams are assumed to be much longer than wide. Under this assumption the flow field varies slowly along the beam and consequently the fluid forces acting on the beam are approximated by those acting on an infinitely long rigid beam performing transverse oscillations. It is found that if dissipative effects are small, the frequency response of the cantilever beam in the vicinity of a resonance peak is well approximated by that of a simple harmonic oscillator. Further it is noted that the hydrodynamic function (function describing the fluid forces) varies only slightly with frequency and therefore can be treated as a constant in the vicinity of the resonance peak of a mode, and can be evaluated at the resonance frequency of that mode in the absence of dissipative effects. Using these considerations, expressions for the resonance frequency and Q-factor are derived.

Flexural and torsional oscillations of circular cylinders and flat blades of zero thickness immersed in an unbounded viscous compressible fluid are considered in Van Eysden and Sader [52]. The considered cylinders are infinitely long and the vibrations can be described by an arbitrary wavenumber along their length. The presented theoretical study provides exact analytical solutions for the three-dimensional flow fields and the resulting hydrodynamic loads. From the inviscid compressible solution for the fluid forces acting on the circular cylinder it is found that energy
dissipation by radiation damping sets in if the acoustic wavenumber in the fluid
is greater than the wavenumber of the beam flexural oscillations. Considering the
viscous compressible solution for the fluid forces acting on the circular cylinder it is
found that the forces contain both features from the compressible inviscid solution
and from the incompressible viscous solution.

Most of the density and viscosity sensors discussed above, except those of Fitzgerald
ald et al. [15] and Ghatkesar et al. [17], make use of only one oscillation mode
(usually the fundamental mode), and do not make use of the varying sensitivities of
different modes on fluid density or viscosity.

Even though it has been found by several authors, that higher flexural modes have
the advantages of better decoupling of the mounting from the environment, and less
susceptibility to external noise, hardly any sensor is operated in a higher flexural osc-
illation mode. Consequently, in the references cited above, the flexural oscillations
of the beams are always modeled using the Euler Bernoulli beam model. For higher
oscillation modes or if the beam length is not much larger than its thickness, the
Euler Bernoulli beam model is not appropriate, and the Timoshenko beam model
is known to give more accurate results. To the author’s knowledge, no theoretical
model that combines the Timoshenko beam model with one of the fluid models pre-
sented above exists. Problems of the application of the Timoshenko beam theory are
certainly the bulky expressions leading to tedious calculations and that it is difficult
to find practical expressions relating the fluid properties with measured quantities.

The influence of the mounting of the vibrating structures, which could have a sig-
nificant effect if for example the lowest flexural oscillation mode is excited, is scarcely
considered. In the models presented above, always ideally mounted structures are
assumed, e.g. a rigid clamping instead of a resilient support. The quantitative effect
on the comparison between model and experiment is not clear. The effect might also
be covered by a calibration step. For example in Ghatkesar et al. [17] the beam
thickness is adapted to match model and experiment.

The sensor designs discussed here are all built from very simple structures, e.g.
cylindrical rods, plates or rectangular beams. Of course this is an advantage because
the number of parameters in the systems is kept small. But still the presented sen-
 sor must usually be calibrated because some geometrical or material parameters are
not known with sufficient accuracy. This highlights the difficulties of an accurate
modeling of vibrating structures.

Considering flexural oscillations of a beam in a viscous incompressible fluid, most
of the fluid models presented above assume a two-dimensional velocity field in the
fluid, with fluid motion in a plane perpendicular to the beam axis, and fluid flow in
axial direction is neglected (e.g. Kremlevskii and Stepichev [31], Retsina et al. [39],
Sader [44]). This simplification is made under the condition that the wave length
greatly exceeds the dimensions of the beam cross-section (Retsina et al. [39], Sader
[44]). The influence of edge effects, for instance the effect of the three-dimensional
nature of the fluid flow at the free end of a cantilever beam, has not been considered yet. The fluid models for higher mode orders presented in Van Eysden and Sader [51], [52] describe the full three-dimensional flow field but can only treat beams of infinite length with sinusoidal deflections.

Fluid density and viscosity sensors presented in recent scientific literature mostly focus on micromechanical or nanomechanical sensors. The advantages of miniaturized sensors are for example the small sample volumes, the small thermal mass, their high sensitivity on changes in fluid properties, and that they possibly can be produced as low-cost disposable sensors. Disadvantages are their fragility or their often very low Q-factors in fluids with increased viscosity, which complicates the evaluation of resonance peaks and might introduce a coupling between different oscillation modes.

1.3. Modeling challenges

The multiphysics simulation capabilities are growing which gives the impression that many of the engineering problems can be understood and solved quite easily. However there are still a lot of challenges. One problem is posed by the need to cover different orders of dimensions in a model. For fluid structure interaction problems between a structure with dimensions in the centimeter range which is immersed in a viscous fluid, where the viscous boundary layer must be resolved, which has typically a dimension in the micrometer range, the models become unmanageably large. If one has also complex geometries, that for example cannot be described by simple beam or shell elements, the resulting models become even larger.

Due to the nonlinearity of the Navier-Stokes equation, models involving fluids described by the Navier-Stokes equation must be solved in time domain. When considering structural vibrations in contact with such a fluid, this means that in order to obtain the steady state solution, one has to wait for the transients to decay, which requires many time steps and simulations take a very long time. The accurate representation of the excitation of structural vibrations in a time domain simulation is also often not trivial.

Especially for thin or small structures it is difficult to obtain a good agreement between a numerical model and the experiment. Even small deviations in geometry can have a large influence in the model (e.g. the mass distribution obtained from modeled geometry and material density can differ significantly between model and experiment), and in numerical simulations small geometric details as chamfers or radii cannot be incorporated.

A drawback of numerical simulations is that dependencies cannot be directly extracted. In order to study the influence of a certain parameter on the system behavior, the model must be evaluated several times with varying parameters, which is time-consuming with larger models and the outcome is specific to the modeled situation. With analytical models it is sometimes easier to recognize the physical dependencies and awareness of underlying assumptions can help to recognize the important physical mechanisms in the system. On the other hand it is clear, that analytical models are incapable to treat complicated geometries and complex three-
1.4. Scope and content of the thesis

An existing viscosity sensor based on a torsionally vibrating cylindrical tube shall be extended to include the ability to perform density measurements using bending vibrations. In this regard, the present work considers the analytical and numerical modeling and experimental characterization of a density sensor based on flexural oscillations of a cylindrical tube.

Instead of focusing on the development of an actual sensor, the work aims at developing an understanding of the important aspects in building and modeling of such a sensor. This understanding covers both the description of the mechanical structure and its interaction with a surrounding fluid. Considering flexural oscillations of cylindrical rods or tubes which are not perfectly symmetric, as is the case in practice, the problem arises that there are two close oscillation modes according to the two principal axes of the cross-section. These close modes can interfere with each other and make their use for density measurement impossible. In a first step a sensor design must be found that allows to separate these modes. A design that allows to separate such flexural oscillation modes has not been found in literature and seems to be new.

Most vibration sensors presented in literature are based on very simple oscillating structures as cylindrical rods, plates or rectangular beams, and still an accurate modeling of such vibrating structures seems difficult. In this work, a complex oscillating structure consisting of several parts is considered. In addition, also higher order flexural modes are considered, in contrast to most flexural oscillation sensors found in literature, that make use of only the lowest bending modes. The use of higher order bending modes requires the use of Timoshenko beam theory instead of the commonly used Euler Bernoulli beam theory, with a consequently higher complexity of the analytical models and an increased number of system parameters. Different modeling approaches (analytical and numerical) are studied in order to describe the flexurally oscillating mechanical structure. Experiments are performed in order to determine the system parameters and to validate the models.

Considering the interaction between the oscillating cylindrical tube and the surrounding fluid, it is important to develop an understanding for the fluid behavior and the aspects that affect the fluid forces acting on the oscillating structure. Dependent not only on fluid parameters as fluid density, viscosity, and compressibility, but also on the oscillation mode, frequency, and amplitude and the dimensions of the oscillating structure and the fluid container, the fluid behavior can be different. Fluid compressibility becomes more important with increasing oscillation frequency, while viscous forces become smaller; the fluid might interact with the walls of the fluid container, acoustic radiation can occur, and amplitude-dependent second order effects as acoustic streaming can be present. Based on non-dimensional parameters and general analytical results from literature, the importance of the various possible fluid effects can be estimated. Fluid forces predicted by several analytical and dimensional fluid behavior. Therefore it seems necessary to pursue both analytical and numerical modeling.
In Chapter 1, an introduction is given to the study of fluid-structure interaction. Experiments are performed to study the effects of different model formulations. The fluid structure interaction on oscillation amplitude, dimension of the fluid cavity, and various fluid properties are compared in order to study the effects of different model formulations. Experiments are performed in order to study the dependence of the fluid structure interaction on oscillation amplitude, dimension of the fluid cavity, and various fluid properties. Based on the experimental studies and comparisons between model predictions and experiments, a better understanding of the fluid behavior and the relevant fluid effects is obtained.

In contrast to building a model that can predict the resonance frequency of the flexurally oscillating tube in a fluid which has been discussed so far, the inverse problem must be solved in order to build a density sensor, i.e., the problem of the determination of the fluid density from the measured resonance frequency must be solved. Based on Euler Bernoulli beam theory and a simplified description of the fluid-structure interaction, a general form of the fluid density-resonance frequency relationship containing two calibration constants is found. The relationship is successfully applied to experimental data. A similar derivation of the fluid density-resonance frequency relationship based on Timoshenko beam theory could not be established and the reasons are explained.

In Chapter 2, the possibility of obtaining a resonator sensor with either good sensitivity on fluid density or fluid viscosity is discussed. The concept of combining the viscosity measurement and the density measurement is presented.

In Chapter 3, the design and mechanical model of the flexural oscillator is considered. The requirements on the sensor design are discussed and the final sensor design used in this work is presented. Two analytical models based on Timoshenko beam theory and an FEM model are set up. For an accurate representation of the flexural oscillator by these models, an accurate determination of the model parameters is crucial. A procedure is presented in order to determine the required parameters. Experiments are performed in order to validate the models, to characterize the influence of the clamping of the sensor, and to determine the influence of temperature variations on resonance frequency and damping of the flexural oscillations.

Chapter 4 considers the modeling of the resonator in a fluid. Based on a dimensional analysis of the Navier-Stokes equation with respect to the present situation, the importance of the various fluid effects can be assessed. For the description of the interaction of a flexurally oscillating beam with a surrounding fluid an adequate fluid model is required. Several analytical fluid models, treating the fluid as 2D/3D, (in)compressible, (in)viscid or (un)bounded, are introduced and compared to each other. A finite element model of the resonator which treats the surrounding fluid as an acoustic medium is built. Experiments with the density sensor in fluids are conducted in order to check for the linearity of the fluid behavior and study the influence of fluid boundaries on the sensor behavior. A series of measurements in various test fluids is performed, and the experimentally determined resonance frequencies and damping values are compared with model predictions.

Chapter 5 considers the inverse problem, i.e., the determination of fluid density and viscosity based on the measured resonance frequency and damping of the flexural oscillator. Based on simplifications of the present fluid-structure interaction problem, a general form of the fluid density-resonance frequency relationship is found, that contains two calibration constants. This relationship is then applied to
the measurements in the test fluids.
2. Measurement principle and concept of combined density/viscosity measurement

2.1. Separation of the influences of fluid density and viscosity

The interaction between an oscillating body and a viscous, incompressible Newtonian fluid is governed by the density and the viscosity of the fluid. Generally, the interaction depends on the combination of both the density and the viscosity of the fluid. Therefore, if one aims to measure the two parameters by vibration experiments, the question arises, how to separate the influences of the two parameters. This can be achieved by considering different oscillation modes, which show different sensitivities on the two fluid parameters.

In order to appreciate the combined effect of fluid density and viscosity, consider an infinite plane in contact with an incompressible, viscous Newtonian fluid according to Fig. 2.1, where the plane performs harmonic oscillations with velocity \( v = \Re\{v_0 \cdot e^{i \omega t}\} \) along a direction \( y \) lying in the plane. The solution of this problem is well known and can be found for example in Landau and Lifshitz [32]. The fluid near the plane executes a shearing motion where the only non-zero velocity component is in \( y \)-direction and is described by

\[
v_f(x) = \Re\{v_0 e^{(-x/\delta)} e^{i(\omega t - x/\delta)}\}
\]

with

\[
\delta = \sqrt{\frac{2 \eta}{\omega}} = \sqrt{\frac{2 \eta}{\rho_f \omega}}
\]

in which \( v_0 \) is the oscillation amplitude of the plane, \( \omega \) is the angular frequency of the oscillation, \( \delta \) is the penetration depth and \( \Re\{\} \) means that the real part of the expression has to be taken. The penetration depth depends on the kinematic
Chapter 2. Measurement principle and concept of combined density/viscosity measurement

viscosity $\nu$, which is related to the dynamic viscosity $\eta$ by the fluid density $\rho_f$ as $\eta = \nu \cdot \rho_f$. The velocity amplitude decays exponentially with distance $x$ from the plane.

The shear stress $\sigma$ acting on the plane is given by

$$\sigma = \Re\{\frac{\rho_f \eta \omega}{2} \cdot (1 + i) \cdot v_0 \cdot e^{i\omega t}\} = \Re\{\sqrt{\frac{\rho_f \eta \omega}{2}} \cdot (v + \frac{\dot{v}}{\omega})\} = \frac{\eta}{\delta} \Re\{v\} + \frac{\rho_f \delta}{2} \Re\{\dot{v}\} \quad (2.1)$$

The shear force acting on the plane can be calculated by integrating the shear stress $\sigma$ from Eq. (2.1) over the plane's surface. From Eq. (2.1) it can be seen that this shear force is composed of two parts that are either proportional to the velocity or to the acceleration of the plane. The part proportional to the plane's velocity is responsible for the energy dissipation and is called the dissipative part (Landau and Lifshitz [32]). The part proportional to the acceleration of the plane is the reaction force to the acceleration of the fluid (Dual [12]) and is called the inertial part (Landau and Lifshitz [32]). The inertial part of the shear force does not lead to energy dissipation.

The existing viscosity sensor is based on the torsional vibrations of a circular tube. Considering a plane perpendicular to the axis of the tube (Fig. 2.2), it is obvious that the fluid near the surface of the tube experiences a shearing motion, while the fluid further away is at rest. The disturbed region of the fluid is characterised by the penetration depth $\delta$, which for moderate viscosities is very small compared to the radius of the tube (e.g. for $f_{res} = 5$ kHz, $\rho_f = 1000 \text{ kg/m}^3$, $\eta = 10 \text{ mPas}$ and a tube radius $R = 4 \text{ mm}$, one obtains $\delta = 25.2 \mu\text{m}$ and $\delta/R = 6.3 \times 10^{-3}$). Regarding the fluid forces acting on the tube, a small penetration depth leads to a small inertial force. If the additional inertia due to the surrounding fluid is small compared to the inertia of the oscillating tube itself, the resonance frequency of the oscillating tube will not change significantly. On the other hand, if the oscillating tube shows little damping, the additional damping force due to the fluid can have a significant effect on the overall damping of the system. Therefore the damping of the torsionally oscillating tube is sensitive to the viscosity of the fluid, but still depends also on the fluid density, as can be inferred from the factor $\sqrt{\rho_f \eta \omega}$ in the expression of the dissipative part of the fluid forces according to Eq. (2.1).

Consider now another fundamental case of the interaction between an oscillating structure and a fluid, which is of importance regarding flexural oscillations of a structure in a compressible fluid. Consider therefore again an infinite plane according to Fig. 2.1, which performs harmonic oscillations with deflections in $x$-direction and has a sinusoidally varying amplitude along the $y$-direction. The considered fluid is

\[\text{If } \delta/R \ll 1 \text{ the curvature of the cylindrical surface of the tube becomes almost negligible for the fluid motion. In this case the fluid motion is approximately one-dimensional and the governing equations describing the problem take the same form as for the case of the oscillating infinite plane, and the shear stress acting on the surface of the tube is described according to Eq. (2.1). The considerations on the oscillating infinite plane can then be transferred approximately to the case of the torsionally oscillating tube. See Dual [12, ch. 2.8.1, and ch. 2.8.2] for a detailed discussion of this aspect.}\]
2.1. Separation of the influences of fluid density and viscosity

Figure 2.2.: Fluid-structure interaction: Torsional vs. flexural oscillator.

The fluid motion around the tube is characterized by a displacement or acceleration of the fluid in the direction of the oscillation of the tube and a shearing motion at right angles to it. Besides the shearing forces acting on the tube surface, there are reaction forces from the acceleration of the surrounding fluid. These fluid forces proportional to the acceleration of the tube are much stronger than in the case of the rotational oscillations considered above, and should strongly depend on the fluid density. The acceleration-proportional fluid forces should significantly affect the oscillation frequency of the flexurally oscillating tube. This expectedly high sensitivity of flexural oscillations on density changes in the surrounding fluid motivates their use for density measurements.
Considering density measurements in a compressible fluid, it is clear from the discussion of Eqs. (2.2) and (2.3) that the case (ii) is sought, where the fluid loading is purely inertial and has a strong effect on the resonance frequency of the density sensor. The case (i) on the other hand could be problematic, because the acoustic radiation can lead to an unwanted interaction with the environment of the sensor. The description of the fluid forces in a compressible fluid requires also an additional parameter, e.g. the speed of sound, in order to describe the compressibility of the fluid. Such an additional parameter is not desirable, since it complicates the determination of the fluid density.

Obviously the real situation of a flexurally oscillating tube in a fluid is more complicated than the simple cases discussed above. Still, some of the features discussed above are present in the real situation and illustrate some basic characteristics of the fluid-structure interaction.

### 2.2. Description as 1 dof oscillator

In this section, the vibrations of a single-degree-of-freedom system ("1 dof oscillator") are considered. Thereby, relationships and terms shall be introduced, which will be used later in this text, especially in the evaluation and interpretation of measurement data.

Many mechanical vibrations of linear continuous systems can be treated as single-degree-of-freedom systems if their oscillation modes are uncoupled. Using the modal analysis solution technique, it can be shown that flexural vibrations of conservative beams can be described by single-degree-of-freedom systems (Meirovitch [34, e.g. p. 135, pp. 387/388], Rao [37]). For damped beams, if the damping is non-proportional, the damping can introduce a coupling of the oscillation modes (Meirovitch [34, p. 483]). If the oscillation modes are well separated (along the frequency axis) and the system is weakly damped, it is usually assumed that the oscillation modes can be described as a 1dof oscillator.

The description of a 1 dof oscillator presented here is taken from Sayir and Kaufmann [46], Glockler [18], Rao [37], Meirovitch [34], Dual [12].

The differential equation describing the forced oscillation of a damped single-degree-of-freedom system under a harmonic excitation force is given by

\[ m \ddot{x} + d \dot{x} + cx = F \cos(\Omega \cdot t) \] (2.4)

in which \( m \) denotes the mass, \( d \) the viscous damping coefficient, \( c \) the stiffness, and \( F \) and \( \Omega \) denote the amplitude and frequency of the harmonic excitation force. By introducing

\[ \delta_1 = \frac{d}{2m}, \quad \omega_0 = \sqrt{\frac{c}{m}}, \quad \bar{f} = \frac{F}{m} \] (2.5)

where \( \omega_0 \) is the natural frequency of the system, the equation is given as
\[ \ddot{x} + 2\delta_1 \dot{x} + \omega_0^2 x = \bar{f} \cos(\Omega \cdot t) \]  
\hspace{1cm} (2.6)

For subcritical damping ($\delta_1 < \omega_0$), the homogeneous solution is an exponentially decaying oscillation. In order to determine the particular solution it is practical to adopt complex notation:

\[ \ddot{z}_p + 2\delta_1 \dot{z}_p + \omega_0^2 z_p = \bar{f} e^{i\Omega t} \]  
\hspace{1cm} (2.7)

The solution corresponding to the excitation force $\bar{f} \cos(\Omega t) = \Re\{\bar{f} e^{i\Omega t}\}$ is then given by $\Re\{z_p(t)\}$. Using the ansatz $z_p(t) = C e^{i\Omega t}$ with $C$ as a complex constant, one obtains

\[ z_p(t) = C e^{i\Omega t} = \frac{\bar{f} e^{i\Omega t}}{\omega_0^2 - \Omega^2 + i 2 \delta_1 \Omega} \]  
\hspace{1cm} (2.8)

Based on Eq. (2.8), it is easy to formulate the transfer function $G_d(\Omega)$ relating the deflection magnitude with the magnitude of the excitation force $\bar{f}$, and the transfer function $G_v(\Omega)$ relating the velocity magnitude with the magnitude of the excitation force $\bar{f}$:

\[ G_d(\Omega) = \frac{C}{\bar{f}} = \frac{1}{\omega_0^2 - \Omega^2 + i 2 \delta_1 \Omega} \]  
\hspace{1cm} (2.9)

\[ G_v(\Omega) = \frac{i \Omega C}{\bar{f}} = \frac{i \Omega}{\omega_0^2 - \Omega^2 + i 2 \delta_1 \Omega} \]  
\hspace{1cm} (2.10)

The magnitude and phase of $G_v(\Omega)$ are plotted in Fig. 2.3 for two levels of damping.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure23.png}
\caption{Magnitude and phase of $G_v(\Omega)$ for two levels of damping.}
\end{figure}

An important measure for the damping of the system is the quality factor ($Q$-factor) (Dual [12]):

\[ Q = \frac{\omega_0}{2\delta_1} \]  
\hspace{1cm} (2.11)

There are other expressions for the $Q$-factor. According to Cremer et al. [10] it can be expressed as

\[ Q = 2\pi \frac{\text{stored energy}}{\text{energy dissipated per cycle}} \]  
\hspace{1cm} (2.12)
in which the stored energy is the energy content of the system, that oscillates between potential and kinetic energy.

In the phase-frequency curve obtained from Eq. (2.10), the resonance frequency can be identified as the frequency at which the velocity is in phase with the excitation force, and the damping is related to the slope of the phase curve at the resonance frequency. Consider Fig. 2.4, in which the phase $\varphi$ of $G_v$ is plotted versus frequency $f$ and where $f(\varphi = \alpha)$ denotes the frequency at which the velocity has a phase difference $\alpha$ with respect to the excitation force. The resonance frequency and the $Q$-factor can then be expressed as

$$f_{\text{res}} = f(\varphi = 0)$$

$$Q = \frac{f_{\text{res}}}{f(\varphi = -\alpha) - f(\varphi = +\alpha)} \cdot \tan \alpha$$

For a phase difference $\alpha = \pm 45^\circ$ the corresponding frequency difference equals the bandwidth $df$ of the resonance peak, and the $Q$-factor can consequently be expressed as

$$Q = \frac{f_{\text{res}}}{df}$$

Figure 2.4.: Phase-frequency curve of $G_v$: Determination of resonance frequency and damping.

An important difference between the transfer function of the deflection or deflection velocity and the excitation force of the 1 dof oscillator considered above is presented in Dinser [11, ch. 2.1.1]. Considering the transfer function for the deflection $G_d(\Omega)$ (Eq. (2.9)), one observes that the maximum amplitude of the transfer function does not occur at the natural frequency $\omega_0$ of the system if the system is damped. More precisely, the maximum amplitude of the deflection transfer function is shifted to frequencies lower than the natural frequency of the system with increasing damping. On the other hand, for the transfer function for the velocity, $G_v(\Omega)$ (Eq. (2.10)), the maximum amplitude of the transfer function occurs at the natural frequency $\omega_0$ of the system, independently of the damping of the system. This aspect should be kept in mind for the evaluation and interpretation of measurements.

For the measurements presented in this text, the evaluation encompasses the determination of the resonance frequency and the damping. For all measurements the
velocity response has been evaluated. The determination of the resonance frequency and the damping is achieved by measuring the phase-frequency curve near the resonance frequency and fitting the phase-frequency curve of a 1 dof oscillator into the measured data.

The phase-frequency curve for the velocity response

\[ \dot{z}_p = i\Omega C e^{i\Omega t} \]  

(2.16)
is given as

\[ \varphi_{\text{os}}(\Omega) = \arg\{i\Omega C\} = \arctan\left(\frac{\omega_0^2 - \Omega^2}{2\delta_1 \Omega}\right) \]  

(2.17)

Therefore the two parameters obtained from the fitting process are \( \omega_0 \) and \( \delta_1 \).

2.3. Concept of combined density/viscosity measurement

The basic idea behind the combined density/viscosity measurement is to employ the high sensitivity of the damping of the torsional oscillation on the fluid viscosity to determine the fluid viscosity, and the high sensitivity of the resonance frequency of the flexural oscillation on the fluid density to determine the fluid density. As explained earlier, the difficulties arise from the combined influence of fluid density and viscosity on the two oscillation modes. In this section, the possibilities to combine the measurement results from the torsional and flexural oscillation experiments shall be considered.

In Dual [12] it is shown that the bandwidth \( df \) of the torsionally oscillating tube is proportional to \( \sqrt{\rho_f \eta \omega} \), if \( \delta/R \ll 1 \) and the influence of the curvature of the cylindrical surface can be neglected, and if edge effects at the free end of the tube are neglected:

\[ df \sim \sqrt{\rho_f \eta \omega} \]

Neglecting the frequency dependence of the damping\(^2\), one can set \( \omega \approx \omega_0 \) and write

\[ df \approx c \cdot \sqrt{\rho_f \eta \omega_0} \]
in which \( c \) is a constant that has to be determined from a calibration measurement, and \( \omega_0 \) is the angular frequency at resonance. Thus an expression for the fluid viscosity can be formulated as

\[ \eta \approx \frac{df^2}{c^2 \rho_f \omega_0} \]  

(2.18)

In Eq. (2.18), the bandwidth \( df \) and the angular resonance frequency \( \omega_0 \) are obtained from the measurement, the constant \( c \) is determined once by a calibration

\(^2\text{From Dual [12, table 2.2] for a fluid with a viscosity of 100 mPas, the value of } df \text{ is approximately } 10 \text{Hz (including } df \text{ of the intrinsic damping), while the resonance frequency is around } 11.330 \text{kHz. The change in oscillation frequency within the bandwidth of the resonance peak is therefore below } 0.1 \%.}
measurement and the fluid density $\rho_f$ must be provided from elsewhere. Note that the angular resonance frequency $\omega_0$ is almost constant, because of the small inertial part of the fluid forces acting on the torsionally vibrating tube compared to the inertia of the tube (see explanation in section 2.1). The expression for the fluid viscosity in Eq. (2.18) then reduces to a relationship between the measured bandwidth $df$ and the fluid density $\rho_f$.

Consider now the flexural oscillations of the tube. The yet unknown relationship between the resonance frequency and the fluid density might depend on the fluid viscosity, the compressibility of the fluid, and on the geometry of the fluid container (if an interaction between the disturbed fluid and the walls of the container takes place). Nonlinear effects like acoustic streaming or relocation of the fluid in axial direction of the tube instead of displacement of the fluid in a plane perpendicular to the axis of the tube could also occur and influence the relationship.

$$f_{res} = f(\rho_f, \eta, compressibility, geometry, \ldots) \quad (2.19)$$

The complexity of the above relationship can depend on the design of the sensor and the container, the oscillation mode, and the types of fluids considered. The degree of complexity that is needed in the model depends on the required accuracy and the measurement range striven for. For example, the design of the sensor affects the resonance frequency, which has an influence on the boundary layer thickness and hence influences the effect of fluid viscosity. Depending on the required accuracy, some of the mentioned effects must be included in the relationship or can be neglected. For small measurement ranges, linearization can simplify the formulation (see for example Rust [43, ch. 2.5]).

In the simplest case, if the relationships in Eqs. (2.18), (2.19) are simple equations relating the measured quantities ($df$ from torsional oscillation, $f_{res}$ from flexural oscillation) with the fluid parameters (density $\rho_f$, viscosity $\eta$), the equations can be solved directly, and the fluid parameters are readily obtained. If the relationships in Eqs. (2.18), (2.19) cannot be solved directly, they can be solved numerically or an iterative evaluation might be useful for a practical implementation. A possible iterative evaluation scheme is given in Fig. 2.5 and could work as follows: starting with an initial guess for the fluid density $\rho_f$, together with the measured bandwidth $df$ of the torsional oscillation, a value for the fluid viscosity $\eta$ can be calculated. Using the measured resonance frequency from the flexural oscillation and the previously calculated viscosity, an updated value for the fluid density can be computed and the next iteration step can begin. Details of the loop and convergence behavior should be discussed for specific cases.

Whether the sensor has to be operated alternately in torsional or flexural oscillation mode or if the two modes can be excited at the same time, should be discussed.

---

3In a practical measurement situation the fluid density may vary between 600 kg/m$^3$ and 1200 kg/m$^3$, while the fluid viscosity may change by several orders of magnitude, e.g. from 1 mPas to 100 mPas or higher. It seems therefore reasonable to start the iterative evaluation scheme with an initial guess for the quantity that shows a smaller spread.
2.3. Concept of combined density/viscosity measurement

Figure 2.5.: Iterative scheme to determine fluid density and viscosity.

later. If the behavior of the mechanical structure and of the surrounding fluid are linear, a superposition of the two motions should in general be possible.

The accuracy of the viscosity measurement discussed above, depends, among other things, on the accuracy of the density measurement. It seems therefore worthwhile to consider the error propagation due to errors in the calculated fluid density. An estimation of the required accuracy for the density measurement can also help to decide how detailed a model for the density calculation must be and which physical effects can be neglected.

Considering Eq. (2.18), the relative error in the viscosity value depends in the following way on the relative errors of the different variables (Busch and Ott [7]):

$$
\frac{\Delta \langle \eta \rangle}{\eta} = \pm \sqrt{\left(2 \cdot \frac{\Delta \langle df \rangle}{df}\right)^2 + \left(2 \cdot \frac{\Delta \langle c \rangle}{c}\right)^2 + \left(\frac{\Delta \langle \rho_f \rangle}{\rho_f}\right)^2 + \left(\frac{\Delta \langle \omega_0 \rangle}{\omega_0}\right)^2}
$$

(2.20)

In which $\langle X \rangle$ is the arithmetic mean value and $\Delta \langle X \rangle$ is the uncertainty of the arithmetic mean value. If the uncertainties in $df$, $c$ and $\omega_0$ are neglected, the relative error in the calculated viscosity value is equal to the relative error in the density measurement:

$$
\frac{\Delta \langle \eta \rangle}{\eta} = \pm \frac{\Delta \langle \rho_f \rangle}{\rho_f}
$$

(2.21)

Relative errors in the calculated viscosity in the order of one percent seem desirable, when compared to existing process viscometers. Consequently the relative errors in the density measurements should be less than one percent$^4$.

While the primary goal is to establish a relationship between the resonance frequency of the flexurally oscillating tube and the fluid density, the relation between the damping and the fluid viscosity/fluid density will also be partly considered in this work. Even though the sensitivity of flexural oscillations on viscosity changes is expected to be low, knowledge of the dependencies could be of use. For instance, if the main interest of an application is the measurement of the fluid density and only a rough estimate of the fluid viscosity is needed, the sensor could be operated

$^4$Note that the limit in the measurement of low viscosities with the torsional oscillation sensor is given by the intrinsic damping of the sensor and its stability with time and temperature. Further it should be kept in mind that Eq. (2.18) is only valid for $\delta/R \ll 1$ and the above error consideration becomes invalid if this condition is not fulfilled, which could occur if the fluid viscosity is too high.
in flexural oscillation mode only. In addition this knowledge could be used as a cross-check for the viscosity measurement.

Considering the measurement of fluid density, an analysis of the error propagation equivalent to Eq. (2.20) can be made, once the relationship in Eq. (2.19) is established. Error considerations for the density measurement are presented in section 5.4.
3. Design and mechanical model of the flexural oscillator

3.1. Sensor design

The starting point of the work presented here, has been a viscosity sensor which emerged from the sensor presented in Dual [12]. In the rest of the text, this sensor will be referred to as the “ETH sensor” because of its development at ETH. The ETH sensor differs from the sensor presented in Dual [12] for example by the excitation and readout mechanism, which changed from a piezoelectric to an electromagnetic mechanism and the design of the inner rod, which changed from a solid rod to a tube. The ETH sensor is depicted schematically in Fig. 3.1.

Figure 3.1.: Schematic representation of the ETH sensor (not to scale). (a) sensor body, (b) outer tube, (c) connecting piece, (d) inner tube, (e) permanent magnet, (f) coil bobbin with coils.

The ETH sensor in Fig. 3.1 consists of a an outer tube (b) which is press-fitted into a sensor body (a). A connecting piece (c) links the outer tube with the inner tube. At the free end of the inner tube a permanent magnet (e) is attached. The coils needed for the electromagnetic excitation and readout of the sensor are held by the coil bobbin (f). All connections between sensor parts are press-fits, and the connections which are in contact with the surrounding fluid are laser welded after mounting, in order to assure tightness.

For the density measurement based on flexural oscillations, it was tried to stay close to the design of the ETH sensor. Thereby the established descriptions and models of the viscosity sensor (based on torsional oscillations) can be transferred easily to the adapted design. A later fusion of the density and viscosity measurements into a combined density/viscosity sensor should then be possible without much difficulty.
Chapter 3. Design and mechanical model of the flexural oscillator

3.1.1. Requirements

The main requirements on the design of the density sensor are:

- Mode separation
- Excitation of flexural and torsional modes
- Measurability (measurement with vibrometer)
- Low mass per length, but not too low
- Low complexity for manageable modeling

These requirements shall be discussed in this section.

For flexural oscillations of a tube with perfect rotational symmetry and isotropic material behavior, the direction of the deflections is arbitrary and given by the excitation. If the tube is not completely rotational symmetric, but shows slightly different bending stiffnesses in two directions, e.g. has a slightly elliptical cross-section, there occur two flexural oscillation modes with slightly different resonance frequencies and with deflections in the directions of the principal axes of the cross-section. In practice, the cross-sections are never perfectly rotational symmetric, which leads to two close flexural resonance frequencies. Because of damping, the resonance peaks have a certain width along the frequency axis and therefore the two resonance peaks can partly overlap (if the resonance frequencies are close enough or damping is large enough). Such a situation can make the evaluation of the resonance peaks (determination of resonance frequency and bandwidth) impossible, when the two peaks cannot be resolved. Damping, when not proportional, can also lead to a coupling of the modes, in which case the system can hardly be described accurately. A solution to this adversity is the deliberate modification of the cross-section such that there are two directions with clearly distinct bending stiffnesses. Thereby the two resonance frequencies can be moved as far apart that no more disturbing interference has to be expected. Some possible modified cross-sections are given in Fig. 3.2.

![Figure 3.2: Possible beam cross-sections suitable for mode separation.](image)

Another idea is to introduce just a section with two distinct bending stiffness directions. Keeping the design of the ETH sensor, one can for example machine a notch into the inner tube (Fig. 3.3). The flexural oscillations, which are combined flexural oscillations of the inner and outer tube, then show two distinct oscillation directions and distinct resonance frequencies. The amount by which the two resonance frequencies are moved apart by the notch depends on its location with respect
3.1. Sensor design

Figure 3.3.: Possible adaption of the inner tube of the ETH sensor (part (d) in Fig. 3.1) in order to achieve mode separation. The depicted notch machined into the surface of the inner tube leads to two distinct stiffness directions regarding the flexural oscillations of the ETH sensor (Fig. 3.1), and therefore enables mode separation.

to the mode shape.

Note that the possible designs presented in Fig. 3.2 and Fig. 3.3, could also be used with torsional oscillations.

The existing electromagnetic excitation and readout mechanism used for the torsional oscillation measurements can easily be modified to suit also flexural oscillations. Dependent on the alignment of the coils in the coil bobbin (part (f) in Fig. 3.1) with the polarization of the permanent magnet (part (e) in Fig. 3.1), either a torsional moment or a bending moment is exerted on the permanent magnet.

The existing readout and evaluation mechanism is based on the gated phase-locked loop as described in patent Goodbread and Dual [19], which eliminates crosstalk between excitation and readout. It works very well for the torsional oscillator, where the resonance frequency of the oscillation is not varying much. In the case of flexural oscillations larger changes in resonance frequency occur and therefore the hardware has to be adjusted depending on the frequency range. Also, a continuous excitation instead of a gated measurement seems easier to be modelled accurately. Therefore, instead of introducing further uncertainties, it seems more safe to base the evaluation of the flexural oscillations on vibrometer measurements, which also eliminates crosstalk. With a vibrometer the velocity of a point on the resonator can be measured with high accuracy, even for very small amplitudes. Vibrometer measurements, when the oscillator is placed in fluid, however, turned out to be difficult. Measurements through the fluid are possible but tedious. The laser beam has to cross a transparent container wall and the fluid, and after reflection on the cylindrical surface of the sensor make its way back to the lens of the vibrometer. Besides the difficulty of proper alignment of laser beam, container wall, and sensor, reflections and scattering occurs at the various surfaces. The preparation of the measurement setup can become very time-consuming, and often the setup is not stable, therefore preventing a reasonable experimental study.

When trying to measure the flexural oscillations on the accessible surface of the inner tube, one faces the obstacle that the amplitudes are very small and the measured signal is noisy. The best possibility seems to be (for the present study) to remove the inner tube of the sensor and measure on the inner surface of the outer tube.
Chapter 3. Design and mechanical model of the flexural oscillator

The resonance frequency of the flexural oscillation changes due to the additional inertia caused by the surrounding fluid. Density changes in the fluid lead to larger changes of the resonance frequency, if the inertia of the sensor is small. Therefore, a tube with low mass per length is beneficial. On the other hand, for the viscosity measurement based on torsional vibrations, small changes in resonance frequency are an advantage (cp. section 2.3; an additional strong dependence on resonance frequency is not desirable). The mass per length of the tube should thus not be as small that the added inertia due to the boundary layer on the surface of the torsionally oscillating tube becomes relevant. Further, the fragility of a very thin tube could be problematic in an industrial application.

It was found that the analytical description of the flexural oscillations of the ETH sensor, with its different parts involved in the oscillations, is difficult. Especially the connection between the inner and outer tube is difficult to represent accurately in a model. Also for an accurate finite element model of the mechanical structure, the various parts of the sensor lead to a large number of elements. If one strives to include also the fluid in the FEM model, the model becomes unmanageably large. In the study of the fluid-structure interaction, a lot of insight could be obtained from a simpler setup, e.g. a setup reduced to the outer tube.

3.1.2. Final sensor design and fabrication

Based on the requirements discussed in section 3.1.1, first experimental and modeling experiences (regarding suitable mode order, required excitation force, measurability), and analytical estimations (regarding resonance frequency, mode separation, required excitation force), a new sensor design as presented in Fig. 3.4 has been chosen.

![Figure 3.4: Final sensor design.](image)

The final sensor design used for this work (Fig. 3.4) consists of a single tube press-fitted into the sensor body. The free end of the tube is closed by a bushing, that also holds a permanent magnet. All connections are press-fits and the connections which are in contact with the surrounding fluid are glued after mounting in order
to assure tightness. The ribs visible in sectional view $A - A$ (Fig. 3.4) ensure the required mode separation. The alignment of the diametrically magnetized permanent magnet and the excitation coil allows to excite flexural oscillations. Note that by adapting the alignment between the permanent magnet and the excitation coil, also torsional vibrations can be excited, and that this sensor design could also be used for viscosity measurements based on torsional oscillations. The inside of the tube is free and thereby enables vibrometer measurements on the inner wall of the tube, at a position with maximal deflection amplitude and consequently a good signal-to-noise ratio. A mirror (provided and held by a Polytec OFV-C-102 Side Exit Head as described in section 3.3.2.5) is necessary to deflect the laser beam to the desired measurement position. The dimension of the outer diameter of the tube stayed the same as the one of the ETH sensor, i.e. 8 mm. The wall thickness of the tube is slightly increased to 0.3 mm instead of 0.25 mm for the ETH sensor. Therefore the mass per length is slightly higher than for the ETH sensor.

The tube is manufactured by turning the outer diameter and the bore for the press fit first and then creating the inner contour by electro-discharge machining. The bushing and the sensor body are manufactured by turning. The permanent magnet has been purchased. All materials are stainless steel, except the magnet which is made of Neodymium Iron Boron.

Details of the final sensor design and the materials can be found in appendix A.1.

3.1.3. Sensor mounting

The mounting of the sensor body should be designed such that the boundary conditions (e.g. geometric position, clamping forces) are clearly defined and reproducible. Ideally it should also minimize the interaction with the environment, e.g. minimize the interaction of clamping forces with the surroundings or minimize the influence of disturbing external shocks and vibrations on the sensor. An interaction of the oscillating tube with the mounting can affect the resonance frequency and the damping of the oscillation. In the case of torsional oscillations the interaction with the surroundings can easily be minimized by introducing a decoupling disk (see Dual [12]). For flexural oscillations, such a decoupling is not readily achieved. Tuning fork configurations are often used for this purpose. Other possibilities are Weisbord or blocking masses, discussed for example in Haueis [26, ch. 6]. Obviously, such means of decoupling are not applicable for the current sensor. The mounting used in this work is a slight modification of a housing obtained from Brookfield Engineering Laboratories Inc. [3], which they use for the clamping of their torsional resonator sensor. The mounting of the sensor body is depicted in Fig. 3.5.

The sensor body (a) is centered and elastically supported in the housing (b) by three rubber O-rings (c), (d), (e). The sensor body is pressed into the housing by a
screwed cover plate (f) and a rubber sheet (g). The housing is centered to the rest of the setup by an O-ring (i) and attached to the rest of the measurement setup by screws (h). All rubber elements are compressed after mounting. The forces are defined by the O-rings and the geometrical dimensions and are reproducible. The decoupling of the flexural oscillations from the surroundings (for different oscillation modes) is not easy to estimate in advance. In section 3.3.2.4, an experimental examination of the influence of the mounting on the flexural oscillations of the sensor is presented.

3.2. Mechanical models of the flexural oscillator

Density measurement could be realized by an empirical approach, for example performing measurements in a set of fluids and a description of the observed behavior by a curve fitted to the data. If one aims to get insight into the physics of the problem, e.g. understand observed phenomena or study dependencies on design and fluid parameters, a model of the system is helpful. This section is dedicated to the modeling of the mechanical structure which forms the basis for the later inclusion of the interaction with the fluid. The main interest lies in the determination of the resonance frequency of the system, because it is the parameter required for the density measurement. Damping of the structure will be considered occasionally. The considered sensor is depicted in Fig. 3.4. The names of the sensor parts introduced in Fig. 3.4 are used throughout the text.

3.2.1. Overview on modeling approaches

Three different models are considered here: two analytical models and a finite element model.

The advantage of analytical models is that the system is described by a mathematical formula, which is usually quickly evaluated. This allows to efficiently carry out parameter studies. The influence of individual geometric and material parameters can easily and quickly be assessed (e.g. influence of the mass and location of the magnet in the sensor). Inclusion or neglect of physical effects can also be studied
3.2. Mechanical models of the flexural oscillator

(e.g. rotary inertia and shear effects in beam bending, clamping of the sensor), thereby allowing to assess the importance of specific physical effects.

FEM models have the advantage that they are in general very flexible and accurate. Many details can be included in an FEM model, which would have to be simplified for an analytical treatment, e.g. the fitting of the magnet in the tube (rigid vs. flexible bodies) or the clamping of the sensor in the sensor body. Inclusion of effects which cannot be accounted for in an analytical model, like the deformation of the beam cross-section under flexure, also lead to more accurate results. On the other hand, accurate FEM models of complex systems lead to high demands on computational power and evaluation is time-consuming. Therefore parameter studies become very time-consuming.

The analytical models presented here, basically describe a beam which is clamped at one end and carries a mass at the free end. The clamping of the beam can be rigid or flexible.

![Sensor and schematic structure of the sensor models. (a) Sensor, (b) Structure of the mobility model, (c) Structure of the DeqD model.](image)

The first analytical model is based on the transfer matrix method. Two matrices are used to model the sensor: one matrix to describe the tube as a beam, and one matrix to describe a rigid mass with rotary inertia which is attached to the free end of the beam (Fig. 3.6, (b)). Note that the rigid mass encompasses the contributions of the magnet, of the bushing and of the part of the tube around the bushing. Taking into account an excitation moment acting on the rigid mass, an expression for the mobility at the free end can be formulated and evaluated. The mobility function allows to determine the resonance frequency and the damping of the system. This model is referred to as mobility model.

The second analytical model considers a beam where a concentrated mass (point mass) and concentrated rotary inertia (point inertia) is introduced at a point near the free end of the beam (Fig. 3.6, (c)). Note that the concentrated mass and concentrated rotary inertia encompasses only the additional mass and additional rotary inertia introduced by the magnet and the bushing compared to the mass and rotary
inertia of the beam. The concentrated mass and concentrated rotary inertia are directly introduced in the differential equations of the beam using distribution theory. Making use of Laplace transforms, a frequency equation is obtained, whose roots give the resonance frequencies of the system. This model is referred to as DeqD model (differential equation with distribution theory).

The finite element model is built in ABAQUS. The solution procedure natural frequency extraction is used to obtain the natural frequencies and the corresponding mode shapes of the system.

It is interesting to compare these three models with each other because they show different levels of detail and complexity. For instance it is interesting to know whether the simple frequency equation from the DeqD model can give similar results for the resonance frequencies as the FEM model or why the different models differ. The two analytical models differ in the way the influence of the magnet and the bushing on the beam vibrations are modeled. In the case of the mobility model, the kinematic transition conditions between the beam and the rigid mass require equal deflection and equal slope. In the case of the DeqD model the concentrated mass/rotary inertia does not cause any kinematic restrictions. Therefore the mode shapes obtained from the two models might differ slightly. Based on the outcome of the models, the sensitivity of the modeling on such details can be assessed.

3.2.2. Flexural oscillations: equations of motion and model parameters

The analytical models rely on bending theory. The Euler Bernoulli beam theory is only applicable for slender beams, i.e. for beams with a thickness which is small compared to a typical wavelength (Sayir and Kaufmann [46], pp. 220-222, Hagedorn and DasGupta [24], p. 144). For thick beams or higher bending modes, the effects of rotary inertia and shear deformation should be taken into account, which are included in the Timoshenko beam theory. For the flexural oscillator considered here, the higher bending modes are of particular interest. The Timoshenko beam theory is therefore adopted in the mobility and the DeqD model.\(^1\)

The derivation of the differential equations describing the Timoshenko beam can be found in many textbooks (Graff [22], Weaver et al. [55], Hagedorn and DasGupta [24], Rao [37]).

In the Timoshenko beam theory the slope of the centroidal axis of the beam \(\frac{\partial y}{\partial x}\) is composed of the slope \(\psi\) due to bending and the slope \(\gamma_0\) due to shearing effects

\[
\frac{\partial y}{\partial x} = \psi + \gamma_0 \tag{3.1}
\]

\(^1\)With an outer diameter of 8 mm and a length of \(L = 91.22\) mm of the oscillating tube, the third bending mode has a thickness-wavelength ratio of about 0.11, because from Euler-Bernoulli beam theory, for a fixed-free beam, and for the third bending mode \((2 \cdot \pi /\text{wavelength}) \cdot L = 7.855\).
The coupled differential equations describing the deflection of the centroidal axis \( y(x,t) \) and the slope due to bending \( \psi(x,t) \) of a uniform beam are given by

\[
AG\kappa \left( \frac{\partial \psi}{\partial x} - \frac{\partial^2 y}{\partial x^2} \right) + \rho A \frac{\partial^2 y}{\partial t^2} = q(x,t) \tag{3.2a}
\]

\[
AG\kappa \left( \frac{\partial y}{\partial x} - \psi \right) + EI \frac{\partial^2 \psi}{\partial x^2} - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{3.2b}
\]

In which \( E \) is Young’s modulus, \( I \) is the area moment of inertia, \( \rho \) is the density of the beam material, \( A \) is the area of the cross-section, \( \kappa \) is the shear correction factor, \( G \) is the shear modulus, \( q(x,t) \) is an external distributed load, \( x \) is the position along the undeformed centroidal axis, and \( t \) denotes the time. The shear correction factor depends on the geometry of the cross-section and Poisson’s ratio (Hutchinson [28], [27]).

From the equations above, a differential equation describing solely the deflection \( y(x,t) \) of the centroidal axis of the beam can be derived

\[
EI \cdot \frac{\partial^4 y}{\partial x^4} + \rho A \cdot \frac{\partial^2 y}{\partial t^2} - \rho I \cdot \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\rho IE}{\kappa G} \cdot \frac{\partial^4 y}{\partial x^2 \partial t^4} + \frac{\rho^2 I}{\kappa G} \cdot \frac{\partial^4 y}{\partial t^4} = q(x,t) + \frac{\rho I}{AG\kappa} \cdot \frac{\partial^2 q(x,t)}{\partial t^2} \tag{3.3}
\]

For a beam without external distributed load, the differential equation reduces to

\[
Euler-Bernoulli \quad \text{rotary inertia} \quad \text{shearing deformation}
\]

\[
EI \cdot \frac{\partial^4 y}{\partial x^4} + \frac{\rho A \cdot \partial^2 y}{\partial t^2} - \frac{\rho I \cdot \partial^4 y}{\partial x^2 \partial t^2} - \frac{\rho IE}{\kappa G} \cdot \frac{\partial^4 y}{\partial x^2 \partial t^4} + \frac{\rho^2 I}{\kappa G} \cdot \frac{\partial^4 y}{\partial t^4} = 0 \tag{3.4}
\]

In Eq. (3.4), the correction terms accounting for the effects of rotary inertia and shear deformation, which improve the Euler-Bernoulli beam theory, are visible.

The expressions for the bending moment and the shear force, which are needed for the formulation of boundary conditions, are given by

\[
M = EI \cdot \frac{\partial \psi}{\partial x} \quad \text{and} \quad Q = AG\kappa \cdot \gamma_0 = AG\kappa \cdot \left( \frac{\partial y}{\partial x} - \psi \right) \tag{3.5}
\]

The accurate determination of the geometrical and material parameters in the Timoshenko beam model is crucial. Therefore the above equations have been reformulated as functions of new parameters that seem more accessible to an experimental parameter identification. The experimental parameter identification is discussed in section 3.3.1. By defining

\[
\gamma_0^2 = \frac{E}{\rho} \quad \gamma_y^2 = \frac{I}{A} \quad \bar{m} = \rho A \quad s^2 = \frac{E}{\kappa G} \tag{3.6}
\]
one can write Eq. (3.4) as
\[ c_0^2 \cdot \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} \cdot \frac{1}{j_y^2} (1 + s^2) + \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{j_y^2 \cdot s^2}{c_0^2} \cdot \frac{\partial^4 y}{\partial t^4} = 0 \] (3.7)

and for the bending moment and shear force
\[ M = c_0^2 j_y^2 \bar{m} \cdot \frac{\partial \psi}{\partial x} \quad \text{and} \quad Q = \frac{c_0^2 \bar{m}}{s^2} \cdot \left( \frac{\partial y}{\partial x} - \psi \right) \] (3.8)

The parameter \( c_0 \) is the longitudinal wave speed, depends only on material parameters, and can easily be determined experimentally. Parameter \( j_y \) is called radius of gyration and depends only on the geometry of the beam cross-section. Parameter \( s \) depends on material parameters \( E \) and \( G \), and by the shear factor \( \kappa \) also on the geometry of the cross-section and the Poisson’s ratio. The parameter \( \bar{m} \) is the mass per length of the beam, which is easy to measure experimentally.

Of course, the parameters in Eq. (3.4) could be grouped differently. An appealing formulation is given in Hagedorn and DasGupta [24, Eq. (3.141)], where the shear wave speed \( c_s^2 = G/\rho \) is introduced
\[ c_0^2 c_s^2 \kappa \cdot \frac{\partial^4 y}{\partial x^4} + \frac{j_y^2 \kappa}{c_0^2} \cdot \frac{\partial^2 y}{\partial t^2} \cdot \left( c_0^2 + c_s^2 \right) \cdot \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial^4 y}{\partial t^4} = 0 \] (3.9)

For circular cylindrical beams the shear wave speed \( c_s \) could easily and accurately be determined from a torsional vibration experiment. However, for general cross-sections the shear wave speed is given as \( c_2 = \sqrt{G \cdot I^* / (\rho \cdot I_p)} \) in which \( I_p \) is the polar moment of area and \( GI^* \) is called torsional rigidity, in which \( I^* \) depends on the geometry of the cross-section (Sayir et al. [45]). An accurate determination of \( I^* \) is complicated.

### 3.2.3. Mobility model (Transfer matrix method and mobility function)

The concept of impedance and mobility in the description of structural vibrations is discussed in Gardonio and Brennan [16], Fahy and Gardonio [14]. They present various impedance and mobility expressions for lumped elements (e.g. spring, damper, mass) and distributed structural elements (e.g. rod, shaft, beam, plate). Gardonio and Brennan [16] presents impedance, mobility, and transfer matrices for Euler-Bernoulli beams. The derivation of various transfer mobilities for Euler-Bernoulli beams is shown in detail in Fahy and Gardonio [14]. A method based on the transfer matrix technique for the analysis of Euler-Bernoulli and Timoshenko beams with non-proportional damping, and which can be connected to lumped elements and other beams, is presented in Sorrentino et al. [48],[47].

The method presented in Sorrentino et al. [48], [47], is similar to the procedure presented here. The procedures differ for instance by the state vector considered. In
Sorrentino et al. [48], [47] the deflection and its first three derivatives are considered, while usually in transfer matrix methods the state vector consists of kinematic quantities (e.g. deflection, slope, or corresponding velocities) and kinetic quantities (e.g. bending moment, shear force) (Gardonio and Brennan [16], Fahy and Gardonio [14], Nashif [35]). Another difference pertains to the angular frequency. In Sorrentino et al.[48], [47] the oscillations are described by a complex angular frequency, which includes the oscillation frequency and the damping of the oscillation. In contrast, in this work the resonance frequency is always treated as a real quantity. Employing a complex angular frequency seems unfavorable when the influence of a surrounding fluid shall later be included in the model. As the descriptions of the fluid forces acting on a continuously oscillating beam are usually expressed using a real angular frequency (Retsina et al. [39], Sader [44], Kadyrov et al. [30]), they cannot readily be combined with a model using a complex angular frequency.

For the derivation of the mobility model, the sensor is first divided in two parts: a beam, and a rigid mass attached to it. The mass and rotary inertia of this rigid mass comprises the contributions of the magnet, of the bushing and of the part of the tube around the bushing. Two transfer matrices are then derived, which relate kinematic and kinetic quantities between both ends of the beam, and between the location of the driving force and the attached end of the rigid mass. The transfer matrices can be combined to get the transfer matrix of the whole system. After introducing the boundary conditions, the desired mobility function can be derived.

The kinematic and force variables between the parts and the environment are defined in Figs. 3.7 and 3.8.

![Figure 3.7.](image)

Figure 3.7.: Definition of kinematic and force variables for the beam in the mobility model. $M_1$: bending moment, $Q_1$: shear force, $\psi_1$: slope due to bending, $y_1$: beam deflection.

Consider first the derivation of the transfer matrix for the beam. Introducing the ansatz $y(x,t) = e^{k_1 x} \cdot e^{i\omega t}$ for harmonic beam vibrations with angular frequency $\omega$ into Eq. (3.4), the solution can be written as

$$y(x,t) = (a_1 \cdot e^{k_1 x} + a_2 \cdot e^{k_2 x} + a_3 \cdot e^{k_3 x} + a_4 \cdot e^{k_4 x}) \cdot e^{i\omega t} \quad (3.10)$$
Figure 3.8.: Definition of kinematic and force variables for the rigid mass in the mobility model. $M_a, Q_a$: moment and shear force acting on the rigid mass at position $x = 0$, $\eta_a$: deflection of the rigid mass at position $x = 0$, $M_s, Q_s$: moment and shear force acting on the center of mass of the rigid mass, $\eta_s$: deflection of the center of mass of the rigid mass, $\varphi$: rotation of the axis of the rigid mass, $x_s$: $x$-coordinate of the center of mass $S$.

with wavenumbers $k$ given by

$$k_{1,2} = \sqrt{-B_1 \pm \sqrt{B_1^2 - 4 \cdot B_2}}$$
$$k_{3,4} = -k_{1,2}$$

in which

$$B_1 = \frac{(1 + s^2)\omega^2}{\nu_0^2} \quad \text{and} \quad B_2 = \frac{s^2\omega^4}{\nu_0^4} - \frac{\omega^2}{\nu_0^2\nu_y^2}$$

Based on this solution the transfer matrix of the beam ($\text{beamTM}$) can be formulated as

$$\begin{pmatrix} Q_2 \\ M_2 \\ \psi_2 \\ y_2 \end{pmatrix} = \begin{bmatrix} \text{beamTM} \end{bmatrix} \begin{pmatrix} Q_1 \\ M_1 \\ \psi_1 \\ y_1 \end{pmatrix}$$

in which $\text{beamTM}$ is a $4 \times 4$-matrix that relates the kinetic and kinematic variables from one end of the beam with the other. The matrix $\text{beamTM}$ is given in appendix A.2.

For the rigid mass, a similar transfer matrix ($\text{rmTM}$) can be formulated, that relates the kinetic and the kinematic variables at the center of mass of the rigid mass to the attached end of the rigid mass

$$\begin{pmatrix} Q_a \\ M_a \\ \varphi_a \\ \eta_a \end{pmatrix} = \begin{bmatrix} \text{rmTM} \end{bmatrix} \begin{pmatrix} Q_a \\ M_a \\ \varphi_a \\ \eta_a \end{pmatrix}$$
The $4 \times 4$-matrix $rmT M$ is given in appendix A.3.

Taking into account the transition conditions between the beam and the rigid mass (i.e. $Q_2 = Q_a$, $M_2 = M_a$, $\psi_2 = \varphi_a$, $y_2 = \eta_a$), considering an excitation moment $M_{exc}$ acting on the rigid mass, and employing the boundary conditions (at the fixed beam end: $\psi_1 = y_1 = 0$; at the rigid mass: $Q_s = 0$), the system can described by

$$
\begin{pmatrix}
0 \\
M_{exc} \\
\varphi_s \\
\eta_s
\end{pmatrix} =
\begin{bmatrix}
rmT M & \text{beamTM} \\
\end{bmatrix}
\begin{pmatrix}
Q_1 \\
M_1 \\
0 \\
0
\end{pmatrix} \quad (3.15)
$$

An advantage of this procedure to obtain the overall system equations lies in its modularity. Several beam elements could easily be connected together, without increasing the dimension of the matrix which describes the whole system. Adaptions on the different elements can be carried out without need to change the evaluation routine.

The system described by Eq. (3.15) can be solved for the unknown variables, from which the following driving point mobility function can be derived

$$
Y_{drive}(\omega) = \frac{i\omega \varphi_s}{M_{exc}} \quad (3.16)
$$

which describes the frequency dependent relationship between the rotational velocity of the rigid mass and the driving moment.

Material damping has been included in the model using a hysteretic damping model, in which simply the Young’s modulus $E$ is replaced by a complex modulus $E_d = E \cdot (1 + i/Q_{mat})$ in which $Q_{mat}$ is the quality factor of the material. Note that a complex Young’s modulus leads to a complex parameter $c_0$ in Eq. (3.6), but does not affect parameter $s$ in Eq. (3.6), because $G = E/(2 \cdot (1 + \nu))$ for a linear elastic isotropic material.

A resilient mounting can be included in the model. If the mounting is modeled by rotational and linear springs with spring stiffnesses $k_r$ and $k_l$ according to Fig. 3.9, the resilient mounting is included by replacing in Eq. (3.13)

$$
\begin{pmatrix}
Q_1 \\
M_1 \\
\psi_1 \\
y_1
\end{pmatrix}
\quad \text{by} \quad
\begin{pmatrix}
k_l \cdot y_1 \\
k_r \cdot \psi_1 \\
\psi_1 \\
y_1
\end{pmatrix} \quad (3.17)
$$

Based on the driving point mobility, the amplitude-frequency and phase-frequency curves of the system can be calculated numerically. The resonance frequencies and damping values can then easily be determined according to Eqs. (2.13) and (2.14).

All calculations are carried out with Wolfram Mathematica.
3.2.4. DeqD model (Differential equation and distribution theory)

The derivation of the DeqD model follows the procedure presented in Grant [23]. In contrast to Grant [23], also the rotary inertia of the concentrated mass is considered, and a resilient clamping can be included in the model.

In the DeqD model the sensor is modeled as a cantilever beam carrying a concentrated mass (point mass) and a concentrated rotary inertia (point inertia), which describe the influence of the magnet and the bushing (Fig. 3.6). The influences of the concentrated mass and the concentrated rotary inertia can directly be included in the differential equations for the Timoshenko beam, Eqs. (3.2), using distribution theory. For a beam without external distributed load, these equations are given as

\[
AGK \left( \frac{\partial \psi}{\partial x} - \frac{\partial^2 y}{\partial x^2} \right) + \left( \rho A + m_c \delta(x - x_c) \right) \frac{\partial^2 y}{\partial t^2} = 0 \quad (3.18a)
\]

\[
AGK \left( \frac{\partial y}{\partial x} - \psi \right) + E I \frac{\partial^2 \psi}{\partial x^2} - \left( \rho I + J_c \delta(x - x_c) \right) \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (3.18b)
\]

in which \( m_c \) and \( J_c \) are the concentrated mass and the concentrated rotary inertia acting at position \( x = x_c \) along the centroidal axis of the beam, and \( \delta() \) represents the Dirac delta function.

The concentrated mass \( m_c \) and the concentrated rotary inertia \( J_c \) describe the additional mass and additional rotary inertia to a uniform beam which is extended over the whole length of the sensor. The concentrated mass \( m_c \) is determined as the mass of the bushing (extended only to the inner radius of the tube) plus the mass of the magnet, and minus the mass of the ribs of the tube (along the length of the bushing). The concentrated rotary inertia \( J_c \) is determined as the mass moment of inertia with respect to the center of mass of the parts included in \( m_c \), i.e. the mass moment of inertia of the bushing (extended only to the inner radius of the tube) plus the mass moment of inertia of the magnet, and minus the mass moment of inertia of the ribs of the tube (along the length of the bushing). Note that there is a clearance of 0.14 mm between the length of the bushing and the length of the hole in the tube which is not accounted for in the model (see detailed drawings of the sensor in appendix A.1, nominal dimension 5.7 in Fig. A.2, and nominal dimension 5.6 in Fig. A.1, where the actual dimensions are 5.74 and 5.60).

In Grant [23] the equations are rewritten in non-dimensional form. Analogously
3.2. Mechanical models of the flexural oscillator to Grant [23], considering harmonic motion \(y = Ye^{i\omega t}, \psi = \Psi e^{i\omega t}\) and introducing the non-dimensional length \(\zeta = x/L\) (\(\omega\): angular frequency of oscillation, \(L\): length of the beam), Eqs. (3.18) can be expressed as

\[
\ddot{\psi} - \frac{1}{\zeta^2} \left[ 1 - b^2 \dot{s}^2 - b^2 \dot{s}^2 Z \right] \psi + \frac{\dot{Y}}{L} = 0 \tag{3.19a}
\]

\[
Y'' + \left[ b^2 \dot{s}^2 + b^2 \dot{s}^2 Z \right] Y - L \ddot{\psi} = 0 \tag{3.19b}
\]

in which the primes denote differentiation with respect to \(\zeta\). The non-dimensional parameters \(b, r, s, a, m, p\) can be expressed using the parameters defined in Eqs. (3.6) and are given below. Note that the parameter \(p\) (which describes the concentrated rotary inertia) is not present in Grant [23].

\[
\begin{align*}
b &= \omega L^2/(c_0 j) \\
r &= jy/L \\
s &= jy/L \\
a &= x_c/L \\
m &= m_c/(\overline{m} L) \\
p &= J_c/(\overline{m} L j_y) \\
\end{align*}
\]

Analogously to Grant [23], Eqs. (3.19) can be solved using Laplace transformation. Taking into account the boundary conditions and consistency equations at the position of the concentrated mass/rotary inertia, leads to a set of four linear homogeneous equations in four unknowns. Putting these equations in matrix form and considering that the determinant of the matrix must be zero for a non-trivial solution, leads to a frequency equation. The roots of this frequency equation represent the natural frequencies of the beam.

The frequency equation depends on the following parameters:

\[
\text{frequency equation} = f(\tilde{\beta}, r, s, m, p, a, L, \gamma)
\]

in which

\[
\tilde{\beta} = \frac{1}{\sqrt{2}} \cdot \sqrt{(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + 4\gamma}}
\]

and \(\gamma = 1\) for a beam in vacuum.

The expression for the frequency equation is given in appendix A.4.

Material damping is not included in the DeqD model.

A resilient foundation can be considered by modifying the boundary conditions. If the mounting is modeled by rotational and linear springs with spring stiffnesses \(k_r\) and \(k_t\) according to Fig. 3.9, the boundary conditions at \(\zeta = 0\) \((x = 0)\) are

\[
\Psi(0) = \frac{f_r}{L} \Psi'(0) \quad \text{and} \quad Y(0) = \frac{f_t}{L} Y'(0) - \frac{f_r f_t}{L} \Psi'(0) \tag{3.20}
\]

in which

\[
\begin{align*}
f_r &= \overline{m} c_0^2 j_y^2/k_r \\
f_t &= \overline{m} c_0^2 s^2 k_y
\end{align*}
\]

Note that setting \(f_r = f_t = 0\) leads to the boundary conditions of a clamped end
(k_r \to \infty, k_i \to \infty).

All calculations are carried out with *Wolfram Mathematica*.

### 3.2.5. FEM model

The density sensor consists of several parts (Fig. 3.4), namely the tube, the bushing, the magnet, and the sensor body. In order to achieve a good agreement between the model and the physically present sensor, the FEM model is built up step by step. Thereby the different stages of the model can be compared with experimental results and model deficiencies can be identified directly. This step by step procedure is also in accordance with the experimental parameter determination described in the next section.

The following (sub-)models are considered:

a) The tube alone, with the cross-section shown in Fig. 3.4 (sectional view A-A) running throughout the whole length of the tube, i.e. without the bore for the bushing. The tube is modeled completely free, without any external constraints.

b) The tube with mounted bushing and magnet. The tube is modeled completely free, without any external constraints.

c) The tube with bushing and magnet mounted in the sensor body. The sensor body is modeled completely free, without any external constraints.

d) The tube with bushing and magnet is shortened by the part of the tube which is lying inside the sensor body. The newly obtained cross-section at the transition to the sensor body is rigidly clamped.

For each additionally included part in the above models, a mesh convergence study has been conducted in order to obtain a sufficient accuracy with a manageable model size (number of degrees of freedom).

The models a), b), c) can be directly compared to experiments. For that purpose the natural frequencies of the lowest bending modes, and for deflections in both principal axes directions of the cross-section, shall be obtained from the FEM model and can later be compared to experimental results (see section 3.3.2).

A priori it is not clear how detailed the clamping of the tube in the sensor body and the support of the sensor body in the housing must be modeled, and how these details affect the resonance frequencies predicted by the FEM model.

The experimental investigation presented in the next section shows, that for higher oscillation modes the influence of the mounting of the sensor body in the housing is negligible. It is also found, that for the higher oscillation modes the agreement between model c) and the experiment is very good. Because the current sensor will be operated with higher bending modes, it is concluded that the support
of the sensor body in the housing will not be included in the FEM models and that no further refinement of the modeled mounting of the tube in the sensor body is necessary.

During later studies (see section 4.2.3) it has been found, that the details of the mounting of the sensor do not matter for the comparison of the resonance frequencies of the density sensor in fluid obtained from model predictions or from measurements, if normalized quantities are compared to each other (normalization of the resonance frequencies in fluid with the resonance frequencies in air/vaccum). Because of this finding, the model d) described above is used for later simulations of the sensor in fluid.

All the sensor parts are discretized by 20-node quadratic brick (hexahedral) elements with reduced integration. In model d) the tube consists of 181184 elements, the bushing of 3328 elements, and the magnet of 1440 elements. Connections between the parts of the sensor are defined as surface-based mesh tie constraints (The two surfaces in contact are divided into a master and a slave surface depending on which mesh is coarser. The nodes of the slave surface are constrained to follow the motion of the master surface.). Fig. 3.10 shows a detail of the meshed cross-section of the tube.

![Figure 3.10.: Detail of the meshed cross-section of the tube.](image)

The material parameters required for the simulation are mass density \( \rho \), Young's modulus \( E \), and Poisson's ratio \( \nu \). The determination of these parameters is discussed in the next section. Material damping is not included in the models.

The analysis procedure 'natural frequency extraction' is used which performs an eigenvalue extraction to determine the natural frequencies and the corresponding mode shapes. Fig. 3.11 shows the mode shapes of the four lowest flexural modes obtained from model c) discussed above.

All FEM modeling presented here is done with the simulation software Abaqus 6.10-EF1.
3.2.6. Treatment as 1 dof oscillator

In section 2.2 the 1 dof oscillator has been discussed as the basis to evaluate and interpret the measurements. Using the modal analysis procedure to solve the differential equations for the Euler Bernoulli or the Timoshenko beam, it can be shown that these beams behave like a 1 dof oscillator, and expressions for the corresponding modal masses and modal stiffnesses can be obtained. The modal analysis procedure will also later prove useful for the solution of the inverse problem (section 5.1.2).

The steps to transform the solution of the Euler Bernoulli beam into the form of a 1 dof oscillator can be found in Fahy and Gardonio [14, ch. 2.3.3]. For convenience and later reference, the main steps of this procedure are given here.

Starting from the differential equation for the Euler Bernoulli beam under the influence of a harmonic excitation force $f_0 e^{i\omega t}$ with amplitude $f_0$ and acting at position $x_e$

$$\rho A y_{tt} + EI y_{xxxx} = f_0 \delta(x - x_e) e^{i\omega t} \tag{3.22}$$

and using the modal summation ansatz (Fahy and Gardonio [14, Eq. (2.36)])

$$y(x, t) = \Re\{\sum_{n=1}^{\infty} \phi_n(x) \tilde{q}_n(\omega) e^{i\omega t}\} \tag{3.23}$$

in which (Fahy and Gardonio [14]): “$\phi_n(x)$ represents a real modal function of the $n$-th natural mode of the beam and $\tilde{q}_n(\omega)$ represents the corresponding complex modal amplitude”. The operation $\Re\{\}$ is omitted in the following equations.

Inserting this ansatz into the differential equation, multiplying both sides of the equation with mode shape $\phi_m(x)$ and integrating both sides over the length $L$ of the beam yields

$$\sum_{n=1}^{\infty} \left( EI \tilde{q}_n \int_0^L \phi_n'' \phi_m dx - \omega^2 \rho A \tilde{q}_n \int_0^L \phi_n \phi_m dx \right) = \int_0^L f_0 \delta(x - x_e) \phi_m dx \tag{3.24}$$

Considering the following orthogonality relations between the natural modes of
3.2. Mechanical models of the flexural oscillator

the beam (which hold for clamped, free, and simply supported boundary conditions)

\[ \int_0^L \phi_i \phi_j \, dx = 0 \quad \text{if } (i \neq j) \quad (3.25) \]

\[ \int_0^L \phi''_i \phi_j \, dx = 0 \quad \text{if } (i \neq j) \quad (3.26) \]

it is clear that the left hand side vanishes if \( m \neq n \). For \( m = n \) one obtains

\[ K_n \ddot{q}_n - \omega^2 M_n \ddot{q}_n = F_n \quad (3.27) \]

in which

\[ K_n = E I \int_0^L \phi''_n \phi_n \, dx \quad (3.28) \]

\[ M_n = \rho A \int_0^L \phi''_n \, dx \quad (3.29) \]

\[ F_n = f_0 \phi_n (x_e) \quad (3.30) \]

where \( K_n, M_n, F_n \) are called the modal stiffness, the modal mass, and the modal force respectively. Eq. (3.27) is of the same form as the equation of a 1 dof oscillator.

By defining the natural frequency as \( \omega_n = \sqrt{K_n/M_n} \), the (Fahy and Gardonio [14]): “complex displacement response of the beam at position \( x \)” is therefore given by (Fahy and Gardonio [14])

\[ \ddot{y}(x, \omega) = \sum_{n=1}^{\infty} \phi_n(x) \ddot{q}_n(\omega) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(x_e)}{M_n(\omega_n^2 - \omega^2)} \cdot f_0 \quad (3.31) \]

This equation expresses the response to the harmonic excitation as a sum of the responses of the individual natural modes of the beam. The response of each single mode is expressed in a form according to the 1 dof oscillator as presented in section 2.2.

According to Fahy and Gardonio [14], the effect of light damping can be taken into account in the above derivation. Describing the structural damping using a hysteretic damping model, the response of the damped beam is simply obtained by replacing the modal stiffness by a complex stiffness \( K_n \cdot (1 + i \eta_{loss}) \) (\( \eta_{loss} \): loss factor). The complex displacement response is then given by

\[ \ddot{y}(x, \omega) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(x_e)}{M_n[\omega_n^2 (1 + i \eta_{loss}) - \omega^2]} \cdot f_0 \quad (3.32) \]

where the description of the contributing modes has the same form as the 1 dof oscillator in Eq. (2.9).

A modal analysis solution procedure, analogously to the one described above for
the Euler Bernoulli beam, is also possible for the case of the Timoshenko beam. However, there is an important difference concerning the orthogonality relations.

Considering the coupled differential equations of the Timoshenko beam Eq. (3.2) and assuming harmonic motion of the form \( y(x, t) = Y(x)e^{i\omega t} \) and \( \psi(x, t) = \Psi(x)e^{i\omega t} \), the equations can be expressed conveniently in operator form (Han et al. [25]).

\[
\begin{bmatrix}
-AG\kappa \frac{\partial^2}{\partial x^2} & AG\kappa \frac{\partial}{\partial x} \\
-AG\kappa \frac{\partial}{\partial x} & -EI\frac{\partial^2}{\partial x^2} + AG\kappa
\end{bmatrix}
\begin{bmatrix}
Y \\
\Psi
\end{bmatrix}
-\omega^2
\begin{bmatrix}
\rho A & 0 \\
0 & \rho I
\end{bmatrix}
\begin{bmatrix}
Y \\
\Psi
\end{bmatrix}
= \begin{bmatrix}
q(x, t) \\
0
\end{bmatrix}
\]

Therefore

\[
L(\bar{W}) - \omega^2 M(\bar{W}) = \bar{Q} \quad \text{in which} \quad \bar{W} = (Y(x) \quad \Psi(x))^T
\]

(3.33)

In contrast to the situation with the Euler Bernoulli beam, the orthogonality relations for the Timoshenko beam are with respect to the operators \( L \) and \( M \). The orthogonality relations (which hold for clamped, free, and simply supported boundary conditions) are given as (Han et al. [25])

\[
\int_0^L \bar{W}_i^T M(\bar{W}_j) \, dx = 0 \quad \text{if} \quad (i \neq j)
\]

(3.35)

\[
\int_0^L \bar{W}_i^T L(\bar{W}_j) \, dx = 0 \quad \text{if} \quad (i \neq j)
\]

(3.36)

Searching again for the response to a harmonic excitation which acts at position \( x_e \), one can set \( q(x, t) = f_0 \delta(x - x_e)e^{i\omega t} \) in Eq. (3.33) and use the modal summation ansatz

\[
u(x, t) = \sum_{n=1}^{\infty} \bar{W}_n(x) \tilde{q}_n(\omega)e^{i\omega t}
\]

(3.37)

in which \( \bar{W}_n(x) = (Y_n(x) \quad \Psi_n(x))^T \) is the vector of the real \( n \)-th eigenfunctions of the beam (without external distributed load) and \( \tilde{q}_n(\omega) \) is the corresponding complex modal amplitude.

Proceeding similarly as for the case of the Euler Bernoulli beam one obtains

\[
K_n \tilde{q}_n - \omega^2 M_n \tilde{q}_n = F_n
\]

(3.38)

in which

\[
K_n = \int_0^L \bar{W}_n^T L(\bar{W}_n) \, dx
\]

(3.39)

\[
M_n = \int_0^L \bar{W}_n^T M(\bar{W}_n) \, dx
\]

(3.40)

\[
F_n = f_0 Y_n(x_e)
\]

(3.41)
where $K_n$, $M_n$, $F_n$ are the modal stiffness, the modal mass, and the modal force of the Timoshenko beam respectively. Eq. (3.38) has the same form as the equation of a 1 dof oscillator.

By defining the natural frequency as $\omega_n = \sqrt{K_n/M_n}$, the complex displacement response of the Timoshenko beam is given by

$$\tilde{w}(x, \omega) = \sum_{n=1}^{\infty} \tilde{W}_n(x) \tilde{\phi}_n(\omega) = \sum_{n=1}^{\infty} \frac{\tilde{W}_n(x) Y_n(x_e)}{M_n(\omega_n^2 - \omega^2)} \cdot f_0$$

Finally, the case of an Euler Bernoulli beam with a concentrated mass $m_c$ and a concentrated rotary inertia $J_c$ at position $x_c$ under the influence of a harmonic excitation force $f_0 e^{i\omega t}$ acting at position $x_e$ shall be considered. According to Pan [36], the differential equation can be written as

$$EIy_{xxxx} - \frac{\partial}{\partial x} \left( j_c \delta(x - x_c) y_{xtt} \right) + (\rho A + m_c \delta(x - x_c)) y_{tt} = f_0 \delta(x - x_c) e^{i\omega t}$$

In order to bring the solution into the form of a 1 dof oscillator, one can follow the procedure presented above. Only the orthogonality relation Eq. (3.25) must be replaced by a modified orthogonality relation (Pan [36])

$$\rho A \int_0^L \phi_i \phi_j dx + m_c \phi_i(x_c) \phi_j(x_c) + j_c \phi_i'(x_c) \phi_j'(x_c) = 0 \quad \text{if} \ (i \neq j)$$

One obtains for the complex displacement response

$$\tilde{y}(x, \omega) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n(x_e)}{(M_n + M_c + J_c)(\omega_n^2 - \omega^2)} \cdot f_0$$

in which the natural frequency is defined as $\omega_{nc} = \sqrt{K_n/(M_n + M_c + J_c)}$ and in which

$$K_n = EI \int_0^L \phi_n'' \phi_n dx$$

$$M_n = \rho A \int_0^L \phi_n^2 dx$$

$$M_c = m_c (\phi_n(x_c))^2$$

$$J_c = j_c (\phi_n'(x_c))^2$$

According to this equation, the influence of the concentrated mass/rotary inertia depends on the position along the beam and on the oscillation mode. The same expression for $\omega_{nc}$ could be obtained from the Rayleigh-Ritz method using the actual modal functions as trial functions and applying the orthogonality relation in Eq. (3.44).
Chapter 3. Design and mechanical model of the flexural oscillator

3.3. Experimental characterization of the flexural oscillator without fluid

3.3.1. Experimental determination of model parameters

The models introduced in the preceding section rely on several geometric and material parameters. An accurate determination of these parameters is crucial, but not all parameters can be determined directly. The determination of geometry parameters showed to be especially error-prone. Because of the small geometric dimensions, even small measurement errors can have a significant influence. The dimensions of the cross-section of the tube are hard to be measured accurately, due to accessibility with measurement instruments, and also elastic deformation of the cross-section by the micrometer caliper can occur. The radii and chamfers of the different parts are hard to capture. As an example, consider the cross-sectional area of the tube. If the outer radius is taken 0.01 mm larger than specified (i.e. 4.01 mm instead of 4 mm), the calculated cross-sectional area is already 3\% higher than according to specification.

The model parameters should therefore be chosen such that they can be determined accurately. In Eqs. (3.6) a suitable set of parameters for the analytical models has been stated. The parameters required for the different models are (see section 3.2):

- Mobility model: \( c_0, j_y, \bar{m}, s, L, m, I_s, x_s \)
- DeqD model: \( c_0, j_y, \bar{m}, s, L, m_c, J_c, x_c \)
- FEM model: \( \rho, E, \nu \) and geometry of the parts

The determination of the above parameters is presented in this section.

As discussed earlier, the sensor can perform flexural oscillations in the directions of both principal axes of the cross-section of the tube. The analytical models considered here (mobility and DeqD model) are used to model only flexural oscillations with deflections in \( Y \)-direction with respect to the coordinate system defined in Fig. 3.12. In contrast, the considered FEM model can describe flexural oscillations in the directions of both principal axes of the tube cross-section. In this text, flexural oscillations with deflections in \( Y \)- or \( X \)-direction are denoted as \( Y \)-bending or \( X \)-bending modes.

Consider first the distribution of mass of the tube. A direct determination of the material density is difficult. For instance when using the relationship "density=mass/volume", the mass of the tube can be determined accurately by weighing, but the determination of the volume is error-prone. As discussed above, the cross-sectional area cannot accurately be determined by measuring the geometrical dimensions. Another attempt to determine the volume of the tube from the buoyancy force on the tube immersed in water, also did not give satisfying results. The
3.3. Experimental characterization of the flexural oscillator without fluid

results showed larger scattering, most probably due to air bubbles. Luckily the analytical models can be formulated dependent on parameter \( \bar{m} = \rho A = m/L \), which can be accurately determined from the weighed mass \( m \) and the length \( L \) of the tube\(^2\). The experimentally determined value is \( \bar{m} = 0.06053 \text{kg/m} \) \((m = 5.993 \text{g}, \ L = 99.02 \text{mm})\).

The longitudinal wave speed \( c_0 \) can accurately be determined from longitudinal vibrations of the tube\(^3\). It should be noted though, that for higher longitudinal modes inertia effects can become important (Graff [22, ch. 2.5.3]). It is also noted in Sayir and Kaufmann [46, ch. 35.2], that the equations for the description of the longitudinal vibrations used below are only valid for thin rods, and consequently longitudinal wavelengths should be larger than 5 to 6 times the thickness of the rod (otherwise, lateral strain effects affect the vibrations). Therefore, only the lowest longitudinal mode is evaluated.

The longitudinal wave speed \( c_0 \) is related to the resonance frequency \( f_n \) of the free-free longitudinal mode \( n \) by \( c_0 = f_n 2L/n \). For the lowest longitudinal mode one obtains \( c_0 = 4869.68 \text{m/s} \) \((f_1 = 24589.4 \text{Hz}, \ L = 99.02 \text{mm})\).

The parameters \( j_y \) and \( s \) have been defined in Eq. (3.6). The parameter \( j_y \) depends on the area moment of inertia \( I \) and the cross-sectional area \( A \), and therefore solely on the geometry of the cross-section. It must be noted though, that the area moment of inertia \( I \) depends on the direction of the flexural oscillations with respect to the principal axes of the cross-section. Consequently the \( Y \)- and \( X \)-bending modes of current sensor have different values for the area moment of inertia \( I \) and therefore also for the parameter \( j_y \). But as mentioned earlier, the analytical models considered here describe only \( Y \)-bending modes, and therefore only one parameter \( j_y \) must be determined. Parameter \( s \) depends on Young's modulus \( E \), shear modulus \( G \) and shear factor \( \kappa \). The shear factor \( \kappa \) is dependent on the geometry of the cross-section and Poisson's ratio \( \nu \) (Hutchinson [28], [27]). Thus the parameters \( j_y \) and \( s \) depend only on geometry and material properties and should stay the same

---

\( ^2 \)A tube according to the tube of the sensor but with the cross-section shown in Fig. 3.4 (sectional view A-A) running throughout the whole length of the sensor, i.e. without the bore for the bushing and the magnet, has been used.

\( ^3 \)A tube according to the tube of the sensor but with the cross-section shown in Fig. 3.4 (sectional view A-A) running throughout the whole length of the sensor, i.e. without the bore for the bushing and the magnet, has been used.
for all \( Y \)-bending modes.

It is very difficult to accurately measure the geometrical and material parameters contained in the parameters \( j_y \) and \( s \), but since parameters \( j_y \) and \( s \) are the same for all \( Y \)-bending modes, these parameters can be determined by parameter fitting. Note therefore that the Timoshenko beam model for a free-free beam is entirely described by the parameters \( c_0, j_y, s, \omega, L \). The determination of \( c_0 \) has been described in the preceding paragraph, and the length \( L \) is easily measured. Based on this Timoshenko beam model for a free-free beam and the measured resonance frequencies of the two lowest bending modes, the parameters \( j_y \) and \( s \) can be found by a root finding function.

The values which are obtained by this parameter fitting process are \( j_y = 2.585 \text{mm} \) and \( s = 2.263 \).

The shear factor \( \kappa \) is not used in any of the models described here. It could be calculated though from the definition of \( s \) and using the relationship \( G = E/(2 \cdot (1 + \nu)) \) as \( \kappa = 2 \cdot (1 + \nu)/s^2 \). Using the above value for \( s \) and inserting a literature value for the Poisson’s ratio, \( \nu = 0.29 \), one obtains \( \kappa = 0.50 \).

In the mobility model the rigid mass is modeled as a separate part which is described by the following parameters: mass \( m \), mass moment of inertia with respect to the center of mass \( I_s \), and the location of the center of mass along the axis of the sensor \( x_s \). The elements of the rigid mass are depicted in the left graph in Fig. 3.13. The masses of the magnet \( m_m \) (medium grey), and the bushing \( m_b \) (light grey) are readily obtained by weighing. The mass \( m_{\text{tab}} \) of the tube around the bushing (dark grey) is determined by the formula \( m_{\text{tab}} = m_{\text{tot}} - m(l_{\text{tot}} - l_{\text{tab}}) \), in which \( m_{\text{tot}} \) is the mass of the tube with the bore for the bushing, \( m \) is the distribution of the mass determined above, and \( l_{\text{tot}} \) and \( l_{\text{tab}} \) are the total length of the tube and the length of the bore respectively. Therefore one has that \( m = m_m + m_b + m_{\text{tab}} \). The parameters \( I_s \) and \( x_s \) are calculated from the modeled geometry and the masses of the individual elements determined before.

![Figure 3.13](image-url)

**Figure 3.13.** Representation of the rigid mass in the analytical models. **Left:** representation in the mobility model. **Right:** representation in the DeqD model. The magnet is depicted in medium grey, the bushing in light grey, and the tube around the bushing in dark grey.

\(^4\)For comparison: The shear factor for a hollow circular cross-section with radii 3.7 mm/4 mm and \( \nu = 0.29 \) is given by Hutchinson \([28, \text{Eq. (44)}]\) as \( \kappa = 0.56 \).
In the DeqD model a continuous beam with a concentrated rigid mass is considered where the concentrated mass is described by the parameters: concentrated mass $m_c$, concentrated mass moment of inertia with respect to the center of mass $J_c$, and the location of the concentrated rigid mass along the axis of the sensor $x_c$. As explained earlier (section 3.2.4), $m_c$ and $J_c$ are the additional mass and mass moment of inertia with respect to those of the beam. Some simplifications are made in determining the parameters. The situation from which the parameters are deduced is depicted in the right graph in Fig. 3.13. The clearance of 0.14 mm between the bottom of the bore in the tube and the bushing is neglected. The bushing is assumed to have a radial extent up to the inner radius $r_i$ of the tube only (tube and bushing are made from the same material and it seems justified to add the oversize of the bushing to the tube). The concentrated mass is determined as $m_c = \text{mass of bushing with outer radius } r_i + \text{mass magnet} - \text{mass of ribs}$. The “mass of the bushing with outer radius $r_i$” is determined from the modeled volume and the weighed mass of the bushing. The “mass of the magnet” can be determined by weighing. The “mass of the ribs” takes into account the mass of the ribs of the cross-section, which is not part of the concentrated mass. This mass is calculated from the modeled geometry and conforming to the mass distribution $m$ of the beam determined earlier. The concentrated mass moment of inertia with respect to the center of mass is determined as $J_c = \text{mass moment of inertia of bushing with outer radius } r_i + \text{mass moment of inertia of magnet} - \text{mass moment of inertia of ribs}$. $J_c$ is calculated from the modeled geometry and the masses of the elements of $m_c$. The determination of the “mass moment of inertia of the ribs” is simplified by considering the ribs as cuboids i.e. neglecting the curved transition to the rest of the beam cross-section. The parameter $x_c$ is calculated from the modeled geometry and the masses of the individual elements determined before.

In the FEM model, many geometrical details can be represented. Still it is not possible to include all radii and chamfers, and the error-proneness in measuring the geometry of the sensor parts exemplified at the beginning of this section is problematic. The material properties for the tube are determined the following way: The density $\rho$ is determined from the experimentally determined mass distribution $m$ and the modeled cross-sectional area $A$ by $\rho = m/A$. The Young’s modulus $E$ is calculated from the experimentally determined longitudinal wave speed $c_0$ and the previously determined material density (using the definition of $c_0$) by $E = \rho c_0^2$. The value of the Poisson’s ratio $\nu$ is taken from literature.\(^5\) The employed material parameters for the tube take the values $\rho = 8317 \text{ kg/m}^3$, $E = 197 \text{ GPa}$, and $\nu = 0.29$. The material of the bushing should have the same properties as the one of the tube. However, due to the uncertainties in the geometrical dimensions, the material density is again calculated from the measured weight and the modeled volume. The

\(^5\) For circular cylindrical beams the Poisson’s ratio could be calculated from the shear wave speed $c_s$, the longitudinal wave speed $c_0$, and the relationship $G = E/(2 \cdot (1 + \nu))$. The shear and longitudinal wave speeds could thereby be determined accurately by torsional and longitudinal vibration experiments. As mentioned in section 3.2.2, this procedure is not possible for the beam with the current cross-section.
Young’s modulus and Poisson’s ratio on the other hand are taken from the material properties of the tube. Such an approach seems justified as magnet, bushing, and surrounding tube build a very stiff entity compared to the tube, and probably could be modeled with good accuracy as a rigid part. Therefore, an accurate representation of the mass distribution seems more important than the accurate representation of the material stiffness. The employed material parameters for the bushing take the values $\rho = 7286 \, \text{kg/m}^3$, $E = 197 \, \text{GPa}$, and $\nu = 0.29$.

For the magnet, the material density is again calculated from the measured weight and the modeled volume. In the FEM model the small bore running along the axis of the magnet (Fig. A.4) is neglected though. The value of the Young's modulus is taken from literature. As no data could be found for the Poisson’s ratio of the magnet material, a value typical for steel is used. The employed material parameters for the magnet take the values $\rho = 6953 \, \text{kg/m}^3$, $E = 155 \, \text{GPa}$, and $\nu = 0.3$. (Note: While writing this thesis, a reference has been found that indicates a value of $\nu = 0.24$ for the magnet material. With the same argument as before, that magnet and bushing and surrounding tube build an almost rigid entity, for which the material stiffness is secondary, it is assumed that knowing the actual value of the Poisson’s ratio of the magnet material is not important.)

The sensor body is made from the same material as the tube and the bushing, and consequently should have the same material properties as these parts. Again, one faces the problem that not all geometric details can accurately be represented in the FEM model. As before, the material density is determined from the measured weight and the modeled volume, and for the material stiffness and Poisson’s ratio the same values as for the tube and the bushing are inserted. Thereby the total mass of the sensor body is accurately represented in the model and the material stiffness, which is important at the clamping of the tube in the sensor body, should also be represented accurately. The employed material parameters for the sensor body take the values $\rho = 7908 \, \text{kg/m}^3$, $E = 197 \, \text{GPa}$, and $\nu = 0.29$.

### 3.3.2. Comparison between models and experiments

The sensor is built up step by step and measurements are conducted at the different stages and compared with model predictions. Thereby the different stages of the models can be verified and sources for model inaccuracies can be identified. The comparison of experimental and modeling results for the different steps are presented here.

The following stages are considered:

- **Tube with continuous cross-section**: Tube alone, with the cross-section shown in Fig. 3.4 (sectional view A-A) running throughout the whole length of the tube, i.e. without the bore for the bushing and the magnet.

- **Tube with bushing and magnet**: Tube with assembled bushing and magnet.

- **Assembled sensor, completely free**: Assembled sensor consisting of sensor body, tube, bushing, and magnet. Completely free mounting.
• **Assembled sensor mounted in the housing:** Assembled sensor (consisting of sensor body, tube, bushing, and magnet) mounted in the housing described in section 3.1.3.

• **Influence of sealing and measurement setup:** Consider assembled sensor mounted in the housing. Study the influences of:
  - Sealing of transitions between the assembled sensor parts.
  - Attachment of reflective tape on the inside of the tube.
  - Introduction of measurement setup for vibrometer measurements on the inside of the tube.

For some configurations, the flexural vibrations in the directions of the two principal axes of the cross-section of the tube are considered. According to the coordinate directions in Fig. 3.12, oscillations with deflections in Y- or X-direction are denoted as Y-bending or X-bending modes.

### 3.3.2.1. Experimental setup

The support of the sensor parts at the different stages and of the free assembled sensor or the assembled sensor mounted in the housing is depicted in Figs. 3.16-3.18. A completely free mounting in the models is mimicked in the experiments by mounting the parts on foam material. In the cases of the ‘tube with continuous cross section’ and of the ‘tube with bushing and magnet’ discussed later, the foam material is additionally positioned at the vibration nodes in order to minimize the interaction with the environment (Exception: for the X-bending modes of the ‘tube with continuous cross section’, the tube has been in contact with the foam material on its full length.). In the case of the sensor with free boundaries (Fig. 3.17), the sensor is excited to have deflections in the direction parallel to the foam pad. Thereby the possible effect of an elastic bedding should be minimized. In the case of the assembled sensor mounted in the housing (Fig. 3.18), the sensor housing is mounted on a heavy and rigid table (cp. section 3.1.3 for the mounting of the sensor in the housing and the fixation of the housing).

The quantities that shall be measured are the resonance frequency and the damping of the sensor (or sensor parts, if the intermediate stages of the sensor are considered). Fig. 3.14 shows a schematic of the measurement procedure. Controlled by a **LabVIEW** routine a frequency generator (**freqGen**) drives an electrical coil or an **EMAT** (Electromagnetic Acoustic Transducer) in order to excite the sensor (or sensor parts) to flexural oscillations. A vibrometer (**Vibro**) is used to measure the velocity response on a point on the sensor (or sensor parts). A lock-in amplifier (**Lock-In**) is used to determine the phase between the excitation signal from the frequency generator and the velocity response. A frequency counter (**Counter**) is used to accurately measure the frequency of the excitation signal. Phase and frequency information is read into LabView. By stepping through a frequency range in the neighborhood of a resonance peak, the data recorded in LabView can be used to describe the phase-frequency curve. From the recorded phase-frequency curve, the
The required resonance frequency and Q-factor can be found as described in section 2.2 by fitting the phase curve of a 1 dof oscillator (Eq. (2.17)) to the measured curve. The single steps of this procedure are explained in more detail below, together with a description of the employed devices.

![Schematic of the measurement procedure.](image)

The excitation of the oscillation modes is realized by an EMAT (Electromagnetic Acoustic Transducer) in the cases of the 'tube with continuous cross section' and of the 'tube with bushing and magnet'. In the cases of the sensor with free boundary conditions and the sensor mounted in the housing (Figs. 3.17, 3.18), the oscillations are excited electromagnetically according to the description in section 3.1.2. The EMAT and the coil are made in-house. The sinusoidal excitation signal is generated by a function generator (Stanford Research Systems, Model DS345). The EMAT is driven via a voltage-controlled current source (Kepco Bipolar Operational Power Supply/Amplifier BOP 36-6M). The coil is driven directly by the function generator.

A single-point vibrometer (sensor head: Polytec OFV-303, controller: Polytec OFV-3001) is used to measure the vibration velocity of a point on the outer surface of the vibrating structure. No reflective tape is used for these measurements. An exception are the measurements with the sensor mounted in the housing, where also measurements on the inside of the tube with a fibre optic vibrometer system together with a side exit head are performed as explained in more detail in section 3.3.2.5.

A LabVIEW routine is used to record the phase-frequency curve in the vicinity of a resonance peak (phase between the excitation signal and the measured velocity signal). Therefore, the function generator is operated at stepwise increasing frequencies within the chosen frequency range. After a waiting time of 6s to 8s, which allows for transients to decay, the corresponding phase and frequency values are measured six times and the mean values are stored. The phase is measured using a lock-in amplifier (EG&G, model 5210). Note that the lock-in amplifier is not in a locked state for these measurements. The oscillation frequency is measured by a frequency counter (Programmable Counter/Timer Keithley 776).

The recorded phase-frequency curve is evaluated by fitting the phase-frequency curve of a 1 dof oscillator in the measured data as described in section 2.2, Eq. (2.17).
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Thereby the values for the resonance frequency and the damping value $\delta_1$, or the Q-factor are obtained. An example of this evaluation process is presented in Fig. 3.15.

![Graph showing phase-frequency curve](image)

Figure 3.15.: Fitting the phase-frequency curve of a 1 dof oscillator in the measured data in order to determine the resonance frequency and the damping. The circles represent the measured data, the solid line shows the fitted curve. The determined resonance frequency $f_{\text{res}}$, quality factor $Q$, and the goodness-of-fit described by the R-square value are printed in the figure. The presented data corresponds to case e) in section 3.3.2.5.

3.3.2.2. Tube with continuous cross-section, completely free

The first case considered is the tube alone, with the cross-section shown in Fig. 3.4 (sectional view A-A) running throughout the whole length of the tube, i.e. without the bore for the bushing and the magnet. In the analytical models, the tube is modeled as a beam with free ends. In the FEM model, no boundary conditions are specified, which leads to a completely free part. In the experiment these boundary conditions are approximated by mounting the sensor on foam material (Fig. 3.16).

![Experimental setup](image)

Figure 3.16.: Experimental setup for the case of the tube with continuous cross-section and with free boundaries.
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The resonance frequencies of the two lowest \(Y\)- and \(X\)-bending modes from measurements and models are presented in Tab. 3.1. The perfect agreement between the analytical models and the measurements is due to the determination of the model parameters \(j_y\) and \(s\) by fitting the model to the experimental data as described in section 3.3.1. \(X\)-bending is not considered by the analytical models. In order to compare the model and the experimental results and to better recognize the deviations, the ratios of the resonance frequencies from the models and the corresponding measurements is also given in Tab. 3.1.

<table>
<thead>
<tr>
<th>(f_{res}) in [Hz]</th>
<th>Y-Bending modes</th>
<th>X-Bending modes</th>
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</thead>
<tbody>
<tr>
<td></td>
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</tr>
<tr>
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<tr>
<td>Mobility model</td>
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<tr>
<td>DeqD Model</td>
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<td>11371</td>
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<td>1.0009</td>
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<tr>
<td>(f_{res,Mobility}/f_{res,Meas})</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(f_{res,DeqD}/f_{res,Meas})</td>
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<td>1</td>
</tr>
</tbody>
</table>

Table 3.1.: Top: Resonance frequencies \(f_{res}\) from measurements and models for the case of the tube with continuous cross-section and with free boundaries. Bottom: Ratios of the resonance frequencies from the models and the corresponding values from measurements.

3.3.2.3. Tube with bushing and magnet, completely free

The experimental setup for the case of the tube with assembled bushing and magnet is the same as depicted in Fig. 3.16. The location of the bushing and magnet is at the beam end opposite to the beam end excited by the EMAT. In the analytical models, again the ends of the assembly are modeled as free ends. In the FEM model, no boundary conditions are specified, therefore modeling a completely free assembly. In the experiment these boundary conditions are approximated by mounting the sensor on foam material.

The resonance frequencies of the three lowest \(Y\)- and \(X\)-bending modes from measurements and models are presented in Tab. 3.2. \(X\)-bending is not considered by the analytical models. In order to compare the model and the experimental results and to better recognize the deviations, the ratios of the resonance frequencies from the models and the corresponding measurements is also given in Tab. 3.2.

3.3.2.4. Influence of sensor mounting

The assembled sensor, consisting of sensor body, tube, bushing, and magnet, is considered in this section. In order to study the influence of sensor mounting on the resonance frequencies and \(Q\)-factors of the flexural oscillations, two different experimental setups are examined: the assembled sensor completely free, and the assembled sensor mounted in the housing.
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<table>
<thead>
<tr>
<th>$f_{res}$ in [Hz]</th>
<th>Y-Bending modes</th>
<th>X-Bending modes</th>
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Table 3.2.: Top: Resonance frequencies $f_{res}$ from measurements and models for the case of the tube with assembled bushing and magnet, and with free boundaries. Bottom: Ratios of the resonance frequencies from the models and the corresponding values from measurements.

Assembled sensor, completely free  The experimental setup is depicted in Fig. 3.17. The mounting on foam material shall mimic a free boundary condition, i.e. no or negligible interaction with the environment. In the corresponding FEM model, no boundary conditions are specified, such that a completely free assembly is considered. No analytical models are built for this case. This is due to the following reason, which will become clear later (section 4.2.3): It is found that for the evaluation of the analytical and FEM models including the influence of a surrounding fluid, the detailed description of the mounting of the tube is not relevant. For the determination of fluid density, only the change in resonance frequency with respect to the resonance frequency without fluid matters. Consequently, in subsequent models the transition of the tube to the sensor body is modeled as a rigid clamping. However, an FEM model including the sensor body can be useful to study the clamping in more detail, e.g. the deformations that occur on the tube and sensor body, and the corresponding forces, which might be represented in an analytical model by springs.

Figure 3.17.: Experimental setups for the case of the assembled sensor with free boundaries.

The resonance frequencies from measurements and FEM model are listed in Tab. 3.3 for the lowest four Y-bending modes and the lowest three X-bending modes. The same table shows also the ratios of the resonance frequencies from the FEM model and the corresponding measurements.
Assembled sensor mounted in the housing An interesting question is whether and how much the mounting of the assembled sensor in the housing affects the resonance frequencies and $Q$-factors of the flexural oscillations, or if the characteristics of the flexural oscillations are determined by the mounting of the tube in the sensor body. The mounting of the assembled sensor in the housing has already been described in section 3.1.3. The experimental setup is depicted in Fig. 3.18. Measurements have been performed for the lowest $Y$-bending modes, which are the modes that shall be used for measurements in fluid. Tab. 3.4 shows the resonance frequencies and the $Q$-factors for the assembled sensor mounted in the housing and for comparison the corresponding values for the free assembled sensor discussed in the preceding paragraph. No measurement values could be obtained for the lowest $Y$-bending mode of the assembled sensor mounted in the housing. Several resonance peaks have been found in the frequency range of this mode, which also partially overlap, and it is not clear which mode corresponds to the lowest $Y$-bending mode.

This observation and the measurement results presented in Tab. 3.4 indicate that there is some coupling between the assembled sensor and the housing, but that the coupling decreases with increasing mode order. For mode orders greater than two,
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<table>
<thead>
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<th>2nd</th>
<th>3rd</th>
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<tr>
<td>Q-factor measured, free [-]</td>
<td>1402</td>
<td>1870</td>
<td>1074</td>
<td>1122</td>
</tr>
<tr>
<td>Q-factor measured, mounted [-]</td>
<td>-</td>
<td>617</td>
<td>913</td>
<td>1042</td>
</tr>
</tbody>
</table>

Table 3.4: *Top*: Resonance frequencies $f_{res}$ from measurements of the free assembled sensor (according to Tab. 3.3) and the assembled sensor mounted in the housing. *Bottom*: Q-factors from measurements of the free assembled sensor and the assembled sensor mounted in the housing.

Tab. 3.4 shows negligible differences between the resonance frequencies measured with the free assembled sensor and the assembled sensor mounted in the housing. Considering the Q-factors, a stronger dependence on the mounting in the housing is found, but which also decreases with increasing mode order.

Considering the resonance frequencies it can be concluded that for mode orders greater than two, the transition between the tube and the sensor body is the dominant part, and the influence of the mounting in the housing can be neglected. Regarding the Q-factors, it can be concluded that the mounting in the housing has some influence even for mode orders greater than two, but that the influence is decreasing with mode order.

3.3.2.5. Influence of sealing and measurement setup

The transitions between the assembled sensor parts that will be in contact with fluid must be sealed in order that no fluid is soaked in, because such fluid could affect the measurements. Therefore, after mounting of the sensor parts (via press-fits), the transitions between the sensor body and tube and between tube and bushing are glued with ZAP CA PT-08 [5] on the surface which will be in contact with fluid.

An interesting question is whether and how much this sealing affects the resonance frequencies and Q-factors of the oscillations.

For the measurements in fluid, the vibrometer measurements shall be conducted from the inside of the tube (as discussed in section 3.1.2), in contrast to the measurements on the outer surface conducted so far. Such a measurement has been realized using a Polytec OFV-C-102 Side Exit Head (Fig. 3.19) in combination with a Polytec OFV-512 Fiber-Optic Sensor Head and a Polytec OFV-3001 Vibrometer Controller. The measurement setup is depicted in Fig. 3.20. As can be seen from the right picture the clearance between the side exit head and the ribs of the cross-section of the tube is very small\(^6\), and alignment is tricky. In order to facilitate the alignment and obtain good reflectivity of the laser beam, a small strip of reflective tape is mounted inside the tube. The presence of this reflective tape might affect resonance frequency or damping of the sensor.

\(^6\)Nominal outer diameter of the side exit head: 5 mm, measured clearance between the ribs of the cross-section: 5.43 mm

53
Experiments showed that the presence of the side exit head in the tube leads to a slightly increased damping of the flexural oscillations, while the influence on the resonance frequency is negligible (see below for a quantitative examination). The increased damping showed also to be slightly dependent on the position of the side exit head with respect to the cross-section of the tube. An experiment with the setup turned upside down and continuously guiding a small stream of helium inside the tube showed to decrease the damping of the oscillations. Therefore the air displaced or squeezed in the small clearance between the side exit head and the tube seems to cause the increased damping. Hence, it should be noted that for comparable damping measurements, the alignment of the side exit head inside the tube should not be altered between measurements.

Four different experimental configurations are considered in order to quantitatively capture the effects discussed above:

a) Unsealed sensor. Measurement with vibrometer on outer surface of the tube. No reflective tape. (Values according to Tab. 3.4.)

b) Sealed sensor. Measurement with vibrometer on outer surface of the tube. No reflective tape.
c) Sealed sensor. Measurement with vibrometer on outer surface of the tube. Reflective tape attached.


Measurement results for the third and fourth Y-bending mode are presented in Tab. 3.5. Note that each measurement result for the sealed sensor is from only one recorded phase-frequency curve. Due to the good repeatability of the foregoing measurements, these measurement results seem nonetheless significant. By recording and evaluating the whole phase-frequency curve, extraordinary behavior of the oscillations would have been detected, but has not been observed. The random errors are minimal, as the single measured points of the phase-frequency curve are averaged values. In the evaluation of the recorded phase-frequency curve by fitting the phase-frequency curve of a 1 dof oscillator into the measured data, an insufficient accordance would also have been seen in the r-square value of the goodness-of-fit statistics, but has not been observed (in the evaluation of the measurements with the sealed sensor, the r-square values have been 0.99999 or 1, except for the fourth bending mode in case e), where an r-square value of 0.99998 has been obtained).

<table>
<thead>
<tr>
<th></th>
<th>Y-Bending modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3rd</td>
</tr>
<tr>
<td>(f_{res})  case a) [Hz]</td>
<td>11071</td>
</tr>
<tr>
<td>(f_{res})  case b) [Hz]</td>
<td>11085</td>
</tr>
<tr>
<td>(f_{res})  case c) [Hz]</td>
<td>11072</td>
</tr>
<tr>
<td>(f_{res})  case d) [Hz]</td>
<td>11086</td>
</tr>
<tr>
<td>(f_{res})  case e) [Hz]</td>
<td>11068</td>
</tr>
<tr>
<td>(Q)-factor case a)</td>
<td>-</td>
</tr>
<tr>
<td>(Q)-factor case b)</td>
<td>-</td>
</tr>
<tr>
<td>(Q)-factor case c)</td>
<td>-</td>
</tr>
<tr>
<td>(Q)-factor case d)</td>
<td>-</td>
</tr>
<tr>
<td>(Q)-factor case e)</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3.5.: Measurements with the assembled sensor mounted in the housing in order to study the influence of sealing and measurement setup. \(Top\): Resonance frequencies \(f_{res}\) from measurements of the configurations a)-e) discussed above. \(Bottom\): \(Q\)-factors from measurements of the configurations a)-e). (Values for unsealed sensor, case a), are according to Tab. 3.4.)

### 3.3.2.6. Influence of mass and rotary inertia of the bushing and the magnet

Based on the analytical models, the sensitivity of the sensor on the mass and rotary inertia of the bushing and magnet can be estimated. The DeqD model, in which
bushing and magnet are modeled as a concentrated rigid mass with rotary inertia, allows to conveniently estimate the sensitivity on uncertainties of the location and magnitude of the rigid mass and the rotary inertia. Tab. 3.6 presents the resonance frequencies predicted by the DeqD model for the four lowest Y-bending modes and the deviations from these resonance frequencies if the rotary inertia is neglected, if the mass and the rotary inertia are neglected, and for varying the location of the concentrated mass/rotary inertia \( x_c \), for varying the magnitude of the rigid mass \( m_c \), and for varying the magnitude of the rotary inertia of the rigid mass \( J_c \). A rigidly clamped tube is considered in the evaluated DeqD model.

<table>
<thead>
<tr>
<th>( f_{\text{res}} ) from DeqD model</th>
<th>Y-Bending modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial state</td>
<td>1st</td>
</tr>
<tr>
<td>( J_c = 0 )</td>
<td>579 Hz</td>
</tr>
<tr>
<td>( J_c = 0, m_c = 0 )</td>
<td>+0.043 %</td>
</tr>
<tr>
<td>( x_c \pm 0.2 \text{mm} )</td>
<td>+18.9 %</td>
</tr>
<tr>
<td>( m_c \cdot (1 \pm 5 %) )</td>
<td>+0.27 %</td>
</tr>
<tr>
<td>( J_c \cdot (1 \pm 5 %) )</td>
<td>+0.35 %</td>
</tr>
</tbody>
</table>

Table 3.6.: Top: Resonance frequencies \( f_{\text{res}} \) of the complete sensor for the four lowest Y-bending modes as predicted by the DeqD model. In the DeqD model the mounting of the tube is modeled as a rigid clamping. Bottom: Deviations of resonance frequencies for varying parameters \( x_c \), \( m_c \), and \( J_c \) in the DeqD model. Deviations are expressed as percentages of the resonance frequencies in the initial state.

3.3.2.7. Influence of temperature changes

Temperature changes affect the geometric dimensions, the elastic material properties and the dissipative material behavior of the sensor parts and the setup. These changes affect the resonance frequency and Q-factor of the system. While the measurements in fluid are carried out at a fixed and controlled temperature, it is still of value to know the temperature dependence of these quantities near the measurement temperature. It allows to assess the influence of small temperature changes occurring despite of the temperature control. Therefore measurements in air at three temperatures (20 °C, 22 °C, and 24 °C) have been conducted. Fig. 3.21 shows the variation of resonance frequency and bandwidth with temperature.

A linear approximation of the variation of the resonance frequency and bandwidth presented in Fig. 3.21 gives a change of about \(-2.3 \text{ Hz/}^\circ\text{C}\) for the resonance frequency and about \(-0.63 \text{ Hz/}^\circ\text{C}\) for the bandwidth. (In this small temperature range and the intention to estimate the influence of small temperature differences from the fixed measurement temperature, a linear approximation seems justified.) It is observed that the Q-factors in Fig. 3.21 are a bit higher than the one presented in Tab. 3.5, case e). The reason for this difference is probably the influence of the side exit head (cp. section 3.3.2.5), which might have had a slightly different position than in earlier measurements (the measurements with temperature variation
3.4. Discussion

For the case of the tube with a continuous cross-section the resonance frequencies from the FEM model show a very good agreement with the measured values. For the results presented in Tab. 3.1, the deviations are about 0.1% for the \( Y \)-bending modes and up to 0.84% for the \( X \)-bending modes. In this case the resonance frequencies from the analytical models coincide with the measured values, because the two model parameters \( j_y \) and \( s \) are determined from the measured resonance frequencies by a fitting process.

Considerable differences can be seen between the resonance frequencies obtained from Timoshenko and Euler Bernoulli beam model. From an Euler Bernoulli beam model using the same geometry and material parameters as considered here, i.e. the same parameters \( c_0 \), \( f_y \), and \( L \), one obtains for the two lowest resonance frequencies 4571 Hz and 12 600 Hz. Compared to the results from one of the Timoshenko beam models given in Tab. 3.1 (mobility or DeqD model), one observes differences of 3.7% and 10.8% for the first and second resonance frequency respectively.

In the case of the tube with assembled bushing and magnet with free boundaries, Tab. 3.2 shows a very good agreement between all models and the measurements. The resonance frequencies of the FEM model deviate from the experimental values by maximally 0.42% for the \( Y \)-bending modes, and by maximally 0.9% for the \( X \)-bending modes. The somewhat higher deviations for the \( X \)-bending modes are not considered to be critical, because the main focus of this work lies on the \( Y \)-bending modes. For the analytical models, the deviations of the resonance frequencies from

Figure 3.21.: Variation of resonance frequency \( f_{\text{res}} \) and bandwidth \( df \) with temperature of the third \( Y \)-bending mode in air. Squares: measured data, dashed line: linear fit.

have been conducted 10 months after the measurements presented in section 3.3.2.5).
the measured values are less than 1 % for the mobility model and less than 0.25 % for the DeqD model. It is interesting to note that both analytical models, which differ in the way the influence of the bushing and magnet is taken into account, give comparable results. The analytical models even show a comparable accuracy as the FEM model. This might be due to the low mode orders considered here, where the curvature at the location of bushing and magnet is small, and the details of the mounting of bushing and magnet in the tube are not so important.

Considering the assembled sensor with free boundaries, a very good agreement between the FEM model and the measurements is found, except for the lowest X- and Y-bending modes. The resonance frequencies of the higher order modes from the FEM model show deviations from the measured values of maximally 0.33 % for the Y-bending modes, and maximally 0.28 % for the X-bending modes. The worse agreement between FEM model and measurements for the lowest modes indicates an interaction of the assembled sensor with the environment, probably via the bedding on the foam material, possibly also via reaction forces to the excitation forces. The good agreement is also seen as a confirmation, that the connections between the sensor parts are well described by the ‘mesh tie constraints’ in the FEM model. It seems not necessary to reproduce the physically present press fits with the consequent prestresses in the FEM model. The comparison of the resonance frequencies for the X- and Y-bending modes shows that the corresponding modes are well separated, which is a major requirement on the sensor design as discussed in section 3.1.1.

Comparison of the free assembled sensor with the assembled sensor mounted in the housing (Tab. 3.4) shows a decreasing influence of the housing with increasing mode order. Considering the resonance frequency, for mode orders greater than two, the mounted sensor behaves like the free sensor. Considering the Q-factors, the dependence on the mounting is stronger, but also decreasing with mode order. Therefore, regarding the resonance frequencies of mode orders greater than two, the transition between the tube and the sensor body is the determinant part. On the other hand, regarding the Q-factors, the mounting of the sensor body in the housing has some influence, but decreasing with mode order. For the lowest bending mode, a strong interaction of the assembled sensor with its environment is observed (either via the mounting in the housing or via reaction forces to the excitation forces). It is not clear for this mode, which parts of the setup are excited.

The comparison presented in Tab. 3.5 shows that the sealing of the sensor leads to slightly increased resonance frequencies and slightly decreased Q-factors (for the third and fourth Y-bending mode). The introduction of the side exit head and the reflective tape into the tube leads to a slight decrease of the resonance frequencies, but a more pronounced decrease of the Q-factors.

In order to estimate the influence of the mass and rotary inertia of the bushing and the magnet on the resonance frequencies of the sensor and in order to estimate the influence of uncertainties of the mass and rotary inertia as represented in the model, the Deq model has been evaluated in section 3.3.2.6 by varying the respective parameters. As can be seen from Tab. 3.6, the influence of the rotary inertia on the resonance frequencies of the four lowest bending modes is much smaller than
the influence of the mass, but while the influence of the mass decreases with mode order, the influence of the rotary inertia is increasing with mode order (consider the cases in Tab. 3.6 where rotary inertia and mass are neglected in the model). For the third bending mode, which is of primary interest in this work, neglecting the rotary inertia would give an estimated error in the resonance frequency of 1.14%.

In the DeqD model the mass and rotary inertia of the bushing and the magnet are represented by a concentrated rigid mass/rotary inertia. Varying the location of the concentrated mass/rotary inertia and varying their magnitude gives an estimation, on how sensitive the model is on uncertainties of these parameters. At least for the considered variations, the influences on the resonance frequency are small. For the third bending mode, uncertainty in the precise location of the concentrated mass/rotary inertia seems to be the primary source for deviations between model and experiment. Uncertainty in the modeled rigid mass should be small, because the mass of the magnet, which determines about 80% of the modeled rigid mass (1.334g out of 1.628g), can be accurately determined by weighing. The variations of the magnitude of the rigid mass considered in Tab. 3.6 suggest that the influence of uncertainties in the modeled rigid mass of 5% have an influence of less than 0.14% on resonance frequency, which seems negligible. The uncertainty in the modeled rotary inertia is difficult to assess. However, because the overall influence of the rotary inertia on the resonance frequency is small (about 1.14%), uncertainties in the modeled rotary inertia should have a negligible influence.

The reasons for the small deviations observed between models and experimental results can be found probably in differences in the geometry and material parameters, in the way the connections between the sensor parts are modeled (mesh tie constraints vs. press fits), and the bedding on foam material in contrast to free boundaries. Comparing the FEM model with the analytical models, it should be noted that deformations of the tube cross-section is not taken into account in the analytical models. Fig. 3.22 shows a sectional view of the deflected tube from the FEM model. The deformation of the cross-section is clearly visible in the scaled representation. Against the background of the good agreement between models and experiments, this effect does not seem to be important for the modes considered here. For higher modes, the effect could become more important though.

![Figure 3.22.: Deformed cross-section of the beam under bending.](image-url)
Based on the results presented above, the third and fourth $Y$-bending modes are best suited for further studies with fluid. For the lower order modes the interaction of the sensor with the environment is problematic. For higher order modes the decoupling from the environment could be better, but the analytical beam model presumably cannot represent the physical situation accurately because of higher order effects (e.g., distortion of the beam cross-section).
4. Modeling of the resonator in fluid

Due to the presence of the fluid, the flexurally oscillating tube experiences damping and inertial forces as discussed in section 2.1. This chapter deals with the description of the fluid forces and how they are implemented in the models. The differences between various fluid descriptions are explored. Measurements in various test fluids are presented and compared to model predictions.

In section 4.1 the modeling of the interaction between the oscillating structure and the surrounding fluid is considered. Section 4.1.2 deals with the analytical description of the fluid forces acting on the oscillating structure. Based on a dimensional analysis, the importance of the various possible fluid effects is assessed. As it is not clear a priori which analytical description of the fluid forces is suitable to describe the situation considered in this work, several analytical models for the description of the fluid forces are introduced, which appear to be appropriate for the current situation. In section 4.1.3 finite element simulations are considered as a further means to model the fluid-structure interaction. The introduction of the analytical descriptions of the fluid forces in the analytical models of the oscillating structure is discussed in section 4.1.4.

In section 4.2 results from experiments and models are presented. After a description of the experimental setup in section 4.2.1, the experimental verification that the fluid behaves linearly is presented in section 4.2.2. Section 4.2.3 considers the influence of the sensor fixation on the comparison between experimental and model results. In section 4.2.4 the influence of the fluid boundaries on the resonance frequency and damping of the sensor is examined experimentally. Section 4.2.5 presents measurements of resonance frequency and damping in various test fluids. These measurements form the basis for the assessment of the suitability of the various fluid models introduced before to describe the current fluid-structure interaction problem. They also form the basis to assess the density measurement capabilities of the current sensor. In section 4.2.6 the fluid forces predicted by the various fluid models introduced before are compared with each other (for fluid properties corresponding to the test fluids used in the experiments). This comparison allows to discuss the different features of the various fluid models and their influences on the magnitudes of the predicted fluid forces. The finite element model allows to study the influence of the speed of sound of the test fluids on the resonance frequencies of the sensor, and results are presented in section 4.2.7. In section 4.2.8 the measurement results for resonance frequency and damping in the test fluids are compared with the predictions of the different models of the density sensor. This comparison allows to assess the suitability of the various fluid models for the description of the present fluid-structure interaction problem. The findings of this chapter are discussed in section 4.3.
4.1. Fluid-structure interaction

4.1.1. Introduction

The forces exerted by the fluid on the sensor depend on the fluid properties and the fluid behavior under the given boundary conditions (e.g. fluid density and viscosity, compressibility, oscillation frequency and amplitude, geometrical dimensions of the fluid boundaries, mode shape of the oscillating tube). The motion of the fluid might lead to an interaction with the fluid boundaries, sound waves could be produced that affect the damping of the sensor or lead to acoustical resonances in the fluid cavity, the fluid motion could be planar or show three-dimensional flow patterns, and second order effects as acoustic streaming could be present. An important step is to identify the significance of the possible fluid effects for the current situation.

The fluids considered in this work are viscous Newtonian fluids, which are only weakly compressible or assumed to be incompressible. The experimentally examined fluids have densities in the range of 700 kg/m$^3$ to 1000 kg/m$^3$, viscosities between 0.5 mPas and 70 mPas, and their speeds of sound are in the range of 1100 m/s to 1500 m/s.

4.1.2. Analytical description of the fluid forces

4.1.2.1. Assessment of importance of various fluid effects

Consider a harmonically oscillating cylindrical tube in a fluid which is otherwise at rest. The significance of the various fluid effects that might occur shall be assessed in this section based on dimensional analysis and dependencies known from literature. The fluid motion of a general compressible fluid under small amplitude harmonic excitation is described by the continuity equation and the Navier-Stokes equation (Wang [54]) (neglecting volume forces)

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho_f + \rho') \mathbf{v}' = 0$$

$$\left(\rho_f + \rho'\right) \left(\frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}' \cdot \nabla' \mathbf{v}'\right) = -\nabla' p' + \eta \nabla'^2 \mathbf{v}' + \left(\frac{1}{3} \eta + \eta_B\right) \nabla' \nabla' \cdot \mathbf{v}'$$

and a constitutive equation for the fluid

$$p' = c_f^2 \rho'$$

in which $\mathbf{v}$ is the fluid velocity, $\rho_f$ is the density of the fluid at rest, $\rho$ and $p$ are the variations of fluid density and pressure, $\eta$ and $\eta_B$ are the fluid’s dynamic viscosity and bulk viscosity, $c_f$ is the speed of sound, $t$ is time, and the primes indicate that dimensional quantities are considered.

According to Wang [54] (see appendix A.5 for details), the above equations can be

---

1By weakly compressible it is meant that the fluid behavior is well described by an incompressible fluid and that compressibility is not the main characteristic of the fluid.
written in non-dimensional form as

\[ M_p^2 \frac{\partial \rho}{\partial t} + \nabla \cdot (1 + \varepsilon M_p^2 \rho) \vec{v} = 0 \]  

(4.4)

and the constitutive equation for the fluid as

\[ p = \rho \]  

(4.6)

in which the non-dimensional parameters are defined as

\[ M_p^2 = \frac{\omega^2 a^2}{c_f^2}, \quad \varepsilon = \frac{U_\infty}{\omega a}, \quad M^2 = \frac{\omega a^2}{\nu} \]  

(4.7)

in which \( \omega \) is the angular frequency of the harmonic oscillation, \( a \) is the length dimension of the solid object\(^2\), \( U_\infty \) is the velocity amplitude of the oscillation, \( \nu \) is the kinematic viscosity, and the dimensionless bulk viscosity is given as \( \eta_b = \eta B/\eta \). The parameter \( M_p \) describes the fluid compressibility, the parameter \( \varepsilon \) the magnitude of the inertial effects, and \( M \) the viscous influences.

Riley [41] gives a physical interpretation of the parameters \( \varepsilon \) and \( M \). Parameter \( M \) describes the ratio of the typical length dimension \( a \) to a viscous length (penetration depth \( \delta = \sqrt{2 \nu/\omega} \)), and parameter \( \varepsilon \) describes the ratio between the deflection amplitude of the oscillating body and length dimension \( a \). Similarly, the parameter \( M_p \) could be seen as the ratio between the length dimension \( a \) and an acoustic length (wavelength \( \lambda_f = 2\pi c_f/\omega \)) in the fluid. Some texts also express the condition for an incompressible fluid as the requirement that \( ka \) is small, in which \( k \) is the acoustic wavenumber in the fluid (e.g. Fahy and Gardonio [14, p. 244], Kremlevskii and Stepinchev [31]). In Van Eysden and Sader [52] it is stated “Compressibility is important when the acoustic wavelength becomes comparable to or smaller than the dominant length scale for the flow.”, and according to Van Eysden and Sader [52] in the case of a flexurally oscillating cylinder, the dominant length scale for the flow is the cylinder radius.

In order to assess the relative importance of the different effects, Tab. 4.1 summarizes the range of the above non-dimensional parameters for the fluids and conditions in the experiments presented later in this text. Details on the calculations of these parameters can be found in appendix A.6. The properties of the considered fluids are also listed in Tab. 4.4. Note, that the Reynolds number can be calculated from the given non-dimensional parameters as \( Re = \varepsilon M^2 = U_\infty a/\nu \). In Tab. 4.1, one observes that the non-dimensional parameters are of the following orders: \( M_p^2 \approx 0.1 \), \( \varepsilon \approx 3 \times 10^{-7} \), and \( 1/M^2 \approx 5 \times 10^{-6} \). Considering the non-dimensional continuity equation Eq. (4.4), one observes that in a first approximation the fluid behaves incompressible, but that the fluid compressibility (via the local derivative of the density) still has a not very small influence. The advective derivative of the den-

\(^2\)The diameter of the cylindrical tube in the case considered here.
Table 4.1.: Values of the non-dimensional parameters in the non-dimensional forms of the continuity and the Navier-Stokes equation according to Wang [54] for the fluids and conditions in the experiments presented later in this text. The two different values for ε reflect the different amplitudes for $U_\infty$ present in the experiments.

The compressibility of the fluid could lead to acoustic radiation, which would affect the damping of the sensor, and could cause standing acoustic waves in the fluid container. A way to assess the occurrence of acoustic radiation is presented in Junger and Feit [29, pp. 172/173]. There, an infinitely extended circular cylinder in an unbounded acoustic fluid, which performs harmonic oscillations with deflections that are sinusoidal in both axial and circumferential direction is considered. The sinusoidal deflections in axial direction are described by an axial wavenumber $k_z$. The sinusoidal deflections in circumferential direction are described by a circumferential wavenumber $n/R$, in which $n$ is a positive integer or zero, and $R$ is the radius of the cylinder.\(^3\) According to Junger and Feit [29, pp. 172/173], acoustic radiation can occur only if the axial wavenumber $k_z$ of the vibrating cylinder is smaller than the wavenumber $k_f$ of the acoustic fluid. Otherwise the fluid forces on the cylinder are purely inertial. According to Junger and Feit [29, pp. 172/173], in the case $k_f < k_z$, the pressure field in the acoustic fluid has the form of standing waves in axial direction, and the pressure is exponentially decaying in radial direction. Junger and Feit [29, pp. 172/173] note also, that standing waves in circumferential direction are of importance for $k_f < n/R$.

For the fluids considered in this work, the wavenumbers $k_f$ are between 37.1/m and 53.1/m, except for air where it takes a value of 2021/m (see appendix A.7). The axial wavenumber of the cylinder can be estimated from Euler Bernoulli beam theory, which gives a value of $k_z = 86.1/m$ for the third bending mode. Consequently, as $k_z > k_f$ (except in air), acoustic radiation should not occur in the present situation. However, it should be noted that this estimation considers an infinitely

\(^3\)The case of $n = 1$ and $k_z \neq 0$ describes flexural oscillations of an infinite cylinder in an inviscid acoustic fluid. (The velocity component in circumferential direction on the surface of the flexurally oscillating cylinder does not matter in an inviscid acoustic fluid.)
extended cylinder, but the present sensor is finite and edge effects are not considered in the estimation. In Van Eysden and Sader [52], where flexural oscillations of circular cylinders of infinite length in a compressible viscous unbounded fluid are considered, it is noted that such a clearly defined onset of radiation damping does not exist for finite beams, because there the flexural vibrations are described by a spectrum of wavenumbers (instead of one wavenumber as in the considered case of an infinitely long beam).

The circumferential wavenumber for the flexural oscillations considered here, takes a value of $1/R = 2501/m$, which is larger than the wavenumbers $k_f$ in the fluids considered in this work. Therefore, standing pressure waves in circumferential direction might occur with the flexural oscillations considered in this work. However, it should be noted that the considered fluids are not ideal acoustic fluids and that in current situation a finite cylinder is considered in contrast to the situation treated in Junger and Feit [29, pp. 172/173].

An interesting question is, under which circumstances the velocity field in the fluid can be treated as planar, with fluid velocities in a plane perpendicular to the axis of the tube. Or if a pronounced fluid flow in axial direction of the tube occurs. For example, the fluid could be relocated in axial direction of the tube rather than being displaced in a plane perpendicular to the axis. Considering a flexurally oscillating tube, axial fluid velocities can be induced by axial viscous stresses in the fluid in the boundary layer along the tube due to the rotation of the cross-section of the tube under bending. As the axial velocity component of the cylinder surface is much smaller than the lateral velocity component, the induced axial fluid velocity is expected to be very small.

Many authors considering flexurally oscillating beams in viscous incompressible fluids treat the flow field as planar, and describe the flow field at a point along the beam according to transverse oscillations of an infinitely extended rigid body with the shape of the beam cross-section (Kremlevskii and Stepichev [31], Retsina et al. [39], Sader [44]). This simplification is made under the condition that the beam length greatly exceeds the dimensions of the beam cross-section (Retsina et al. [39], Sader [44]). In contrast, in Van Eysden and Sader [52] (where flexural oscillations of circular cylinders of infinite length in a compressible viscous unbounded fluid are considered) it is shown that the flow field is three-dimensional.

Acoustic streaming has often been considered in the context of oscillating cylinders. As a second order effect, its influence on the current sensor is expected to be small, at least for the inertial fluid forces. As maintaining the streaming motion requires energy, which is taken from the oscillating cylinder, this effect could have an influence on the damping of the sensor. According to Riley [41, p. 420] and Squires and Quake [50, Eq. (53)] the second order steady streaming velocity is of the order $\varepsilon U_{\infty}$, which is very small for the present case. Further, because the associated streaming Reynolds number $Re_s = \varepsilon^2 M^2$ is very small ($Re_s \ll 1$), it can be concluded (Riley [41, p. 420], Squires and Quake [50, p. 1012]), that the streaming is extended over a larger region around the oscillating body. This might lead to an
interaction of the steady streaming with the container walls.

4.1.2.2. Overview of analytical fluid models

The discussion in the foregoing section allows to assess the importance of different fluid effects in the current fluid-structure interaction problem. Still it is not clear yet, which analytical fluid model is best suited to accurately describe the fluid forces acting on the oscillating structure. For instance it is not clear, how pronounced the effect of fluid compressibility is for the sensor behavior, if a planar fluid velocity field can be assumed or if a three-dimensional flow field must be considered, or whether the influence of fluid boundaries must be considered. Therefore, different fluid models are considered in this chapter, which seem appropriate to describe the fluid-structure interaction of current density sensor. The comparison between measurement results and model predictions presented later in this chapter will allow to assess the suitability of the different fluid models for the modeling of the density sensor. Tab. 4.2 gives an overview of the fluid models considered in this chapter.

<table>
<thead>
<tr>
<th>2D Unbounded</th>
<th>Bounded</th>
<th>3D Unbounded</th>
<th>Bounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incompressible</td>
<td>Kremlevskii and Stepichen [31]</td>
<td>Chen et al. [9]</td>
<td></td>
</tr>
<tr>
<td>Compressible</td>
<td>Kadyrov et al. [30]</td>
<td>Van Eysden and Sader [52]</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2.: Overview of the analytical fluid models used in this work. The labeling '2D' and '3D' refers to whether a two- or three-dimensional fluid velocity field is considered in the respective fluid model. The division into 'unbounded' and 'bounded' refers to whether the fluid is considered as an unbounded fluid or if fluid boundaries are included in the model. The division into 'incompressible' and 'compressible' refers to whether the fluid is described as incompressible or compressible in the respective model. The listed models all consider viscous fluids.

In the simplest description of the interaction between the flexurally oscillating tube and the surrounding fluid, the tube is considered to be composed of infinitesimal rigid slices which perform transversal oscillations. The fluid forces acting on these slices are obtained from the consideration of a fluid velocity field which is assumed to be planar with velocities lying in the plane of such a slice and thereby lying in a plane perpendicular to the axis of the tube. Equivalently, the fluid forces can be obtained from the consideration of an infinitely long rigid cylinder which performs transversal oscillations within a surrounding fluid. As discussed earlier, such a description based on the assumption of a 2D velocity field seems justified if the beam wavelength is much larger than the dimension of the cross-section of the beam.
As the dimensional analysis in the foregoing section showed that fluid compressibility should have a small effect on the fluid behavior, fluid models considering an incompressible fluid are considered first. The fluid model presented in Kremlevskii and Stepichev [31] considers an infinitely long rigid cylinder which performs transversal oscillations within an incompressible viscous unbounded fluid, and therefore considers a 2D fluid velocity field. The expression for the fluid forces obtained from this model can be simplified if the penetration depth is small. In this text, this simplified expression for the fluid forces is referred to as ‘fluid model from Kremlevskii and Stepichev [31] for the case of small penetration depths’.

Fluid boundaries could have an effect on the fluid forces. Clearly such an effect is undesirable as it would introduce further dependencies which would complicate the density measurement. Therefore it is reasonable to consider a model including fluid boundaries in order to examine the influences of fluid boundaries and estimate the required distance between oscillating structure and fluid boundary in order to avoid an interaction. The fluid model presented in Chen et al. [9] considers an infinitely long rigid cylinder which performs transversal oscillations within an incompressible viscous bounded fluid (and therefore considers also a 2D fluid velocity field).

Even though fluid compressibility should have a small effect on the fluid behavior according to the dimensional analysis presented in the foregoing section, its influence seems not negligible and its effect on the fluid forces should be considered. If considering again an infinitely long rigid cylinder which performs transversal oscillations, a description of the fluid forces in a compressible viscous bounded fluid can be found in Kadyrov et al. [30] (as an infinitely long rigid cylinder is considered, the considered fluid velocity field is again two-dimensional).

According to the discussions in sections 2.1 and 4.1.2.1, the fluid velocity field could be three-dimensional. It seems interesting to compare the differences between the fluid forces predicted by models considering a 2D or 3D fluid velocity field. Analytical fluid models exist that consider the 3D velocity field of infinitely long circular cylinders with sinusoidally varying deflections along the axis of the cylinder. Such models cannot exactly represent the situation of the current density sensor, because the tube of the density sensor is finite in length. However, such a fluid model can give an estimation of the effect of a three-dimensional fluid velocity field on the fluid forces, and indicate if it is necessary to incorporate such effects in a model of the density sensor.4

The fluid model from Van Eysden and Sader [52] considers transverse oscillations of an infinite cylinder with sinusoidally varying deflections along its axis, which is immersed in a compressible viscous unbounded fluid and considers a 3D fluid velocity field.

The fluid models discussed here and listed in Tab. 4.2 are presented in the next

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4Of course, an FEM model could represent the effects of a 3D velocity field (both along the tube and at the free end of the tube of the density sensor) more accurately, but the FEM model introduced in section 4.1.3 can only represent an acoustic fluid. Also, in a numerical model, the study of parameter dependencies is less practical.
sections, together with the respective expressions for the fluid forces.

4.1.2.3. 2D, incompressible fluid models

Fluid forces in an unbounded fluid cavity An analytical description of the forces of an incompressible viscous unbounded fluid on a transversely oscillating rigid cylinder of infinite extent can be found in Kremlevskii and Stepichev [31], Rosenhead [42, Ch. VII.12, Eq. (102)].

In Kremlevskii and Stepichev [31], the fluid is described by the linearized Navier-Stokes equation (justification for neglecting the non-linear term is given in section 4.1.2.1) and the equation of continuity for an incompressible fluid

\[
\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \eta \Delta \vec{v} \tag{4.8}
\]

\[
\nabla \cdot \vec{v} = 0 \tag{4.9}
\]

The motion of the harmonically oscillating cylinder with radius \( R \) is described by its velocity \( \Re \{u\} = \Re \{u_0 e^{i\omega t}\} = u_0 \cos(\omega t) \) in which \( u_0 \) is the velocity amplitude and \( \omega \) is the angular oscillation frequency.\(^5\) Consequently the boundary conditions for the fluid motion are given by the coincidence with the motion of the cylinder on the cylinder surface and zero velocity far away from the cylinder. The resulting force per unit length \( f \) acting on the cylinder is obtained by integration of the normal stresses \( \sigma_{rr} \) and tangential stress \( \sigma_{r\theta} \) (expressed in polar coordinates \( r, \theta \) as introduced in Fig. 4.1) acting on the surface of the cylinder

\[
f = \int_0^{2\pi} (\sigma_{rr} \cos(\theta) - \sigma_{r\theta} \sin(\theta)) r \, d\theta \tag{4.10}
\]

in which the stresses are determined as

\[
\sigma_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r} \tag{4.11}
\]

\[
\sigma_{r\theta} = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \tag{4.12}
\]

in which \( p \) is the pressure, \( v_r \) and \( v_\theta \) are the radial and tangential fluid velocities, the motion of the cylinder is in the direction of \( \theta = 0 \), and the fluid stresses are evaluated at the surface of the cylinder.

The final expression for the resulting force per unit length \( f \) is given by

\[
f = \pi R \text{Re} k_s \eta \left( -4 \frac{\tilde{H}_1^{(1)}(ik_s R)}{\tilde{H}_0^{(1)}(ik_s R)} + ik_s R \right) \tag{4.13}
\]

\(^5\)Note that Kremlevskii and Stepichev [31] use a time-dependence of the form \( e^{-i\omega t} \), which is changed here to a time-dependence of the form \( e^{+i\omega t} \).
4.1. Fluid-structure interaction

Figure 4.1.: Polar coordinates $r, \theta$ with respect to the cross-section of the oscillating rigid cylinder. The oscillatory motion of the cylinder (with velocity $u_0 e^{i\omega t}$) is in the direction of $\theta = 0$.

in which $H_0^{(1)}$ and $H_1^{(1)}$ are zeroth- and first-order Hankel functions of the first kind, and $k_s = \sqrt{i\omega \rho_f / \eta}$ is denoted as the shear wave number in Kremlevskii and Stepičev [31].

The same expression for the resulting fluid force $f$ can be found in Rosenhead [42, Ch. VII.12, Eq. (102)] and Chen et al. [9], but expressed with modified Bessel functions.

Fluid forces in an unbounded fluid cavity for the case of small penetration depths According to Kremlevskii and Stepičev [31], the expression for the fluid force $f$ from Eq. (4.13) can be simplified if the penetration depth $\delta$ is much less than the radius of the rod, which is stated in Kremlevskii and Stepičev [31] as the condition $k_s R \gg 1$, in which case the ratio between the Hankel functions simplifies to $H_1^{(1)}(ik_s R) / H_0^{(1)}(ik_s R) \approx -i$. The fluid force from Eq. (4.13) can then be written for $\delta / R \ll 1$ as

$$f = (\alpha + i\beta)u$$  \hspace{1cm} (4.14)

with

$$\alpha = -2\sqrt{2}\pi R \sqrt{\omega \rho_f / \eta}$$ \hspace{1cm} (4.15)

$$\beta = -2\sqrt{2}\pi R \sqrt{\omega \rho_f / \eta} - \pi R^2 \rho_f \omega$$ \hspace{1cm} (4.16)

in which $\alpha$ and $\beta$ describe the dissipative and the inertial part of the fluid force respectively. It can be observed that the inertial part of the fluid force consists of two parts, one dependent on fluid viscosity and density, and the other independent of the fluid viscosity. By rewriting the inertial part of the fluid force as

$$\beta = -\pi R^2 \rho_f \omega (1 + 2 \frac{\delta}{R})$$ \hspace{1cm} (4.17)

The positive root of $k_s$ must be used. This can be seen for instance from the case of small penetration depths discussed in the next paragraph. In this case, the ratio between the Hankel functions in Eq. (4.13) can be simplified to $H_1^{(1)}(ik_s R) / H_0^{(1)}(ik_s R) \approx -i$, and the sign of the root of $k_s$ in Eq. (4.13) must be chosen such that the fluid forces have a physically meaningful sign.
it can be seen that the part depending on fluid viscosity is a factor \(2\delta/R\) smaller than the part which is not dependent on viscosity. It can also be seen from Eq. (4.17) that the part of the inertial force which is related to fluid viscosity depends on the fluid mass (per length) contained in a layer of thickness \(\delta\) around the cylinder. On the other hand, the part of the inertial force which is independent of fluid viscosity, depends on the fluid mass (per length) which is displaced by the cylinder. If \(\delta/R \ll 1\), which is the case if the penetration depth is much smaller than the radius of the cylinder, the dependence of the inertial force on the fluid viscosity becomes negligible.

**Fluid forces in a finite fluid cavity** In Chen et al. [9] (consider errata given in Wambgsanss et al. [53]), the case of a rigid cylinder of infinite extent oscillating in a fluid bounded by a concentric cylindrical rigid wall is considered. The radii of the oscillating cylinder and the cylindrical wall are denoted by \(d\) and \(D\) respectively.\(^7\) The velocity of the harmonically oscillating cylinder is given by \(\Re\{u_0 e^{i\omega t}\} = u_0 \cos(\omega t)\). The resulting fluid force per unit length acting on the oscillating cylinder is given as

\[
f = Mu_0\omega \left[\Re\{H\} \sin(\omega t) + \Im\{H\} \cos(\omega t)\right]
\]

in which \(M = \rho \pi d^2\) and \(H(k_s, d, D)\) is a complex-valued function given in Chen et al. [9] with \(k_s = \sqrt{i\omega \rho_f / \eta}\) defined identically to the model of Kremlevskii and Stepichev [31]. The force \(f\) consists of two parts. The part \(Mu_0\omega \Re\{H\} \sin(\omega t)\) is in phase with the acceleration and represents the inertial part. The part \(Mu_0\omega \Im\{H\} \cos(\omega t)\) is in anti-phase to the velocity and represents the dissipative part. Fig. 4.2 shows the dependence of the real and imaginary parts of the function \(H\) on the ratio of the cylinder radii \(D/d\) for several values of parameter \(S = \omega d^2 \rho_f / \eta = 2d^2 / \delta^2\). These graphs indicate the dependence of the fluid forces on the dimension of the fluid boundary for given oscillation frequency and fluid properties.

### 4.1.2.4. 2D and 3D, compressible fluid models

As seen from the dimensional analysis in section 4.1.2.1, fluid compressibility is expected to have a small influence on the fluid forces for the situation considered in this text. A description of the forces of a compressible viscous fluid on a harmonically oscillating rigid cylinder of infinite extent and the fluid bounded by a concentric rigid wall is given in Kadyrov et al. [30]. An expression for the fluid forces of a compressible viscous unbounded fluid on a cylinder of infinite length which performs harmonic oscillations with sinusoidally varying amplitude along the cylinder axis can be found in Van Eysden and Sader [52].

In most practical measurement situations the fluid will be bounded and acoustic waves reflected at the fluid boundaries could lead to acoustic resonances in the fluid chamber. It seems reasonable therefore to consider the case of a bounded fluid. The situation with the current density sensor does not correspond to the situation

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\(^7\)The letter 'd' is indeed used to denote a radius in Chen et al. [9].
4.1. Fluid-structure interaction

of a cylinder of infinite extent, but sufficiently far away from the free end of the tube, the velocity field of the fluid is expected to be two-dimensional. Therefore the description of the fluid forces presented in Kadyrov et al. [30] is considered here in more detail.

In contrast to the situation of an oscillating rigid infinitely long cylinder, it would be interesting to study a cylinder with (sinusoidally) varying deflections along its axis and recognize possible differences between the two- and three-dimensional descriptions of the fluid velocity field. At this time, no fluid model considering a finite cylindrical beam in a bounded fluid seems to be available. In order to recognize all the same some of the features of a three-dimensional fluid velocity field, the fluid model of Van Eysden and Sader [52] is also employed in this work.

Fluid forces in a 2D bounded fluid  In Kadyrov et al. [30] the situation of an infinitely long rigid cylinder performing transverse oscillations in a concentric cylindrical rigid fluid cavity filled with a compressible viscous fluid is considered. Expressions for the two-dimensional velocity field in the fluid and the fluid forces acting on the oscillating cylinder are derived. The motion of the harmonically oscillating cylinder with radius $R$ is described by its velocity $\Re\{u\} = \Re\{u_0 e^{i\omega t}\} = u_0 \cos(\omega t)$ in which $u_0$ is the velocity amplitude and $\omega$ is the angular oscillation frequency. The fluid is described by the linearized Navier-Stokes equation and the equation of
continuity for a viscous compressible fluid

\[ \rho_f \frac{\partial \vec{v}}{\partial t} = -\nabla \rho' + \frac{\eta}{3} \nabla \nabla \cdot \vec{v} + \eta \Delta \vec{v} \]  
(4.19)

\[ \frac{\partial \rho'}{\partial t} + \rho_f \nabla \cdot \vec{v} = 0 \]  
(4.20)

and the following constitutive law for the fluid

\[ \frac{\partial \rho'}{\partial t} = c_f^2 \]  
(4.21)

in which \( \vec{v} \) is the fluid velocity vector, \( \rho_f \) is the density of the fluid at rest, \( \rho' \) and \( p' \) are the variations of fluid density and pressure, \( \eta \) is the dynamic viscosity of the fluid, \( c_f \) is the speed of sound in the fluid, and \( t \) is time.

The formula for the fluid forces is not explicitly given in Kadyrov et al. [30], but in applied form as a loading term of the cylinder. By repeating the calculations in Kadyrov et al. [30], the following expression for the resulting fluid force per unit length \( f \) is obtained

\[ f = \rho_f \pi R^2 i \omega \left[ (m_{11} + m_{13}) J_1(\chi_1) + (m_{21} + m_{23}) Y_1(\chi_1) - (m_{31} + m_{33}) J_1(\chi_2) - (m_{41} + m_{43}) K_1(\chi_2) \right] \]  
(4.22)

with

\[ \chi_1 = \sqrt{\frac{\bar{\omega}^2}{(1 + 2i\bar{\omega}/Re)}} \quad \chi_2 = \sqrt{2i\bar{\omega} Re/3} \]  
(4.23)

in which \( \bar{\omega} = \omega R/c_f \) is the non-dimensional angular oscillation frequency of the cylinder, \( Re = 3 \rho_f c_f R/(2\eta) \), and \( J_1, Y_1, I_1, K_1 \) are (modified) Bessel functions, and \( m_{rs} \) are entries of the matrix \( \mathbf{M}^{-1} \) (inverse of the matrix \( \mathbf{M} \)) given in the appendix, Eq. (A.9) with row index \( r \) and column index \( s \).

**Fluid forces in a 3D unbounded fluid** In Van Eysden and Sader [52] transverse time-harmonic oscillations of an infinite cylinder with sinusoidally varying deflections along its axis and immersed in an unbounded viscous compressible fluid is considered. For the cylinder with radius \( R \), it is assumed that the cylinder axis is in \( z \)-direction and that the deflections of the cylinder axis are described as \( \Re \{ \mathbf{Y} \} = \Re \{ Y_0 e^{ikz} e^{-i\omega t} \} \) in which \( Y_0 \) is the deflection amplitude, \( k_z \) is the wavenumber along the cylinder axis, and \( \omega \) is the angular oscillation frequency.9 Expressions for the three-dimensional velocity field in the fluid and the fluid forces acting on the oscillating cylinder are derived in Van Eysden and Sader [32]. The derivation is based on a description of the fluid by the compressible linearized Navier-Stokes

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9Some typos seem to be present in the paper. See explanation in appendix A.8.
9Note that Van Eysden and Sader [52] use a time-dependence of the form \( e^{-i\omega t} \).
4.1. Fluid-structure interaction equation, the equation of continuity, and a linearized equation of state for the fluid

\[ \tilde{\rho} \frac{\partial \tilde{v}}{\partial t} = -\nabla p' + \eta \nabla^2 \tilde{v} + \left( \frac{\eta}{3} + \eta_B \right) \nabla (\nabla \cdot \tilde{v}) \] (4.24)

\[ \frac{\partial \tilde{p}}{\partial t} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0 \] (4.25)

\[ \tilde{\rho} = \rho_f + \rho' = \rho_f (1 + K p') \] (4.26)

in which \( \tilde{v} \) is the fluid velocity vector, \( \rho_f \) is the density of the fluid at rest, \( \rho' \) and \( p' \) are the variations of fluid density and pressure, \( \eta \) and \( \eta_B \) are the fluid’s dynamic viscosity and bulk viscosity. \( K \) denotes the fluid compressibility, which is defined as

\[ K = \frac{1}{\rho_f} \left( \frac{\partial \tilde{p}}{\partial \tilde{\rho}} \right) \] (4.27)

According to Landau and Lifshitz [32], this fluid compressibility is related to the speed of sound \( c_f \) by \( K = 1/(\rho_f c_f^2) \).

In Van Eysden and Sader [52] the resulting fluid force per unit length \( f \) acting on the cylinder is expressed as

\[ f = \pi \rho_f \omega^2 R^2 \Gamma(\omega) Y(z, \omega) \] (4.28)

in which \( \Gamma \) is called the hydrodynamic function and represents a normalized force per length.

The hydrodynamic function \( \Gamma = \Gamma(Re, \kappa, \zeta, \eta_B/\eta) \) is given in the appendix A.9 for reference. It depends on the Reynolds number \( Re \), the normalized wavenumber of the cylinder \( \kappa \), the normalized wavenumber of the fluid \( \zeta \), and the ratio of the bulk and dynamic viscosity of the fluid. These parameters are defined as

\[ Re = \frac{\rho_f \omega R^2}{\eta} \quad \kappa = k_z R \quad \zeta = \omega R \sqrt{\rho_f K} = \frac{\omega R}{c_f} \] (4.29)

In this text, the bulk viscosity \( \eta_B \) of the fluid is always neglected. Also in the results and discussion of Van Eysden and Sader [52], the Stokes hypothesis has been applied, and the bulk viscosity has been neglected.

4.1.3. Finite element simulations

As already discussed in section 1.3, analytical models have several advantages. They allow recognizing physical dependencies directly, e.g. the dependency of acoustic radiation from the relationship between the wavenumbers of the flexurally oscillating beam and in the fluid (section 4.1.2.1, Van Eysden and Sader [52]), studies of parameter dependencies are quickly performed, and the underlaying assumptions and simplifications as well as the exploration of their validity help getting a physical understanding of the system. On the other hand, analytical models are often limited in the geometric complexity they can handle and the ability to cover all physical aspects involved, and expressions quickly become bulky and impractical for further
evaluation, e.g. the description of a vibrating cantilever beam of finite length in a fluid or the effect of cross-sectional distortion of the beam under bending can hardly be modeled analytically, and already the inclusion of rotary inertia and shearing effects in the beam model leads to rather bulky equations. Finite element models can be useful to explore such geometries and physical effects that cannot be described analytically and recognize the limitations of analytical models or enable to introduce suitable corrections. They also allow observing and understanding effects that are hard to capture by other analytical or experimental means, e.g. the fluid motion in the vicinity of the free end of a vibrating cantilever beam immersed in fluid.

As mentioned in section 1.3, numerical simulations of a fluid-structure interaction problem involving a fluid described by the Navier-Stokes equation are challenging. It is hard to cover the different dimensions in the model, e.g. resolve the boundary layer which has a dimension in the micrometer range, over the length of the sensor which has a dimension in the centimeter range. The nonlinearity of the Navier-Stokes equation dictates a simulation in time domain. If a vibrating structure immersed in fluid is considered and the steady state solution is sought, one has to await the decay of transients, which usually requires many simulation steps, and extremely long simulation times are required. Despite a tremendous effort a trustworthy numerical fluid-structure interaction model based on the Navier-Stokes equation has not been achieved in this work. The two commercially available softwares explored during this work could either not handle the large models or gave no trustworthy results. FEM simulations of fluid-structure interaction that treat the fluid as an acoustic medium are easier to perform and proved to give useful results. There is no boundary layer to resolve and as the wavelengths in the fluids considered in this work are quite large compared to the other dimensions in the system, the required number of elements to discretize the fluid is moderate. The ability to perform simulations in the frequency domain allows directly obtaining the steady state solution for an oscillation mode of a vibrating structure. A disadvantage of this type of simulation is the absence of a possibility to represent the viscous fluid damping in the system.

The fact that fluid-structure interaction simulations with an acoustic medium give useful results for the density sensor considered in this work is not too surprising. The dimensional analysis presented in section 4.1.2.1 revealed that for the situation considered here (frequency range of sensor, choice of test fluids), pressure forces are the prominent forces in the system and viscous forces are small. Taking the linearized compressible Navier-Stokes equation as presented in Eq. (4.19) and neglecting viscosity leads to

\[ \nabla p' + \rho_f \frac{\partial \tilde{u}}{\partial t} = 0 \]  

(4.30)

which is already the momentum equation by which the acoustic medium is described in the FEM model, together with the constitutive equation for an inviscid, linear, compressible fluid and the equation of continuity, which (combined) can be expressed as

\[ p' = -K_f \nabla \cdot \tilde{u} \]  

(4.31)

in which \( K_f \) is the bulk modulus of the fluid, and \( \tilde{u} \) is the fluid particle displacement. The bulk modulus of the fluid is related to the speed of sound and the density of the fluid by \( K_f = \rho_f c_f^2 \).
4.1. Fluid-structure interaction

All FEM simulations in this text are performed with the Abaqus 6.10-EF1 software. The structural FEM model for the sensor has already been described in section 3.2.5. Here, only the inclusion of the acoustic fluid surrounding the sensor is considered. In the experiments the test fluids are held in a glass beaker. Consequently the fluid in the FEM model is bounded by a cylindrical cavity with dimensions according to Fig. 4.3.

![Figure 4.3. Dimensions of the cylindrical fluid cavity in the FEM model.](image)

The fluid boundaries in contact with the cylindrical cavity and the sensor housing (hatched edges in Fig. 4.3) are modeled as rigid walls. The circular ring-shaped fluid boundary on the top (around the sensor housing) is modeled as a free surface. In Abaqus 6.10-EF1 the default fluid boundary condition is a rigid wall. Actually, the pressure gradient normal to the boundary divided by the fluid density is set to zero in the absence of any other boundary conditions, which represents a stationary rigid wall. A free surface is defined in Abaqus 6.10-EF1 by setting the pressure to zero on the respective surface. The coupling between the oscillating parts of the sensor and the fluid is automatized in the applied coupling procedure (interface elements are computed internally). The fluid is discretized by 20-node quadratic acoustic brick elements (element type AC3D20), which have acoustic pressure as the only degree of freedom (except coupled acoustic-structural elements that have also displacement degrees of freedom). Regarding the mesh refinement of the acoustic medium, it is stated in the Abaqus Analysis User’s Manual, that for a reasonable accuracy the shortest wavelength in the analysis should be resolved with at least six internodal intervals (distance between neighboring nodes in an element), and that a resolution with ten or more internodal intervals would substantially improve the accuracy. It is also noted that if the details near the fluid-structure interface are important, the two regions should show a similarly high refinement.

From the test fluids considered in this work (excluding air) and the resonance frequencies measured therein, it follows that the shortest acoustic wavelength present in the experiments has a dimension of 118.9 mm and occurs in 2,2,4-Trimethylpentane. A mesh convergence study has been conducted and the finally chosen mesh for all FEM models considered in this work is depicted in Fig. 4.4. The mesh shows a much higher resolution of the aforementioned wavelength in 2,2,4-Trimethylpentane.
than the aspired ten internodal intervals, but also shows a high refinement at the interface between the tube and the fluid. Note that quadratic elements are used and the internodal interval corresponds to half of the element length. The total number of elements in the fluid cavity is 94080.

The analysis procedure ‘natural frequency extraction’ is used which performs an eigenvalue extraction and provides for example the natural frequencies, and the corresponding mode shapes of the sensor, and the pressure distribution in the fluid.

From the simulation results the pressures at the fluid nodes which are at the interface between the sensor and the fluid are extracted. A Matlab routine has been written that allows to visualize the resulting pressure forces along the tube of the sensor (the pressures at the nodes along the front surface at the free end of the tube are excluded though).

As mentioned earlier, a disadvantage of this type of simulation is that fluid damping cannot be included. A volumetric drag coefficient could be defined in the model, that models fluid velocity-dependent pressure amplitude losses, and the model could be evaluated using a complex eigenvalue extraction procedure. Such a model is used for example if an acoustic medium flowing through a porous matrix is considered. It cannot represent the physics of the damping of the current density sensor and has therefore not been used in this work.

4.1.4. Introduction of fluid forces in the sensor models

In section 3.2.2 where the beam model implemented in the analytical models of the sensor is discussed, a general external distributed load has been introduced and has been denoted as \( q(x,t) \). The analytical descriptions of the fluid forces in section 4.1.2
all consider time-harmonic motion and the fluid forces can always be brought in the
form \( f(\omega) = (\alpha + i \beta)u \) as in Eq. (4.14). Therein, \( \Re\{u\} = \Re\{u_0 e^{i \omega t}\} = u_0 \cos(\omega t) \)
describes the velocity of the oscillating beam (with velocity amplitude \( u_0 \) and angular
oscillation frequency \( \omega \)), and \( \alpha \) and \( \beta \) describe the dissipative and inertial parts
of the fluid forces, respectively. As the analytical models for the sensor presented in
section 3.2 also assume time-harmonic motion, the fluid forces can simply be incor-
porated in these models by setting \( q(x, t) = f(\omega) \).

Considering the mobility model for the density sensor in fluid, the transfer matrix
of the beam in fluid can be derived as in section 3.2.3 from the ansatz \( y(x, t) = e^{i k x} \cdot e^{i \omega t} \)
for the deflection of the centroidal axis of the beam, which again leads to
a solution of the form given in Eq. (3.10) with wavenumbers of the form given in
Eqs. (3.11), but in which the expressions for \( B_1 \) and \( B_2 \) must now be replaced by

\[
B_1 = \frac{(1 + s^2)\omega^2}{c_0^2} + \frac{s^2}{m c_0^2} (\alpha + i \beta) i \omega \\
B_2 = \frac{s^2 \omega^4}{c_0^4} - \frac{\omega^2}{c_0^2 j_y^2} - \frac{(\alpha + i \beta) i \omega}{m c_0^2 j_y^2} + \frac{s^2}{m c_0^2} (\alpha + i \beta) i \omega^3
\]

in which \( \alpha \) and \( \beta \) describe the dissipative and inertial parts of the fluid forces re-
spectively.
The derivation of the transfer matrix for the rigid mass in fluid is presented in ap-
pendix A.10. The combination of the transfer matrices for the beam and the rigid
mass in fluid in order to obtain the description of the whole system, and the deriv a-
tion of the driving point mobility function is done according to the description in
section 3.2.3.

In the DeqD model for the density sensor in fluid, only inertial fluid forces are
considered, and dissipative fluid forces are neglected. Therefore, only the inertial
part \( \beta \) of the fluid forces can be taken into account, and the dissipative part \( \alpha \)
of the fluid forces is neglected. The frequency equation of the DeqD model for
the density sensor in fluid is given in appendix A.4. The DeqD model represents
a simplified model of the density sensor, in which all damping effects (including
material damping) are neglected, and only a frequency equation is evaluated. The
motivation for this model is, as stated earlier, to explore the level of detail and com-
plexity that is required in a model in order to accurately describe the density sensor.

A specialty of the mobility model is that it allows to adapt the magnitude of
the fluid forces acting near the free end of the oscillating beam. This property
is interesting since fluid forces that stem from 2D fluid models overestimate the
fluid forces at the free end of the beam. Because the fluid cannot effectively be
displaced by the lateral motion of the flexurally oscillating beam near the free end,
but can shun towards the front end of the beam, the fluid forces acting on the
beam are smaller than the model would predict from the beam velocity at its free
end. The effect can be seen for instance from the resulting fluid forces extracted
from the FEM simulations. This decrease in the magnitude of the fluid forces near
Chapter 4. Modeling of the resonator in fluid

the free end can be represented in the mobility model. This is possible because the mobility model consists of two parts, the cantilever beam and a rigid mass. The description of this rigid mass is based on evaluating the linear and angular momentum balances, in which the fluid forces enter in integrated form (see appendix A.10). Before integrating the fluid forces they can be multiplied by an arbitrary function in order to mimic decreasing fluid forces towards the free end. Even though the effective variation of the fluid forces near the free end of the beam is not known to date¹⁰, this procedure allows to estimate the influence of this edge effect on the resonance frequencies predicted by the analytical models. Fig. 4.5 shows the magnitude of the fluid forces along the sensor obtained from the mobility model with/without a ‘free end correction’. The considered ‘free end correction’ along the length \( l_{rm} \) of the rigid mass \( (l_{rm} = 5.74\, \text{mm}) \) is implemented by multiplying the fluid forces along the rigid mass with \( 1 - \frac{2}{3}(\frac{x}{l_{rm}})^2 \). Thereby the fluid forces according to the fluid model from Eqs. (4.14)-(4.16) are quadratically decreased along the axial coordinate \( x \) of the rigid mass and reach 1/3 of the magnitude predicted by the fluid model at the free end of the sensor (at \( x = l_{rm} \)). The quadratic decrease of the fluid forces and the decrease to 1/3 of the originally predicted magnitude at the free end of the sensor are chosen arbitrarily, in order to qualitatively mimic the distribution of the fluid forces observed in FEM simulations (cp. Fig. 4.13).

![Figure 4.5: Distribution of the magnitude of the fluid forces along the sensor immersed in DI water, from the mobility model, and using the fluid model from Eqs. (4.14)-(4.16). Left: Without free end correction. Right: Free end correction of the form \( 1 - \frac{2}{3}(\frac{x}{l_{rm}})^2 \). Solid line: magnitude of fluid forces along the tube, dotted line: magnitude of fluid forces along the rigid mass.](image)

4.2. Experimental and model results

In this section results from experiments and modeling are presented. Measurements have been conducted in order to check the linear fluid behavior, that has been

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¹⁰The decrease of the fluid forces could be modeled according to the results from the FEM simulations. As the FEM models describe the fluid as an acoustic medium and cannot account for the viscous flow behavior, they cannot represent the real situation, and have not been used to extract a description of the decreasing fluid forces.
assumed in the analytical models for the fluid forces presented in the preceding section. The influence of fluid boundaries is examined experimentally. Measurements in various test fluids are performed for the later assessment of the sensor models and the density measurement capabilities. The different fluid models presented in the preceding section are compared to each other. Finally, model predictions for the resonance frequencies and damping in the test fluids are compared with experimental results.

In chapter 3 it has been found that higher order flexural oscillation modes give a better decoupling of the sensor from the mounting than lower order modes. For too high mode orders, the analytical beam model presumably cannot represent the physical situation accurately anymore because of higher order effects. Therefore, in all experiments presented in this section, the third flexural oscillation mode of the density sensor is used.

4.2. Experimental setup

In this section the measurement setup used for the experimental study of the density sensor in fluids is considered. The design and the mounting of the density sensor have been introduced in section 3.1, and parts of the experimental setup have already been presented in sections 3.3.2.1 and 3.3.2.5, where the experimental characterization of the flexural oscillator without fluid has been considered. For the measurements in fluids, the sensor housing is mounted on a heavy and rigid table as shown in Fig. 4.6. Additional holders for the excitation coil and for the beaker with the test fluid are screwed directly on the table. The beaker is clamped in the holder via rubber elements and mounted such that it has no contact to the sensor or the sensor housing. The dimensions of the beaker and the distances between the sensor and the beaker walls are given in Fig. 4.3. The beaker is filled with test fluids such that the entire tube and the base of the sensor housing are immersed in the fluid.

The quantities that shall be measured are the resonance frequency and the damping of the sensor in fluid. The measurement procedure is the same as described in section 3.3.2.1, with the oscillation velocity measured on the inside of the tube by a fiber optic vibrometer system and a side exit head as described in section 3.3.2.5.

Measurements at a fixed temperature are sought and therefore the experimental setup is located in a climatized room. Temperature measurements in the test fluids are conducted using a PT-100 element with an Ametek Jofra Instruments DTI-1000 Digital Temperature Indicator.

4.2.2. Verification of linear fluid response

The dimensional analysis in section 4.1.2.1, considering the non-dimensional Navier-Stokes equation Eq. (4.5), showed that the contribution of the advective derivative of the velocity (which is the non-linear term in the equation) is very small. It has also
been shown that the magnitude of this contribution depends on parameter \( \varepsilon = \frac{V_0}{a} \), which represents the ratio between the deflection amplitude and the length dimension \( a \) of the oscillating body. As an experimental check that these amplitude dependent non-linear forces are negligible for the present measurements, experiments with varying velocity amplitudes of the flexural oscillations of the density sensor have been conducted. If the measured resonance frequencies and Q-factors do not depend on the velocity amplitudes, it can be concluded that the system behaves linearly and the non-linear term in the Navier-Stokes equation is indeed negligible.

Fig. 4.7 shows the measured resonance frequencies and Q-factors for varying velocity amplitudes of the flexural oscillations of the density sensor. Results are shown for the test fluids DI-water, N35, and 2,2,4-Trimethylpentane for which the fluid properties are given in Tab. 4.4. The indicated velocities are the maximally measured velocity amplitudes when recording the phase-frequency curve\(^{11}\), and are measured with the vibrometer at a point inside the sensor tube and approximately 22 mm from the clamped end of the sensor tube. The laser beam is aligned manually to point in the direction of the deflections of the sensor tube.

The variations of the resonance frequencies with velocity amplitudes presented in Fig. 4.7 seem to show a slight trend towards lower resonance frequencies for increasing velocity amplitudes. However, the observed variations in the resonance

\(^{11}\)The maximum amplitude of a resonance peak has been sought manually by adapting the excitation frequency. The frequencies that have been measured to obtain the phase-frequency curve contain this frequency at which maximum amplitude has been found.
4.2. Experimental and model results

Figure 4.7.: Measured resonance frequencies and Q-factors for varying velocity amplitudes of the flexural oscillations of the density sensor.

frequencies are extremely small and considered to be negligible. The variations of the Q-factors with velocity amplitudes presented in Fig. 4.7 do not show a trend, and the observed variations of the Q-factors are small. It is therefore concluded that the resonance frequencies and Q-factors are not dependent on velocity amplitude in
Chapter 4. Modeling of the resonator in fluid

the considered experimental situation. Consequently, it seems justified to describe the fluid behavior by the linearized Navier-Stokes equation.

4.2.3. Influence of sensor fixation

When comparing measured resonance frequencies of the density sensor in fluid with model predictions, either normalized resonance frequencies are compared with each other, or the resonance frequencies from the models are scaled for direct comparison with their experimental counterparts. The resonance frequencies in fluid obtained from the analytical models \( f_{res,fluid;analytic} \) and from measurements \( f_{res,fluid;meas} \) are normalized with the corresponding resonance frequency in air. The resonance frequencies in fluid obtained from the FEM model \( f_{res,fluid;FEM} \) are normalized with the corresponding value in vacuum. Accordingly the scaled resonance frequencies are defined for the analytical models as

\[
\frac{f_{res,scaled}}{f_{res,air;analytic}} = \frac{f_{res,fluid;analytic}}{f_{res,air;analytic}} \cdot \frac{f_{res,air;meas}}{f_{res,air;meas}}
\]  

(4.33)

and for the FEM model as

\[
\frac{f_{res,scaled}}{f_{res,vacuum,FEM}} = \frac{f_{res,fluid,FEM}}{f_{res,vacuum,FEM}} \cdot \frac{f_{res,air;meas}}{f_{res,air;meas}}
\]  

(4.34)

in which \( f_{res,air;analytic} \) and \( f_{res,vacuum,FEM} \) are the resonance frequencies in air from the analytical model and in vacuum for the FEM model.

By this normalization, only the influence of the fluid forces relative to the unloaded sensor can be compared. For instance, influences of the mounting that are not represented in the models, and lead to differing absolute values of the resonance frequencies, are eliminated by this procedure. That this is indeed the case, is visualized in Fig. 4.8, which shows the resonance frequencies in test fluids predicted by the original mobility model and a calibrated mobility model. The calibrated mobility model is obtained by replacing the rigid clamping in the original mobility model with a resilient clamping by introducing a torsional spring according to section 3.2.3. The spring constant of the torsional spring is then adapted such that the resonance frequency in air predicted by the calibrated mobility model coincides with the experimentally determined value.

It can readily be seen from the right graph in Fig. 4.8 that the scaled resonance frequencies in the various test fluids obtained from the original mobility model coincide with the ones obtained with the calibrated model (the maximum difference between the normalized resonance frequencies from the rigid and the resiliently mounted mobility model is \( 2.2 \times 10^{-4} \)). Therefore, a calibration of the analytical models seems obsolete, if normalized or scaled resonance frequencies are considered.
4.2. Experimental and model results

Figure 4.8.: Comparison between resonance frequencies obtained from the mobility model with a rigidly clamped end and with a resiliently mounted end, for the various test fluids with properties given in Tab. 4.4. The resilient mounting is described according to section 3.2.3 with \( k_r = 4069 \text{ Nm} \) and \( k_t = 1 \times 10^{30} \text{ N/m} \), which corresponds to a mounting with a torsional spring and a rigid mounting considering lateral deflections. The rotational spring stiffness is adapted such that the resonance frequency in air predicted by the mobility model coincides with the measured resonance frequency in air. The mobility models use a fluid description according to Kremlevskii and Stepichev [31], Eqs. (4.14)-(4.16). Left: Resonance frequency versus fluid density. Right: Normalized resonance frequency versus fluid density (normalization with the respective resonance frequencies in air predicted by the mobility models).

4.2.4. Influence of fluid boundaries

In order to experimentally determine the influence of the fluid boundaries, experiments with three different beaker sizes and DI-Water as test fluid are conducted. The small, medium, and large beakers have an inner diameter of 67 mm, 88 mm, 102 mm. The distance between the free end of the density sensor and the top edge of the excitation coil which is about 10 mm for measurements in the small and medium beakers, had to be increased for the measurements in the large beaker to about 22.5 mm for the larger beaker to fit into the setup.

The measurements in the small beaker showed three resonance peaks in the frequency range where the resonance peak of the third bending mode of the sensor is expected. The nature of these peaks is not clear. The additional peaks are possibly fluid cavity resonances (acoustic resonances occurring due to reflecting and interfering sound waves between oscillating cylinder and fluid container). The measurements in the medium and large beaker showed no anomalies. Tab. 4.3 shows the measured resonance frequencies and Q-factors from the measurements in the medium and large beaker, which have been determined according to the usual procedure. The recorded phase-frequency and amplitude-frequency curves of the three resonance peaks in the small beaker seem distorted, probably due the mutual interaction be-
between the resonances. Therefore the resonance frequencies reported in Tab. 4.3 for the small beaker could not be determined according to the usual procedure, but are the manually determined frequencies at the maximum amplitudes of the resonance peaks. Q-factors could not be determined for the resonance peaks found in the small beaker.

<table>
<thead>
<tr>
<th></th>
<th>( f_{\text{res}} ) [Hz]</th>
<th>( Q ) [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small beaker</td>
<td>( 8710, 9010, 8455 )</td>
<td>N/A</td>
</tr>
<tr>
<td>Medium beaker</td>
<td>( 8752.8 )</td>
<td>356.7</td>
</tr>
<tr>
<td>Large beaker</td>
<td>( 8726.3 )</td>
<td>504.1</td>
</tr>
</tbody>
</table>

Table 4.3.: Resonance frequencies \( f_{\text{res}} \) and Q-factors \( Q \) measured in DI-Water in three different beaker sizes with inner diameters of 67 mm, 88 mm, 102 mm. In the measurements with the small beaker, three resonance peaks are detected. (No Q-factors are determined for the resonance peaks found in the small beaker.)

The fluid temperature has been measured after experiments in one beaker size were complete, and varied between 21.8 °C and 22.1 °C.

Obviously, the measurements in the small beaker show a strong influence of the fluid boundaries. When comparing between measurements in the medium and the large beaker, the fluid boundaries show to have a small influence on the resonance frequencies, but a strong influence on the damping of the sensor. Comparing the measured resonance frequencies and Q-factors given in Tab. 4.3, it is found that in the large beaker the resonance frequency is 0.3 % lower than in the medium beaker, and that the Q-factor in the larger beaker is 41 % larger than in the medium beaker. For density measurements where the resonance frequency shall be evaluated, the medium beaker seems to be an adequate choice, since the resonance frequencies show small differences between measurements in the medium and the large beaker, and a smaller amount of test fluid is required. If the damping of the sensor shall be evaluated, a more careful study of the influence of the fluid boundaries should be undertaken first. Thereby also different test fluids should be considered.

In all measurements presented in the other sections of this work, the medium beaker has been used.

4.2.5. Measurement of resonance frequency and damping in various test fluids

Measurements have been conducted in various test fluids covering a density range of 700 kg/m\(^3\) to 1000 kg/m\(^3\), a viscosity range of 0.5 mPas to 70 mPas, and speeds of sound in the range of 1100 m/s to 1500 m/s. The fluids and their properties are listed in Tab. 4.4.

Fluids N4, S6, and N35 are CANNON \( \text{R} \), certified viscosity reference standards
4.2. Experimental and model results

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( \rho_f ) [kg/m(^3)]</th>
<th>( \eta ) [mPas]</th>
<th>( c_f ) [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air, 22 °C</td>
<td>1.2</td>
<td>0.018</td>
<td>344.6</td>
</tr>
<tr>
<td>DI-Water, 21.1 °C</td>
<td>997.97</td>
<td>0.977</td>
<td>1483.9</td>
</tr>
<tr>
<td>N4, 21.5 °C</td>
<td>813.68</td>
<td>5.220</td>
<td>1364.3</td>
</tr>
<tr>
<td>S6, 21.3 °C</td>
<td>874.34</td>
<td>9.085</td>
<td>1406.5</td>
</tr>
<tr>
<td>N35, 21.5 °C</td>
<td>860.54</td>
<td>68.51</td>
<td>1445.1</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane, 20.9 °C</td>
<td>691</td>
<td>0.50</td>
<td>1100.7</td>
</tr>
</tbody>
</table>

Table 4.4.: Test fluids and their properties (density \( \rho_f \), viscosity \( \eta \), speed of sound \( c_f \)). The indicated temperatures are the reference temperatures for the determination of the fluid properties. See further explanations in the text.

(N4 is composed of Poly-Alfa-Olefin, S6 and N35 are composed of mineral oil).

The temperature in the test fluids has not been monitored continuously, but after a series of measurements. The temperature differences before and after a series of measurements in a single test fluid have not exceeded 0.1 °C in the recorded cases. While the set temperature for the climatized room has always been 22 °C, the temperatures in the various test fluids for which results are presented in section 4.2.5 have been between 20.9 °C and 21.5 °C.\(^\text{12}\) The uncertainty of the measured temperatures is therefore conservatively estimated to be ±0.5 °C. The temperatures indicated in Tab. 4.4 are the temperatures measured in the fluids after the measurements in the respective fluids were completed. An exception is air, for which the set temperature of the air conditioning system is indicated.

The densities and viscosities of air, DI-Water, and 2,2,4-Trimethylpentane are taken from literature. For the fluids N4, S6, and N35, the manufacturer provides a program called VisDisk that allows to interpolate the viscosities and densities stated on the certificate of analysis in order to obtain the properties at the desired temperature.

The speeds of sound in air and 2,2,4-Trimethylpentane are taken from literature. The speeds of sound in the other fluids have been determined experimentally. Therefore, an ultrasound transducer has been used to send sound pulses through the fluid and detect the pulses reflected at the fluid boundary. Thereby the fluid is contained in a double-walled beaker and the ultrasound transducer is slightly dipped into the fluid. The ultrasound transducer is oriented such that the emitted sound pulses travel perpendicular to the fluid boundary (beaker wall) at which they are reflected. By detecting the travel time of the ultrasound pulse from emission to detection (after reflection) at two positions which are a known distance apart, the speed of sound can be calculated. The employed ultrasound transducer is from Panametrics (Accuscan “S”, A543S), the two measurement positions have been reached using a micrometer screw, the ultrasound pulses have been recorded from the ultrasound transducer us-

\(^\text{12}\)The installed air conditioning system can only cool, but not heat the room.
ing an oscilloscope, and the evaluation is done in Matlab. Measurements have been conducted in the aforementioned double-walled beaker through which temperature-controlled water (from a Viscotherm VT2 water bath from Paar Physica) has been pumped in order to achieve the reference temperatures indicated in Tab. 4.4. In order to check the described method to determine the speed of sound, the speed of sound measured in DI-Water has been compared with a value from literature. From [4], for water at 1 bar, and by linearly interpolating between the tabulated values for 20 °C and 25 °C, one obtains a speed of sound of 1485.5 m/s at 21.1 °C, which is in very good agreement with the experimentally determined value presented in Tab. 4.4.

The results of the measurements in the test fluids from Tab. 4.4 are presented in Fig. 4.9. The left graph shows the measured resonance frequencies versus fluid density (blue squares, left axis), and the fluid viscosities of the test fluids plotted in the same graph (red dotted line, right axes). The right graph shows the measured bandwidths versus the square root of the fluid density-fluid viscosity product \( \sqrt{\rho_f \eta} \). Note that these bandwidths contain both the structural damping and the fluid damping of the sensor. The motivation to plot the bandwidths versus \( \sqrt{\rho_f \eta} \) stems from the observation that the fluid damping forces in the incompressible viscous unbounded fluid model according to Eq. (4.15) depend on this expression.

![Figure 4.9: Results of the measurements in the test fluids of Tab. 4.4. Left: Resonance frequency vs. fluid density (blue squares, left axis) and viscosities of the test fluids (red dotted line, right axes). Right: Bandwidth vs. the square root of the fluid density-fluid viscosity product. The plotted bandwidths contain both the structural damping and the fluid damping of the sensor.](image)

Note that the measured resonance frequency/fluid density data points in the left graph of Fig. 4.9 build a quite smooth curve, even though there are large differences between the viscosities of the test fluids. This supports the expectation that fluid viscosity effects have a negligible influence on the inertial fluid forces (for the test fluids considered in this work) and therefore on the measured resonance frequencies.
In order to assess the precision of the conducted measurements, Tab. 4.5 lists the mean values and standard deviations of the measured resonance frequencies and bandwidths. Measurements in each test fluid have been conducted three times, except 2,2,4-Trimethylpentane which has been measured 4 times, and air which has been measured 5 times. During all measurements, the mounting of the sensor, and the mounting and positioning of the side exit head has not been changed. This ensures an unchanging initial state of the sensor. The measurements in air have been conducted after the measurements in one fluid have been completed, and at the end of the measurements. With this procedure, changes in the initial state of the sensor would be readily detected, for instance from the standard deviation of the measurements in air.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>(f_{\text{res,mean}}) [Hz]</th>
<th>(f_{\text{res, std.dev.}}) [Hz]</th>
<th>(df_{\text{mean}}) [Hz]</th>
<th>(df_{\text{std.dev.}}) [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>11 069</td>
<td>1.3</td>
<td>16.4</td>
<td>1.2</td>
</tr>
<tr>
<td>DI-Water</td>
<td>8766.7</td>
<td>0.15</td>
<td>25.1</td>
<td>0.04</td>
</tr>
<tr>
<td>N4,</td>
<td>9043</td>
<td>0.1</td>
<td>40.5</td>
<td>0.063</td>
</tr>
<tr>
<td>S6</td>
<td>8960.1</td>
<td>0.058</td>
<td>53.5</td>
<td>0.032</td>
</tr>
<tr>
<td>N35</td>
<td>8965.6</td>
<td>0.2</td>
<td>93.6</td>
<td>0.32</td>
</tr>
<tr>
<td>Trimethylpentane</td>
<td>9254.5</td>
<td>0.17</td>
<td>18.5</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Table 4.5.: Mean values and standard deviations of measured resonance frequencies \(f_{\text{res}}\) and bandwidths \(\Delta f\) of the measurements presented in section 4.2.5. Trimethylpentane stands for 2,2,4-Trimethylpentane.

Tab. 4.5 shows that the standard deviations of the measured resonance frequencies are very small with respect to their mean values and a precise measurement has been achieved. The standard deviations of the measured bandwidths are small with respect to their mean values, except for air where a quite large standard deviation is observed. As mentioned in section 3.3.2.5, the damping of the sensor is influenced by the positioning and the alignment of the side exit head inside the tube. Even though the fixation of the side exit head has not been changed for all measurements presented here, some slight variations in its alignment inside the tube could have occurred due to forces from the vibrometer sensor head attached to it. This could have led to a not entirely stable internal damping of the sensor, which would explain the observed larger standard deviation in air.

### 4.2.6. Comparison between fluid forces from analytical and finite element models

In this section the fluid forces predicted by the different fluid models introduced in section 4.1.2 are compared to each other in order to estimate the influence of the different aspects included in these models and to assess the significance of these aspects. The considered fluid models are: the model from Kremlevskii and Stepichev [31] in which the fluid forces are described by Eq. (4.13); the model from Kremlevskii and Stepichev [31] for the case of small penetration depths, in which the fluid forces are described by Eqs. (4.14)-(4.16); the model from Chen et al. [9] in which the fluid
forces are described by Eq. (4.18); the model from Kadyrov et al. [30] in which the fluid forces are described by Eq. (4.22); the model from Van Eysden and Sader [52] in which the fluid forces are described by Eq. (4.28); the FEM model described in section 4.1.3.

Two models have been introduced that describe the fluid forces of an incompressible viscous fluid on a harmonically oscillating rigid cylinder of infinite extent: the model of Kremlevskii and Stepichev [31] which considers an unbounded fluid and the model of Chen et al. [9] which describes a fluid bounded by a concentric cylindrical rigid boundary. In order to discuss the predicted influence of the fluid boundary for the experimental setup of this work, Fig. 4.10 shows the real and imaginary parts of function \( H \) defined in the fluid model of Chen et al., Eq. (4.18), for the fluid properties and resonance frequencies of the experiments presented in section 4.2.5. Function \( H \) is a normalized fluid force per length, where normalization is done with the fluid force for an incompressible, inviscid, unbounded fluid. The real part \( \Re \{H\} \) describes the inertial part, and the imaginary part \( \Im \{H\} \) describes the dissipative part of the fluid forces. The dashed lines in Fig. 4.10 indicate the solution for an incompressible viscous unbounded fluid given in Chen et al. [9], which is equal to the description of the fluid forces according to Kremlevskii and Stepichev, Eq. (4.13). For the experimental setup one has \( D/d = 11 \). A value of \( \Re \{H\} = 1 \) and \( \Im \{H\} = 0 \) corresponds to the case of an incompressible, inviscid, unbounded fluid. Therefore the relative influence of viscosity and fluid boundary on the inertial force can directly be read from the ordinate in the left graph of Fig. 4.10. Considering an unbounded fluid (dashed lines in Fig. 4.10), the influence of fluid viscosity on the inertial fluid forces is below 1% compared to the inviscid (unbounded) case, except for the fluid N35 where it is below 3%.

According to the left graph of Fig. 4.10, the fluid model of Chen et al. [9] predicts lower inertial forces in larger fluid containers. With respect to the density sensor, larger resonance frequencies are expected in larger fluid containers therefore. Ac-
4.2. Experimental and model results

According to the right graph of Fig. 4.10, a negligible influence of the fluid boundaries on the dissipative forces is predicted by the fluid model of Chen et al. [9] for increasing values of $D/d$.

Experiments with varying beaker sizes presented in section 4.2.4 showed a slight decrease in resonance frequency between the medium and large beaker sizes (experimentally determined decrease of $-0.3\%$ for an increase from $D/d = 11.0$ to $D/d = 12.8$) and a considerable increase of the Q-factor between the medium and large beaker sizes (experimentally determined increase of $41\%$ for an increase from $D/d = 11.0$ to $D/d = 12.8$). Obviously, the fluid model of Chen et al. [9] cannot explain this experimentally observed behavior. For the small beaker, three resonance peaks were found which is not expected from the above incompressible fluid descriptions, but could be caused by fluid cavity resonances of a compressible fluid. Also in measurements with the fourth bending mode (not shown here) disturbing resonance peaks were found.

The dimensional analysis presented in section 4.1.2.1 actually suggests that compressibility can have a small influence on the fluid forces. For increasing oscillation frequencies the fluid should actually become more compressible (cp. parameter $M_p$ in section 4.1.2.1), which could explain the disturbing modes observed in measurements with the fourth bending mode (not shown here). It seems therefore worthwhile to consider also models describing compressible fluids and discuss the differences to the incompressible fluid descriptions discussed so far.

In section 4.1.2.4 the fluid model from Kadyrov et al. [30] has been introduced, which considers an infinite rigid cylinder in a compressible viscous fluid which is bounded by a concentric cylindrical rigid wall. Fig. 4.11 shows a comparison between the fluid forces according to Kadyrov et al. and the fluid forces according to Kremlevskii and Stepichev for small penetration depths, Eqs. (4.14)-(4.16), which are for an incompressible viscous unbounded fluid. Fig. 4.11 shows the parameters $\alpha$ and $\beta$ versus oscillation frequency, where $\alpha$ and $\beta$ respectively describe the dissipative and inertial parts of the fluid force, described by $f = (\alpha + i\beta)u$ in which $u$ is the velocity of the oscillating cylinder. The plots show the solutions according to Kadyrov et al. for three different speeds of sound viz. 1100 m/s, 1400 m/s, and 1500 m/s. The considered fluid has a density of 1000 kg/m$^3$, a viscosity of 10 mPas and vanishing bulk viscosity. The radii of the oscillating cylinder and the cylindrical boundary correspond to the situation in the experiments viz. 3.985 mm and 44 mm.

In the fluid forces according to Kadyrov et al. in Fig. 4.11 one observes peaks that correspond to fluid cavity resonances, which move to higher frequencies for increasing speed of sound of the fluid.\footnote{That these peaks are fluid cavity resonances can also be seen from plots of the 2D fluid pressure field (not shown here).} It is important to note that these fluid resonances affect the fluid forces in a relatively large frequency region. Resonances of the density sensor that lie near such fluid resonances could be highly influenced.
by such fluid resonances.\textsuperscript{14} On the other hand the experimental situation is not a 2D situation as considered in the above model. Deviations between the description according to Kadyrov et al. and the experimental situation have to be expected therefore.

The model from Van Eysden and Sader [52] presented in section 4.1.2.4 considers the fluid forces of a compressible viscous unbounded fluid on an infinite cylinder with sinusoidal deflections along the cylinder axis. The sinusoidal deflections are thereby described by the axial wavenumber $k_z$. This model cannot exactly describe the flexurally oscillating tube of the density sensor, because the tube of the density sensor is finite in length, and therefore the deflections are described by a spectrum of wavenumbers. However, it seems reasonable in a first approximation to consider an axial wavenumber that resembles the value present in the experiments. This allows to study the influence of three-dimensional fluid effects, which are not included in the 2D fluid models discussed so far in this section.

If a Timoshenko beam in a viscous fluid is considered, the deflections are described by two complex wavenumbers (cp. section 4.1.4, Eq. (4.32)), and it is not clear which value to choose in order to approximate the axial wavenumber of the sensor. A simple but reasonable approximation can be obtained from the Euler Bernoulli beam model (which is contained in the Timoshenko beam model). In this work, the inserted wavenumber $k_z$ is therefore calculated from Euler Bernoulli bending theory and for vacuum (appendix A.7). For Euler Bernoulli bending theory, and if only inertial fluid forces are considered, this is also the wavenumber which is present for the beam in fluid (compare section 5.1.1).

\textsuperscript{14}Dependent on whether the resonance frequency of the density sensor is lower or higher than a fluid cavity resonance, the inertial and dissipative fluid forces can be increased or decreased compared to the case of the incompressible viscous unbounded fluid depicted in Fig. 4.11.
Fig. 4.12 shows a comparison between the fluid forces according to Van Eysden and Sader (compressible viscous unbounded fluid), Kadyrov et al. (compressible viscous bounded fluid), and Kremlevskii and Stepichev (incompressible viscous unbounded fluid and for the case of small penetration depths, Eqs. (4.14)-(4.16)). Fluid forces according to Van Eysden and Sader are plotted for zero axial wavenumber (2D situation of an infinite cylinder) and an axial wavenumber in the order of the one present in the experiments with the third bending mode of the density sensor (as discussed above). The value of the inserted normalized wavenumber is \( \kappa = k_z R = 0.343 \) (\( k_z \): axial wavenumber, \( R \): cylinder radius). The fluid properties are the ones of fluid N35 according to Tab. 4.4 and bulk viscosity is neglected. Again, the parameters \( \alpha \) and \( \beta \) graphed in Fig. 4.12 describe the dissipative and inertial parts of the fluid force expressed as \( f = (\alpha + i\beta)u \) (\( u \): velocity of the oscillating cylinder).

According to section 4.1.2.1, an infinitely extended circular cylinder performing harmonic oscillations with deflections varying sinusoidally in axial direction can radiate acoustic energy if the axial wavenumber \( k_z \) is smaller than the acoustic wavenumber \( k_f \) in the fluid. In the case of an increased axial wavenumber the onset of acoustic radiation is moved to a higher oscillation frequency. The presence of acoustic radiation leads to an increased damping of the oscillation. These effects can be observed in the right graph of Fig. 4.12. In Fig. 4.12 the dissipative part \( \alpha \) of the fluid force according to Van Eysden and Sader for zero axial wavenumber is larger than the solution from Kremlevskii and Stepichev. It is clear that for this case the fluid wavenumber is always larger than the axial wavenumber of the beam and acoustic radiation is always present. In the case of a normalized axial beam wavenumber of \( \kappa = 0.343 \) the onset of acoustic radiation should be at 19 803 Hz (axial and acoustic wavenumber are equal) which agrees well with the situation depicted in the right graph of Fig. 4.12. For frequencies below this onset of acoustic radiation the damping forces are slightly smaller than in the solution from Kremlevskii and Stepichev, but are becoming much larger above this point. It is obvious that in the solution
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according to Kadyrov et al. the dissipative fluid force near the depicted fluid resonance largely differs from the ones from the other models (note that the fluid model from Kadyrov et al. considers a bounded fluid, whereas the other fluid models consider an unbounded fluid).

Considering the inertial part $\beta$ of the fluid forces in the left graph of Fig. 4.12 it can be observed that the solution from Van Eysden and Sader for zero axial wavenumber shows a slightly higher force than the solution from Kremlevskii and Stepičev. In contrast the solution for a normalized axial beam wavenumber of $\kappa = 0.343$ shows a smaller force below the onset of acoustic radiation. The force according to Kadyrov et al. shows large deviations from the other models in the neighbourhood of the depicted fluid cavity resonance (note that the fluid model from Kadyrov et al. considers a bounded fluid, whereas the other fluid models consider an unbounded fluid).

FEM simulations yield the pressure forces of the inviscid acoustic fluid acting on the tube of the density sensor in resonance. These pressure forces can be integrated around the circumference and the distribution of the resulting force along the tube can be calculated.\textsuperscript{15} The mode shapes of the resonating tube can also be extracted from the FEM simulations. Thereby it is possible to calculate the fluid forces predicted by the 2D analytical fluid models for the deflection velocities given by the extracted mode shapes. Fig. 4.13 shows a comparison between the resulting pressure forces along the tube obtained from the FEM model, and obtained from the corresponding (inertial) fluid force according to the fluid model from Kremlevskii and Stepičev for small penetration depths and neglected fluid viscosity\textsuperscript{16}, for the deflection velocity described by the extracted mode shape. The mode shape has been extracted from the nodal deflections of the FEM mesh on the outer surface of the tube, and in direction of the beam deflection along the tube. The fluids considered in Fig. 4.13 are DI-water and 2,2,4-Trimethylpentane with properties from Tab. 4.4 (viscosity neglected).

From Fig. 4.13 it is found that the distribution of the resulting pressure force obtained from the FEM simulation is qualitatively quite well described by the description according to the fluid model from Kremlevskii and Stepičev for small penetration depths. The incompressible inviscid unbounded fluid model according to Kremlevskii and Stepičev for small penetration depths generally predicts higher fluid forces, except near the clamped end of the tube. Pressure forces in the FEM simulations must not vanish at the clamped end of the tube because there is no free surface but the surface of the sensor housing constrains the fluid motion. At the free end of the tube, the fluid cannot be effectively compressed by the lateral motion of the tube surface but shunts towards the front surface of the sensor, which leads to reduced pressure forces at the free end of the sensor. Consequently the 2D model according to Kremlevskii and Stepičev for small penetration depths predicts too...\textsuperscript{15}The pressure acting on the nodes of the front surface of the tip of the tube is not included in this discussion.

\textsuperscript{16}The fluid model of Kremlevskii and Stepičev for small penetration depths, Eqs. (4.14)-(4.16), considers fluid viscosity, but it is neglected here to enable a direct comparison with FEM model results.
4.2. Experimental and model results

Figure 4.13.: Inertial fluid forces along the tube of the density sensor in resonance: Pressure forces obtained from FEM simulations (magenta, dotted), and inertial fluid forces calculated according to the fluid model from Kremlevskii and Stepichev for small penetration depths and neglected viscosity, based on the mode shapes extracted from FEM simulations (blue, dotted). The mode shapes have been extracted from the nodal deflections of the FEM mesh on the outer surface of the tube, and in direction of the beam deflection along the tube. Left: Fluid properties of DI-Water (viscosity neglected). Right: Fluid properties of 2,2,4-Trimethylpentane (viscosity neglected).

Large forces compared to the FEM model which accounts for such 3D effects. At the positions of maximal beam deflection (except at the free end), the deviations of the fluid forces according to Kremlevskii and Stepichev for small penetration depths and neglected fluid viscosity from the ones obtained from the FEM model are below 11%.

Fig. 4.14 obtained from the FEM model shows the pressure distribution in the fluid chamber for the density sensor in 2,2,4-Trimethylpentane and for the third flexural oscillation mode of the sensor. A sectional view along the symmetry plane is depicted, which shows both the fluid and the sensor. Fig. 4.15 shows the situation for two fluid cavity resonances for which the deformation of the sensor resembles the deformation of the sensor in the third flexural oscillation mode. These fluid cavity resonances seem therefore likely to be excited in practice by the density sensor. Note that the simulation software normalizes the eigenvectors such that the maximum fluid pressure is one. Consequently the maximum pressure amplitudes observed in Figs. 4.14 and 4.15 is one. The mode shapes of the density sensor depicted in these figures are scaled in order to be visible. In the case of the fluid cavity resonances (Fig. 4.15) the scale factor has been five times larger than in the case of the third flexural oscillation mode of the sensor (Fig. 4.14).

It is interesting to compare the resonance frequencies of the fluid cavity resonances predicted by the fluid model according to Kadyrov et al. (2D compressible viscous bounded fluid) and the fluid cavity resonance found in the FEM model (3D acoustic fluid). As the FEM model cannot account for fluid viscosity, results for 2,2,4-Trimethylpentane which has a very small viscosity is used for the comparison. In 2,2,4-Trimethylpentane the fluid model according to Kadyrov et al. predicts
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4.2.7. Influence of speed of sound

The FEM model described in section 4.1.3 is used in order to study the influence of the speed of sound of a fluid on the resonance frequency of the density...
4.2. Experimental and model results

sensor. Therefore simulations are conducted for three different speeds of sound ($c_1 = 1100 \text{ m/s}$, $c_2 = 1400 \text{ m/s}$, $c_3 = 1500 \text{ m/s}$), and four different fluid densities ($700 \text{ kg/m}^3$, $800 \text{ kg/m}^3$, $900 \text{ kg/m}^3$, $1000 \text{ kg/m}^3$) for each value of the speed of sound. These speeds of sound and fluid densities cover the range of the properties of the test fluids presented in Tab. 4.4. Note that the FEM model considers a finite fluid volume with fluid boundaries as discussed in section 4.1.3 (Fig. 4.3).

The left graph in Fig. 4.16 shows the normalized resonance frequencies from the FEM model ($f_{res,vacuum,FEM} = 11285 \text{ Hz}$ used for normalization) versus fluid density for the various speeds of sound. According to the presented results from the FEM model, an increase of the speed of sound from 1100 m/s to 1500 m/s leads to an increase of the resonance frequency (for a fixed density) that is not larger than 0.7 %. Note that this number does not indicate the overall influence of fluid compressibility on the resonance frequencies of the the density sensor, e.g. compared to the consideration of an incompressible fluid. Note also, that for resonance frequencies of the density sensor which lie close to fluid cavity resonances, the influence of fluid compressibility could be much more pronounced, and could show a much stronger influence on the resonance frequencies of the density sensor than depicted in Fig. 4.16. As the considered FEM model results are from an eigenvalue extraction procedure, and an undamped situation is considered, the possible excitation of nearby fluid cavity resonances (which could occur in the experiments) is not accounted for in the presented FEM model results.

![Graph showing normalized resonance frequencies and change in resonance frequency per change in speed of sound](image)

Figure 4.16.: Left: Normalized resonance frequencies from FEM model for fluids with three different speeds of sound (1100 m/s, 1400 m/s, 1500 m/s), and four different fluid densities (700 kg/m$^3$, 800 kg/m$^3$, 900 kg/m$^3$, 1000 kg/m$^3$). Right: Change in resonance frequency per change in speed of sound as calculated from the left graph. The dashed lines connecting the data points are just for better visibility.

The right graph in Fig. 4.16 shows the change in resonance frequency per change in speed of sound. For instance the ‘x’ denoted ‘c2-c1’ represent the change in resonance frequency (for fixed density) if the speed of sound is increased from $c_1 = 1100 \text{ m/s}$ to $c_2 = 1400 \text{ m/s}$, and accordingly for the other data points. From this...
graph it can be concluded that the change in resonance frequency for a given change in speed of sound is not linear and depends also on the fluid density.

4.2.8. Comparison between model predictions and experiments

In this section measurement results for resonance frequencies and damping in test fluids presented in section 4.2.5 are compared with predictions of the various models introduced so far.

The following models are taken for the comparison:

- **DeqD**: DeqD model with fluid description according to Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), but taking into account only the inertial part \( \beta \) of the fluid forces, and neglecting the dissipative part \( \alpha \) of the fluid forces.

- **Mobility**: Mobility model with fluid description according to Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16).

- **Mobility, corr**: Mobility model with fluid description according to Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16) and a 'free end correction' as discussed in section 4.1.4 and depicted in Fig. 4.5.

- **Mobility, 3D**: Mobility model with 3D compressible fluid description according to Van Eysden and Sader, Eq. (4.28) for a normalized axial wavenumber of the beam of \( \kappa = 0.343 \) as before (section 4.2.6).

- **FEM**: FEM model described in section 4.1.3.

As shown below, the DeqD model and the mobility model give quite similar results. The DeqD model has the disadvantage that damping is not included in this model. Therefore the mobility model is mainly used in the comparison. The fluid model describing an incompressible viscous bounded fluid according to Chen et al. (Eq. (4.18)) is not used in the comparison because the inertial fluid forces would be larger than those for the corresponding unbounded fluid model from Kremlevskii and Stepichev and therefore only increase the discrepancies between measurements and predictions from the Mobility model. The fluid model for the 2D compressible viscous bounded fluid according to Kadyrov et al. (Eq. (4.22)) is also not used in the comparison because it shows fluid cavity resonances that make an evaluation of the mobility model with this fluid model impossible (fluid cavity resonances are near the resonance frequencies of the bending mode of the density sensor and the interference with these fluid cavity resonances makes an evaluation impossible).

The normalized axial wavenumber \( \kappa = k_z R = 0.343 \) (\( k_z \): axial wavenumber, \( R \): cylinder radius) in the Mobility, 3D model is chosen to resemble the value present in the experiments as discussed in section 4.2.6. The inserted wavenumber \( k_z \) is calculated from Euler Bernoulli bending theory and for vacuum (appendix A.7). For
4.2. Experimental and model results

Euler Bernoulli bending theory, and if only inertial fluid forces are considered, this is also the wavenumber which is present for the beam in fluid (compare section 5.1.1).

In the mobility models, material damping has been included by using a hysteretic damping model as described in section 3.2.3, in which simply the Young's modulus $E$ is replaced by a complex modulus $E_d = E \cdot (1 + i/Q_{mat})$ in which $Q_{mat}$ is the quality factor of the material. The value of $Q_{mat}$ is chosen such that the value of the Q-factor in air predicted by the model corresponds to the measured Q-factor in air.

Fig. 4.17 shows the resonance frequencies from measurements in test fluids presented in section 4.2.5 and scaled resonance frequencies (scaling according to Eqs. (4.33) and (4.34)) predicted by various models plotted versus fluid density. The deviations of the scaled resonance frequencies predicted by the various models from the measured resonance frequencies expressed as percentages of the measured resonance frequencies are also presented in the figure.

A detailed view of the resonance frequencies versus fluid density is given in Fig. 4.18 which allows to better distinguish between the different model results.

Figure 4.17: Left: Resonance frequencies from measurements in test fluids presented in section 4.2.5 and scaled resonance frequencies predicted by various models plotted versus fluid density. Right: Deviations of the scaled resonance frequencies predicted by the various models from the measured resonance frequencies expressed as percentage of the measured resonance frequencies.
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Fig. 4.18.: Detailed view of the left graph in Fig. 4.17.

Fig. 4.19 shows the bandwidths due to fluid influence (without influence of structural damping), obtained from measurements in test fluids from section 4.2.5 and from the predictions of the various models, plotted versus the square root of the fluid density-fluid viscosity product. In order to determine the bandwidths due to fluid influence, first the Q-factors due to the fluid influence \( Q_{fluid} \) have to be computed by subtracting the influence of the structural or material damping \( Q_{mat} \) from the measured Q-factors or the corresponding model predictions \( Q_{total} \)

\[
Q_{fluid} = \left( \frac{1}{Q_{total}} - \frac{1}{Q_{mat}} \right)^{-1} \quad (4.35)
\]

For the calculations, the structural or material damping \( Q_{mat} \) (which should be measured in vacuum) is approximated by the values measured in air or predicted by the models in air. From the Q-factors due to the fluid influence, the bandwidths due to the fluid influence can then be calculated as

\[
df_{fluid} = \frac{f_{res, fluid}}{Q_{fluid}} \quad (4.36)
\]

in which \( f_{res, fluid} \) is the measured resonance frequency or the unscaled resonance frequency predicted by the model.

Fig. 4.19 shows also the deviations of the bandwidths due to fluid influence predicted by the various models from the corresponding measured bandwidths expressed as percentages of these measured bandwidths.

Considering Figs. 4.17 and 4.18 it is found that the DeqD, Mobility, and Mobility, corr models give almost identical results. The ‘free end correction’ included in the Mobility, corr model has a visible but negligible influence. All three models though show a slightly wrong trend with increasing differences of the predicted resonance frequencies from the measured resonance frequencies with increasing fluid density. The deviations of these three models are in the range of \(-2.2\%\) to \(-3.6\%\).

The predicted resonance frequencies from the Mobility, 3D and FEM model show a better agreement with the measurement results. The deviations show no trend
4.3. Discussion

Figure 4.19: *Left:* Bandwidths due to fluid influence (without influence of structural damping), obtained from measurements in test fluids from section 4.2.5 and from predictions of various models, plotted versus the square root of the fluid density-fluid viscosity product. *Right:* Deviations of the bandwidths due to fluid influence predicted by the various models from the corresponding measured bandwidths expressed as percentage of these measured bandwidths. (Deviations for air not plotted.)

with respect to fluid density. The deviations of the Mobility, 3D model are in the range of $-0.7\%$ to $-1.1\%$ and the deviations of the FEM model are in the range of $-0.9\%$ to $-1.3\%$.

In Fig. 4.19 the bandwidths in fluid predicted by the models show larger deviations from the measurements, especially for small values of the square root of the fluid density-fluid viscosity product. Even though the general trend is predicted right. The deviations for the Mobility model are in the range of $56.7\%$ to $-19.5\%$ and for the Mobility, 3D model in the range of $-0.7\%$ to $-70.4\%$ (deviations in air are not considered). If the values in 2,2,4-Trimethylpentane would be excluded, the deviations of the Mobility model would be in the range of $10.3\%$ to $-19.5\%$.

4.3. Discussion

The dimensional analysis at the beginning of this chapter revealed that pressure forces are the significant forces in the considered fluid-structure interaction problem. Viscous forces are expected much smaller than the pressure forces and the influence of the advective derivative of the velocity (which is the non-linear term in the Navier-Stokes equation) should be negligible in the momentum balance. The influence of fluid compressibility in the continuity equation is seen to be small, but probably not negligible. From the comparison between the acoustic wavenumber in the fluid and the wavenumber describing the flexural oscillations of the sensor it is
found that no acoustic radiation should be present.

Various analytical fluid models are considered in order to explore the influences of fluid viscosity, fluid boundaries, fluid compressibility, and the difference between treating the fluid velocity field as a two-dimensional or a three-dimensional field. In the existing FEM model of the density sensor, the influence of a surrounding acoustic medium is introduced. The FEM model provides a further means to study the fluid forces, especially considering the effects at the free end of the tube of the sensor, and the possible occurrence of fluid cavity resonances.

Considering the 2D incompressible unbounded fluid model from Kremlevskii and Stepichev, Eq. (4.13) (or the equivalent description given in Chen et al. [9]), the predicted influence of fluid viscosity on the inertial fluid forces for the present experimental situation is below 1% compared to the inviscid (unbounded) case, except for the fluid N35 where it is below 3%.

The fluid model from Chen et al. [9] considers a 2D incompressible viscous bounded fluid, and therefore allows to study the influence of fluid boundaries. However, the variation of the fluid forces predicted by the fluid model of Chen et al. [9] for varying sizes of the fluid container are contradictory to findings from experiments with varying beaker sizes presented in section 4.2.4. Additionally, for a small beaker size and measurements with the fourth bending mode (not shown in this work), disturbing resonance peaks were found (in addition to the expected resonance peak of the bending mode of the density sensor), which are not expected from an incompressible fluid description, but could be caused by fluid cavity resonances of a compressible fluid.

The fluid model from Kadyrov et al., which considers a 2D compressible viscous bounded fluid, predicts fluid cavity resonances with distinct influences on inertial and dissipative fluid forces in a broad frequency range in the vicinity of such fluid cavity resonances. FEM model results show that fluid cavity resonances have a three-dimensional pressure distribution in the fluid cavity. Therefore, the significance of the two-dimensional fluid cavity resonances predicted by the fluid model from Kadyrov et al. must be questioned for the present situation.

The fluid model from Van Eysden and Sader considers a 3D compressible viscous unbounded fluid surrounding an infinitely long cylinder with sinusoidal deflections along the cylinder axis. The onset of acoustic radiation if the wavenumber in fluid is larger than the axial wavenumber of the cylinder can be observed in the damping forces predicted by the model. If acoustic radiation occurs, the damping forces increase significantly. For the test fluids considered in this work, acoustic radiation should not occur though. For the situation considered in this work the inertial forces predicted by the fluid model from Van Eysden and Sader are found to be very similar to the predictions from the fluid model from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16). It should be kept in mind though, that the fluid model from Van Eysden and Sader considers an infinite beam, whereas the tube of the density sensor is finite, and consequently the fluid model from Van Eysden and Sader can only give an approximation of the real situation.

From the FEM model the distribution of the inertial fluid force along the tube of the density sensor can be extracted. The extracted mode shape of the tube allows to
calculate the fluid forces which the fluid model from Kremlevskii and Stepichev for the case of small penetration depths and neglected fluid viscosity would predict for the velocity distribution prescribed by this mode shape. The comparison between these fluid forces shows a qualitative agreement, but differences in the magnitudes (differences below 15% at the positions of maximum deflection). This comparison shows also that the fluid force near the free end of the sensor is overpredicted by the model from Kremlevskii and Stepichev for the case of small penetration depths. In a three-dimensional situation the fluid cannot be effectively compressed by the lateral motion of the tube surface but shuns towards the front surface of the sensor, which leads to reduced forces at the free end of the sensor. The FEM model has been used to study the influence of the speed of sound of a fluid on the predicted resonance frequency (for a given fluid density). The considered speeds of sound cover the range of the test fluids used in this work. It is found that the resonance frequency depends in a non-linear and fluid density-dependent way on the speed of sound. The considered variations in speed of sound lead to differences in the resonance frequencies that are not larger than 0.7%. However, it should be noted that for resonance frequencies of the density sensor which lie close to fluid cavity resonances, the influence of fluid compressibility could be much more pronounced, and could show a much stronger influence on the resonance frequencies of the density sensor than expected from this study.

From the comparison of the various fluid models it can be concluded that the fluid model from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), gives a good description of the fluid forces for the current situation, except near the free end of the tube of the sensor.

Experiments with varying oscillation amplitudes of the density sensor in the test fluids showed that the measured resonance frequencies and Q-factors are not amplitude-dependent for the considered situation. Thereby it is verified that the influence of the advective derivative of the fluid velocity in the Navier-Stokes equation is indeed negligible, and the fluid behaviour can be described by the linearized Navier-Stokes equation.

The influence of the dimension of the fluid container on the resonance frequency and Q-factor measured with the density sensor has been investigated experimentally. Measurements have been done in DI-Water. Three different beaker sizes have been tested, of which the medium beaker size is the one used for the rest of the measurements presented in this work. While the resonance frequency is little affected if the medium beaker is replaced by the large beaker, the Q-factor shows a significant change. Measurements in the small beaker showed three resonance peaks instead of the one expected. The additional peaks are probably fluid cavity resonances. If the evaluation of the resonance frequency of the density sensor is of primary interest, measurements in the medium beaker are adequate, because the influence of fluid boundaries seems small and the required amount of test fluid is limited. If one aims for damping measurements, a more careful study of fluid boundary influences should be undertaken first, also considering other test fluids.
Measurements in various test fluids have been conducted and the measured resonance frequencies and bandwidths have been compared to the predictions from analytical and FEM models. The resonance frequencies predicted by the FEM model and by the mobility model with the 3D fluid model from Van Eysden and Sader agree very well with the measured resonance frequencies. The deviations of the predicted from the measured resonance frequencies are in the range of $-0.9\%$ to $-1.3\%$ for the FEM model, and in the range of $-0.7\%$ to $-1.1\%$ for the mobility model with the 3D fluid model from Van Eysden and Sader.

Model predictions based on the fluid model of Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), showed a good agreement between the predicted and the measured resonance frequencies, but somewhat higher deviations (in the range of $-2.2\%$ to $-3.6\%$) and a slightly wrong trend (increasing differences with increasing fluid density).

An arbitrary introduced ‘free end correction’ employed in the mobility model with the fluid model from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), which should take into account the reduced fluid forces at the free end of the sensor, showed a visible but negligible influence on the predicted resonance frequencies.

The above observations suggest that it is important to take the fluid compressibility and the three-dimensional nature of the fluid velocity field into account. The small differences between the inertial fluid forces predicted by the fluid models from Van Eysden and Sader and from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), observed in Fig. 4.12 seem to matter. However, the influence of fluid boundaries is not taken into account in the fluid model from Van Eysden and Sader, and should be considered in the future. Probably a similar problem as with the fluid model from Kadyrov et al. can arise, with fluid resonances that do not reflect the exact physical behavior in the experiments.

Both the mobility model with the fluid model from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16), as well as the mobility model with the fluid model from Van Eysden and Sader can predict the general trend of the bandwidth versus the square root of the fluid density-fluid viscosity product, but the predicted values show large deviations from the measured values, especially for small values of the square root of the fluid density-fluid viscosity product. The question arises why the damping values are not well predicted by the models. First, it should be noted that it has never been the intention to use the density sensor for damping or viscosity measurements and that the density sensor has not been designed to do so. For instance the dimensional analysis from section 4.1.2.1 shows that the viscous forces are small for the sensor and the current operation conditions. In order to accurately measure small fluid damping, the sensor should show a much smaller internal damping than the one to be measured, and stable conditions regarding the damping of the sensor are required. It has been found in section 3.3.2.5 that the measurement setup with the side exit head inside the tube influences the damping of the sensor. Even though the fixation of the side exit head has not been changed for all measurement results presented here, some slight variations in its alignment inside the tube could have occurred due to the forces from the vibrometer
sensor head attached to it, which could have led to a not entirely stable internal damping of the sensor during the measurements. Actually the standard deviations of the measured bandwidths listed in Tab. 4.5 show a large value in air compared to those in other fluids. Thereby it should be noted that a measurement in air has been conducted after each measurement series in a test fluid, and the presented standard deviation of the bandwidth in air reflects the stability of the sensor with respect to internal damping during the whole measurement process.

The comparison between resonance frequencies predicted by the DeqD model and the mobility model (with the fluid model from Kremlevskii and Stepichev for the case of small penetration depths, Eqs. (4.14)-(4.16)) showed that both models give very similar results. The DeqD model cannot give predictions for the damping though.
5. Inverse problem

The determination of fluid density and viscosity from measured resonance frequency and damping is called the inverse problem. Different approaches are possible to solve the inverse problem. One could perform a series of measurements with test fluids whose properties cover the required range. A curve fitting could then relate the measured quantities with the fluid properties, e.g. the resonance frequencies with the fluid densities. For the measurements presented in the foregoing chapter, which showed only small dependence of the resonance frequencies on fluid viscosity (or speed of sound), a relationship between measured resonance frequency and fluid density could easily be obtained. In the case of strong dependence on other fluid properties, more measurements are needed to cover the range of fluid properties, which could become very time-consuming. Also, an adaption of the measurement setup could require a repetition of the measurements, or a temperature dependence of the setup could require measurements at different temperatures. If an accurate model of the system is available, the measurements in fluids with different fluid properties could be replaced by model predictions. These calculated supporting points could again be evaluated by curve fitting. The obvious benefits of this approach are the quick and flexible evaluation for various ranges of fluid parameters and measurement configurations. The challenge is of course to build such an accurate model. A third approach, standing between the two just presented approaches, would be to search for general dependencies between measured quantities and involved fluid parameters. Based on model results, such dependencies could be found, and fitting constants could be introduced to cover effects not accounted for in the models. The undetermined constants could then be found by parameter fitting based on experimental data. The expected benefits of such an approach are the identification of physical relationships and a good agreement between predictions and experiments with little calibration measurements and a small number of unknown parameters. Comparison of resonance frequencies from experiments and from predictions of the mobility model with the fluid model from Van Eysden and Sader in section 4.2.8 showed deviations of up to 1.1%. The predictions from the FEM model showed deviations up to 1.3%. Although a good agreement between measurements and model predictions is obtained, the accuracy seems not sufficient to derive the relationships between measured quantities and fluid parameters directly from these models (see section 2.3, where it was found that a relative error in the density measurement of less than one percent is desirable). Therefore the third approach discussed above will be followed here.
Chapter 5. Inverse problem

5.1. Model-based density and viscosity calculation

5.1.1. Direct evaluation of the frequency equation

Consider first the simple case of an Euler Bernoulli beam under the influence of a surrounding inviscid fluid. The differential equation of an Euler Bernoulli beam under an external distributed load \( q(x, t) \) is given as

\[
\rho Ay_{tt} + EI y_{xxxx} = q(x, t)
\]  

(5.1)

If there is no external distributed load, \( q(x, t) = 0 \), imposing the boundary conditions of a cantilever beam leads to the frequency equation

\[
1 + \cos(k L) \cosh(k L) = 0
\]  

(5.2)

in which \( k \) is the wavenumber, \( k^4 = (\rho A \omega^2)/(EI) \), and \( L \) is the length of the beam. The solution for oscillation mode \( n \) is given by \( k_n L = C_n \), in which \( C_n \) is a mode dependent constant.

If the fluid forces due to an incompressible inviscid fluid are modelled according to the fluid model from Kremlevskii and Stepipev for small penetration depths, Eqs. (4.14)-(4.16), the external distributed load is described by \( q(x, t) = i \beta y_x \) and \( \beta = -\pi R^2 \rho_f \omega \). Considering the boundary conditions of a cantilever beam leads to the same frequency equation as in Eq. (5.2), but with a different wavenumber \( k_f \) with \( k_f^4 = (\rho A + \pi R^2 \rho_f) \omega^2/(EI) \). Of course the solutions of the frequency equations for the cases with or without fluid are the same

\[
k_n L_{\text{vacuum}} = C_n = k_n L_{\text{fluid}}
\]  

(5.3)

This equation leads to the following relationship between the fluid density and the angular resonance frequencies in fluid \( \omega_f \) and in vacuum \( \omega_{\text{vac}} \)

\[
\frac{\omega_f}{\omega_{\text{vac}}} = \left(1 + \frac{\rho_f \pi R^2}{\rho A}\right)^{-1/2}
\]  

(5.4)

which can be solved for the fluid density

\[
\rho_f = \frac{\rho A}{\pi R^2 \left[\left(\frac{\omega_{\text{vac}}}{\omega_f}\right)^2 - 1\right]}
\]  

(5.5)

Based on this equation, the fluid density can be calculated from the measured resonance frequencies in fluid and in vacuum.

5.1.2. Description as 1 dof oscillator

Consider an Euler Bernoulli cantilever beam in a viscous fluid. Analogously to the procedure presented in section 3.2.6, a modal summation ansatz can be used which, together with appropriate orthogonality relations, leads to a description corresponding to a 1 dof system. Starting from Eq. (3.22) and introducing the fluid forces as
an external distributed load \( q(x, t) \) as in Eq. (5.1), where

\[
q(x, t) = (\alpha + i \beta) y_t = -c_f y_t - m_f y_{tt} \tag{5.6}
\]

the differential equation of the problem is

\[
(\rho A + m_f) y_{tt} + c_f y_t + EI y_{xxxx} = f_0 \delta(x - x_e) e^{i \omega t} \tag{5.7}
\]

in which \( c_f \) and \( m_f \) are viscous damping and added mass coefficients. For a description of the fluid forces according to the fluid model from Kremlevskii and Stepichev for small penetration depths, Eqs. (4.14)-(4.16), these coefficients can be expressed as

\[
c_f(\omega) = -\alpha = 2\sqrt{2} \pi R \sqrt{\omega \rho \eta} \tag{5.8}
\]

\[
m_f(\omega) = -\frac{\beta}{\omega} = \pi R^2 \rho f \left( 1 + 2 \frac{\delta}{R} \right) \tag{5.9}
\]

in which \( \delta = \sqrt{2\eta/\rho_f \omega} \) is the boundary layer thickness.

Two simplifications of the fluid forces will be made in this section. As the ratio \( \delta/R \) is very small for the measurements considered in this text, the influence of viscosity on the added mass coefficient \( m_f \) will be neglected. Also the frequency dependence of the viscous damping coefficient \( c_f \) will be neglected, because the variation of \( c_f \) is very small within the bandwidth of the resonance peak\(^1\) and the same simplification is also made by Sader [44]. Therefore, the parameter \( c_f \) is taken as a constant, evaluated at the resonance frequency of the respective oscillation mode.

Note that for the the modal function \( \phi_n(x) \) in the modal summation ansatz Eq. (3.23), the modal function of the beam in vacuum or the modal function of the beam in fluid (which is not known a priori) can be inserted. In both cases, the orthogonality relations Eqs. (3.25)-(3.26) stay the same. Following the procedure in section 3.2.6, the following expressions for the resonance frequency \( \omega_n \) and the Q-factor \( Q_n \) of mode \( n \) are obtained

\[
\omega_n = \sqrt{\frac{K_n}{M_{nf}}} \quad Q_n = \frac{(\rho A + m_f) \omega_n}{c_f(\omega_n)} \tag{5.10}
\]

in which

\[
K_n = EI \int_0^L \phi_n''' \phi_n^2 dx \tag{5.11}
\]

\[
M_{nf} = (\rho A + m_f) \int_0^L \phi_n^2 dx \tag{5.12}
\]

\(^1\)For a resonance frequency of \( f_{res} = 9000 \text{ Hz} \) and a bandwidth of \( df = 100 \text{ Hz} \), the variation of \( c_f \) within the bandwidth is \( \Delta \alpha \approx (\partial \alpha/\partial \omega) \cdot \Delta \omega \approx (\alpha df)/(2 f_{res}) = \alpha/180. \)
If the modal function of the beam in vacuum is used in the modal summation ansatz, the expression for $K_n$ can be rewritten

$$K_n = EI k_n^4 \int_0^L \phi_n^2 \, dx \quad (5.13)$$

in which $k_n$ is the wavenumber of mode $n$ in vacuum and $k_n^4 = \rho A \omega_n^2 / (EI)$ with $\omega_n$ as the angular resonance frequency of mode $n$ in vacuum. This allows to relate the expression for the resonance frequency in the fluid to the one in vacuum

$$\frac{\omega_n}{\omega_{0n}} = \left(1 + \frac{m_f}{\rho A}\right)^{-1/2} \quad (5.14)$$

Using the expressions for $c_f$ and $m_f$ discussed above, one obtains

$$\frac{\omega_n}{\omega_{0n}} = \left(1 + \frac{\rho_f \pi R^2}{\rho A}\right)^{-1/2} \quad (5.15)$$

$$Q_n = \frac{(\rho A + \rho_f \pi R^2) \omega_n}{2 \sqrt{2} \pi R \sqrt{\omega_n m_f^2}} \quad (5.16)$$

$$df_n = \frac{\omega_n}{2 \pi Q_n} = \frac{\sqrt{2} R \sqrt{\omega_n m_f^2}}{\rho A} \left(\frac{\omega_n}{\omega_{0n}}\right)^2 \quad (5.17)$$

or when solved for the fluid density and fluid viscosity

$$\rho_f = \frac{\rho A}{\pi R^2} \left[\left(\frac{\omega_{0n}}{\omega_n}\right)^2 - 1\right] \quad (5.18)$$

$$\eta = \frac{1}{\omega_n m_f} \cdot \left[\frac{\rho A}{\sqrt{2} R} \cdot df_n \cdot \left(\frac{\omega_{0n}}{\omega_n}\right)^2\right]^2 \quad (5.19)$$

in which $df_n$ is the bandwidth of mode $n$ measured in the fluid.

Expectedly, the expressions relating fluid density and resonance frequency are the same as in the foregoing section.

The expressions for fluid density and viscosity in Eqs. (5.18)-(5.19) could also be obtained with the Galerkin Method, using the modal functions in vacuum as comparison functions, making use of the orthogonality relations Eqs. (3.25)-(3.26), and using the relation which led from Eq. (5.11) to Eq. (5.13).

As shown in section 3.2.6, the description of an Euler Bernoulli cantilever beam with a concentrated mass and a concentrated rotary inertia can also be brought in the form of a 1 dof system. If the effect of the added mass due to the surrounding fluid shall be incorporated, the same procedure can be followed, one has simply to include the added mass coefficient $m_f$ by replacing $\rho A$ with $(\rho A + m_f)$ in the equations in section 3.2.6. Compared to the case of the simple cantilever beam discussed before, a difficulty arises. The orthogonality relation Eq. (3.44) now contains also the added mass coefficient $m_f$. Therefore, Eq. (3.44) describes orthogonality only between modal functions of the beam in the respective fluid. As a consequence, the
modal function in vacuum cannot be used in the modal summation ansatz, and the resonance frequency in fluid cannot directly be related to the resonance frequency in vacuum. Of course the actual modal functions in fluid can be used to obtain a description as a 1 dof system, but these modal functions are not known a priori.

Similar difficulties are encountered, if one aims for incorporating the influence of the surrounding fluid in the Timoshenko beam model. Following the procedure in section 3.2.6, the beam equations containing the viscous damping and added mass coefficients $c_f$ and $m_f$ can be written as

$$L(\tilde{W}) + i\omega C(\tilde{W}) - \omega^2 M_f(\tilde{W}) = \tilde{Q}$$

(5.20)

in which

$$M_f = \begin{bmatrix} (\rho A + m_f) & 0 \\ 0 & \rho I \end{bmatrix}, \quad C = \begin{bmatrix} c_f & 0 \\ 0 & 0 \end{bmatrix}$$

(5.21)

and operator $L$ is according to Eq. (3.33).

Considering first the undamped case ($C(\tilde{W}) = 0$), it is found that similar orthogonality relations as in Eqs. (3.35)-(3.36) apply. But as the orthogonality relations are with respect to the operators $M_f$ and $L$, and operator $M_f$ contains the added mass coefficient $m_f$, these relations are valid only for the vectors of eigenfunctions $\tilde{W}$ corresponding to the beam in the respective fluid. These vectors are not known a priori. The vectors of eigenfunctions in vacuum cannot be used in the modal summation ansatz and no direct relation between the resonance frequencies in fluid and in vacuum can be derived.

Considering the damped case, the above procedure is only possible if the damping operator $C$ can be expressed as a linear combination of the operators $L$ and $M$ or $M_f$.

5.2. Application to experimental data

5.2.1. Density calculation

According to Eqs. (5.4) and (5.15), the relationship between the fluid density $\rho_f$ and the angular resonance frequencies in fluid $\omega_f$ and in vacuum $\omega_{\text{vac}}$ is of the form

$$\frac{\omega_f}{\omega_{\text{vac}}} = (1 + a \cdot \rho_f)^{-1/2} + b$$

(5.22)

in which $a$ and $b$ are two constants. Theoretically, if the system would be accurately described by the models in section 5.1 based on Euler Bernoulli beam theory, parameter $b$ should be zero. Of course, the Timoshenko beam model deviates from the Euler Bernoulli beam model, especially for higher order modes. Also the influence of the concentrated mass/rotary inertia at the tip of the beam, which leads to a decrease of the resonance frequency (cp. $\omega_{\text{nc}}$ in Eq. (3.45)) is not taken into account. It has also been observed in section 4.2.8 that the analytical models (that are based on Timoshenko beam theory and consider a concentrated mass/rotary inertia) which use the fluid model from Kremlevskii and Stepichev for the case of
small penetration depths, Eqs. (4.14)-(4.16), give a quite good description of the resonance frequency-fluid density curve, but show a slightly wrong trend. Therefore, Eq. (5.22) cannot accurately describe the entire possible fluid density range, but shows to be accurate for a density range of practical interest, e.g. 700 kg/m$^3$ to 1000 kg/m$^3$. The deviations of the model from the real system are captured by the parameters $a$ and $b$, which are determined by calibration measurements.

The nonlinear least squares method in Matlab is used to fit the function in Eq. (5.22) to the measurement data from section 4.2.5. Only the data from measurements in fluids is used for the fit; the measurement in air is excluded. The measured resonance frequency in air is inserted into $\omega_{vac}$ in Eq. (5.22) instead of the resonance frequency in vacuum (because no value in vacuum is available, and the added mass due to the surrounding air is negligibly small). Figure 5.1 shows the measurement data and the fitted curve. The errorbars show the standard deviations of the measurements. The fitting leads to the parameter values $a = 4.80 \times 10^{-4}$ m$^3$/kg and $b = -0.0306$ with a goodness-of-fit described by a R-square value of 0.9986. According to Eq. (5.4), the expected value of parameter $a$ would be $8.24 \times 10^{-4}$ m$^3$/kg. As discussed above, deviations from this value had to be expected, because Eq. (5.4) is based on Euler Bernoulli beam theory which cannot accurately describe the considered flexural oscillations, and the influence of the concentrated mass/rotary inertia at the tip of the beam are not taken into account in the derivation of Eq. (5.4). Of course insufficiencies of the description of the fluid forces employed in the derivation of Eq. (5.4) can also be covered in the parameter $a$ determined by calibration.

Measurements in at least two calibration fluids are necessary to determine the fitting parameters $a$ and $b$. If the two fluids in Fig. 5.1 with extremal fluid densities, corresponding to 2,2,4-Trimethylpentane (691 kg/m$^3$) and DI-water (998 kg/m$^3$), are used for the fitting process, one obtains $a = 4.77 \times 10^{-4}$ m$^3$/kg and $b = -0.0311$ with a goodness-of-fit described by a R-square value of 1. The measurement data and the fitted curve are depicted in Fig. 5.2. Using this calibrated description, the

![Figure 5.1](image-url)
5.2. Application to experimental data

Figure 5.2.: Model-based description of the resonance frequency-fluid density relationship. Two fluids are used for calibration.

Fluid densities of the remaining fluids can be calculated as

\[
\rho_f = \frac{1}{a} \left[ \left( \frac{\omega_f}{\omega_{\text{vac}}} - b \right)^{-2} - 1 \right]
\]  \hspace{1cm} (5.23)

Tab. 5.1 shows the calculated fluid densities together with the known fluid densities (from Tab. 4.4), and the deviations of the calculated densities from the known densities, expressed as percentage of the known fluid densities. It is found that the deviations are smaller than 0.75%.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( \rho_{\text{calc}} ) [kg/m(^3)]</th>
<th>( \rho_{\text{known}} ) [kg/m(^3)]</th>
<th>Deviation [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>N4</td>
<td>818.3</td>
<td>813.7</td>
<td>0.57</td>
</tr>
<tr>
<td>S6</td>
<td>870.4</td>
<td>874.3</td>
<td>-0.45</td>
</tr>
<tr>
<td>N35</td>
<td>866.9</td>
<td>860.5</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Table 5.1.: Density values \( \rho_{\text{calc}} \) calculated using Eq. (5.23) with parameters \( a \) and \( b \) from calibration with 2,2,4-Trimethylpentane and DI-water. Comparison with the known fluid densities \( \rho_{\text{known}} \) (from Tab. 4.4). The deviations are expressed as percentage of the known fluid densities.

5.2.2. Viscosity calculation

According to Eq. (5.17) the relation between the bandwidth due to fluid influence \( df_{\text{fluid}} \) and fluid density \( \rho_f \), fluid viscosity \( \eta \), and angular resonance frequencies in fluid \( \omega_f \) and in vacuum \( \omega_{\text{vac}} \) is of the form

\[
df_{\text{fluid}} = a \cdot \sqrt{\omega_f \rho_f \eta} \cdot \left( \frac{\omega_f}{\omega_{\text{vac}}} \right)^2
\]  \hspace{1cm} (5.24)
in which \( a \) is a constant. Note that \( df_{\text{fluid}} \) is the bandwidth due to fluid influence alone (without the influence of structural damping) and is calculated according to Eqs. (4.35), (4.36).

The nonlinear least squares method in Matlab is used to fit the function in Eq. (5.24) into the measurement data from section 4.2.5. Only the data from measurements in fluids is used for the fit; the measurement in air is excluded. The measured resonance frequency in air is inserted into \( \omega_{\text{vac}} \) in Eq. (5.24) instead of the resonance frequency in vacuum (because no value in vacuum is available, and the added mass due to the surrounding air is negligibly small). Figure 5.3 shows the measurement data and the fitted curve. The fitting leads to a value of \( a = 0.002245 \text{ m}^2/\text{kg} \) with a goodness-of-fit described by a R-square value of 0.9644. According to Eq. (5.17), the expected value of parameter \( a \) would be \( 93.1 \times 10^{-3} \text{ m}^2/\text{kg} \).

The fitted curve describes the general trend of the measurements right, but there is a larger scattering of the measurement data along this curve. A poor agreement of the bandwidths due to fluid influence predicted by the analytical models with those from measurements has already been recognized in section 4.2.8 and discussed in section 4.3. The larger scattering observed in Fig. 5.3 is therefore not surprising.

Eq. (5.24) together with the value of \( a = 0.002245 \text{ m}^2/\text{kg} \) determined by the above fitting can be used to calculate the viscosities predicted from the measured bandwidths. The calculated viscosities, the true values of the viscosities, and the deviations are listed in Tab. 5.2.
5.3. Sensitivity on fluid density and accuracy

<table>
<thead>
<tr>
<th>Fluid</th>
<th>(\eta_{\text{calc}}) [mPas]</th>
<th>(\eta_{\text{known}}) [mPas]</th>
<th>Deviation [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DI-Water</td>
<td>1.3</td>
<td>0.977</td>
<td>37.9</td>
</tr>
<tr>
<td>N4</td>
<td>7.1</td>
<td>5.22</td>
<td>35.5</td>
</tr>
<tr>
<td>S6</td>
<td>15.3</td>
<td>9.085</td>
<td>67.9</td>
</tr>
<tr>
<td>N35</td>
<td>61.5</td>
<td>68.51</td>
<td>-10.2</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane</td>
<td>0.23</td>
<td>0.5</td>
<td>-53.5</td>
</tr>
</tbody>
</table>

Table 5.2.: Viscosity values \(\eta_{\text{calc}}\) calculated using Eq. (5.24) with parameter \(a\) from calibration presented in Fig. 5.3. Comparison with the known fluid viscosities \(\eta_{\text{known}}\) (from Tab. 4.4). The deviations are expressed as percentage of the known fluid viscosities.

5.3. Sensitivity on fluid density and accuracy

From the resonance frequency-fluid density relationship in Eq. (5.22), which has been successfully used to describe the experimental data, the sensitivity \(s_\rho\) of the resonance frequency with respect to fluid density can be calculated as

\[
s_\rho = \frac{\partial f_{\text{res, fluid}}}{\partial \rho_f} = -(1/2) f_{\text{res, vac}} a (1 + a \cdot \rho_f)^{-3/2}
\]  

(5.25)

The sensitivity \(s_\rho\) changes with fluid density, and decreases with increasing fluid density. Further it depends on the resonance frequency in vacuum, and the fitting parameter \(a\). Using the parameter \(a\) found in section 5.2.1 for the calibration with 2,2,4-Trimethylpentane and DI-water \((a = 4.77 \times 10^{-4} \text{m}^3/\text{kg})\) and the angular resonance frequency in air instead of the one in vacuum, the numerical values of the sensitivity \(s_\rho\) can be calculated. For \(\rho_f = 700 \text{ kg/m}^3\) one obtains \(s_\rho = -1.71 \text{ Hz m}^3/\text{kg}\), and for \(\rho_f = 1000 \text{ kg/m}^3\) one obtains \(s_\rho = -1.47 \text{ Hz m}^3/\text{kg}\).

In order to estimate the accuracy of the density calculation presented in section 5.2.1, consider the agreement between the measured resonance frequencies and the fitted curve presented in Fig. 5.2. Fig. 5.4 represents (schematically) a detailed view of Fig. 5.2. It shows an averaged measured resonance frequency (□) together with errorbars that indicate the standard deviation of the measurements, and the fitted resonance frequency-fluid density curve. It is assumed that the measured resonance frequency lies close to the fitted curve.

For the time being neglect the standard deviation of the measurements. In Fig. 5.4 the measured resonance frequency \(f_{\text{res, meas}}\) is related to the true fluid density \(\rho_{\text{meas}}\) present during measurements. For the measured resonance frequency, the model predicts a fluid density \(\rho_{\text{model}}\). The absolute error of the predicted fluid density is then \(\Delta \rho_f = |\rho_{\text{model}} - \rho_{\text{meas}}|\). This absolute density error depends on the difference \(\Delta f_{\text{res}}\) between the measured resonance frequency \(f_{\text{res, meas}}\) and the resonance frequency predicted by the model for \(\rho_{\text{meas}}\), and it depends on the slope of the fitted resonance frequency-fluid density curve (if the measured resonance frequency lies close to the fitted curve, the fitted curve can locally be considered as a straight line). This slope is described by the sensitivity \(s_\rho\) introduced previously. The absolute density
error can then approximately be described as

\[ \frac{\Delta f_{\text{res}}}{\Delta \rho_e} \approx s_p \quad \text{i.e.} \quad \Delta \rho_e \approx \frac{\Delta f_{\text{res}}}{s_p} \]  

(5.26)

The maximum absolute density error \( \Delta \rho_e \) determines the accuracy of the density measurement that can be obtained.

Using the frequency difference \( \Delta f_{\text{res}} \) and the sensitivity \( s_p \) observed in Fig. 5.2, the accuracy of the density calculation presented in section 5.2.1 can be estimated. The maximum value of \( \Delta f_{\text{res}} \) found in Fig. 5.2 is 10.1 Hz, and the minimum sensitivity \( s_p \) in the considered density range is \(-1.47 \text{ Hz m}^3/\text{kg} \) (for \( \rho_f = 1000 \text{ kg/m}^3 \)). The calculated maximum absolute density error is then \( \Delta \rho_e = 6.9 \text{ kg/m}^3 \).

Consider now the influence of the standard deviation of the measurements. The calculation of the maximum absolute density error \( \Delta \rho_e \) due to the standard deviation of the measurements is analogous to the calculation before. For the maximum value of \( \Delta f_{\text{res}} \), now twice the maximum standard deviation found in the measurements is inserted in Eq. (5.26), and the same value for \( s_p \) used before. The standard deviations of the measured resonance frequencies in fluid according to Tab. 4.5 are smaller than 0.2 Hz. Therefore one obtains \( \Delta \rho_e = 0.27 \text{ kg/m}^3 \).

Taking into account both the systematic and random absolute density errors \( \Delta \rho_e \), the accuracy of the density calculation presented in section 5.2.1 is estimated to be about 7 kg/m³.

The maximum absolute density error due to the standard deviation of the measurements determined above (\( \Delta \rho_e = 0.27 \text{ kg/m}^3 \)) gives an estimation of the obtainable density resolution. In view of a process application which requires only a relative density measurement (i.e. measurement of density changes with respect to a reference density), the density resolution is the essential quantity.

5.4. Error considerations (density calculation)

In sections 5.1 and 5.2 a procedure to determine the fluid density from the measured resonance frequency has been presented and applied to the measurements in the test
5.4. Error considerations (density calculation)

Fluids from section 4.2.5. According to Tab. 5.1, the maximum relative error in the calculated densities is 0.74%, and the maximum absolute error in the calculated densities is 6.4 kg/m³. Possible reasons for the discrepancies between the calculated and the true densities are discussed here.

The predicted densities are calculated from Eq. (5.23) which stems from Eq. (5.22), and contains two fitting constants. Eq. (5.22) is derived from both a simplified description of the oscillating beam and a simplified description of the fluid forces acting on it. Therefore it is not clear what physical inadequateness of the applied model to describe the measurements is covered by these calibration parameters, which complicates the discussion of possible sources of error. On the other hand an accurate prediction of fluid densities has been shown using Eq. (5.23), which suggests that the main physical aspects are well covered by this simplified description.

Comparing the influence of possible sources of error on the measured resonance frequencies with the differences observed between the measured resonance frequencies and the fitted resonance frequency-fluid density curve according to Eq. (5.22), can give some hints which sources of error might be most important.

From Fig. 5.2 it is found that the mean values of the measured resonance frequencies show deviations from the fitted curve between \(-10.1\) Hz and \(6.1\) Hz (compared to mean values of the measured resonance frequencies between \(8766.7\) Hz and \(9254.5\) Hz).

The following aspects are possible reasons for the discrepancies between the measured resonance frequencies and their description by the fitted resonance frequency-fluid density curve:

- Errors in determining the resonance frequencies (and Q-factors) by fitting the phase curve of a 1 dof oscillator into the measured phase curve.

- Temperature related errors: Constant measurement temperature has been assumed. Temperature variations affect the measured resonance frequencies and the assumed fluid properties.

- Positioning of the side exit head and alignment inside the tube can affect the measurements.

- Fluid cavity resonances in the vicinity of the resonance mode of the density sensor can lead to interferences.

- The applied procedure for density calculation is based on considerations of a simplified model. For instance the influence of viscosity on the added mass is neglected, fluid compressibility is not included, the influence of fluid boundaries is not taken into account, fluid forces from a 2D model are considered.

- Description of the oscillating structure might be insufficient, as the procedure applied for the density calculation is based on considerations of the Euler
Bernoulli beam model and neglecting the mass/rotary inertia at the free end of the beam.

These aspects are discussed below. Errors introduced by the measuring devices are not considered here because their accuracy is assumed to be much better than the errors discussed here.

The standard deviations of the measured resonance frequencies are below 0.2 Hz in the fluids (except in air where it is 1.34 Hz). The random errors described by the standard deviations of the measured resonance frequencies are much smaller than the maximum difference between the measured resonance frequencies and the fitted curve and cannot explain the observed difference.

The fitting of the phase curve of a 1 dof oscillator to the measured phase curve in order to determine the resonance frequency and Q-factor worked very well. For the fluids Air, DI-Water, S6, N35, 2,2,4-Trimethylpentane the goodness-of-fit described by the R-square value showed values in the range 0.99998 to 1. For the fluid N4 the goodness-of-fit described by the R-square value showed values in the range 0.99992 to 0.99995. Based on the good agreement, no relevant errors are expected from this procedure.

A deviation of the assumed constant measurement temperature has two effects. It influences the resonance frequency of the sensor in air as shown in section 3.3.2.7, and it affects the assumed fluid density (which has been determined for reference temperature). As discussed in section 4.2.5, the uncertainty of the measured temperatures is conservatively estimated to be ±0.5 °C. From section 3.3.2.7 it follows that such a temperature uncertainty leads to an uncertainty of the resonance frequency in air of about ±1.13 Hz. The resulting uncertainty of the resonance frequency in the fluids is smaller than this value (which follows from Eq. (5.22) and noting that the right hand side of this equation is smaller than 1). A temperature variation of ±0.5 °C leads to a maximum variation of the density of the considered test fluids of only ±0.4 kg/m³. Considering the maximum sensitivity \( s_p = -1.71 \text{ Hz m}^3/\text{kg} \) determined in section 5.3, the influence of this density variation on the measured resonance frequency can be estimated to be about ±0.7 Hz. The combined temperature related uncertainty is therefore expected to be smaller than ±2 Hz which is small compared to the differences between the measured resonance frequencies and the fitted curve. Of course the temperature uncertainty affects also the fitted curve, which is not considered here.

In section 3.3.2.5 is has been found that the positioning of the side exit head and its alignment inside the tube has little effect on the resonance frequency of the sensor in air, but has a pronounced influence on its damping. The repeated measurements in air in between the measurements in the test fluids showed a maximum difference in the resonance frequencies in air of 3.6 Hz, which shows that the setup has been stable during the measurements. As before, the consequent uncertainty of the resonance frequency in the fluids is smaller than this value, and seems to be small compared to the differences between the measured resonance frequencies and
the fitted curve.

From above considerations follows that the most likely cause for the discrepancies between the measured resonance frequencies and the fitted curve is the limited capability of Eq. (5.22) to describe the measurements. At first sight it is tempting to use Eq. (5.14) for a discussion of the influences of the different fluid models on the inertial fluid force and the consequent influence on the density calculation. However, as noted before it is not clear what modeling deficiencies are captured by the fitted parameters $a$ and $b$, and as they differ quite a bit from the theoretical values, it seems not reasonable to discuss these influences based on Eq. (5.14).

In order to improve present description of the fluid density-resonance frequency relationship, a more elaborate model should be considered. Possibly further parameters should be included, for instance the speed of sound in the fluid or the fluid viscosity. Even though Fig. 4.10 suggests that the influence of fluid viscosity is small in the present situation (considering an incompressible fluid and a 2D fluid velocity field), and Fig. 4.12 shows that the inertial fluid forces are very similar for both compressible and incompressible fluids (at least for an unbounded compressible fluid), it might be necessary to include the effects of fluid viscosity and compressibility in a more elaborate model.

Fluid cavity resonances could affect the measured resonance frequencies and have an influence on the agreement between the measurements and the model predictions. The presence of disturbing fluid cavity resonances should be visible in the measured phase-frequency curves. No anomalies have been observed in the evaluated phase-frequency curves, except for the measurements in the small beaker presented in Tab. 4.3.

5.5. Discussion

In order to calculate the fluid density from the resonance frequency measured by the density sensor, the strategy is followed to search for a simplified analytic dependency between the fluid density and the measured resonance frequency, and introduce fitting parameters to account for aspects not covered by this simplified dependency. The major benefit of this approach is that a small number of fitting parameters is required because part of the physics is adequately described, and therefore the effort for calibration is kept small.

The coupled system of an Euler Bernoulli cantilever beam surrounded by a viscous fluid, where the fluid forces are described according to the fluid model from Kremlevskii and Stepičev for small penetration depths, can be brought into the form of a damped 1 dof oscillator. From this description, a relationship between the fluid density and the measured resonance frequencies in fluid and in vacuum, and a relationship between the fluid density-fluid viscosity product, the measured bandwidth, and the measured resonance frequencies in fluid and in vacuum can be obtained. General forms of these expressions can be formulated, that contain two
fitting constants in the expression related to fluid density calculation, and one fitting constant in the expression related to fluid viscosity calculation. Unfortunately it has been found that this procedure is not possible if a cantilever beam with a concentrated mass/rotary inertia or a Timoshenko beam is considered.

Measurements in at least two fluids are required to calibrate the fluid density-resonance frequency relationship. Using the test fluid with the lowest and the highest density for calibration, the fluid densities of the other test fluids have accurately been predicted with relative errors smaller than 0.74%. Based on this exemplified density calculation, the accuracy of the density measurement is estimated to be 7 kg/m³. Viscosity calculation has also been considered, but expectedly is not as successful as the density calculation. This result is not worrying though, as the current density sensor has never been intended to be used for viscosity measurements, but should be combined with a viscosity measurement based on torsional oscillations.

Various sources of error have been considered that could explain the remaining differences between the measurements and the calibrated fluid density-resonance frequency relationship. The considered error sources, e.g. random errors, fitting errors, temperature influence, influence of the measurement setup, could not fully explain the observed differences. This suggests that the current form of the fluid density-resonance frequency relationship should be replaced by a more elaborate description, possibly taking into account other fluid parameters like viscosity or speed of sound.

According to the simplified model description in Eq. (5.4) (which should still capture the main characteristics of the system), the change in resonance frequency due to density of the surrounding fluid depends on the ratio of the “displaced fluid mass per length by the presence of the tube” ($\rho_f \pi R^2$) to the “mass per length of the tube” ($\rho A$). The higher this ratio, the larger is the change in resonance frequency compared to the resonance frequency in vacuum. It can also be seen that for higher resonance frequencies in vacuum (also for higher modes) the change in resonance frequency increases and consequently is easier to be detected. If the sensitivity of the resonance frequency on fluid density shall be increased, the above considerations suggest that one should either decrease the mass per length of the tube or increase the outer radius of the tube. A small mass per length of the tube is realized in the current setup. The practical limitations are found in the stability of the sensor and the limited range of densities of the materials which are suitable to build the sensor from. An increase of the outer radius of the tube could still be conducted, with the practical limitation, that flexural modes become harder to excite due to the higher forces required.
6. Conclusions and outlook

The objective of this work is the study of different modeling approaches for the structure and the fluid-structure interaction of a flexural vibration sensor for density measurements, as well as the experimental characterization of the sensor, and the validation of the models.

A critical point in the design of a cylindrical flexural vibration sensor is the occurrence of very close flexural modes with deflections along the two principal directions of the beam cross-section. A sensor design is proposed here that avoids this difficulty.

Two analytical models and an FEM model have been developed, which describe the structural vibrations of the sensor. In the analytical models, the structure of the density sensor is modeled as a cantilever beam with a rigid mass/rotary inertia at the free end. The two analytical models differ in the way they describe the attached mass. In the case of the mobility model, the kinematic transition conditions between the beam and the rigid mass require equal deflection and equal slope. In the case of the DeqD model the concentrated mass/rotary inertia does not cause any kinematic restrictions. Comparison of these two approaches shows the sensitivity of an accurate model on this detail. It is found that for the current sensor and the considered operation conditions, both models are equally well suited to describe the structural vibrations of the sensor.

The third flexural oscillation mode of the density sensor is used in this work. The advantages of the higher oscillation mode are the good decoupling from the environment and, considering the later application, the larger absolute changes in resonance frequency due to density differences of the surrounding fluid, which are consequently easier detected. In turn, the use of a higher oscillation mode requires to use the Timoshenko beam theory instead of the simpler Euler Bernoulli beam theory in order to accurately describe the structure. This introduces further parameters in the models, which have to be determined.

The considered models rely on several geometric and material parameters. An accurate determination of these parameters is crucial, but not all parameters can be determined directly. The determination of geometric parameters showed to be especially error-prone. Because of the small geometric dimensions, even small measurement errors can have a significant influence. The presented procedure to determine the model parameters has led to accurate models of the oscillating structure.

The influence of the sensor mounting on the resonance frequency and damping of the sensor has been studied experimentally for the lowest four bending modes.
Considering the resonance frequency, it is found that for mode orders greater than two, the mounted sensor behaves like the free sensor. Considering the $Q$-factor, the dependence on the mounting is stronger, but also decreasing with mode order.

The analytical description of the fluid-structure interaction relies on an adequate fluid model which can predict the inertial and dissipative forces acting on the oscillating structure. Various analytical fluid models have been considered in order to explore the influences of fluid viscosity, fluid boundaries, fluid compressibility, and the difference between treating the fluid velocity field as a two-dimensional or a three-dimensional field.

In the FEM model the fluid is described as an acoustic medium. As pressure forces are the dominant forces in the current fluid-structure interaction problem, a description of the fluid as an acoustic medium is justified. The FEM model is especially suited to study effects at the free end of the tube of the sensor, and the possible occurrence of fluid cavity resonances.

Comparison of the fluid forces predicted by the various analytical fluid models suggests that for the considered test fluids and operation conditions the fluid forces are well described by a 2D incompressible viscous unbounded fluid. However, considering the resonance frequencies predicted by the mobility model for different fluid models showed a much better agreement with the experimentally determined resonance frequencies if a 3D compressible viscous unbounded fluid model is employed. The small differences between the fluid models seem to be important, and it seems to be important to take the fluid compressibility and the three-dimensional nature of the fluid velocity field into account, in particular near the clamping and the free end of the tube of the density sensor.

From experiments with DI-water and different dimensions of the fluid container it can be concluded that for the current setup the fluid boundary has a small influence on the inertial fluid forces. In contrast the dissipative forces show to be considerably influenced by the presence of the fluid boundaries. An accurate representation of the damping forces in the models seems delicate.

Measurements in various test fluids have been conducted and the measured resonance frequencies and bandwidths have been compared to the predictions from analytical and FEM models. The resonance frequencies predicted by the FEM model and by the mobility model with a 3D compressible viscous unbounded fluid model from Van Eysden and Sader [52] agree very well with the measured resonance frequencies. The deviations of the predicted from the measured resonance frequencies are in the range of $-0.9\%$ to $-1.3\%$ for the FEM model, and in the range of $-0.7\%$ to $-1.1\%$ for the mobility model with the fluid model from Van Eysden and Sader [52].

Model predictions based on the 2D incompressible viscous unbounded fluid model from Kremlevskii and Stepichev [31] for small penetration depths showed a good agreement between the predicted and the measured resonance frequencies, but somewhat higher deviations (in the range of $-2.2\%$ to $-3.6\%$) and a slightly wrong trend.
(increasing differences with increasing fluid density). Expectedly, the damping of the density sensor due to the surrounding fluid is predicted less successfully by the (analytical) models.

Based on the considerations of a simple Euler Bernoulli beam and fluid forces according to Kremlevskii and Stepichev [31] for the case of small penetration depths, a general form of the fluid density-resonance frequency relationship has been found, that contains two calibration constants. Using the test fluid with the lowest and the highest density for calibration, the fluid densities of the other test fluids have accurately been predicted with relative errors smaller than 0.74%. Based on the data of the exemplified density calculation, the accuracy of the density measurement is estimated to be 7 kg/m$^3$ for the given fluid container.

Finally it can be concluded that two analytical models and an FEM model have been developed that are able to accurately predict the resonance frequencies of the density sensor in the considered test fluids, and that an accurate density measurement has been achieved.

If in the future density measurements with higher accuracy are sought, the currently employed fluid density-resonance frequency relationship should be replaced by a more elaborate description, possibly taking into account other fluid parameters like viscosity or speed of sound. This requires an adaption of the applied expression for the inertial fluid force acting on the sensor. While the analytical fluid model from Van Eysden and Sader [52] showed to accurately describe the fluid forces, the expressions for the fluid forces are not in a form that can be easily incorporate in the existing model.

The analytical models consider a Timoshenko beam model and include the influence of the attached mass/rotary inertia, but for the solution of the inverse problem a simple Euler Bernoulli beam model has been considered. The presented procedure to find the fluid density-resonance frequency relationship does not work for the Timoshenko beam model or if the attached mass/rotary inertia shall be included. Therefore, a different approach to establish a fluid density-resonance frequency relationship will be required, if a more accurate description of the mechanical structure shall be included.

The analytical fluid model from Van Eysden and Sader [52] showed to accurately describe the fluid forces. However it does not account for the fluid boundaries and considers an infinitely long beam. It would be interesting to include the effect of fluid boundaries in the model, and compare predictions from both formulations with experimental data.
A. Appendix

A.1. Drawings of sensor parts

The drawings for the tube, the bushing, and the sensor body, based on which the parts have been manufactured, are presented in Figs. A.1-A.3. The dimensions of the magnet, which has been purchased from [1], are presented in Fig. A.4. The tube, the bushing, and the sensor body are made of stainless steel (X5CrNi18-10, 1.4301, AISI 304). The magnet is made of Neodymium Iron Boron.

The transitions between the assembled sensor parts which are in contact with fluid must be sealed in order that no fluid is soaked in. These transitions are glued using the glue ZAP CA PT-08 [5].

Figure A.1.: Drawing of the bushing.

\( \theta \): Press fit with part Magnet. Allowance for interference: 0.02 mm.
\( \theta ** \): Press fit with part Tube. Allowance for interference: 0.02 mm.
Figure A.2.: Drawing of the tube.
A.1. Drawings of sensor parts

(*): Press fit with part Tube. Allowance for interference: 0.02mm

Figure A.3.: Drawing of the sensor body.

Figure A.4.: Drawing of the magnet. The magnet is polarized diametrically. The surface of the magnet is nickel-plated.
A.2. Transfer matrix for a Timoshenko beam in vacuum

The expression for the transfer matrix of the Timoshenko beam \((\text{beam}TM)\) used for this work, which has been discussed in section 3.2.3, is too large to be represented here. However, it was found that the matrix \(\text{beam}TM\) can be described in a manageable form, if the notation from Sorrentino et al. [47] is adopted, and the state vector used in Sorrentino et al. [47] (consisting of the beam deflection and its first three derivatives) is translated to the state vector used in section 3.2.3 (consisting of bending moment, shear force, slope due to bending, and beam deflection) by the matrix \(D\) introduced below.

\[
\text{beam}TM = D \cdot \Phi \cdot e^{\lambda L} \cdot \Phi^{-1} \cdot D^{-1}
\]  

(A.1)

where

\[
D = \begin{bmatrix}
-\frac{\rho \omega^2}{\beta} & 0 & \frac{\alpha \rho \omega^2}{
\beta} & 0 \\
0 & EI & 0 & -EI\lambda \\
-\lambda & 0 & \frac{\nu}{\beta} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(A.2)

and

\[
\Phi = \begin{bmatrix}
\lambda_1 & \lambda_2 & -\lambda_1 & -\lambda_2 \\
\lambda_1 & \lambda_2 & \lambda_1 & \lambda_2 \\
\lambda_1 & \lambda_2 & -\lambda_1 & -\lambda_2 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]  

(A.3)

\[
e^{\lambda L} = \begin{bmatrix}
e^{\lambda_1 L} & 0 & 0 & 0 \\
0 & e^{\lambda_2 L} & 0 & 0 \\
0 & 0 & e^{-\lambda_1 L} & 0 \\
0 & 0 & 0 & e^{-\lambda_2 L}
\end{bmatrix}
\]  

(A.4)

with

\[
\lambda_{1,2} = \sqrt{\frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}}
\]  

(A.5)

and

\[
\alpha = -\rho \omega^2 \left( \frac{1}{\kappa G} + \frac{1}{E} \right) \\
\beta = \frac{\rho \omega^2}{EI} \left( 1 - \frac{\rho I \omega^2}{\kappa G A} \right) \\
\chi = -\frac{\rho \omega^2}{\kappa G} \\
\psi = \chi - \frac{\kappa G A}{EI}
\]

in which \(E\) is the Young’s modulus, \(I\) is the area moment of inertia, \(\rho\) is the
density of the beam material, \( A \) is the area of the beam cross-section, \( \kappa \) is the shear correction factor, \( G \) is the shear modulus, \( L \) is the length of the beam, and \( \omega \) is the angular frequency of the oscillation. Note that the eigenvalue \( s \) in Sorrentino et al. [47] is replaced by \( i\omega \) in above equations.

A.3. Transfer matrix for the rigid mass

The transfer matrix describing the rigid mass in the mobility model (section 3.2.3) reads as follows:

\[
rmTM = \begin{bmatrix}
1 & 0 & -m\omega^2 x_s & -m\omega^2 \\
-x_s & 1 & -I_s\omega^2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & x_s & 1
\end{bmatrix}
\]  

(A.6)

in which \( m \) is the mass, \( I_s \) is the mass moment of inertia with respect to the center of mass, \( \omega \) is the angular frequency of the oscillation, and \( x_s \) is the distance of the center of mass from the attached end of the rigid mass (Fig. 3.8).

A.4. Frequency equation of the DeqD model

The frequency equation of the DeqD model for a rigidly clamped beam \( (f_r = f_t = 0) \) can be expressed as
\[
\frac{\alpha L (\beta^2 - \gamma)}{\alpha L^2} \sin b + \alpha L (\beta^2 - \gamma) \cos \theta + \alpha L (\beta^2 - \gamma^2) \cos \theta \delta \sin \theta \delta - 1 + \delta \sin \theta \delta + \delta \sin \theta \delta - 1 + \delta \sin \theta \delta \]
\]
A.5. Non-dimensional forms of the continuity equation and the Navier-Stokes equation according to Wang [54]

in the above equation \( b \) and \( \tilde{\alpha} \) can be replaced by

\[
\begin{align*}
\bar{b} &= \sqrt{\frac{\gamma}{(\bar{b}^2 - r^2) \cdot (\bar{b}^2 - s^2 \gamma)}} \\
\tilde{\alpha} &= \sqrt{\bar{b}^2 - r^2 - s^2 \gamma}
\end{align*}
\]

such that the frequency equation is solely expressed by the parameters \( \tilde{\beta}, r, s, m, p, a, L, \) and \( \gamma \). The parameter \( \tilde{\beta} \) is given as

\[
\tilde{\beta} = \frac{1}{\sqrt{2}} \cdot \sqrt{(r^2 + s^2 \gamma) + \sqrt{(r^2 - s^2 \gamma)^2 + \frac{4 \gamma}{b^2}}}
\]

The added mass effect due to a fluid surrounding the oscillating beam is taken into account by the parameter \( \gamma \) (damping forces due to a surrounding fluid cannot be taken into account in the DeqD model):

Considering fluid forces of the form \( f(\omega) = (\alpha + i \beta) u \), in which \( \Re\{u\} = \Re\{u_0 e^{i\omega t}\} = u_0 \cos(\omega t) \) describes the velocity of the oscillating beam (with velocity amplitude \( u_0 \) and angular oscillation frequency \( \omega \)), and in which \( \alpha \) and \( \beta \) describe the dissipative and inertial parts of the fluid forces respectively, the parameter \( \gamma \) is given as

\[
\gamma = 1 - \frac{\beta}{m_0}.
\]

If there is no fluid: \( \gamma = 1. \)

Note that the expressions \( b \tilde{\alpha}, b \tilde{\beta} \) are non-dimensional wavenumbers, similar to the wavenumbers in Eqs. (3.11). The parameter \( b \) corresponds to a non-dimensional wavenumber of an Euler Bernoulli beam.

A.5. Non-dimensional forms of the continuity equation and the Navier-Stokes equation according to Wang [54]

In section 4.1.2.1 the non-dimensional forms of the continuity equation and the Navier-Stokes equation for a compressible fluid under small amplitude harmonic excitation according to Wang [54] are considered in order to assess the importance of the various fluid effects. As these equations stem from a private communication and would not be accessible to the public otherwise, they are represented below.

Note that in the quotation below the nomenclature of Wang [54] is used, which differs from the nomenclature used in the rest of this work.

Begin quotation.

Nondimensionalization of the governing equations for a oscillatory fluid
Jingtao Wang
(Dated: February 18, 2013)
Appendix A. Appendix

The fluid motion without heat conduction subjected to a small amplitude excitation are generally governed by the equation of continuity and the Navier-Stokes equation

\[
\frac{\partial \rho'}{\partial t'} + \nabla' \cdot ((\rho_0 + \rho')\mathbf{v}') = 0
\]

\[
(\rho_0 + \rho') \left( \frac{\partial \mathbf{v}'}{\partial t'} + \mathbf{v}' \cdot \nabla \mathbf{v}' \right) = -\mathbf{v}' \rho' + \mu \nabla^2 \mathbf{v}' + \left( \frac{1}{3} \mu + \mu_B \right) \nabla' \mathbf{v}' \cdot \mathbf{v}'
\]

(1)

with the linear equation of state

\[
p' = \frac{c_0^2 \rho'}{\rho_0}
\]

(2)

Taking the dimensionless parameters without primes and defined as follows

\[
\mathbf{v} = \frac{\mathbf{v}}{U_\infty}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\omega \partial t'}, \quad \nabla = a \nabla', \quad \rho = \frac{\rho'}{\rho_0 U_\infty \omega a}, \quad p = \frac{p'}{\rho_0 U_\infty \omega a}
\]

(3)

where \( \rho_0 \) is the undisturbed fluid density, \( c_0 \) the speed of sound in the fluid, \( a \) the length dimension of the solid object, \( U_\infty \) the amplitude of oscillation, and \( \omega \) the circular frequency of oscillation. Inserting Eq. (3) into (1), we have

\[
\frac{\alpha U_\infty \omega a}{c_0^2} \frac{\partial \rho}{\partial t} + \frac{1}{a} \nabla \cdot \left( \rho_0 + \frac{\alpha U_\infty \omega a}{c_0^2} \rho \right) U_\infty \mathbf{v} = 0
\]

\[
(\rho_0 + \frac{\alpha U_\infty \omega a}{c_0^2} \rho) \left( \omega U_\infty \frac{\partial}{\partial t} + \frac{U_\infty^2}{a} \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\frac{\rho_0 U_\infty \omega a}{a} \nabla p + \frac{\alpha U_\infty \omega a}{a} \left[ \nabla^2 \mathbf{v} + \left( \frac{1}{3} + \mu_b \right) \nabla \mathbf{v} \cdot \mathbf{v} \right]
\]

(4)

where the dimensionless bulk viscous coefficient \( \mu_b = \mu_B / \mu \), which yields

\[
M^2 p \frac{\partial \rho}{\partial t} + \nabla \cdot (1 + \varepsilon M^2 \rho) \mathbf{v} = 0
\]

\[
(1 + \varepsilon M^2 \rho) \left( \frac{\partial \mathbf{v}}{\partial t} + \varepsilon \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \frac{1}{M^2} \left[ \nabla^2 \mathbf{v} + \left( \frac{1}{3} + \mu_b \right) \nabla \mathbf{v} \cdot \mathbf{v} \right]
\]

(5)

with the dimensionless parameters introduced as

\[
M^2 = \frac{\omega^2 a^2}{c_0^2}, \quad \varepsilon = \frac{U_\infty}{\omega a}, \quad \frac{M^2}{\varepsilon} = \frac{\omega a^2}{c_0^2}
\]

The dimensionless equation of state is obtained from Eqs. (2) and (3)

\[
p = \rho
\]

(6)

End quotation.
A.6. Determination of the non-dimensional parameters in the governing equations for the fluid

In section 4.1.2.1 the non-dimensional forms of the continuity equation and the Navier-Stokes equation for a compressible fluid under small amplitude harmonic excitation according to Wang [54] (reproduced in appendix A.5) are considered in order to assess the importance of the various fluid effects. Three non-dimensional parameters occur in these equations. Their numerical values for the fluids and conditions in the experiments performed in this work are presented in Tab. 4.1. In this section the details on the calculation of these parameters are given. Additionally the values for the Reynolds numbers are listed.

The non-dimensional parameters in Wang [54] are defined as

\[ M_p^2 = \frac{\omega^2 a^2}{c_f^2} \quad \epsilon = \frac{U_\infty}{\omega a} \quad M^2 = \frac{\omega^2}{\nu} \]  \hspace{1cm} (A.8)

in which \( \omega \) is the angular frequency of the harmonic oscillation, \( a \) is the length dimension of the solid object, \( c_f \) is the speed of sound, \( U_\infty \) is the velocity amplitude of the oscillation, and \( \nu \) is the kinematic viscosity. The parameter \( M_p \) describes the fluid compressibility, the parameter \( \epsilon \) the magnitude of the inertial effects, and \( M \) the viscous influences.

For the length dimension \( a \) of the solid object, the diameter of the cylindrical tube is inserted, which is \( a = 8 \text{ mm} \).

Tab. A.1 lists the test fluids considered in this work and their material properties. They are the same as presented in Tab. 4.4, but the kinematic viscosity \( \nu \) is listed additionally.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( \rho_f ) [kg/m³]</th>
<th>( \eta ) [mPas]</th>
<th>( c_f ) [m/s]</th>
<th>( \nu ) [m²/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DI-Water, 21.1 °C</td>
<td>997.97</td>
<td>0.977</td>
<td>1485.9</td>
<td>9.79 \times 10^{-7}</td>
</tr>
<tr>
<td>N4, 21.5 °C</td>
<td>813.68</td>
<td>5.220</td>
<td>1364.3</td>
<td>6.42 \times 10^{-6}</td>
</tr>
<tr>
<td>S6, 21.3 °C</td>
<td>874.34</td>
<td>9.085</td>
<td>1406.5</td>
<td>1.04 \times 10^{-5}</td>
</tr>
<tr>
<td>N35, 21.5 °C</td>
<td>860.54</td>
<td>68.51</td>
<td>1445.1</td>
<td>7.96 \times 10^{-5}</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane, 20.9 °C</td>
<td>691</td>
<td>0.50</td>
<td>1100.7</td>
<td>7.24 \times 10^{-5}</td>
</tr>
</tbody>
</table>

Table A.1.: Test fluids and their properties (density \( \rho_f \), dynamic viscosity \( \eta \), speed of sound \( c_f \), and kinematic viscosity \( \nu = \eta / \rho_f \)). The indicated temperatures are the reference temperatures for the determination of the fluid properties.

Tab. A.2 lists the resonance frequencies \( f_{res} \) and the velocity amplitudes of the oscillations \( U_\infty \) measured in the test fluids. The presented values for \( U_\infty \) are the
maximally measured velocity amplitudes when recording the phase-frequency curve\(^1\), and are measured with the vibrometer at a point inside the sensor tube and approximately 22 mm from the clamped end of the sensor tube. The laser beam is aligned manually to point in the direction of the deflections of the sensor tube. Note that the indicated values for \(U_\infty\) must not be the maximum velocity amplitudes in the system because the measurement point must not be the point on the tube with maximum deflection, even though it has been tried to get close to this point, and the laser beam is only aligned manually into the direction of the tube deflections. However, the presented values for \(U_\infty\) should give a good estimation for the maximum velocity amplitudes in the system.

\(U_{1;\text{min}}\) and \(U_{1;\text{max}}\) are the maximally measured velocity amplitudes at the minimum and the maximum excitation amplitudes used in the experiments.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>(f_{\text{res}}) [Hz]</th>
<th>(U_{\infty;\text{min}}) [mm/s]</th>
<th>(U_{\infty;\text{max}}) [mm/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DI-Water</td>
<td>8766.7</td>
<td>0.030</td>
<td>0.264</td>
</tr>
<tr>
<td>N4</td>
<td>9043.0</td>
<td>0.029</td>
<td>0.239</td>
</tr>
<tr>
<td>S6</td>
<td>8960.1</td>
<td>0.029</td>
<td>0.192</td>
</tr>
<tr>
<td>N35</td>
<td>8965.6</td>
<td>0.029</td>
<td>0.145</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane</td>
<td>9254.5</td>
<td>0.028</td>
<td>0.288</td>
</tr>
</tbody>
</table>

Table A.2.: Measured resonance frequencies \(f_{\text{res}}\) in the test fluids. Maximally measured velocity amplitudes \(U_{\infty;\text{min}}\) and \(U_{\infty;\text{max}}\) at the minimum and the maximum excitation amplitudes used in the measurements in the test fluids.

Using the values presented in tables A.1 and A.2, the nondimensional parameters from Eq. (A.8) can be calculated and are presented in Tab. A.3.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>(M_p^2)</th>
<th>(\varepsilon_{\text{min}})</th>
<th>(\varepsilon_{\text{max}})</th>
<th>(M^2)</th>
<th>(1/M^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DI Water</td>
<td>0.088</td>
<td>(6.8 \times 10^{-8})</td>
<td>(6.0 \times 10^{-7})</td>
<td>(3.6 \times 10^6)</td>
<td>(0.28 \times 10^{-6})</td>
</tr>
<tr>
<td>N4</td>
<td>0.11</td>
<td>(6.4 \times 10^{-8})</td>
<td>(5.3 \times 10^{-7})</td>
<td>(0.57 \times 10^6)</td>
<td>(1.8 \times 10^{-6})</td>
</tr>
<tr>
<td>S6</td>
<td>0.10</td>
<td>(6.4 \times 10^{-8})</td>
<td>(4.3 \times 10^{-7})</td>
<td>(0.35 \times 10^6)</td>
<td>(2.9 \times 10^{-6})</td>
</tr>
<tr>
<td>N35</td>
<td>0.097</td>
<td>(6.4 \times 10^{-8})</td>
<td>(3.2 \times 10^{-7})</td>
<td>(0.045 \times 10^6)</td>
<td>(22 \times 10^{-6})</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane</td>
<td>0.18</td>
<td>(6.0 \times 10^{-8})</td>
<td>(6.2 \times 10^{-7})</td>
<td>(5.1 \times 10^6)</td>
<td>(0.19 \times 10^{-6})</td>
</tr>
</tbody>
</table>

Table A.3.: Values of the non-dimensional parameters in the non-dimensional forms of the continuity equation and of the Navier-Stokes equation according to Wang [54] for the fluids and conditions in the experiments presented in this work. The two different values for \(\varepsilon\) reflect the different amplitudes for \(U_\infty\) present in the experiments.

The Reynolds number represents the ratio between the inertial and the viscous forces in the fluid. The Reynolds number can be calculated from the non-dimensional

\(^1\)The maximum amplitude of a resonance peak has been sought manually by adapting the excitation frequency. The frequencies at which measurements have been conducted in order to obtain the phase-frequency curve contain this frequency at which maximum amplitude has been found.
A.7. Determination of wavenumbers in order to assess the occurrence of acoustic radiation

parameters given above as $Re = \varepsilon M^2 = U_\infty a / \nu$. The values are presented in Tab. A.4.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>$Re_{\min}$ [-]</th>
<th>$Re_{\max}$ [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DI-Water</td>
<td>0.25</td>
<td>2.2</td>
</tr>
<tr>
<td>N4</td>
<td>0.036</td>
<td>0.30</td>
</tr>
<tr>
<td>S6</td>
<td>0.022</td>
<td>0.15</td>
</tr>
<tr>
<td>N35</td>
<td>0.003</td>
<td>0.015</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane</td>
<td>0.31</td>
<td>3.2</td>
</tr>
</tbody>
</table>

Table A.4.: Reynolds numbers for the fluids and conditions in the experiments presented in this work. The two different values $Re_{\min}$, $Re_{\max}$ reflect the different amplitudes for $U_\infty$ present in the experiments.

A.7. Determination of wavenumbers in order to assess the occurrence of acoustic radiation

As discussed in section 4.1.2.1, acoustic radiation can occur only if the axial wavenumber $k_z$ of the vibrating cylinder is smaller than the acoustic wavenumber $k_f$ of the fluid. In this section, the details of the determination of the wavenumbers are presented.

The acoustic wavenumbers $k_f$ in the test fluids are determined from the speed of sound $c_f$ of the fluids and the measured resonance frequency $f_{res}$ of the flexural oscillation mode of the density sensor according to $k_f = \omega / c_f = (2\pi f_{res}) / c_f$. Tab. A.5 summarizes the speeds of sound, the resonance frequencies, and the calculated wavenumbers for the test fluids used in this work.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>$c_f$ [m/s]</th>
<th>$f_{res}$ [Hz]</th>
<th>$k_f$ [1/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>344.6</td>
<td>11 069.4</td>
<td>201.8</td>
</tr>
<tr>
<td>DI-Water</td>
<td>1485.9</td>
<td>8766.7</td>
<td>37.1</td>
</tr>
<tr>
<td>N4</td>
<td>1364.3</td>
<td>9043.0</td>
<td>41.6</td>
</tr>
<tr>
<td>S6</td>
<td>1406.5</td>
<td>8960.1</td>
<td>40.0</td>
</tr>
<tr>
<td>N35</td>
<td>1445.1</td>
<td>8965.6</td>
<td>39.0</td>
</tr>
<tr>
<td>2,2,4-Trimethylpentane</td>
<td>1100.7</td>
<td>9254.5</td>
<td>52.8</td>
</tr>
</tbody>
</table>

Table A.5.: Speed of sound $c_f$ of the test fluids used in this work, measured resonance frequency $f_{res}$ of the flexural oscillation mode of the density sensor in these fluids, and corresponding acoustic wavenumber $k_f$ in these test fluids.

The axial wavenumber $k_z$ of the density sensor oscillating in the third flexural oscillation mode can be estimated from Euler Bernoulli beam theory. For a fixed-free Euler Bernoulli beam, the axial wavenumber fulfills the equation (Rao [37, p. 333]) $k_z \cdot L = 7.8547$, in which $L$ is the length of the beam. With $L = 91.22$ mm for the
Appendix A. Appendix

free length of the tube of the density sensor, one obtains \( k_z = 86.11 \text{m} \).
Note that for a (fixed-free) Euler Bernoulli beam, and if only inertial fluid forces
are considered, the wavenumber \( k_z \) of the beam in fluid is the same as in vacuum
(compare section 5.1.1).

As \( k_z > k_f \) for the fluids considered above (except for air), acoustic radiation is
expected to be zero for most of the sensor surface.

A.8. Matrix \( M \) in the expression for the fluid forces
according to Kadyrov et al. [30]

In section 4.1.2.4 the fluid forces in a 2D compressible viscous bounded fluid ac-
cording to Kadyrov et al. [30] are discussed. The expression for the fluid forces
according to Kadyrov et al. [30] depends on entries of the inverse of a Matrix \( M \),
which arises from the evaluation of boundary conditions of the fluid velocity field.
As reference Kadyrov et al. [30] seems to contain some typos, the calculations have
been repeated, which led to a slightly different expression for \( M \) than given in reference Kadyrov et al. [30]. The following expression for matrix \( M \) has been used
in present work

\[
M = \begin{bmatrix}
\chi_1 J'_1(\chi_1) & \chi_1 Y'_1(\chi_1) & -I'_1(\chi_2) & -K'_1(\chi_2) \\
\chi_1 J'_2(\chi_1) & \chi_1 Y'_2(\chi_1) & -I'_1(\chi_2) & -K'_1(\chi_2) \\
J'_1(\chi_1) & Y'_1(\chi_1) & -\chi_2 I'_{1\delta}(\chi_2) & -\chi_2 K'_{1\delta}(\chi_2) \\
J'_2(\chi_1) & Y'_2(\chi_1) & -\chi_2 I'_{2\delta}(\chi_2) & -\chi_2 K'_{2\delta}(\chi_2)
\end{bmatrix}
\]  

(A.9)

in which \( \delta = R_o / R \) is the ratio between the radius \( R_o \) of the cylindrical fluid bound-
ary and the radius \( R \) of the oscillating cylinder. The prime denotes differentiation
with respect to the argument of the function, e.g.

\[
J'_1(\chi_1) = \left[ \frac{\partial J_1(z)}{\partial z} \right]_{z = \chi_1}
\]  

(A.10)

and accordingly for the other expressions.

The above expression for the matrix \( M \) differs from the one presented in reference
Kadyrov et al. [30] in that there are additional factors \( \chi_1 \) and \( \chi_2 \) in front of the
differentiated (modified) Bessel functions. This discrepancy stems from the fact,
that the interior derivatives of the differentiated (modified) Bessel functions are
missing in reference Kadyrov et al. [30].
A.9. Hydrodynamic function $\Gamma$ in the expression for the fluid forces according to Van Eysden and Sader [52]

In section 4.1.2.4 the description of the fluid forces according to Van Eysden and Sader [52] is discussed. It depends on a hydrodynamic function $\Gamma$. For convenience, the expression for the hydrodynamic function $\Gamma$ from Eq. (20) in Van Eysden and Sader [52] is presented here.

$$\Gamma(\kappa, \zeta, Re) = \alpha^2 K_1(-i\alpha)\{\alpha(\kappa^2 + iRe)K_0(-i\alpha)K_1(-i\beta) + K_1(-i\alpha)[\alpha^2 \beta K_0(-i\beta)$$
$$- 4ReK_1(-i\beta)]\}/[ReD(\kappa, \zeta, Re)] \quad (A.11)$$

in which

$$D(\kappa, \zeta, Re) = \kappa^2 \alpha^2 K_0^2(-i\alpha)K_1(-i\beta) + i\alpha K_0(-i\alpha)K_1(-i\alpha) \times [(\kappa^2 + iRe)K_1(-i\beta)$$
$$- i\alpha^2 \beta K_0(-i\beta)] + i\alpha^2 \beta K_0(-i\beta)K_1^2(-i\alpha) \quad (A.12)$$

and

$$\alpha = \sqrt{iRe - \kappa^2} \quad \beta = \sqrt{\frac{Re \zeta^2}{Re - i\zeta^2(4/3 + \eta_B/\eta) - \kappa^2}} \quad (A.13)$$

and in which

$$Re = \frac{\rho f \omega R^2}{\eta} \quad \kappa = k_z R \quad \zeta = \omega R \sqrt{\rho f K} = \frac{\omega R}{c_f} \quad (A.14)$$

In these equations $\rho_f$ is the density, $\eta$ is the dynamic viscosity, $\eta_B$ is the bulk viscosity, $c_f$ the speed of sound, and $K$ is the compressibility (cp. Eq. (4.27)) of the fluid. Further, $R$ is the cylinder radius, $k_z$ is the wavenumber along the cylinder axis, and $\omega$ is the angular oscillation frequency.

A.10. Introduction of the fluid forces acting on the rigid mass, and free end correction

In this section the description of the rigid mass in the mobility model is considered for the case where the rigid mass is surrounded by a fluid. The fluid forces acting on the rigid mass are introduced in the model as a distributed load $q(x, t)$. The description of the rigid mass is based on evaluating the linear and angular momentum balances, in which the fluid forces enter in integrated form. It is possible to multiply the distribution of the fluid forces along the rigid mass with an arbitrary function before integrating them, and thereby mimic the decreasing fluid forces towards the free end of the rigid mass as described in section 4.1.4. The details of the derivation are presented in this section.
The considered situation is depicted in Fig. A.5, where also the kinematic and force variables for the description of the rigid mass are defined.

Figure A.5.: Definition of kinematic and force variables for the rigid mass in the mobility model, if a distributed load \( q(x,t) \) acts on the rigid mass. \( M_a, Q_s \): moment and shear force acting on the rigid mass at position \( x = 0 \), \( \eta_a \): deflection of the rigid mass at position \( x = 0 \), \( M_s, Q_s \): moment and shear force acting on the center of mass of the rigid mass, \( \eta_s \): deflection of the center of mass of the rigid mass, \( \varphi \): Rotation of the axis of the rigid mass, \( x_s \): \( x \)-coordinate of the center of mass \( S \). \( l_{rm} \) denotes the length of the rigid mass.

The kinematics is described by

\[
\varphi_a = \varphi_s = \varphi \quad \text{and} \quad \varphi = \frac{\partial \eta}{\partial x} \quad (A.15)
\]
\[
\eta_s = \eta_a + \varphi x_s \quad (A.16)
\]

If the mass and the mass moment of inertia with respect to the center of mass are denoted by \( m \) and \( I_s \) respectively, the linear and angular momentum balances yield

\[
m \ddot{\eta}_s = Q_s - Q_a + \int_0^{l_{rm}} q(x,t) dx \quad (A.17)
\]
\[
I_s \ddot{\varphi} = M_s - M_a + Q_a x_s + \int_0^{l_{rm}} (x - x_s) q(x,t) dx \quad (A.18)
\]

Consider only time-harmonic motion (time dependence expressed by \( e^{i\omega t} \)) and express the distributed load as \( q(x,t) = (\alpha + i \beta) f(x) \) in which \( f(x) \) is an arbitrary function of the position \( x \) along the rigid mass. The function \( f(x) \) allows to mimic the decreasing fluid forces towards the free end of the rigid mass.

Defining

\[
\int_0^{l_{rm}} f(x) dx = J_1 \quad \int_0^{l_{rm}} x f(x) dx = J_2 \quad \int_0^{l_{rm}} x^2 f(x) dx = J_3 \quad (A.19)
\]
the kinematic relations and the momentum balances can be written as

\begin{align}
\varphi_s &= \varphi_a \\
\eta_s &= \eta_a + \varphi_a x_s \\
Q_s &= Q_a + [-m\omega^2 x_s - (\alpha + i\beta)i\omega J_2]\varphi_a + [-m\omega^2 - (\alpha + i\beta)i\omega J_1]\eta_a \\
M_s &= -x_s Q_a + M_a + [-I_s \omega^2 - (\alpha + i\beta)i\omega (J_3 - J_2 x_s)]\varphi_a - (\alpha + i\beta)i\omega (J_2 - J_1 x_s)\eta_a
\end{align}

Putting these equations in matrix form, the transfer matrix for the rigid mass in fluid according to Eq. (3.14) is obtained.
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[21] GOODWIN, A. R. H., JAKEWAYS, C. V., AND DE LARA, M. M. A mems vibrating edge supported plate for the simultaneous measurement of density and viscosity: Results for nitrogen, methylbenzene, water, 1-propene,1,1,2,3,3-trifluoro-oxidized-polymel, and polydimethylsiloxane and four certified reference materials with viscosities in the range (0.038 to 275) mpa.s and densities between (408 and 1834) kg.m(-3) at temperatures from (313 to 373) k and pressures up to 60 mpa. Journal of Chemical and Engineering Data 53, 7 (2008), 1436–1443.


List of publications