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QUANTUM INVARIANTS AND LAGRANGIAN TOPOLOGY

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To my parents

Abstract

Let (M, ω) be a symplectic manifold and $L \subset M$ a Lagrangian submanifold. In this thesis we construct invariants of embedded closed monotone Lagrangian submanifolds, termed quantum invariants, and study their properties and relations to other existing invariants in symplectic topology. These are invariant under symplectomorphisms and are derived from the Lagrangian quantum homology of L. The thesis consists of two parts.

The Lagrangian quantum homology $QH(L)$ of L has rich algebraic structures compatible with the ring structure on the ambient quantum homology $QH(M)$. In certain situations $QH(L)$ is isomorphic (non canonically) to the singular homology $H(L)$ and the algebraic structures of $QH(L)$ correspond to a class of deformations of the classical algebraic structures on $H(L)$. In the first part we develop a deformation theory of the quantum algebraic structures in Lagrangian quantum homology. We then construct invariants associated to the classes of deformations. We obtain cohomological invariants which arise from the quantum product, the quantum module action and the quantum inclusion on $QH(L)$.

The second part presents joint work with Paul Biran. This part is concerned with the study of a specific invariant of the quantum product, termed the discriminant, and describes the implications and properties in the ambient quantum homology and under Lagrangian cobordisms. Under certain assumptions we prove that the homology class of L satisfies a cubic equation in the ambient quantum homology $QH(M)$. Furthermore, using the coefficients of this cubic equation we can compute the discriminant of the Lagrangian L. We also study the relation between this invariant and Lagrangian cobordisms. We pay special attention to the case of Lagrangian spheres and provide several examples of computations of the invariant.

Zusammenfassung

Sei (M, ω) eine symplektische Mannigfaltigkeit und $L \subset M$ eine Lagrange-Untermannigfaltigkeit. In dieser Dissertation konstruieren wir Invarianten von eingebetteten, kompakten und monotonen Lagrange-Untermannigfaltigkeiten, genannt Quanteninvarianten, und erforschen ihre Eigenschaften und die Zusammenhänge zu anderen bestehenden Invarianten der symplektischen Topologie. Diese Quanteninvarianten sind invariant unter Symplektomorphismen und werden mithilfe der Lagrange-Quantenhomologie von L konstruiert. Die Arbeit besteht aus zwei Teilen.

Die Lagrange-Quantenhomologie $QH(L)$ von L besitzt vielfältige algebraische Strukturen, die mit der Ring-Struktur der Quantenhomologie $QH(M)$ kompatibel sind. In bestimmten Situationen entsprechen die algebraischen Strukturen von $QH(L)$ einer Klasse von Deformationen der klassischen Strukturen auf der singulären Homologie $H(L)$. Im ersten Teil der Arbeit entwickeln wir eine Theorie der Deformationen von algebraischen Strukturen in der Lagrange-Quantenhomologie und konstruieren daraus Invarianten. Wir erhalten Invarianten kohomologischer Natur, welche vom Quantenprodukt, von der Quantenmodulstruktur und von der Quanteninklusion in $QH(L)$ abstammen.

Der zweite Teil der Arbeit stellt ein Projekt mit Paul Biran vor. In diesem Teil untersuchen wir eine Invariante des Quantenprodukts, die sogenannte Diskriminante, und beschreiben ihre Eigenschaften im Zusammenhang mit der Quantenhomologie und mit Lagrange-Kobordismen. Unter gewissen Annahmen erfüllt die Homologie-Klasse von L eine kubische Gleichung in $QH(M)$. Zudem können wir mit den Koeffizienten dieser Gleichung die Diskriminante von L berechnen. Den Fall einer Lagrange-Sphäre betrachten wir besonders und berechnen die Diskriminante in verschiedenen Beispielen.

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Contents

Introduction

The theory of pseudoholomorphic curves has been an essential tool in symplectic topology since Gromov applied these ideas in his seminal paper [40]. In particular, by using these techniques Gromov proved the non-existence of exact Lagrangian submanifolds in linear symplectic space $(\mathbb{R}^{2n}, \omega_{\text{std}})$. This was achieved by finding a topological condition on Lagrangian submanifolds of $\mathbb{R}^{2n}$ using the existence of pseudoholomorphic disks with boundary on the Lagrangian submanifold. Since then questions concerning the topology and geometry of Lagrangian submanifolds have been of great interest and have far-reaching consequences to other fields, such as mirror symmetry.

The focal point of this thesis lies in the construction of invariants of embeddings of Lagrangian submanifolds, termed quantum invariants, and the study of their properties and relations to other existing invariants. These are invariant under symplectomorphisms and are constructed using Lagrangian quantum homology, which is a variant of Floer homology. The thesis consists of two parts. The first part explains the deformation theory of the algebraic structures in Lagrangian quantum homology and derives quantum invariants from this approach. The second part is devoted to the study of a specific invariant of the quantum product, termed the discriminant, and describes the implications and properties in the ambient quantum homology and under Lagrangian cobordisms. We provide an overview of the results.

Deformation theory of quantum structures

Floer theory and Lagrangian quantum homology Lagrangian Floer homology was originally introduced by Andreas Floer [30] in his approach to solve the Arnold conjecture. It can be seen as an infinite dimensional version of Morse homology and relies heavily on the theory of pseudoholomorphic curves initiated by Gromov.

Let (M, ω) be a connected and compact symplectic manifold. Assume $L \subset M$ is a closed monotone Lagrangian submanifold with minimal Maslov number $N_L \geq 2$. Furthermore assume L is spin with a fixed spin structure. We restrict ourselves to the monotone case, although Floer homology can be defined in a more general setting by using advanced methods, see [31].

For a fixed ground ring k we take the ring of Laurent polynomials $k[t, t^{-1}]$ as our coefficient ring. Let $H : M \times [0, 1] \to \mathbb{R}$ be a Hamiltonian with Hamiltonian vector field X_t^H and flow ψ_t^H. We assume that $\psi_1^H(L)$ is transverse to L and denote by $\mathcal{O}(L, L)$ the set of Hamiltonian chords, i.e. the set of paths $\gamma : [0, 1] \to M$ such that $\dot{\gamma}(t) = X_t^H(\gamma(t))$ and $\gamma(0), \gamma(1) \in L$. Choose a family $\mathbf{J} = \{J_t\}_{t \in [0,1]}$ of generic almost complex structures compatible with ω. The Floer complex $CF(L, L; H, \mathbf{J})$ is then defined as the $k[t, t^{-1}]$-module generated by the elements of $\mathcal{O}(L, L)$. The Floer differential $\partial_F : CF(L, L; H, \mathbf{J}) \to CF(L, L; H, \mathbf{J})$ is given by counting perturbed pseudoholomorphic strips connecting two generators, i.e. by maps $u : [0, 1] \times \mathbb{R} \to M$ satisfying Floer's equation with a Hamiltonian perturbation

$$\partial_s u + J_t(u) \left(\partial_t u - X_t^H(u)\right) = 0,$$

with the boundary conditions $u(0, s), u(1, s) \in L$. These strips are weighted by their symplectic area in the definition of ∂_F and we have $(\partial_F)^2 = 0$. The resulting homology

$$HF(L, L; H, \mathbf{J})$$

is then the (self-) Floer homology of L. For a different choice of auxiliary data $(H', \mathbf{J}')$ one can construct canonical isomorphisms $HF(L, L; H, \mathbf{J}) \to HF(L, L; H', \mathbf{J}')$. Thus the homology is independent of choices up to isomorphism and we may write $HF(L, L)$. $HF(L, L)$ may also be given a cyclic grading.

In general Floer homology groups are difficult to compute, although in certain cases the Floer homology is known. Specifically, if $L_0 \subset M$ is displaceable, then the Floer homology vanishes,

$$HF(L_0, L_0) = 0.$$

On the other hand, if a Lagrangian L_1 satisfies $[\omega] \cdot \pi_2(M, L_1) = 0$, then we have $HF(L_1, L_1) \simeq H(L_1; k) \otimes k[t, t^{-1}]$.

Lagrangian quantum homology is an alternative construction of Floer homology which was introduced in order to make calculations in Floer theory possible. This approach was originally suggested by Fukaya [32] and Oh [58]. The full theory was then implemented by Biran-Cornea [13, 14, 15]. In contrast to Floer homology, the main objects of study are pseudoholomorphic disks with boundary on the Lagrangian submanifold as opposed to solutions of Floer's equation with a Hamiltonian perturbation and Lagrangian boundary conditions.

The chain complex in Lagrangian quantum homology is constructed using a coefficient ring $\mathcal{R}$ and choice of data triple $\mathcal{D} = (f, (\cdot, \cdot), J)$ consisting of a Morse function $f : L \to \mathbb{R}$, a Riemannian metric $(\cdot, \cdot)$ on L and an ω-compatible almost complex structure J. For a generic choice of $\mathcal{D}$ the chain complex is well-defined and we obtain the Lagrangian quantum homology $QH(L; \mathcal{R})$ of L. For two generic data triples $\mathcal{D}$, $\mathcal{D}'$ there are canonical isomorphisms between the respective quantum homologies. If $\mathcal{R} = k[t, t^{-1}]$, then there is a canonical isomorphism to the Floer homology of L, $HF(L, L) \simeq QH(L; k[t, t^{-1}])$. For the coefficient ring $k[t]$ no such isomorphism exists, since the Floer homology is not well-defined for these coefficients.

In certain cases we have $QH(L; \mathcal{R}) \simeq H(L; \mathcal{R})$. We call such Lagrangians $\mathcal{R}$-wide. This isomorphism is not canonical and depends on the choice of data triple $\mathcal{D}$ used in the construction of $QH(L; \mathcal{R})$. Changes in data then result in maps $H(L; \mathcal{R}) \to H(L; \mathcal{R})$ which are deformations of the identity, we refer to §1.1.3 for more details.

Quantum product and enumerative invariants The Donaldson product in $HF(L, L)$ corresponds to the quantum product in Lagrangian quantum homology. For a $k[t]$-wide Lagrangian L, the quantum product transfers via the isomorphism to $H(L; k) \otimes k[t]$ and defines a deformation of the intersection product in singular homology. Since the isomorphism is not canonical, the quantum product on $H(L; k) \otimes k[t]$ may differ for different choices of generic data triple $\mathcal{D}$. Nevertheless one can still extract an invariant of the Lagrangian embedding, as was previously shown by Biran and Cornea in [16]. This is done by taking the discriminant of a quadratic form on $H_*(L; k)$ defined via the quantum product, we recollect this approach in §1.2.2.

In the case of a Lagrangian torus $L \simeq T^2$ Biran-Cornea offer an enumerative interpretation of this discriminant, which we recall here. Choose a generic almost complex structure J compatible with ω and fix three points p, q, r on L. The number of J-holomorphic disks $u : (D^2, \partial D^2) \to (M, L)$ with Maslov index 4 that pass through p, q, r is not an invariant of L and depends on the choices of p, q, r as well as J. Nevertheless, it is still possible to obtain an enumerative invariant of L by redefining the count of J-holomorphic disks.

Theorem (An enumerative invariant in [16]). *Fix three generic points $p, q, r \in L$ and choose an oriented path $\overline{pq}$ in L joining p to q and similarly for q to r and r to p. Let n_{pqr} be the number of Maslov-4 disks passing through p, q and r in this order. Let n_p be the signed number of J-holomorphic Maslov-2 disks passing through p and crossing the edge $\overline{qr}$. Define the numbers n_q and n_r similarly. If $HF(L, L) \neq 0$, then*

$$\Delta = 4n_{pqr} + n_p^2 + n_q^2 + n_r^2 - 2n_p n_q - 2n_q n_r - 2n_r n_p$$

is independent of the choices p, q, r and J.

Deformations of quantum structures There are further algebraic structures in Floer homology which have counterparts in Lagrangian quantum homology. Denote by $(QH(M), *)$ the quantum homology ring of the ambient symplectic manifold M. The quantum cap product $QH(M) \to HF(L, L)$ (see [4], also known as a special case of the closed-open map in the setting of Fukaya categories) corresponds to the quantum module action of $QH(M)$ on $QH(L)$. This defines a $QH(M)$-module structure on $QH(L)$, and combined with the quantum product this makes $QH(L)$ into a $QH(M)$-algebra. The open-closed map $HF(L, L) \to QH(M)$ in Floer homology corresponds to the quantum inclusion map $QH(L) \to QH(M)$. From these rich algebraic structures we construct new quantum invariants, extending the approach given in [16].

For a $k[t]$-wide Lagrangian we transfer the quantum module action to $H(L; k)\otimes k[t]$. On $H(L; k)\otimes k[t]$ it is given by a deformation of the external intersection product of $H(M; k)$ on $H(L; k)$. On $H(M; k)$ the quantum product $*$ is a deformation of the intersection product. Motivated by Hochschild cohomology, we define a general algebraic theory for graded deformations of module actions over deformed algebras. These deformations are called coupled deformations. For an algebra A and an A-module N we define cohomology groups of degree d, $HE^{*,d}(A, N)$, which we use to characterize the first order terms of graded coupled deformations of degree d. We then obtain an invariant of the quantum module action of $QH(M)$ on $QH(L)$ for an embedding $L \hookrightarrow M$. We summarize this in the result of §2.2.6:

Theorem A. *Let $L \subset M$ be a monotone Lagrangian with $N_L \geq 2$. Assume L is $k[t]$-wide. Let $QH(L)$ be the Lagrangian quantum homology of L and denote by $\circledast$ the quantum module action of $QH(M)$ on $QH(L)$. Then $(QH(L), \circledast, QH(M), *)$ defines a nontrivial cohomology class*

$$c(QH(L), \circledast, QH(M), *) \in HE^{1,-N_L}(H(M), H(L)),$$

which is independent of choice of generic data $\mathcal{D}$.

We take a similar approach to the quantum algebra structure. Since $QH(L)$ is a $QH(M)$ algebra, by transferring the algebraic maps to $H(L; k)\otimes k[t]$ we obtain a class of algebraic maps on $H(L; k)\otimes$

$k[t]$ which represent deformations of the intersection product on L, the external intersection product and the intersection product on M. We call such deformations triple deformations and we define a general algebraic theory for such graded deformations. For algebras A and X, with X also an A-module, we define cohomology groups of degree d, $HG^{*,d}(A, X)$, which we use to characterize first order terms of graded triple deformations of degree d. We then obtain an invariant of the quantum algebra structure of $QH(L)$ over $QH(M)$ for an embedding $L \hookrightarrow M$. We state the result of §2.3.5:

Theorem B. *Let $L \subset M$ be a monotone Lagrangian with $N_L \geq 2$. Assume L is $k[t]$-wide. Let $QH(L)$ be the Lagrangian quantum homology of L and denote by $(QH(L), \circ, \circledast, QH(M), *)$ the quantum algebra $(QH(L), \circ)$ over $(QH(M), *)$ defined by the Lagrangian embedding. Then $(QH(L), \circ, \circledast, QH(M), *)$ defines a nontrivial cohomology class*

$$c(QH(L), \circ, \circledast, QH(M), *) \in HG^{1,-N_L}(H(M), H(L)),$$

which is independent of choice of generic data $\mathcal{D}$.

This section is then succeeded by a short discussion on algebraic deformations of module morphisms in §2.4.

The Lagrangian cubic equation

This part presents joint work with Paul Biran. Here we construct another invariant of a Lagrangian submanifold using the quantum product and study the implications and the properties of this invariant. In this part we assume our Lagrangian L to satisfy certain conditions, which we summarize to Assumption $\mathscr{L}$:

1. L is closed (i.e. compact without boundary). Furthermore L is monotone with minimal Maslov number N_L that satisfies $N_L \mid n$.

2. L is oriented. Moreover we assume that L is spinable (i.e. can be endowed with a spin structure).

3. $HF_n(L, L)$ has rank 2.

4. Write $\chi = \chi(L)$ for the Euler characteristic of L. We assume that $\chi \neq 0$.

A cubic equation Denote by $(QH(M), *)$ the quantum homology of M with quantum product $*$ and coefficients in the ring $\mathbb{Z}[q]$ graded by $|q| = -2$. For a Lagrangian submanifold $L \subset M$ denote by $[L] \in QH_n(M)$ its homology class in M. Let $\varepsilon = (-1)^{n(n-1)/2}$.

Theorem C (The Lagrangian cubic equation)**.** *Let $L \subset M$ be a Lagrangian submanifold satisfying Assumption $\mathscr{L}$. Then there exist unique rational constants $\sigma_L \in \frac{1}{\chi^2}\mathbb{Z}$, $\tau_L \in \frac{1}{\chi^3}\mathbb{Z}$ such that the following equation holds in $QH(M)$:*

$$[L]^{*3} - \varepsilon\chi\sigma_L[L]^{*2}q^{n/2} - \chi^2\tau_L[L]q^n = 0. \tag{1}$$

If χ is square-free, then $\sigma_L \in \frac{1}{\chi}\mathbb{Z}$, $\tau_L \in \frac{1}{\chi^2}\mathbb{Z}$. Moreover, the constant σ_L can be expressed in terms of genus 0 Gromov-Witten invariants as follows:

$$\sigma_L = \frac{1}{\chi^2}\sum_A GW^M_{A,3}([L], [L], [L]),$$

where the sum is taken over all classes $A \in H_2(M)$ with $\langle c_1, A\rangle = n/2$.

The discriminant For a Lagrangian L that satisfies (1) – (3) of Assumption $\mathscr{L}$ we define an invariant Δ_L of L, called the discriminant, using the quantum product. In dimension two the discriminant Δ_L coincides with the enumerative invariant Δ previously described. The cubic equation then provides a method of computation for Δ_L.

Theorem D. *Let $L \subset M$ be a Lagrangian submanifold satisfying Assumption $\mathscr{L}$. Let $\sigma_L, \tau_L \in \mathbb{Q}$ be the constants from the cubic equation* (1)*. Then*

$$\Delta_L = \sigma_L^2 + 4\tau_L.$$

The discriminant and Lagrangian cobordisms The theory of Lagrangian cobordisms was introduced by Arnold [5, 6] and further studied by other authors [28, 7, 19]. An *elementary Lagrangian cobordism* $V : L \rightsquigarrow L'$ of two Lagrangians L, $L' \subset (M, \omega)$ is given by a Lagrangian submanifold $V \subset (M \times T^*[0,1]), \omega \oplus dp \wedge dq)$ such that $\partial V = L \times \{(0, t_0)\} \sqcup L' \times \{(1, t_1)\}$. This definition extends to the case where the cobordism has many ends $V : (L_1, \ldots, L_r) \rightsquigarrow (L'_1, \ldots, L'_s)$, see [17].

In [17, 18] Biran and Cornea revisited the theory of Lagrangian cobordisms and studied the relation to Lagrangian quantum homology. One result demonstrates that the existence of a cobordism (V, L, L') implies $QH(L) \simeq QH(L')$. In certain cases we can describe the behaviour of the discriminant under Lagrangian cobordisms.

Theorem E. *Let $L_1, \ldots, L_r \subset M$ be Lagrangian submanifolds, each satisfying conditions* (1) – (3) *of Assumption $\mathscr{L}$. Let $V^{n+1} \subset \mathbb{R}^2 \times M$ be a connected monotone Lagrangian cobordism whose ends correspond to $L_1, \ldots, L_r$ and assume that V admits a spin structure. Denote by N_V the minimal Maslov number of V and assume that:*

1. *$N_V \mid n$;*
2. *$H_{jN_V}(V, \partial V) = 0$ for every j;*
3. *$H_{1+jN_V}(V) = 0$ for every j.*

Then $\Delta_{L_1} = \cdots = \Delta_{L_r}$. Moreover if $r \geq 3$, then Δ_{L_i} is a perfect square for every i.

As a corollary we obtain:

Corollary F. *Let (M, ω) be a monotone symplectic manifold with $2C_M \mid n$, where C_M is the minimal Chern number of M. Let $L_1, L_2 \subset M$ be two Lagrangian spheres that intersect transversely at exactly one point. Then $\Delta_{L_1} = \Delta_{L_2}$ and moreover this number is a perfect square.*

Outline

This thesis consists of two parts, which can be read independently of each other. Overlaps occur only in §1.1 of Part I and §3.2 of Part II and are limited to basic definitions within the settings and the descriptions of the algebraic structures in Lagrangian quantum homology.

In Part I we begin with a recollection of the construction of Lagrangian quantum homology and the essential algebraic structures (§1.1). We then discuss the deformation viewpoint and the relation to invariants of pseudoholomorphic disks (§1.2). The second chapter presents the deformation theory of the quantum structures. It begins with a description of previous work done on the quantum product (§2.1), this is followed by the development of the deformation theory for the quantum module action (§2.2), the quantum algebra structure (§2.3) and quantum module morphisms (§2.4). In every section we explain how to derive invariants of Lagrangian embeddings using these structures.

In Part II we first recall the main results and provide more details on the construction of Δ_L and the results together with an overview of the examples (§3.1). Subsequently we describe the necessary ingredients from Floer theory and Lagrangian quantum homology (§3.2). In the next chapter we prove more general versions of the main statements and derive additional corollaries from them (§4.1 and §4.2). The last chapter is dedicated to the computation of examples. We briefly explain how to construct Lagrangian spheres in symplectic Del Pezzo surfaces and compute the discriminant of various Lagrangians (§5.1). We then explain an extension of the discriminant and the Lagrangian cubic equation over a more general coefficient ring and recalculate the discriminants in this case as well (§5.2).

Part I.

Deformation theory of quantum structures

1. Lagrangian quantum homology and disk invariants

1.1. Algebraic structures in Lagrangian quantum homology

1.1.1. Setting Let (M, ω) be a connected symplectic manifold. We assume M to be compact, but under suitable adjustments our results can be extended to the case when M is a tame symplectic manifold (see [3]). Lagrangian submanifolds will be assumed to be connected and closed. Denote by $H_2^D(M, L) \subset H_2(M, L)$ the image of the Hurewicz homomorphism $\pi_2(M, L) \to H_2(M, L)$. For a given Lagrangian submanifold $L \subset (M, \omega)$ there are two homomorphisms

$$\omega : H_2^D(M, L) \to \mathbb{R}, \quad \mu : H_2^D(M, L) \to \mathbb{Z},$$

the former is given by integration of ω and the latter is the Maslov index. A Lagrangian submanifold is *monotone*, if there exists a constant $\rho > 0$ such that

$$\omega(A) = \rho\mu(A), \quad \forall\, A \in H_2^D(M, L).$$

The minimal Maslov number of L is defined as

$$N_L := \min\{\, \mu(A) \mid \mu(A) > 0, A \in H_2^D(M, L)\}.$$

The minimal Chern number of M is defined as

$$C_M := \min\{\, c_1(B) \mid c_1(B) > 0, B \in \pi_2(M)\},$$

where c_1 is the first Chern class of M.

In the following constructions we use a ground ring k. k will usually be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ or possibly $\mathbb{Z}_2$. In the event that $k \neq \mathbb{Z}_2$, we assume that our Lagrangians are orientable and spinnable and equip these with a fixed orientation and spin structure. Next consider the group ring $k[H_2^D(M,L)]$. Elements of the group ring can be written as polynomials in the variable T, i.e. $P(T) = \sum_{A \in H_2^D} a_A T^A$ with coefficients $a_A \in k$. This ring is graded by setting $|T^A| = -\mu(A)$. An important subring of the group ring is given by

$$\widetilde{\Lambda}^+ = \Big\{ P(T) \in k[H_2^D(M,L)] \ \Big| \ P(T) = a_0 + \sum_{A, \mu(A)>0} a_A T^A \Big\}.$$

The coefficient rings we use later will be $\widetilde{\Lambda}^+$-algebras. By this we mean commutative, graded rings $\mathcal{R}$ which are also graded algebras over $\widetilde{\Lambda}^+$. This structure is typically specified by a graded morphism of rings $\widetilde{\Lambda}^+ \to \mathcal{R}$. In the sequel we will frequently use the coefficient rings $\Lambda = k[t, t^{-1}]$ and $\Lambda^+ = k[t]$. These are $\widetilde{\Lambda}^+$-algebras defined by the morphism $T^A \mapsto t^{\mu(A)/N_L}$. The grading is then $|t| = -N_L$.

1.1.2. Lagrangian quantum homology From now on we assume $L \subset (M, \omega)$ is a monotone Lagrangian submanifold with $N_L \geq 2$. The Lagrangian quantum homology of L is described in detail in [13, 14, 15], we recall the basic notions here.

The pearl complex The construction of Lagrangian quantum homology relies on a choice of data triple $\mathcal{D} = (f, (\cdot, \cdot), J)$, where $f : L \to \mathbb{R}$ is a Morse function, $(\cdot, \cdot)$ is a Riemannian metric on L and J is an ω-compatible almost complex structure on M. For a $\widetilde{\Lambda}^+$-algebra $\mathcal{R}$ the chain complex

$$C(\mathcal{D}; \mathcal{R}) = k\langle \mathrm{Crit}(f) \rangle \otimes \mathcal{R}$$

is called the pearl complex. It obtains its grading from the Morse indices of the critical points and the grading on $\mathcal{R}$. The map

$$d_p : C_*(\mathcal{D}; \mathcal{R}) \longrightarrow C_{*-1}(\mathcal{D}; \mathcal{R}),$$

defined by counting pearly trajectories, which are configurations of Morse flow lines connected by pseudoholomorphic disks, is called the pearly differential. d_p can be written as a sum of operators, where the first summand is the classical Morse differential d_M on $C_*(\mathcal{D};\mathcal{R})$.

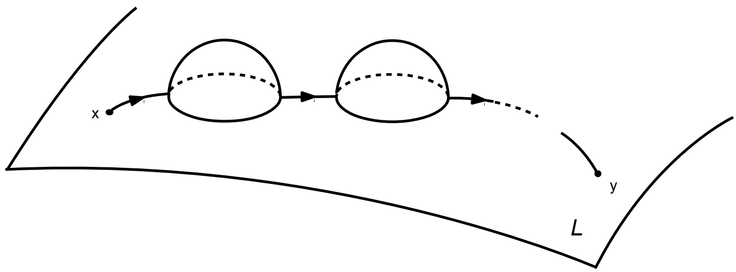

Figure 1.1.: Pearly trajectory connecting two critical points x and y

For a coefficient ring $\mathcal{R}$ and generic choice of data $\mathcal{D}$ the map d_p of the pearl complex $C(\mathcal{D};\mathcal{R})$ is well-defined and a differential. The resulting homology is called the *Lagrangian quantum homology* of L and denoted by $QH_*(L;\mathcal{R})$. We briefly recall the associated algebraic structures.

Quantum product $QH_*(L;\mathcal{R})$ has the structure of an associative (but not necessarily commutative) ring with unity,

$$QH_i(L;\mathcal{R}) \otimes_{\mathcal{R}} QH_j(L;\mathcal{R}) \longrightarrow QH_{i+j-n}(L;\mathcal{R}), \quad x \otimes y \longmapsto x \circ y.$$

Here $n = \dim(L)$. The unity lies in $QH_n(L;\mathcal{R})$ and is denoted by e_L.

Quantum module map Denote by $(QH_*(M;\mathcal{R}), *)$ the quantum homology of M with quantum product $*$. The extension of coefficient ring to $\mathcal{R}$ is induced by the composition of morphisms $\pi_2(M) \to$

$\pi_2(M, L) \to H_2^D(M, L)$. Then $QH_*(L; \mathcal{R})$ becomes a module over $QH_*(M; \mathcal{R})$ in the sense that there exists a canonical map

$$QH_i(M; \mathcal{R}) \otimes_{\mathcal{R}} QH_j(L; \mathcal{R}) \longrightarrow QH_{i+j-2n}(L; \mathcal{R}), \quad a \otimes x \longmapsto a \circledast x,$$

such that the following identity holds

$$(a * b) \circledast x = a \circledast (b \circledast x),$$

for all classes $a, b \in QH_*(M; \mathcal{R})$ and $x \in QH_*(L; \mathcal{R})$. Furthermore, the ring $QH_*(L; \mathcal{R})$ with the quantum product $\circ$ is an algebra over $QH_*(M; \mathcal{R})$, i.e. the following identities hold (up to signs)

$$a \circledast (x \circ y) = (a \circledast x) \circ y = x \circ (a \circledast y),$$

for all classes $a \in QH_*(M; \mathcal{R})$ and $x, y \in QH_*(L; \mathcal{R})$. Strictly speaking, the product in $QH_*(M; \mathcal{R})$ is not commutative (however, skew-commutative), but we can restrict ourselves to $QH_{\text{even}}(M; \mathcal{R})$ to obtain a genuine algebra. Similar identities then hold over $QH_{\text{odd}}(M; \mathcal{R})$.

Quantum inclusion There is a morphism

$$i_L : QH_*(L; \mathcal{R}) \longrightarrow QH_*(M; \mathcal{R})$$

which comes from a chain level extension of the classical map in homology $H_*(L) \to H_*(M)$ induced by $L \hookrightarrow M$. The map i_L is a $QH_*(M; \mathcal{R})$-module morphism, i.e. the following identity holds

$$i_L(a \circledast x) = a * i_L(x)$$

for all classes $a \in QH_*(M; \mathcal{R})$ and $x \in QH_*(L; \mathcal{R})$.

On chain level these operations correspond to deformations of the classical operations on the Morse chain complex. More precisely, on chain level we can see that the quantum product can be written as a sum of operators, where the first (classical) summand is given precisely by the Morse-theoretic definition of the intersection product. The quantum module structure is on chain level also a deformation of the classical module map, given by the Morse-theoretic definition of the external intersection product (intersection of cycles in M with cycles in L). The same holds for the quantum inclusion, which is a deformation of the classical inclusion map on chain level. In certain situations (e.g. when L is Λ^+-wide, see §1.1.3) these algebraic quantum structures are deformations of the corresponding structures on the homology of L, this will be made more precise later.

Coefficient rings A remark on the choice of coefficient ring $\mathcal{R}$ is in order. Note that for $\mathcal{R} = \Lambda$ and $\mathcal{R} = \Lambda^+$ the two quantum homologies $QH_*(L;\Lambda)$ and $QH_*(L;\Lambda^+)$ may be genuinely different. The ring $QH_*(L;\Lambda)$ is isomorphic to the Floer homology $HF(L,L)$ (see §1.1.2) and thus vanishes for displaceable Lagrangian submanifolds L. On the other hand $QH_*(L;\Lambda^+)$ is always non-zero. This can be seen as follows. Choose a Morse function $f : L \to \mathbb{R}$ in the data triple $\mathcal{D}$ such that f has a single maximum $x_n \in C_n(\mathcal{D};\Lambda^+)$. We know the Morse differential $d_M(x_n) = 0$ and for degree reasons no quantum terms in the pearly differential d_p show up. Thus $d_p(x_n) = 0$ and again for degree reasons within the coefficient ring Λ^+, x_n cannot be a coboundary (for the coefficient ring Λ this is however possible). Thus $QH_*(L;\Lambda^+)$ always contains the unity $[x_n] = e_L \neq 0$.

Example. The difference between these two homologies is also observable for $L \simeq S^1 \subset (\mathbb{R}^2, dx \wedge dy)$. Since L is displaceable we have $HF(L,L) = QH_*(L;\Lambda) = 0$. Now choose a generic data triple $\mathcal{D} = (f, (\cdot,\cdot), J)$ such that f has two critical points x_1, x_0 with Morse indices equal to 1 and 0. The minimal Maslov number $N_L = 2$ and there is a single (unparametrized) pseudoholomorphic disk with boundary on L.

On $C_*(\mathcal{D};\Lambda^+)$ we have

$$d_p(x_1) = 0, \qquad d_p(x_0) = x_1 t,$$

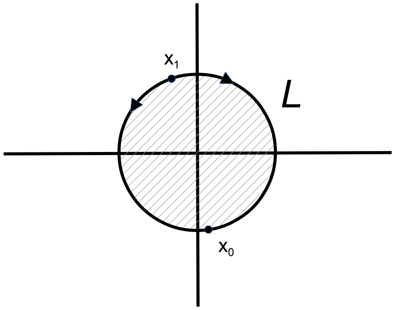

Figure 1.2.: $L \subset \mathbb{R}^2$

where the second differential comes from the disk joining x_0 to x_1. Therefore x_1 is our only cycle and we obtain

$$QH_1(L;\Lambda^+) \simeq k[x_1], \qquad QH_0(L;\Lambda^+) = 0.$$

Note also that our Λ^+-module $QH_*(L;\Lambda^+)$ contains a torsion element, since $t[x_1] = 0$.

There are other relations between the quantum homologies with these choices of coefficient rings. The inclusion of chain complexes $C_*(\mathcal{D};\Lambda^+) \hookrightarrow C_*(\mathcal{D};\Lambda)$ induces a natural map $QH_*(L;\Lambda^+) \to QH_*(L;\Lambda)$, for degree reasons we then have an isomorphism in degree 0,

$$QH_0(L;\Lambda^+) \xrightarrow{\simeq} QH_0(L;\Lambda),$$

and a surjection in degree 1

$$QH_1(L;\Lambda^+) \twoheadrightarrow QH_1(L;\Lambda).$$

Furthermore, we say that L is $\mathcal{R}$-*wide* if there exists an isomorphism $QH_*(L;\mathcal{R}) \simeq H_*(L;k) \otimes_k \mathcal{R}$ between the homologies of L. This isomorphism need not be canonical and may depend on the choice of initial data. If $QH_*(L;\mathcal{R}) = 0$, we say L is $\mathcal{R}$-*narrow*.

Relation to Floer homology It is shown in [13, 14] that the Lagrangian quantum homology of L with coefficients in Λ is isomorphic to the self-Floer homology of L,

$$QH_*(L;\Lambda) \simeq HF_*(L,L;\Lambda).$$

This isomorphism is constructed using a version of the PSS isomorphism in the Lagrangian setting. It is also a ring isomorphism and identifies the quantum product on $QH_*(L;\Lambda)$ with the Donaldson product in Floer homology.

In the monotone case $HF_*(L,L)$ is defined using a Hamiltonian isotopy and the Laurent polynomial ring Λ, which is a subring of the universal Novikov ring. Note that Floer homology cannot be defined using the polynomial ring Λ^+ or the positive Novikov ring, since there are no well-defined continuation maps of the corresponding chain complexes for these coefficient rings.

1.1.3. Deformations of algebraic structures in singular homology

Lagrangian quantum homology Here we describe when the algebraic structures of Lagrangian quantum homology can be seen as deformations of their counterparts in singular homology.

Recall that a Lagrangian L is Λ^+-wide, if there is an isomorphism $QH_*(L;\Lambda^+) \simeq H_*(L;k) \otimes_k \Lambda^+$. This isomorphism is not canonical and depends on the choice of data triple $\mathcal{D}$, $\Psi_{\mathcal{D}} : QH_*(L;\Lambda^+) \xrightarrow{\sim} H_*(L;k) \otimes_k \Lambda^+$. In [16] we have a description for changes in choices of $\mathcal{D}$.

Proposition 1.1.1 ([16]). *Let L be a monotone Lagrangian which is Λ^+-wide. Let $\mathcal{D}$, $\mathcal{D}'$ be two data triples and denote by $\Psi_{\mathcal{D}}, \Psi_{\mathcal{D}'} : QH_*(L;\Lambda^+) \to H_*(L;k) \otimes_k \Lambda^+$ their induced isomorphisms. Then we have:*

1. *Every isomorphism $\Psi_{\mathcal{D}}$ sends the unity e_L of $QH_*(L;\Lambda^+)$ to the unity $[L]$ of $H_*(L;k)$.*

2. *For $\Psi_{\mathcal{D}}$ and $\Psi_{\mathcal{D}'}$ we have*

$$\Psi_{\mathcal{D}} \circ \Psi_{\mathcal{D}'}^{-1} = \mathrm{Id} + \psi_1 t + \psi_2 t^2 + \dots,$$

where $\psi_j : H_*(L;k) \to H_{*+jN_L}(L;k)$, $j \geq 1$. In other words, $\Psi_{\mathcal{D}} \circ \Psi_{\mathcal{D}'}^{-1}$ is a deformation of the identity.

Therefore, when L is Λ^+-wide, we can transfer the quantum product on $QH_*(L;\Lambda^+)$ to $H_*(L;k) \otimes \Lambda^+$ using the map $\Psi_{\mathcal{D}}$. More precisely, consider the diagram

$$\begin{tikzcd}
(H_*(L;k) \otimes \Lambda^+) \otimes_{\Lambda^+} (H_*(L;k) \otimes \Lambda^+) \arrow[r, "\tilde{\circ}"] \arrow[d, "\Psi_{\mathcal{D}}^{-1} \otimes \Psi_{\mathcal{D}}^{-1}"'] & H_*(L;k) \otimes \Lambda^+ \\
QH_*(L;\Lambda^+) \otimes_{\Lambda^+} QH_*(L;\Lambda^+) \arrow[r, "\circ"] & QH_*(L;\Lambda^+) \arrow[u, "\Psi_{\mathcal{D}}"']
\end{tikzcd}$$

The quantum product $\tilde{\circ}$ on $H_*(L;k) \otimes \Lambda^+$ is obtained from the quantum product and is a deformation of the classical intersection product. This can be seen by the previous theorem and by looking at the quantum product on chain level, where it is a deformation of the intersection product in Morse homology. By changing the choice of data triple $\mathcal{D}$ we obtain another quantum product on $H_*(L;k) \otimes \Lambda^+$ which lies in the same deformation equivalence class as $\tilde{\circ}$. This notion will be further explained in §2.1.

Similar remarks apply to the quantum module action. Thus, when L is Λ^+-wide, we can transfer the quantum module action to $H_*(L;k) \otimes \Lambda^+$ and obtain a deformation of the classical module action of $H_*(M)$ on $H_*(L)$. This can be seen by looking at the quantum module action on chain level, where it is a deformation of the external intersection product in Morse homology. Changes in choice of data triple then result in equivalent quantum module actions, this notion will be further explained in §2.2.

From these structures on $H_*(L;k) \otimes \Lambda^+$ we will construct invariants. These invariants depend on the Lagrangian embedding and are related to invariants of pseudoholomorphic disks.

Ambient quantum homology Denote $H_*(M) = H_*(M;\mathbb{Z})/T$, where T is the torsion submodule. The quantum homology ring of M with coefficients in $\mathcal{R}$ is a ring structure on

$$QH_*(M;\mathcal{R}) := H_*(M) \otimes_{\mathbb{Z}} \mathcal{R},$$

called the ambient quantum product. This product is a deformation of the classical intersection product on the homology of M. The structure constants of the product are determined by the genus zero three-point Gromov-Witten invariants. Note that, in this case, the ambient quantum product corresponds to a single deformation of the ring $(H_*(M) \otimes \mathcal{R}, \cdot)$. This stands in contrast with the situation described previously, where the Lagrangian quantum product corresponds to an equivalence class of deformations of the ring $(H_*(L) \otimes \mathcal{R}, \cdot)$.

1.2. Pseudoholomorphic disk invariants

One possible consequence of invariants associated to the quantum structures is the construction of enumerative invariants of pseudoholomorphic disks. This has been a subject of much interest in symplectic topology, which we outline here. Subsequently we describe how the quantum structures prove useful in finding enumerative invariants for such disks.

1.2.1. Gromov-Witten invariants

Closed Gromov-Witten invariants Let (M,ω) be a compact symplectic manifold of dimension $2n$ and fix a spherical homology class $A \in H_2(M;\mathbb{Z})$ together with a nonnegative integer k. Choose a generic almost complex structure J that is compatible with ω. Take k homology classes $a_1, \ldots, a_k \in H_*(M)$ such that the degrees satisfy

$$\sum_{i=1}^{k} \deg(a_i) = 2n + 2c_1(A) + 2k - 6.$$

In favourable situations (e.g. when M is monotone), one can define the closed Gromov-Witten invariant

$$GW^M_{A,k}(a_1, \ldots, a_k) \in \mathbb{Z},$$

which is an invariant of the symplectic manifold (M, ω). Roughly speaking, this invariant counts the number of J-holomorphic spheres representing the class A and intersecting the cycles $a_1, \ldots, a_k$.

Open Gromov-Witten invariants For a Lagrangian submanifold $L \subset M$ we study the space of J-holomorphic maps $u : (D^2, \partial D^2) \to (M, L)$. In the past it was postulated by physicists that these J-holomorphic maps also lead to invariants, thus generalizing the previous closed invariants to the open case where the domain has a boundary. Attempts to generalize the approach in the closed case failed at the beginning, this was due to the following reasons.

For a class $\beta \in \pi_2(M, L)$ and integers $k, l > 0$ denote by $\mathcal{M}_{k,l}(\beta; J)$ the moduli space of J-holomorphic maps $(D^2, \partial D^2) \to (M, L)$ with k boundary marked points and l interior marked points representing the class β. The compactification $\overline{\mathcal{M}}_{k,l}(\beta; J)$ of the moduli space is given by the Gromov compactification.

The problems that arise when one attempts to proceed as in the closed case come from the differences in the moduli spaces of J-holomorphic spheres and disks. The moduli space of disks has boundary components and is in certain cases non-orientable, these occurrences are then an obstruction to defining an invariant similar to the one given for J-holomorphic spheres.

The issue concerning the orientation of the moduli space is solved by considering an additional structure on the Lagrangian L. Namely, if L is orientable and relatively spin, then the moduli space $\overline{\mathcal{M}}_{k,l}(\beta; J)$ is orientable. The second issue related to the boundary can be addressed in several ways and many authors have succeeded in defining open Gromov-Witten invariants, see for example [69, 56, 21]. In these examples invariants were defined for a special class of Lagrangians L, namely those that arise as the fixed point set of an anti-symplectic involution, i.e. for a map $c : M \to M$ such that $c^*\omega = -\omega$. One approach to the boundary problem in this case then uses the involution c to identify codimension one components of the boundary of different disk moduli spaces, such that the resulting space has codimension two (and higher) components. This provides invariants in certain cases, but relies strongly on the involution c.

1.2.2. Pseudoholomorphic disk invariants in the monotone case Consider a monotone Lagrangian $L \subset (M, \omega)$. The monotonicity of L implies the monotonicity of M. As explained before, there are no general approaches in defining invariants for pseudo-holomorphic disks. In particular, there are examples that show the dependency of the existence of a pseudholomorphic disk on the incidence conditions imposed, see [22].

Nevertheless, one can construct disk invariants in the monotone case. We start by describing two elementary disk invariants first.

Disks of Maslov-index two Assume $N_L \geq 2$ and let $\beta \in \pi_2(M, L)$ such that $\mu(\beta) = 2$. The number $d \in \mathbb{Z}$ of J-holomorphic disks representing the class β and passing through a generic point $p \in L$ is an invariant of the Lagrangian L. Indeed, monotonicity implies that the space $\mathcal{M}_{1,0}(\beta; J)$ is compact and has dimension $n+\mu(\beta)-2 = n$. One can show that the evaluation map

$$\mathrm{ev}_1 : \mathcal{M}_{1,0}(\beta; J) \to L,$$

is transverse and that the degree of the map is independent of J. The preimage of a generic point $\mathrm{ev}_1^{-1}(p)$ is then an invariant giving a count of disks through $p \in L$.

Disks with an interior and boundary marked point Assume L is a Lagrangian submanifold of dimension 2 with $N_L \geq 2$ and that $QH_*(L) \neq 0$. Choose two generic points $p \in M \backslash L$ and $q \in L$. Then the number of J-holomorphic disks with one interior marked point mapped to p and one boundary marked point mapped to q is independent of the choices of p, q and J. This can be seen by examining the moduli space involved together with the evaluation map or alternatively, by using the quantum inclusion map in Lagrangian quantum homology, see [16].

The discriminant In [16] Biran-Cornea were able to construct other disk invariants using the quantum product in Lagrangian quantum homology. We briefly describe the construction in the case $N_L = 2$. This will serve as motivation for the upcoming ideas in the next chapter.

We choose $\mathcal{R} = \Lambda^+$ and assume L is Λ^+-wide. For degree reasons we have a canonical isomorphism

$$QH_{n-1}(L;\Lambda^+) \simeq H_{n-1}(L;k) \otimes_k \Lambda^+,$$

and a canonical short exact sequence

$$0 \longrightarrow k[L]t \xrightarrow{i} QH_{n-2}(L;\Lambda^+) \xrightarrow{\pi} H_{n-2}(L;k) \longrightarrow 0.$$

For an element $x \in QH_{n-1}(L;\Lambda^+)$ we consider the quantum product $x \circ x \in QH_{n-2}(L;\Lambda^+)$. The quantum product is a deformation of the classical product and since the classical intersection $x \cdot x = 0$, we have $\pi(x \circ x) = 0$. From the short exact sequence we obtain an element $\phi(x) \in k$ such that

$$x \circ x = \phi(x)[L]t.$$

The map ϕ is obviously homogeneous of degree 2 over k and thus defines a quadratic form $\phi : H_{n-1}(L;k) \to k$. Now fix a basis in $H_{n-1}(L;\mathbb{Z})$, this is a finitely generated free abelian group (in case $H_{n-1}(L;\mathbb{Z})$ has torsion we simply mod out the torsion ideal). The quadratic form is represented by a symmetric matrix S in this basis, by taking the determinant of S we obtain an invariant called the discriminant of the quadratic form

$$\Delta_\phi = \det(S).$$

The discriminant is invariant under a change of basis of $H_{n-1}(L;\mathbb{Z})$, since any automorphism A of a finitely generated free abelian group has $\det(A) = \pm 1$ and transforms the quadratic form by $A^T SA$, therefore leaving the discriminant unchanged. With suitable adjustments one can generalize this invariant to the case $N_L > 2$.

Thus we obtain an invariant associated to the quantum product $\circ$. On chain level the product is constructed using pseudoholomorphic disks, and in [16] the authors were able to find an enumerative invariant of pseudoholomorphic disks by exploiting the discriminant in the case T^2 as described in the introduction.

2. Deformation theory of quantum structures

2.1. Deformations of products

2.1.1. Basic notation We fix our notation and conventions for the basic algebraic objects here. All rings are assumed to be commutative. Let k be a ring. Unless stated otherwise, tensor products will be taken over the ground ring k, $\otimes = \otimes_k$. For maps of tensor products of k-modules X_i, $\varphi : X_1 \otimes \ldots \otimes X_n \to X_{n+1}$, we use the notation $\varphi(x_1, \ldots, x_n) = \varphi(x_1 \otimes \ldots \otimes x_n)$. The space of k-linear maps from X to Y is denoted by $\mathrm{Hom}_k(X, Y)$.

Definition 2.1.1. A k-algebra (A, η_0) is a (left) k-module together with a map $\eta_0 : A \otimes A \to A$, $(a, b) \mapsto \eta_0(a, b) =: ab$, such that

1. $a(b + c) = ab + ac, \quad (a + b)c = ac + bc$;
2. $\lambda(ab) = (\lambda a)b = a(\lambda b)$,

for $\lambda \in k$ and $a, b, c \in A$. Furthermore, if $a(bc) = (ab)c$ for all $a, b, c \in A$, we say the algebra is *associative*. If A has an element 1_A such that $a1_A = 1_A a = a$ for all $a \in A$, we say the algebra is *unital*.

Equivalently, an associative k-algebra is a ring A (not necessarily unital or commutative) together with a ring homomorphism $f : k \to A$ such that $f(k)$ is contained in the center of A. We recall the notion of a module over a k-algebra. In our notation the module action is an explicit map μ_0.

Definition 2.1.2. Let (A, η_0) be a unital, associative k-algebra. A (left)A-*module* (N, μ_0) is a k-module N together with a k-linear map $\mu_0 : A \otimes N \to N$, $a \otimes n \mapsto \mu_0(a, n) =: an$, such that

1. $(a+b)n = an + bn, \quad a(m+n) = am + an;$
2. $a(bn) = (ab)n;$
3. $1_A n = n,$

for $a, b \in A$ and $m, n \in N$. A similar definition holds for right A-modules. An *A-bimodule* N is a left and right A-module such that both module actions are compatible, i.e. $(an)b = a(nb)$ for $a, b \in A$ and $n \in N$.

2.1.2. Deformations of associative algebras The classical deformation theory of associative algebras was developed by Gerstenhaber in [33, 34, 35]. An essential tool in the classification of such deformations is Hochschild cohomology. We offer an overview of the relevant results here, see [50] for a detailed introduction to the subject.

Definition 2.1.3. Let R be a ring and $\iota : k \hookrightarrow R$ a subring. An *augmentation* of R is a homomorphism $\epsilon : R \to k$ such that $\epsilon \circ \iota = \mathrm{Id}_k$, i.e. the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\epsilon"] & k \\ k \arrow[u, "\iota"] \arrow[ur, "\mathrm{Id}_k"'] & \end{tikzcd}$$

We call the ring R an *augmented ring* over k and use the notation (R, ϵ).

The ideal $I := \mathrm{Ker}(\epsilon)$ is called the *augmentation ideal* and we have $R/I \simeq k$. If the ideal I is maximal, the ring k is a field and vice versa.

Examples. 1. Consider the polynomial ring $R = k[t]$. For a given $\lambda \in k$ define the map

$$\epsilon_\lambda : k[t] \longrightarrow k, \qquad p(t) \longmapsto p(\lambda).$$

This is a ring homomorphism and defines an augmentation via the usual inclusion $k \hookrightarrow k[t]$. The augmentation ideal is $\mathrm{Ker}(\epsilon_\lambda) = (t - \lambda)$ and every element in k corresponds to an augmentation.

2. The ring of dual numbers $R = k[t]/(t^2)$ is an augmented ring by setting

$$\epsilon : k[t]/(t^2) \longrightarrow k, \qquad x + yt \longmapsto x.$$

If k is a field, then R is a local Artin ring with unique maximal ideal (t).

3. Consider the ring of formal power series $R = k[[t]]$. Define an augmentation by

$$\epsilon : k[[t]] \longrightarrow k, \qquad \sum_{i \geq 0} \lambda_i t^i \longmapsto \lambda_0.$$

If k is Noetherian, then so is $k[[t]]$. If k is a field, then R is a local Noetherian ring with the maximal ideal (t).

We now proceed to give the general definition of deformations of associative algebras over a chosen augmented ring (R, ϵ). The augmentation ϵ is used to distinguish the additional terms of the deformed algebra from the classical part.

Definition 2.1.4. Let (A, η_0) be a unital associative algebra over k and let (R, ϵ) be an augmented ring over k. An R-*deformation* of A is a unital associative R-algebra $(A \otimes R, \eta)$ such that the map

$$\mathrm{Id}_A \otimes \epsilon : A \otimes R \longrightarrow A$$

is an algebra homomorphism. The *trivial R-deformation* of A is the tensor product $A \otimes R$ with multiplication

$$(a \otimes r)(b \otimes s) = ab \otimes rs$$

extended R-linearly to $R \otimes_k A$. We denote the trivial deformation by $(A \otimes R, \eta_0)$. The space of deformations is denoted by $\widetilde{\mathrm{Def}}_R(A)$ or simply $\widetilde{\mathrm{Def}}(A)$.

If $R = k[t]$ (a similar statement holds for the ring of dual numbers $k[t]/(t^2)$ or ring of formal power series $k[[t]]$), we can rewrite the definition as follows: a deformation of (A, η_0) is a unital associative $k[t]$-algebra $(A \otimes k[t], \eta)$ with product

$$\eta = \eta_0 + \eta_1 t + \eta_2 t^2 + \dots, \qquad \eta_i \in \mathrm{Hom}_k(A \otimes_k A, A),$$

which satisfies $\eta(1_A, a) = \eta(a, 1_A) = a$ for $a \in A$. Note that the associativity of $(A \otimes k[t], \eta)$ implies that the maps η_i must satisfy

$$\sum_{i=0}^{n} \eta_i(\eta_{n-i}(a,b),c) = \sum_{i=0}^{n} \eta_i(a, \eta_{n-i}(b,c)) \tag{2.1}$$

for $n \in \mathbb{N}$ and $a, b, c \in A$.

Remark. Several authors (see [33, 50]) construct deformations using rings R which are local, complete and Noetherian with residue field k. Examples of such rings include $k[t]/(t^2)$ and $k[[t]]$ whenever k is a field. One problem that shows up when using general rings R, e.g. $k[t]$, is that the homomorphism

$$\mathrm{Id}_A + \phi_1 t + \phi_2 t^2 + \dots, \quad \phi_i \in \mathrm{Hom}_k(A, A),$$

might not be invertible. This is necessary in order to apply cohomological tools to deformation theory. In our theory we will define deformations of algebraic structures over the ring $R = k[t]$, which is possible under suitable adjustments. In particular, the class of algebraic objects we consider later on are graded finitely generated algebras and modules, which simplifies the theory in certain aspects.

For $R = k[t]/(t^2)$, $k[[t]]$ (this also extends to the case where R is a local complete Noetherian ring with residue field k) we have the following notion of equivalence of R-deformations.

Definition 2.1.5. Let $(A \otimes R, \eta)$ and $(A \otimes R, \eta')$ be two R-deformations of A. These deformations are *equivalent*, if there exists an algebra homomorphism

$$\Phi : (A \otimes R, \eta) \longrightarrow (A \otimes R, \eta'),$$

such that Φ restricted to the inclusion $A \hookrightarrow A \otimes R$ is the identity, i.e. $\Phi(a \otimes 1) = a \otimes 1$ for $a \in A$. The set of equivalence classes of R-deformations of A will be denoted by $\mathrm{Def}_R(A)$ or simply $\mathrm{Def}(A)$.

For $R = k[t]/(t^2)$ or $k[[t]]$ we can write the homomorphism Φ as

$$\Phi = \mathrm{Id}_A + \phi_1 t + \phi_2 t^2 + \ldots, \qquad \phi_i \in \mathrm{Hom}_k(A, A).$$

Note that in this case Φ is always invertible, hence an isomorphism. In the case $R = k[t]$ we first define the monoid

$$\widetilde{\mathrm{Iso}}(A) = \{\Phi = \mathrm{Id}_A + \phi_1 t + \phi_2 t^2 + \ldots + \phi_n t^n \bigm| \phi_i \in \mathrm{Hom}_k(A, A)\},$$

and then restrict ourselves to the group of invertible elements,

$$\mathrm{Iso}(A) = \{\Phi \in \widetilde{\mathrm{Iso}}(A) \bigm| \exists\, \Phi^{-1} \in \widetilde{\mathrm{Iso}}(A)\}.$$

We then define the set of equivalence classes of $k[t]$-deformations to be $\mathrm{Def}(A) := \widetilde{\mathrm{Def}}(A)/\mathrm{Iso}(A)$.

2.1.3. Hochschild cohomology and deformation theory

We describe how Hochschild cohomology is used in characterizing deformations of associative algebras.

Definition 2.1.6. The *Hochschild cohomology* of a k-algebra (A, η_0) with coefficients in a bimodule N is the cohomology of the Hochschild complex,

$$0 \longrightarrow N \xrightarrow{d^H} C^1(A, N) \xrightarrow{d^H} C^2(A, N) \xrightarrow{d^H} \ldots \xrightarrow{d^H} C^n(A, N) \xrightarrow{d^H} \ldots$$

The cochain groups are given by

$$C^n(A, N) := \mathrm{Hom}_k(A^{\otimes n}, N),$$

with Hochschild coboundary operator $d^H : C^n(A,N) \to C^{n+1}(A,N)$

$$\begin{aligned} d^H\varphi(a_1,\dots,a_{n+1}) &= a_1\varphi(a_2,\dots,a_{n+1}) \\ &\quad + \sum_{i=1}^{n}(-1)^i\varphi(a_1,\dots,a_ia_{i+1},\dots,a_{n+1}) \\ &\quad + (-1)^{n+1}\varphi(a_1,\dots,a_n)a_{n+1}. \end{aligned}$$

We denote the resulting Hochschild cohomology group of degree n by

$$HH^n(A,N) := H^n(C^*(A,N),d^H).$$

Calculations of the Hochschild cohomology groups for $n = 0,1$ yield

$$HH^0(A,N) = \{m \in N \mid am = ma,\ \forall\, a \in A\}$$

and

$$HH^1(A,N) = \mathrm{Der}(A,N)/\{\text{ inner derivations }\},$$

where $\mathrm{Der}(A,N)$ is given by maps $f : A \to N$ satisfying $f(ab) = af(b) + f(a)b$ and the inner derivations are maps $a \mapsto am - ma$ for an $m \in N$.

In what follows we will usually take the bimodule N to be A. The resulting Hochschild cohomology is then related to the associative deformations of A over $R = k[t], k[t]/(t^2)$, or $k[[t]]$. One can see this as follows. Writing out the definition for Hochschild 1- and 2-cochains we have for $\phi \in C^1(A,A)$ and $\varphi \in C^2(A,A)$

$$\begin{aligned} d^H\phi(a,b) &= a\phi(b) - \phi(ab) + \phi(a)b, \\ d^H\varphi(a,b,c) &= a\varphi(b,c) - \varphi(ab,c) + \varphi(a,bc) - \varphi(a,b)c. \end{aligned}$$

Using equation (2.1), we see that the first non-classical term η_1 in the product η is a cocycle, i.e. $d^H\eta_1 = 0$. Furthermore, for equivalent deformations we have $\eta_1 - \eta_1' = d^H\phi_1$. Thus to every equivalence class of deformations we can associate a cohomology class and this gives us a well-defined nontrivial map

$$\Xi : \mathrm{Def}_R(A) \longrightarrow HH^2(A,A), \qquad [(R \otimes_k A, \eta)] \longmapsto [\eta_1].$$

We quote the classical results, see [33, 50].

Theorem 2.1.1. *Let $R = k[t]/(t^2)$. Then the map Ξ is an isomorphism.*

Theorem 2.1.2. *Let k be a field and $R = k[[t]]$. Assume A is an associative k-algebra with $HH^2(A, A) = 0$. Then all R-deformations of A are equivalent to the trivial deformation.*

2.1.4. Graded Hochschild cohomology We are mostly interested in graded algebras, therefore it will be necessary to describe the deformation theory and Hochschild cohomology in the graded case. Due to our motivation coming from symplectic topology, we restrict ourselves to deformations over the polynomial ring $R = k[t]$. We use cohomological grading in this section to simplify notation, in contrast to the homological grading used in the symplectic setting.

Let $(A = \bigoplus_{i\geq 0} A^i, \eta_0)$ be a graded k-algebra, i.e. $\eta_0(A^i, A^j) \subset A^{i+j}$. We grade the ring $k[t]$ by setting $|t| = d \in \mathbb{Z}$. A map $\varphi \in \mathrm{Hom}_k(A^{\otimes n}, A)$ has degree $l \in \mathbb{Z}$, if for n homogeneous elements $a_1, \ldots, a_n$ we have

$$|\varphi(a_1, \ldots, a_n)| = l + \sum_{i=1}^{n} |a_i|.$$

A graded $k[t]$-deformation of A is then a deformation as in Definition 2.1.4, where the maps $\eta_j \in \mathrm{Hom}_k(A \otimes_k A, A)$ of $\eta = \eta_0 + \eta_1 t + \ldots$ now have degree $|\eta_j| = -jd$. An equivalence of graded deformations is then given by an algebra isomorphism $\Phi \in \mathrm{Iso}(A)$ as above, where the maps $\phi_j \in \mathrm{Hom}_k(A, A)$ of $\Phi = \mathrm{Id}_A + \phi_1 t + \ldots$ have degree $|\phi_j| = -jd$. Note that in the graded case, for finitely generated k-algebras $A = \bigoplus_{i=0}^{n} A^i$ and $d \neq 0$, deformations of the identity are always invertible, i.e. $\widetilde{\mathrm{Iso}}(A) = \mathrm{Iso}(A)$. The set of equivalence classes of graded $k[t]$-deformations will be denoted in this case by $\mathrm{Def}_d(A)$.

The graded Hochschild cochain complex is given by

$$C^{n,l}(A, A) := \{\, \varphi \in \mathrm{Hom}_k(A^{\otimes n}, A) \;\big|\; |\varphi| = l \,\}.$$

The coboundary operator d^H maps $C^{n,l}(A,A)$ to $C^{n+1,l}(A,A)$ and the graded Hochschild cohomology groups are denoted by

$$HH^{n,l}(A,A).$$

2.1.5. The Lagrangian quantum product Let L be a monotone Lagrangian that is Λ^+-wide with minimal Maslov number N_L. By the discussion in section 1.1.3 we can associate a graded Hochschild cohomology class to $(QH_*(L;\Lambda^+),\circ)$.

For a generic choice of data triple $\mathcal{D}$ there is an isomorphism $\Psi_D : QH_*(L;\Lambda^+) \to H_*(L;k)\otimes\Lambda^+$. Denote $A := H_*(L;k)$ for the homology ring of L, we grade A by setting $A^i = H_{n-i}(L;k)$. The unit has degree 0. In the cohomological grading we have $|t| = N_L$. By transferring the quantum product to $A\otimes\Lambda^+$ we obtain a deformation of the intersection product on A. If we write

$$\circ = \rho_0 + \rho_1 t + \rho_2 t^2 + \dots,$$

where ρ_0 is the intersection product, then the map ρ_1 lies in $C^{2,-N_L}(A,A)$. Changes in choice of generic data triple $\mathcal{D}$ give rise to equivalent deformations of A and thus we have a well-defined class in $\mathrm{Def}_{N_L}(A)$. We obtain a purely algebraic invariant of L, the cohomology class associated to the quantum product

$$\mathrm{Def}_{N_L}(A) \longrightarrow HH^{2,-N_L}(A,A), \quad (QH_*(L;\Lambda^+),\circ) \longmapsto [\rho_1].$$

This provides us with an invariant of L, possibly distinguishing two Lagrangian embeddings of the same manifold. If two Lagrangian embeddings can be mapped symplectomorphically to each other, the Lagrangian quantum product is preserved and the associated cohomology classes must agree.

The space $\mathrm{Def}_{N_L}(A)$ of equivalence classes of deformations can be enlarged as follows. In $\widetilde{\mathrm{Def}}_{N_L}(A)$, instead of taking the equivalence relation given by $\mathrm{Iso}(A)$, we consider $\widetilde{\mathrm{Def}}_{N_L}(A)/\mathrm{Iso}_g(A)$, where

$$\mathrm{Iso}_g(A) := \{\Phi \in \mathrm{Iso}(A) \mid \Phi = \Psi_{\mathcal{D}} \circ \Psi_{\mathcal{D}}^{-1} \text{ for a data triple } \mathcal{D}\}.$$

The subscript g stands for the subgroup of geometric isomorphisms. This gives us a larger group of equivalence classes of Lagrangian embeddings $L \hookrightarrow M$, since the subgroup $\mathrm{Iso}_g(A)$ depends strongly on the embedding.

2.2. Deformations of module structure

2.2.1. Coupled deformations The following objects motivate our development for a deformation theory of module structures. Let k be our ground ring and denote by $H_*(M;k) =: A$ the algebra over k and by $H_*(L;k) =: N$ the module over k (we omit the algebra structure of N for the moment). N is also a module over A with the module action given by the external intersection product. From the quantum module action and ambient quantum product (see section 1.1.2) we obtain a deformation of both structures that are compatible with each other. We study the abstract algebraic situation.

From now on we fix the ring $R = k[t]$. Let (A, η_0) be a unital, associative k-algebra and let (N, μ_0) be an A-module. We consider first deformations of the module structure μ_0.

Definition 2.2.1. An *R-deformation* $(N \otimes R, \mu)$ of the A-module (N, μ_0) is given by an R-linear map

$$\mu : (A \otimes R) \otimes_R (N \otimes R) \longrightarrow N \otimes R, \qquad a \otimes m \longmapsto \mu(a, m),$$

that satisfies

1. $(N \otimes R, \mu)$ is a module over $(A \otimes R, \eta_0)$;
2. $\mu(1_A, m) = m$ for all $m \in N$;
3. μ reduces to μ_0 for $t = 0$.

This concept was previously studied by Yau [70], where he constructed a deformation theory similar to the one in the case of associative algebras.

We now introduce the concept of a coupled deformation, which can be seen as simultaneous deformations of the algebra and module structure which are compatible with each other.

Definition 2.2.2. Let (A, η_0) be a unital associative k-algebra and (N, μ_0) an A-module. A *coupled deformation* (N, μ, A, η) is given by an R-deformation $(A \otimes R, \eta)$ of A and an R-linear map

$$\mu : (A \otimes R) \otimes_R (N \otimes R) \longrightarrow N \otimes R, \qquad a \otimes m \longmapsto \mu(a, m),$$

that satisfies

1. $(N \otimes R, \mu)$ is a module over $(A \otimes R, \eta)$;
2. $\mu(1_A, m) = m$ for all $m \in N$;
3. μ reduces to μ_0 for $t = 0$.

We denote by $\widetilde{\mathrm{Def}}(N, A)$ the set of coupled deformations of N and A.

This definition also extends to the case where $R = k[t]/(t^2)$ or $k[[t]]$. We call the coupled deformation (N, μ, A, η) given by the R-linear extension of $\mu = \mu_0$ and $\eta = \eta_0$ the *trivial* coupled deformation.

Using the definition we can rewrite the product on A as

$$\eta = \eta_0 + \eta_1 t + \eta_2 t^2 + \ldots + \eta_l t^l, \qquad \eta_i \in \mathrm{Hom}_k(A \otimes A, A),$$

and the module action on N as

$$\mu = \mu_0 + \mu_1 t + \mu_2 t^2 + \ldots + \mu_n t^n, \qquad \mu_j \in \mathrm{Hom}_k(A \otimes N, N).$$

From condition (1) of Definition 2.2.2 we obtain the equation

$$\mu(\eta(a, b), m) = \mu(a, \mu(b, m)),$$

for $a, b \in A$, $m \in N$ and computing both sides we have

$$\begin{aligned}\mu(\eta(a,b),m) = (ab)m + \eta_1(a,b)mt + \eta_2(a,b)mt^2 &\quad + \dots \\ + \mu_1(ab,m)t + \mu_1(\eta_1(a,b),m)t^2 &+ \dots \\ + \mu_2(ab,m)t^2 &\quad + \dots \\ &\quad\ddots\end{aligned}$$

and

$$\begin{aligned}\mu(a,\mu(b,m)) = a(bm) + a\mu_1(b,m)t + a\mu_2(b,m)t^2 &\quad + \dots \\ + \mu_1(a,bm)t + \mu_1(a,\mu_1(b,m))t^2 &+ \dots \\ + \mu_2(a,bm)t^2 &\quad + \dots \\ &\quad\ddots\end{aligned}$$

Comparing the terms of order n we find the following conditions for η_i and μ_j:

$$\begin{aligned} n = 0: &\quad (ab)m = a(bm) &(2.2)\\ n = 1: &\quad \eta_1(a,b)m + \mu_1(ab,m) = a\mu_1(b,m) + \mu_1(a,bm) &(2.3)\\ &\vdots \\ n \quad : &\quad \sum_{i=0}^{n} \mu_i(\eta_{n-i}(a,b),m) = \sum_{i=0}^{n} \mu_i(a,\mu_{n-i}(b,m)) &(2.4)\end{aligned}$$

Equation (2.2) is fulfilled by the definition of the module action of (A, η_0) on (N, μ_0). Equation (2.3) will be used in the upcoming definition of a cochain complex, similar to the equation relating deformations of associative algebras to Hochschild cocycles.

2.2.2. Equivalence of coupled deformations We introduce an appropriate notion of equivalence for coupled deformations. Similar to the discussion in §2.1.3, we must pay attention to the case $R =$

$k[t]$. We first define the monoid of deformations of the identity map of a module,

$$\widetilde{\mathrm{Iso}}(N) := \{\Lambda = \mathrm{Id}_N + \lambda_1 t + \ldots + \lambda_n t^n \mid \lambda_i \in \mathrm{Hom}_k(N, N)\}.$$

Then the group of isomorphisms of the R-module $N \otimes R$ is the group of invertible elements of $\widetilde{\mathrm{Iso}}(N)$,

$$\mathrm{Iso}(N) := \{\Lambda \in \widetilde{\mathrm{Iso}}(N) \mid \exists\, \Lambda^{-1} \in \widetilde{\mathrm{Iso}}(N)\}.$$

Definition 2.2.3. Two coupled deformations (N, μ, A, η) and (N, μ', A, η') are *equivalent*, if there exists a $\Phi \in \mathrm{Iso}(A)$ and $\Lambda \in$ Iso(N) such that

$$\begin{aligned} \eta(a, b) &= \Phi^{-1}(\eta'(\Phi(a), \Phi(b))), \\ \mu(a, m) &= \Lambda^{-1}\left(\mu'(\Phi(a), \Lambda(m))\right), \end{aligned}$$

for $a, b \in A$ and $m \in N$.

We say the coupled deformation (N, μ, A, η) is *trivial* when it is equivalent to the trivial deformation (N, μ_0, A, η_0). In this case there exists a $\Lambda \in$ Iso(N) and $\Phi \in \mathrm{Iso}(A)$ such that

$$am = \Lambda^{-1}(\mu'(\Phi(a), \Lambda(m))).$$

The set of equivalence classes of coupled deformations is denoted by

$$\mathrm{Def}(N, A) = \widetilde{\mathrm{Def}}(N, A)/(\mathrm{Iso}(N) \times \mathrm{Iso}(A)).$$

2.2.3. Coupled deformation complex We now define a cochain complex associated to a coupled deformation. First we set

$$D^n(A, N) := \mathrm{Hom}_k(\underbrace{A \otimes \ldots \otimes A}_{n \text{ copies}} \otimes N, N),$$

the space of n-cochains. We define the coboundary operator

$$d^D : D^n(A, N) \longrightarrow D^{n+1}(A, N),$$

by setting

$$
\begin{aligned}
&(d^D\varphi)(a_1,\dots,a_{n+1},m)\\
:=\quad & a_1\varphi(a_2,\dots,a_{n+1},m)\\
&+\sum_{i=1}^{n}(-1)^i\varphi(a_1,\dots,a_ia_{i+1},\dots,a_{n+1},m)\\
&+(-1)^{n+1}\varphi(a_1,\dots,a_n,a_{n+1}m).
\end{aligned}
$$

Proposition 2.2.1. *d^D is a coboundary operator, i.e. $(d^D)^2=0$.*

For the sake of the exposition we defer the proof to the appendix. For $\psi\in D^0(A,N)=\mathrm{Hom}_k(N,N)$ we see that the image of the coboundary operator

$$(d^D\psi)(a,m)=a\psi(m)-\psi(am)$$

is the obstruction to ψ being a module morphism. For $\varphi\in D^1(A,N)$ $=\mathrm{Hom}_k(A\otimes N,N)$ we have

$$(d^D\varphi)(a,b,m)=a\varphi(b,m)-\varphi(ab,m)+\varphi(a,bm).$$

The cochain complex $(D^*(A,N),d^D)$ has an even richer structure, it is notably a differential graded algebra. We define the product

$$
\begin{aligned}
\bullet: D^k(A,N)\times D^l(A,N) &\longrightarrow D^{k+l}(A,N),\\
(\varphi,\tau) &\longmapsto \varphi\bullet\tau,
\end{aligned}
$$

by setting

$$(\varphi\bullet\tau)(a_1,\dots,a_{k+l},m):=\varphi(a_1,\dots,a_k,\tau(a_{k+1},\dots,a_{k+l},m)).$$

This product is non-commutative and associative with the identity

$$\mathrm{Id}_N\in D^0(A,N).$$

Proposition 2.2.2. *The cochain complex $(D^*(A,N),d^D)$ with the product $\bullet$ is a differential graded algebra, i.e.*

$$d^D(\varphi\bullet\tau)=(d^D\varphi)\bullet\tau+(-1)^{|\varphi|}\varphi\bullet(d^D\tau).$$

We defer the proof to the appendix. In degree 0 this product corresponds to the usual composition of endomorphisms in $D^0(A,N) = \mathrm{End}_k(N,N)$. The cohomology therefore equals the endomorphism ring,

$$H^0(D^*(A,N), d^D) \simeq \mathrm{End}_A(N,N).$$

There is a non-trivial map from the Hochschild complex of A to the complex $(D^*(A,N), d^D)$. We define

$$\iota : C^*(A,A) \longrightarrow D^*(A,N)$$

by setting

$$\iota\varphi(a_1,\ldots,a_n,m) := \varphi(a_1,\ldots,a_n)m$$

for $\varphi \in C^n(A,A)$. This map is an embedding of complexes in the case when N is torsion-free.

Proposition 2.2.3. *The map $\iota : C^*(A,A) \longrightarrow D^*(A,N)$ is a cochain map.*

Proof. We have

$$\begin{aligned}
&d^D(\iota\varphi)(a_1,\ldots,a_{n+1},m)\\
= \quad & a_1(\iota\varphi)(a_2,\ldots,a_{n+1},m)\\
& + \sum_{i=1}^{n}(-1)^i(\iota\varphi)(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1},m)\\
& + (-1)^{n+1}(\iota\varphi)(a_1,\ldots,a_n,a_{n+1}m).\\
= \quad & a_1\varphi(a_2,\ldots,a_{n+1})m\\
& + \sum_{i=1}^{n}(-1)^i\varphi(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})m\\
& + (-1)^{n+1}\varphi(a_1,\ldots,a_n)a_{n+1}m\\
= \quad & \iota(d^H\varphi)(a_1,\ldots,a_{n+1},m)
\end{aligned}$$

□

2.2.4. Cohomology of coupled deformations Recalling equation (2.3) on page 27, we can rewrite the equation as

$$\eta_1(a,b)m = a\mu_1(b,m) - \mu_1(ab,m) + \mu_1(a,bm).$$

In terms of the cochain map ι this is equivalent to

$$\iota\eta_1(a,b,m) = d^D\mu_1(a,b,m) \tag{2.5}$$

for $a, b \in A$, $m \in N$. Using this as motivation, we define the cochain complex associated to a coupled deformation to be the mapping cone of the map ι. We have

$$\mathrm{Cone}(\iota)^n = C^{n+1}(A,A) \oplus D^n(A,N),$$

(a direct sum of k-modules) with the coboundary operator

$$d^E(\eta,\mu) = (-d^H\eta, -\iota\eta + d^D\mu)$$

for $(\eta,\mu) \in C^{n+1}(A,A) \oplus D^n(A,N)$.

Definition 2.2.4. We denote the cohomology of the complex $(\mathrm{Cone}(\iota)^*, d^E)$ by $HE^*(A,N)$. We call $HE^*(A,N)$ the *coupled deformation cohomology.*

There is a short exact sequence of cochain complexes given by

$$0 \longrightarrow D^n(A,N) \longrightarrow \mathrm{Cone}(\iota)^n \longrightarrow C^{n+1}(A,A) \longrightarrow 0,$$

where the first map sends μ to $(0,\mu)$ and the second map sends (η,μ) to $-\eta$. Note that the coboundary operator on $C^*(A,A)$ is now $-d^H$. We obtain the cohomology long exact sequence

$$\begin{aligned} \ldots &\to HC^n(A,A) \to HD^n(A,N) \to HE^n(A,N) \\ &\to HC^{n+1}(A,A) \xrightarrow{\delta} HD^{n+1}(A,N) \to HE^{n+1}(A,N) \to \ldots, \end{aligned}$$

where the connectant δ equals ι_*. The main result of this section is given by the following characterization of coupled deformations in terms of cohomology.

Theorem 2.2.4. *Let (N, μ, A, η) be a coupled deformation given by*

$$\begin{aligned} \eta(a,b) &= ab + \eta_1(a,b)t + \eta_2(a,b)t^2 + \dots, \\ \mu(a,m) &= am + \mu_1(a,m)t + \mu_2(a,m)t^2 + \dots \end{aligned}$$

for $a, b \in A$, $m \in N$. Then (η_1, μ_1) is a 1-cocycle in the complex $(\mathrm{Cone}(\iota)^*, d^E)$. Furthermore, if (N, μ', A, η') is another coupled deformation equivalent to (N, μ, A, η), then $(\eta_1, \mu_1) - (\eta_1', \mu_1')$ is a 1-coboundary.

The theorem associates to every equivalence class of a coupled deformation a non-trivial cohomology class in the coupled deformation cohomology by setting

$$\begin{aligned} \mathrm{Def}(N, A) &\longrightarrow HE^1(A, N), \\ [(N, \mu, A, \eta)] &\longmapsto [(\eta_1, \mu_1)] \in HE^1(A, N). \end{aligned}$$

Proof of Theorem 2.2.4. The chain (η_1, μ_1) is a cocycle due to equation (2.5) and the fact that η_1 is a Hochschild cocycle, i.e. $d^H\eta_1 = 0$. Now assume that two coupled deformations (N, μ, A, η) and (N, μ', A, η') are equivalent through the isomorphisms $(\Lambda, \Phi) \in \mathrm{Iso}(N) \times \mathrm{Iso}(A)$. For $a \in A$ and $m \in N$ we have

$$\begin{aligned} \mu'\left(\Phi(a), \Lambda(m)\right) &= am + a\lambda_1(m)t + \phi_1(a)mt + \mu_1'(a,m)t \\ &\quad +\phi_1(a)\lambda_1(m)t^2 + \mu_1'(a, \lambda_1(m))t^2 \\ &\quad +\mu_1'(\phi_1(a), m)t^2 + \dots. \end{aligned}$$

Using $\Lambda^{-1} \circ \Lambda(m) = m$, we see that for $\Lambda^{-1} = \mathrm{Id}_N + \lambda_1^{(-1)}t + \lambda_2^{(-1)}t^2 + \dots$ we have

$$\lambda_1^{(-1)}(m) = -\lambda_1(m).$$

Therefore,

$$\begin{aligned} \Lambda^{-1}\left(\mu'\left(\Phi(a), \Lambda(m)\right)\right) &= am + a\lambda_1(m)t + \phi_1(a)mt + \mu_1'(a,m)t \\ &\quad -\lambda_1(am)t + \mu_1'(a, \lambda_1(m))t^2 \\ &\quad -\lambda_1(\phi_1(a)m)t^2 + \dots \\ &= \mu(a,m). \end{aligned}$$

Comparing the terms of order n we obtain:

$$\begin{array}{ll} n=0: & am = am \\ n=1: & \mu_1(a,m) = a\lambda_1(m) + \phi_1(a)m + \mu_1'(a,m) - \lambda_1(am) \\ \vdots & \\ n \quad : & \mu_n(a,m) = \displaystyle\sum_{i+j+k+l=n} \lambda_i^{(-1)} \left(\mu_j' \left(\phi_k(a), \lambda_l(m)\right)\right) \end{array}$$

Thus for equivalent coupled deformations we have

$$\mu_1(a,m) - \mu_1'(a,m) = \iota\phi_1(a,m) + d^D\lambda_1(a,m),$$

and since Φ is an equivalence of deformations of A we also have

$$\eta_1(a,b) - \eta_1'(a,b) = d^H\phi_1(a,b).$$

In the mapping cone $(\mathrm{Cone}(\iota)^*, d^E)$ we see that

$$\begin{aligned} (\eta_1, \mu_1) - (\eta_1', \mu_1') &= (d^H\phi_1, \iota\phi_1 + d^D\lambda_1) \\ &= d^E(-\phi_1, \lambda_1), \end{aligned}$$

for $(-\phi_1, \lambda_1) \in \mathrm{Cone}(\iota)^0$. Thus the difference of both classes is given by a coboundary. □

2.2.5. Graded cohomology of coupled deformations We consider the case of graded algebras and modules as well. Again we use cohomological grading in this section for clarity. Let $(A = \bigoplus_{i\geq 0} A^i, \eta_0)$ be a graded k-algebra and $(N = \bigoplus_{i\geq 0} N^i, \mu_0)$ a graded (left) A-module, i.e. $\mu_0(A^i, N^j) \subset N^{i+j}$. We grade the ring $R = k[t]$ by setting $|t| = d \in \mathbb{Z}$. A map $\varphi \in \mathrm{Hom}_k(A^{\otimes n} \otimes N, N)$ has degree $l \in \mathbb{Z}$, if for $n+1$ homogeneous elements $a_1, \ldots, a_n \in A$ and $m \in N$ we have

$$|\varphi(a_1, \ldots, a_n, m)| = l + \sum_{i=1}^{n} |a_i| + |m|.$$

A graded coupled deformation (N, μ, A, η) is then a deformation in the sense of Definition 2.2.2, where the maps η_j of $\eta = \eta_0 + \eta_1 t + \dots$ have degree $|\eta_j| = -jd$ and the maps μ_j of $\mu = \mu_0 + \mu_1 t + \dots$ have degree $|\mu_j| = -jd$. The set of equivalence classes of graded couple deformations will be denoted in this case by $\mathrm{Def}_d(N, A)$.

The graded coupled deformation complex is given by

$$D^{n,l}(A, N) := \{ \varphi \in \mathrm{Hom}_k(A^{\otimes n} \otimes N, N) \bigm| |\varphi| = l \}.$$

The coboundary operator d^D maps $D^{n,l}(A, A)$ to $D^{n+1,l}(A, A)$ and the cochain map $\iota : C^{n,l}(A, A) \to D^{n,l}(A, A)$ also preserves grading, thus the same holds for the coboundary operator d^E on the graded mapping cone $\mathrm{Cone}(\iota)^{n,l}$. The graded coupled deformation cohomology groups are denoted by

$$HE^{n,l}(A, N).$$

2.2.6. The quantum module action in Lagrangian quantum homology Let $L \subset (M, \omega)$ be a monotone Lagrangian that is Λ^+-wide with minimal Maslov number N_L. Similar to the discussion in 2.1.5 we can associate a graded cohomology class to the module $(QH_*(L; \Lambda^+), \circledast)$ over $(QH_*(M; \Lambda^+), *)$.

Denote by $N := H_*(L; k)$ and $A = H_*(M; k)$ the respective homologies of L and M with grading given by $N^i := H_{n-i}(L; k)$ and $A^j = H_{2n-j}(M; k)$. We grade the ring $k[t]$ by setting $|t| = N_L$. For a generic choice of data triple $\mathcal{D}$ there is an isomorphism $\psi_{\mathcal{D}} : QH_*(L; \Lambda^+) \to N \otimes \Lambda^+$ and by transferring the quantum module action to $N \otimes \Lambda^+$ we obtain a deformation of the classical module action μ_0 of A on N,

$$a \circledast m = \mu(a, m) = \mu_0(a, m) + \mu_1(a, m)t + \dots.$$

Changes in choice of generic data triple $\mathcal{D}$ give rise to equivalent deformations of the module action. Note that the deformation of A, $(QH_*(M; \Lambda^+), *)$, remains fixed in our case. Thus we obtain a purely algebraic invariant of $L \subset M$, the cohomology class associated to the coupled deformation via the map

$$\mathrm{Def}_{N_L}(N, A) \longrightarrow HE^{1,-N_L}(A, N),$$

defined by

$$(QH_*(L;\Lambda^+), \circledast, QH_*(M;\Lambda^+), *) \longmapsto [(\eta_1, \mu_1)].$$

This proves Theorem A of the introduction.

We can enlarge $\mathrm{Def}_{N_L}(N, A)$ by considering a new equivalence relation in our space of coupled deformations $\widetilde{\mathrm{Def}}_{N_L}(N, A)$. In our case the deformation of A remains fixed and by further restricting to the subgroup of geometric isomorphisms $\mathrm{Iso}_g(N)$ of N, the space $\widetilde{\mathrm{Def}}_{N_L}(N, A)/(\{\mathrm{Id}_A\} \times \mathrm{Iso}_g(N))$ gives us a larger space of equivalence classes for Lagrangian embeddings.

We finish this section with an example for the computations involved.

Example. Let $L \simeq T^2$ be a monotone Lagrangian torus in the complex projective space $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$ and assume $N_L = 2$. We compute the cohomology group $HE^{1,-2}(N, A)$ associated to a degree 2 coupled deformation of $H_*(\mathbb{C}P^2; k)$ and $H_*(T^2; k)$.

We use cohomological grading and set $N^i := H_{2-i}(T^2; k)$ and $A^j = H_{4-j}(\mathbb{C}P^2; k)$. Then $N = \bigoplus_{i=0}^2 N^i$ is a module over $A = \bigoplus_{j=0}^4 A^j$. Picking a basis we may write the generators of N over k as $\{1_N, e_1, e_2, e_1 \wedge e_2\}$ and the generators of A as $\{1_A, a, a^2\}$. The generator a is represented by the class of a line $[\mathbb{C}P^1]$. The degrees of the generators are $|e_i| = 1$, $|e_1 \wedge e_2| = 2$, $|a| = 2$ and $|a^2| = 4$. The external intersection product is given by $a\,1_N = \gamma e_1 \wedge e_2$ for $\gamma \in k$ and corresponds to the intersection of $[\mathbb{C}P^1]$ with the fundamental class of L.

The deformation of A is then

$$(A \otimes k[t], \eta) = (QH_*(M;\Lambda^+), *) \simeq k[a, t]/(a^3 = t^3)$$

Note in our grading we have $|t| = 2$ and due to our grading the ambient quantum product becomes

$$\eta = \eta_0 + \eta_3 t^3,$$

where $\eta_3(a, a^2) = \eta_3(a^2, a) = 1_A$. Hence $\eta_1 = 0$ and a cocycle $(\eta_1, \mu_1) \in C^{2,-2}(A, A) \oplus D^{1,-2}(A, N)$ of our coupled deformation is given by $(0, \mu_1)$ such that

$$d^E(0, \mu_1) = (0, d^D\mu_1) = (0, 0).$$

We write down the possible value table for $\mu_1 \in D^{1,-2}(A, N)$, recalling that $\mu(1_A, m) = m$ and using the fact that $|\mu_1| = -2$:

$\mu_1(\cdot,\cdot)$	1_N	e_1	e_2	$e_1 \wedge e_2$
a	$\xi_1 1_N$	$\xi_2 e_1 + \xi_3 e_2$	$\xi_4 e_1 + \xi_5 e_2$	$\xi_6 e_1 \wedge e_2$
a^2	$\xi_7 e_1 \wedge e_2$	0	0	0

The coefficients $\xi_i \in k$ for $i = 1, \ldots, 7$. A cocycle $\mu_1 \in D^{1,-2}(A, N)$ must satisfy $d^D\mu_1(a_1, a_2, m) = 0$, since $\mu_1(1_A, m) = 0$ and for degree reasons the only non-trivial equation we get is then

$$d^D\mu_1(a, a, 1_N) = a\mu_1(a, 1_N) - \mu_1(a^2, 1_N) + \mu_1(a, a1_N) = 0. \quad (2.6)$$

Using the value table we rewrite equation (2.6) as

$$(\gamma\,\xi_1 - \xi_7 + \gamma\,\xi_6)e_1 \wedge e_2 = 0. \quad (2.7)$$

The coboundaries are given by $d^D\lambda_1$ with $\lambda_1 \in D^{0,-2}(A, N) = \mathrm{Hom}_k^{-2}(N, N)$. The only non-trivial term is then $\lambda_1(e_1 \wedge e_2) =: \delta\, 1_N$ for $\delta \in k$. Then the coboundary is

$$d^D\lambda_1(a_1, m) = a_1\lambda_1(m) - \lambda_1(a_1 m)$$

and has nontrivial values for

$$d^D\lambda_1(a, 1_N) = -\gamma\,\delta\, 1_N, \qquad d^D\lambda_1(a, e_1 \wedge e_2) = \gamma\,\delta\, e_1 \wedge e_2. \quad (2.8)$$

Equation (2.7) determines $\xi_7 = \gamma\,\xi_1 + \gamma\,\xi_6$ and with the coboundary values in (2.8) we see that the expression $\xi_1 + \xi_6$ is constant. Therefore the group $HE^{1,-2}(N, A)$ for coupled deformations (N, μ, A, η) with $\eta_1 = 0$ has rank 5.

2.2.7. Infinitesimals of coupled deformations In this section we provide deformation theoretic results of coupled deformations. Let (N, μ, A, η) be a coupled deformation given by

$$\begin{aligned}\eta(a,b) &= ab + \eta_l(a,b)t^l + \eta_{l+1}(a,b)t^{l+1} + \dots, \\ \mu(a,m) &= am + \mu_l(a,m)t^l + \mu_{l+1}(a,m)t^{l+1} + \dots,\end{aligned}$$

where $\eta_i = 0$ and $\mu_i = 0$ for $0 < i < l$. We call (η_l, μ_l) the *infinitesimal* of the coupled deformation. Using the defining equation (2.4) for coupled deformations for $n = l$ on page 27,

$$\sum_{i=0}^{l} \mu_i(\eta_{l-i}(a,b), m) = \sum_{i=0}^{l} \mu_i(a, \mu_{l-i}(b,m)),$$

we see that (η_l, μ_l) is a cocycle.

The following theorem holds in the case when $R = k[t]/(t^2)$ or $k[[t]]$. It is also true for $R = k[t]$ when we take graded coupled deformations of a finitely generated algebra A and module N over k.

Theorem 2.2.5. *Let (N, μ, A, η) be a coupled deformation with infinitesimal (η_l, μ_l). Assume (η_l, μ_l) is a 1-coboundary. Then (N, μ, A, η) is equivalent to a coupled deformation (N, μ', A, η') with infinitesimal (η'_n, μ'_n) of higher degree, i.e. $n > l$.*

Proof. We have

$$(\eta_l, \mu_l) = d^E(-\phi_l, \lambda_l) = \left(d^H\phi_l, \iota\phi_l + d^D\lambda_l\right).$$

We define the maps

$$\Phi = \mathrm{Id}_A - \phi_l t^l, \qquad \Lambda = \mathrm{Id}_N - \lambda_l t^l.$$

By the assumptions on R or on A and N, these maps are invertible with respect to the ring R. We calculate the new infinitesimal of A,

$$\begin{aligned}
&\Phi^{-1}\left(\eta(\Phi(a),\Phi(b))\right)\\
&= \quad ab + \left(\eta_l(a,b) - a\phi_l(b) + \phi_l(ab) - \phi_l(a)b\right)t^l\\
&\qquad + \eta_{l+1}(a,b)t^{l+1} + \ldots + \phi_l(a)\phi_l(b)t^{2l} + \ldots\\
&= \quad \eta'(a,b).
\end{aligned}$$

The infinitesimal η'_n of η' has degree $n > l$, since $\eta_l = d^H\phi_l$. Now we compute the infinitesimal of the module action under the chosen isomorphisms,

$$\begin{aligned}
&\Lambda^{-1}\left(\mu(\Phi(a),\Lambda(m))\right)\\
&= \quad \Lambda^{-1}\left(\mu(a - \phi_l(a)t^l, m - \lambda_l(m)t^l)\right)\\
&= \quad am + (\mu_l(a,m) - \phi_l(a)m - a\lambda_l(m) + \lambda_l(am))t^l\\
&\qquad + \mu_{l+1}(a,m)t^{l+1} + \ldots + \phi_l(a)\lambda_l(m)t^{2l} + \ldots\\
&= \quad \mu'(a,m).
\end{aligned}$$

The infinitesimal μ'_n of μ' has degree $n > l$, since $\mu_l = \iota\phi_l + d^D\lambda_l$. Thus (N,μ,A,η) is equivalent to a coupled deformation (N,μ',A,η') with infinitesimal (η'_n,μ'_n) of degree $n > l$. □

We call a pair (N,A) *rigid*, if every coupled deformation (N,μ,A,η) is equivalent to the trivial deformation. A consequence of the previous theorem is a cohomological criterion for the rigidity of a coupled deformation.

Corollary 2.2.6. *If the cohomology group $HE^1(A,N)$ is trivial, then (N,A) is rigid.*

Proof. Since $HE^1(A,N) = 0$, we can apply Theorem 2.3.4 to the infinitesimal of the coupled deformation (N,μ,A,η) to obtain an equivalent coupled deformation (N,μ',A,η') of higher degree. Iterating the process gives us an equivalent coupled deformation of arbitrary high degree. □

2.3. Triple deformations

2.3.1. Triple deformations Let (A, η_0) and (X, ρ_0) be two unital, associative k-algebras. Furthermore assume that the product of A is commutative and that X is also an A-algebra. This structure is given by a ring homomorphism $A \to X$ such that the image of A lies in the center of X. Alternatively, one can also postulate a module action $\mu_0 : A \otimes X \to X$, $ax := \mu_0(a, x)$, that satisfies certain conditions for the algebra structure. Let $R = k[t]$.

We now define simultaneous compatible deformations of all three structures.

Definition 2.3.1. Let (A, η_0) be a unital, commutative and associative k-algebra and (X, ρ_0) a unital, associative k-algebra that is also an A-algebra. Let $(A \otimes R, \eta)$ be a commutative deformation of A. A *triple deformation* (X, ρ, μ, A, η) is given by a coupled deformation (X, μ, A, η) and an R-linear map

$$\rho : (X \otimes R) \otimes_R (X \otimes R) \longrightarrow X \otimes R, \qquad x \otimes y \longmapsto \rho(x, y),$$

that satisfies

1. $(X \otimes R, \rho)$ is an associative R-algebra;
2. $\mu(\eta(a, b), \rho(x, y)) = \rho(\mu(a, x), \mu(b, y))$, for $a, b \in A$, $x, y \in X$;
3. $\rho(1_X, x) = \rho(x, 1_X) = x$, for $x \in X$;
4. ρ reduces to ρ_0 for $t = 0$.

The product ρ can then be written as

$$\rho = \rho_0 + \rho_1 t + \rho_2 t^2 + \ldots,$$

Writing out condition (2) of the definition gives us

$$\begin{aligned}
&\mu(\eta(a, b), \rho(x, y)) \\
&= \quad \mu\left(ab + \eta_1(a, b)t + \ldots, xy + \rho_1(x, y)t + \ldots\right) \\
&= \quad (ab)(xy) + (\eta_1(a, b)xy + ab\rho_1(x, y) + \mu_1(ab, xy))t \\
&\qquad + \eta_1(a, b)\rho_1(x, y)t^2 + \ldots
\end{aligned}$$

and

$$\begin{aligned}
&\rho\left(\mu(a,x),\mu(b,y)\right)\\
&= \quad \rho\left(ax+\mu_1(a,x)t+\ldots,by+\mu_1(b,y)t+\ldots\right)\\
&= \quad (ax)(by)+(ax\mu_1(b,y)+\mu_1(a,x)by+\rho_1(ax,by))t\\
&\quad\quad +\mu_1(a,x)\mu_1(b,y)t^2+\ldots
\end{aligned}$$

This results in the equation for coefficients of t^n for triple deformations

$$\sum_{i+j+k=n}\mu_i(\eta_j(a,b),\rho_k(x,y))=\sum_{i+j+k=n}\rho_i(\mu_j(a,x),\mu_k(b,y)). \quad (2.9)$$

The coefficient of t therefore gives us the equation

$$\begin{aligned}
&\eta_1(a,b)xy+ab\rho_1(x,y)+\mu_1(ab,xy)\\
&= \quad ax\mu_1(b,y)+\mu_1(a,x)by+\rho_1(ax,by). \quad (2.10)
\end{aligned}$$

2.3.2. Equivalence of triple deformations We introduce an appropriate notion of equivalence for triple deformations.

Definition 2.3.2. Two triple deformations (X,ρ,μ,A,η) and (X,ρ',μ',A,η') are *equivalent* if there exist isomorphisms $\Phi\in\mathrm{Iso}(A)$ and $\Lambda\in\mathrm{Iso}(X)$ that satisfy

$$\begin{aligned}
\eta(a,b) &= \Phi^{-1}(\eta'(\Phi(a),\Phi(b))),\\
\mu(a,x) &= \Lambda^{-1}(\mu'(\Phi(a),\Lambda(x))),\\
\rho(x,y) &= \Lambda^{-1}(\rho'(\Lambda(x),\Lambda(y))),
\end{aligned}$$

for $a,b\in A$ and $x,y\in X$.

When a triple deformation (X,ρ,μ,A,η) is equivalent to $(X,\rho_0,\mu_0,A,\eta_0)$ we say the deformation is *trivial*. The set of equivalence classes of triple deformations is denoted by

$$\mathrm{Def}^{tr}(X,A).$$

2.3.3. Cohomology of triple deformations Consider the k-algebra $A \otimes X$ with the product

$$(a \otimes x)(b \otimes y) = ab \otimes xy.$$

The algebra X is then an $A \otimes X$-bimodule by setting

$$(a \otimes x)z = (ax)z, \qquad z(b \otimes y) = z(by)$$

for $z \in X$.

Now recall that for a k-algebra B and a B-bimodule N we can define the Hochschild cochain complex with coefficients in N, $(C^*(B, N), d^H)$. We will use multiple versions of this complex in order to define a complex associated to a triple deformation.

The module action μ_0 induces three cochain maps of Hochschild complexes (we use the same notation for the first and second map),

$$\begin{aligned} \mu_{0*} : C^n(A, A) &\longrightarrow C^n(A \otimes X, X), \\ \mu_{0*} : C^n(X, X) &\longrightarrow C^n(A \otimes X, X), \\ \mu_0^* : C^n(X, X) &\longrightarrow C^n(A \otimes X, X), \end{aligned}$$

given by

$$\begin{aligned} \mu_{0*}\eta(a_1 \otimes x_1, \ldots, a_n \otimes x_n) &:= \eta(a_1, \ldots, a_n)x_1 \cdots x_n, \\ \mu_{0*}\rho(a_1 \otimes x_1, \ldots, a_n \otimes x_n) &:= a_1 \cdots a_n \rho(x_1, \ldots, x_n), \\ \mu_0^*\rho(a_1 \otimes x_1, \ldots, a_n \otimes x_n) &:= \rho(a_1x_1, \ldots, a_nx_n), \end{aligned}$$

for $\eta \in C^n(A, A)$ and $\rho \in C^n(X, X)$.

Lemma 2.3.1. *The maps μ_{0*} and μ_0^* are cochain maps.*

We defer the proof of the lemma to the appendix. We now define the cochain complex

$$G^n(A, X) := C^{n+1}(X, X) \oplus C^{n+1}(A, A) \oplus C^n(A \otimes X, X)$$

with coboundary operator

$$d^G(\rho, \eta, \mu) := (-d^H\rho, -d^H\eta, -\mu_{0*}\eta - \mu_{0*}\rho + \mu_0^*\rho + d^H\mu)$$

for $(\rho, \eta, \mu) \in G^n(X, A)$.

Lemma 2.3.2. *d^G is a coboundary operator, i.e. $d^G \circ d^G = 0$.*

Proof.

$$\begin{aligned} \left(d^G\right)^2(\rho,\eta,\mu) &= d^G(-d^H\rho, -d^H\eta, -\mu_{0*}\eta - \mu_{0*}\rho + \mu_0^*\rho + d^H\mu) \\ &= ((d^H)^2\rho,\ (d^H)^2\eta,\ \mu_{0*}d^H\eta + \mu_{0*}d^H\rho - \mu_0^*d^H\rho \\ &\qquad -d^H\mu_{0*}\eta - d^H\mu_{0*}\rho + d^H\mu_0^*\rho + (d^H)^2\mu)) \\ &= (0,0,0) \end{aligned}$$

□

The complex can be seen as a generalisation of a mapping cone as follows: for two cochain maps $f : A \to C$ and $g : B \to C$ we take the complex $A^{*+1} \oplus B^{*+1} \oplus C^*$ and the differential d given by

$$d(a,b,c) = (-d^A a, -d^B b, -f(a) - g(b) + d^C c).$$

For A or B equal to zero we obtain the mapping cone of the other cochain map.

Definition 2.3.3. We define the triple deformation cohomology to be the cohomology of the cochain complex $(G^*(X,A), d^G)$. We denote the cohomology by $HG^*(X,A)$.

Theorem 2.3.3. *Let (X,ρ,μ,A,η) be a triple deformation given by*

$$\begin{aligned} \eta(a,b) &= ab + \eta_1(a,b)t + \eta_2(a,b)t^2 + \dots, \\ \mu(a,x) &= ax + \mu_1(a,x)t + \mu_2(a,x)t^2 + \dots, \\ \rho(x,y) &= xy + \rho_1(x,y)t + \rho_2(x,y)t^2 + \dots \end{aligned}$$

for all $a,b \in A$ and $x,y \in X$. Then (ρ_1,η_1,μ_1) is a 1-cocycle in the cochain complex $(G^*(X,A), d^G)$. Furthermore, if (X,ρ',μ',A,η') is a triple deformation equivalent to (X,ρ,μ,A,η), then $(\rho_1,\eta_1,\mu_1) - (\rho_1',\eta_1',\mu_1')$ is a 1-coboundary.

The theorem associates to every equivalence class of a triple deformation a non-trivial cohomology class, i.e.

$$\mathrm{Def}^{tr}(X,A) \longrightarrow HG^1(A,X), \quad [(X,\rho,\mu,A,\eta)] \longmapsto [(\rho_1,\eta_1,\mu_1)].$$

Proof. We see that (ρ_1, η_1, μ_1) is a cocycle since η_1 and ρ_1 are Hochschild cocycles, i.e. $d^H\eta_1 = 0$, $d^H\rho_1 = 0$, and the equation

$$-\mu_{0*}\eta_1 - \mu_{0*}\rho_1 + \mu_0^*\rho_1 + d^H\mu_1 = 0$$

is equivalent to equation (2.10).

Now assume that two triple deformations (X, ρ, μ, A, η) and $(X, \rho', \mu', A, \eta')$ are equivalent through the isomorphisms $\Phi \in \mathrm{Iso}(A)$ and $\Lambda \in \mathrm{Iso}(X)$. If we write $\Phi = \mathrm{Id}_A + \phi_1 t + \ldots$ and $\Lambda = \mathrm{Id}_X + \lambda_1 t + \ldots$, then we see that

$$\begin{aligned} \rho_1 - \rho_1' &= d\lambda_1, \\ \eta_1 - \eta_1' &= d\phi_1, \\ \mu_1 - \mu_1' &= \mu_{0*}\phi_1 + \mu_{0*}\lambda_1 - \mu_0^*\lambda_1. \end{aligned}$$

Thus we obtain

$$\begin{aligned} (\rho_1, \eta_1, \mu_1) - (\rho_1', \eta_1', \mu_1') &= (d^H\lambda_1, d^H\phi_1, \mu_{0*}\phi_1 + \mu_{0*}\lambda_1 - \mu_0^*\lambda_1) \\ &= d^G(-\lambda_1, -\phi_1, 0). \end{aligned}$$

□

2.3.4. Graded cohomology of triple deformations We consider the graded case as well. Again we use cohomological grading in this section and as before, the algebras $A = \bigoplus_{i\geq 0} A^i$, $X = \bigoplus_{j\geq 0} X^j$ are graded and the ring $k[t]$ is graded with $|t| = d \in \mathbb{Z}$. A graded triple deformation (X, ρ, μ, A, η) is then a deformation in the sense of Definition 2.3.1, where the maps η_j, μ_j and ρ_j all have degree $-jd$. The set of equivalence classes of graded couple deformations will be denoted in this case by $\mathrm{Def}_d^{\mathrm{tr}}(X, A)$.

The coboundary operator d^G of the graded cochain complex $G^{n,l}(A, X)$ preserves the grading and the graded cohomology groups are denoted by

$$HG^{n,l}(A, X).$$

2.3.5. The quantum algebra structure Let $L \subset (M, \omega)$ be a monotone Lagrangian that is Λ^+-wide with minimal Maslov number N_L. By the discussion in Chapter 1, section 1.1.3, we can associate a graded cohomology class to the $(QH_*(M;\Lambda^+), *)$-algebra structure of $(QH_*(L;\Lambda^+), \circ)$.

Denote by $X := H_*(L;k)$ and $A = H_*(M;k)$ with grading given by $X^i := H_{n-i}(L;k)$ and $A^j = H_{2n-j}(M;k)$. We grade the ring $k[t]$ by setting $|t| = N_L$. For a generic choice of data triple $\mathcal{D}$ there is an isomorphism $\Psi_{\mathcal{D}} : QH_*(L;\Lambda^+) \to X \otimes \Lambda^+$ and by transferring the quantum structures we obtain a triple deformation of the classical structures on X and A,

$$\begin{aligned} a * b &= ab + \eta_1(a,b)t + \eta_2(a,b)t^2 + \ldots, \\ a \circledast x &= ax + \mu_1(a,x)t + \mu_2(a,x)t^2 + \ldots. \\ x \circ y &= xy + \rho_1(x,y)t + \rho_2(x,y)t^2 + \ldots. \end{aligned}$$

Changes in choices of generic data triple $\mathcal{D}$ give rise to equivalent deformations of triple deformations. Once again, the deformation of A remains fixed. We obtain an invariant through the map

$$\mathrm{Def}^{\mathrm{tr}}_{N_L}(X, A) \longrightarrow HG^{1,-N_L}(A, X),$$

defined by

$$(QH_*(L;\Lambda^+), \circ, \circledast, QH_*(M;\Lambda^+), *) \longmapsto [(\rho_1, \eta_1, \mu_1)].$$

This proves Theorem B of the introduction.

We finish this section with an elementary example for the computations involved.

Example. Let $L \simeq S^1$ be a monotone Lagrangian in $(\mathbb{C}P^1, \omega_{\mathrm{FS}})$ with $N_L = 2$. L is given by the equator (or a circle Hamiltonianly isotopic to the equator) on the round 2-sphere. We compute the cohomology group $HG^{1,-N_L}(A, X)$ associated to a degree 2 triple deformation of $H_*(\mathbb{C}P^1;k)$ and $H_*(S^1;k)$.

We use cohomological grading and set $X^i := H_{1-i}(S^1;k)$ and $A^j := H_{2-j}(\mathbb{C}P^1;k)$. Then $X = X^0 \oplus X^1$ and $A = A^0 \oplus A^2$ are

k-algebras and X is a module over A. Picking a basis we may write the generators of X over k as $\{1_L, p_L\}$ and the generators of A over k as $\{1_M, p_M\}$, where the generators 1_M and 1_L are the unities and p_L and p_M are represented by classes of points in L and M. The degrees of the generators are $|p_L| = 1$, $|p_M| = 2$. The external intersection product is given by $1_M x = x$ and $p_M x = 0$ for all $x \in X$.

The deformation of A is then

$$(A \otimes k[t], \eta) = (QH_*(\mathbb{C}P^1; \Lambda^+), *) \simeq k[p_M, t]/(p_M^2 = t^2).$$

Note in our grading we have $|t| = 2$ and due to this the ambient quantum product becomes

$$\eta = \eta_0 + \eta_2 t^2,$$

where $\eta_2(p_M, p_M) = 1_M$ and $\eta_1 = 0$.

For degree reasons the deformation of X is then

$$(X \otimes k[t]) = (QH_*(L; \Lambda^+), *) \simeq k[p_L, t]/(p_L^2 = \gamma t),$$

with the quantum product $\rho = \rho_0 + \rho_1 t$ and $\rho_1(p_L, p_L) = \gamma\, 1_L$ for $\gamma \in k$.

A cochain in $G^{1,-N_L}(A, X)$ is given by

$$(\rho_1, \eta_1, \mu_1) \in C^{2,-2}(X, X) \oplus C^{2,-2}(A, A) \oplus C^{1-2}(A \otimes X, X).$$

A triple deformation (X, ρ, μ, A, η) in our case then maps to a cocycle $(\rho_1, 0, \mu_1) \in G^{1,-N_L}(A, X)$. We write down the possible value table for $\rho_1 \in C^{2,-2}(X, X)$ and $\mu_1 \in C^{1,-2}(A \otimes X, X)$. For degree reasons we have

$\rho_1(\cdot, \cdot)$	1_L	p_L
1_L	0	0
p_L	0	$\gamma 1_L$

with $\gamma \in k$ and

$\mu_1(\cdot,\cdot)$	1_L	p_L
1_M	0	0
p_M	$\delta_1 1_L$	$\delta_2 p_L$

with $\delta_i \in k$. Thus the space of cochains has rank 3 in this case. A cocycle $(\rho_1, 0, \mu_1)$ must satisfy $d^H\rho_1 = 0$ and equation (2.10) in our case becomes

$$ab\rho_1(x,y) + \mu_1(ab, xy) = ax\mu_1(b,y) + \mu_1(a,x)by + \rho_1(ax, by)$$

for all $a, b \in A$ and $x, y \in X$. Using this equation and the fact that $|\rho_1| = |\mu_1| = -2$ we obtain only one nontrivial equation,

$$\mu_1(p_M, p_L) = p_L\mu_1(p_M, 1_L).$$

Together with the value table this implies $\delta_2 = \delta_1$. The coboundaries are given by $d^G(\lambda_1, \phi_1, 0) = (-d^H\lambda_1, -d^H\phi_1, -\mu_{0*}\phi_1 - \mu_{0*}\lambda_1 + \mu_0^*\lambda_1)$ for $\lambda_1 \in C^{1,-2}(X,X)$ and $\phi_1 \in C^{1,-2}(A,A)$. Due to degree reasons $\lambda_1 = 0$ and since $\eta_1 = 0$ we choose $\phi_1 = 0$. We obtain no other relations on the cocycles. Therefore $HG^{1,-2}(A,X)$ has rank 2 for triple deformations (X, ρ, μ, A, η) with $\eta_1 = 0$.

Coming back to our example for the equator $S^1 \subset \mathbb{C}P^1$, one can show (see [15]) that $\gamma = 1$, $\delta_1 = \delta_2 = 1$, i.e. the quantum terms are given by

$$p_L \circ p_L = 1_L t, \quad p_M \circledast 1_L = 1_L t, \quad p_M \circledast p_L = p_L t.$$

2.3.6. Infinitesimals of triple deformations Let (X, ρ, μ, A, η) be a triple deformation given by

$$\begin{aligned}
\rho(x,y) &= xy + \rho_l(x,y)t^l + \rho_{l+1}(x,y)t^{l+1} + \dots, \\
\eta(a,b) &= ab + \eta_l(a,b)t^l + \eta_{l+1}(a,b)t^{l+1} + \dots, \\
\mu(a,m) &= am + \mu_l(a,m)t^l + \mu_{l+1}(a,m)t^{l+1} + \dots,
\end{aligned}$$

where $\rho_i = 0$, $\eta_i = 0$ and $\mu_i = 0$ for $0 < i < l$. We call as before (ρ_l, η_l, μ_l) the *infinitesimal* of the coupled deformation. Using the defining equation (2.9) for $n = l$,

$$\sum_{i+j+k=n} \mu_i(\eta_j(a,b), \rho_k(x,y)) = \sum_{i+j+k=n} \rho_i(\mu_j(a,x), \mu_k(b,y)),$$

we see that (ρ_l, η_l, μ_l) is a cocycle.

The following theorem holds in the case when $R = k[t]/(t^2)$ or $k[[t]]$. It is also true for $R = k[t]$ when we take graded triple deformations of finitely generated algebras A and X over k.

Theorem 2.3.4. *Assume (X, ρ_0) is a commutative k-algebra. Let (X, ρ, μ, A, η) be a triple deformation with infinitesimal (ρ_l, η_l, μ_l). Assume (ρ_l, η_l, μ_l) is a 1-coboundary. Then (X, ρ, μ, A, η) is equivalent to a triple deformation $(X, \rho', \mu', A, \eta')$ with infinitesimal $(\rho'_n, \eta'_n, \mu'_n)$ of higher degree, i.e. $n > l$.*

Proof. We have

$$\begin{aligned} (\rho_l, \eta_l, \mu_l) &= d^G(-\lambda_l, -\phi_l, \xi_l) \\ &= (d^H\lambda_l, d^H\phi_l, \mu_{0*}\phi_l + \mu_{0*}\lambda_l - \mu_0^*\lambda_l + d^H\xi_l). \end{aligned}$$

We define the isomorphisms

$$\Phi = \mathrm{Id}_A - \phi_l t^l, \qquad \Lambda = \mathrm{Id}_X - \lambda_l t^l.$$

These maps are invertible with respect to R under our assumptions. We see that under the isomorphism Φ the infinitesimal of A has degree greater than l. The infinitesimal of X also has degree greater than l and we have

$$\mu'_l(a,x) = (ax)z - z(ax) = a(xz - zx),$$

for $z := \xi_l(1)$. Since X is commutative, this expression vanishes and the infinitesimal of the module action has degree greater than l. $\square$

We call a pair (X, A) rigid, if every triple deformation is equivalent to the trivial deformation. Once again the theorem provides us with a criterion for the rigidity of a triple deformation.

Corollary 2.3.5. *Let (X, ρ, μ, A, η) be a triple deformation and assume that (X, ρ_0) is commutative. If the cohomology group $HG^1(A, X)$ is trivial, then (X, A) is rigid.*

2.4. Deformations of module morphisms

In this section we develop the theory of deformations of module morphisms.

2.4.1. Deformed module morphisms Let (A, η_0) be a unital, associative k-algebra. Let (X, μ_0^X), (Y, μ_0^Y) be two A-modules and let $\psi_0 : X \to Y$ be an A-module morphism, i.e. a k-linear map such that $\psi_0(\mu_0^X(a, x)) = \mu_0^Y(a, \psi_0(x))$ for $a \in A$ and $x \in X$.

Definition 2.4.1. Let (X, μ^X, A, η) and (Y, μ^Y, A, η) be two coupled deformations and $\psi_0 : X \to Y$ an A-module morphism. A *deformation* of ψ_0 is a $k[t]$-module morphism

$$\psi : X \otimes k[t] \longrightarrow Y \otimes k[t],$$

that satisfies

1. $\psi(\mu^X(a, x)) = \mu^Y(a, \psi(x))$,
2. ψ reduces to ψ_0 for $t = 0$.

The morphism ψ can be written as

$$\psi(x) = \psi_0(x) + \psi_1(x)t + \psi_2(x)t^2 + \dots,$$

with $\psi_i \in \mathrm{Hom}_k(X, Y)$. We compute

$$\begin{aligned} \psi(\mu^X(a, x)) &= \psi_0(ax) + \psi_0(\mu_1^X(a, x))t + \psi_1(ax)t \\ &\quad + \psi_0(\mu_2^X(a, x))t^2 + \dots \end{aligned}$$

and

$$\begin{aligned} \mu^Y(a, \psi(x)) &= a\psi_0(x) + \mu_1^Y(a, \psi_0(x))t + a\psi_1(x)t \\ &\quad + a\psi_2(x)t^2 + \dots \end{aligned}$$

This results in the defining equation for coefficients of t^n of deformed module morphisms

$$\sum_{i+j=n} \psi_i(\mu_j^X(a,x)) = \sum_{i+j=n} \mu_i^Y(a,\psi_j(x)).$$

In particular, we have for $n=1$

$$\psi_0(\mu_1^X(a,x)) + \psi_1(ax) = a\psi_1(x) + \mu_1^Y(a,\psi_0(x)). \tag{2.11}$$

2.4.2. Equivalence of deformed module morphisms Let $\Phi \in \mathrm{Iso}(A)$ be an isomorphism of $(A \otimes R, \eta)$ and $(A \otimes R, \eta')$. Let (X, μ^X, A, η) and (X, μ'^X, A, η') be two equivalent coupled deformations via Φ and $\Lambda_X \in \mathrm{Iso}(X)$, let (Y, μ^Y, A, η) and (Y, μ'^Y, A, η') be two equivalent coupled deformations via Φ and $\Lambda_Y \in \mathrm{Iso}(Y)$.

Definition 2.4.2. Let $\psi : (X, \mu^X, A, \eta) \to (Y, \mu^Y, A, \eta)$ and $\psi' : (X, \mu'^X, A, \eta') \to (Y, \mu'^Y, A, \eta')$ be two deformations of the module morphism ψ_0. These are called *equivalent*, if

$$\psi = \Lambda_Y^{-1}\left(\psi'\left(\Lambda_X(x)\right)\right).$$

2.4.3. Cohomology of deformed module morphisms Recall that $\mathrm{Hom}_k(X,Y)$ is an A-bimodule with action given by

$$(a\varphi b)(x) = a\varphi(bx),$$

for $\varphi \in \mathrm{Hom}_k(X,Y)$. We consider the Hochschild cochain complex of A with coefficients in $\mathrm{Hom}_k(X,Y)$, i.e. the complex

$$C^n(A;X,Y) := C^n(A, \mathrm{Hom}_k(X,Y)) = \mathrm{Hom}_k(A^{\otimes n}, \mathrm{Hom}_k(X,Y))$$

with Hochschild coboundary map

$$\begin{aligned}
& d^H\varphi(a_1,\ldots,a_{n+1})(x) \\
= \quad & a_1\varphi(a_2,\ldots,a_{n+1})(x) \\
& + \sum_{i=1}^{n}(-1)^i\varphi(a_1,\ldots,a_ia_{i+1},\ldots,a_{n+1})(x) \\
& + (-1)^{n+1}\varphi(a_1,\ldots,a_n)(a_{n+1}x)
\end{aligned}$$

The A-module morphism ψ_0 defines two maps

$$\begin{array}{rcl} \psi_{0*} : D^n(A,X) & \longrightarrow & C^n(A;X,Y), \\ \psi_0^* : D^n(A,Y) & \longrightarrow & C^n(A;X,Y), \end{array}$$

given by

$$\begin{array}{rcl} (\psi_{0*}\xi)(a_1,\ldots,a_n)(x) & := & \psi_0(\xi(a_1,\ldots,a_n,x)), \\ (\psi_0^*\zeta)(a_1,\ldots,a_n)(x) & := & \zeta(a_1,\ldots,a_n,\psi_0(x)). \end{array}$$

Lemma 2.4.1. *The maps ψ_{0*} and ψ_0^* are cochain maps.*

The proof is deferred to the appendix. We also have the two known cochain maps

$$\begin{array}{rcl} \iota_X : C^{n+1}(A,A) & \longrightarrow & D^n(A,X), \\ \iota_Y : C^{n+1}(A,A) & \longrightarrow & D^n(A,Y). \end{array}$$

We define a cochain complex

$$I^n(A,X,Y) := C^{n+1}(A,A) \oplus D^n(A,X) \oplus D^n(A,Y) \oplus C^{n-1}(A;X,Y)$$

with operator

$$d^I(\eta,\xi,\zeta,\delta) = (-d^H\eta,\ -\iota_X\eta + d^D\xi,\ -\iota_Y\eta + d^D\zeta,\ \psi_{0*}\xi - \psi_0^*\zeta - d^H\delta).$$

Lemma 2.4.2. *d^I is a coboundary operator, i.e. $(d^I)^2 = 0$.*

Proof. The cochain complex can be seen as an iteration of mapping cones. We compute

$$\begin{array}{rl} \multicolumn{2}{l}{(d^I)^2(\eta,\xi,\zeta,\delta)} \\ = & d^I(-d^H\eta, -\iota_X\eta + d^D\xi, -\iota_Y\eta + d^D\zeta, \psi_{0*}\xi - \psi_0^*\zeta - d^H\delta) \\ = & ((d^H)^2\eta, \iota_X d^H\eta + d^D(-\iota_X\eta + d^D\xi), \iota_Y d^H\eta + d^D(-\iota_Y\eta + d^D\zeta), \\ & \psi_{0*}(-\iota_X\eta + d^D\xi) - \psi_0^*(-\iota_Y\eta + d^D\zeta) - d^H(\psi_{0*}\xi - \psi_0^*\zeta - d\delta)) \\ = & (0,0,0,0) \end{array}$$

The last component is equal to zero because ψ_{0*} and ψ_0^* are cochain maps and $\psi_{0*}(\iota_X\eta) = \psi_0^*(\iota_Y\eta)$. ☐

We denote the cohomology of this complex by

$$HI^n(A; X, Y) := H^n(I^*(A; X, Y), d^I).$$

Theorem 2.4.3. *Let $\psi : (X, \mu^X, A, \eta) \longrightarrow (Y, \mu^Y, A, \eta)$ be a deformed module morphism of $\psi_0 : X \to Y$ given by*

$$\begin{aligned} \eta(a,b) &= ab + \eta_1(a,b)t + \eta_2(a,b)t^2 + \dots, \\ \mu^X(a,x) &= ax + \mu_1^X(a,x)t + \mu_2^X(a,x)t^2 + \dots, \\ \mu^Y(a,y) &= ay + \mu_1^Y(a,y)t + \mu_2^Y(a,y)t^2 + \dots, \\ \psi(x) &= \psi_0(x) + \psi_1(x)t + \psi_2(x)t^2 + \dots \end{aligned}$$

for $a, b \in A$, $x \in X$ and $y \in Y$. Then $(\eta_1, \mu_1^X, \mu_1^Y, \psi_1)$ is a 1-cocycle in the cochain complex $(I^*(A; X, Y), d^I)$. Furthermore if ψ' is an equivalent deformed module morphism, then $(\eta_1, \mu_1^X, \mu_1^Y, \psi_1) - (\eta_1', \mu_1'^X, \mu_1'^Y, \psi_1')$ is a 1-coboundary.

The theorem associates to every equivalence class of a deformed module morphism a non-trivial cohomology class, i.e.

$$[\psi : (X, \mu^X, A, \eta) \to (Y, \mu^Y, A, \eta)] \longmapsto [(\eta_1, \mu_1^X, \mu_1^Y, \psi_1)],$$

where $[(\eta_1, \mu_1^X, \mu_1^Y, \psi_1)] \in HI^1(A; X, Y)$.

Proof. We see that $(\eta_1, \mu_1^X, \mu_1^Y, \psi_1)$ is a cocycle by our previous work on the cohomology of coupled deformations and equation (2.11), which is equivalent to

$$\psi_{0*}\mu_1^X - \psi_0^*\mu_1^Y - d^H\psi_1 = 0.$$

Assuming that $(X, \mu^X, A, \eta) \backsim (X, \mu'^X, A, \eta')$ via the isomorphisms $\Phi = \mathrm{Id}_A + \phi_1 t + \dots$ and $\Lambda_X = \mathrm{Id}_X + \lambda_1^X t + \dots$ and $(Y, \mu^Y, A, \eta) \backsim (Y, \mu'^Y, A, \eta')$ via the isomorphisms Φ and $\Lambda_Y = \mathrm{Id}_Y + \lambda_1^Y t + \dots$, then we already know using Theorem 2.2.4 that

$$(\eta_1, \mu_1^X, \mu_1^Y) - (\eta_1', \mu_1'^X, \mu_1'^Y) = (d^H\phi_1,\ \iota_X\phi_1 + d^D\lambda_1^X,\ \iota_Y\phi_1 + d^D\lambda_1^Y).$$

Writing out the equation $\psi = \Lambda_Y^{-1}(\psi'(\Lambda_X(x)))$ we see that the first order terms are

$$\psi_1(x) = \psi_1'(x) + \psi_0(\lambda_1^X(x)) - \lambda_1^Y(\psi_0(x)).$$

Therefore

$$\begin{aligned}
&(\eta_1, \mu_1^X, \mu_1^Y, \psi_1) - (\eta_1', \mu_1'^X, \mu_1'^Y, \psi_1') \\
&= \ (d^H\phi_1,\ \iota_X\phi_1 + d^D\lambda_1^X,\ \iota_Y\phi_1 + d^D\lambda_1^Y, \psi_0(\lambda_1^X(x)) - \lambda_1^Y(\psi_0(x))) \\
&= \ d^I(-\phi_1, \lambda_1^X, \lambda_1^Y, 0),
\end{aligned}$$

which completes our proof. □

2.4.4. Graded cohomology of deformed module morphisms We consider the graded theory as well. As before, the algebra A and modules X, Y are graded. We grade the ring $k[t]$ by setting $|t| = d \in \mathbb{Z}$. A graded deformation of a module morphism is given as in Definition 2.4.1 and all cochain complexes are graded as well. The graded cohomology groups are denoted by

$$HI^{n,l}(A; X, Y).$$

2.4.5. The quantum inclusion map Let $L \subset (M, \omega)$ be a monotone Lagrangian that is Λ^+-wide with minimal Maslov number N_L. The same discussion as in the previous sections apply to the quantum inclusion $i_L : QH_*(L; \Lambda^+) \to QH_*(M; \Lambda^+)$.

Denote by $X := H_*(L; k)$ and $A := H_*(M; k)$ the respective homologies of L and M. For a generic choice of data triple $\mathcal{D}$ the isomorphism $\Psi_{\mathcal{D}} : QH_*(L; \Lambda^+) \to X \otimes \Lambda^+$ transfers the quantum inclusion to $X \otimes \Lambda^+$ and we obtain a deformation of the classical inclusion of $X \to A$,

$$i_L(x) = i_0(x) + i_1(x)t + i_2(x)t^2 + \dots.$$

The quantum inclusion is a $(QH_*(M; \Lambda^+), *)$-module map, i.e.

$$i_L(a \circledast x) = a * i_L(x).$$

Therefore in the language of deformed module morphisms, the two coupled deformations (X, μ^X, A, η) and (Y, μ^Y, A, η) are given by (X, μ, A, η) and (A, η, A, η). Changes in the choice of generic data triple $\mathcal{D}$ give rise to equivalent deformations of the quantum inclusion map. Thus we obtain an invariant of $L \subset M$, namely the cohomology class associated to the quantum inclusion defined by

$$[i_L : QH_*(L; \Lambda^+) \to QH_*(M; \Lambda^+)] \longmapsto [(\eta_1, \mu_1, \eta_1, i_1)],$$

where $[(\eta_1, \mu_1, \eta_1, i_1)] \in HI^{1,-N_L}(A; X, A)$. We finish this section with an elementary example for the computations involved.

Example. Let $L \simeq S^1$ be a monotone Lagrangian in $(\mathbb{C}P^1, \omega_{\mathrm{FS}})$ with $N_L = 2$. L is given by the equator on the Riemann sphere, this simple example should illustrate the computations needed for deformations of module morphisms. We compute the cohomology group $HI^{1,-N_L}(A; X, A)$ associated to a degree 2 deformation of the map $i_0 : H_*(S^1; k) \to H_*(\mathbb{C}P^1; k)$ induced by the inclusion $L \hookrightarrow M$.

We use cohomological grading and set $X^i := H_{1-i}(S^1; k)$ and $A^j := H_{2-j}(\mathbb{C}P^1; k)$. Then $A = A^0 \oplus A^2$ is a k-algebra and $X = X^0 \oplus X^1$ is a module over k and A. Picking a basis we may write the generators of X over k as $\{1_L, p_L\}$ and the generators of A over k as $\{1_M, p_M\}$, where the generators 1_M and 1_L are the fundamental classes and p_L, p_M are represented by classes of points in L and M. The degrees of the generators are $|1_L| = |1_M| = 0$ and $|p_L| = 1$, $|p_M| = 2$. The external intersection product is given by $1_M x = x$ and $p_M x = 0$ for all $x \in X$. The module morphism i_0 is given by $i_0(1_L) = 0$ and $i_0(p_L) = p_M$. In cohomological grading we have $|i_0| = n = 1$.

We now describe deformations of the module morphism. First the deformation of A is given by

$$(A \otimes k[t], \eta) = (QH_*(\mathbb{C}P^1; \Lambda^+), *) \simeq k[p_M, t]/(p_M^2 = t^2).$$

Note in our grading we have $|t| = 2$ and due to this the ambient quantum product becomes

$$\eta = \eta_0 + \eta_2 t^2,$$

where $\eta_2(p_M, p_M) = 1_M$ and $\eta_1 = 0$. $(A \otimes k[t], \eta)$ is also a module over itself with module action η.

A deformation of i_0 is then given by a $k[t]$-linear map

$$i_L : X \otimes k[t] \longrightarrow A \otimes k[t],$$

such that $i_L(a \circledast x) = a * i_L(x)$. We may write $i_L = i_0 + i_1 t$ and we have the following value tables:

	1_L	p_L
$i_0(\cdot)$	0	p_M

	1_L	p_L
$i_1(\cdot)$	0	$\zeta\, 1_M$

for $\zeta \in k$. Note in cohomological grading we have $|i_1| = n - 2 = -1$. The map μ_1^X of the coupled deformation (X, μ^X, A, η) is given by

$\mu_1^X(\cdot,\cdot)$	1_L	p_L
1_M	0	0
p_M	$\delta_1\, 1_L$	$\delta_2\, p_L$

for $\delta_i \in k$. The map μ_1^Y of the coupled deformation (A, μ^Y, A, η) is equal to η_1 and hence zero.

A cochain in $I^{1,-N_L}(A; X, A)$ is given by $(\eta_1, \mu_1^X, \mu_1^Y, i_1) \in C^{2,-2}(A, A) \oplus D^{1,-2}(A, X) \oplus D^{1,-2}(A, Y) \oplus C^{0,-2}(A; X, Y)$. A deformation of the module morphism i_0 in our case then corresponds to a cocycle $(0, \mu_1^X, 0, i_1)$. The cocycle must satisfy

$$d^I(0, \mu_1^X, 0, i_1) = (0,\ d^D\mu_1^X,\ 0,\ i_{0*}\mu_1^X - d^H i_1) = (0, 0, 0, 0).$$

The equation $d^D\mu_1^X = 0$ gives us no further restrictions. We consider the equation

$$(i_{0*}\mu_1^X - d^H i_1)(a, x) = i_0(\mu_1^X(a, x)) - a\, i_1(x) + i_1(ax) = 0$$

for $a \in A$ and $x \in X$. Using this equation and the value tables above we obtain only one nontrivial equation,

$$i_0(\mu_1^X(p_M, p_L)) - p_M\, i_1(p_L) + i_1(p_M\, p_L) = \delta_2 p_M - \zeta p_M = 0.$$

This implies $\delta_2 = \zeta$. The coboundaries are given by $d^I(\phi_1, \lambda_1^X, \lambda_1^Y, 0)$. Since $\eta_1 = 0$ we choose $\phi_1 = 0$ and $\lambda_1^Y = 0$. For degree reasons $\lambda_1^X = 0$. Therefore $HI^{1,-2}(A; X, A)$ has rank 2 for deformations of the module morphism i_0 with $\eta_1 = 0$.

Coming back to our example for the equator $S^1 \subset \mathbb{C}P^1$, one can show (see [15]) that $\zeta = 1$, $\delta_1 = \delta_2 = 1$, i.e. the quantum terms are given by

$$p_M \circledast 1_L = 1_L t, \quad p_M \circledast p_L = p_L t, \quad i_L(p_L) = p_M + 1_M t.$$

Part II.

The Lagrangian cubic equation

3. Main results and Floer theory setting

3.1. Main results

Let M^{2n} be a closed symplectic manifold and $L^n \subset M^{2n}$ a Lagrangian submanifold, where $n = dimL = \frac{1}{2}\dim M$. Denote by $HF_*(L, L)$ the self Floer homology of L with coefficients in $\mathbb{Z}$. See §3.2 for the Floer theoretical setting. In what follows we will recurringly appeal to the following set of assumptions or to a subset of it:

Assumption $\mathscr{L}$.

1. L is closed (i.e. compact without boundary). Furthermore L is monotone with minimal Maslov number N_L that satisfies $N_L \mid n$ (see §3.2.1 for the definitions). Set $\nu = n/N_L$.
2. L is oriented. Moreover we assume that L is spinable (i.e. can be endowed with a spin structure).
3. $HF_n(L, L)$ has rank 2.
4. Write $\chi = \chi(L)$ for the Euler-characteristic of L. We assume that $\chi \neq 0$.

Note that conditions (1) and (2) together imply that $n =$ even, since orientable Lagrangians have $N_L =$ even. Independently conditions (2) and (4) also imply that $n =$even. As we will see later there are many Lagrangian submanifolds that satisfy Assumption $\mathscr{L}$. For example, even dimensional Lagrangian spheres which satisfy condition (1) of Assumption $\mathscr{L}$ automatically satisfy the other three conditions. See §3.1.3 and §5.1 for more examples.

3.1.1. The Lagrangian cubic equation Denote by $QH(M)$ the quantum homology of M with coefficients in the ring $\mathbb{Q}[q]$, where the degree of the variable q is $|q| = -2$. We denote by $*$ the quantum product on $QH(M)$ and for a class $a \in QH(M)$, $k \in \mathbb{N}$, we write a^{*k} for the k'th power of a with respect to this product. Given an oriented Lagrangian submanifold $L \subset M$ denote by $[L] \in QH_n(M)$ its homology class in the quantum homology of the ambient manifold M. We will also make use of the notation $\varepsilon = (-1)^{n(n-1)/2}$. Our first result is the following.

Theorem 1 (The Lagrangian cubic equation)**.** *Let $L \subset M$ be a Lagrangian submanifold satisfying assumption $\mathscr{L}$. Then there exist unique rational constants $\sigma_L \in \frac{1}{\chi^2}\mathbb{Z}$, $\tau_L \in \frac{1}{\chi^3}\mathbb{Z}$ such that the following equation holds in $QH(M)$:*

$$[L]^{*3} - \varepsilon\chi\sigma_L[L]^{*2}q^{n/2} - \chi^2\tau_L[L]q^n = 0. \tag{3.1}$$

If χ is square-free, then $\sigma_L \in \frac{1}{\chi}\mathbb{Z}$, $\tau_L \in \frac{1}{\chi^2}\mathbb{Z}$. Moreover, the constant σ_L can be expressed in terms of genus 0 Gromov-Witten invariants as follows:

$$\sigma_L = \frac{1}{\chi^2}\sum_A GW^M_{A,3}([L],[L],[L]), \tag{3.2}$$

where the sum is taken over all classes $A \in H_2(M)$ with $\langle c_1, A\rangle = n/2$.

In §4.1 we will prove a more general result concerning a Lagrangian submanifold L and an arbitrary class $C \in QH_n(M)$ which satisfies $C \cdot [L] \neq 0$. We will prove that they satisfy a mixed equation of order three involving $[L]$ and C. Equation (3.1) is the special case $C = [L]$.

Here is an immediate corollary of Theorem 1:

Corollary 2. *Let $L \subset M$ be a Lagrangian submanifold satisfying assumption $\mathscr{L}$. Assume in addition that there exists a symplectic diffeomorphism $\varphi : M \longrightarrow M$ such that $\varphi_*([L]) = -[L]$. Then $\sigma_L = 0$, hence equation (3.1) reads in this case:*

$$[L]^{*3} - \chi^2\tau_L[L]q^n = 0.$$

In particular, when L is a Lagrangian sphere, by taking φ to be the Dehn twist along L we obtain $\sigma_L = 0$.

Proof of Corollary 2. Applying φ_* to the equation (3.1) and comparing the result to (3.1) yields $\varepsilon\chi\sigma_L[L]^{*2} = 0$. Since $\chi \neq 0$ it follows that $\sigma_L[L]^{*2} = 0$. But $[L] \cdot [L] = \varepsilon\chi \neq 0$, hence $[L]^{*2} \neq 0$. This implies that $\sigma_L = 0$.

Turning to the case when L is a Lagrangian sphere, let φ be the Dehn twist along L. The Picard-Lefschetz formula (see e.g. [27, 1]) gives $\varphi_*([L]) = -[L]$ since $n = \dim L$ is even and $\chi = 2$. $\square$

3.1.2. The discriminant Let A be a quadratic algebra over $\mathbb{Z}$. By this we mean that A is a commutative unital ring such that $\mathbb{Z}$ embeds as a subring of A, $\mathbb{Z} \hookrightarrow A$, and furthermore that $A/\mathbb{Z} \cong \mathbb{Z}$. Thus the underlying additive abelian group of A is a free abelian group of rank 2. Pick a generator $p \in A/\mathbb{Z}$ so that $A/\mathbb{Z} = \mathbb{Z}p$. We have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \xrightarrow{\epsilon} \mathbb{Z}p \longrightarrow 0, \tag{3.3}$$

where the first map is the ring embedding and ϵ is the obvious projection. Choose a lift $x \in A$ of p, i.e. $\epsilon(x) = p$. Then additively we have $A \cong \mathbb{Z}x \oplus \mathbb{Z}$. With these choices there exist $\sigma(p, x), \tau(p, x) \in \mathbb{Z}$ such that

$$x^2 = \sigma(p, x)x + \tau(p, x).$$

The integers $\sigma(p, x), \tau(p, x)$ depend on the choices of p and of x. However, a simple calculation (see §3.2.5) shows that the following expression

$$\Delta_A := \sigma(p, x)^2 + 4\tau(p, x) \in \mathbb{Z} \tag{3.4}$$

is independent of p and x, hence is an invariant of the isomorphism type of A. We call Δ_A the discriminant of A.

Remarks. 1. Another description of Δ_A is the following. Write A as $A \cong \mathbb{Z}[T]/(f(T))$, where $f(T) \in \mathbb{Z}[T]$ is a monic quadratic polynomial. Then Δ_A is the discriminant of $f(T)$ (and is independent of the choice of $f(T)$). In particular $A_{\mathbb{C}} := A \otimes \mathbb{C}$ is semi-simple iff $\Delta_A \neq 0$.

2. When Δ_A is not a square, $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$ is a quadratic number field. The discriminant Δ_A is related, but not necessarily equal to the discriminant of $A_{\mathbb{Q}}$ as defined in number theory.

Let L be a Lagrangian submanifold satisfying conditions $(1) - (3)$ of Assumption $\mathscr{L}$ and choose a spin structure on L compatible with its orientation. Consider $A = HF_n(L, L)$ endowed with the Donaldson product

$$* : HF_n(L, L) \otimes HF_n(L, L) \longrightarrow HF_n(L, L), \quad a \otimes b \longmapsto a * b.$$

Recall that A is a unital ring with a unit which we denote by $e_L \in HF_n(L, L)$. The conditions $(1) - (3)$ of Assumption $\mathscr{L}$ ensure that A is a quadratic algebra over $\mathbb{Z}$. (In case A has torsion we just replace it by A/T, where T is its torsion ideal.) Denote by Δ_L the discriminant of A, $\Delta_L := \Delta_A$ as defined in (3.4). (We suppress here the dependence on the spin structure, as we will soon see that in our case Δ_L does not depend on it.)

The following theorem shows that the discriminant Δ_L depends only on the class $[L] \in QH_n(M)$ and can be computed by means of the ambient quantum homology of M.

Theorem 3. *Let $L \subset M$ be a Lagrangian submanifold satisfying Assumption $\mathscr{L}$. Let $\sigma_L, \tau_L \in \mathbb{Q}$ be the constants from the cubic equation* (3.1) *in Theorem 1. Then*

$$\Delta_L = \sigma_L^2 + 4\tau_L.$$

The proof appears in §4.1.

Remarks. 1. **Warning:** The pair of coefficients σ_L, τ_L and $\sigma(p, x)$, $\tau(p, x)$ should not be confused. The first pair is always uniquely determined by $[L]$ and can be read off the ambient quantum homology of M via the cubic equation (3.1). In contrast, the second pair $\sigma(p,x), \tau(p,x)$ is defined via Lagrangian Floer homology and strongly depend on the choice of the lift x of p. For example, we have seen that if L is a sphere then $\sigma_L = 0$, but as we will see later (e.g. in §4.2) for some (useful) choices of x we have $\sigma(p,x) \neq 0$. Apart from that, $\sigma(p,x), \tau(p,x) \in \mathbb{Z}$ while $\sigma_L, \tau_L \in \mathbb{Q}$. Still, the two pairs of coefficients are related in that $\sigma(p,x)^2 + 4\tau(p,x) = \sigma_L^2 + 4\tau_L = \Delta_L$.

As we will see in the proof of Theorem 3, the coefficients σ_L, τ_L do occur as $\sigma(p,x_0), \tau(p,x_0)$ but for a special choice of x_0, which however requires working over $\mathbb{Q}$.

2. A different version of the discriminant Δ_L was previously defined and studied by Biran-Cornea in [16]. In that paper the discriminant occurs as an invariant of a quadratic form defined on $H_{n-1}(L)$ via Floer theory, as recalled in §1.2.2. In the case L is a 2-dimensional Lagrangian torus the discriminant from [16] and Δ_L, as defined above, happen to coincide due to the associativity of the product of $HF_n(L,L)$. Moreover, in dimension 2, Δ_L has an enumerative description in terms of counting holomorphic disks with boundary on L which satisfy certain incidence conditions, as mentioned in the introduction. This description continues to hold also for 2-dimensional Lagrangian spheres with $N_L = 2$ (or more generally for all 2-dimensional Lagrangian submanifolds satisfying Assumption $\mathscr{L}$) and the proof is the same as in [16].

3. Since σ_L, τ_L do not depend on the spin structure chosen for L it follows from Theorem 3 that Δ_L does not depend on that choice either. As for the orientation on L, if we denote by $\bar{L}$ the Lagrangian L with the opposite orientation, then it follows from Theorem 1 that $\sigma_{\bar{L}} = -\sigma_L$ and $\tau_{\bar{L}} = \tau_L$. In particular $\Delta_{\bar{L}} = \Delta_L$.

The next theorem is concerned with the behavior of the discriminant under Lagrangian cobordism. We refer the reader to [17] for the definitions.

Theorem 4. *Let $L_1, \dots, L_r \subset M$ be monotone Lagrangian submanifolds, each satisfying conditions* (1) *–* (3) *of Assumption $\mathscr{L}$. Let $V^{n+1} \subset \mathbb{R}^2 \times M$ be a connected monotone Lagrangian cobordism whose ends correspond to $L_1, \dots, L_r$ and assume that V admits a spin structure. Denote by N_V the minimal Maslov number of V and assume that:*

1. $N_V | n$.
2. $H_{jN_V}(V, \partial V) = 0$ *for every* j.
3. $H_{1+jN_V}(V) = 0$ *for every* j.

Then $\Delta_{L_1} = \cdots = \Delta_{L_r}$. Moreover if $r \geq 3$ then Δ_{L_i} is a perfect square for every i.

The proof is given in §4.2. As a corollary we obtain:

Corollary 5. *Let (M, ω) be a monotone symplectic manifold with $2C_M \mid n$, where C_M is the minimal Chern number of M. Let $L_1, L_2 \subset M$ be two Lagrangian spheres that intersect transversely at exactly one point. Then $\Delta_{L_1} = \Delta_{L_2}$ and moreover this number is a perfect square.*

We will in fact prove a stronger result in §4.2.1 (see Corollary 4.2.3).

3.1.3. Examples We begin with a topological criterion that assures that condition (3) in Assumption $\mathscr{L}$ is satisfied.

Proposition 6. *Let $L \subset M$ be an oriented Lagrangian submanifold satisfying condition* (1) *of Assumption $\mathscr{L}$. Assume in addition that:*

1. $[L] \neq 0 \in H_n(M; \mathbb{Q})$ *(this is satisfied e.g. when $\chi(L) \neq 0$).*
2. $H_{jN_L}(L) = 0$ *for every* $0 < j < \nu$.

Then condition (3) *in Assumption $\mathscr{L}$ is satisfied too. In particular Lagrangian spheres L that satisfy condition* (1) *of Assumption $\mathscr{L}$ satisfy the other three conditions in assumption $\mathscr{L}$.*

The proof appears in §3.2.3.

We now provide a sample of examples. More details will be given in §5.1

Lagrangian spheres in blow-ups of $\mathbb{C}P^2$ Let (M_k, ω_k) be the monotone symplectic blow-up of $\mathbb{C}P^2$ at $2 \leq k \leq 6$ points. We normalize ω_k so that it is cohomologous to c_1. Denote by $H \in H_2(M_k)$ the homology class of a line not passing through the blown up points and by $E_1, \ldots, E_k \in H_2(M_k)$ the homology classes of the exceptional divisors over the blown up points. With this notation the Poincaré dual of the cohomology class of the symplectic form $[\omega_k] \in H^2(M_k)$ satisfies

$$PD[\omega_k] = PD(c_1) = 3H - E_1 - \cdots - E_k.$$

The Lagrangian spheres $L \subset M_k$ lie in the following homology classes (see §5.1.1 for more details):

1. For $k = 2$: $\pm(E_1 - E_2)$.
2. For $3 \leq k \leq 5$: $\pm(E_i - E_j)$, $i < j$, and $\pm(H - E_i - E_j - E_l)$ with $i < j < l$.
3. For $k = 6$ we have the same homology classes as in (2) and in addition the classes $\pm(2H - E_1 - \cdots - E_6)$.

Note that all these Lagrangian spheres satisfy Assumption $\mathscr{L}$ since $N_L = 2$.

The discriminants of these Lagrangian spheres are gathered in Table 3.1, the detailed computations being postponed to §5.1. The column under λ_L will be explained in §3.2.4.

The Lagrangian spheres in the three homology classes $E_i - E_j$, $i < j$, of M_3 all have the same discriminant. This can also be seen by noting that one can choose three Lagrangian spheres L_1, L_2, L_3, one in each of these homology classes, so that every pair of them intersects transversely at exactly one point. The equality of their

	$[L]$	Δ_L	λ_L
M_2	$\pm(E_1 - E_2)$	5	-1
M_3	$\pm(E_i - E_j)$	4	-2
	$\pm(H - E_1 - E_2 - E_3)$	-3	-3
M_4	$\pm(E_i - E_j)$	1	-3
	$\pm(H - E_i - E_j - E_l)$	1	-3
M_5	$\pm(E_i - E_j)$	0	-4
	$\pm(H - E_i - E_j - E_l)$	0	-4
M_6	$\pm(E_i - E_j)$	0	-6
	$\pm(H - E_i - E_j - E_l)$	0	-6
	$\pm(2H - E_1 - \ldots - E_6)$	0	-6

Table 3.1.: Classes representing Lagrangian spheres and their discriminants.

discriminants (as well as the fact that they are perfect squares) follows then by Corollary 5. We elaborate more on these examples in §5.1.

Lagrangian spheres in Fano hypersurfaces Let $M^{2n} \subset \mathbb{C}P^{n+1}$ be a Fano hypersurface of degree $d \leq n+1$ endowed with the induced symplectic form. Note that when $d \geq 2$, M contains Lagrangian spheres. The minimal Chern number C_M is $n + 2 - d$. If n is a multiple of $2C_M = 2(n+2-d)$ then the Lagrangian spheres $L \subset M$ satisfy Assumption $\mathscr{L}$, hence the discriminant Δ_L is defined. Using the description of the quantum homology of a Fano hypersurface [23, 38] we obtain $\Delta_L = 0$. More details on this calculation are given in §5.1.

3.2. Floer theory setting

3.2.1. Monotone symplectic manifolds and Lagrangians We briefly recall here some ingredients from Floer theory that are relevant for this part. These include Lagrangian Floer homology and especially its realization as Lagrangian quantum homology (a.k.a pearl homology). The reader is referred to [57, 59, 31, 15, 16] for more details.

Let (M, ω) be a symplectic manifold. Denote by $c_1 \in H^2(M)$ the first Chern class of the tangent bundle $T(M)$ of M. Denote by $H_2^S(M)$ the image of the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M)$. We say that (M, ω) is monotone if there exists a constant $\kappa > 0$ such that

$$A_\omega = \kappa I_{c_1},$$

where $A_\omega : H_2^S(M) \longrightarrow \mathbb{R}$ is the homomorphism defined by integrating ω over spherical classes and I_{c_1} is viewed as a homomorphism $H_2^S(M) \longrightarrow \mathbb{Z}$. We denote by C_M the positive generator of the subgroup image $(I_{c_1}) \subset \mathbb{Z}$ so that image $(I_{c_1}) = C_M\mathbb{Z}$. If image $(I_{c_1}) = 0$ we put $C_M = \infty$.

$L \subset M$ a Lagrangian submanifold. Denote by $H_2^D(M, L)$ the image of the Hurewicz homomorphism $\pi_2(M, L) \longrightarrow H_2(M, L)$. We say that L is monotone if there exists a constant $\rho > 0$ such that

$$A_\omega = \rho\mu,$$

where $A_\omega : H_2^D(M, L) \longrightarrow \mathbb{R}$ is the homomorphism defined by integrating ω over homology classes and $\mu : H_2^D(M, L) \longrightarrow \mathbb{Z}$ is the Maslov index homomorphism. We denote by N_L the positive generator of the subgroup image $(\mu) \subset \mathbb{Z}$ so that image $(\mu) = N_L\mathbb{Z}$.

Finally, denote by $j : H_2^S(M) \longrightarrow H_2^D(M, L)$ the obvious homomorphism. Then we have $\mu(j(A)) = 2I_{c_1}(A)$ for every $A \in H_2^S(M)$. Therefore, if L is a monotone Lagrangian and $I_{c_1} \neq 0$, then (M, ω) is also monotone and we have $N_L \mid 2C_M$. When $\pi_1(L) = \{1\}$ we actually have $N_L = 2C_M$.

3.2.2. Floer homology and Lagrangian quantum homology

Let $L \subset M$ be a closed monotone Lagrangian submanifold with $2 \leq N_L \leq \infty$. Under the additional assumptions that L is spin one can define the self Floer homology $HF(L, L)$ with coefficients in $\mathbb{Z}$. This group is cyclically graded, with grading in $\mathbb{Z}/N_L\mathbb{Z}$.

From the point of view of the present paper it is more natural to work with the Lagrangian quantum homology $QH(L)$ rather than with the Floer homology $HF(L, L)$. This is justified by the fact that for an appropriate choice of coefficients we have an isomorphism of rings $QH(L) \cong HF(L, L)$. The advantage of $QH(L)$ in our context is that it bears a simple and explicit relation to the singular homology $H(L)$ of L. For example, under certain circumstances (relevant for our considerations) and with the right coefficient ring, $QH(L)$ can be viewed as a deformation of the singular homology ring $H(L)$ endowed with the intersection product.

We will now summarize the most basic properties of Lagrangian quantum homology. The reader is referred to [15, 16] for the foundations of the theory.

Denote by $\Lambda = \mathbb{Z}[t^{-1}, t]$ the ring of Laurent polynomials over $\mathbb{Z}$, graded so that the degree of t is $|t| = -N_L$. We denote by $QH^{\#}(L)$ the Lagrangian quantum homology of L with coefficients in $\mathbb{Z}$ and by $QH(L; \Lambda)$ the one with coefficients in Λ. Thus $QH^{\#}(L)$ is cyclically graded modulo N_L and $QH(L; \Lambda)$ is $\mathbb{Z}$-graded and N_L-periodic, i.e. $QH_i(L; \Lambda) \cong QH_{i-N_L}(L; \Lambda)$, the isomorphism being given by multiplication with t. And we have $QH_i(L; \Lambda) \cong QH^{\#}_{i \pmod{N_L}}(L)$, hence the grading on $QH(L; \Lambda)$ is an unwrapping of the cyclic grading of $QH^{\#}(L)$. Sometimes, when the context is clear, we will write $QH(L)$ for $QH(L; \Lambda)$.

The Lagrangian quantum homology has the following algebraic structures. There exists a quantum product

$$QH_i(L; \Lambda) \otimes QH_j(L; \Lambda) \longrightarrow QH_{i+j-n}(L; \Lambda), \quad \alpha \otimes \beta \longmapsto \alpha * \beta,$$

which turns $QH(L; \Lambda)$ into a unital associative ring with unity $e_L \in QH_n(L; \Lambda)$.

We now briefly recall relations between the Lagrangian and ambient quantum homologies. Denote by $R = \mathbb{Z}[q^{-1}, q]$ the ring of Laurent polynomials in the variable q, whose degree we set to be

$|q| = -2$. Denote by $QH(M;R)$ the quantum homology of M with coefficients in R, endowed with the quantum product $*$. The Lagrangian quantum homology $QH(L;\Lambda)$ is a module over the subring $QH(M;\Lambda) \subset QH(M;R)$, where Λ is embedded in R by $t \mapsto q^{N_L/2}$. We denote this operation by

$$QH_i(M;\Lambda) \otimes QH_j(L;\Lambda) \longrightarrow QH_{i+j-2n}(L;\Lambda), \quad a \otimes \alpha \longmapsto a * \alpha.$$

The reason for using the same notation $*$ as for the quantum product on L is that the module operation is compatible with the latter in the following sense:

$$c * (\alpha * \beta) = (c * \alpha) * \beta = \alpha * (c * \beta), \tag{3.5}$$

for all $c \in QH_{\text{even}}(M;\Lambda)$ and $\alpha, \beta \in QH(L;\Lambda)$. Put in other words, $QH(L;\Lambda)$ is an algebra over $QH_{\text{even}}(M;\Lambda)$. (A similar relation holds when $|c|$ is odd but we will not need it here.)

There is also a quantum inclusion map

$$i_L : QH_i(L;\Lambda) \longrightarrow QH_i(M;\Lambda),$$

which is linear over the ring $QH(M;\Lambda)$, i.e. $i_L(c*\alpha) = c * i_L(\alpha)$ for every $c \in QH(M;\Lambda)$ and $\alpha \in QH(L;\Lambda)$. An important property of i_L is that $i_L(e_L) = [L]$, see [16].

Next there is an augmentation morphism

$$\epsilon_L : QH(L;\Lambda) \longrightarrow \Lambda,$$

which is induced from a chain level extension of the classical augmentation. The augmentation satisfies the following identity:

$$\langle PD(h), i_L(\alpha)\rangle = \epsilon_L(h * \alpha), \quad \forall h \in H_*(M),\ \alpha \in QH(L;\Lambda), \tag{3.6}$$

where PD stands for Poincaré duality and $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing extended over Λ in an obvious way. Sometimes it will be more convenient to view the augmentation as a map

$$\widetilde{\epsilon}_L : QH(L;\Lambda) \longrightarrow H_0(L;\Lambda) = \Lambda[\text{point}].$$

These augmentations ϵ_L and $\widetilde{\epsilon}_L$ descend also to $QH^{\#}(L)$ and by slight abuse of notation we denote them in the same way:

$$\epsilon_L : QH^{\#}(L) \longrightarrow \mathbb{Z}, \quad \widetilde{\epsilon}_L : QH^{\#}(L) \longrightarrow H_0(L).$$

As mentioned earlier we will not really use Floer homology in this paper, but Lagrangian quantum homology instead. The justification for replacing $HF(L, L)$ by $QH^{\#}(L)$ is due to the PSS isomorphism

$$PSS : HF_*(L, L) \longrightarrow QH^{\#}_*(L).$$

This is a ring isomorphism which intertwines the Donaldson product and the quantum product on $QH^{\#}(L)$. A version of PSS works with coefficients in Λ too. For more details on the PSS isomorphism see [2, 8, 24, 15]. See also [41, 42] for the extension to $\mathbb{Z}$-coefficients.

Finally, we remark that everything mentioned above in this section continues to hold (with obvious modifications) also with other choices of base rings, replacing $\mathbb{Z}$ by $\mathbb{Q}$ or $\mathbb{C}$. For $K = \mathbb{Q}$ or $\mathbb{C}$ we write $\Lambda_K = K[t^{-1}, t]$, $R_K = K[q^{-1}, q]$ for the associated rings of Laurent polynomials and $HF(L, L; \Lambda_K)$, $QH(L; \Lambda_K)$ and $QH(M; R_K)$ for the corresponding homologies. Sometimes it will be useful to drop the Laurent polynomial rings Λ_K and R_K and simply work with $HF(L, L; K)$, $QH(L; K)$ and $QH(M; K)$. Another variation that will be used in the sequel is to replace Λ_K and R_K by polynomial rings (rather than Laurent polynomials), i.e. work with coefficients in $\Lambda^+_K = K[t]$ and $R^+_K = K[q]$. See [14, 15, 16] for a detailed account on this choice of coefficients. When the base ring K is obvious we will abbreviate $Q^+H(L) := QH(L; \Lambda^+_K)$ and similarly for $Q^+H(M)$. (There has been only one exception to this notation. In the introduction §3.1 we denoted by $QH(M)$ the quantum homology $QH(M; R^+)$ in order to facilitate the notation, but henceforth we will stick to the notation we have just described.) The homologies of the type Q^+H will be called *positive quantum homologies*. Again, everything described above continues to work for the positive versions of quantum homologies with one important exception: the PSS isomorphism does not hold over Λ^+_K (at least not for a straightforward version of Floer homology).

3.2.3. Proof of Proposition 6 By a spectral sequence argument (see [59, 12, 14, 15]) it easily follows that the dimension of $QH_n(L; \Lambda_{\mathbb{Q}})$ is at most 2. We will now show that the dimension of this vector space is exactly 2.

We first claim that the unity is not trivial, $e_L \neq 0 \in QH_n(L; \Lambda_{\mathbb{Q}})$. To see this consider the quantum inclusion map $i_L : QH_n(L; \Lambda_{\mathbb{Q}}) \longrightarrow QH_n(M; R_{\mathbb{Q}})$ from §3.2.2. It is well known [16] that $i_L(e_L) = [L]$. As $[L] \neq 0$ it follows that $e_L \neq 0$.

By Poincaré duality there exists a class $c \in H_n(M; \mathbb{Q})$ such that $c \cdot [L] \neq 0$. Put $x := c * e_L \in QH_0(L; \Lambda_{\mathbb{Q}})$. From (3.6) we get that $\epsilon_L(x) \neq 0$. This implies that the two elements $xt^{-\nu}, e_L \in QH_n(L; \Lambda_{\mathbb{Q}})$ are linearly independent. From this it follows that $\dim QH_n(L; \Lambda_{\mathbb{Q}}) = 2$.

From the above it now follows that the rank of $QH_n^{\#}(L)$ is 2. Finally, from the PSS isomorphism we obtain that $HF_n(L, L)$ has rank 2. $\square$

3.2.4. Eigenvalues of c_1 and Lagrangian submanifolds Let $L \subset M$ be a closed spin monotone Lagrangian submanifold with $QH(L; \mathbb{C}) \neq 0$. Assume in addition that $N_L = 2$. With these assumptions one can define an invariant $\lambda_L \in \mathbb{Z}$ which counts the number of Maslov-2 pseudo-holomorphic disks $u : (D, \partial D) \longrightarrow (M, L)$ whose boundary $u(\partial D)$ pass through a generic point $p \in L$, see §1.2.2. The value of λ_L turns out to be independent of the almost complex structure as well as of the generic point p. See [16] for more details. We extend the definition of λ_L to the case $N_L > 2$ by setting $\lambda_L = 0$.

Consider now the following operator

$$P : QH(L; \Lambda_{\mathbb{C}}) \longrightarrow QH(L; \Lambda_{\mathbb{C}}), \quad \alpha \longmapsto PD(c_1) * \alpha,$$

where PD stands for Poincaré duality. By abuse of notation we have denoted here by $c_1 \in H^2(M; \mathbb{C})$ the image of the first Chern class of TM under the change of coefficients map $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{C})$.

The following is well known:

1. If $N_L = 2$, then $P(\alpha) = \lambda_L \alpha t$ for every $\alpha \in QH(L; \Lambda_{\mathbb{C}})$.
2. If $N_L > 2$, then $P \equiv 0$.

For the proof of (1), see [4] for a special case (where the statement is attributed to folklore, in particular also to Kontsevich and to Seidel) and [63] for the general case. As for (2), it follows immediately from the fact that the restriction of c_1 to L vanishes, $c_1|_L = 0 \in H^2(L; \mathbb{C})$, together with degree reasons.

Denote by $\mathcal{I}_L \subset QH(M; R_{\mathbb{C}})$ the image of the quantum inclusion map $i_L : QH(L; \Lambda_{\mathbb{C}}) \longrightarrow QH(M; R_{\mathbb{C}})$. Note that $\mathcal{I}_L$ is an ideal of the ring $QH(M; R_{\mathbb{C}})$.

Proposition 3.2.1. *$\mathcal{I}_L \neq 0$ iff $QH(L; \Lambda_{\mathbb{C}}) \neq 0$ and in this case λ_L is an eigenvalue of the operator*

$$Q : QH(M; R_{\mathbb{C}}) \longrightarrow QH(M; R_{\mathbb{C}}), \quad a \longmapsto PD(c_1) * aq^{-1}.$$

Moreover, $\mathcal{I}_L$ is a subspace of the eigenspace of Q corresponding to the eigenvalue λ_L. In particular if $[L] \neq 0 \in H_n(M; \mathbb{C})$, then $[L]$ is an eigenvector of Q corresponding to λ_L.

Remark 3.2.2. Denote by $Q' : QH(M; \mathbb{C}) \longrightarrow QH(M; \mathbb{C})$ the same operator as Q but acting on $QH(M; \mathbb{C})$ instead of $QH(M; R_{\mathbb{C}})$. Similarly, denote by $\mathcal{I}'_L \subset QH(M; \mathbb{C})$ the image of i_L. The statement of Proposition 3.2.1 continues to hold for Q' and $\mathcal{I}'_L$. Moreover, if $[L] \neq 0$ then

$$\dim_{\mathbb{C}} \mathcal{I}'_L \geq 2,$$

hence the multiplicity of the eigenvalue λ_L with respect to the operator Q' is at least 2. Indeed, $[L] = i_L(e_L) \in \mathcal{I}'_L$. Now take $c \in H_n(M; \mathbb{C})$ with $c \cdot [L] \neq 0$. As $\mathcal{I}'_L$ is an ideal we have $c * [L] \in \mathcal{I}'_L$. But $c * [L] = \#(c \cdot [L])[\text{point}] + (\text{other terms})$, hence $c * [L]$ is not proportional to $[L]$. (Here $\#(c \cdot [L])$ stands for the intersection number of c and $[L]$.)

Proof of Proposition 3.2.1. Assume that $QH(L; \Lambda_{\mathbb{C}}) \neq 0$. By duality for Lagrangian quantum homology there exists $x \in QH_0(L; \Lambda_{\mathbb{C}})$

with $\epsilon_L(x) \neq 0$. (See [15], Proposition 4.4.1. The proof there is done over $\mathbb{Z}_2$ but the extension to any field is straightforward in view of [16]).

From (3.6) (with $h = [M]$ and $\alpha = x$) it follows that $i_L(x) \neq 0$, hence $\mathcal{I}_L \neq 0$. The opposite assertion is obvious.

The statement about the eigenspace of Q follows immediately from the discussion about the operator P and the fact that i_L is a $QH(M; R_{\mathbb{C}})$-module map.

Finally, note that $[L] \in \mathcal{I}_L$ since $[L] = i_L(e_L)$. $\square$

The following observation shows that the eigenvalues corresponding to different Lagrangians coincide under certain circumstances.

Proposition 3.2.3. *Let $L, L' \subset M$ be two closed monotone spin Lagrangian submanifolds. Assume that $[L]\cdot[L]' \neq 0$. Then $\lambda_L = \lambda_{L'}$.*

Proof. We view $[L], [L']$ as elements of $QH_n(M; \mathbb{C})$. We have

$$\mathrm{PD}(c_1) * ([L] * [L']) = (\mathrm{PD}(c_1) * [L]) * [L'] = \lambda_L [L] * [L'].$$

At the same time, since $|\mathrm{PD}(c_1)| = \text{even}$ we also have

$$\mathrm{PD}(c_1) * ([L] * [L']) = [L] * (\mathrm{PD}(c_1) * [L']) = \lambda_{L'} [L] * [L'].$$

Since $[L] \cdot [L'] \neq 0$ we have $[L] * [L'] \neq 0$ and the result follows. $\square$

3.2.5. More on the discriminant

Well-definedness We start with showing that the discriminant, as defined in §3.1.2, is independent of the choices of p and x. We first fix p and show independence of its lift x. Indeed if y is another lift of p then $y = x + r$ for some $r \in \mathbb{Z}$. A straightforward calculation shows that

$$\sigma(p, y) = \sigma(p, x) + 2r, \quad \tau(p, y) = \tau(p, x) - \sigma(p, x)r - r^2.$$

Another direct calculation shows that

$$\sigma(p, y)^2 + 4\tau(p, y) = \sigma(p, x)^2 + 4\tau(p, x).$$

Assume now that $p' \in A/\mathbb{Z}$ is a different generator. We then have $p' = -p$ and so we can choose $x' = -x$ as a lift of p'. It easily follows that

$$\sigma(p', x') = -\sigma(p, x), \quad \tau(p', x') = \tau(p, x),$$

hence again $\sigma(p', x')^2 + 4\tau(p', x') = \sigma(p, x)^2 + 4\tau(p, x)$. □

A useful extension over other rings Let A be a quadratic algebra over $\mathbb{Z}$ as described in §3.1.2. Let K be a commutative ring which extends $\mathbb{Z}$, i.e. we have $\mathbb{Z} \subset K$ as a subring. For simplicity we will assume that K is torsion free. We will mainly consider $K = \mathbb{Q}$ or $K = \mathbb{C}$. Write $A_K = A \otimes K$.

For practical purposes it will be sometimes useful to calculate Δ_A using A_K rather than via A itself. This can be done as follows. From the sequence (3.3) we obtain the following exact sequence:

$$0 \longrightarrow K \longrightarrow A_K \xrightarrow{\epsilon} Kp \longrightarrow 0, \tag{3.7}$$

where as before, ϵ is the projection to the quotient and p stands for a generator of $A/\mathbb{Z} \subset A_K/K$. Pick a lift $x \in A_K$ of p and define $\sigma(p, x), \tau(p, x)$ by the same recipe as in §3.1.2, only that now these two numbers belong to K rather than to $\mathbb{Z}$. A simple calculation, similar to §3.2.5 above shows that we still have $\Delta_A = \sigma(p, x)^2 + 4\tau(p, x)$ (and of course despite the calculation being done in K we still have $\Delta_A \in \mathbb{Z}$).

Remark 3.2.4. It is essential here that the generator p is integral, i.e. that $p \in A_K$ was chosen to come from A. If we allow to replace p by any non-trivial element of A_K/K then the corresponding discriminant will depend on that choice, but not on the choice of the lift x. In fact, if $p' = cp$, $c \in K$, then the discriminants corresponding to p' and p are related by $\Delta(p') = c^2\Delta(p)$. Therefore, when $K = \mathbb{Q}$ for example, the sign of the discriminant is an invariant of $A_{\mathbb{Q}}$. In fact it determines whether $A_{\mathbb{Q}}$ and $A_{\mathbb{C}}$ are semi-simple algebras.

The case of $A = QH_n^{\#}(L)$ Let $L \subset M$ be a Lagrangian submanifold satisfying conditions (1) – (3) of Assumption $\mathscr{L}$. Fix a spin structure on L. Denote by $e_L \in QH_n^{\#}(L)$ the unity. Without loss of generality we may assume that $QH_n^{\#}(L)$ is torsion-free, otherwise we just replace it by $QH_n^{\#}(L)/T$, where T is the torsion ideal. Thus $QH_n^{\#}(L)$ is a quadratic algebra over $\mathbb{Z}$.

By duality for Lagrangian quantum homology [15, 16], the augmentation $\widetilde{\epsilon}_L : QH_0^{\#}(L) \longrightarrow H_0(L;\mathbb{Z})$ is surjective. Keeping in mind that in our case $QH_0^{\#}(L) = QH_n^{\#}(L)$ (since $N_L \mid n$) we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{Z}e_L \longrightarrow QH_n^{\#}(L) \xrightarrow{\widetilde{\epsilon}_L} H_0(L;\mathbb{Z}) \longrightarrow 0.$$

Let K be a torsion-free commutative ring that contains $\mathbb{Z}$. Let $p =$ [point] $\in H_0(L;\mathbb{Z})$ be the homology class of a point. Tensoring the last sequence by K we obtain:

$$0 \longrightarrow Ke_L \longrightarrow QH_n^{\#}(L;K) \xrightarrow{\widetilde{\epsilon}_L} Kp \longrightarrow 0. \tag{3.8}$$

In order to calculate Δ_L, choose a lift $x \in QH_n^{\#}(L;K)$ of p with respect to $\widetilde{\epsilon}_L$. Then we have

$$x * x = \sigma(p,x)x + \tau(p,x)e_L, \tag{3.9}$$

with some $\sigma(p,x), \tau(p,x) \in K$. The discriminant can then be calculated by

$$\Delta_L = \sigma(p,x)^2 + 4\tau(p,x).$$

In the following we will need to use the equality (3.9) but in $QH_n(L;\Lambda_K)$ rather than in $QH_n^{\#}(L;K)$. We have $QH_0(L;\Lambda_K) = t^{\nu}QH_n(L;\Lambda_K)$, with $\nu = n/N_L$. The lift x of p has now to be chosen in $QH_0(L;\Lambda_K)$ and the previous equation now takes place in $QH_{-n}(L;\Lambda_K)$ and has the following form:

$$x * x = \sigma(p,x)xt^{\nu} + \tau(p,x)e_Lt^{2\nu}. \tag{3.10}$$

Finally, we mention that sometimes it is more convenient to define the discriminant using the positive Lagrangian quantum homology

$QH(L;\Lambda_K^+)$ rather than $QH(L;\Lambda_K)$. The resulting discriminant is obviously the same.

4. Proofs of main results

4.1. The Lagrangian cubic equation

We begin by proving the following result that generalizes Theorems 1 and 3.

Theorem 4.1.1. *Let $L \subset M$ be a Lagrangian submanifold satisfying conditions (1) – (3) of Assumption $\mathscr{L}$. Assume in addition that $[L] \neq 0 \in H_n(M; \mathbb{Q})$. Let $c \in H_n(M; \mathbb{Z})$ be a class satisfying $\xi := \#(c \cdot [L]) \neq 0$. Then there exist unique rational constants $\sigma_{c,L} \in \frac{1}{\xi^2}\mathbb{Z}$, $\tau_{c,L} \in \frac{1}{\xi^3}\mathbb{Z}$ such that the following equation holds in $QH(M; R^+)$:*

$$c * c * [L] - \xi \sigma_{c,L}\, c * [L] q^{n/2} - \xi^2 \tau_{c,L}\, [L] q^n = 0. \qquad (4.1)$$

The coefficients $\sigma_{c,L}, \tau_{c,L}$ are related to the discriminant of L by $\Delta_L = \sigma_{c,L}^2 + 4\tau_{c,L}$. If ξ is square-free, then $\sigma_{c,L} \in \frac{1}{\xi}\mathbb{Z}$, $\tau_{c,L} \in \frac{1}{\xi^2}\mathbb{Z}$. Moreover, $\sigma_{c,L}$ can be expressed in terms of genus 0 Gromov-Witten invariants as follows:

$$\sigma_{c,L} = \frac{1}{\xi^2} \sum_A GW_{A,3}(c, c, [L]), \qquad (4.2)$$

where the sum is taken over all classes $A \in H_2(M)$ with $\langle c_1, A \rangle = n/2$.

Proof. Fix a spin structure on L. In view of §3.2.2 we replace $HF_n(L, L; \mathbb{Q})$ by $QH_n(L; \Lambda_{\mathbb{Q}})$. By assumption, this is a 2-dimensional vector space over $\mathbb{Q}$. Recall also that $QH_0(L; \Lambda_{\mathbb{Q}}) \cong QH_n(L; \Lambda_{\mathbb{Q}})$. Put

$$x := \tfrac{1}{\xi} c * e_L \in QH_0(L; \Lambda_{\mathbb{Q}}),$$

where c is viewed here as an element of $QH_n(M;R_{\mathbb{Q}})$ and $*$ is the module operation mentioned in §3.2.2. Let $p = [\text{point}] \in H_0(L;\mathbb{Q})$ be the class of a point. We have

$$\widetilde{\epsilon}_L(x) = \tfrac{1}{\xi}\#(c \cdot [L])p = p.$$

It follows that $\{x, e_L t^\nu\}$ is a basis for $QH_0(L;\Lambda_{\mathbb{Q}})$. Following the recipe in §3.2.5 and formula (3.10) there exist $\sigma_{c,L}, \tau_{c,L} \in \mathbb{Q}$ such that

$$x * x = \sigma_{c,L} x t^\nu + \tau_{c,L} e_L t^{2\nu}, \tag{4.3}$$

where $*$ stands here for the Donaldson (or Lagrangian quantum product) on $QH(L)$.

We now apply the quantum inclusion map i_L (see §3.2.2) to both sides of (4.3). We have

$$i_L(x * x) = \tfrac{1}{\xi^2} i_L((c * e_L) * (c * e_L)) = \tfrac{1}{\xi^2} c * c * i_L(e_L) = \tfrac{1}{\xi^2} c * c * [L].$$

Here we have used properties of the operations described in §3.2.2, and in particular identity (3.5). We also have

$$i_L(x) = \tfrac{1}{\xi} c * i_L(e_L) = \tfrac{1}{\xi} c * [L].$$

Recall also that we view Λ as a subring of R via the embedding $t \longmapsto q^{N_L/2}$, so that under this embedding we have $t^\nu \longmapsto q^{n/2}$. Therefore by applying i_L to (4.3) we immediately obtain the equation claimed by the theorem. The statement on Δ_L follows at once from §3.2.5.

Next we claim that $\xi^2\sigma_{c,L}, \xi^3\tau_{c,L} \in \mathbb{Z}$ and moreover, if ξ is square-free, then in fact $\xi\sigma_{c,L}, \xi^2\tau_{c,L} \in \mathbb{Z}$. To this end we will denote Λ by $\Lambda_{\mathbb{Z}}$ to emphasize that the ground ring is $\mathbb{Z}$. To prove the claim, set $y := \xi x$ and note that $y \in QH_0(L;\Lambda_{\mathbb{Z}})$. For y we obtain the resulting equation in $QH_{-n}(L;\Lambda_{\mathbb{Z}})$ using (4.3)

$$y * y = \xi\sigma_{c,L} y t^\nu + \xi^2\tau_{c,L} e_L t^{2\nu}. \tag{4.4}$$

We apply the augmentation morphism $\epsilon_L : QH(L;\Lambda_{\mathbb{Z}}) \to \Lambda_{\mathbb{Z}}$ and obtain

$$\epsilon_L(y * y) = \xi\sigma_{c,L}\epsilon_L(y)t^\nu = \xi^2\sigma_{c,L}t^\nu.$$

Since the left-hand side lies in $\Lambda_{\mathbb{Z}}$ it follows that $\xi^2\sigma_{c,L} \in \mathbb{Z}$. Multiplying equation (4.4) with ξ we see that $\xi^3\tau_{c,L} \in \mathbb{Z}$. We now write $\sigma_{c,L} = u/\xi^2$ and $\tau_{c,L} = v/\xi^3$ with $u, v \in \mathbb{Z}$. The discriminant is then

$$\Delta_L = \frac{u^2}{\xi^4} + 4\frac{v}{\xi^3} \in \mathbb{Z}$$

and thus we have $\xi^4\Delta_L = u^2 + 4\xi v$. Since $\xi|(u^2 + 4\xi v)$ it follows that $\xi|u^2$. If ξ is square-free then $\xi|u$ and hence $\xi\sigma_{c,L} = u/\xi \in \mathbb{Z}$. Now using equation (4.4) we see that $y * y - \xi\sigma_{c,L} y t^{\nu} \in QH_{-n}(L;\Lambda_{\mathbb{Z}})$ and therefore $\xi^2\tau_{c,L} \in \mathbb{Z}$.

It remains to prove the statement on the relation between $\sigma_{c,L}$ and the Gromov-Witten invariants. For this purpose we will need the following Lemma:

Lemma 4.1.2. *Let $p_M = [\text{point}] \in H_0(M)$ be the class of a point. Denote by $\widetilde{\epsilon}_M : QH_*(M;R) \longrightarrow H_0(M;R) = R\,p_M$ the classical augmentation extended linearly over R. Let $a, b \in H_*(M)$ be two (classical) classes of pure degree. Then*

$$\widetilde{\epsilon}_M(a * b) = \widetilde{\epsilon}_M(a \cdot b),$$

where $\cdot$ is the classical intersection product. In particular, the class p_M appears as a summand in $a * b$ if and only if $|a| + |b| = 2n$ and $a \cdot b \neq 0$.

We postpone the proof of the Lemma and proceed with the proof of the theorem.

Denote by $k = C_M$ the minimal Chern number of M (see §3.2.1). Write

$$c * [L] = c \cdot [L] + \sum_{j \geq 1} \alpha_{2jk} q^{jk},$$

with $\alpha_{2jk} \in H_{2jk}(M)$. (The choice of the sub-indices was made to reflect the degree in homology.) Then we have

$$c * c * [L] = \#(c \cdot [L]) c * p_M + \sum_{j \geq 1} c * \alpha_{2jk} q^{jk},$$

which together with (4.1) give:

$$\xi\sigma_{c,L}c*[L]q^{n/2}+\xi^2\tau_{c,L}[L]q^n = \#(c\cdot[L])c*p_M+\sum_{j\geq 1} c*\alpha_{2jk}q^{jk}. \quad (4.5)$$

Applying $\widetilde{\epsilon}_M$ to (4.5) we obtain using Lemma 4.1.2 that

$$\xi^2\sigma_{c,L}p_M q^{n/2} = \widetilde{\epsilon}_M(c\cdot\alpha_n)q^{n/2} = \#(c\cdot\alpha_n)p_M q^{n/2}. \quad (4.6)$$

By the definition of the quantum product we have:

$$\#(c\cdot\alpha_n) = \sum_A GW^M_{A,3}(c,c,[L]),$$

where the sum goes over $A \in H_2(M)$ with $\langle c_1, A\rangle = n/2$. (Note that since n=even the order of the classes $(c,c,[L])$ in the Gromov-Witten invariant does not make a difference.) Substituting this in (4.6) yields the desired identity.

Note that we have carried out the proof above for the quantum homology $QH(M;R)$ with coefficients in the ring $R = \mathbb{Z}[q^{-1},q]$, but since (M,ω) is monotone, it is easy to see that equation (4.1) involves only positive powers of q hence it holds in fact in $QH(M;R^+)$, where $R^+ = \mathbb{Z}[q]$.

To complete the proof of the theorem we still need the following.

Proof of Lemma 4.1.2. Write

$$a*b = a\cdot b + \sum_{j\geq 1}\gamma_j q^{jk},$$

where $a\cdot b \in H_{|a|+|b|-2n}(M)$ is the classical intersection product of a and b, k is the minimal Chern number, and $\gamma_j \in H_{|a|+|b|-2n+2jk}(M)$. In order to prove the lemma we need to show that $\gamma_{j_0} = 0$, where $2j_0k = 2n-|a|-|b|$.

Suppose by contradiction that $\gamma_{j_0} \neq 0$. Then there exists $A \in H_2(M)$ with

$$2\langle c_1, A\rangle = 2j_0k = 2n-|a|-|b|$$

such that $GW_{A,3}(a, b, [M]) \neq 0$, where $[M] \in H_{2n}(M)$ is the fundamental class. Since $[M]$ poses no additional incidence conditions on GW-invariants, this implies that for a generic almost complex structure there exists a pseudoholomorphic rational curve passing through generic representatives of the classes a and b. More precisely denote by $\mathcal{M}_{0,2}(A, J)$ the space of simple rational J-holomorphic curves with 2 marked points in the class A. Denote by $ev : \mathcal{M}_{0,2}(A, J) \longrightarrow M \times M$ the evaluation map. Since $GW_{A,3}(a, b, [M]) \neq 0$, then for a generic choice of (pseudo) cycles D_a, D_b representing a, b and for a generic choice of J the map ev is transverse to $D_a \times D_b$ and moreover $ev^{-1}(D_a \times D_b) \neq \emptyset$. However this is impossible because

$$\begin{aligned}
&\dim \mathcal{M}_{0,2}(A, J) + \dim(D_a \times D_b) = \\
&\big(2n + 2\langle c_1, A\rangle - 2\big) + |a| + |b| = 4n - 2 < \dim(M \times M).
\end{aligned}$$

$\square$

The proof of Theorem 4.1.1 is now complete.

$\square$

4.1.1. Proof of Theorems 1 and 3 The proof follows immediately from Theorem 4.1.1. Indeed, since $\#([L] \cdot [L]) = \varepsilon\chi \neq 0$ we can take $c = [L]$, $\xi = \varepsilon\chi$ in Theorem 4.1.1. The constants σ_L, τ_L from Theorem 1 are now $\sigma_{L,L}, \tau_{L,L}$ respectively, and we have $\Delta_L = \sigma_{L,L}^2 + 4\tau_{L,L}$. $\square$

4.1.2. Further results We present here a few other results that follow from the same ideas as in the proof of Theorem 4.1.1.

Proposition 4.1.3. *Let $L_1, L_2 \subset M$ be two Lagrangian submanifolds satisfying conditions* (1) – (3) *of Assumption $\mathscr{L}$ (possibly with different minimal Maslov numbers). Assume that $[L_1] \cdot [L_2] = 0$. Then one of the following two (non exclusive) possibilities occur:*

1. *either* $[L_1]$ *and* $[L_2]$ *are proportional in* $H_n(M;\mathbb{Q})$ *and moreover we have the relation* $[L_1]*[L_1] = \gamma[L_1]q^{n/2}$ *in* $QH(M;R^+_{\mathbb{Q}})$ *for some* $\gamma \in \mathbb{Z}$*;*
2. *or* $[L_1] * [L_2] = 0$.

Remark. Note that if possibly (1) occurs in the Proposition and moreover $N_{L_1} = N_{L_2} = 2$, then $\lambda_{L_1} = \lambda_{L_2}$. This is so because by the Proposition $[L_1]$ and $[L_2]$ are proportional and $[L_i]$ is an eigenvector of the operator P with eigenvalue λ_{L_i} (see §3.2.4).

Here is a simple example of Lagrangians L_1, L_2 satisfying the conditions of Proposition 4.1.3. We take M to be the monotone blow-up of $\mathbb{C}P^2$ at 3 points and L_1, L_2 Lagrangian spheres in the classes $[L_1] = H - E_1 - E_2 - E_3$, $[L_2] = E_2 - E_3$ (using the notation of §3.1.3). See §5.1.1 for more details on how to actually construct these spheres. Clearly $[L_1] \cdot [L_2] = 0$, hence the Proposition implies that $[L_1] * [L_2] = 0$ (which can of course be confirmed also by direct calculation). One can construct many other examples of this type in monotone blow-ups of $\mathbb{C}P^2$ at $3 \leq k \leq 8$ points.

On the other hand, if $L \subset M$ is a Lagrangian satisfying conditions (1) – (3) of Assumption $\mathscr{L}$ and we assume in addition that $\chi(L) = 0$, then we can take $L = L_1 = L_2$. Proposition 4.1.3 then implies that $[L] * [L] = [L]\gamma q^{n/2}$ for some $\gamma \in \mathbb{Z}$. The simplest example should be when L is a 2-torus, however we are not aware of any example of a monotone Lagrangian 2-torus satisfying conditions (1) – (3) of Assumption $\mathscr{L}$ and with $[L] \neq 0$. An easy (algebraic) argument shows that such tori cannot exist in a symplectic 4-manifold with $b_2^+ = 1$ (e.g. in blow-ups of $\mathbb{C}P^2$). It would be interesting to know if this holds in greater generality.

Finally, we remark that if one replaces the condition $[L_1] \cdot [L_2] = 0$ by the stronger assumption that $L_1 \cap L_2 = \emptyset$, and drops conditions (3), (4) of Assumption $\mathscr{L}$, then it still follows that $[L_1] * [L_2] = 0$. This is proved in [15]-Theorem 2.4.1 (see also §8 in [14]).

Proof of Proposition 4.1.3. Without loss of generality assume that $[L_1], [L_2] \neq 0 \in H_n(M;\mathbb{Q})$, otherwise possibility (2) obviously holds.

Define $y_1 = [L_2] * e_{L_1} \in QH_0(L_1; \Lambda^1_{\mathbb{Q}})$ and $y_2 = [L_1] * e_{L_2} \in QH_0(L_2; \Lambda^2_{\mathbb{Q}})$. Here we have denoted $\Lambda^1_{\mathbb{Q}} = \mathbb{Q}[t_1^{-1}, t_1]$ with $|t_1| = -N_{L_1}$ and $\Lambda^2_{\mathbb{Q}} = \mathbb{Q}[t_2^{-1}, t_2]$ with $|t_2| = -N_{L_2}$ since we have to distinguish between the coefficient rings of the Lagrangians L_1 and L_2. Note that under the embeddings of $\Lambda^1_{\mathbb{Q}}$ and $\Lambda^2_{\mathbb{Q}}$ into $R_{\mathbb{Q}} = \mathbb{Q}[q^{-1}, q]$ we have $t_1^{\nu_1} = q^{n/2} = t_2^{\nu_2}$. (See §3.2.2.)

Since $[L_1] \cdot [L_2] = 0$ and due to condition (3) of Assumption $\mathscr{L}$ we have

$$y_1 = \gamma_1 e_{L_1} t_1^{\nu_1}, \quad y_2 = \gamma_2 e_{L_2} t_2^{\nu_2},$$

for some $\gamma_1, \gamma_2 \in \mathbb{Q}$ and where $\nu_1 = n/N_{L_1}$, $\nu_2 = n/N_{L_2}$. At the same time we also have

$$i_{L_1}(y_1) = i_{L_2}(y_2) = [L_1] * [L_2].$$

Here we have used the fact that n must be even, hence $[L_1] * [L_2] = [L_2] * [L_1]$.

It follows that $\gamma_1 [L_1] q^{n/2} = [L_1] * [L_2] = \gamma_2 [L_2] q^{n/2}$ and the result follows. (As in the proof of Theorem 4.1.1, note that here the identities proved involve only positive powers of q, hence they hold in $QH(M; R^+)$ too.) $\square$

4.2. The discriminant and Lagrangian cobordisms

This section provides the proofs of Theorem 4 and a generalization of Corollary 5. We start with:

Proof of Theorem 4. Before going into the details of the proof, here is the rationale behind it. To the Lagrangian cobordism V we can associate a (relative) quantum homology $QH(V, \partial V)$ which has a quantum product. The quantum product on $QH(V, \partial V)$ is related to the quantum products for the ends of V via a quantum connectant $\delta : QH(V, \partial V) \longrightarrow QH(\partial V) = \oplus_{i=1}^r QH(L_i)$. This makes it possible to find relations between the products on the quantum homologies $QH(L_i)$ of different ends of V and the quantum product on $QH(V, \partial V)$. In particular this gives the desired relation between the discriminants of the different ends.

We now turn to the details of the proof. We will use here several versions of the pearl complex and its homology (also called Lagrangian quantum homology) both for Lagrangian cobordisms as well as for their ends. We refer the reader to [14, 15, 16] for the foundations of the theory in the case of closed Lagrangians and to §5 of [17] in the case of cobordisms.

Throughout this proof we will work with $\mathbb{Q}$ as the base field and with $\Lambda = \mathbb{Q}[t^{-1}, t]$ or $\Lambda^+ = \mathbb{Q}[t]$ as coefficient rings. We denote by $\mathcal{C}$ and $\mathcal{C}^+$ the pearl complexes with coefficients in Λ and Λ^+ respectively, and by QH and Q^+H their homologies. The latter is sometimes called the positive Lagrangian quantum homology.

Before we go on, a small remark regarding the coefficients is in order. Throughout this proof we grade the variable $t \in \Lambda$ as $|t| = -N_V$. This is the standard grading for $QH(V)$ and $QH(V, \partial V)$ and their positive versions. We use the same coefficient rings (and grading) also for $QH(L_i)$ and its positive version. This is possible since $N_V | N_{L_i}$, hence our ring Λ^+ is an extension of the corresponding ring in which the degree of t is $-N_{L_i}$.

Recall that (for any Lagrangian submanifold) the positive quantum homology Q^+H admits a natural map $Q^+H \longrightarrow QH$ induced by the inclusion $\mathcal{C}^+ \longrightarrow \mathcal{C}$. Again, for degree reasons the induced map in homology is an isomorphism in degree 0 and surjective in degree 1:

$$Q^+H_0 \xrightarrow{\cong} QH_0, \quad Q^+H_1 \twoheadrightarrow QH_1. \tag{4.7}$$

In fact, the last map is an isomorphism whenever the minimal Maslov number is > 2. We also have $Q^+H_n(K) \cong H_n(K)$ for every n-dimensional Lagrangian submanifold K.

Coming back to the proof of the theorem, we first claim there is a commutative diagram

$$\begin{tikzcd}
Q^+H_1(V) \arrow[r, "j_Q"] \arrow[d, "s"'] & Q^+H_1(V,\partial V) \arrow[r, "\delta"] \arrow[d, "s"'] & Q^+H_0(\partial V) \arrow[r, "i_Q"] \arrow[d, "s"'] & Q^+H_0(V) \arrow[d, "s"'] \\
H_1(V) \arrow[r, "j"] & H_1(V,\partial V) \arrow[r, "\partial"] \arrow[d] & H_0(\partial V) \arrow[r, "i"] \arrow[d] & H_0(V) \\
& 0 & 0 &
\end{tikzcd} \tag{4.8}$$

with exact rows and columns. The second row of the diagram is the classical homology sequence for the pair $(V, \partial V)$ with ∂ being the connecting homomorphism (we use $\mathbb{Q}$ coefficients here). The first row is its quantum homology analogue, and we remark that the quantum connectant δ is multiplicative with respect to the quantum product (see §5 of [17] and [65]). The vertical maps s come from the following general exact sequence of chain complexes:

$$0 \longrightarrow t\mathcal{C}^+ \xrightarrow{\iota} \mathcal{C}^+ \xrightarrow{s} CM \longrightarrow 0, \tag{4.9}$$

where CM stand for the Morse complex (defined using the same Morse function and metric as used for the pearl complex, but with coefficients in $\mathbb{Q}$ rather than Λ^+). The second map in this exact sequence, $s : \mathcal{C}^+ \longrightarrow CM$, is induced by $t \mapsto 0$ (i.e. it sends a pearly chain to its classical part, omitting the t's), and ι stands for the inclusion. We now explain why the two middle s maps in (4.8) are surjective. We start with the third s map (i.e. the one before the rightmost s). We have:

$$H_0(\partial V) = \bigoplus_{i=1}^{r} H_0(L_i), \quad Q^+H_0(\partial V) = \bigoplus_{i=1}^{r} Q^+H_0(L_i). \tag{4.10}$$

Next, note that the composition of $s : Q^+H_0(L_i) \longrightarrow H_0(L_i)$ with the inclusion $H_0(L_i) \subset H_0(L_i; \Lambda^+)$ coincides with the augmentation $\widetilde{\epsilon}_{L_i} : Q^+H_0(L_i) \longrightarrow H_0(L; \Lambda^+)$. The fact that s is surjective now follows easily from §3.2.5 and (4.7).

The surjectivity of the second from the left s map requires a different argument. Consider the chain complex $\mathcal{D}_* = (t\mathcal{C}^+)_*$, viewed as a subcomplex of $\mathcal{C}^+$. In view of the exact sequence (4.9) the surjectivity of the second from the left s map in (4.8) would follow if we show that $H_0(\mathcal{D}) = 0$. To this end consider the following filtration $\mathcal{F}_\bullet\mathcal{D}$ of $\mathcal{D}$ by subcomplexes, defined by:

$$\mathcal{F}_m\mathcal{D} := t^{-m}\mathcal{D} = t^{-m+1}\mathcal{C}^+ \quad \forall\, m \leq 0,$$
$$\mathcal{F}_k\mathcal{D} := \mathcal{D} \quad \forall\, k \geq 0.$$

A simple calculation (in the spirit of [12, 13, 59]) shows that the first page of the spectral sequence associated to this filtration satisfies:

$$E^1_{p,q} \cong t^{-p+1}H_{p+q+N_V-pN_V}(V, \partial V) \quad \forall\, p \leq 0,$$
$$E^1_{p,q} = 0 \quad \forall\, p \geq 1.$$

It follows from the assumption of the theorem that for all p, q with $p + q = 0$ we have $E^1_{p,q} = 0$, hence also $E^\infty_{p,q} = 0$. Since this spectral sequence converges to $H_*(\mathcal{D})$ this implies that $H_0(\mathcal{D}) = 0$. This completes the proof of the surjectivity of the second from the left s map in (4.8).

We now proceed with the proof of the theorem, based on the diagram (4.8) and its properties. We first remark that due to the assumptions of the theorem the number of ends of V must be $r \geq 2$. Indeed, by the results of [17] if a Lagrangian submanifold L_1 is Lagrangian null-cobordant (i.e. there exists a monotone Lagrangian cobordism V with only one end being L_1) then $HF(L_1, L_1) = 0$, in contrast with the assumption that L_1 satisfies condition (3) of Assumption $\mathscr{L}$. We therefore assume from now on that $r \geq 2$.

Denote by $p_i \in H_0(L_i) \subset H_0(\partial V)$ the class corresponding to a point in L_i. Let $\alpha_2, \ldots, \alpha_r \in H_1(V, \partial V)$ be classes with $\partial\alpha_i = p_1 - p_i$. Choose lifts $\overline{p}_i \in Q^+H_0(\partial V)$ of the p_i's under the map s as well as lifts $\overline{\alpha}_2, \ldots, \overline{\alpha}_r \in Q^+H_1(V, \partial V)$ of $\alpha_2, \ldots, \alpha_r$. Denote by $e_V \in Q^+H_{n+1}(V, \partial V)$ the unity and by $e_{L_i} \in Q^+H_n(L_i)$ the unities corresponding to the L_i's. Note that $\delta(e_V) = e_{L_1} + \cdots + e_{L_r}$. Finally, put $\nu = n/N_V$. Since the Lagrangians L_i satisfy conditions (1) – (3) of Assumption $\mathscr{L}$ and in view of §3.2.5, we have:

$$Q^+H_0(\partial V) \cong QH_0(\partial V) = \mathbb{Q}\overline{p}_1 \oplus \cdots \oplus \mathbb{Q}\overline{p}_r \oplus \mathbb{Q}e_{L_1}t^{\nu} \oplus \cdots \oplus \mathbb{Q}e_{L_r}t^{\nu}.$$

Proposition 4.2.1. $\dim_{\mathbb{Q}}(\text{image}\,(\delta)) = r$. *Moreover, for every choice of α_i's and $\overline{\alpha}_i$'s the elements*

$$\delta(\overline{\alpha}_2), \ldots, \delta(\overline{\alpha}_r), (e_{L_1} + \cdots + e_{L_r})t^{\nu}$$

form a basis (over $\mathbb{Q}$) of the vector space $\text{image}\,(\delta) \subset QH_0(\partial V)$.

We defer the proof of the lemma and continue with the proof of our theorem.

Denote by $\mathcal{B} \subset Q^+H_1(V, \partial V)$ the kernel of $\delta : Q^+H_1(V, \partial V) \longrightarrow Q^+H_0(\partial V)$. By Proposition 4.2.1 the elements

$$\overline{\alpha}_2, \ldots, \overline{\alpha}_r, e_V t^{\nu}$$

induce a basis for the vector space $Q^+H_1(V, \partial V)/\mathcal{B}$.

We now continue by proving that $\Delta_{L_1} = \Delta_{L_2}$. The other equalities follow by the same recipe. Using the preceding basis we can write:

$$\begin{aligned} \overline{\alpha}_2 * \overline{\alpha}_2 &= \sum_{j=2}^{r} \xi_j \overline{\alpha}_j t^{\nu} + Bt^{\nu} + \rho e_V t^{2\nu}, \\ \delta(\overline{\alpha}_2) &= \overline{p}_1 - \overline{p}_2 + \sum_{k=1}^{r} a_k e_{L_k} t^{\nu}, \end{aligned} \tag{4.11}$$

for some $\xi_j, a_k, \rho \in \mathbb{Q}$ and $B \in \mathcal{B}$. For the first equality we have used the fact that $\overline{\alpha}_2 * \overline{\alpha}_2 \in Q^+H_{1-n}(V, \partial V) \cong t^{\nu}Q^+H_1(V, \partial V)$.

We will also need a similar equality to the second one in (4.11), but for $\delta(\overline{\alpha}_i)$:

$$\delta(\overline{\alpha}_i) = \overline{p}_1 - \overline{p}_i + \sum_{k=1}^{r} a_k^{(i)} e_{L_k} t^{\nu}, \quad \forall\, 2 \leq i \leq r, \tag{4.12}$$

where $a_k^{(i)} \in \mathbb{Q}$. (Note that according to our notation $a_k = a_k^{(2)}$.)

At this point we need to separate the arguments to the cases $r \geq 3$ and $r = 2$. (As we have already remarked, $r = 1$ is impossible under the assumptions of the theorem.) We assume first that $r \geq 3$. The case $r = 2$ will be treated after that.

We now perform a little change in the basis and the choice of the lift $\overline{p}_i$ as follows:

$$\begin{aligned}
&\overline{\alpha}_2 \longrightarrow \overline{\alpha}_2 - a_3 e_V t^{\nu}, \quad \overline{\alpha}_i \longrightarrow \overline{\alpha}_i \quad \forall i \geq 3, \\
&\overline{p}_1 \longrightarrow \overline{p}_1 + (a_1 - a_3) e_{L_1} t^{\nu}, \quad \overline{p}_2 \longrightarrow \overline{p}_2 - (a_2 - a_3) e_{L_2} t^{\nu}, \\
&\overline{p}_i \longrightarrow \overline{p}_i \quad \forall i \geq 3.
\end{aligned}$$

To simplify notation we continue to denote the new basis elements by $\overline{\alpha}_i$ and similarly for the $\overline{p}_i$'s. By abuse of notation we also continue to denote the new coefficients a_k, $a_k^{(i)}$, ξ_j and ρ resulting from the basis change by the same symbols, and similarly for the term $B \in \mathcal{B}$. The outcome of the basis change is that now the second equality in (4.11) becomes:

$$\delta(\overline{\alpha}_2) = \overline{p}_1 - \overline{p}_2 + \sum_{k=4}^{r} a_k e_{L_k} t^{\nu}. \tag{4.13}$$

(Of course, if $r = 3$ then the third term in the last equation is void.) We now use the fact that δ is multiplicative (see [17]):

$$\delta(\overline{\alpha}_2 * \overline{\alpha}_2) = \delta(\overline{\alpha}_2) * \delta(\overline{\alpha}_2) = \overline{p}_1^{*2} + \overline{p}_2^{*2} + \sum_{k=4}^{r} a_k^2 e_{L_k} t^{2\nu}. \tag{4.14}$$

We now express $\overline{p}_1^{*2} \in Q^+ H_{-n}(L_1) \cong t^{\nu} Q^+ H_0(L_1)$ in terms of the basis $\{\overline{p}_1 t^{\nu}, e_{L_1} t^{2\nu}\}$ and similarly for $\overline{p}_2^{*2}$:

$$\overline{p}_1^{*2} = \sigma_1 \overline{p}_1 t^{\nu} + \tau_1 e_{L_1} t^{2\nu}, \qquad \overline{p}_2^{*2} = \sigma_2 \overline{p}_2 t^{\nu} + \tau_2 e_{L_2} t^{2\nu},$$

where $\sigma_1, \sigma_2 \in \mathbb{Q}$ and $\tau_1, \tau_2 \in \mathbb{Q}$. (In fact, by choosing the α_i's, $\overline{\alpha}_i$'s and $\overline{p}_i$'s carefully, over $\mathbb{Z}$, the coefficients $\sigma_1, \sigma_2, \tau_1, \tau_2$ will in fact be in $\mathbb{Z}$, but we will not need that.) Substituting this into (4.14) we obtain:

$$\delta(\overline{\alpha}_2 * \overline{\alpha}_2) = \sigma_1 \overline{p}_1 t^{\nu} + \sigma_2 \overline{p}_2 t^{\nu} + \tau_1 e_{L_1} t^{2\nu} + \tau_2 e_{L_2} t^{2\nu} + \sum_{k=4}^{r} a_k^2 e_{L_k} t^{2\nu}. \tag{4.15}$$

Applying δ to the first equality in (4.11) and using (4.13) and (4.15) we obtain:

$$\begin{aligned} &\xi_2 \Big(\overline{p}_1 - \overline{p}_2 + \sum_{k=4}^{r} a_k e_{L_k} t^{\nu} \Big) t^{\nu} + \sum_{i=3}^{r} \xi_i \Big(\overline{p}_1 - \overline{p}_i + \sum_{q=1}^{r} a_q^{(i)} e_{L_q} t^{\nu} \Big) t^{\nu} \\ &+ \rho(e_{L_1} + \cdots + e_{L_r}) t^{2\nu} \\ &= \ \sigma_1 \overline{p}_1 t^{\nu} + \sigma_2 \overline{p}_2 t^{\nu} + \tau_1 e_{L_1} t^{2\nu} + \tau_2 e_{L_2} t^{2\nu} + \sum_{k=4}^{r} a_k^2 e_{L_k} t^{2\nu}. \end{aligned}$$

Comparing the coefficients of $\overline{p}_3, \ldots, \overline{p}_r$ we deduce that $\xi_3 = \cdots = \xi_r = 0$. The last equation thus becomes:

$$\begin{aligned} &\xi_2 \Big(\overline{p}_1 - \overline{p}_2 + \sum_{k=4}^{r} a_k e_{L_k} t^{\nu} \Big) t^{\nu} + \rho(e_{L_1} + \cdots + e_{L_r}) t^{2\nu} \\ &= \sigma_1 \overline{p}_1 t^{\nu} + \sigma_2 \overline{p}_2 t^{\nu} + \tau_1 e_{L_1} t^{2\nu} + \tau_2 e_{L_2} t^{2\nu} + \sum_{k=4}^{r} a_k^2 e_{L_k} t^{2\nu}. \end{aligned} \tag{4.16}$$

Comparing the coefficients of $\overline{e}_3$ on both sides of (4.16) (recall that $r \geq 3$) we deduce that $\rho = 0$. It easily follows now that $\tau_1 = \tau_2 = 0$ and that $\sigma_1 = \xi_2 = -\sigma_2$. By the definition of the discriminant it follows that

$$\Delta_{L_1} = \sigma_1^2 = \sigma_2^2 = \Delta_{L_2}.$$

Note that the relation between our σ_i's and τ_i's and the notation used in §3.1.2 and in §3.2.5 is $\sigma_1 = \sigma_1(p_1, \overline{p}_1)$, $\sigma_2 = \sigma_2(p_2, \overline{p}_2)$ and similarly for τ_1, τ_2. Finally we remark that since $\Delta_{L_1} = \sigma_1^2 \in \mathbb{Z}$ we must have $\sigma_1 \in \mathbb{Z}$, hence Δ_{L_1} is a perfect square.

We now turn to the case $r = 2$. In that case we can write (4.11) as

$$\begin{aligned} \overline{\alpha}_2 * \overline{\alpha}_2 &= \xi \overline{\alpha}_2 t^{\nu} + B t^{\nu} + \rho e_V t^{2\nu}, \\ \delta(\overline{\alpha}_2) &= \overline{p}_1 - \overline{p}_2 + a_1 e_{L_1} t^{\nu} + a_2 e_{L_2} t^{\nu}, \end{aligned} \tag{4.17}$$

By an obvious basis change (among $\overline{p}_1$, $\overline{p}_2$) we may assume that $a_1 = a_2 = 0$. Then the identity $\delta(\overline{\alpha}_2 * \overline{\alpha}_2) = \delta(\overline{\alpha}_2) * \delta(\overline{\alpha}_2)$ becomes:

$$\xi(\overline{p}_1 - \overline{p}_2) t^{\nu} + \rho(e_{L_1} + e_{L_2}) t^{2\nu} = \sigma_1 \overline{p}_1 t^{\nu} + \sigma_2 \overline{p}_2 t^{\nu} + \tau_1 e_{L_1} t^{2\nu} + \tau_2 e_{L_2} t^{2\nu}.$$

It follows immediately that $\sigma_1 = -\sigma_2$ and $\tau_1 = \tau_2$. Consequently $\Delta_{L_1} = \Delta_{L_2}$.

To complete the proof of the theorem it remains to prove Proposition 4.2.1. For this purpose we will need the following Lemma.

Lemma 4.2.2. *Let $j \geq 0$ and consider the connecting homomorphism*

$$\delta : Q^+ H_{1+jN_V}(V, \partial V) \longrightarrow Q^+ H_{jN_V}(\partial V).$$

Let $\eta \in Q^+ H_{1+jN_V}(V, \partial V)$ and assume that $\delta(\eta)$ is divisible by t. Then η is also divisible by t.

Proof of the lemma. The connecting homomorphism δ is part of the following diagram:

$$\begin{array}{ccccc} & & Q^+ H_{1+jN_V}(V, \partial V) & \xrightarrow{\delta} & Q^+ H_{jN_V}(\partial V) \\ & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle s} \\ H_{1+jN_V}(V) & \xrightarrow{j} & H_{1+jN_V}(V, \partial V) & \xrightarrow{\partial} & H_{jN_V}(\partial V) \end{array} \tag{4.18}$$

where the vertical s-maps are induced by (4.9). Since $\delta(\eta)$ is divisible by t we have $s(\delta(\eta)) = 0$ hence $\partial(s(\eta)) = 0$. By assumption $H_{1+jN_V}(V) = 0$ hence the bottom map ∂ is injective, and therefore we have $s(\eta) = 0$. Looking again at (4.9) it follows that

$$\eta \in \text{image}\Big(H_{1+jN_V}(t\mathcal{C}^+) \xrightarrow{\iota_*} Q^+ H_{1+jN_V}(V, \partial V)\Big),$$

where $\mathcal{C}^+$ stands for the positive pearl complex of $(V, \partial V)$. But

$$H_{1+jN_V}(t\mathcal{C}^+) \cong tQ^+H_{1+(j+1)N_V}(V, \partial V)$$

via an isomorphism for which ι_* becomes the inclusion

$$tQ^+H_{1+(j+1)N_V}(V, \partial V) \subset Q^+H_{1+jN_V}(V, \partial V).$$

This proves that η is divisible by t. $\square$

We are finally in position to prove the preceding proposition.

Proof of Proposition 4.2.1. Note that

$$\{\overline{p}_1, \delta(\overline{\alpha}_2), \ldots, \delta(\overline{\alpha}_r), \delta(e_V)t^\nu, e_{L_2}t^\nu, \ldots, e_{L_r}t^\nu\}$$

is a basis for $Q^+H_0(\partial V)$ (recall that $\delta(e_V) = e_{L_1} + \cdots + e_{L_r}$). Therefore it is enough to show that the subspace of $Q^+H_0(\partial V)$ generated by $\overline{p}_1, e_{L_2}t^\nu, \ldots, e_{L_r}t^\nu$ has trivial intersection with image(δ).

Let $\gamma = c\overline{p}_1 + \sum_{j=2}^r b_j e_{L_j} t^\nu$, where $c, b_j \in \mathbb{Q}$ and assume that $\gamma = \delta(\beta)$ for some $\beta \in Q^+H_1(V, \partial V)$. We have $s(\gamma) = cp_1$, where the map s is the third vertical map from diagram (4.8). It follows from that diagram that $\partial(s(\beta)) = cp_1$. But this is possible only if $c = 0$ since $p_1 \notin$ image(∂).

Thus $\gamma = \sum_{j=2}^r b_j e_{L_j} t^\nu$ and we have to show that $\gamma = 0$. Recall that $\gamma = \delta(\beta)$. We claim that β is divisible by t^ν, i.e. there exists $\beta' \in Q^+H_{n+1}(V, \partial V)$ such that $\beta = t^\nu\beta'$. To prove this we first note that γ is divisible by t. By Lemma 4.2.2, β is also divisible by t. Thus there exists $\beta_1 \in Q^+H_{1+N_V}(V, \partial V)$ with $\beta = t\beta_1$. In particular $\delta(\beta_1) = \sum_{j=2}^r b_j e_{L_j} t^{\nu-1}$. Continuing by induction, using Lemma 4.2.2 repeatedly, we obtain elements $\beta_j \in Q^+H_{1+jN_V}(V, \partial V)$ with $t\beta_{j+1} = \beta_j$ for every $1 \leq j \leq \nu - 1$. Take $\beta' = \beta_\nu$.

We have $t^\nu\delta(\beta') = \sum_{j=2}^r b_j e_{L_j} t^\nu$ for some $\beta' \in Q^+H_{n+1}(V, \partial V)$. As $Q^+H_{n+1}(V, \partial V) = \mathbb{Q}e_V$ we have $\beta' = ae_V$ for some $a \in \mathbb{Q}$. But $\delta(e_V) = e_{L_1} + \cdots + e_{L_r}$ hence $a(e_{L_1} + \cdots + e_{L_r})t^\nu = (\sum_{j=2}^r b_j e_{L_j})t^\nu$. Since by condition (3) of Assumption $\mathscr{L}$ we know that the element

$e_{L_1} \in Q^+ H_n(\partial V)$ is not torsion (over Λ^+), it follows that $a = 0$. Consequently $b_2 = \cdots = b_r = 0$ and so $\gamma = 0$. This concludes the proof of Proposition 4.2.1. □

Having proved Proposition 4.2.1, the proof of Theorem 4 is now complete. □

4.2.1. Lagrangians intersecting at one point We start with a stronger version of Corollary 5 from §3.1.2.

Corollary 4.2.3. *Let (M, ω) be a monotone symplectic manifold. Let $L_1, L_2 \subset M$ be two Lagrangian submanifolds that satisfy conditions (1) – (3) of Assumption $\mathscr{L}$ and such that $N_{L_1} = N_{L_2}$. Denote by $N = N_{L_i}$ their mutual minimal Maslov number and assume further that:*

1. *$H_{1+jN}(L_1) = H_{1+jN}(L_2) = 0$ for every j;*
2. *$H_{jN-1}(L_1) = H_{jN-1}(L_2) = 0$ for every j;*
3. *either $\pi_1(L_1 \cup L_2) \to \pi_1(M)$ is injective, or $\pi_1(L_i) \to \pi_1(M)$ is trivial for $i = 1, 2$.*

Finally, suppose that L_1 and L_2 intersect transversely at exactly one point. Then

$$\Delta_{L_1} = \Delta_{L_2}$$

and moreover this number is a perfect square.

Note that if L_1, L_2 are even dimensional Lagrangian spheres then conditions (1) – (3) of Corollary 4.2.3 are obviously satisfied, hence Corollary 5 follows from Corollary 4.2.3.

We now turn to the proof of Corollary 4.2.3. We will need the following Proposition.

Proposition 4.2.4. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangian submanifolds intersecting transversely at one point. Then there exists a Lagrangian cobordism $V \subset \mathbb{R}^2 \times M$ with three ends, corresponding to*

L_1, L_2 and $L_1 \# L_2$ and such that V has the homotopy type of $L_1 \vee L_2$. If L_1 and L_2 are monotone with the same minimal Maslov number N and they satisfy assumption (3) from Corollary 4.2.3 then V is also monotone with minimal Maslov number $N_V = N$. Moreover, if L_1 and L_2 are spin then V admits a spin structure that extends those of L_1 and L_2.

Before proving this proposition we show how to deduce Corollary 4.2.3 from it.

Proof of Corollary 4.2.3. Consider the Lagrangian cobordism provided by Proposition 4.2.4. Since V is homotopy equivalent to $L_1 \vee L_2$ and L_i satisfy assumptions (1) and (2) of Corollary 4.2.3 a simple calculation shows that

$$H_{jN}(V, \partial V) = 0, \quad H_{1+jN}(V) = 0, \quad \forall j.$$

The result now follows immediately from Theorem 4. □

We now turn to the proof of the Proposition.

Proof of Proposition 4.2.4. The proof is based on a version of the Pol-te-ro-vich Lagrangian surgery [61] adapted to the case of cobordisms [17]. We briefly outline those parts of the construction that are relevant here. More details can be found in [17].

Consider two plane curves γ_1, γ_2 as in Figure 4.1. Consider the Lagrangian submanifolds $\gamma_1 \times L_1, \gamma_2 \times L_2 \subset \mathbb{R}^2 \times M$. The surgery construction from [17] produces a Lagrangian cobordism $V \subset \mathbb{R} \times M$ with two negative ends which coincide with negative ends of $\gamma_i \times L_i$ and with whose positive end looks like the positive end of $\gamma_3 \times (L_1 \# L_2)$, where the curve γ_3 is depicted in Figure 4.2 and $L_1 \# L_2$ stands for the Polterovich surgery (in M) of L_1 and L_2 (which coincides with the connected sum of the L_i's because they intersect transversely at exactly one point). The projection of V to $\mathbb{R}^2$ is depicted in Figure 4.2.

Next we determine the topology of V. Consider the curves $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ (which are extensions of the γ_i's to curves with positive ends as in

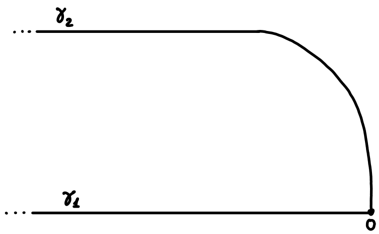

Figure 4.1.

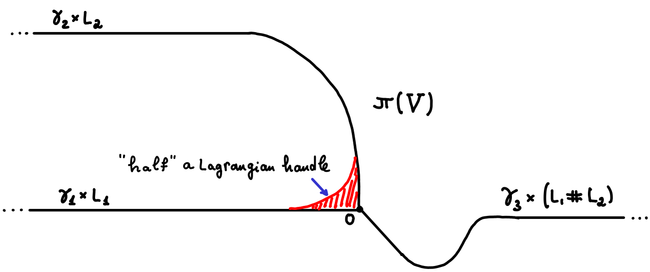

Figure 4.2.

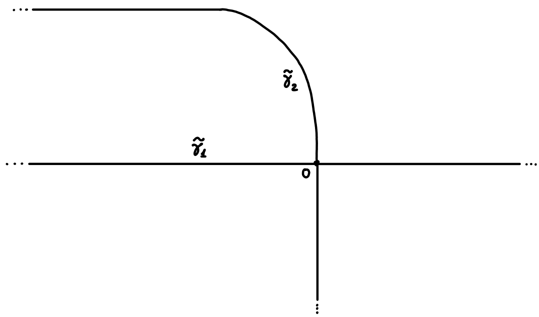

Figure 4.3.

Figure 4.3.) Consider the Polterovich surgery $W = (\widetilde{\gamma}_1 \times L_1)\#(\widetilde{\gamma}_2 \times L_2) \subset \mathbb{R}^2 \times M$ (note that the latter two Lagrangians also intersect transversely at a single point), see Figure 4.4.

Denote by $\pi : \mathbb{R}^2 \times M \longrightarrow \mathbb{R}^2$ the projection, and by $S \subset \mathbb{R}^2$ the strip depicted in Figure 4.5. Put $V_0 = W \cap \pi^{-1}(S)$. According to [17], V_0 is a manifold with boundary, with two obvious boundary components corresponding to the L_i's and a third boundary component which is $W \cap \pi^{-1}(0)$. The latter is exactly the Polterovich surgery $L_1\#L_2$. Moreover V_0 is homotopy equivalent to V (in fact $V_0 \subset V$ and is a deformation retract of V). A straightforward calculation shows that there is an embedding $L_1 \vee L_2 \subset V_0$ and moreover that $L_1 \vee L_2$ is a deformation retract of V_0. (In fact, one can show that V_0 is diffeomorphic to the boundary connected sum of $[0,1] \times L_1$ and $[0,1] \times L_2$, where the connected sum occurs among the boundary components $\{1\} \times L_i$, $i = 1, 2$.)

The statement on monotonicity follows from the Seifert - Van Kampen theorem (see also [17]).

Assume now that L_1, L_2 are spin. Then $\widetilde{\gamma}_1 \times L_1$ and $\widetilde{\gamma}_2 \times L_2$ are also spin, with a spin structure extending those of the ends. Recall

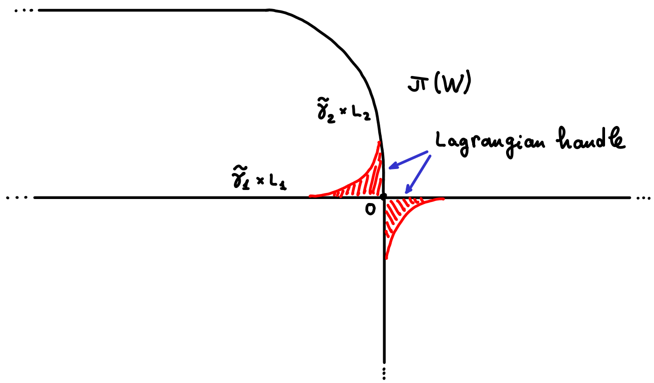

Figure 4.4.

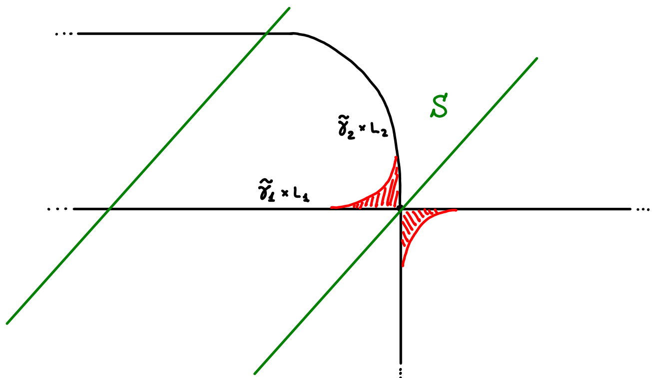

Figure 4.5.

that the connected sum of spin manifolds is also spin [47]. Thus $W = (\widetilde{\gamma}_1 \times L_1)\#(\widetilde{\gamma}_2 \times L_2)$ is spin too and by standard arguments it follows that the spin structure on W can be chosen so that it extends those given on the ends. By restriction we obtain a spin structure on $V_0 \subset W$ and consequently also the desired one on V. $\square$

5. Examples

5.1. Examples over the Laurent polynomial ring

This section is a continuation of §3.1.3 in which we provide more details to the examples. We will work here with the following setting. (M, ω) will be a monotone symplectic manifold with minimal Chern number C_M. To keep the notation short we will denote here by $QH(M)$ the quantum homology of M with coefficients in the ring $R = \mathbb{Z}[q^{-1}, q]$ (with $|q| = -2$), instead of writing $QH(M; R)$.

5.1.1. Lagrangian spheres in symplectic blow-ups of $\mathbb{C}P^2$ Denote as in §3.1.3 by M_k the blow-up of $\mathbb{C}P^2$ at $k \leq 6$ points endowed with a Kähler symplectic structure ω_k in the cohomology class of $c_1 \in H^2(M_k)$. Note that $-K_{M_k}$ is ample, hence c_1 represents a Kähler class. Note that $C_{M_k} = 1$.

We first claim that the set of classes in $H_2(M_k)$ which are represented by Lagrangian spheres are precisely those that appear in Table 3.1. This is well known and there are many ways to prove it (see e.g. [62, 29, 49, 64]). For the classes $A = E_i - E_j \in H_2(M_k)$ when $k = 2$ and $k = 3$ it is easy to find Lagrangian spheres in the class A by an explicit construction which we outline below (see [29] for more details). For $k \geq 4$, as well as $k = 3$ with $A = H - E_1 - E_2 - E_3$, it seems less trivial to perform explicit constructions and we could appeal instead to less transparent methods such as (relative) inflation like in [49, 64] (we will briefly outline this in a special case below). Another approach which works for some of the k's is to realize M_k as a fiber in a Lefschetz pencil and obtain the Lagrangian spheres as vanishing cycles (e.g. M_6 is the cubic surface in $\mathbb{C}P^3$ and M_5 is a complete intersection of two quadrics in $\mathbb{C}P^4$). Yet another approach comes from real algebraic geometry, where one can obtain

Lagrangian spheres in some of the M_k's as a component of the fixed point set of an anti-symplectic involution. This works for $k = 5, 6$ and all classes A, and for $k = 3$ with $A = E_i - E_j$. See [43] for more details. Finally note that for $2 \leq k \leq 8$, $k \neq 3$, the group of symplectomorphisms of M_k acts transitively on the set of classes that can be represented by Lagrangian spheres [26, 49], hence it is enough to construct one Lagrangian sphere in each M_k. (This also explains why the invariants in Table 3.1 coincide for different classes within each of the M_k's with the exception $k = 3$.)

Despite the many ways to establish Lagrangian spheres in the M_k's the shortest (albeit not the most explicit) path to this end is to appeal to the work Li-Wu [49]. According to [49] a homology class $A \in H_2(M_k)$ can be represented by a Lagrangian sphere iff it satisfies the following two conditions:

(LS-1) A can be represented by a smooth embedded 2-sphere.

(LS-2) $\langle [\omega_k], A \rangle = 0$.

(LS-3) $A \cdot A = -2$.

We remark again that we have assumed that $[\omega_k] = c_1$ (otherwise one has to assume in addition that $\langle c_1, A \rangle = 0$).

It is straightforward to see that all the classes in Table 3.1 satisfy conditions (LS-2) and (LS-3) above. As for condition (LS-1), note that if $C', C'' \subset M^4$ are two *disjoint* embedded smooth 2-spheres in a 4-manifold M^4, then by performing the connected sum one obtains a new smooth embedded 2-sphere in the class $[C'] + [C'']$. From this it follows that any non-trivial class of the form $\sum_{i=1}^{k} \epsilon_i E_i$ with $\epsilon_i \in \{-1, 0, 1\}$ can be represented by a smooth embedded 2-sphere. This settles the cases $\pm(E_i - E_j)$. For the other type of classes, note that H and $2H$ can both be represented by smooth embedded 2-spheres (e.g. a projective line and a conic respectively) hence the same holds also for classes of the form $\pm(H - E_i - E_j - E_l)$ and $\pm(2H - \sum_{i=1}^{6} E_i)$.

We remark that in fact there are no other classes but the ones in Table 3.1 that can be represented by Lagrangian spheres in M_k. This can be proved by elementary means using conditions (LS-2) and (LS-3) above.

Construction of Lagrangian spheres in M_2 and M_3 We now outline a more explicit way to construct Lagrangian spheres in some of the M_k's (c.f. [29]). Consider $Q = \mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with the symplectic form $\omega = 2\omega_{\mathbb{C}P^1} \oplus 2\omega_{\mathbb{C}P^1}$, where $\omega_{\mathbb{C}P^1}$ is the standard Kähler form on $\mathbb{C}P^1$ normalized so that $\mathbb{C}P^1$ has area 1. Note that the first Chern class of Q satisfies $c_1 = [\omega]$. The symplectic manifold Q contains a Lagrangian sphere $\overline{\Delta}$ in the class $[\mathbb{C}P^1 \times \mathrm{pt}] - [\mathrm{pt} \times \mathbb{C}P^1]$ (i.e. the class of the anti-diagonal). For example, we can write $\overline{\Delta}$ as the graph of the antipodal map, given in homogeneous coordinates by

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1, \qquad [z_0 : z_1] \longmapsto [-\overline{z_1} : \overline{z_0}].$$

Next, we claim that Q admits a symplectic embedding of two disjoint closed balls B_1, B_2 of capacity 1 whose images are disjoint from $\overline{\Delta}$. This can be easily seen from the toric picture. Indeed the image of the moment map of Q is the square $[0, 2] \times [0, 2]$ and the image of $\overline{\Delta}$ under that map is given by the anti-diagonal $\{(x, y) \mid x, y \in [0, 2], x + y = 2\}$. By standard arguments in toric geometry we can symplectically embed in Q a ball B_1 of capacity 1 whose image under the moment map is $\{(x, y) \mid x, y \in [0, 2], x+y \leq 1\}$. Similarly we can embed another ball B_2 whose image is $\{(x, y) \mid x, y \in [0, 2], x + y \geq 3\}$. Clearly B_1, B_2 and $\overline{\Delta}$ are mutually disjoint. Denote by $\widetilde{Q}_1$ the blow-up of Q with respect to B_1 and by $\widetilde{Q}_2$ the blow-up of Q with respect to both balls B_1 and B_2. It is well known that $\widetilde{Q}_1$ is symplectomorphic to M_2 via a symplectomorphism that sends the class $\overline{\Delta}$ to $E_1 - E_2$. And $\widetilde{Q}_2$ is symplectomorphic to M_3 by a similar symplectomorphism. It follows that $E_1 - E_2$ represents Lagrangian spheres both in M_2 and in M_3. Construction of Lagrangian spheres in the other classes of the type $E_i - E_j$ in M_3 can be done in a similar way.

Lagrangian spheres in the class $H - E_1 - E_2 - E_3$ in M_3 We start with the complex blow-up of $\mathbb{C}P^2$ at three points that *lie on the same projective line*. Denote by E_i the exceptional divisors over the blown-up points. The result of the blow up is a complex algebraic surface X which contains an embedded holomorphic rational curve Σ in the class $H - E_1 - E_2 - E_3$. Note also that there are three embedded holomorphic curves $C_i \subset X$, $i = 1, 2, 3$, in the classes $[C_i] = H - E_i$. Since $[C_i] \cdot [\Sigma] = 0$ the curves C_i are disjoint from Σ. Pick a Kähler symplectic structure ω_0 on X. After a suitable normalization we can write $[\omega_0] = h - \lambda_1 e_1 - \lambda_2 e_2 - \lambda_3 e_3$, where h, e_1, e_2, e_3 are the Poincaré duals to H, E_1, E_2, E_3 respectively. It is easy to check that $\lambda_i \geq 0$ and that $\lambda_1 + \lambda_2 + \lambda_3 < 1$. We now change ω_0 to a new symplectic form ω' such that:

1. ω' coincides with ω_0 outside a small neighborhood $\mathcal{U}$ of Σ, where $\mathcal{U}$ is disjoint from the curves C_1, C_2, C_3.

2. $\omega'|_{T(\Sigma)} \equiv 0$, i.e. Σ becomes a Lagrangian sphere with respect to ω'.

3. ω' and ω are in the same deformation class of symplectic forms on X (i.e. they can be connected by a path of symplectic forms).

This can be achieved for example using the *deflation* procedure [64] (see also [48]). Alternatively, one can construct ω' using Gompf fiber-sum surgery [39] with respect to $\Sigma \subset X$ and the diagonal in $\mathbb{C}P^1 \times \mathbb{C}P^1$:

$$(Y, \omega'') = (X, \omega_0)\,{}_{\Sigma}\#_{\mathrm{diag}}\,(\mathbb{C}P^1 \times \mathbb{C}P^1, a\omega_{\mathbb{C}P^1} \oplus a\omega_{\mathbb{C}P^1}),$$

where $a = \frac{1}{2}\int_{\Sigma} \omega_0$, and S^2 is symplectically embedded in X as Σ and in $\mathbb{C}P^1 \times \mathbb{C}P^1$ as the diagonal. Since the anti-diagonal $\overline{\Delta}$ is a Lagrangian sphere in $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is disjoint from the diagonal it gives rise to a Lagrangian sphere $L'' \subset Y$. Finally observe that the surgery has not changed the diffeomorphism type of X, namely there exists a diffeomorphism $\phi : Y \longrightarrow X$ and moreover ϕ can be chosen in such a way that $\phi(L'') = \Sigma$. Take now $\omega' = \phi_* \omega''$. To obtain a symplectic deformation between ω' and ω_0 one can perform the preceding surgery in a suitable one-parametric family, where the symplectic form on $\mathbb{C}P^1 \times \mathbb{C}P^1$ is rescaled so that the area of one of the factors becomes smaller and smaller and the area of the other increases so that the area of the diagonal stays constant.

Having replaced the form ω_0 by ω' we have a Lagrangian sphere in the desired homology class $H - E_1 - E_2 - E_3$ but the form ω' might not be in the cohomology class of c_1. We will now correct that using inflation.

After a normalization we can assume that $[\omega'] = h - \lambda_1' e_1 - \lambda_2' e_2 - \lambda_3' e_3$. Since Σ is Lagrangian with respect to ω' we have $\lambda_1' + \lambda_2' + \lambda_3' = 1$. Recall also that the surfaces C_1, C_2, C_3 are symplectic with respect to ω', hence $\lambda_i \leq 1$ for every i. Moreover, by construction, the surfaces C_1, C_2, C_3 can be made simultaneously J-holomorphic for some ω'-compatible almost complex structure J. Since the C_i's are disjoint from Σ we can find neighborhoods U_i of C_i such that the U_i's are disjoint from Σ. We now perform inflation simultaneously along the three surfaces C_1, C_2, C_3. More specifically, by the results of [10, 11] there exist closed 2-forms ρ_i supported in U_i, representing the Poincaré dual of $[C_i]$ (i.e. $[\rho_i] = h - e_i$) and such that the 2-form

$$\omega_{t_1,t_2,t_3} = \omega' + t_1\rho_1 + t_2\rho_2 + t_3\rho_3$$

is symplectic for every $t_1, t_2, t_3 \geq 0$. See Lemma 2.1 in [10] and Proposition 4.3 in [11] (see also [44, 45, 46, 53, 54]). The cohomology class of ω_t' is:

$$[\omega_t'] = (1 + t_1 + t_2 + t_3)h - (\lambda_1' + t_1)e_1 - (\lambda_2' + t_2)e_2 - (\lambda_3' + t_3)e_3.$$

Choosing $t_i^0 = 1 - \lambda_i'$ we have $t_i^0 > 0$ and $1 + t_1^0 + t_2^0 + t_3^0 = 4 - (\lambda_1' + \lambda_2' + \lambda_3') = 3$, hence:

$$[\omega'_{t_1^0,t_2^0,t_3^0}] = 3h - e_1 - e_2 - e_3 = c_1.$$

Due to the support of the forms ρ_i the surface Σ remains Lagrangian for $\omega'_{t_1^0,t_2^0,t_3^0}$. Finally note that $\omega'_{t_1^0,t_2^0,t_3^0}$ is in the same symplectic deformation class of ω_0, hence by standard results $(X, \omega'_{t_1^0, t_2^0, t_3^0})$ is symplectomorphic to M_3.

Calculation of the discriminant for M_k, $2 \leq k \leq 6$ We now give more details on the calculation of the discriminant Δ_L for each of the examples in Table 3.1. In what follows, for a symplectic manifold M, we denote by $p \in H_0(M)$ the homology class of a point. As before we write $QH(M)$ for the quantum homology ring of M with coefficients in $R = \mathbb{Z}[q^{-1}, q]$ where $|q| = -2$. The calculations below make use of the "multiplication table" of the quantum homology of the M_k's which can be found in [25].

Before we begin we recall that on M_k with $4 \leq k \leq 6$ the group of symplectomorphisms of M_k acts transitively on the set of classes that can be represented by Lagrangian spheres [26, 49]. Therefore, for $k \geq 4$ we will perform explicit calculations only for Lagrangians in the class $E_1 - E_2$.

2-point blow-up of $\mathbb{C}P^2$ $QH(M_2)$ has the following ring structure:

$$\begin{aligned}
p * p &= Hq^3 + [M_2]q^4 \\
p * H &= (H - E_1)q^2 + (H - E_2)q^2 + [M_2]q^3 \\
p * E_i &= (H - E_i)q^2 \\
H * H &= p + (H - E_1 - E_2)q + 2[M_2]q^2 \\
H * E_i &= (H - E_1 - E_2)q + [M_2]q^2 \\
E_1 * E_2 &= (H - E_1 - E_2)q \\
E_1 * E_1 &= -p + (H - E_2)q + [M_2]q^2 \\
E_2 * E_2 &= -p + (H - E_1)q + [M_2]q^2.
\end{aligned}$$

Consider Lagrangian spheres $L \subset M_2$ in the class $E_1 - E_2$. A straightforward calculation shows that:

$$(E_1 - E_2)^{*3} - 5(E_1 - E_2)q^2 = 0,$$

and thus we obtain $\Delta_L = 5$. Multiplication of c_1 with $[L]$ gives: $c_1 * (E_1 - E_2) = (-1)(E_1 - E_2)q$, hence $\lambda_L = -1$. The associated ideal (see §3.2.4) $\mathcal{I}_L \subset QH_*(M_2)$ is:

$$\mathcal{I}(E_1 - E_2) = R(-2p + (E_1 + E_2)q + 2[M_2]q^2) \oplus R(E_1 - E_2).$$

3-point blow-up of $\mathbb{C}P^2$ $QH(M_3)$ has the following ring structure:

$$\begin{aligned}
&p * p = (3H - E_1 - E_2 - E_3)q^3 + 3[M_3]q^4 \\
&p * H = (3H - E_1 - E_2 - E_3)q^2 + 3[M_3]q^3 \\
&p * E_i = (H - E_i)q^2 + [M_3]q^3 \\
&H * H = p + (3H - 2E_1 - 2E_2 - 2E_3)q + 3[M_3]q^2 \\
&H * E_i = (2H - 2E_i - E_j - E_k)q + [M_3]q^2, \quad i \neq j \neq k \neq i \\
&E_i * E_i = -p + (2H - E_1 - E_2 - E_3)q + [M_3]q^2 \\
&E_i * E_j = (H - E_i - E_j)q, \quad i \neq j.
\end{aligned}$$

Consider Lagrangians $L, L' \subset M_3$ in the classes $[L] = E_i - E_j$ and $[L'] = H - E_1 - E_2 - E_3$. The corresponding Lagrangian cubic equations are given by:

$$\begin{aligned}
&(E_i - E_j)^{*3} - 4(E_i - E_j)q^2 = 0, \\
&(H - E_1 - E_2 - E_3)^{*3} + 3(H - E_1 - E_2 - E_3)q^2 = 0,
\end{aligned}$$

and thus obtain $\Delta_L = 4$ and $\Delta_{L'} = -3$. Multiplication with c_1 gives:

$$\begin{aligned}
&c_1 * (E_i - E_j) = (-2)(E_i - E_j)q, \\
&c_1 * (H - E_1 - E_2 - E_3) = (-3)(H - E_1 - E_2 - E_3)q,
\end{aligned}$$

hence $\lambda_L = -2$ and $\lambda_{L'} = -3$. The associated ideals in $QH(M_3)$ are:

$$\begin{aligned}
\mathcal{I}_L &= R(-2p + 2(H - E_3)t + 2[M_3]q^2) \oplus R(E_1 - E_2), \\
\mathcal{I}_{L'} &= R(-2p + (3H - E_1 - E_2 - E_3)q + 4[M_3]q^2) \\
&\qquad \oplus R(H - E_1 - E_2 - E_3).
\end{aligned}$$

The Lagrangian spheres in different homology classes of the type $E_i - E_j$ in M_3 have the same discriminant and the same eigenvalue λ_L. This is so because for every $i < j$ there is a symplectomorphism $\varphi : M_3 \longrightarrow M_3$ such that $\varphi_*(E_1 - E_2) = E_i - E_j$. In contrast, note that there exists no symplectomorphism of M_3 sending $E_1 - E_2$ to $H - E_1 - E_2 - E_3$. This was previously shown in [68].

4-point blow-up of $\mathbb{C}P^2$ $QH(M_4)$ has the following ring structure:

$$
\begin{aligned}
p * p &= (9H - 3E_1 - 3E_2 - 3E_3 - 3E_4)q^3 + 10[M_4]q^4 \\
p * H &= (8H - 3E_1 - 3E_2 - 3E_3 - 3E_4)q^2 + 9[M_4]q^3 \\
p * E_i &= (3H - 2E_i - \sum_{j \neq i} E_j)q^2 + 3[M_4]q^3 \\
H * H &= p + (6H - 3E_1 - 3E_2 - 3E_3 - 3E_4)q + 8[M_4]q^2 \\
H * E_i &= (3H - 3E_i - \sum_{j \neq i} E_j)q + 3[M_4]q^2 \\
E_i * E_i &= -p + (3H - 2E_i - \sum_{j \neq i} E_j)q + 2[M_4]q^2 \\
E_i * E_j &= (H - E_i - E_j)q + [M_4]q^2
\end{aligned}
$$

As explained above it is enough to calculate our invariants for Lagrangians in the class $E_1 - E_2$. A straightforward calculation shows that:

$$(E_1 - E_2)^{*3} = (E_1 - E_2)q^2, \quad c_1 * (E_1 - E_2) = -3(E_1 - E_2)q,$$

hence $\Delta_L = 1$ and $\lambda_L = -3$. The associated ideals for Lagrangians L, L' with $[L] = E_1 - E_2$ and $L' = H - E_1 - E_2 - E_3$ are:

$$
\begin{aligned}
\mathcal{I}_L = R(&-2p + (4H - E_1 - E_2 - 2E_3 - 2E_4)q + 2[M_4]q^2) \\
&\oplus R(E_1 - E_2), \\
\mathcal{I}_{L'} = R(&-2p + (3H - E_1 - E_2 - E_3)q + 2[M_4]q^2) \\
&\oplus R(H - E_1 - E_2 - E_3).
\end{aligned}
$$

5-point blow-up of $\mathbb{C}P^2$ $QH(M_5)$ has the following ring structure:

$$
\begin{aligned}
p * p &= (36H - 12E_1 - 12E_2 - 12E_3 - 12E_4 - 12E_5)q^3 + 52[M_5]q^4 \\
p * H &= (25H - 9E_1 - 9E_2 - 9E_3 - 9E_4 - 9E_5)q^2 + 36[M_5]q^3 \\
p * E_i &= (9H - 5E_i - 3\sum_{j\neq i} E_j)q^2 + 12[M_5]q^3 \\
H * H &= p + (18H - 8E_1 - 8E_2 - 8E_3 - 8E_4 - 8E_5)q + 25[M_5]q^2 \\
H * E_i &= (8H - 6E_i - 3\sum_{j\neq i} E_j)q + 9[M_5]q^2 \\
E_i * E_i &= -p + (6H - 4E_i - 2\sum_{j\neq i} E_j)q + 5[M_5]q^2 \\
E_i * E_j &= (3H - 2E_i - 2E_j - \sum_{k\neq i,j} E_k)q + 3[M_5]q^2
\end{aligned}
$$

As before, it is enough to consider only the case $[L] = E_1 - E_2$. A direct calculation gives:

$$(E_1 - E_2)^{*3} = 0, \quad c_1 * (E_1 - E_2) = -4(E_1 - E_2)q,$$

hence $\Delta_L = 0$, $\lambda_L = -4$. The associated ideals for Lagrangians L, L' with $[L] = E_1 - E_2$ and $[L'] = H - E_1 - E_2 - E_3$ are:

$$
\begin{aligned}
\mathcal{I}_L &= R(-2p + (6H - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5)q + 4[M_5]q^2) \\
&\quad \oplus R(E_1 - E_2), \\
\mathcal{I}_{L'} &= R(-2p + (6H - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5)q + 4[M_5]q^2) \\
&\quad \oplus R(H - E_1 - E_2 - E_3).
\end{aligned}
$$

6-point blow-up of $\mathbb{C}P^2$ $QH(M_6)$ has the following the ring structure:

$$\begin{aligned}
p * p &= (252H - 84E_1 - 84E_2 - 84E_3 - 84E_4 - 84E_5 - 84E_6)q^3 \\
&\qquad + 540[M_6]q^4 \\
p * H &= (120H - 42E_1 - 42E_2 - 42E_3 - 42E_4 - 42E_5 - 42E_6)q^2 \\
&\qquad + 252[M_6]q^3 \\
p * E_i &= (42H - 20E_i - 14\sum_{j\neq i} E_j)q^2 + 84[M_6]q^3 \\
H * H &= p + (63H - 25E_1 - 25E_2 - 25E_3 - 25E_4 - 25E_5 - 25E_6)q \\
&\qquad + 120[M_6]q^2 \\
H * E_i &= (25H - 15E_i - 9\sum_{j\neq i} E_j)q + 42[M_6]q^2 \\
E_i * E_i &= -p + (15H - 9E_i - 5\sum_{j\neq i} E_j)q + 20[M_6]q^2 \\
E_i * E_j &= (9H - 5E_i - 5E_j - 3\sum_{k\neq i,j} E_j)q + 14[M_6]q^2
\end{aligned}$$

Again, we may assume without loss of generality that $[L] = E_1 - E_2$. A direct calculation gives:

$$(E_1 - E_2)^{*3} = 0, \quad c_1 * (E_1 - E_2) = -6(E_1 - E_2)q,$$

hence $\Delta_L = 0$, $\lambda_L = -6$.

Interestingly, the associated ideals $\mathcal{I}_L$ for Lagrangians L in any of the classes: $E_i - E_j$, $2H - E_i - E_j - E_l$, $2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6$ all coincide:

$$\mathcal{I}_L = R(-2p + (12H - 4\sum_{j=1}^{6} E_j)q + 12[M_6]q^2) \bigoplus R(2H - \sum_{j=1}^{6} E_j).$$

5.1.2. Lagrangian spheres in Fano hypersurfaces Let $M^{2n} \subset \mathbb{C}P^{n+1}$ be a Fano hypersurface of degree d, where $n \geq 3$. We endow

M with the symplectic structure induced from $\mathbb{C}P^{n+1}$. It is easy to check that M is monotone and that the minimal Chern number is $C_M = n+2-d$.

We view the homology $H_*(M;\mathbb{Q})$ as a ring, endowed with the intersection product which we denote by $a \cdot b$ for $a, b \in H_*(M;\mathbb{Q})$. Write $h \in H_{2n-2}(M;\mathbb{Q})$ for the class of a hyperplane section. The homology $H_*(M;\mathbb{Q})$ is generated as a ring by the class h and the subspace of primitive classes, denoted by $H_n(M;\mathbb{Q})_0$. (Recall that the latter is by definition the kernel of the map $H_n(M;\mathbb{Q}) \longrightarrow H_{n-2}(M;\mathbb{Q})$, $a \longmapsto a \cdot h$).

Assume that $d \geq 2$. Then by Picard-Lefschetz theory M contains Lagrangian spheres (that can be realized as vanishing cycles of the Lefschetz pencil associated to the embedding $M \subset \mathbb{C}P^{n+1}$). Assume further that $2C_M|n$. Then the Lagrangian spheres $L \subset M$ have minimal Maslov number $N_L = 2C_M$ and it is easy to see that they satisfy Assumption $\mathscr{L}$ (see e.g. Proposition 6). Therefore in this case the discriminant Δ_L is defined and there is a cubic equation too.

In order to calculate these, we appeal to the work of Collino-Jinzenji [23] (see also [38, 9, 67] for related results). We set $x := h + d![M]q$ if $C_M = 1$, and $x := h$, if $C_M \geq 2$. Specifically, we will need the following:

Theorem 5.1.1 (Collino-Jinzenji [23]). *In the quantum homology ring of M with coefficients in $\mathbb{Q}[q]$ we have the following identities:*

1. $x * a = 0$ *for every* $a \in H_n(M;\mathbb{Q})_0$.
2. $a * b = \frac{1}{d}\#(a \cdot b)(x^{*n} - d^d x^{*(d-2)} q^{n+2-d})$ *for every* $a, b \in H_n(M;\mathbb{Q})_0$.

Coming back to our Lagrangian spheres $L \subset M$, we clearly have $[L] \in H_n(M;\mathbb{Q})_0$. Therefore we obtain from Theorem 5.1.1:

$$[L] * [L] * [L] = \frac{1}{d}\#([L] \cdot [L])(x^{*n} * [L] - d^d x^{*(d-2)} * [L] q^{n+2-d}) = 0, \tag{5.1}$$

where in the last equality we have used that $d > 2$ (hence $x^{*(d-2)} * [L] = 0$). The fact that we must have $d > 2$ follows easily from the assumption that $2C_M | n$.

It follows immediately from (5.1) that $\Delta_L = 0$.

An example which is not a sphere All our examples so far were for Lagrangians that are spheres. However, our theory is more general and applies to other topological types of Lagrangians (see e.g. Assumption $\mathscr{L}$, Proposition 6 and Theorem 1). Here is such an example with $L \simeq S^m \times S^m$.

Let $Q \subset \mathbb{C}P^{m+1}$ be the complex m-dimensional quadric $Q = \{[z_0 : \ldots : z_{m+1}] \mid -z_0^2 + \ldots + z_{m+1}^2 = 0\}$ endowed with the symplectic structure induced from $\mathbb{C}P^{m+1}$. Then $S := \{[z_0 : \ldots : z_{m+1}] \mid -z_0^2 + \ldots + z_{m+1}^2 = 0, z_i \in \mathbb{R}\}$ is a Lagrangian sphere. The first Chern class c_1 of Q equals the Poincaré dual of mh, where h is a hyperplane section of Q associated to the projective embedding $Q \subset \mathbb{C}P^{m+1}$. The minimal Chern number is $C_Q = m$ and S has minimal Maslov number $N_S = 2m$. Note that S does not satisfy Assumption $\mathscr{L}$ (since N_S does not divide m). *Henceforth we will assume that $m =$ even.*

Put $M = Q \times Q$ endowed with the split symplectic structure induced from both factors and consider the Lagrangian submanifold $L \subset M$ which is the product of two copies of S:

$$L := S \times S \subset Q \times Q.$$

Put $2n = \dim_{\mathbb{R}} M$ so that $\dim L = n = 2m$.

The symplectic manifold $Q \times Q$ has minimal Chern number $C_M = m$ and the minimal Maslov number of L is $N_L = 2m = n$. By Proposition 6, L satisfies assumption $\mathscr{L}$.

For our calculations the following identities in the quantum homology ring of Q will be relevant (see e.g. [9]):

1. $h * [S] = 0$.
2. $a * b = \frac{1}{2}\#(a \cdot b)(h^{*m} - 4[Q]q^m)$ for every $a, b \in H_m(Q; \mathbb{Q})_0$.

To calculate Δ_L we compute $[L]^{*3}$ in $QH(Q \times Q)$. By the Künneth formula in quantum homology [55] we have $QH(Q \times Q; \mathbb{Z}[q]) \cong QH(Q; \mathbb{Z}[q]) \otimes_{\mathbb{Z}[q]} QH(Q; \mathbb{Z}[q])$. Together with the previous identities (with $a = b = [S]$) this gives:

$$[L] * [L] = ([S] * [S]) \otimes ([S] * [S]) = (h^{*m} - 4[Q]q^m) \otimes (h^{*m} - 4[Q]q^m),$$

and therefore

$$\begin{aligned} [L]^{*3} &= (h^{*m} * [S] - 4[S]q^m) \otimes (h^{*m} * [S] - 4[S]q^m) \\ &= 16[S] \otimes [S]q^{2m} \ = \ 16[L]q^{2m}. \end{aligned}$$

It follows that $\sigma_L = 0$ and $\tau_L = 1$ (in the notation of Theorem 1), hence $\Delta_L = 4\tau_L = 4$.

5.2. Finer invariants over more general rings

Much of the theory developed in the previous sections can be enriched so that the discriminant Δ_L and the cubic equation take into account the homology classes of the holomorphic curves involved in their definition. The result is clearly a finer invariant.

We now briefly explain this generalization. Let $L \subset (M, \omega)$ be a monotone Lagrangian submanifold. Denote by $H_2^D(M, L) \subset H_2(M, L; \mathbb{Z})$ the image of the Hurewicz homomorphism $\pi_2(M, L) \longrightarrow H_2(M, L; \mathbb{Z})$. We abbreviate $H_2^D = H_2^D(M, L)$ when L is clear from the discussion.

We will use here the ring $\widetilde{\Lambda}^+$, introduced in [15], which is the universal ring of coefficients for Lagrangian quantum homology. Denote by $\widetilde{\Lambda}^+$ the following ring:

$$\widetilde{\Lambda}^+ = \Big\{ p(T) \mid p(T) = c_0 + \sum_{\substack{A \in H_2^D \\ \mu(A) > 0}} c_A T^A, \quad c_0, c_A \in \mathbb{Z} \Big\}. \tag{5.2}$$

We grade $\widetilde{\Lambda}^+$ by assigning to the monomial T^A degree $|T^A| = -\mu(A)$. Note that the degree-0 component of $\widetilde{\Lambda}^+$ is just $\mathbb{Z}$ (not linear combinations of T^A with $\mu(A) = 0$). As explained in [15] we can define $QH(L; \widetilde{\Lambda}^+)$, and in fact $QH(L; \mathcal{R})$ for rings $\mathcal{R}$ which are $\widetilde{\Lambda}^+$-algebras.

Similarly to $\widetilde{\Lambda}^+$ we associate to the ambient manifold the ring $\widetilde{\Gamma}^+$. This ring is defined in the same way as $\widetilde{\Lambda}^+$ but with H_2^D replaced by $H_2^S := \text{image}\,(\pi_2(M) \longrightarrow H_2(M; \mathbb{Z}))$ and with $\mu(A) > 0$ replaced by $\langle c_1, A \rangle > 0$ in (5.2). To avoid confusion we will denote the formal variable in $\widetilde{\Gamma}^+$ with S and we grade $|S^A| = -2\langle c_1, A \rangle$. Similarly to $QH(L; \widetilde{\Lambda}^+)$ we can define the ambient quantum homology $QH(M; \widetilde{\Gamma}^+)$ with coefficients in $\widetilde{\Gamma}^+$ and in fact with coefficients in any ring $\mathcal{A}$ which is a $\widetilde{\Gamma}^+$-algebra. In particular, since the map $H_2^S \longrightarrow H_2^D$ gives $\widetilde{\Lambda}^+$ the structure of an $\widetilde{\Gamma}^+$-algebra and we can define $QH(M; \widetilde{\Lambda}^+) = QH(M; \widetilde{\Gamma}^+) \otimes_{\widetilde{\Gamma}^+} \widetilde{\Lambda}^+$.

Assume for simplicity that L satisfies the assumptions of Proposition 6. Then the conclusion of Proposition 6 holds with $HF(L, L)$ replaced by $QH(L; \widetilde{\Lambda}^+)$ in the sense that $\text{rank}_{\mathbb{Z}}\, QH(L; \widetilde{\Lambda}^+) = 2$. Assume further that L is oriented and spinable. Again, the main example satisfying all these assumptions is L being a Lagrangian sphere in a monotone symplectic manifold M with $2C_M | \dim L$.

The definition of the discriminant Δ_L carries over to this setting as follows. Pick an element $x \in QH_0(L; \widetilde{\Lambda}^+)$ which lifts $[\text{point}] \in H_0(L)$ as in §3.2.5. Write

$$x * x = \widetilde{\sigma} x + \widetilde{\tau} e_L,$$

where $\widetilde{\sigma}, \widetilde{\tau} \in \widetilde{\Lambda}^+$ are elements of degrees $|\widetilde{\sigma}| = -n$ and $|\widetilde{\tau}| = -2n$ respectively. As before, the elements $\widetilde{\sigma}$ and $\widetilde{\tau}$ depend on x. Define

$$\widetilde{\Delta}_L = \widetilde{\sigma}^2 + 4\widetilde{\tau} \in \widetilde{\Lambda}^+.$$

The same arguments as in §3.2.5 show that $\widetilde{\Delta}_L$ is independent of the choice of x.

Theorem continues to hold but the cubic equation (3.1) has now the form:

$$[L]^{*3} - \varepsilon\chi\widetilde{\sigma}_L [L]^{*2} - \chi^2 \widetilde{\tau}_L [L] = 0, \tag{5.3}$$

where $\widetilde{\sigma}_L \in \frac{1}{\chi}\widetilde{\Lambda}^+$, $\widetilde{\tau}_L \in \frac{1}{\chi^2}\widetilde{\Lambda}^+$ are uniquely determined. (Note that in (5.3) we do not have the variable q anymore since the elements $\chi\widetilde{\sigma}_L, \chi^2\widetilde{\tau}_L$ are assumed in advance to be in the ring $\widetilde{\Lambda}^+$.) As for identity (3.2), it now becomes:

$$\widetilde{\sigma}_L = \frac{1}{\chi^2}\sum_A GW_{A,3}([L],[L],[L])T^{j(A)}, \tag{5.4}$$

where $j : H_2^S \longrightarrow H_2^D$ is the map induced by inclusion.

Analogous versions of Theorem 4.1.1 hold over $\widetilde{\Lambda}^+$ too.

Denoting by $\bar{L}$ the Lagrangian L with the opposite orientation, it is easy to check that

$$\widetilde{\sigma}_{\bar{L}} = -\widetilde{\sigma}_L, \quad \widetilde{\tau}_{\bar{L}} = \widetilde{\tau}_L, \quad \widetilde{\Delta}_{\bar{L}} = \widetilde{\Delta}_L. \tag{5.5}$$

We now discuss the action of symplectic diffeomorphisms on these invariants. Let $\varphi : M \longrightarrow M$ be a symplectomorphism. The action $\varphi_* : H_2^S \longrightarrow H_2^S$ of φ on homology induces an isomorphism of rings $\varphi_\Gamma : \widetilde{\Gamma}^+ \longrightarrow \widetilde{\Gamma}^+$. Put $L' = \varphi(L)$. Instead of the preceding ring $\widetilde{\Lambda}^+$ we now have two rings $\widetilde{\Lambda}^+_L$ and $\widetilde{\Lambda}^+_{L'}$ associated to L and to L' respectively. The action $\varphi_D : H_2^D \longrightarrow H_2^D$ of φ on homology induces an isomorphism of rings $\varphi_\Lambda : \widetilde{\Lambda}^+_L \longrightarrow \widetilde{\Lambda}^+_{L'}$. Moreover, writing an $\mathcal{R}$-algebra $\mathcal{A}$ as ${}_{\mathcal{R}}\mathcal{A}$, the pair of maps $(\varphi_\Lambda, \varphi_\Gamma)$ gives rise to an isomorphism of algebras ${}_{\widetilde{\Gamma}^+}\widetilde{\Lambda}^+_L \longrightarrow {}_{\widetilde{\Gamma}^+}\widetilde{\Lambda}^+_{L'}$.

Turning to quantum homologies, standard arguments together with the previous discussion yield two ring isomorphisms (both denoted φ_Q by abuse of notation):

$$\begin{aligned} \varphi_Q &: QH(L;\widetilde{\Lambda}^+_L) \longrightarrow QH(L';\widetilde{\Lambda}^+_{L'}), \\ \varphi_Q &: QH(M;\widetilde{\Lambda}^+_L) \longrightarrow QH(M;\widetilde{\Lambda}^+_{L'}), \end{aligned}$$

which are linear over $\widetilde{\Gamma}^+$ via φ_Γ and also $(\widetilde{\Lambda}^+_L, \widetilde{\Lambda}^+_{L'})$ linear via φ_Λ. Most of the theory from §3.2.2 extends, with suitable modifications, to the present setting.

The following follows immediately from the preceding discussion and (5.5) above (c.f. Corollary 2):

Theorem 5.2.1. *Let $\varphi : M \longrightarrow M$ be a symplectomorphism. Then:*

$$\widetilde{\sigma}_{\varphi(L)} = \varphi_{\Lambda}(\widetilde{\sigma}_L), \quad \widetilde{\tau}_{\varphi(L)} = \varphi_{\Lambda}(\widetilde{\tau}_L), \quad \widetilde{\Delta}_{\varphi(L)} = \varphi_{\Lambda}(\widetilde{\Delta}_L).$$

In particular $\widetilde{\tau}_L$ and $\widetilde{\Delta}_L$ are invariant under the action of the group $\mathrm{Symp}(M, L)$ *of symplectomorphisms $\varphi : (M, L) \longrightarrow (M, L)$ and $\widetilde{\sigma}_L$ is invariant under the action of the subgroup* $\mathrm{Symp}^+(M, L) \subset \mathrm{Symp}(M, L)$ *of those φ's that preserve the orientation on L. If $\varphi \in \mathrm{Symp}(M, L)$ reverses orientation on L then $\varphi_{\Lambda}(\widetilde{\sigma}_L) = -\widetilde{\sigma}_L$.*

Before we move on to examples we explain how to obtain the older invariants $\sigma_L, \tau_L, \Delta_L$ from the ones described here. Denote by $\eta : \widetilde{\Lambda}^+ \longrightarrow \mathbb{Z}[t]$ the ring homomorphism defined by $\eta(T^A) = t^{\mu(A)/N_L}$ for every $A \in H_2^D$. An analogous map can be defined also for $\widetilde{\Gamma}^+ \longrightarrow \mathbb{Z}[q]$. These homomorphisms induce ring maps $QH(L; \widetilde{\Lambda}^+) \longrightarrow Q^+H(L)$ and $\eta_Q : QH(M; \widetilde{\Lambda}^+) \longrightarrow Q^+H(M)$ with obvious compatibility properties. The following theorem easily follows.

Theorem 5.2.2. *Applying η_Q to the cubic equation* (5.3) *we obtain the equation* (3.1)*. In particular $\eta(\widetilde{\sigma}_L) = \sigma_L t^{n/N_L}$, $\eta(\widetilde{\tau}_L) = \tau_L t^{2n/N_L}$, $\eta(\widetilde{\Delta}_L) = \Delta_L t^{2n/N_L}$.*

5.2.1. Examples revisited Here we briefly present the outcome of the calculation of our invariants $\widetilde{\sigma}_L$, $\widetilde{\tau_L}$ and $\widetilde{\Delta}_L$ on blow-ups of $\mathbb{C}P^2$ at $2 \leq k \leq 6$ points. We use similar notation as in §3.1.3. For simplicity we denote by $u \in H_4(M_k)$ the fundamental class viewed as the unity of $QH(M_k)$. As before we appeal to [25] for the calculation of the quantum homology of the ambient manifolds. Since the explicit calculations in $QH(M_k)$ turn out to be very lengthy we often omit the details and present only the end results (more details can be found in the appendix). We recall again that in $QH(M; \widetilde{\Gamma}^+)$ the quantum variables are denoted now by S^A where $A \in H_2^S$.

2-point blow-up of $\mathbb{C}P^2$ $QH(M_2; \widetilde{\Gamma}^+)$ has the following ring structure:

$$
\begin{aligned}
&p * p = HS^H + uS^{2H-E_1-E_2} \\
&p * H = (H - E_1)S^{H-E_1} + (H - E_2)S^{H-E_2} + uS^H \\
&p * E_1 = (H - E_1)S^{H-E_1} \\
&p * E_2 = (H - E_2)S^{H-E_2} \\
&H * H = p + (H - E_1 - E_2)S^{H-E_1-E_2} + u(S^{H-E_1} + S^{H-E_2}) \\
&H * E_1 = (H - E_1 - E_2)S^{H-E_1-E_2} + uS^{H-E_1} \\
&H * E_2 = (H - E_1 - E_2)S^{H-E_1-E_2} + uS^{H-E_2} \\
&E_1 * E_1 = -p + (H - E_1 - E_2)S^{H-E_1-E_2} + E_1S^{E_1} + uS^{H-E_1} \\
&E_2 * E_2 = -p + (H - E_1 - E_2)S^{H-E_1-E_2} + E_2S^{E_2} + uS^{H-E_2} \\
&E_1 * E_2 = (H - E_1 - E_2)S^{H-E_1-E_2}.
\end{aligned}
$$

Let $L \subset M_2$ be a Lagrangian sphere in the class $[L] = E_1 - E_2$. Then $H_2^D = H_2(M, L) \cong H_2(M)/H_2(L)$ and as a basis for H_2^D we can choose $\{H, E\}$, where E stands for the image of both E_1 and E_2 in $H_2(M)/H_2(L)$. (Thus in $\widetilde{\Lambda}^+$ we have $S^{E_1} = S^{E_2} = T^E$.)

A straightforward calculation shows that the Lagrangian cubic equation becomes:

$$(E_1 - E_2)^{*3} = (T^{2E} + 4T^{H-E})(E_1 - E_2),$$

and therefore $\widetilde{\sigma}_L = 0$ and

$$\widetilde{\Delta}_L = 4\widetilde{\tau}_L = T^{2E} + 4T^H.$$

3-point blow-up of $\mathbb{C}P^2$ The multiplication table for $QH(M_3; \widetilde{\Gamma}^+)$ is rather long hence we omit it here (see the appendix for these details).

Consider first Lagrangian spheres $L \subset M_3$ in the class $[L] = E_1 - E_2$. We choose $\{H, E, E_3\}$ for a basis for H_2^D where E stands for the image of both of E_1 and E_2 in H_2^D. A straightforward calculation using the Lagrangian cubic equation gives $\widetilde{\sigma}_L = 0$ and

$$\widetilde{\Delta}_L = 4\widetilde{\tau}_L = 4T^{H-E} + T^{2E} - 2T^{H-E_3} + T^{2H-2E-2E_3}.$$

Next we consider Lagrangian $L \subset M_3$ with $[L] = H - E_1 - E_2 - E_3$. We work with the basis $\{E_1, E_2, E_3\}$ for H_2^D. Direct calculation shows that $\widetilde{\sigma}_L = 0$ and

$$\widetilde{\Delta}_L = 4\widetilde{\tau}_L = T^{2E_1} + T^{2E_2} + T^{2E_3} - 2T^{E_1+E_2} - 2T^{E_1+E_3} - 2T^{E_2+E_3}.$$

4-point blow-up of $\mathbb{C}P^2$ Consider Lagrangian spheres in the class $[L] = E_1 - E_2$ and work with the basis $\{H, E, E_3, E_4\}$, where $E = [E_1] = [E_2] \in H_2^D$. Omitting the details of a rather long calculation we obtain $\widetilde{\sigma}_L = 0$ and

$$\begin{aligned}\widetilde{\Delta}_L = 4\widetilde{\tau}_L &= T^{2E} + 4T^{H-E} - 2T^{H-E_3} - 2T^{H-E_4} \\ &\quad + T^{2H-2E-2E_3} + T^{2H-2E-2E_4} - 2T^{2H-2E-E_3-E_4}.\end{aligned}$$

For Lagrangian spheres in the class $[L] = H - E_1 - E_2 - E_3$ we obtain $\widetilde{\sigma}_L = 0$ and

$$\begin{aligned}\widetilde{\Delta}_L = 4\widetilde{\tau}_L &= T^{2E_1} + T^{2E_2} + T^{2E_3} - 2T^{E_1+E_2} - 2T^{E_1+E_3} \\ &\quad - 2T^{E_2+E_3} + 4T^{E_1+E_2+E_3-E_4},\end{aligned}$$

where we have worked here with the basis $\{E_1, E_2, E_3, E_4\}$ for H_2^D.

5-point blow-up of $\mathbb{C}P^2$ Consider Lagrangian spheres in the class $[L] = E_1 - E_2$ and work with the basis $\{H, E, E_3, E_4, E_5\}$, where $E = [E_1] = [E_2] \in H_2^D$. Omitting the details of a rather long calculation (see the appendix) we obtain $\widetilde{\sigma}_L = 0$ and

$$\begin{aligned}\widetilde{\Delta}_L = 4\widetilde{\tau}_L &= T^{2E} + 4T^{H-E} - 2T^{H-E_3} - 2T^{H-E_4} - 2T^{H-E_5} \\ &\quad + T^{2H-2E-2E_3} + T^{2H-2E-2E_4} + T^{2H-2E-2E_5} \\ &\quad - 2T^{2H-2E-E_3-E_4} - 2T^{2H-2E-E_3-E_5} \\ &\quad - 2T^{2H-2E-E_4-E_5} + 4T^{2H-E-E_3-E_4-E_5}.\end{aligned}$$

Consider now a Lagrangian sphere in the class $[L] = H - E_1 - E_2 - E_3$. We work with the basis $\{E_1, E_2, E_3, E_4, E_5\}$ for H_2^D. We obtain again $\widetilde{\sigma}_L = 0$ and

$$\begin{aligned}\widetilde{\Delta}_L = 4\widetilde{\tau}_L = {} & T^{2E_1} + T^{2E_2} + T^{2E_3} - 2T^{E_1+E_2} - 2T^{E_1+E_3} \\ & - 2T^{E_2+E_3} + 4T^{E_1+E_2+E_3-E_4} + 4T^{E_1+E_2+E_3-E_5} \\ & + T^{2(E_1+E_2+E_3-E_4-E_5)} - 2T^{2E_1+E_2+E_3-E_4-E_5} \\ & - 2T^{E_1+2E_2+E_3-E_4-E_5} - 2T^{E_1+E_2+2E_3-E_4-E_5}.\end{aligned}$$

6-point blow-up of $\mathbb{C}P^2$ Due to the complexity of the calculation we restrict here to Lagrangians in the class $[L] = E_1 - E_2$. We work with the basis $\{H, E, E_3, E_4, E_5, E_5, E_6\}$ for H_2^D, where $E = [E_1] = [E_2]$.

$$\begin{aligned}\widetilde{\Delta}_L = {} & T^{2E} + 4T^{H-E} - 2T^{H-E_3} - 2T^{H-E_4} - 2T^{H-E_5} - 2T^{H-E_6} \\ & + T^{2H-2E-2E_3} + T^{2H-2E-2E_4} + T^{2H-2E-2E_5} + T^{2H-2E-2E_6} \\ & - 2T^{2H-2E-E_3-E_4} - 2T^{2H-2E-E_3-E_5} - 2T^{2H-2E-E_3-E_6} \\ & - 2T^{2H-2E-E_4-E_5} - 2T^{2H-2E-E_4-E_6} \\ & - 2T^{2H-2E-E_5-E_6} - 2T^{2H-E_3-E_4-E_5-E_6} \\ & + 4T^{2H-E-E_3-E_4-E_5} + 4T^{2H-E-E_3-E_4-E_6} \\ & + 4T^{2H-E-E_3-E_5-E_6} + 4T^{2H-E-E_4-E_5-E_6} \\ & - 2T^{3H-2E-2E_3-E_4-E_5-E_6} - 2T^{3H-2E-E_3-2E_4-E_5-E_6} \\ & - 2T^{3H-2E-E_3-E_4-2E_5-E_6} - 2T^{3H-2E-E_3-E_4-E_5-2E_6} \\ & + 4T^{3H-3E-E_3-E_4-E_5-E_6} + T^{4H-2E-2E_3-2E_4-2E_5-2E_6}\end{aligned}$$

Appendix

A. Various proofs of algebraic statements

A.1. Coupled deformation cohomology

Proposition A.1.1. *d^D is a coboundary operator, i.e.* $(d^D)^2 = 0$.

For the proof we define the operators

$$b_i : D^n(A, M) \longrightarrow D^{n+1}(A, M), \quad 0 \leq i \leq n+1,$$

by setting for $\varphi \in D^n(A, M)$

$$\begin{aligned} b_0(\varphi)(a_1, \dots, a_{n+1}, m) &= a_1 \varphi(a_2, \dots, a_{n+1}, m), \\ b_j(\varphi)(a_1, \dots, a_{n+1}, m) &= \varphi(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}, m), \\ b_{n+1}(\varphi)(a_1, \dots, a_{n+1}, m) &= \varphi(a_1, \dots, a_n, a_{n+1} m). \end{aligned}$$

for $1 \leq j \leq n$. The operator d^D is then given by

$$d^D = \sum_{i=0}^{n+1} (-1)^i b_i.$$

Lemma A.1.2. *For $0 \leq i < j \leq n+2$ we have*

$$b_j \circ b_i = b_i \circ b_{j-1}.$$

Proof.

$$\begin{aligned} & b_j \circ b_i(\varphi)(a_1, \dots, a_{n+2}, m) \\ = \ & b_i(\varphi)(a_1, \dots, a_j a_{j+1}, \dots, a_{n+2}, m) \\ = \ & \varphi(a_1, \dots, a_i a_{i+1}, \dots, a_j a_{j+1}, \dots, a_{n+2}, m) \\ = \ & b_i \circ b_{j-1}(\varphi)(a_1, \dots, a_{n+2}, m) \end{aligned}$$

□

Proof of A.1.1.

$$\begin{aligned}
(d^D)^2 &= \sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-1)^{i+j} b_j \circ b_i \\
&= \sum_{j=1}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j} b_j \circ b_i + \sum_{j=0}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j} b_j \circ b_i \\
&= \sum_{i=0}^{n+1}\sum_{j=i+1}^{n+2}(-1)^{i+j} b_j \circ b_i + \sum_{j=0}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j} b_j \circ b_i \\
&= \sum_{i=0}^{n+1}\sum_{j=i+1}^{n+2}(-1)^{i+j} b_i \circ b_{j-1} + \sum_{j=0}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j} b_j \circ b_i \\
&= \sum_{i=0}^{n+1}\sum_{j=i}^{n+1}(-1)^{i+j+1} b_i \circ b_j + \sum_{j=0}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j} b_j \circ b_i \\
&= 0.
\end{aligned}$$

□

Proposition A.1.3. *The complex $(D^*(A,X), d^D)$ with the product $\bullet$ is a differential graded algebra, i.e.*

$$d^D(\varphi \bullet \tau) = (d^D\varphi) \bullet \tau + (-1)^{|\varphi|}\varphi \bullet (d^D\tau).$$

Proof. We compute for $\varphi \in D^k(A,X)$ and $\tau \in D^l(A,X)$,

$$\begin{aligned}
&d^D(\varphi \bullet \tau)(a_1, \ldots, a_{k+l+1}, x) \\
&= a_1\varphi(a_2, \ldots, a_{k+1}, \tau(a_{k+2}, \ldots, a_{k+l+1}, x)) \\
&\quad + \sum_{i=1}^{k}(-1)^i \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{k+1}, \tau(a_{k+2}, \ldots, a_{k+l+1}, x)) \\
&\quad + \sum_{i=k+1}^{k+l}(-1)^i \varphi(a_1, \ldots, a_k, \tau(a_{k+1}, \ldots, a_j a_{j+1}, \ldots, a_{k+l+1}, x)) \\
&\quad + (-1)^{k+l+1}\varphi(a_1, \ldots, a_k, \tau(a_{k+1}, \ldots, a_{k+l}, a_{k+l+1}x)).
\end{aligned}$$

The two terms on the right hand side are

$$\begin{aligned}
&(d^D\varphi \bullet \tau)(a_1, \ldots, a_{k+l+1}, x) \\
&= \; a_1\varphi(a_2, \ldots, a_{k+1}, \tau(a_{k+2}, \ldots, a_{k+l+1}, x)) \\
&\quad + \sum_{j=1}^{k}(-1)^j\varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{k+1}, \tau(a_{k+2}, \ldots, a_{k+l+1}, x)) \\
&\quad + (-1)^{k+1}\varphi(a_1, \ldots, a_k, a_{k+1}\tau(a_{k+2}, \ldots, a_{k+l+1}, x)),
\end{aligned}$$

and

$$\begin{aligned}
&(\varphi \bullet d^D\tau)(a_1, \ldots, a_{k+l+1}, x) \\
&= \; \varphi(a_1, \ldots, a_k, d^D\tau(a_{k+1}, \ldots, a_{k+l+1}, x)) \\
&= \; \varphi(a_1, \ldots, a_k, a_{k+1}\tau(a_{k+2}, \ldots, a_{k+l+1}, x)) \\
&\quad + \sum_{j=1}^{l}(-1)^j\varphi(a_1, \ldots, a_k, \tau(a_{k+1}, \ldots, a_{k+j}a_{k+j+1}, \ldots, a_{k+l+1}, x)) \\
&\quad + (-1)^{l+1}\varphi(a_1, \ldots, a_k, \tau(a_{k+1}, \ldots, a_{k+l}, a_{k+l+1}x)).
\end{aligned}$$

Multiplying $(\varphi \bullet d^D\tau)$ with $(-1)^k = (-1)^{|\varphi|}$ we see that

$$d^D(\varphi \bullet \tau) = (d^D\varphi) \bullet \tau + (-1)^k\varphi \bullet (d^D\tau),$$

which finishes the proof. □

A.2. Triple deformation cohomology

The module action $\mu_0 : A \otimes X \to X$ induces three cochain maps, which we list as follows (we use the same notation for the first and second map):

$$\begin{aligned}
\mu_{0*} : C^n(A,A) &\longrightarrow C^n(A\otimes X, X),\\
\eta &\longmapsto \eta(a_1,\dots,a_n)x_1\cdots x_n,\\
\mu_{0*} : C^n(X,X) &\longrightarrow C^n(A\otimes X, X),\\
\rho &\longmapsto a_1\cdots a_n\rho(x_1,\dots,x_n),\\
\mu_0^* : C^n(X,X) &\longrightarrow C^n(A\otimes X, X),\\
\rho &\longmapsto \rho(a_1x_1,\dots,a_nx_n),
\end{aligned}$$

Lemma A.2.1. *The maps $(\mu_0)_*$ and $(\mu_0)^*$ are cochain maps.*

Proof. For $\eta \in C^n(A,A)$ we

$$\begin{aligned}
&d(\mu_{0*}\eta)(a_1\otimes x_1,\dots,a_{n+1}\otimes x_{n+1})\\
&= (a_1\otimes x_1)(\mu_{0*}\eta)(a_2\otimes x_2,\dots,a_{n+1}\otimes x_{n+1})\\
&\quad + \sum_{i=1}^{n}(-1)^i(\mu_{0*}\eta)(a_1\otimes x_1,\dots,a_ia_{i+1}\otimes x_ix_{i+1},\dots,a_{n+1}\otimes x_{n+1})\\
&\quad + (-1)^{n+1}(\mu_{0*}\eta)(a_1\otimes x_1,\dots,a_n\otimes x_n)(a_{n+1}\otimes x_{n+1})\\
&= (a_1x_1)\eta(a_2,\dots,a_{n+1})x_2\cdots x_{n+1}\\
&\quad + \sum_{i=1}^{n}(-1)^i\eta(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})x_1\cdots x_{n+1}\\
&\quad + (-1)^{n+1}\eta(a_1,\dots,a_n)(x_1\cdots x_n)(a_{n+1}x_{n+1})\\
&= (d\eta)(a_1,\dots,a_{n+1})x_1\cdots x_{n+1}\\
&= \mu_{0*}(d\eta)(a_1\otimes x_1,\dots,a_{n+1}\otimes x_{n+1}),
\end{aligned}$$

since A lies in the center of X. For $\rho \in C^n(X,X)$ we compute

$$\begin{aligned}
&d(\mu_{0*}\rho)(a_1 \otimes x_1, \ldots, a_{n+1} \otimes x_{n+1}) \\
&= (a_1 \otimes x_1)(\mu_{0*}\rho)(a_2 \otimes x_2, \ldots, a_{n+1} \otimes x_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i (\mu_{0*}\rho)(a_1 \otimes x_1, \ldots, a_i a_{i+1} \otimes x_i x_{i+1}, \ldots, a_{n+1} \otimes x_{n+1}) \\
&\quad + (-1)^{n+1} (\mu_{0*}\rho)(a_1 \otimes x_1, \ldots, a_n \otimes x_n)(a_{n+1} \otimes x_{n+1}) \\
&= (a_1 x_1)(a_2 \cdots a_{n+1})\rho(x_2, \ldots, x_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i (a_1 \cdots a_{n+1})\rho(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\
&\quad + (-1)^{n+1} (a_1 \cdots a_{n+1})\rho(x_1, \ldots, x_n)(a_{n+1} x_{n+1}) \\
&= (a_1 \cdots a_{n+1}) d\rho(x_1, \ldots, x_{n+1}) \\
&= \mu_{0*}(d\rho)(a_1 \otimes x_1, \ldots, a_{n+1} \otimes x_{n+1})
\end{aligned}$$

Now for $\rho \in C^n(X, X)$ we compute

$$\begin{aligned}
&d(\mu_0^*\rho)(a_1 \otimes x_1, \ldots, a_{n+1} \otimes x_{n+1}) \\
&= (a_1 \otimes x_1)(\mu_0^*\rho)(a_2 \otimes x_2, \ldots, a_{n+1} \otimes x_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i (\mu_0^*\rho)(a_1 \otimes x_1, \ldots, a_i a_{i+1} \otimes x_i x_{i+1}, \ldots, a_{n+1} \otimes x_{n+1}) \\
&\quad + (-1)^{n+1} (\mu_0^*\rho)(a_1 \otimes x_1, \ldots, a_n \otimes x_n)(a_{n+1} \otimes x_{n+1}) \\
&= (a_1 x_1)(a_2 \cdots a_{n+1})\rho(x_2, \ldots, x_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i (a_1 \cdots a_{n+1})\rho(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\
&\quad + (-1)^{n+1} (a_1 \cdots a_n)\rho(x_1, \ldots, x_n)(a_{n+1} x_{n+1}) \\
&= (a_1 \cdots a_{n+1}) d\rho(x_1, \ldots, x_{n+1}) \\
&= \mu_0^*(d\rho)(a_1 \otimes x_1, \ldots, a_{n+1} \otimes x_{n+1})
\end{aligned}$$

$\square$

A.3. Cohomology of deformations of module morphisms

Let $\psi_0 : X \to Y$ be an A-module morphism. Then ψ_0 defines two maps

$$\begin{aligned} \psi_{0*} : D^n(A,X) &\longrightarrow C^n(A;X,Y), \\ \psi_0^* : D^n(A,Y) &\longrightarrow C^n(A;X,Y), \end{aligned}$$

given by

$$\begin{aligned} (\psi_{0*}\xi)(a_1,\dots,a_n)(x) &:= \psi_0(\xi(a_1,\dots,a_n,x)), \\ (\psi_0^*\zeta)(a_1,\dots,a_n)(x) &:= \zeta(a_1,\dots,a_n,\psi_0(x)). \end{aligned}$$

Lemma A.3.1. *The maps ψ_{0*} and ψ_0^* are cochain maps.*

Proof. For $\xi \in D^n(A,X)$ we compute

$$\begin{aligned} &\left[d(\psi_{0*}\xi)(a_1,\dots,a_{n+1})\right](x) \\ = \;& \left[a_1(\psi_{0*}\xi)(a_2,\dots,a_{n+1})\right](x) \\ &+ \left[\sum_{i=1}^{n}(-1)^i(\psi_{0*}\xi)(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})\right](x) \\ &+ \left[(-1)^{n+1}(\psi_{0*}\xi)(a_1,\dots,a_n)a_{n+1}\right](x) \\ = \;& a_1\psi_0(\xi(a_1,\dots,a_{n+1},x)) \\ &+ \sum_{i=1}^{n}(-1)^i\psi_0(\xi(a_1,\dots,a_ia_{i+1},\dots,a_{n+1},x)) \\ &+ (-1)^{n+1}\psi_0(\xi(a_1,\dots,a_n,a_{n+1}x)) \\ = \;& \psi_0(d\xi(a_1,\dots,a_{n+1},x)) \\ = \;& \left[\psi_{0*}(d\xi)(a_1,\dots,a_{n+1})\right](x). \end{aligned}$$

For $\zeta \in D^n(A, Y)$ we compute

$$\begin{aligned}
&[d(\psi_0^*\zeta)(a_1, \ldots, a_{n+1})]\,(x)\\
&= \;[a_1(\psi_0^*\zeta)(a_2, \ldots, a_{n+1})]\,(x)\\
&\quad + \left[\sum_{i=1}^{n}(-1)^i(\psi_0^*\zeta)(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1})\right](x)\\
&\quad + \left[(-1)^{n+1}(\psi_0^*\zeta)(a_1, \ldots, a_n)a_{n+1}\right](x)\\
&= \;a_1\zeta(a_2, \ldots, a_{n+1}, \psi_0(x))\\
&\quad + \sum_{i=1}^{n}(-1)^i\zeta(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}, \psi_0(x))\\
&\quad + (-1)^{n+1}\zeta(a_1, \ldots, a_n, \psi_0(a_{n+1}x))\\
&= \;d\zeta(a_1, \ldots, a_{n+1}, \psi_0(x))\\
&= \;[\psi_0^*(d\zeta)(a_1, \ldots, a_{n+1})]\,(x)
\end{aligned}$$

□

B. Quantum homology ring of Del Pezzo surfaces

B.1. 3-point blow-up of $\mathbb{C}P^2$

$QH(M_3;\widetilde{\Gamma}^+)$ has the following ring structure:

$$
\begin{aligned}
p * p = \;& HS^H + (2H - E_1 - E_2 - E_3)S^{2H-E_1-E_2-E_3} \\
& + u(S^{2H-E_1-E_2} + S^{2H-E_1-E_3} + S^{2H-E_2-E_3}) \\
p * H = \;& (H - E_1)S^{H-E_1} + (H - E_2)S^{H-E_2} + (H - E_3)S^{H-E_3} \\
& + uS^H + 2uS^{2H-E_1-E_2-E_3} \\
p * E_1 = \;& (H - E_1)S^{H-E_1} + uS^{2H-E_1-E_2-E_3} \\
p * E_2 = \;& (H - E_2)S^{H-E_2} + uS^{2H-E_1-E_2-E_3} \\
p * E_3 = \;& (H - E_3)S^{H-E_3} + uS^{2H-E_1-E_2-E_3} \\
H * H = \;& p + (H - E_1 - E_2)S^{H-E_1-E_2} + (H - E_1 - E_3)S^{H-E_1-E_3} \\
& + (H - E_2 - E_3)S^{H-E_2-E_3} + u(S^{H-E_1} + S^{H-E_2} + S^{H-E_3}) \\
H * E_1 = \;& (H - E_1 - E_2)S^{H-E_1-E_2} + (H - E_1 - E_3)S^{H-E_1-E_3} + uS^{H-E_1} \\
H * E_2 = \;& (H - E_1 - E_2)S^{H-E_1-E_2} + (H - E_2 - E_3)S^{H-E_2-E_3} + uS^{H-E_2} \\
H * E_3 = \;& (H - E_1 - E_3)S^{H-E_1-E_3} + (H - E_2 - E_3)S^{H-E_2-E_3} + uS^{H-E_3} \\
E_1 * E_1 = \;& -p + E_1S^{E_1} + (H - E_1 - E_2)S^{H-E_1-E_2} \\
& + (H - E_1 - E_3)S^{H-E_1-E_3} + uS^{H-E_1} \\
E_2 * E_2 = \;& -p + E_2S^{E_2} + (H - E_1 - E_2)S^{H-E_1-E_2} \\
& + (H - E_2 - E_3)S^{H-E_2-E_3} + uS^{H-E_2} \\
E_3 * E_3 = \;& -p + E_3uS^{E_3} + (H - E_1 - E_3)S^{H-E_1-E_3} \\
& + (H - E_2 - E_3)S^{H-E_2-E_3} + uS^{H-E_3} \\
E_1 * E_2 = \;& (H - E_1 - E_2)S^{H-E_1-E_2} \\
E_1 * E_3 = \;& (H - E_1 - E_3)S^{H-E_1-E_3} \\
E_2 * E_3 = \;& (H - E_2 - E_3)S^{H-E_2-E_3}
\end{aligned}
$$

B.2. 4-point blow-up of $\mathbb{C}P^2$

$QH(M_4;\widetilde{\Gamma}^+)$ has the following ring structure:

$$\begin{aligned}
p*p = \;& (2H-E_1-E_2-E_3)S^{2H-E_1-E_2-E_3} + (2H-E_1-E_2-E_4)S^{2H-E_1-E_2-E_4}\\
& + (2H-E_1-E_3-E_4)S^{2H-E_1-E_3-E_4} + (2H-E_2-E_3-E_4)S^{2H-E_2-E_3-E_4}\\
& + u(S^{2H-E_1-E_2} + S^{2H-E_1-E_3} + S^{2H-E_1-E_4} + S^{2H-E_2-E_3} + S^{2H-E_2-E_4}\\
& + S^{2H-E_3-E_4} + S^{3H-2E_1-E_2-E_3-E_4} + S^{3H-E_1-2E_2-E_3-E_4}\\
& + S^{3H-E_1-E_2-2E_3-E_4} + S^{3H-E_1-E_2-E_3-2E_4}) + HS^H\\
p*H = \;& (H-E_1)S^{H-E_1} + (H-E_2)S^{H-E_2} + (H-E_3)S^{H-E_3} + (H-E_4)S^{H-E_4}\\
& + 2(2H-E_1-E_2-E_3-E_4)S^{2H-E_1-E_2-E_3-E_4}\\
& + u(S^H + 2S^{2H-E_1-E_2-E_3} + 2S^{2H-E_1-E_2-E_4}\\
& + 2S^{2H-E_1-E_3-E_4} + 2S^{2H-E_2-E_3-E_4})\\
p*E_1 = \;& (H-E_1)S^{H-E_1} + (2H-E_1-E_2-E_3-E_4)S^{2H-E_1-E_2-E_3-E_4}\\
& + u(S^{2H-E_1-E_2-E_3} + S^{2H-E_1-E_2-E_4} + S^{2H-E_1-E_3-E_4})\\
p*E_2 = \;& (H-E_2)S^{H-E_2} + (2H-E_1-E_2-E_3-E_4)S^{2H-E_1-E_2-E_3-E_4}\\
& + u(S^{2H-E_1-E_2-E_3} + S^{2H-E_1-E_2-E_4} + S^{2H-E_2-E_3-E_4})\\
p*E_3 = \;& (H-E_3)S^{H-E_3} + (2H-E_1-E_2-E_3-E_4)S^{2H-E_1-E_2-E_3-E_4}\\
& + u(S^{2H-E_1-E_2-E_3} + S^{2H-E_1-E_3-E_4} + S^{2H-E_2-E_3-E_4})\\
p*E_4 = \;& (H-E_4)S^{H-E_4} + (2H-E_1-E_2-E_3-E_4)S^{2H-E_1-E_2-E_3-E_4}\\
& + u(S^{2H-E_1-E_2-E_4} + S^{2H-E_1-E_3-E_4} + S^{2H-E_2-E_3-E_4})\\
H*H = \;& p + (H-E_1-E_2)S^{H-E_1-E_2} + (H-E_1-E_3)S^{H-E_1-E_3}\\
& + (H-E_1-E_4)S^{H-E_1-E_4} + (H-E_2-E_3)S^{H-E_2-E_3}\\
& + (H-E_2-E_4)S^{H-E_2-E_4} + (H-E_3-E_4)S^{H-E_3-E_4}\\
& + u(S^{H-E_1} + S^{H-E_2} + S^{H-E_3} + S^{H-E_4} + 4S^{2H-E_1-E_2-E_3-E_4})\\
H*E_1 = \;& (H-E_1-E_2)S^{H-E_1-E_2} + (H-E_1-E_3)S^{H-E_1-E_3}\\
& + (H-E_1-E_4)S^{H-E_1-E_4} + u(S^{H-E_1} + 2S^{2H-E_1-E_2-E_3-E_4})\\
H*E_2 = \;& (H-E_1-E_2)S^{H-E_1-E_2} + (H-E_2-E_3)S^{H-E_2-E_3}\\
& + (H-E_2-E_4)S^{H-E_2-E_4} + u(S^{H-E_2} + 2S^{2H-E_1-E_2-E_3-E_4})\\
H*E_3 = \;& (H-E_1-E_3)S^{H-E_1-E_3} + (H-E_2-E_3)S^{H-E_2-E_3}\\
& + (H-E_3-E_4)S^{H-E_3-E_4} + u(S^{H-E_3} + 2S^{2H-E_1-E_2-E_3-E_4})\\
H*E_4 = \;& (H-E_1-E_4)S^{H-E_1-E_4} + (H-E_2-E_4)S^{H-E_2-E_4}\\
& + (H-E_3-E_4)S^{H-E_3-E_4} + u(S^{H-E_4} + 2S^{2H-E_1-E_2-E_3-E_4})
\end{aligned}$$

$$
\begin{aligned}
E_1 * E_1 = \ & -p + E_1 S^{E_1} + (H - E_1 - E_2) S^{H-E_1-E_2} \\
& + (H - E_1 - E_3) S^{H-E_1-E_3} + (H - E_1 - E_4) S^{H-E_1-E_4} \\
& + u(S^{H-E_1} + S^{2H-E_1-E_2-E_3-E_4}) \\
E_2 * E_2 = \ & -p + E_2 S^{E_2} + (H - E_1 - E_2) S^{H-E_1-E_2} \\
& + (H - E_2 - E_3) S^{H-E_2-E_3} + (H - E_2 - E_4) S^{H-E_2-E_4} \\
& + u(S^{H-E_2} + S^{2H-E_1-E_2-E_3-E_4}) \\
E_3 * E_3 = \ & -p + E_3 S^{E_3} + (H - E_1 - E_3) S^{H-E_1-E_3} \\
& + (H - E_2 - E_3) S^{H-E_2-E_3} + (H - E_3 - E_4) S^{H-E_3-E_4} \\
& + u(S^{H-E_3} + S^{2H-E_1-E_2-E_3-E_4}) \\
E_4 * E_4 = \ & -p + E_4 S^{E_4} + (H - E_1 - E_4) S^{H-E_1-E_4} \\
& + (H - E_2 - E_4) S^{H-E_2-E_4} + (H - E_3 - E_4) S^{H-E_3-E_4} \\
& + u(S^{H-E_4} + S^{2H-E_1-E_2-E_3-E_4}) \\
E_1 * E_2 = \ & (H - E_1 - E_2) S^{H-E_1-E_2} + u S^{2H-E_1-E_2-E_3-E_4} \\
E_1 * E_3 = \ & (H - E_1 - E_3) S^{H-E_1-E_3} + u S^{2H-E_1-E_2-E_3-E_4} \\
E_1 * E_4 = \ & (H - E_1 - E_4) S^{H-E_1-E_4} + u S^{2H-E_1-E_2-E_3-E_4} \\
E_2 * E_3 = \ & (H - E_2 - E_3) S^{H-E_2-E_3} + u S^{2H-E_1-E_2-E_3-E_4} \\
E_2 * E_4 = \ & (H - E_2 - E_4) S^{H-E_2-E_4} + u S^{2H-E_1-E_2-E_3-E_4} \\
E_3 * E_4 = \ & (H - E_3 - E_4) S^{H-E_3-E_4} + u S^{2H-E_1-E_2-E_3-E_4}
\end{aligned}
$$

B.3. 5-point blow-up of $\mathbb{C}P^2$

To abbreviate notation we write for classes $A \in H_2^S(M)$,

$$A = dH - a_i E_i - a_j E_j - a_k E_k - a_l E_l - a_m E_m,$$

where the indices $i, j, k, l, m \in \{1, 2, 3, 4, 5\}$ are always distinct and the sums are always taken over all permutations of indices. For example the notation: "4 classes: $H - E_1 - E_i$" denotes the sum over $H - E_1 - E_2$, $H - E_1 - E_3$, $H - E_1 - E_4$ and $H - E_1 - E_5$. In the ring structure of $QH(M_5; \widetilde{\Gamma}^+)$ we restrict ourselves to the products relevant to the computation of the discriminant. These products are symmetric with respect to E_i, we write only the products with E_1 explicitly.

$$\begin{aligned} p * H = \ & \underbrace{(H - E_1)S^{H-E_1} + \ldots + (H - E_5)S^{H-E_5}}_{\text{5 classes: } H - E_i} \\ & + \underbrace{2(2H - E_1 - E_2 - E_3 - E_4)S^{2H-E_1-E_2-E_3-E_4} + \ldots}_{\text{5 classes: } 2H - E_i - E_j - E_k - E_l} \\ & + uS^H + 2u(\underbrace{S^{2H-E_1-E_2-E_3} + \ldots + S^{2H-E_3-E_4-E_5}}_{\text{10 classes: } 2H - E_i - E_j - E_k}) \\ & + 3u(\underbrace{S^{3H-2E_1-E_2-\ldots-E_5} + \ldots + S^{2H-E_1-\ldots-E_4-2E_5}}_{\text{5 classes: } 3H - 2E_i - E_j - E_k - E_l - E_m}) \end{aligned}$$

$$\begin{aligned}
p * E_1 = \;& (H - E_1)S^{H-E_1} \\
&+ \underbrace{(2H - E_1 - E_2 - E_3 - E_4)S^{2H-E_1-E_2-E_3-E_4} + \ldots}_{\text{4 classes: } 2H-E_1-E_i-E_j-E_k} \\
&+ u(\underbrace{S^{2H-E_1-E_2-E_3} + \ldots + S^{2H-E_1-E_4-E_5}}_{\text{6 classes: } 2H-E_1-E_i-E_j}) \\
&+ 2uS^{3H-2E_1-E_2-E_3-E_4-E_5} \\
&+ u(\underbrace{S^{3H-E_1-2E_2-E_3-E_4-E_5} + \ldots + S^{3H-E_1-E_2-E_3-E_4-2E_5}}_{\text{4 classes: } 3H-E_1-2E_i-E_j-E_k-E_l}) \\
E_1 * E_1 = \;& -p + E_1 S^{E_1} \\
&+ \underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_1 - E_5)S^{H-E_1-E_5}}_{\text{4 classes: } H-E_1-E_i} \\
&+ (2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} \\
&+ u(S^{H-E_1} + \underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_1-E_3-E_4-E_5}}_{\text{4 classes: } 2H-E_1-E_i-E_j-E_k}) \\
E_1 * E_2 = \;& (H - E_1 - E_2)S^{H-E_1-E_2} \\
&+ (2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} \\
&+ u(S^{2H-E_1-E_2-E_3-E_4} + S^{2H-E_1-E_2-E_3-E_5} + S^{2H-E_1-E_2-E_4-E_5}) \\
H * H = \;& p + (\underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_4 - E_5)S^{H-E_4-E_5}}_{\text{10 classes: } H-E_i-E_j}) \\
&+ 4(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} \\
&+ u(\underbrace{S^{H-E_1} + \ldots + S^{H-E_5}}_{\text{5 classes: } H-E_i}) \\
&+ u(\underbrace{4S^{2H-E_1-E_2-E_3-E_4} + \ldots + 4S^{2H-E_2-E_3-E_4-E_5}}_{\text{5 classes: } 2H-E_i-E_j-E_k-E_l}) \\
H * E_1 = \;& \underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_1 - E_5)S^{H-E_1-E_5}}_{\text{4 classes: } H-E_1-E_i} \\
&+ 2(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} \\
&+ uS^{H-E_1} \\
&+ 2u(\underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_1-E_3-E_4-E_5}}_{\text{4 classes: } 2H-E_1-E_i-E_j-E_k})
\end{aligned}$$

B.4. 6-point blow-up of $\mathbb{C}P^2$

To abbreviate notation we write for classes $A \in H_2^S(M)$,

$$A = dH - a_iE_i - a_jE_j - a_kE_k - a_lE_l - a_mE_m - a_nE_n,$$

where the indices $i, j, k, l, m, n \in \{1, 2, 3, 4, 5, 6\}$ are always distinct and the sums are always taken over all permutations of indices. For example the notation: "5 classes: $H - E_1 - E_i$" denotes the sum over $H - E_1 - E_2$, $H - E_1 - E_3$, $H - E_1 - E_4$, $H - E_1 - E_5$ and $H - E_1 - E_6$. In the ring structure of $QH(M_6; \widetilde{\Gamma}^+)$ we restrict ourselves to the products relevant to the computation of the discriminant. These products are symmetric with respect to E_i, we write only the products with E_1 explicitly.

$$\begin{aligned}
p * H = \; & \underbrace{(H - E_1)S^{H-E_1} + \ldots + (H - E_6)S^{H-E_6}}_{\text{6 classes: } H-E_i} \\
& + \underbrace{2(2H - E_1 - E_2 - E_3 - E_4)S^{2H-E_1-E_2-E_3-E_4} + \ldots}_{\text{15 classes: } 2H-E_i-E_j-E_k-E_l} \\
& + \underbrace{3(3H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6)S^{3H-2E_1-E_2-E_3-E_4-E_5-E_6} + \ldots}_{\text{6 classes: } 3H-2E_i-E_j-E_k-E_l-E_m-E_n} \\
& + uS^H + 2u(\underbrace{S^{2H-E_1-E_2-E_3} + \ldots + S^{2H-E_4-E_5-E_6}}_{\text{20 classes: } 2H-E_i-E_j-E_k}) \\
& + 3u(\underbrace{S^{3H-2E_1-E_2-E_3-E_4-E_5} + \ldots + S^{2H-E_2-E_3-E_4-E_5-2E_6}}_{\text{30 classes: } 3H-2E_i-E_j-E_k-E_l-E_m}) \\
& + 36uS^{3H-E_1-E_2-E_3-E_4-E_5-E_6} + 5uS^{5H-2E_1-2E_2-2E_3-2E_4-2E_5-2E_6} \\
& + 4u(\underbrace{S^{4H-2E_1-2E_2-2E_3-E_4-E_5-E_6} + \ldots + S^{4H-E_1-E_2-E_3-2E_4-2E_5-2E_6}}_{\text{20 classes: } 4H-2E_i-2E_j-2E_k-E_l-E_m-E_n})
\end{aligned}$$

$$
\begin{aligned}
p * E_1 = \ & (H - E_1)S^{H-E_1} \\
& + \underbrace{(2H - E_1 - E_2 - E_3 - E_4)S^{2H-E_1-E_2-E_3-E_4} + \ldots}_{\text{10 classes: } 2H-E_1-E_i-E_j-E_k} \\
& + 2(3H - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6)S^{3H-2E_1-E_2-E_3-E_4-E_5-E_6} \\
& + \underbrace{(3H - E_1 - 2E_2 - E_3 - E_4 - E_5 - E_6)S^{3H-E_1-2E_2-E_3-E_4-E_5-E_6} + \ldots}_{\text{5 classes: } 3H-E_1-2E_i-E_j-E_k-E_l-E_m} \\
& + u(\underbrace{S^{2H-E_1-E_2-E_3} + \ldots + S^{2H-E_1-E_5-E_6}}_{\text{10 classes: } 2H-E_1-E_i-E_j}) \\
& + 12uS^{3H-E_1-E_2-E_3-E_4-E_5-E_6} + 2uS^{5H-2E_1-2E_2-2E_3-2E_4-2E_5-2E_6} \\
& + 2u(\underbrace{S^{3H-2E_1-E_2-E_3-E_4-E_5} + \ldots + S^{3H-2E_1-E_3-E_4-E_5-E_6}}_{\text{5 classes: } 3H-2E_1-E_i-E_j-E_k-E_l}) \\
& + u(\underbrace{S^{3H-E_1-2E_2-E_3-E_4-E_5} + \ldots + S^{3H-E_1-E_3-E_4-E_5-2E_6}}_{\text{20 classes: } 3H-E_1-2E_i-E_j-E_k-E_l}) \\
& + 2u(\underbrace{S^{4H-2E_1-2E_2-2E_3-E_4-E_5-E_6} + \ldots + S^{4H-2E_1-E_2-E_3-E_4-2E_5-2E_6}}_{\text{10 classes: } 4H-2E_1-2E_i-2E_j-E_k-E_l-E_m}) \\
& + u(\underbrace{S^{4H-E_1-2E_2-2E_3-2E_4-E_5-E_6} + \ldots + S^{4H-E_1-E_2-E_3-2E_4-2E_5-2E_6}}_{\text{10 classes: } 4H-E_1-2E_i-2E_j-2E_k-E_l-E_m}) \\
E_1 * E_1 = \ & -p + E_1S^{E_1} \\
& + \underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_1 - E_6)S^{H-E_1-E_6}}_{\text{5 classes: } H-E_1-E_i} \\
& + \underbrace{(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} + \ldots}_{\text{5 classes: } 2H-E_1-E_i-E_j-E_k-E_l} \\
& + u(S^{H-E_1} + \underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_1-E_4-E_5-E_6}}_{\text{10 classes: } 2H-E_1-E_i-E_j-E_k}) \\
& + 4uS^{3H-2E_1-E_2-E_3-E_4-E_5-E_6} \\
& + u(\underbrace{S^{3H-E_1-2E_2-E_3-E_4-E_5-E_6} + \ldots + S^{3H-E_1-E_2-E_3-E_4-E_5-2E_6}}_{\text{5 classes: } 3H-E_1-2E_i-E_j-E_k-E_l-E_m}) \\
E_1 * E_2 = \ & (H - E_1 - E_2)S^{H-E_1-E_2} \\
& + \underbrace{(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} + \ldots}_{\text{4 classes: } 2H-E_1-E_2-E_i-E_j-E_k} \\
& + u(\underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_1-E_2-E_5-E_6}}_{\text{6 classes: } 2H-E_1-E_2-E_i-E_j}) \\
& + 2uS^{3H-2E_1-E_2-E_3-E_4-E_5-E_6} + 2uS^{3H-E_1-2E_2-E_3-E_4-E_5-E_6} \\
& + u(\underbrace{S^{3H-E_1-E_2-2E_3-E_4-E_5-E_6} + \ldots + S^{3H-E_1-E_2-E_3-E_4-E_5-2E_6}}_{\text{4 classes: } 3H-E_1-E_2-2E_i-E_j-E_k-E_l})
\end{aligned}
$$

$$
\begin{aligned}
H * H = \;& p + (\underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_5 - E_6)S^{H-E_5-E_6}}_{\text{15 classes: } H-E_i-E_j})\\
& + \underbrace{4(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} + \ldots}_{\text{6 classes: } 2H-E_i-E_j-E_k-E_l-E_m}\\
& + u(\underbrace{S^{H-E_1} + \ldots + S^{H-E_6}}_{\text{6 classes: } H-E_i})\\
& + 4u(\underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_3-E_4-E_5-E_6}}_{\text{15 classes: } 2H-E_i-E_j-E_k-E_l})\\
& + 9u(\underbrace{S^{3H-2E_1-E_2-E_3-E_4-E_5-E_6} + \ldots + S^{3H-E_1-E_2-E_3-E_4-E_5-2E_6}}_{\text{6 classes: } 3H-2E_i-E_j-E_k-E_l-E_m-E_n})\\
H * E_1 = \;& \underbrace{(H - E_1 - E_2)S^{H-E_1-E_2} + \ldots + (H - E_1 - E_6)S^{H-E_1-E_6}}_{\text{5 classes: } H-E_1-E_i}\\
& + \underbrace{2(2H - E_1 - E_2 - E_3 - E_4 - E_5)S^{2H-E_1-E_2-E_3-E_4-E_5} + \ldots}_{\text{5 classes: } 2H-E_1-E_i-E_j-E_k-E_l}\\
& + uS^{H-E_1}\\
& + 2u(\underbrace{S^{2H-E_1-E_2-E_3-E_4} + \ldots + S^{2H-E_1-E_4-E_5-E_6}}_{\text{10 classes: } 2H-E_1-E_i-E_j-E_k})\\
& + 6uS^{3H-2E_1-E_2-E_3-E_4-E_5-E_6}\\
& + 3u(\underbrace{S^{3H-E_1-2E_2-E_3-E_4-E_5-E_6} + \ldots + S^{3H-E_1-E_2-E_3-E_4-E_5-2E_6}}_{\text{5 classes: } 3H-E_1-2E_i-E_j-E_k-E_l-E_m})
\end{aligned}
$$

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