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Rasmussen invariants of Whitehead doubles and other satellites

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Abstract. We prove formulae for the $\mathbb{F}_2$-Rasmussen invariant of satellite knots of patterns with wrapping number 2, using the multicurve technology for Khovanov and Bar-Natan homology developed by Kotelskiy, Watson, and the second author. A new concordance homomorphism, which is independent of the Rasmussen invariant, plays a central role in these formulae. We also explore whether similar formulae hold for the Ozsváth–Szabó invariant τ.

1. Main results

In this article, we study the values that Rasmussen invariants take on satellite knots with wrapping number two, such as Whitehead doubles and 2-cables. We begin by stating our main theorem for winding number zero satellites.

Theorem 1.1. *For every knot $K \subset S^3$, there exists a unique integer $\vartheta_2(K)$ such that*

$$s_2(P(K)) = s_2(P_{-\vartheta_2(K)}(U))$$

for all patterns P with wrapping number two and winding number zero.

Theorem 1.2. *The knot invariant ϑ_2 induces a homomorphism from the smooth knot concordance group to the integers, which is linearly independent of the one given by s_2.*

Let us explain our notation. For p a prime, s_p denotes the Rasmussen invariant over the field $\mathbb{F}_p$ (see [28, 35]), while s_0 denotes the original Rasmussen invariant over $\mathbb{Q}$ (see [42]). By a *pattern* P, we mean a knot in the standard solid torus. We write $P(K)$ for the satellite knot with pattern P and companion knot K. For $t \in \mathbb{Z}$, we denote by P_t the pattern obtained from P by applying t right-handed full twists to the standard torus. We denote the unknot by U.

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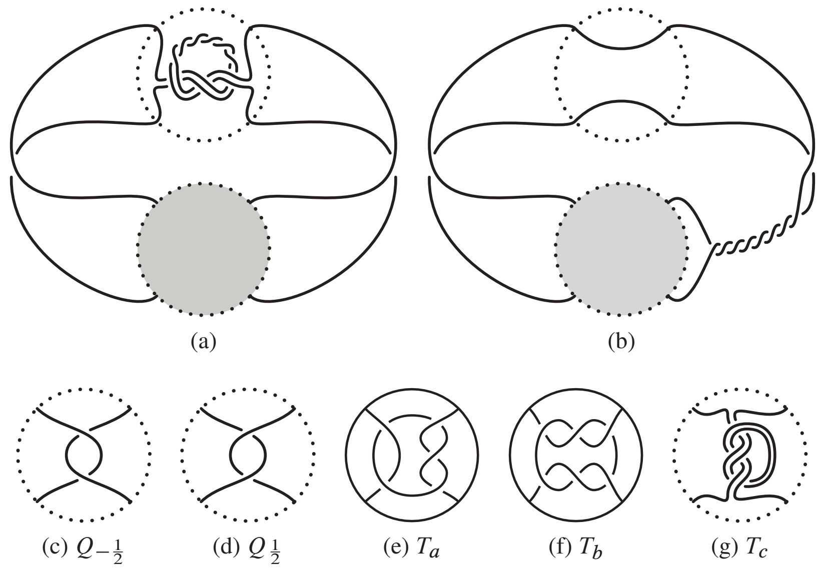


(a) (b)

(c) $Q_{-\frac{1}{2}}$ (d) $Q_{\frac{1}{2}}$ (e) T_a (f) T_b (g) T_c

Figure 1. An application of Theorem 1.1: plugging a suitable pattern tangle T_P into the grey discs in (a) and (b) results in the two knots $P(T_{2,3})$ and $P_{-4}(U)$, respectively, where $T_{2,3}$ denotes the right-handed trefoil. Some possibilities for T_P are shown in (c)–(g). For a fixed P, those two knots have the same Rasmussen invariant s_2 according to Theorem 1.1 since $\vartheta_2(T_{2,3}) = 4$.

For many classical knot invariants, such as the signature or the Alexander polynomial, one can express the invariant of a satellite as a function of the pattern and the invariant of the companion. The Rasmussen invariant s_2, on the other hand, displays a more complicated behaviour: indeed, the independence of s_2 and ϑ_2 claimed in Theorem 1.2 implies that the pair $(P, s_2(K))$ does not determine $s_2(P(K))$. However, Theorem 1.1 tells us that, for all winding number zero, wrapping number two patterns P, the value $s_2(P(K))$ is instead determined by the pair $(P, \vartheta_2(K))$. This is the raison d'être of the knot invariant ϑ_2.

The proof of the theorems in a nutshell. Observe that one may cut the satellite knot $P(K)$ along a sphere into two tangles, as can be seen in Figure 1 (a): one *pattern tangle* T_P associated with P and one *companion tangle* T_K that is determined by K. Now, we use Bar-Natan's extension of Khovanov homology to tangles [2]. Bar-Natan associates with a tangle T a chain complex $[\![T]\!]$ over a certain additive category $\mathrm{Cob}_{/l}$, which depends on the number of endpoints of T. The complexes $[\![T]\!]$ are well-defined up to homotopy equivalence and well-behaved under gluing. Thus, from $[\![T_P]\!]$ and $[\![T_K]\!]$, one may calculate the Khovanov homology of $P(K)$, and from that homology (precisely, the reduced Khovanov homology over $\mathbb{F}_2[X]/(X^2+X)$), one may read off $s_2(P(K))$.

Since P has wrapping number two, T_P and T_K are Conway tangles, i.e. tangles with four endpoints. Chain complexes over $\mathrm{Cob}_{/l}$ for four endpoints are well-understood by the work of Kotelskiy, Watson, and the second author [16, Theorems 1.1 and 1.5]. Indeed, such chain complexes can be encoded geometrically as certain sets of immersed curves (so-called *multicurves*) on the four-punctured sphere. The multicurve of $[\![T]\!]$ consists of a finite number of immersed circles and exactly one immersed interval, which is called the *non-compact component* and is denoted by $\widetilde{\mathrm{BN}}_a(T)$. It turns out that the Rasmussen invariant s_2 of $P(K) = T_P \cup T_K$ is fully determined by $\widetilde{\mathrm{BN}}_a(T_P)$ and $\widetilde{\mathrm{BN}}_a(T_K)$. The crucial step, which relies on the multicurve technology, is to show that $\widetilde{\mathrm{BN}}_a(T_K)$ is, in a suitable sense, approximated by $\widetilde{\mathrm{BN}}_a(Q_{2\vartheta_2(K)})$ for some $\vartheta_2(K) \in \mathbb{Z}$, where $Q_{2\vartheta_2(K)}$ is the (rational) Conway tangle consisting of $2\vartheta_2(K)$ left-handed half-twists. It follows that

$$s_2(P(K)) = s_2(T_P \cup T_K) = s_2(T_P \cup Q_{2\vartheta_2(K)}) = s_2(P_{-\vartheta_2(K)}(U)),$$

as claimed in Theorem 1.1. A detailed proof will be given in Section 6.

The proof of Theorem 1.2 consists of two parts. Firstly, by applying Theorem 1.1 with varying patterns, one can deduce that ϑ_2 is a concordance homomorphism, as we shall see in Proposition 4.3. Secondly, the linear independence of s_2 and ϑ_2 can be seen from their values on the torus knots $T_{2,3}$ and $T_{3,4}$; see Table 1. This and further independence results will be proven in Proposition 7.3 in Section 7.

Knot	s_2	ϑ_2
$T_{2,3}$	2	4
$T_{3,4}$	6	8

Table 1

2. Complementary results and speculations

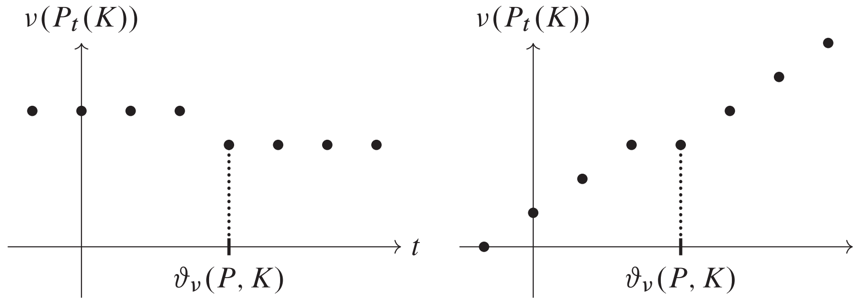
In this extended introduction, we put Theorems 1.1 and 1.2 into the context of other slice-torus invariants. We also explore formulae for Rasmussen invariants of satellite knots with winding number ± 2 patterns and discuss some applications of our results. The reader eager to peruse the proofs of Theorems 1.1 and 1.2 may proceed directly to Section 2.9, where a summary of the paper's structure will help to decide where to continue.

2.1. Pattern-twisting behaviour of slice-torus invariants. It is remarkable that, while the methods that go into proving Theorem 1.1 are very particular to Khovanov homology, the proof that ϑ_2 is a concordance homomorphism, as claimed in Theorem 1.2, does not require any properties of s_2 beyond the fact that it satisfies Theorem 1.1; any such invariant gives rise to a potentially new concordance homomorphism. Thus, one is led to wonder: *Do variations of Theorem* 1.1 *hold for other knot invariants?*

We will discuss this question later for various different knot invariants in Section 4.3. For now, we restrict to *slice-torus invariants* (Definition 4.1). This family of knot invariants was first studied by Livingston [31] and contains the Rasmussen invariants, normalised as $\frac{s_c}{2}$ for any characteristic c, as well as the Ozsváth–Szabó invariant τ coming from Heegaard Floer theory. In the following, let ν denote a slice-torus invariant, K a knot, and P a pattern of wrapping number two and winding number zero. The following result describes how $\nu(P(K))$ is affected by twisting P.

Proposition 2.1. *For any fixed triple* (ν, P, K)*, the function* $\mathbb{Z} \to \mathbb{Z}$, $t \mapsto \nu(P_t(K))$*, is either constant or there exists a unique jump point* $\vartheta \in \mathbb{Z}$ *such that, for all* $t \in \mathbb{Z}$,

$$\nu(P_t(K)) = \nu(P_{t-1}(K)) - \begin{cases} 1 & \text{if } t = \vartheta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$



(a) For winding number 0 patterns (b) For winding number ± 2 patterns

Figure 2. The behaviour of the functions $t \mapsto \nu(P_t(K))$ for wrapping number 2 patterns in the case $\vartheta_\nu(P, K) \neq \infty$.

This explains the role played in Theorem 1.1 by the value $\vartheta_2(K)$, which we can now generalise as follows.

Definition 2.2. For any fixed triple (ν, P, K), we define $\vartheta_\nu(P, K) \in \mathbb{Z} \cup \{\infty\}$ as the unique integer ϑ satisfying (2.1) if such an integer exists and ∞ otherwise; see Figure 2 (a) for an illustration. Moreover, we set

$$\vartheta_\nu(P) := \vartheta_\nu(P, U) \quad \text{and} \quad \vartheta_\nu(K) := \vartheta_\nu(W^+, K),$$

where W^+ is the positive Whitehead pattern, i.e. the pattern corresponding to the tangle in Figure 1 (c). For $c = 0$ or a prime, we write

$$\vartheta_c(K) := \vartheta_{\frac{s_c}{2}}(K) \quad \text{and} \quad \vartheta_c(P) := \vartheta_{\frac{s_c}{2}}(P).$$

Livingston and Naik [33] proved Proposition 2.1 for $P = W^+$ and defined an invariant $t_\nu(K)$, which is related to $\vartheta_\nu(K)$ by a mere translation: $t_\nu(K) = \vartheta_\nu(K) - 1$. Our reason for introducing this shift is given by Theorem 1.2: for $\nu = \frac{s_2}{2}$, $\vartheta_\nu(K) = \vartheta_2(K)$ induces a concordance homomorphism, whereas $t_\nu(K)$ does not.

The invariant $\vartheta_{\frac{s_2}{2}}(K)$ does indeed agree with the integer $\vartheta_2(K)$ from Theorem 1.1. Before justifying this statement, we first make an observation and a computation.

Proposition 2.3. *For any slice-torus invariant ν and knot K, $\vartheta_\nu(K)$ is finite.*

Proof. This is a reformulation of the first part of [33, Theorem 2], where our invariant $\vartheta_\nu(K) = \vartheta_\nu(W^+, K)$ corresponds to $t_\nu(K) + 1$ in their notation. □

Example 2.4. Observe that $W_0^+(U) = U$ and $W_{-1}^+(U) = T_{2,3}$. So, for all slice-torus invariants ν, $\nu(W_0^+(U)) = 0$ and $\nu(W_{-1}^+(U)) = 1$. By Proposition 2.1, this implies

$$\nu(W_t^+(U)) = \begin{cases} 1 & \text{if } t < 0, \\ 0 & \text{otherwise,} \end{cases}$$

and hence $\vartheta_\nu(U) = \vartheta_\nu(W^+, U) = \vartheta_\nu(W^+) = 0$. Similarly, one can determine $\vartheta_\nu(W^-) = 1$, where W^- is the negative Whitehead pattern (corresponding to the tangle in Figure 1 (d)).

Now, suppose ν is a slice-torus invariant and K a knot such that

$$\nu(W_t^+(K)) = \nu(W_{t-\vartheta(K)}^+(U))$$

for some integer $\vartheta(K)$ independent of t. Then it follows that $\vartheta(K) = \vartheta_\nu(K)$. In particular, the integer $\vartheta_2(K)$ from Theorem 1.1 agrees with $\vartheta_{\frac{s_2}{2}}(K)$. The preceding discussion also allows us to make the question from the beginning of this subsection more precise.

Question 2.5. Let ν be a slice-torus invariant. Does $\nu(P(K)) = \nu(P_{-\vartheta_\nu(K)}(U))$ hold for all knots K and all patterns P with wrapping number two and winding number zero?

The following is a variation of Theorem 1.1, which allows us to compute $s_2(P_t(K))$ for any $t \in \mathbb{Z}$ from the three invariants $s_2(P(U))$, $\vartheta_2(K)$, and $\vartheta_2(P)$.

Proposition 2.6. *Suppose ν is a slice-torus invariant for which the answer to Question* 2.5 *is "yes". Then, for all knots K, integers t, and patterns P of wrapping number two and winding number zero, the following holds:*

$$\nu(P_t(K)) = \nu(P(U)) + \begin{cases} 1 & \text{if } \vartheta_\nu(P) \neq \infty \text{ and } t - \vartheta_\nu(K) < \vartheta_\nu(P) \leq 0, \\ -1 & \text{if } \vartheta_\nu(P) \neq \infty \text{ and } t - \vartheta_\nu(K) \geq \vartheta_\nu(P) > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By hypothesis, $\nu(P_t(K)) = \nu(P_{t-\vartheta_\nu(K)}(U))$. The identity now follows from Proposition 2.1, observing that $\nu(P_{t-\vartheta_\nu(K)}(U))$ is equal to $\nu(P(U))$ unless the jump point $\vartheta_\nu(P)$ is finite and lies between $t - \vartheta_\nu(K)$ and 0. □

In Sections 6 and 7, we will see two different methods of computing the invariants $\vartheta_c(K)$ and $\vartheta_c(P)$. Firstly, they can be computed directly from definition, using the computability of the Rasmussen invariants; see Proposition 7.1. Secondly, they can also be easily read off the multicurves associated with pattern and companion tangles. By applying Proposition 6.6 to the curves in Figure 15, we compute the following values for the patterns P^T associated with the tangles T from Figure 1:

$$\vartheta_2(W^+) = \vartheta_2(P^{T_a}) = \vartheta_2(P^{T_c}) = 0, \quad \vartheta_2(W^-) = 1, \quad \text{and} \quad \vartheta_2(P^{T_b}) = \infty.$$

Similarly, using Corollary 6.14, we can easily determine the following values:

$$\vartheta_2(3_1) = 4, \quad \vartheta_2(4_1) = 0, \quad \text{and} \quad \vartheta_2(8_{19}) = 8.$$

Here $3_1 = T_{2,3}$ and $8_{19} = T_{3,4}$ denote the positive torus knots. We have computed the invariants $\vartheta_c(K)$ for a range of small knots K. The results are available online [26]. For more details about computations, see Sections 6 and 7.

Thanks to Theorem 1.1, once we know $\vartheta_2(K)$ and $\vartheta_2(P)$, we also know $\vartheta_2(P, K)$.

Proposition 2.7. *Suppose ν is a slice-torus invariant for which the answer to Question* 2.5 *is "yes". Then, for all knots K and patterns P of wrapping number two and winding number zero,*

$$\vartheta_\nu(P, K) = \vartheta_\nu(P) + \vartheta_\nu(K).$$

Proof. By hypothesis, the functions $t \mapsto \nu(P(K))$ and $t \mapsto \nu(P(U))$ are either both constant (in which the asserted identity becomes $\infty = \infty +$ integer) or the difference

$$\vartheta_\nu(P, U) - \vartheta_\nu(P, K)$$

between their jump points is indeed precisely $-\vartheta_\nu(K)$. □

Livingston–Naik showed that, for all knots K and $t \in \mathbb{Z}$, the values $\nu(W_t^+(K))$ and $\nu(W_t^-(K))$ cannot both be non-zero [33, Theorem 1]. We can push this result a little further.

Corollary 2.8. *Suppose ν is a slice-torus invariant for which the answer to Question* 2.5 *is "yes". Then, for each knot K, there is a unique integer t with*

$$\nu(W_t^+(K)) = \nu(W_t^-(K)) = 0.$$

Indeed, we have

$$(\nu(W_t^+(K)), \nu(W_t^-(K))) = \begin{cases} (0, -1), & t > \vartheta_\nu(K), \\ (0, 0), & t = \vartheta_\nu(K), \\ (1, 0), & t < \vartheta_\nu(K). \end{cases}$$

Proof. By Example 2.4, $\vartheta_\nu(W^+) = 0$ and $\vartheta_\nu(W^-) = 1$ for all slice-torus invariants ν. Also, note that $W^\pm(U)$ is the unknot, so we have $\nu(W^\pm(U)) = 0$. Now apply Proposition 2.6 to $P = W^\pm$. □

2.2. Comparison with the behaviour of τ. This paper was inspired by results of Hedden, who computed the Ozsváth–Szabó invariant τ for twisted Whitehead doubles [9]. From the perspective of the previous subsection, Hedden's results say that, for any knot K,

$$\vartheta_\tau(K) = 2\tau(K). \tag{2.2}$$

Interestingly, twisted Whitehead doubles were the first known examples of knots K with

$$\tau(K) \neq \frac{s_0}{2}(K).$$

In fact, it was Hedden and Ording [10] who found the first such example by showing that

$$\tau(W_2^+(T_{2,3})) = 0 \;\neq\; 1 = \frac{s_0}{2}(W_2^+(T_{2,3})). \tag{2.3}$$

This inequality can now be seen as a consequence of the fact that $\vartheta_\tau(T_{2,3}) = 2\tau(T_{2,3}) = 2$, whereas $\vartheta_0(T_{2,3}) = 3$; see (7.1) in Section 7. In view of identity (2.2), even if Question 2.5 has a positive answer for $\nu = \tau$, the concordance homomorphism ϑ_τ thus obtained is not new. However, identity (2.2) allows us to reformulate Question 2.5 as follows.

Question 2.9. Does the Ozsváth–Szabó invariant τ satisfy

$$\tau(P(K)) = \tau(P_{-2\tau(K)}(U))$$

for all knots $K \subset S^3$ and patterns P with wrapping number 2 and winding number 0?

In Section 4.2, we discuss a result of Levine [21] which gives a positive answer to Question 2.9 for certain patterns P.

A common way to show that two slice-torus invariants ν, ν' are not equal is to evaluate them on a well-chosen twisted Whitehead double W: see e.g. (2.3) above for $s_0 \neq \tau$ or [25] for $s_0 \neq s_3$. However, in this way, one can only ever find knots W with $|\nu(W) - \nu'(W)| = 1$; see Example 2.4. Does there exist a knot K with three-genus 1 such that $|\nu(K) - \nu'(K)| = 2$? We prove the following in Section 4.2.

Proposition 2.10. *There exists a knot K with three-genus 1, Alexander polynomial 1, $\tau(K) = 1$ and $s_2(K) = -2$. More generally, for any prescribed triple (g, a, b) of integers with $|a| \leq g$, $|b| \leq g$, there exists a knot L with three-genus g, Alexander polynomial 1, $\tau(L) = a$ and $s_2(L) = 2b$.*

2.3. Geometric applications. Let us showcase two corollaries to Theorem 1.1. The first answers an open question of Hedden and Pinzón-Caicedo, to "find any knot for which the Whitehead double of both K and $-K$ are non-zero in concordance" [11].

Corollary 2.11. *There exists a knot K such that $W^+(K)$ and $W^+(-K)$ generate a summand of the smooth concordance group $\mathcal{C}$ that is isomorphic to $\mathbb{Z}^2$.*

Proof. Let $K = T_{3,4}^{\#2} \# T_{-2,3}^{\#5}$. Using the additivity of τ and ϑ_2 with respect to the connected sum operation # and Table 1, one computes $\tau(K) = 1$ and $\vartheta_2(K) = -4$, and thus finds the values shown in Table 2. So the split epimorphism $(\tau, \frac{s_2}{2})\colon \mathcal{C} \to \mathbb{Z}^2$ admits the right-sided inverse $(W^+(K), W^+(-K))$, proving the existence of the desired summand. □

Knot	τ	s_2
$W^+(K)$	1	0
$W^+(-K)$	0	2

Table 2

The second corollary concerns a conjecture of Hedden [11]. Let P be a pattern with wrapping number two and winding number zero. The conjecture then says that if $K \mapsto P(K)$ induces a homomorphism $\mathcal{C} \to \mathcal{C}$, then it is the zero homomorphism, i.e. $P(K)$ is slice for all K. Some evidence for this conjecture is provided by the fact that $\nu(P(K)) = 0$ holds for all knots K and slice-torus invariants ν. Indeed, this follows because, by assumption on P, $\nu(P(K^{\#n})) = \nu(P(K)^{\#n})$, which in turn is equal to $n \cdot \nu(P(K))$ for all $n \in \mathbb{Z}$, but the set $\{\nu(P(K)) \mid K \text{ a knot}\}$ contains at most two elements by Proposition 2.1. Using Theorem 1.1, we find another restriction for such P.

Corollary 2.12. *Let P be a pattern with wrapping number two and winding number zero such that the function $K \mapsto P(K)$ induces an endomorphism on the smooth concordance group. Then $s_2(P_t(K)) = 0$ holds for all knots K and all integers t. In particular, $\vartheta_2(P) = \infty$.*

Proof. Let P be a pattern as in the hypothesis, and assume $s_2(P_t(K)) > 0$ for some knot K and some $t \in \mathbb{Z}$. Let u be a positive integer such that $-4u \leq t - \vartheta_2(K)$. Then

$$0 < s_2(P_t(K)) = s_2(P_{t-\vartheta_2(K)}(U)) \leq s_2(P_{-4u}(U)) = s_2(P(T_{2,3}^{\#u})) = u \cdot s_2(P(T_{2,3})).$$

Here, the first two equalities follow from Theorem 1.1 and the last equality from the assumption on P. Consequently, $s_2(P(T_{2,3})) \neq 0$. Therefore, the value $s_2(P(T_{2,3}^{\#u}))$ is unbounded as $u \to \infty$. This contradicts the fact that the knots $P(T_{2,3}^{\#u})$ and $P(T_{2,3}^{\#u'})$ are related by a smooth cobordism of genus one for any $u, u' \in \mathbb{Z}$. Hence we have led the assumption $s_2(P_t(K)) > 0$ ad absurdum. One proceeds similarly for $s_2(P_t(K)) < 0$. So it follows that $s_2(P_t(K)) = 0$ for all knots K and integers t. □

2.4. Does Theorem 1.1 hold over arbitrary coefficients? As discussed in Section 1, our proof of Theorem 1.1 heavily relies on the immersed multicurves technology. At the moment, this technology is the furthest developed for ground fields of characteristic 2. However, multicurves exist for fields of all characteristics, and we expect our proof of Theorem 1.1 to extend without major changes to all ground fields, thus giving a positive answer to Question 2.5 for $\nu = s_c$.

Conjecture 2.13. *For all knots $K \subset S^3$, all c equal to 0 or a prime, and all patterns P with wrapping number two and winding number zero, we have*

$$s_c(P(K)) = s_c(P_{-\vartheta_c(K)}(U)),$$

where $\vartheta_c(K)$ is the integer from Definition 2.2.

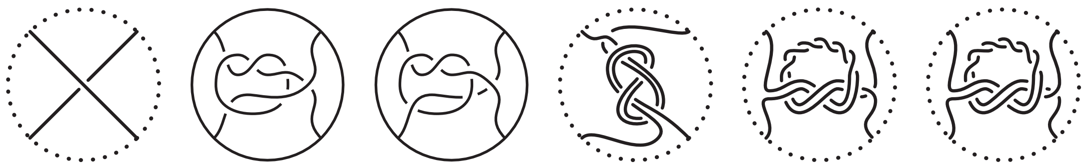
2.5. Winding number ±2 patterns and ϑ_ν-rationality. The techniques for the proof of Theorem 1.1 can be adapted to patterns with wrapping number 2 and winding number ± 2. As before, this allows us to compute the Rasmussen invariant of any satellite knot (with such a pattern) as a function of invariants extracted independently from the pattern and the companion. Interestingly, the technical difficulty that tied us to working over characteristic $c = 2$ in Theorem 1.1 disappears: we can now work over arbitrary characteristics c.

However, compared to the setting considered previously, there is one fundamental difference, which is not particular to the Rasmussen invariants, but which appears in the behaviour the Ozsváth–Szabó invariant τ, too. In fact, this structural difference can be understood for all slice-torus invariants. So we defer the statement of our winding number ± 2 pattern results for the Rasmussen invariants to Section 2.6 and first consider the following general setting: as in Section 2.1, let ν denote a slice-torus invariant and K a knot, but now P denotes a pattern of wrapping number two and winding number ± 2. In analogy with Proposition 2.1, we have the following result, which is illustrated in Figure 2 (b).

Proposition 2.14. *For any fixed triple (ν, P, K), the function $\mathbb{Z} \to \mathbb{Z}$, $t \mapsto \nu(P_t(K))$, is either affine of slope 1 or it has a single “stationary point”, in the sense that there exists a unique integer $\vartheta \in \mathbb{Z}$ such that, for all $t \in \mathbb{Z}$,*

$$\nu(P_t(K)) = \nu(P_{t-1}(K)) + \begin{cases} 0 & \text{if } t = \vartheta, \\ 1 & \text{otherwise}. \end{cases} \tag{2.4}$$

This result was shown for cables by van Cott [47, Theorem 2], and then generalised by Roberts. It is a special case of [43, Theorem 2], but for the sake of completeness, we will provide a quick proof in Section 4.1. In analogy to Definition 2.2, we have the following definition.



(a) Q_{-1} (b) T'_a (c) T'_b (d) T'_c (e) T'_d (f) T'_e

Figure 3. Some tangles for patterns with winding number ± 2.

Definition 2.15. For any fixed triple (ν, P, K), we define $\vartheta_\nu(P, K) \in \mathbb{Z} \cup \{\infty\}$ as the unique integer ϑ satisfying (2.4) if such an integer exists and ∞ otherwise; see Figure 2 (b) for an illustration. Moreover, we set

$$\vartheta_\nu(P) := \vartheta_\nu(P, U) \quad \text{and} \quad \vartheta'_\nu(K) := \vartheta_\nu(C_{2,1}, K),$$

where $C_{2,1}$ is the $(2, 1)$-cable pattern, i.e. the pattern corresponding to the rational tangle Q_{-1} in Figure 3 (a). For $c = 0$ or prime, we use the subscript c in place of the subscript $\nu = \frac{s_c}{2}$.

The key difference between winding number 0 and winding number ± 2 patterns is the following: unlike $\vartheta_\nu(K)$, the invariant $\vartheta'_\nu(K)$ need not be finite for all knots K; in other words, there is no analogue of Proposition 2.3. This motivates the following terminology.

Definition 2.16. If $\vartheta'_\nu(K)$ is finite, we call K *ϑ_ν-rational*, or *ϑ_c-rational* if $\nu = \frac{s_c}{2}$ for $c = 0$ or prime.

Remark 2.17. Since, for all patterns P, $P(K)$ and $P(J)$ are concordant if K and J are, it follows that $\vartheta_\nu(P, K)$ is a concordance invariant, for winding number 0 and ± 2. So $\vartheta'_\nu(K)$ and ϑ_ν-rationality are also concordance invariants. In particular, slice knots are ϑ_ν-rational with $\vartheta'_\nu = \vartheta_\nu = 0$, since $\nu(C_{2,\pm 1}(U)) = 0$ implies that $\vartheta'_\nu(U) = 0$, and $\vartheta_\nu(U) = 0$ was computed in Example 2.4.

Example 2.18. For $K = T_{2,3}$ the right-handed trefoil knot, one may compute that $s_2(C_{2,2t+1}(K)) = 4 + 2t$ holds for all integers t. So the trefoil knot is not ϑ_2-rational. On the other hand, the figure eight knot is ϑ_ν-rational with $\vartheta'_\nu = 0$ for all ν, as a consequence of the following proposition.

Proposition 2.19. *Any knot K with order two in the smooth concordance group, such as an amphicheiral knot, is ϑ_ν-rational and satisfies $\vartheta'_\nu(K) = 0$.*

Proof. The knot $-C_{2,1}(K)$ is isotopic to $C_{2,-1}(-K)$, which is concordant to $C_{2,-1}(K)$. So we have $-\nu(C_{2,1}(K)) = \nu(C_{2,-1}(K))$. But since

$$\nu(C_{2,1}(K)) - \nu(C_{2,-1}(K)) \in \{1, 0\}$$

by Proposition 2.14, it follows that $\nu(C_{2,1}(K)) = \nu(C_{2,-1}(K)) = 0$, and thus $\vartheta'_\nu(K) = 0$. □

Question 2.20. Does $\vartheta'_\nu(K) = \vartheta_\nu(K) = 0$ hold for all slice-torus invariants ν and ϑ_ν-rational knots K?

For $\nu = \tau$, Question 2.20 is already known to admit a positive answer. The answer, which relates ϑ_ν-rationality to Hom's concordance invariant $\varepsilon(K) \in \{0, \pm 1\}$, is an immediate consequence of [12, Theorem 1].

Proposition 2.21. *For any knot K, the following conditions are equivalent: $\vartheta'_\tau(K) = 0$; K is ϑ_τ-rational; $\varepsilon(K) = 0$. Moreover, each of them implies $\vartheta_\tau(K) = 2\tau(K) = 0$.* □

Computational evidence so far seems to suggest that ϑ_c-rationality is independent of the characteristic c and in fact agrees with ϑ_τ-rationality [26].

Question 2.22. Is there a ϑ_c-rational knot K with $\varepsilon(K) \neq 0$? Or conversely, a non-ϑ_c-rational knot K with $\varepsilon(K) = 0$?

The condition $\varepsilon(K) = 0$ admits a geometric description in terms of Hanselman, Rasmussen, and Watson's multicurve invariant $\widehat{\mathrm{HF}}(S^3 \smallsetminus \mathring{\nu}(K))$ of the knot exterior $S^3 \smallsetminus \mathring{\nu}(K)$ (see [8]). It is equivalent to the condition that a particular component of the multicurve invariant $\widehat{\mathrm{HF}}(S^3 \smallsetminus \mathring{\nu}(K))$ agrees with $\widehat{\mathrm{HF}}(S^3 \smallsetminus \mathring{\nu}(U))$ (see [7, discussion after Example 48]), which is a huge restriction. Therefore, the case $\varepsilon(K) \neq 0$ should be understood as the generic case.

For the same reason, "most" knots should not be ϑ_c-rational: in Definition 2.16 and Proposition 6.18, we will characterise ϑ_c-rational knots K in terms of their associated tangle invariants $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$. Specifically, a knot is ϑ_c-rational if and only if the component $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ of the multicurve invariant $\widetilde{\mathrm{BN}}(T_K; \mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}_a(Q_n; \mathbb{F}_c)$ for some rational tangle Q_n, $n \in 2\mathbb{Z}$. So there might actually be a deeper reason for the similarity between ϑ_c- and ϑ_τ-rationality.

2.6. Winding number ±2 patterns, Rasmussen invariants, and τ. As discussed in the previous subsection, the behaviour of Rasmussen invariants of cables with winding number ± 2 exhibits a phenomenon that is absent from the setting of patterns with winding number 0: we need to distinguish between ϑ_c-rational and non-ϑ_c-rational knots. This dichotomy is also reflected in the behaviour of the Rasmussen invariants of satellites with general patterns of winding number ± 2.

First, we consider ϑ_c-rational companions. For these, the formula looks almost the same as for winding number 0 patterns: all the information about the companion is captured by the integer $\vartheta'_c(K)$. And, in fact, $\vartheta_c(K) = \vartheta'_c(K)$.

Theorem 2.23. *Let K be a ϑ_c-rational knot. Then $\vartheta_c(K) = \vartheta'_c(K)$, and for any pattern P with wrapping number 2 and winding number ± 2,*

$$s_c(P(K)) = s_c(P_{-\vartheta_c(K)}(U)) + 6\vartheta_c(K).$$

This gives a positive answer to half of Question 2.20 for $\nu = \frac{s_c}{2}$. We conjecture that the other half holds true as well.

Conjecture 2.24. *For any ϑ_c-rational knot K, $\vartheta_c(K) = 0$.*

Note that if this conjecture holds, then the formula in Theorem 2.23 becomes identical to the formula in Theorem 1.1.

Remark 2.25. As for τ and ε, the converse of Conjecture 2.24 is false: there are many knots K with $\vartheta_c(K) = 0$ that are not ϑ_c-rational, e.g. $K = T_{2,3}\#T_{2,3}\# - T_{3,4}$ for $c = 2$.

Proposition 2.19 and Theorem 2.23 imply a special case of Conjecture 2.24.

Corollary 2.26. *Let K be a knot with order two in the concordance group, e.g. an amphicheiral knot. Then K is ϑ_c-rational and $\vartheta_c(K) = \vartheta'_c(K) = 0$.* □

Next, let us turn to non-ϑ_c-rational knots. Again, the information about the companion knot is contained in a single integer. However, in this case, this integer is simply the Rasmussen invariant $s_c(C_{2,1}(K))$ of the $(2,1)$-cable of the companion.

Theorem 2.27. *Let K be a knot that is not ϑ_c-rational. Then, for any pattern P with wrapping number 2 and winding number ± 2 with $\vartheta_c(P) \neq \infty$,*

$$s_c(P(K)) = s_c(P(U)) + s_c(C_{2,1}(K)) - \begin{cases} 2 & \text{if } \vartheta_c(P) > 0, \\ 0 & \text{if } \vartheta_c(P) \leq 0. \end{cases}$$

There is a version of Theorem 2.27 (namely Theorem 6.21) for patterns P with

$$\vartheta_c(P) = \infty.$$

In this case, the formula for $s_c(P(K))$ looks identical to the one above, except that the condition for whether the last summand is equal to 0 or -2 is more subtle, taking into account also a certain order on the multicurve invariants associated with the pattern P and the companion K. This subtlety is illustrated in Example 6.22.

In analogy to Proposition 2.7, we have the following results.

Proposition 2.28. *Let K be a knot and P a pattern with wrapping number 2 and winding number ± 2. If K is ϑ_c-rational, then $\vartheta_c(P, K) = \vartheta_c(P) + \vartheta_c(K)$. In particular, for such knots K, $\vartheta_c(P, K) = \infty$ if and only if $\vartheta_c(P) = \infty$. If K is not ϑ_c-rational and $\vartheta_c(P) \neq \infty$, then $\vartheta_c(P, K) = \infty$.*

The invariant $\vartheta_c(P)$ is just as computable for winding number ± 2 patterns as for winding number 0 patterns. Again, there are two methods: either directly from definition or indirectly from the multicurve invariants. For example, using Proposition 6.6, we can read off the following values for the patterns P^T associated with the tangles T in Figure 3 from the multicurves in Figure 15:

$$\vartheta_2(P^{T'_a}) = \vartheta_2(P^{T'_d}) = \vartheta_2(P^{T'_e}) = \infty, \quad \vartheta_2(P^{T'_b}) = 1, \quad \text{and} \quad \vartheta_2(P^{T'_c}) = 0.$$

Also, ϑ_c-rationality can be determined either directly using Corollary 6.19 or indirectly using the multicurves; see Definition 2.16. For instance, the trefoil knot it not ϑ_2-rational, but the figure-eight knot is, as we have already seen in Example 2.18. Check our online table [26] for further examples.

Proof of Proposition 2.28. Applying Theorem 2.23 to the pattern P_t for $t \in \mathbb{Z}$, we see that $s_c(P_t(K))$ and $s_c(P_{t-\vartheta_c(K)}(U))$ have the same stationary points if they exist. This implies the first identity. For the second part, we apply Theorem 2.27 to the pattern P_t for $t \in \mathbb{Z}$ and observe that

$$t \mapsto s_c(P_t(U)) - \begin{cases} 2 & \text{if } \vartheta_c(P_t) > 0, \\ 0 & \text{if } \vartheta_c(P_t) \leq 0 \end{cases}$$

is an affine function, since $\vartheta_c(P_t) = \vartheta_c(P) - t$. □

Remark 2.29. If $\vartheta_c(P) = \infty = \vartheta_c(K)$, $\vartheta_c(P, K)$ may be finite or infinite. Examples can be easily constructed using the aforementioned Theorem 6.21.

Based on the similarity between τ and the Rasmussen invariants, it is natural to compare Theorems 2.23 and 2.27 to the behaviour of τ. The following is a special case of a result of Hom [13, Theorem 5].

Theorem 2.30. *For any knot K with $\varepsilon(K) = 0$ and any pattern P (of arbitrary winding or wrapping number), $\tau(P(K)) = \tau(P(U))$.*

Sketch proof. Since Hom's theorem is more general, we describe the basic idea of the proof of Theorem 2.30: for any knot K, $\varepsilon(K) = 0$ is equivalent to the condition that the type D structure $\widehat{\mathrm{CFD}}(S^3 \smallsetminus \mathring{\nu}(K))$ in bordered Heegaard Floer homology contains a direct summand identical to $\widehat{\mathrm{CFD}}(S^3 \smallsetminus \mathring{\nu}(U))$. Since the box-tensor product commutes with the direct sum, [27, Theorem 11.19] implies that the homology group $\mathrm{HFK}^-(P(K))$ contains a direct summand isomorphic to $\mathrm{HFK}^-(P(U))$, and this summand contains the invariant τ (see [27, equation 11.16]). □

This result is much stronger than Theorem 2.23. As we have just seen, the proof relies on bordered Heegaard Floer homology due to Lipshitz, Ozsváth, and Thurston [27]. The multicurve theory from [16] is just a first step towards better understanding similar cut-and-paste technology in Khovanov homology, namely the cobordism complexes due to Bar-Natan [2]. Unfortunately, his theory (like Khovanov homology) is less geometric and therefore harder to apply to this type of problem. Still, by studying Bar-Natan's invariants of companion tangles on more strands, one might hope to show analogous behaviour for Rasmussen invariants.

Conjecture 2.31. *For any ϑ_c-rational knot K and any pattern P (of arbitrary winding or wrapping number), $s_c(P(K)) = s_c(P(U))$.*

Proving a result for τ that is analogous to Theorem 2.27 seems less straightforward than the proof of Theorem 2.30, but might well be within the limits of current technology. Combining Theorem 2.27 with Hom's cabling formula for τ (see [12]), we make the following conjecture.

Conjecture 2.32. *Let K be a knot with $\varepsilon(K) = \pm 1$. Then, for any pattern P with wrapping number 2 and winding number ± 2 with $\vartheta_\tau(P) \neq \infty$,*

$$\tau(P(K)) = \tau(P(U)) + 2\tau(K) + \begin{cases} -1 & \textit{if } \varepsilon(K) = 1 \textit{ and } \vartheta_\tau(P) > 0, \\ 1 & \textit{if } \varepsilon(K) = -1 \textit{ and } \vartheta_\tau(P) \leq 0, \\ 0 & \textit{otherwise}. \end{cases}$$

2.7. Properties of the knot invariants ϑ_ν. One may see easily that ϑ_ν induces a knot concordance invariant (see Remark 2.17). However, it is not known whether ϑ_ν induces a knot concordance homomorphism $\mathcal{C} \to \mathbb{Z}$ in general, i.e. whether $\vartheta_\nu(K\#J) = \vartheta_\nu(K) + \vartheta_\nu(J)$ holds for all knots K and J. We do have the following inequalities due to Park.

Theorem 2.33 ([41, Theorem 1.2]). *For all slice-torus invariants ν and knots K and J,*

$$\vartheta_\nu(K) + \vartheta_\nu(J) - 1 \leq \vartheta_\nu(K\#J) \leq \vartheta_\nu(K) - \vartheta_\nu(-J) + 1.$$

We now discuss the effect of crossing changes on ϑ_ν. Suppose K_- is a knot and λ an unknot in the complement of K_- such that $\mathrm{lk}(K_-, \lambda) = 0$. Performing a $(+1)$-framed Dehn surgery on λ transforms K_- into another knot K_+ in S^3. We say that K_+ (or K_-) arises from K_- (or K_+) by a *negative (or positive) generalised crossing change* [30]. If λ bounds a disc D that intersects K_- transversely in $2n$ points, then K_+ arises from K_- by tying a left-handed full-twist into the $2n$ strands of K_- close to D. We say that the generalised crossing change is *$2n$-stranded*. Note that a 2-stranded generalised crossing change is simply a crossing change in the usual sense.

Let us say that a knot invariant y satisfies the *($2n$-stranded) generalised/classical crossing change inequality* if $y(K_-) \leq y(K_+)$ holds whenever the knot K_+ arises from K_- by a negative generalised/classical crossing change (on $2n$ strands). For example, Levine–Tristram signatures satisfy the generalised crossing change inequality; slice-torus invariants satisfy the classical crossing change inequality; τ (by Ozsváth–Szabó [38, Theorem 1.1]) and s_c with $c \neq 2$ (by Manolescu–Marengon–Sarkar–Willis [36, Theorem 1.11]) satisfy the generalised crossing change inequality, but it is unknown whether all slice-torus invariants do.

Proposition 2.34. *Let $n \geq 1$. If ν is a slice-torus invariant satisfying the $4n$-stranded generalised crossing change inequality, then*

$$\vartheta_\nu(K_-) \leq \vartheta_\nu(K_+) \leq \vartheta_\nu(K_-) + 4n^2$$

holds if K_+ arises from K_- by a $2n$-stranded negative generalised crossing change. In particular, if ν satisfies the generalised crossing change inequality, then so does ϑ_ν.

The case $n = 1$ of Proposition 2.34 is implicit in [15]. We will prove Proposition 2.34 and variations of it in Section 4.4. As a consequence of Proposition 2.34, if ν is a slice-torus invariant satisfying the generalised crossing change inequality, then $\vartheta_\nu(K) \geq 0$ holds for all knots that can be transformed into a slice knot by positive generalised crossing changes, and $\vartheta_\nu(K) \leq 0$ holds for all knots that can be transformed into a slice knot by negative generalised crossing changes. Combined, this already implies $\vartheta_\nu(K) = 0$ for many knots (cf. also [3]), e.g. for the $(2, 1)$-cable of the figure-eight knot.

We suspect that the results of [36] extend to characteristic 2, i.e. that s_2 satisfies the generalised crossing change inequality as well. This would in particular imply the following.

Conjecture 2.35. *If K_+ arises from K_- by a negative generalised crossing change on $2n$ strands, then we have $\vartheta_2(K_-) \leq \vartheta_2(K_+) \leq \vartheta_2(K_-) + 4n^2$.*

Finally, let us mention an inequality proven by Livingston–Naik.

Theorem 2.36 ([33, Theorem 2]). *For all slice-torus invariants ν and all knots K,*

$$\mathrm{TB}(K)+1 \leq \vartheta_\nu(K) \leq -\mathrm{TB}(-K),$$

where $\mathrm{TB}(K)$ *denotes the maximal Thurston–Bennequin number of* K.

If $\vartheta_\nu(-K)=-\vartheta_\nu(K)$ holds, as is the case for $\nu=s_2$, then one may clearly improve the upper bound of Theorem 2.36 by 1.

Proposition 2.37. *For all knots K, we have*

$$\mathrm{TB}(K)+1 \leq \vartheta_2(K) \leq -\mathrm{TB}(-K)-1. \qquad \square$$

2.8. Further open questions. Aside from being a concordance homomorphism, what other properties does ϑ_2 have in common with s_2? For example, is ϑ_2 also an even integer, like s_2 is? First off, note that the divisibility of s_2 by 2 merely comes from a somewhat arbitrary choice of normalisation, and one could equally well work with $\frac{s_2}{2}$. For ϑ_2, on the other hand, in light of Theorem 1.1, our chosen normalisation appears to be the natural one (up to sign). From our discussion so far, there is no apparent reason that $\{\vartheta_2(K) \mid K \text{ a knot}\} \subset n\mathbb{Z}$ should hold for any $n \geq 2$. Yet, we make the following conjecture.

Conjecture 2.38. *For all knots K, $\vartheta_2(K)$ is divisible by* 4.

Divisibility of $\vartheta_2(K)$ by 2 would follow from a more general conjecture about multicurve geography over $\mathbb{F}_2$; see [17, Conjecture 3.10 and the preceding discussion]. In contrast, $\vartheta_c(K)$ need not be divisible by 4 (or any integer $n \geq 2$) for $c \neq 2$; see Section 7.

A lower bound for the smooth slice-genus $g_4(K)$ is provided by $|s_2(K)|/2$. This inspires the following question.

Question 2.39. Is $|\vartheta_2(K)|/4$ a lower bound for the smooth slice-genus $g_4(K)$?

On the one hand, we have not found a proof for this bound, and we have no deeper reason to assume that Question 2.39 should hold, except that s_2 and ϑ_2 might be assumed to behave similarly. On the other hand, we have not found a counterexample.

It would be of interest to calculate the values of s_2 on iterated satellites, as it is possible for τ. This brings us to the following question.

Question 2.40. Is $\vartheta_2(W^+(K))$ determined by $\vartheta_2(K)$ and $s_2(K)$?

2.9. Structure of the paper. In Section 3, we fix our conventions and notation regarding tangles, satellite knots, and Rasmussen invariants. Section 4 contains those geometric proofs that do not rely on the multicurve technologies, namely the proof that ϑ_2 is a concordance homomorphism as claimed in Theorem 1.2, as well as the proofs of Propositions 2.1, 2.10, 2.14, and 2.34. In Section 5, we review the construction and some properties of the multicurve invariants for Bar-Natan homology. We use those in Section 6 to prove Theorem 1.1 (the winding number 0 case) and Theorems 2.23, 2.27, and 6.21 (the winding number ± 2 case). Finally, Section 7 discusses the results of our computer calculations and the conjectures they invite.

3. Basic facts about tangles, satellites, and Rasmussen invariants

3.1. Conway tangles and rational tangles.

Definition 3.1. An *oriented Conway tangle* is an embedding of two intervals into a 3-dimensional ball B^3 such that the preimage of ∂B^3 consists of precisely the four endpoints of the intervals. We consider such tangles up to isotopy fixing the boundary pointwise. If we also identify tangles that are obtained by precomposition with orientation reversing diffeomorphisms of the intervals, we obtain the notion of *unoriented* Conway tangles.

We will usually think of Conway tangles in terms of their diagrams: an *oriented diagram of a Conway tangle* is an immersion of two intervals into a closed 2-dimensional disc D^2 such that the preimage of D^2 consists of precisely the four endpoints of the intervals and such that all self-intersections are transverse double points, together with over-under information at each such double point. We consider such diagrams up to isotopies fixing the boundary pointwise. An *unoriented* version of this notion is obtained in the same way as above.

Identifying D^2 with the unit disc in $\mathbb{C}$, we call a tangle diagram *standard* if the four tangle ends lie at the four equidistant points $\pm e^{\pm i\pi/4}$. Identifying this unit disc with

$$D^2 \times \{0\} \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3,$$

we map these four points into the unit ball B^3 and call a Conway tangle *standard* if the four tangle ends are equal to those four points. Standard Conway tangles and diagrams up to the usual Reidemeister moves are in one-to-one correspondence, via suitable perturbations of the standard projection $\mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ onto the first two factors.

In the following, we will use the notions of Conway tangles and their diagrams interchangeably and assume that they are standard. By default, all tangles and their diagrams are unoriented.

Since we only study Rasmussen invariants of knots (as opposed to links), our definition of Conway tangles is more restrictive than the usual one, which also allows closed tangle components.

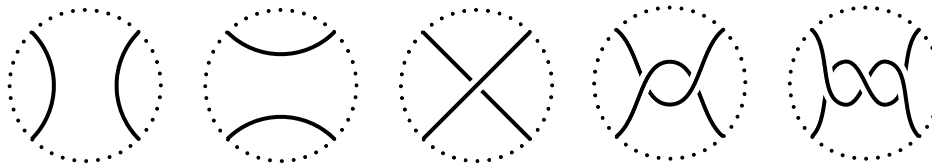
Definition 3.2. The *linking number* $\mathrm{lk}(T)$ of an oriented Conway tangle T is defined as half the signed count of positive ( ) and negative crossings ( ) between the two components of the tangle T (see [16, Definition 4.7]). The linking number of an unoriented Conway tangle is well-defined up to sign. In particular, the property $\mathrm{lk}(T) = 0$ is independent of orientations.

Definition 3.3. The *connectivity* $\mathrm{x}(T)$ of a Conway tangle T is an element of

$$\{\ ,\ ,\ \}$$

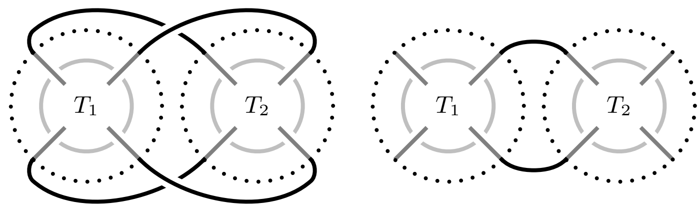
and defined as follows:

$$\mathrm{x}(T) = \begin{cases} & \text{if } T \text{ connects the tangle ends as in the tangle } , \\ & \text{if } T \text{ connects the tangle ends as in the tangle } , \\ & \text{if } T \text{ connects the tangle ends as in the tangle } . \end{cases}$$



(a) Q_∞ (b) Q_0 (c) Q_1 (d) Q_2 (e) Q_{-3}

Figure 4. Diagrams of some rational tangles $Q_{\frac{p}{q}}$.



(a) $T_1 \cup T_2$ (b) $T_1 + T_2$

Figure 5. (a) The union and (b) the sum of two Conway tangles T_1 and T_2.

Definition 3.4. A (standard) Conway tangle T is *rational* if there exists an isotopy, not necessarily fixing the boundary, to the trivial tangle Q_∞ from Figure 4 (a). A theorem of Conway [4] puts such tangles in one-to-one correspondence with elements of $\mathbb{Q}\mathrm{P}^1$. For each slope $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1$, we denote the corresponding rational tangle by $Q_{\frac{p}{q}}$.

Figure 4 gives some examples of rational tangles and serves to fix our conventions.

Definition 3.5. Given two Conway tangles T_1 and T_2, their *union* $T_1 \cup T_2$ is the link defined by the diagram in Figure 5 (a) and their *sum* $T_1 + T_2$ is the Conway tangle by the diagram in Figure 5 (b). Then, for any Conway tangle T, we define the $\frac{p}{q}$*-rational filling* of T by

$$T\left(\frac{p}{q}\right) := Q_{-\frac{p}{q}} \cup T.$$

We call a Conway tangle T *cap-trivial* if $T(\infty)$ is the unknot. We write $\mathrlap{\,|}{+}(T)$ for the tangle obtained by rotating T by π around the y-axis. Moreover, we write $-T$ for the *mirror* of a Conway tangle T, i.e. the tangle obtained by switching over- and under-strands at all crossings in a diagram of T.

We now collect a few identities that will be useful later.

Lemma 3.6. *Let T_1, T_2, and T be Conway tangles, $n \in \mathbb{Z}$, and $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1$. Then*

(a) $T_1 \cup T_2 = T_2 \cup T_1$.

(b) $(T_1 + T) \cup T_2 = T_1 \cup (T_2 + \mathrlap{\,|}{+}(T))$*; in particular,*

$$(T_1 + Q_{\frac{p}{q}}) \cup T_2 = T_1 \cup (T_2 + Q_{\frac{p}{q}}).$$

(c) $T(\frac{p}{q} + n) = (T + Q_{-n})(\frac{p}{q})$.

Proof. The first identity follows from

$$T_1 \cup T_2 = \big(\text{⦶}(T_1) + T_2\big)(0) = \big(\text{⦶}(T_2) + T_1\big)(0) = T_2 \cup T_1.$$

The second follows from

$$\begin{aligned}(T_1 + T) \cup T_2 &= \big(\text{⦶}(T_1 + T) + T_2\big)(0) = \big(\text{⦶}(T) + \text{⦶}(T_1) + T_2\big)(0) \\ &= \big(\text{⦶}(T_1) + T_2 + \text{⦶}(T)\big)(0) = T_1 \cup \big(T_2 + \text{⦶}(T)\big)\end{aligned}$$

and the observation that $\text{⦶}(Q_{\frac{p}{q}}) = Q_{\frac{p}{q}}$ for all $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1$. The last identity follows from

$$\begin{aligned}T\Big(\frac{p}{q} + n\Big) &= Q_{-\frac{p}{q}-n} \cup T = (Q_{-\frac{p}{q}} + Q_{-n}) \cup T \overset{\text{(b)}}{=} Q_{-\frac{p}{q}} \cup (T + Q_{-n}) \\ &= (T + Q_{-n})\Big(\frac{p}{q}\Big).\end{aligned}$$

□

3.2. Satellite knots and Conway tangles.

Definition 3.7. A *pattern* is an oriented knot in the solid torus $S^1 \times D^2$ with a fixed orientation, i.e. an embedding $\iota_P\colon S^1 \hookrightarrow S^1 \times D^2$ considered up to isotopy. The *wrapping number of a pattern* P is equal to the minimal number of intersection points between any representative of the equivalence class of the pattern and a meridional disc $\{*\} \times D^2$, where $* \in S^1$. Likewise, we define the *winding number of a pattern* P as the algebraic intersection number between P and such a disc.

Given an oriented knot $K \subset S^3$, we can consider an embedding $\iota_K\colon S^1 \times D^2 \hookrightarrow S^3$ which identifies the solid torus with a closed tubular neighbourhood of K such that, for any point $* \in \partial D^2$, $S^1 \times \{*\}$ is mapped to a zero-framed longitude λ_K of K and, for any point $* \in S^1$, $\{*\} \times \partial D^2$ is mapped to a meridian μ_K of K. (Both of these identifications preserve orientations.) We then define the *satellite knot* $P(K)$ *of the pattern* P *and companion* K as the knot given by the composition $\iota_K \circ \iota_P$.

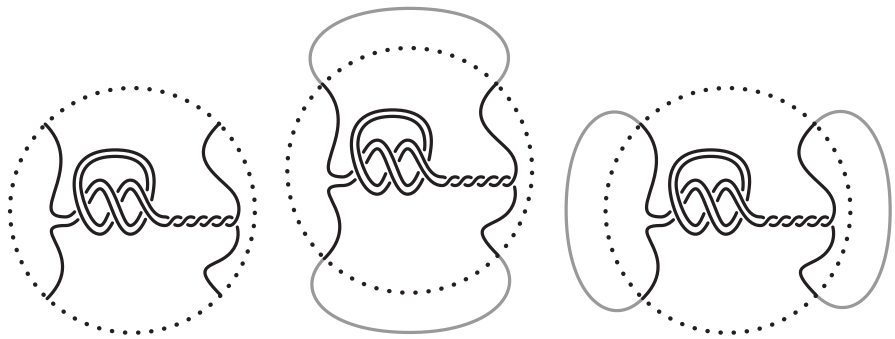
Definition 3.8. Given an oriented Conway tangle T with connectivity $\mathrm{x}(T) \neq$, we can define the *pattern* P^T *of the tangle* T as follows. Consider the embedding

$$\varphi\colon B^3 \hookrightarrow [-2, 2] \times D^2 \to S^1 \times D^2,$$

where the first map is the embedding of the unit 3-ball B^3 into $[-2, 2] \times D^2$ as subspaces of $\mathbb{R}^3$ and the second map is the identity on the second factor and the map $t \mapsto e^{\pi i t/2}$ on the first factor. The subspace $S^1 \times \{\pm \sin(\pi/4)\} \times \{0\}$ defines a 2-component link whose restriction to B^3 is the trivial tangle Q_0. The pattern P^T is now obtained by replacing this tangle by T. The condition $\mathrm{x}(T) \neq$ ensures that the pattern consists of a single component.

Conversely, given a pattern P with wrapping number 2, we can find a tangle T_P with connectivity $\mathrm{x}(T_P) \neq$ such that $P = P^{T_P}$. We call any such tangle a *pattern tangle of* P. Furthermore, given an integer $t \in \mathbb{Z}$, we define the t*-twisted pattern* P_t *of* P as the pattern $P^{T_P + Q_{-2t}}$, where T_P is some pattern tangle of P. Finally, suppose $P(K)$ is a satellite knot with companion K and pattern P with wrapping number 2 and $t \in \mathbb{Z}$. We then call $P_t(K)$ the t*-twisted satellite knot of the pattern* P *and companion* K.

Note that pattern tangles T_P are not necessarily unique. For instance, the tangle T_a and its Conway mutant, obtained from T_a by rotation in the plane by π, both induce the same



(a) T_{3_1} (b) $T_{3_1}(0)$ (c) $T_{3_1}(\infty) = U$

Figure 6. Illustration of Definition 3.9 for $K = 3_1 = T_{2,3}$ the right-handed trefoil knot.

pattern, although they are different tangles. However, the twisted pattern P_t does not depend on the choice of pattern tangle, since the extra twists simply amount to a reparametrisation of the solid torus. In particular, $P_0 = P$.

Definition 3.9. Let K be an oriented knot. The *double tangle T_K of K* is the unoriented Conway tangle that is defined as follows. Let L be the oriented link obtained as union of K with one of its zero-framed longitudes, regarded as a small push-off of K. Then L bounds an embedded unoriented annulus in S^3, namely the image of a suitably chosen homotopy between K and the longitude. Now choose a small open three-ball which intersects this annulus in a trivially embedded band. Then T_K is defined as the intersection of L with the complement of this three-ball. We frame T_K such that $T_K(\infty)$ is the meridional filling that produces the unknot and $T_K(0) = L$ as oriented links if T_K is oriented such that the top and bottom left tangle ends point outwards. In particular, the connectivity of T_K is and the linking number $\mathrm{lk}(T_K)$ vanishes.

In the context of a satellite knot $P(K)$ with wrapping number 2 pattern P, we also sometimes call T_K the *pattern tangle*; this terminology is justified by Lemma 3.12 (a). We define the *pattern P^K of a knot K* as the pattern $P^{T_K + Q_{-1}}$.

Remark 3.10. The key property of T_K that we use in this paper is that it is cap-trivial. Recall from [48, Proposition 9] that, up to adding twists, i.e. replacing T by $T + Q_n$ for some $n \in \mathbb{Z}$, cap-trivial tangles are in one-to-one correspondence with strongly invertible knots. This correspondence is given by taking two-fold branched covers. In the present case, the two-fold cover of B^3 branched along the tangle T_K is the exterior of $K\#K$ for any knot K. So the double tangle of a knot K may alternatively be defined as the quotient tangle of $K\#K$ under the obvious strong inversion that interchanges the two summands. With the conventions as in [17], this even recovers the same framing.

For example, the double tangle T_U of the unknot U is $Q_0 =$. The double tangle T_{3_1} of the right-handed trefoil knot is shown in Figure 6, along with its two closures to the unknot and a two-component link with linking number zero. The number of these twists is determined by the condition $\mathrm{lk}(T_{3_1}) = 0$. The orientation of K is only required in the above definition to distinguish between T_K and $\mathrm{\phi}(T_K)$. In fact, we have the following proposition.

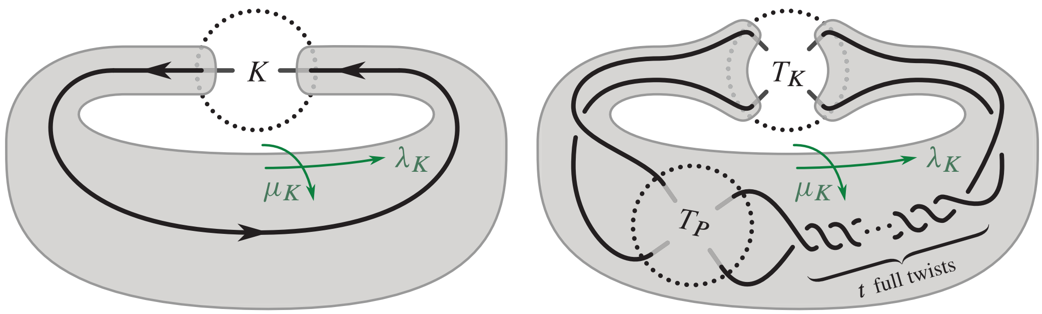


Figure 7. Conventions for double tangles T_K and twisted satellites $P_t(K)$ with $t > 0$. On the left, a closed tubular neighbourhood of K is highlighted in grey, which, on the right, is replaced by the solid torus containing the pattern P.

Proposition 3.11. *For any oriented knot K, $T_{K^r} = \Phi(T_K)$, where K^r denotes the knot K with the opposite orientation. Moreover, two oriented knots K and J are isotopic if and only if the unoriented tangles T_K and T_J are equivalent. In particular, K is a reversible knot if and only if $T_K = \Phi(T_K)$. Finally, mutation around the x-axis always preserves T_K.*

Proof. The first statement follows from the definition of the double tangle. So does the only-if direction of the second statement. For the if direction of the second statement, orient the tangles T_K and T_J such that the top and bottom left tangle ends point outwards. Taking the 0-closures of the tangles and removing one component gives the knots K and J, respectively. So $T_K = T_J$ as unoriented Conway tangles implies $K = J$ as oriented knots. The third statement follows from the previous two. The final statement follows from the observation that twists between the two tangle strands near the tangle ends on the left can be pushed to the tangle ends on the right (this move is commonly called a flype). □

Again, we collect a few properties and relations for later.

Lemma 3.12. *Let P be a pattern with wrapping number 2, T_P a pattern tangle of P, K_1, K_2, and K oriented knots, and $t \in \mathbb{Z}$. Then*

(a) *$P(K) = T_P \cup T_K$. More generally, $P_t(K) = (T_P + Q_{-2t}) \cup T_K$.*

(b) *$P_t(U) = T_P(2t)$.*

(c) *$P_t^{K_1^r}(K_2) = T_{K_1 \# K_2}(2t+1) = P_t^{K_2^r}(K_1)$.*

Proof. In the construction of the satellite knot $P(K)$ in Definition 3.7, we identify a tubular neighbourhood of the companion knot K with the solid torus $S^1 \times D^2$. Using this identification, we can regard $S^1 \times \{\pm \sin(\pi/4)\} \times \{0\}$, the subspace of the solid torus considered in Definition 3.8, as a link in S^3. By comparison with the definition of the double tangle, Definition 3.9, we see that this link in fact coincides with $T_K(0) = Q_0 \cup T_K$. The pattern P is obtained from $S^1 \times \{\pm \sin(\pi/4)\} \times \{0\}$ by replacing the trivial tangle Q_0 by the pattern tangle T_P. Hence $P(K)$ is obtained from $T_K(0) = Q_0 \cup T_K$ by replacing Q_0 by the pattern tangle T_P, which proves the first identity in (a). The second identity follows from the first, since $T_P + Q_{-2t}$ is, by definition, a pattern tangle for P_t. See Figure 7 for an illustration of our conventions.

The identity in (b) follows from (a) together with the fact that $T_U =$. The first equality in (c) follows from

$$\begin{aligned} P_t^{K_1^r}(K_2) &= (T_{K_1^r} + Q_{-2t-1}) \cup T_{K_2} \\ &= Q_{-2t-1} \cup (T_{K_2} + T_{K_1}) \\ &= Q_{-2t-1} \cup T_{K_2 \# K_1} \\ &= T_{K_1 \# K_2}(2t+1), \end{aligned}$$

where the second step uses Proposition 3.11 and Lemma 3.6 (b) and the last step uses

$$K_1 \# K_2 = K_2 \# K_1.$$

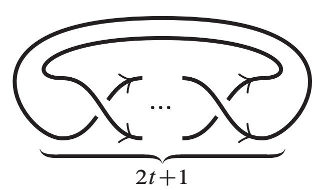
Using this last symmetry, the second equality now follows from the first. □

Definition 3.13. For $t \in \mathbb{Z}$, define the pattern

$$C_{2,2t+1} := P^{Q_{-1-2t}} = P_t^{Q_{-1}}.$$

If K is a knot, then $C_{2,2t+1}(K)$ is called the $(2, 2t+1)$-*cable knot of* K. Setting $K = U$, $C_{2,2t+1}(U) = Q_0(2t+1)$ is the $(2, 2t+1)$-*torus knot*, which we also denote by $T_{2,2t+1}$.

Note that if $t \geq 0$, $T_{2,2t+1}$ has the following diagram with $2t+1$ positive crossings:

$T_{2,2t+1} =$ 

.

Lemma 3.14. *For any knot* K, $C_{2,2t+1}(K) = T_K(2t+1) = P_t^{K^r}(U)$.

Proof. This follows from Lemma 3.12 (c) with $K_1 = U$ and $K_2 = K$. □

Remark 3.15. Based on our conventions, the positive Whitehead double $W^+(K)$ of a knot K equals $T_K(\frac{1}{2})$, and the negative Whitehead double $W^-(K)$ equals $T_K(\frac{-1}{2})$.

3.3. Rasmussen invariants. Rasmussen invariants are integer invariants that one can extract from Bar-Natan homology. The Bar-Natan homology $\widetilde{\mathrm{BN}}(K)$ of a knot K takes the form of a bigraded $\mathbb{Z}[H]$-module, where H is a free variable with homological grading $h(H) = 0$ and quantum grading $q(H) = -2$. Moreover, $\widetilde{\mathrm{BN}}(K)$ is the homology of a chain complex $\widetilde{\mathrm{CBN}}(K)$ of free $\mathbb{Z}[H]$-modules whose chain homotopy type is a knot invariant. The restriction $\widetilde{\mathrm{CBN}}(K)|_{H=0}$ recovers Khovanov's reduced complex computing $\widetilde{\mathrm{Kh}}(K)$. If $\mathbf{k}$ is a field, we define

$$\widetilde{\mathrm{CBN}}(K; \mathbf{k}) := \widetilde{\mathrm{CBN}}(K) \otimes \mathbf{k}$$

and $\widetilde{\mathrm{BN}}(K; \mathbf{k})$ as its homology. Usually, one takes $\mathbf{k} = \mathbb{F}_0 := \mathbb{Q}$ or $\mathbb{F}_c := \mathbb{Z}/c\mathbb{Z}$, where c is a prime. Similar to other knot homology theories, such as knot Floer homology, we have the following structure theorem.

Theorem 3.16. *For any knot K and any field $\mathbf{k}$, $\widetilde{\mathrm{CBN}}(K;\mathbf{k})$ is bigraded chain homotopic to*

$$[q^s h^0 \mathbf{k}[H]] \oplus \bigoplus_i \left[q^{a_i} h^{b_i} \mathbf{k}[H] \xrightarrow{H^{c_i}} q^{a_i+2c_i} h^{b_i+1} \mathbf{k}[H]\right] \tag{3.1}$$

for some uniquely determined $s, a_i \in 2\mathbb{Z}$, $b_i, c_i \in \mathbb{Z}$ with $c_i > 0$. □

This structure theorem has appeared in various guises in [14, 20, 35, 46] and [16, Section 3.3.2]. As an upshot, there exists some integer $s \in \mathbb{Z}$ such that

$$\widetilde{\mathrm{BN}}(K;\mathbf{k}) \cong q^s h^0 \mathbf{k}[H] \oplus (H\text{-torsion}). \tag{3.2}$$

Definition/Proposition 3.17. Given c a prime number or 0, we define the *Rasmussen invariant $s_c(K)$ of K* as the integer s in a decomposition of $\widetilde{\mathrm{BN}}(K;\mathbf{k})$ as in (3.2), where $\mathbf{k}$ is any field of characteristic c.

Proof. Clearly, the integer s is independent of the particular decomposition (3.2). To see that the Rasmussen invariant is the same for fields of the same characteristic, observe that we may write $\widetilde{\mathrm{CBN}}(K;\mathbf{k}) = \widetilde{\mathrm{CBN}}(K;\mathbb{F}_c) \otimes \mathbf{k}$ if $\mathbf{k}$ is a field of characteristic c. Therefore, any decomposition of the form (3.1) for $\mathbb{F}_c$ gives rise to a similar decomposition for $\mathbf{k}$ with the same value of s. □

Remark 3.18. Originally, the Rasmussen invariants were defined in terms of spectral sequences starting at Khovanov homology. The definition of the Rasmussen invariants that we have given above can be shown to be equivalent to those original definitions [14]; see also [16, Section 3.4]. The advantage of working with Bar-Natan homology is two-fold. First, $\mathbf{k}[H]$-modules are conceptually easier to work with than spectral sequences. Second, and more importantly, this approach works over fields of arbitrary characteristic, because it is based on Khovanov homology over $\mathbb{Q}[X]/(p(X))$ with $p(X) = X^2 - X$. It appears to us that the reasons to work with $p(X) = X^2 - 1$ are mainly historic: this is the polynomial Rasmussen [42] used when he constructed s_0 using Lee homology [20]. This approach only generalises to fields of characteristic $c \neq 2$, but cannot be used to define s_2. This is because the construction of the Rasmussen invariant requires $p(X)$ to have two distinct roots over an algebraic closure of $\mathbf{k}$. With the choice $p(X) = X^2 - 1$, that condition is satisfied if and only if $c \neq 2$. The approach via Bar-Natan homology gives a unified definition for all characteristics.

Example 3.19. For all characteristics c and odd integers n,

$$s_c(T_{2,n}) = n + \begin{cases} -1, & n > 0, \\ 1, & n < 0. \end{cases}$$

4. Geometric proofs of twisting behaviours

4.1. Pattern-twisting behaviour of slice-torus invariants. In this subsection, we continue the discussion begun in Sections 2.1 and 2.5 of the behaviour of $\nu(P_t(K))$ for slice-torus invariants ν and patterns P with wrapping number two.

Definition 4.1 ([22, 31]). A knot invariant $\nu(K) \in \mathbb{Z}$ is called a *slice-torus invariant* if the following three conditions are satisfied:

(a) the map from the smooth knot concordance group $\mathcal{C}$ to $\mathbb{Z}$ defined by $[K] \mapsto \nu(K)$ is a well-defined homomorphism;

(b) for all knots K, $\nu(K)$ is less than or equal to the smooth four-genus $g_4(K)$; and

(c) this bound is sharp for positive torus knots, i.e.

$$\nu(T_{p,q}) = g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

for all positive coprime integers p, q.

We will require some well-known properties of slice-torus invariants, which we summarise in the following lemma.

Lemma 4.2. *Let ν be a slice-torus invariant.*

(a) *If two knots K and J are related by a smooth cobordism of genus g, then*

$$|\nu(K) - \nu(J)| \leq g.$$

(b) *If a knot K_+ arises from a knot K_- by a negative crossing change, then*

$$\nu(K_-) \leq \nu(K_+) \leq \nu(K_-) + 1.$$

(c) *If a knot J arises from a knot K by inserting $2n > 0$ positive crossings into two parallely oriented strands, then*

$$\nu(J) - \nu(K) \in \{n, n-1\}.$$

(d) *If a knot K is the plumbing of two unknotted bands with x and y left-handed full twists, respectively, where $x, y \in \mathbb{Z} \smallsetminus \{0\}$, then $\nu(K) = (\operatorname{sgn} x + \operatorname{sgn} y)/2$.*

Sketch of proof. Part (a) follows quickly from the definitions, since

$$\nu(K) - \nu(J) = \nu(K \# -J)$$

and a smooth cobordism of genus g between K and J gives rise to a smooth slice surface of genus g of $K\# - J$. For (b), see [31, Corollary 3] or [42, Corollary 4.3]. It is also a special case of (c). To show (c), one uses the fact that there are smooth cobordisms of genus 1 between J and $K\#T(2, 2n+1)$, and between J and $K\#T(2, 2n-1)$. The values of ν for the knots in (d) can, for example, be computed with the sharper slice-Bennequin inequality, since the knots in question are alternating; see e.g. [22, Theorem 5], noting the different normalisation convention for slice-torus invariants. □

Let us now prove Proposition 2.1 and Proposition 2.14.

Proof of Proposition 2.1. It suffices to show the following two claims.

(i) For all $s, t \in \mathbb{Z}$, $|\nu(P_t(K)) - \nu(P_s(K))| \leq 1$.

(ii) For all $t \in \mathbb{Z}$, $0 \leq \nu(P_t(K)) - \nu(P_{t+1}(K)) \leq 1$.

By Lemma 4.2 (a), it suffices for (i) to show that there is a genus 1 cobordism between $P_t(K)$ and $P_s(K)$ for any $s, t \in \mathbb{Z}$. This is easy to construct: let L be the two-component link obtained as the oriented resolution of $P_t(K)$ at some crossing in the twisting region. The link L may be viewed as the result of attaching a single band to $P_t(K)$ near the resolved crossing and thus gives rise to a cobordism c_t from $P_t(K)$ to L. Clearly, L does not depend on the number of twists t, so we similarly get a cobordism c_s from $P_s(K)$ to L. Composing c_t with the inverse of c_s gives the desired cobordism of genus 1 from $P_t(K)$ to $P_s(K)$.

The inequalities in (ii) follow from Lemma 4.2 (b) and the fact that increasing the number of full twists by 1 is equivalent to changing a positive crossing in the twisting region to a negative one. $\square$

Proof of Proposition 2.14. Let $s, t \in \mathbb{Z}$ with $s \leq t$. Note that $P_t(K)$ arises from $P_s(K)$ by inserting $2(t-s)$ positive crossings into two oriented strands with parallel orientation. So, by Lemma 4.2 (c), we have

$$\nu(P_t(K)) - \nu(P_s(K)) \in \{t-s, t-s-1\}.$$

From this, the proposition follows. $\square$

Finally, let us prove the following statement, which implies the first half of Theorem 1.2.

Proposition 4.3. *Suppose ν is a slice-torus invariant for which the answer to Question* 2.5 *is "yes". Then ϑ_ν induces a homomorphism from the smooth knot concordance group $\mathcal{C}$ to $\mathbb{Z}$.*

Proof. Let knots K, J be given. Let P be the pattern $W^+ + \Phi(T_K)$, where T_K is the double tangle of K (see Definition 3.9) and Φ is the operation that rotates tangles around their vertical axis. Then we have the following equalities for all $t \in \mathbb{Z}$:

$$\begin{aligned}\nu(W^+_{t-\vartheta_\nu(K\#J)}(U)) &= \nu(W^+_t(K\#J)) = \nu(P_t(J)) \\ &= \nu(P_{t-\vartheta_\nu(J)}(U)) = \nu(W^+_{t-\vartheta_\nu(J)}(K)) \\ &= \nu(W^+_{t-\vartheta_\nu(K)-\vartheta_\nu(J)}(U)),\end{aligned}$$

where the first, third, and fifth equality use the assumption that the answer to Question 2.5 is "yes" for ν, and the second and fourth equality use the definition of P. Setting t to $\vartheta_\nu(K\#J)$ and $\vartheta_\nu(K\#J) - 1$ now respectively yields

$$\begin{aligned}\nu(W^+_{\vartheta_\nu(K\#J)-\vartheta_\nu(K)-\vartheta_\nu(J)}(U)) &= \nu(W^+(U)) = 0, \\ \nu(W^+_{\vartheta_\nu(K\#J)-\vartheta_\nu(K)-\vartheta_\nu(J)-1}(U)) &= \nu(W^+_{-1}(U)) = 1;\end{aligned}$$

see Example 2.4. Because of the uniqueness of the jump point (see Proposition 2.1), it follows that $\vartheta_\nu(K\#J) - \vartheta_\nu(K) - \vartheta_\nu(J) = \vartheta_\nu(U) = 0$, and thus ϑ_ν is additive with respect to #.

So it just remains to show that ϑ_ν is a concordance invariant. Let K and J be concordant knots. It follows that, for all $t \in \mathbb{Z}$, the knots $W^+_t(K)$ and $W^+_t(J)$ are also concordant, and thus $\nu(W^+_t(K)) = \nu(W^+_t(J))$. Again using the uniqueness of the jump point, we find $\vartheta_\nu(K) = \vartheta_\nu(J)$, as desired. This concludes the proof. $\square$

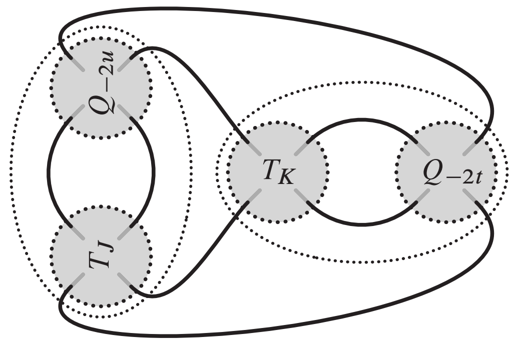


Figure 8. The knots $D_t^{J,u}(K)$ discussed in Remark 4.4.

4.2. Pattern-twisting behaviour of τ. Let us now resume the investigation started in Section 2.2 of the behaviour of $\tau(P(K))$.

Remark 4.4. Levine [21] has established the following generalisation of Hedden's result (2.2): $\vartheta_\tau(K) = 2\tau(K)$. Consider knots as shown in Figure 8, denoted by $D_t^{J,u}(K)$ for knots K, J and integers t, u. Here, we think of $D^{J,u}$ as a pattern, which has wrapping number 2 and winding number 0. Indeed, in the language of Definition 3.8, $D^{J,u}$ is the pattern of the tangle T, where T is $T_J + Q_{-2u}$ rotated by $\pi/2$. Note that $D^{U,\pm 1} = W^{\mp}$ and

$$D_t^{J,u}(K) = D_u^{K,t}(J)^r, \tag{4.1}$$

where the superscript r denotes taking the reverse of a knot. Levine shows that

$$\tau(D_t^{J,u}(K)) = \begin{cases} 1 & \text{if } t < 2\tau(K) \text{ and } u < 2\tau(J), \\ -1 & \text{if } t > 2\tau(K) \text{ and } u > 2\tau(J), \\ 0 & \text{else.} \end{cases} \tag{4.2}$$

Assume that ν is a slice-torus invariant for which the answer to Question 2.5 is "yes", such as $\nu = s_2$, i.e. for all knots K and patterns P with wrapping number two and winding number zero, we have

$$\nu(P(K)) = \nu(P_{-\vartheta_\nu(K)}(U)). \tag{4.3}$$

Then an analogous formula to (4.2) can be deduced for ν. First note that

$$\nu(D_t^{J,u}(K)) \overset{(4.3)}{=} \nu(D^{J,u}_{t-\vartheta_\nu(K)}(U)) \overset{(4.1)}{=} \nu(D_u^{U,t-\vartheta_\nu(K)}(J)^r) \overset{(4.3)}{=} \nu(D^{U,t-\vartheta_\nu(K)}_{u-\vartheta_\nu(J)}(U)^r).$$

Now,

$$D^{U,t-\vartheta_\nu(K)}_{u-\vartheta_\nu(J)}(U)^r$$

is just a Hopf plumbing of two unknotted annuli with $t - \vartheta_\nu(K)$ and $u - \vartheta_\nu(J)$ right-handed full twists, respectively, for which ν can be easily determined as follows, using Lemma 4.2 (d):

$$\nu(D_t^{J,u}(K)) = \begin{cases} 1 & \text{if } t < \vartheta_\nu(K) \text{ and } u < \vartheta_\nu(J), \\ -1 & \text{if } t > \vartheta_\nu(K) \text{ and } u > \vartheta_\nu(J), \\ 0 & \text{else.} \end{cases} \tag{4.4}$$

In particular, a positive answer to Question 2.9 would recover Levine's formula (4.2).

Let us now prove Proposition 2.10, which we restate for the reader's convenience.

Proposition 2.10. *There exists a knot K with three-genus* 1*, Alexander polynomial* 1*, $\tau(K) = 1$ and $s_2(K) = -2$. More generally, for any prescribed triple (g, a, b) of integers with $|a| \le g$, $|b| \le g$, there exists a knot L with three-genus g, Alexander polynomial* 1*, $\tau(L) = a$ and $s_2(L) = 2b$.*

Proof. Take again $J = T_{3,4}^{\#2} \# T_{-2,3}^{\#5}$, as in the proof of Corollary 2.11. Let $K = D^{J,0}(J)$. This knot is obtained from the Hopf plumbing Σ of two unknotted untwisted bands by tying the knot J into each of the two bands, with framing zero. So Σ' and Σ have the same genus and the same Seifert form. Since Σ is a genus one Seifert surface of the unknot, it follows that K has three-genus 1 and Alexander polynomial 1. The values of τ and s_2 for K may be computed from (4.2) and (4.4), respectively, using that $\tau(J) = 1$ and $\vartheta_2(J) = -4$. So we have constructed a knot K, as desired.

Knot	τ	s_2
$W^+(T_{2,3})$	1	2
$W^+(J)$	1	0
K	1	-2
$W^+(-T_{2,3})$	0	0
$W^+(-J)$	0	2

Table 3

The knot L in the second claim may be obtained as a connected sum of g summands, with each summand equal to one of the five knots listed in Table 3, or their mirror images. □

4.3. Pattern-twisting behaviour of other knot invariants. While we have focused on slice-torus invariants and patterns with wrapping number two so far, one could also pursue the following more general question: for which triples $(y, \mathcal{P}, \vartheta)$ of a knot invariant y, a set of patterns $\mathcal{P}$, and an integer-valued knot invariant ϑ does

$$y(P(K)) = y(P_{-\vartheta(K)}(U)) \tag{4.5}$$

hold for all knots K and patterns $P \in \mathcal{P}$? Let us reformulate some previously stated results in these terms.

- Theorem 1.1 says that (4.5) holds for $y = s_2$, $\vartheta = \vartheta_2$, and $\mathcal{P}$ the set $\mathcal{P}_1$ of all patterns with wrapping number two and winding number zero.
- Proposition 2.1 says that (4.5) holds for y equal to a slice-torus invariant ν,

  $$\mathcal{P} = \{P_t \mid t \in \mathbb{Z}\} \subset \mathcal{P}_1$$

  for a single fixed pattern $P \in \mathcal{P}_1$, and $\vartheta(K) = \vartheta_\nu(P, K)$ if $\vartheta_\nu(P, K) \ne \infty$, and $\vartheta(K)$ arbitrary if $\vartheta_\nu(P, K) = \infty$.
- Hedden's result (2.2) $\vartheta_\tau(K) = 2\tau(K)$ implies that (4.5) is satisfied for $y = \tau$, $\vartheta = 2\tau$ and $\mathcal{P} = \mathcal{P}_2 = \{W_t^+ \mid t \in \mathbb{Z}\} \cup \{W_t^- \mid t \in \mathbb{Z}\} \subset \mathcal{P}_1$.

- Levine's result discussed in Remark 4.4 means that (4.5) holds for $y = \tau$, $\vartheta = 2\tau$ and $\mathcal{P} = \mathcal{P}_3 = \{D_t^{J,u} \mid u, t \in \mathbb{Z},\ J \text{ any knot}\}$. Note that $\mathcal{P}_2 \subset \mathcal{P}_3 \subset \mathcal{P}_1$.

Let us also cast a quick glance at some further knot invariants.

- The Upsilon invariant Υ_K, a piecewise linear function $[0,2] \to \mathbb{R}$, is another knot invariant coming from knot Floer homology [40]. If the three-genus $g(K)$ of K is at most 1, then $\Upsilon_K(t) = \tau(K) \cdot (-1 + |1-t|)$ for all $t \in [0,2]$. Since $g(P(K)) \leq 1$ holds for all knots K and $P \in \mathcal{P}_3$, it follows from Levine's result above that (4.5) is also satisfied for $y(K) = \Upsilon_K$, $\vartheta = 2\tau$ and $\mathcal{P} = \mathcal{P}_3$. This was remarked first in [5, Corollary 3.4].
- For $n \geq 3$, the Khovanov–Rozansky $\mathfrak{sl}_n$-concordance homomorphism $\mathfrak{s}_n$ (see [34, 49]) is, properly normalised, a lower bound for $g_4(K)$ and satisfies $\mathfrak{s}_n(T) = g_4(T)$ for positive torus knots T. But in the sense of this article, it is not a slice-torus invariant, since it takes values in $\frac{1}{n-1}\mathbb{Z}$ instead of $\mathbb{Z}$. So it is conceivable that, for some $n \geq 3$ and some knot K, the expression $\mathfrak{s}_n(W_t^+(K))$ as a function of t has more than one jump point. We do not know whether (4.5) holds for $y = \mathfrak{s}_n$ and any interesting set $\mathcal{P}$ of patterns.
- The classical Levine–Tristram signatures σ_ω, $\omega \in S^1$, satisfy $\sigma_\omega(P(K)) = \sigma_\omega(P(U))$ for all P in the set $\mathcal{P}_4$ of patterns with winding number zero (without restrictions on the wrapping number) [29]. In other words, (4.5) holds for $y = \sigma_\omega$, $\mathcal{P} = \mathcal{P}_4$, and $\vartheta(K) = 0$ for all knots K.

4.4. Crossing changes and twists of the satellite companion. In this subsection, we pursue the question how a satellite $P(K)$ changes when a crossing change is applied to K. This will lead us to a proof of Proposition 2.34. First, let us further generalise the notion of generalised crossing change from Section 2.7.

Definition 4.5. Suppose K is a knot and λ is an unknot in the complement of K (the orientation of λ is irrelevant). Performing a $(+1)$-framed Dehn surgery on λ transforms K into another knot J in S^3. Let $h = |\mathrm{lk}(K,\lambda)| = |\mathrm{lk}(J,\lambda)|$ and suppose λ bounds a disc D that intersects K (or, equivalently, J) transversely in n points. Then we say that J arises from K (K from J) by a *left-handed (right-handed) h-homologous twist on n strands*.

One may achieve a left-handed twist on n strands by tying a left-handed full twist into the n strands of K close to D. A generalised negative crossing change is simply a null-homologous left-handed twist.

Remark 4.6. Note that a negative crossing change may be achieved by a null-homologous left-handed twist on two strands, but also by a 2-homologous right-handed twist on two strands. Figure 11 contains a version of this observation for generalised crossing changes.

Lemma 4.7. *Let the knot J arise from the knot K by a left-handed h-homologous twist on n strands. Let P be a pattern with winding number v and wrapping number r. Then $P_{-h^2}(J)$ arises from $P(K)$ by a left-handed $(h \cdot |v|)$-homologous twist on $(n \cdot r)$ strands.*

Proof. By the definition of a twist, there is an unknot λ in the complement of K with $|\mathrm{lk}(K,\lambda)| = h$ such that $(+1)$-Dehn surgery along λ transforms K into J. Denote by L the knot arising from $P(K)$ by $(+1)$-Dehn surgery along λ. Note that L arises from $P(K)$ by a left-

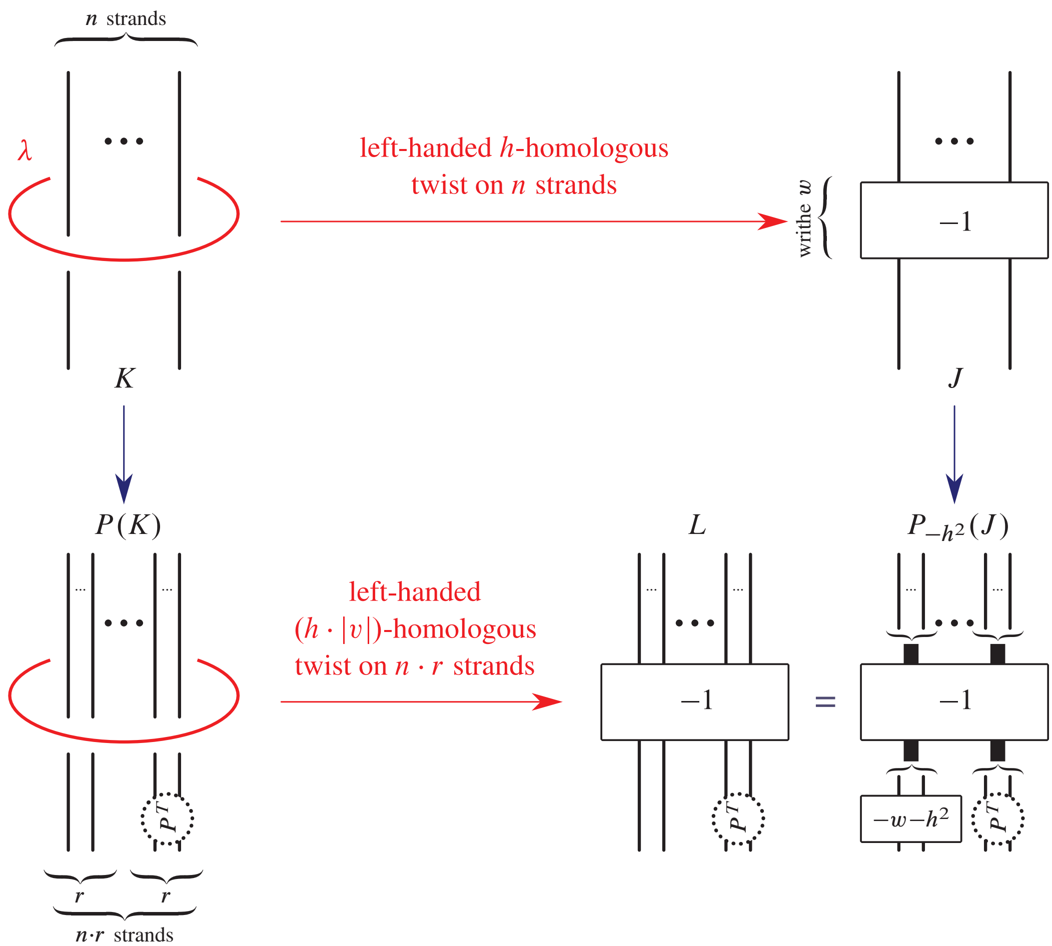


Figure 9. The essence of the proof of Lemma 4.7. The figure shows five knots, each intersected with a neighbourhood of λ. Away from this neighbourhood, the two knots K and J in the top row are identical, and so are the three knots $P(K)$, L, $P_{-h^2}(J)$ in the bottom row. Numbers in boxes indicate right-handed full twists. In the diagram of $P_{-h^2}(J)$, bottom right, the -1 full twist is understood to twist the n bunches of r strands each (drawn thick), but not twist each individual bunch; cf. Figure 10. We assume that P is a pattern with wrapping number r and winding number v, and $h = |\mathrm{lk}(K, \lambda)|$.

handed $(h \cdot |v|)$-homologous twist on $(n \cdot r)$ strands. To prove the statement of the lemma, we have to show $L = P_{-h^2}(J)$.

Let us take a diagrammatic approach. The top row of Figure 9 shows the intersection of diagrams of K and J with a neighbourhood of λ. From these diagrams, we construct diagrams of $P(K)$ and $P_{-h^2}(J)$ as follows. First, we apply an isotopy to the pattern $P \subset S^1 \times D^2$ so that it equals the union of a $2r$-ended tangle P^T sitting inside a ball with $2r$ curves that are parallel to the core curve of $S^1 \times D^2$ (as described in Definition 3.8 for $r = 2$). We then do the following:

- we replace every strand with a bunch of r parallel strands;
- we insert P^T into a bunch of r strands somewhere; and
- we tie left-handed full twists into a bunch of r strands somewhere, as many as the writhe of the original diagram.

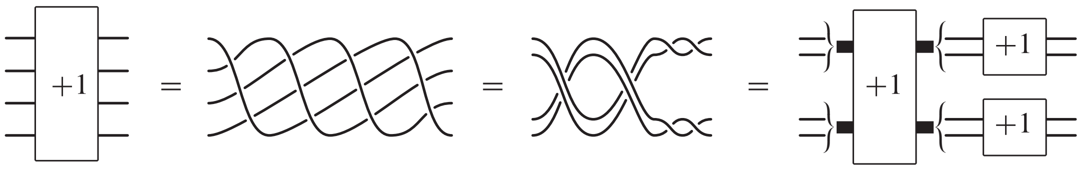


Figure 10. A full twist on four strands equals a full twist on two bunches of two strands each, followed by a full twist on each of the individual bunches.

From the diagram of $P(K)$, we obtain a diagram of L by tying a left-handed full twist into nr strands.

Let us now show that our diagrams of L and $P_{-h^2}(J)$ represent the same knot. Since these diagrams are identical outside from the neighbourhood of λ, we just have to compare them inside of that neighbourhood, as drawn in Figure 9.

In the diagram of L, the full twist on nr strands (consisting of $nr(nr-1)$ crossings) is isotopic to a full twist tied into n bunches of r strands each (consisting of $n(n-1)r^2$ crossings), followed by a full twist in each of the individual bunches (consisting of a total of $nr(r-1)$ crossings). This is illustrated for $n = 2 = r$ in Figure 10. Since K is a knot, the full twists in the individual bunches may be slid around the knot so that they lie next to each other, giving a total of n left-handed full twists tied into one of the bunches.

So the diagrams of L and of $P_{-h^2}(J)$ both consist of a left-handed full twist tied into n bunches of r strands each, and a number of left-handed full twists tied into one of the bunches. To show the equality of L and $P_{-h^2}(J)$, it thus only remains to show that these numbers of twists agree. For L, this number is n, as determined above. For $P_{-h^2}(J)$, this number equals $h^2 + w$, where w is the writhe of the tangle diagram D' given as intersection of the knot diagram of J with the neighbourhood of λ, as shown in the top right of Figure 9.

Let us verify that $w = n - h^2$. The tangle diagram D' has $n(n-1)$ crossings. Of the n strands, $(n+h)/2$ are oriented upwards, and $(n-h)/2$ are oriented downwards (or the other way around). In a full twist, every pair of strands crosses twice, and since the full twist is left-handed, parallely oriented strands cross negatively, while oppositely oriented strands cross positively. One thus computes

$$w(D') = -2\binom{(n+h)/2}{2} - 2\binom{(n-h)/2}{2} + 2 \cdot \frac{n+h}{2} \cdot \frac{n-h}{2} = n - h^2.$$

This concludes the proof. □

Proposition 4.8. *Let the knot J arise from the knot K by a left-handed h-homologous twist on n strands. Let ν be a slice-torus invariant satisfying the $2n$-stranded generalised crossing change inequality. Let P be a pattern with wrapping number two and winding number zero. Assume that neither $\vartheta_\nu(J,P)$ nor $\vartheta_\nu(K,P)$ are equal to ∞. Then we have*

$$\vartheta_\nu(K,P) \le \vartheta_\nu(J,P) + h^2.$$

In particular, if ν satisfies the generalised crossing change inequality, then so does the knot invariant $\vartheta_\nu(\cdot\,,P)$ for knots $K_\pm$ with $\vartheta_\nu(K_\pm,P) \ne \infty$.

Proof. Let $t = \vartheta_\nu(K,P) - 1$. By Lemma 4.7, $P_{t-h^2}(J)$ arises from $P_t(K)$ by a left-handed null-homologous twist on $2n$ strands. Since ν satisfies the generalised crossing change

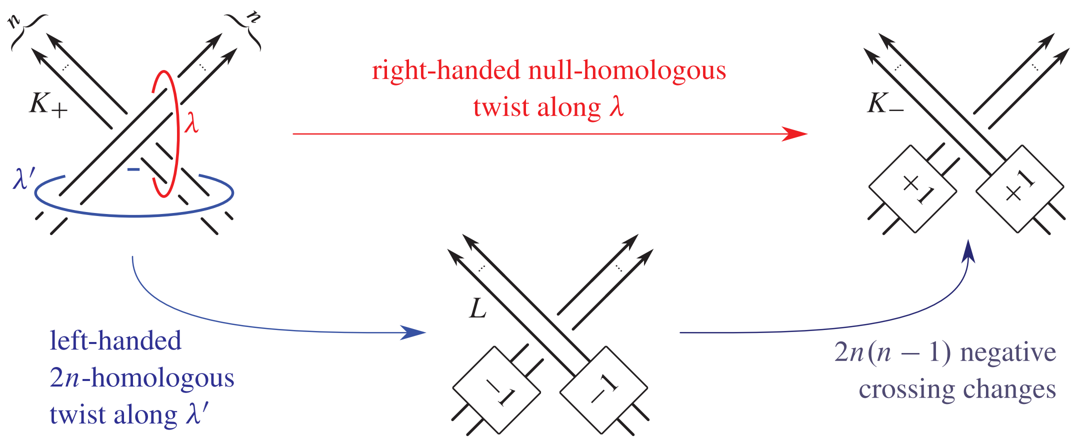


Figure 11. A generalised positive crossing change on $2n$ strands may also be achieved by a left-handed $2n$-homologous $2n$-stranded twist followed by $2n(n-1)$ negative crossing changes.

inequality, it follows that $\nu(P_t(K)) \leq \nu(P_{t-h^2}(J))$. For all $s \leq t-h^2$, we have

$$\nu(P_{t-h^2}(J)) \leq \nu(P_s(J))$$

by Proposition 2.1. By choice of t, we have $\nu(P_t(K)) = \nu(P_{t+1}(K)) + 1$. All combined, we obtain

$$\nu(P_{t+1}(K)) + 1 \leq \nu(P_s(J)) \tag{4.6}$$

for all $s \leq t-h^2$. However, since $P_{t+1}(K)$ and $P_s(J)$ are related by smooth cobordism of genus 1, (4.6) is in fact an equality. So the function $s \mapsto \nu(P_s(J))$ is constant for $s \leq t-h^2$. Therefore, $\vartheta_\nu(J,P) \geq t-h^2+1 = \vartheta_\nu(K,P)-h^2$, as desired. The last statement follows from setting $h=0$. □

Proof of Proposition 2.34. Applying Proposition 4.8 to the case

$$h=0, \quad K=K_-, \quad J=K_+, \quad \text{and} \quad P=W^+$$

yields the first inequality $\vartheta_\nu(K_-) \leq \vartheta_\nu(K_+)$ to be proven. For the second inequality, let L be the knot obtained from K_+ by a left-handed $2n$-homologous $2n$-stranded twist as indicated in Figure 11. Applying Proposition 4.8 to $h=2n$, $K=K_+$, $J=K_-$, and $P=W^+$, we have $\vartheta_\nu(K_+) \leq \vartheta_\nu(L)+4n^2$. Since K_- arises from L by a sequence of $2n(n-1)$ negative crossing changes, we have

$$\vartheta_\nu(L) \leq \vartheta_\nu(K_-),$$

implying $\vartheta_\nu(K_+) \leq \vartheta_\nu(K_-)+4n^2$, as desired. □

5. Review of multicurves for Bar-Natan homology

In [16], Artem Kotelskiy, Liam Watson, and the second author introduced the multicurve invariant $\widetilde{\mathrm{BN}}$ for Conway tangles. In this section, we give a brief overview of this invariant, discussing only those of its properties that we need in subsequent sections. More elaborate introductions to this invariant can be found in [17, 19].

5.1. The construction of the multicurve invariant. Given some field of coefficients $\mathbf{k}$ and a (standard) oriented Conway tangle T, the invariant $\widetilde{\mathrm{BN}}(T;\mathbf{k})$ is defined in two steps. First, following a construction due to Bar-Natan, one defines a bigraded chain complex $[\![T]\!]$ over a certain cobordism category, whose objects are crossingless tangle diagrams [2]. Any such complex can be rewritten as a chain complex $Д(\mathcal{D}_T;\mathbf{k})$ over the following quiver algebra [16, Theorem 1.1]:

$$\mathcal{B} := k\left[\begin{tikzcd} \bullet \arrow[loop left, "D_\bullet"] \arrow[r, bend right, "S_\bullet"'] & \circ \arrow[l, bend right, "S_\circ"'] \arrow[loop right, "D_\circ"] \end{tikzcd}\right] \Big/ \begin{pmatrix} D_\bullet \cdot S_\circ = 0 = S_\circ \cdot D_\circ \\ D_\circ \cdot S_\bullet = 0 = S_\bullet \cdot D_\bullet \end{pmatrix}. \tag{5.1}$$

We read algebra elements from right to left. So for instance, $S_\circ = \iota_\bullet \cdot S_\circ \cdot \iota_\circ$, where $\iota_\bullet$ and $\iota_\circ$ are the two idempotent algebra elements corresponding to the constant paths at $\bullet$ and $\circ$, respectively. The objects $\bullet$ and $\circ$ correspond to the crossingless tangles and , respectively. We write

$$D := D_\circ + D_\bullet \quad \text{and} \quad S := S_\circ + S_\bullet.$$

The algebra $\mathcal{B}$ carries a quantum grading q, which is determined by

$$q(D_\bullet) = q(D_\circ) = -2 \quad \text{and} \quad q(S_\bullet) = q(S_\circ) = -1.$$

The homological grading h vanishes on $\mathcal{B}$. Differentials of bigraded chain complexes over $\mathcal{B}$ are defined to preserve quantum grading and increase the homological grading by 1. More explicitly, if there is an arrow $x \xrightarrow{a} y$ in the differential, then

$$q(y) + q(a) - q(x) = 0, \quad h(y) - h(x) = 1.$$

We indicate these gradings on generators $\bullet$ by super- and subscripts like so: ${}^{q}\bullet_{h}$. The bigraded chain homotopy type of $Д(T;\mathbf{k})$ is an invariant of the tangle T (see [16, Section 4.2]).

Example 5.1. The chain complex corresponding to the trivial tangle $Q_0 =$ (with any orientation) consists of exactly one object and the differential vanishes,

$$Д(Q_0) = [{}^{0}\bullet_{0}].$$

The invariant of the $(-2,3)$-pretzel tangle T_a shown in Figure 1 (e) is equal to

$$Д(T_a) = [{}^{-2}\circ_{-2} \xrightarrow{S} {}^{-1}\bullet_{-1} \xrightarrow{D} {}^{1}\bullet_{0} \xrightarrow{S^2} {}^{3}\bullet_{1} \xrightarrow{D} {}^{5}\bullet_{2} \xrightarrow{S} {}^{6}\circ_{3} \xleftarrow{D} {}^{4}\circ_{2} \xleftarrow{S} {}^{3}\bullet_{1} \xleftarrow{D} {}^{1}\bullet_{0}].$$

(This and other computations for this paper can be found at [24].) Here, the orientation on T_a is chosen to be pointing outwards at the punctures on the top left and bottom right; altering this choice only affects the bigrading (see below).

In the second step, one interprets $Д(T;\mathbf{k})$ geometrically. This relies on the following classification result: chain homotopy classes of bigraded chain complexes over $\mathcal{B}$ are in one-to-one correspondence with certain geometric objects, called multicurves [16, Theorem 1.5]. Using this correspondence between homological algebra and geometry, one now defines the invariant $\widetilde{\mathrm{BN}}(T;\mathbf{k})$ as the multicurve corresponding to $Д(T;\mathbf{k})$.

Multicurves are, by definition, collections of certain immersed curves on $S^2_{4,*}$, a four-punctured sphere with one distinguished/special puncture. The curves also carry a bigrading,

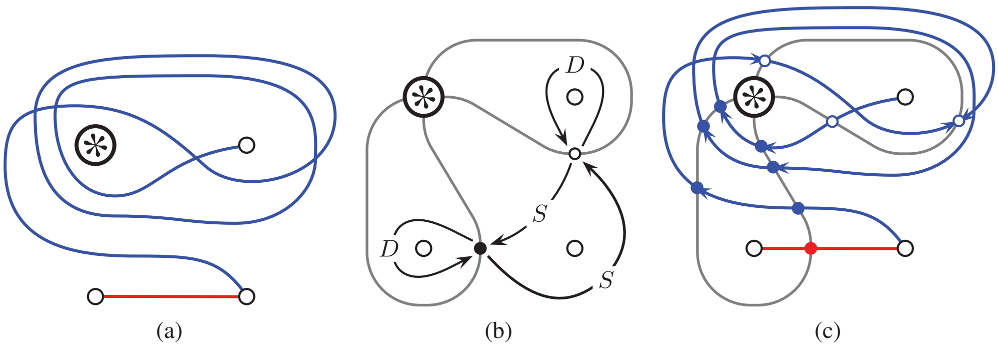


Figure 12. (a) Two non-compact curves on $S^2_{4,*}$: the short one in red shows $\widetilde{\mathrm{BN}}_a(\;)$, the long, blue curve is $\widetilde{\mathrm{BN}}_a(T_a)$; (b) the parametrisation of $S^2_{4,*}$ with the embedded quiver from equation (5.1); and (c) the curves from (a) half-way translated into chain complexes over $\mathcal{B}$.

but we will ignore this for a moment. We distinguish between compact and non-compact curves. A non-compact curve in $S^2_{4,*}$ is a non-null-homotopic immersion of an interval, with ends on the three non-special punctures of $S^2_{4,*}$ (see [16, Definition 1.4]). We consider these curves up to homotopy relative boundary. For the purpose of this paper, we can completely ignore compact curves, thus avoiding the discussion of local systems. Since our notion of Conway tangles does not allow for closed tangle components, we know that, for any Conway tangle T, $\widetilde{\mathrm{BN}}(T;\mathbf{k})$ contains exactly one non-compact component [16, Proposition 6.15]. We denote this component by $\widetilde{\mathrm{BN}}_a(T;\mathbf{k})$; it is this invariant that we will focus on below.

Example 5.2. Figure 12 (a) shows two non-compact immersed curves on the four-punctured sphere $S^2_{4,*}$, which is drawn as the plane plus a point at ∞, minus the special puncture $*$ and the three non-special punctures $\circ$. The short, horizontal curve (drawn in red) shows the invariant $\widetilde{\mathrm{BN}}(Q_0;\mathbf{k}) = \widetilde{\mathrm{BN}}_a(Q_0;\mathbf{k})$, which consists of a single component. The other curve (drawn in blue) is equal to the invariant $\widetilde{\mathrm{BN}}(T_a;\mathbf{k}) = \widetilde{\mathrm{BN}}_a(T_a;\mathbf{k})$. The invariants of both tangles are independent of the field of coefficients $\mathbf{k}$; this is not true in general.

At multiple points in this paper, we need to apply the correspondence between chain complexes and multicurves very explicitly, so we now briefly describe how this works. Consider the two dotted grey arcs in Figure 12 (b), which start and end on the special puncture of $S^2_{4,*}$ and separate the three non-special punctures. We call these arcs the *parametrisation of* $S^2_{4,*}$. Furthermore, fix an embedding of the quiver (5.1) into $S^2_{4,*}$ as shown also in Figure 12 (b). Now, given an immersed curve γ (say, non-compact), we homotope it such that it intersects the parametrisation of $S^2_{4,*}$ minimally. Then each intersection point corresponds to a generator of the associated chain complex, namely an object $\circ$ if the intersection point lies on the grey arc at the top and an object $\bullet$ if it lies on the arc on the left, as prescribed by the vertices of the embedded quiver. The paths between those intersection points that meet the parametrisation only in their two endpoints correspond to the differentials of the chain complex. More explicitly, suppose $\alpha\colon [a_0, a_1] \to S^2_{4,*}$ is a subpath of γ such that the preimage of the parametrisation is equal to $\{a_0, a_1\}$. Then the chain complex contains an arrow

$$\alpha(a_0) \xrightarrow{b_\alpha} \alpha(a_1) \quad \text{or} \quad \alpha(a_1) \xrightarrow{b_\alpha} \alpha(a_0),$$

where the direction of the arrow and $b_\alpha \in \mathcal{B}$ are determined by the unique path in the embedded quiver that is homotopic to α (relative to the parametrisation).

Example 5.3. The curve $\widetilde{\mathrm{BN}}_a(Q_0)$ from Example 5.2 intersects the parametrisation in exactly one point $\bullet$. So the corresponding complex consists of a single object $\circ$ and the differential vanishes. This is precisely the complex $\text{Д}(Q_0)$ from Example 5.1. The curve $\widetilde{\mathrm{BN}}_a(T_a)$ intersects the left arc of the parametrisation three times and the other arc six times. So the corresponding complex contains three generators $\circ$ and six generators $\bullet$. The directions of the differentials are shown in Figure 12 (c). The corresponding complex agrees with $\text{Д}(T_a)$ from Example 5.1.

The reverse direction of the correspondence between chain complexes and multicurves, i.e. translating algebra into geometry, is generally much more involved and relies on the arrow pushing algorithm from [8]. Notice, however, that any chain complex represented by a directed graph with only 0-, 1-, or 2-valent vertices whose arrows are labelled alternately by powers of $S, D \in \mathcal{B}$ arises directly from a multicurve in the way described above. So, in practice, it suffices to find such a representative in the homotopy class of a given chain complex. If the number of objects is small, this is usually straightforward.

Remark 5.4. The multicurve corresponding to a direct sum of chain complexes is equal to the disjoint union of the multicurves corresponding to the individual summands.

The parametrisation of $S^2_{4,*}$ not only plays a key role for translating between chain complexes and multicurves, it is also important for the definition of the bigrading on immersed curves. The bigrading takes the form of a $\mathbb{Z}^2$-valued function $\mathrm{gr} = (q, h)$ on the set of intersection points with the parametrisation that is compatible with how these intersection points are connected by the curve, in the sense that the bigrading on the corresponding chain complex be well-defined. Thus, the bigrading of any given component of a multicurve is determined by the bigrading of a single generator of this component.

In general, the bigrading on $\text{Д}(T)$ depends on an orientation of the tangle T. If T is an unoriented tangle, $\widetilde{\mathrm{BN}}(T)$ only carries *relative* bigradings, i.e. bigradings that are well-defined up to an overall shift that is determined by the linking number of the tangle (Definition 3.9).

Proposition 5.5. *If T is a Conway tangle with linking number* 0*, the (absolute) bigrading on $\widetilde{\mathrm{BN}}(T)$ is independent of the orientation of T.*

Proof. This follows directly from [16, Proposition 4.8]. $\square$

This proposition applies in particular to the trivial tangle and, more generally, to the double tangle T_K for any knot K.

5.2. Properties of the multicurve invariant. The reader may have already noticed the appearance of two conspicuously distinct four-punctured spheres in the construction of the multicurve invariants above: the surface $S^2_{4,*}$, on which the curves live, and the boundary of the unit 3-ball B^3 with the four marked points $\{\pm e^{\pm i\pi/4}\} \times \{0\}$, which are the tangle ends ∂T. The parametrisation of $S^2_{4,*}$ used for translating between chain complexes and multicurves distinguishes one of the four punctures from the other three. A choice of a distinguished point $*$ among the four marked points on ∂B^3 is in fact also required in the definition of the homological invariant $Д(T;\mathbf{k})$. (We always choose the top left tangle end $(-e^{-i\pi/4}, 0)$ as the point $*$.)

This coincidence suggests that one should identify $S^2_{4,*}$ with $\partial B^3 \smallsetminus \partial T$. We do this as follows. Assume that the four punctures in Figure 12 (b) are the points $\pm e^{\pm i\pi/4}$ on the unit circle in $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} = S^2$. We identify this circle with $S^1 \times \{0\} \subset \partial B^3$ via the identity map. We then extend this map to the entire spheres such that the interior of the unit disc in $\mathbb{C} \subset S^2$ is mapped to those points on ∂B^3 with positive z-coordinate (i.e. the front). Of course, identifying these spheres begs the question if the identification is natural with respect to the action of the mapping class group $\mathrm{Mod}(S^2_{4,*})$: does adding a twist to the tangle ends correspond to twisting the curves? At least if $\mathbf{k} = \mathbb{F}_2$, the answer is yes.

Theorem 5.6 ([16, Theorem 1.13]). *For all oriented Conway tangles T and for all mapping classes $\tau \in \mathrm{Mod}(S^2_{4,*})$,*

$$\widetilde{\mathrm{BN}}(\tau(T;\mathbb{F}_2)) = \tau(\widetilde{\mathrm{BN}}(T;\mathbb{F}_2)). \qquad \square$$

We expect that Theorem 5.6 generalises to arbitrary fields, but we will not prove it in this article. We only need to understand how the mapping class group acts on non-compact curves $\widetilde{\mathrm{BN}}_a(T;\mathbf{k})$.

Proposition 5.7. *For all oriented Conway tangles T, fields $\mathbf{k}$, and $\tau \in \mathrm{Mod}(S^2_{4,*})$,*

$$\widetilde{\mathrm{BN}}_a(\tau(T;\mathbf{k})) = \tau(\widetilde{\mathrm{BN}}_a(T;\mathbf{k})).$$

Sketch proof. Following the same strategy of the proof of Theorem 5.6, it suffices to consider how single twists act on the multicurves. Using Bar-Natan's gluing formalism for cobordism complexes of tangles, we compute how a single twist acts on the objects and morphisms in $\mathcal{B}$. The action on objects is given by

$$\circ \longmapsto [\circ \xleftarrow{D} \circ \xrightarrow{1} \circ], \quad \bullet \longmapsto [\bullet \xrightarrow{D} \circ]$$

with the appropriate grading shifts. The action on morphisms is the same as [16, Figure 45a], except that some of the vertical arrows carry minus signs. From this, one can compute the action on type D structures over $\mathcal{B}$ corresponding to elementary curve segments. Again, the answer differs from [16, Figure 47] only in a number of minus signs. From this, we conclude that the underlying curves of $\widetilde{\mathrm{BN}}(T;\mathbf{k})$ behave under twisting as expected. Only the local systems may change. Since non-compact components carry no interesting local systems – all minus signs can be "pushed off the ends", so to speak – the claim follows. $\square$

Remark 5.8. A more careful analysis of the signs in the sketch proof above shows that the local systems of curves corresponding to reduced type D structures whose differentials only contain linear combinations of D, S, and $S^2 \in \mathcal{B}$ do not change under adding twists. However, there do exist tangles T, for which the local systems on the associated multicurves seem to change under twisting, suggesting that the definition of local systems is unnatural. As will be shown in forthcoming work, in which Theorem 5.6 is generalised in full, this is because of the unfortunate choice of the basis

$$\{\iota_\bullet, \iota_\circ, H^n \cdot S_\bullet, H^n \cdot S_\circ, H^n \cdot S_\circ S_\bullet, H^n \cdot S_\bullet S_\circ, H^n \cdot D_\bullet, H^n \cdot D_\circ \mid n \geq 0\}$$

of the algebra $\mathcal{B}$ used in [16] in the definition of local systems. Replacing H by $(-H)$ in this basis makes local systems become natural under twisting.

We note a few more useful facts about the invariant $\widetilde{\mathrm{BN}}_a(T;\mathbf{k})$.

Theorem 5.9. *For any oriented Conway tangle T, the non-compact curve $\widetilde{\mathrm{BN}}_a(T;\mathbf{k})$ connects the two non-special tangle ends that are connected via the tangle T.* □

Theorem 5.10. *For all oriented Conway tangles T, $\widetilde{\mathrm{BN}}_a(\text{⌽}(T);\mathbf{k}) = \widetilde{\mathrm{BN}}_a(T;\mathbf{k})$.* □

Theorem 5.9 is a special case of [16, Proposition 6.15]. Theorem 5.10 is the key observation used in the proof of invariance of the Rasmussen invariant under Conway mutation [16, Theorem 9.17].

Corollary 5.11. *For any oriented knot K, $\widetilde{\mathrm{BN}}_a(T_{K^r};\mathbf{k}) = \widetilde{\mathrm{BN}}_a(T_K;\mathbf{k})$.*

Proof. This follows from Proposition 3.11 and Theorem 5.10. □

Definition 5.12. Given a non-compact curve γ, let $-\gamma$ denote its mirror, i.e. the curve obtained by applying the involution that multiplies the z-coordinate by (-1) and reverses all gradings. Similarly, given a chain complex X over $\mathcal{B}$, let $-X$ be the complex obtained by reversing the directions of all differentials and reversing all gradings.

The following is [16, Propositions 4.26 and 7.1].

Proposition 5.13. *For any oriented Conway tangle T,*

$$\text{Д}(-T;\mathbf{k}) = -\text{Д}(T;\mathbf{k}) \quad \textit{and} \quad \widetilde{\mathrm{BN}}(-T;\mathbf{k}) = -\widetilde{\mathrm{BN}}(T;\mathbf{k}).$$ □

5.3. The pairing theorem for multicurves. The following is a specialisation of [16, Proposition 4.31 and Theorem 7.2] to knots.

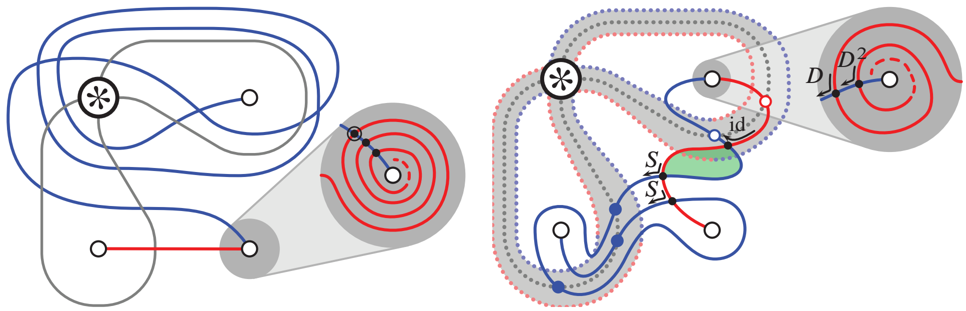
Theorem 5.14 (Pairing Theorem). *Let $K = T \cup T'$ be an oriented knot, where T and T' are two oriented Conway tangles. Then $\widetilde{\mathrm{BN}}(K;\mathbf{k})$ is bigraded isomorphic to both*

$$h^0q^1\,\mathrm{H}_*\big(\mathrm{Mor}(-\text{Д}(T;\mathbf{k}), \text{Д}(T';\mathbf{k}))\big) \quad \textit{and} \quad h^0q^1\,\mathrm{HF}\big(-\widetilde{\mathrm{BN}}(T;\mathbf{k}), \widetilde{\mathrm{BN}}(T';\mathbf{k})\big), \tag{5.2}$$

where HF *denotes the wrapped Lagrangian Floer homology.* □

We now explain how the two expressions in (5.2) are defined.

The morphism space between two chain complexes C and C' over $\mathcal{B}$ is a chain complex over $\mathbf{k}$. Its underlying vector space is generated by homomorphisms from C to C'. These do not necessarily preserve the bigrading, nor do they necessarily commute with the differentials. The differential ∂ on $\mathrm{Mor}(C, C')$ is defined by pre- and post-composition with the differentials of C and C' in the usual way [16, Section 2].



(a) (b)

Figure 13. (a) The pairing of the curves from Figure 12 (a); the first intersection point in the winding region around the lower right puncture is highlighted; (b) two curves in pairing position; see paragraph before Theorem 5.18. The labelled arrows at the intersection points indicate the algebra elements used in the definition of the quantum grading of the intersections points.

To define wrapped Lagrangian Floer homology, let γ and γ' be two immersed curves on $S^2_{4,*}$. For simplicity, let us assume they are both non-compact. Let us further assume that γ and γ' connect different pairs of non-special tangle ends. With notation as in Theorem 5.14, this assumption holds for $\gamma = \widetilde{\mathrm{BN}}_a(T)$ and $\gamma' = \widetilde{\mathrm{BN}}_a(T')$ by Theorem 5.9, since K is a knot. So the endpoints of γ and γ' only meet in one tangle end; let us denote this tangle end by p. By removing bigons between γ and γ', we can make sure that they intersect each other in a minimal number of points, like the two arcs in Figure 12 (a). We call this the *minimal position* for the two curves. Now modify the curve γ in a small neighbourhood of the tangle end p, which we call the *winding region of* p, such that it wraps around p infinitely many times in counterclockwise direction. The vector space $\mathrm{HF}(\gamma, \gamma')$ is now freely generated by the intersection points between γ and γ'.

Example 5.15. Figure 13 (a) shows the Lagrangian Floer homology $\mathrm{HF}(\gamma, \gamma')$ of the curves $\gamma = \widetilde{\mathrm{BN}}(Q_0)$ and $\gamma' = \widetilde{\mathrm{BN}}(T_a)$. Note that $T_a(0) = U$, so $\mathrm{HF}(\gamma, \gamma') \cong \widetilde{\mathrm{BN}}(U)$.

The grading on $\mathrm{HF}(\gamma, \gamma')$ is defined as follows [16, Section 7.2]. Let $\bullet$ be an intersection point between γ and γ'. Consider a path on $\gamma \cup \gamma'$ that starts at an intersection point a of γ with the parametrisation of $S^2_{4,*}$, turns right at $\bullet$ and ends on an intersection point b of γ' with the parametrisation. We can choose a and b such that the path only meets the parametrisation at the endpoints a and b. This path corresponds to some algebra element $s \in \mathcal{B}$ as in the translation between curves and their associated complexes. We then define

$$q(\bullet) := q(b) - q(a) + q(s).$$

One can show that this grading is independent of any choices.

Definition 5.16. For γ, γ', and p as above, let $s(\gamma, \gamma')$ denote the quantum grading of the first generator of $\mathrm{HF}(\gamma, \gamma')$ in the winding region around p.

Recall from Section 3.3 that the Bar-Natan homology $\widetilde{\mathrm{BN}}(K; \mathbf{k})$ of a knot K is not just a bigraded vector space, but in fact a module over the ring $\mathbf{k}[H]$. If we identify the variable H

with the central algebra element

$$H := D + S^2 = D_\bullet + D_\circ + S_\circ S_\bullet + S_\bullet S_\circ \in \mathcal{B},$$

any morphism space $\mathrm{Mor}(C, C')$ can be equipped with an action by $\mathbf{k}[H]$ in the obvious way and this action respects the differential. The first isomorphism from Theorem 5.14 respects these two actions by $\mathbf{k}[H]$.

The action of $\mathbf{k}[H]$ on the Lagrangian Floer homology $\mathrm{HF}(\gamma, \gamma')$ of two multicurves γ, γ' is defined indirectly, namely by transferring the action of H on the homology $\mathrm{H}_*(\mathrm{Mor}(C, C'))$ of the morphism space between the chain complexes C and C' associated with γ and γ'. This works as follows (see [16, Section 5.4]).

Definition 5.17. Let C and C' be two chain complexes over the algebra $\mathcal{B}$. If C and C' are fully cancelled, i.e. if their differentials contain no identity component, we can write the morphism space $\mathrm{Mor}(C, C')$ as a mapping cone

$$(\mathrm{Mor}(C, C'), \partial) \cong [(\mathrm{Mor}^\times(C, C'), 0) \xrightarrow{\beta} (\mathrm{Mor}^+(C, C'), \partial^+)],$$

where $\mathrm{Mor}^\times(C, C')$ consists of the morphisms that only contain idempotent components, $\mathrm{Mor}^+(C, C')$ consists of the morphisms that do not contain any idempotent components, and β and ∂^+ are the restrictions of ∂. The map β induces a map

$$[\mathrm{Mor}^\times(C, C') \xrightarrow{\beta_*} \mathrm{H}_*(\mathrm{Mor}^+(C, C'), \partial^+)].$$

If we understand this map and its domain and codomain well enough, this allows us to compute the homology of the morphism space,

$$\mathrm{H}_*(\mathrm{Mor}(C, C'), \partial) \cong \mathrm{H}_*[\mathrm{Mor}^\times(C, C') \xrightarrow{\beta_*} \mathrm{H}_*(\mathrm{Mor}^+(C, C'), \partial^+)].$$

Let γ and γ' be the two multicurves corresponding to the type D structures C and C'. If one puts these multicurves into a certain position, called the *pairing position* in [16, Section 5.4], we can define the Lagrangian Floer homology of $\mathrm{HF}(\gamma, \gamma')$ combinatorially.

The pairing position of two multicurves γ and γ' is generally different from the minimal position of γ and γ'. It is defined as follows. Consider a small tubular neighbourhood of each arc of the parametrisation of $S^2_{4,*}$. Each neighbourhood has two boundary components, which we may regard as push-offs of the corresponding arc. Divide each of boundary component into a left and a right half when viewed from the interior of the neighbourhood. We now homotope the two multicurves such that γ avoids the left halves of these boundary components and γ' avoids the right halves and such that the curves intersect minimally under these restrictions. (Explicit models for the curves are described in [16, Section 5.4].) Finally, we modify γ in the winding region as before. This is illustrated in Figure 13 (b).

The intersection points of two multicurves in pairing position can be partitioned into two: those intersection points that lie in the neighbourhoods of the arcs are called *upper intersection points* and they freely generate a vector space $\mathrm{CF}^\times(C, C')$; the remaining intersection points are called *lower intersection points* and they freely generate the vector space $\mathrm{CF}^+(C, C')$. By counting bigons connecting upper intersection points to lower ones, such as the bigon shaded green in Figure 13 (b), we can define a map

$$d\colon \mathrm{CF}^\times(C, C') \to \mathrm{CF}^+(C, C').$$

We denote its mapping cone by $\mathrm{CF}(C, C')$ and its homology is $\mathrm{HF}(\gamma, \gamma')$.

The reader may have already noticed that each upper intersection point $\bullet$ corresponds to a pair of intersection points of γ and γ' with the arc in whose neighbourhood the point $\bullet$ lies. This identification of $\mathrm{CF}^\times(C, C')$ and $\mathrm{Mor}^\times(C, C')$ can be extended to the entirety of the two constructions [16, Theorem 5.22].

Theorem 5.18. *Let γ and γ' be two multicurves in pairing position and C and C' their associated type D structures. Then there exist bigraded isomorphisms*

$$\mathrm{Mor}^\times(C, C') \cong \mathrm{CF}^\times(\gamma, \gamma') \quad \textit{and} \quad \mathrm{Mor}^+(C, C') \cong \mathrm{CF}^+(\gamma, \gamma')$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Mor}^\times(C, C') \arrow[r, "\beta_*"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{H}_*(\mathrm{Mor}^+(C, C'), \partial^+) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{CF}^\times(\gamma, \gamma') \arrow[r, "d"] & \mathrm{CF}^+(\gamma, \gamma').
\end{tikzcd}$$

In particular, $\mathrm{HF}(\gamma, \gamma') \cong \mathrm{H}_*(\mathrm{Mor}(C, C'), \partial)$. □

The morphism space $\mathrm{Mor}(C, C')$ between two type D structures C and C' over $\mathcal{B}$ carries a natural action H which is defined by multiplying a morphism by the central element $H \in \mathcal{B}$. Of course, this action commutes with the differential, so we obtain an induced (but not necessarily canonical) action on the mapping cone

$$[\mathrm{Mor}^\times(C, C') \xrightarrow{\beta_*} \mathrm{H}_*(\mathrm{Mor}^+(C, C'), \partial^+)]$$

and both induce the same action on $\mathrm{H}_*(\mathrm{Mor}(C, C'), \partial)$. We can now use the isomorphisms from Theorem 5.18 to transfer this action to $\mathrm{CF}(\gamma, \gamma')$ and ultimately to $\mathrm{HF}(\gamma, \gamma')$.

We will now assume that γ and γ' are two non-compact curves. For computing $\mathrm{HF}(\gamma, \gamma')$, minimal positions of γ and γ' are generally more convenient than pairing positions. Theorem 5.25 from [16], or rather its proof (see [50, Theorem 4.45]), tells us how to pass from one to the other. Namely, we can identify each intersection point x in a minimal position with either a lower intersection point $x_0 \in \mathrm{CF}(\gamma, \gamma')$ or a formal sum $\sum \varepsilon_i x_i$, where $\varepsilon_i \in \{\pm 1\}$ and x_i are upper generators that are connected via a chain of bigons.

In the first case, the action can be easily understood (see [16, Remark 5.23]), and such an intersection point has infinite order exactly if it lies in the winding region of γ and γ'.

The second case requires a bit more care: under the isomorphism from Theorem 5.18, $\sum \varepsilon_i x_i$ corresponds to some element $\sum \varepsilon_i f_i$, where $f_i \in \mathrm{Mor}^\times(C, C')$ is a certain morphism with a single idempotent component. By construction, $\beta_*(\sum \varepsilon_i f_i) = 0$, i.e. there exists some $f_+ \in \mathrm{Mor}^+(C, C')$ such that $\partial^+(f_+) = \beta(\sum \varepsilon_i f_i)$. So the intersection point $x \in \mathrm{HF}(\gamma, \gamma')$ corresponds to the element

$$\left[\sum \varepsilon_i f_i - f_+\right] \in \mathrm{H}_*(\mathrm{Mor}(C, C')).$$

Thus, by definition, $H^n.x \in \mathrm{HF}(\gamma, \gamma')$ is the element corresponding to $[\sum \varepsilon_i H^n.f_i - H^n.f_+]$ which we may regard as an element in $\mathrm{H}_*(\mathrm{Mor}^+(C, C'))$. If n is large enough (namely strictly greater than the length of any differential in C and C'), this element is non-zero if and only if x lies in the winding region of γ and γ', and in this case, it has infinite order.

This argument shows the following lemma.

Lemma 5.19. *Let γ and γ' be two non-compact curves on $S^2_{4,*}$ connecting different pairs of punctures. Then*

$$s(\gamma,\gamma') = \max\{q(x) \mid \textit{homogeneous } x \in \mathrm{HF}(\gamma,\gamma') : H^n.x \neq 0 \textit{ for all } n > 0\}. \qquad \square$$

Proposition 5.20. *With notation as in Theorem* 5.18*,*

$$s_c(K) = s(-\widetilde{\mathrm{BN}}_a(T), \widetilde{\mathrm{BN}}_a(T')) + 1.$$

Proof. Recall from Definition/Proposition 3.17 that $s_c(K)$ is defined as the quantum grading of the generator of the free summand $\mathbb{F}_c[H]$ in any decomposition of $\widetilde{\mathrm{BN}}(K;\mathbb{F}_c)$ as in (3.2) for any knot K. Equivalently,

$$s_c(K) = \max\{q(x) \mid \text{homogeneous } x \in \widetilde{\mathrm{BN}}(K;\mathbb{F}_c) : H^n.x \neq 0 \text{ for all } n > 0\}.$$

By the Pairing Theorem, $\widetilde{\mathrm{BN}}(K;\mathbb{F}_c)$ is bigraded isomorphic to the Lagrangian Floer homology of $-\widetilde{\mathrm{BN}}(T)$ and $\widetilde{\mathrm{BN}}(T')$, shifted by 1 in quantum grading. Furthermore, the wrapped Lagrangian Floer homology of two multicurves is equal to the direct sum of the wrapped Lagrangian Floer homologies between their individual components. This identity naturally respects the $\mathbf{k}[H]$-module structures. The wrapped Lagrangian Floer homology between two curves, of which at least one is compact, is finite-dimensional as a vector space over $\mathbf{k}$, and hence it is H-torsion. So we may replace $-\widetilde{\mathrm{BN}}(T)$ and $\widetilde{\mathrm{BN}}(T')$ by $-\widetilde{\mathrm{BN}}_a(T)$ and $\widetilde{\mathrm{BN}}_a(T')$, respectively, without changing the free part of their wrapped Lagrangian Floer homology. Now apply Lemma 5.19. $\square$

6. Rasmussen invariants and Conway spheres

This section forms the technical heart of this paper and contains the proofs of the main results.

Definition 6.1. Let p be a non-special puncture of $S^2_{4,*}$. We define a cyclic order on the set C_p of non-compact curves on $S^2_{4,*}$ that have exactly one end on p. We do this as follows: given three curves $\gamma_1, \gamma_2, \gamma_3 \in C_p$ that are pairwise distinct as ungraded curves, we may assume without loss of generality that each pair of curves intersects minimally. (This can always be achieved by homotopies that remove innermost bigons.) Then $[\gamma_1, \gamma_2, \gamma_3]$ is cyclically ordered with respect to p, or *p-ordered* for short, if it induces the counterclockwise cyclic order on the intersection points of the curves with a sufficiently small circle around p.

Given four non-compact curves $\gamma_1, \gamma'_1, \gamma_2, \gamma'_2 \in C_p$ with $\gamma_i \neq \gamma'_j$ for all $i, j \in \{1,2\}$ (as ungraded curves), we define

$$i_p(\gamma_1,\gamma'_1,\gamma_2,\gamma'_2) := \begin{cases} 1 & \text{if } \gamma_1 \neq \gamma_2 \text{ and } \gamma'_1 \neq \gamma'_2 \text{ (as ungraded curves)} \\ & \text{and } [\gamma_1,\gamma'_1,\gamma_2,\gamma'_2] \text{ is } p\text{-ordered,} \\ 0 & \text{otherwise.} \end{cases}$$

For the next result, recall the definition of an expression $s(\gamma,\gamma')$ from Definition 5.16.

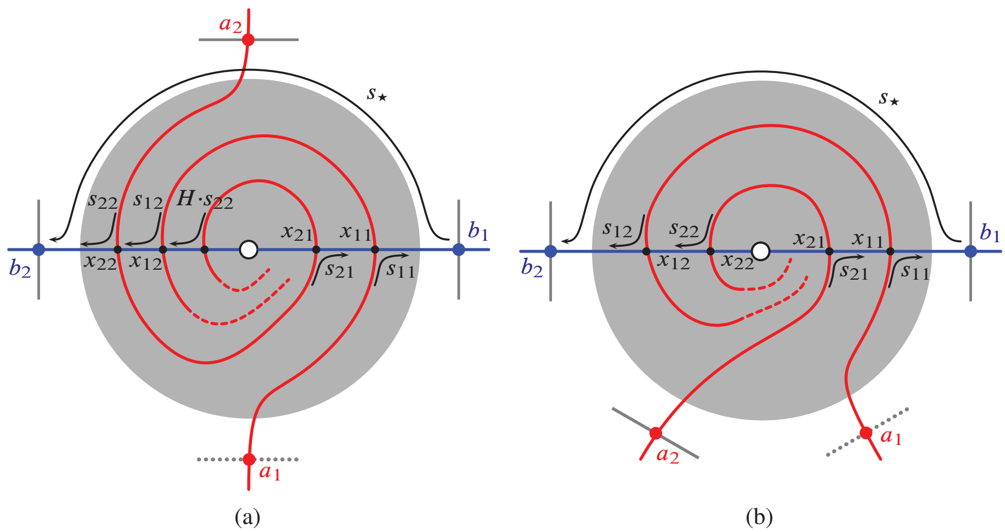


Figure 14. An illustration of the curves in the winding region around the puncture p in the two cases (a) and (b) in the proof of Lemma 6.2.

Lemma 6.2 (Four Curves Lemma). *Let p, p', and p'' be the three non-special punctures of $S^2_{4,*}$. Suppose γ_1 and γ_2 are two curves with ends on the punctures p and p', and γ'_1 and γ'_2 two curves with ends on the punctures p and p''. Then*

$$\begin{aligned} s(\gamma_1,\gamma'_1)+s(\gamma_2,\gamma'_2)-2\cdot i_p(\gamma_1,\gamma'_1,\gamma_2,\gamma'_2) \\ = s(\gamma_1,\gamma'_2)+s(\gamma_2,\gamma'_1)-2\cdot i_p(\gamma_1,\gamma'_2,\gamma_2,\gamma'_1). \end{aligned} \tag{6.1}$$

Note that the condition on the punctures at which the curves end ensures that the quantities $i_p(\gamma_1,\gamma'_1,\gamma_2,\gamma'_2)$ and $i_p(\gamma_1,\gamma'_2,\gamma_2,\gamma'_1)$ are well-defined. Note also that at most one of them is non-zero.

Proof of Lemma 6.2. If $\gamma_1=\gamma_2$ or $\gamma'_1=\gamma'_2$ as ungraded curves, the statement follows from the gluing theorem (Theorem 5.14) combined with the fact that, for any two curves γ and γ',

$$\mathrm{HF}(q^x\gamma,q^y\gamma')=q^{y-x}\,\mathrm{HF}(\gamma,\gamma'),$$

as for morphism spaces [16, Definition 2.2]. Since the punctures p' and p'' are distinct, this leaves the case that all four curves are distinct as ungraded curves. We introduce some notation. For $i,j\in\{1,2\}$, let x_{ij} be the first generator in the winding region around p for the pair (γ_i,γ'_j). Then, by definition of the quantum grading,

$$s(\gamma_i,\gamma'_j)=q(x_{ij})=q(b_j)-q(a_i)+q(s_{ij}),$$

where b_j is the first generator near the end p of γ'_j, a_i is the first generator near the end p of γ_i, and $s_{ij}\in\mathcal{B}$ is the algebra element that labels the morphism corresponding to x_{ij}. Since equation (6.1) is symmetric in γ_1 and γ_2 as well as γ'_1 and γ'_2, it suffices to consider the following two subcases.

Case (a): $[\gamma_1,\gamma'_1,\gamma_2,\gamma'_2]$ *is p-ordered.* In this case, equation (6.1) is equivalent to

$$q(s_{11})+q(s_{22})-2=q(s_{12})+q(s_{21}).$$

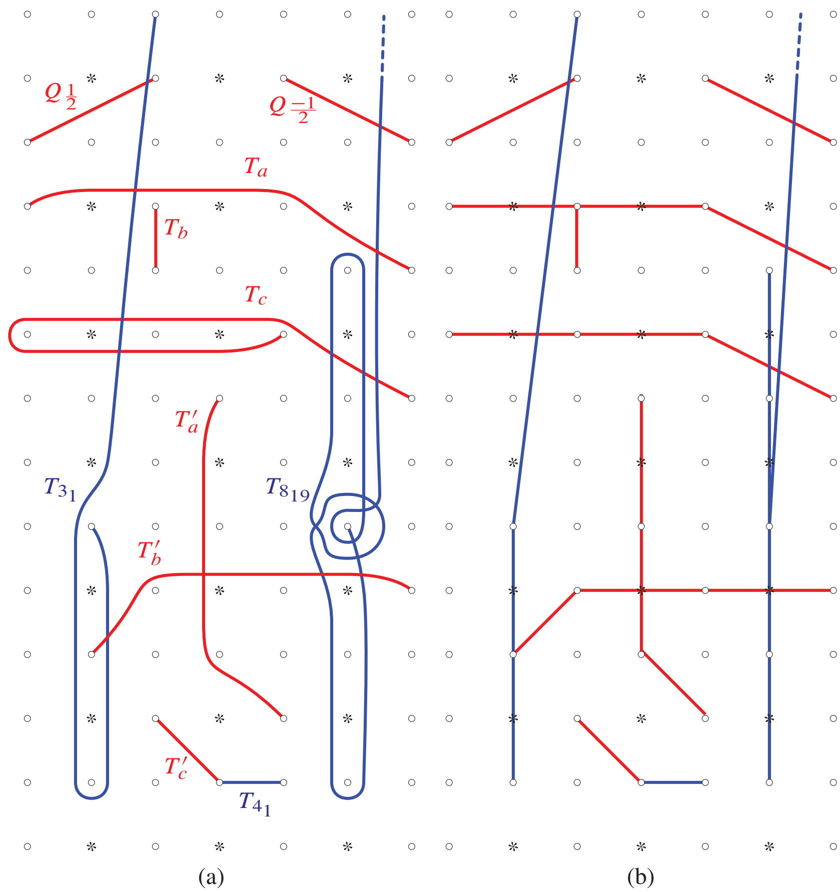


Figure 15. (a) The lifts of the curves $\widetilde{\mathrm{BN}}_a(T)$ for some companion tangles (blue) and the pattern tangles (red) from Figures 1 and 3, and (b) their singular peg-board representatives. $3_1 = T_{2,3}$ and $8_{19} = T_{3,4}$ denote the positive torus knots. The slope of the finite slope segment of the curve in (b) for 8_{19} is 16. The invariants of T'_d and T'_e agree with the mirror of the invariant of T_{3_1} up to single vertical half-twists.

Let $s_\star \in \mathcal{B}$ correspond to the path of minimal length from b_1 to b_2. Then

$$q(s_{12}) - q(s_{11}) = q(s_\star) = q(H \cdot s_{22}) - q(s_{21}) = q(s_{22}) - 2 - q(s_{21})$$

as Figure 14 (a) illustrates. This is equivalent to the desired identity.

Case (b): $[\gamma_1, \gamma'_1, \gamma'_2, \gamma_2]$ *is p-ordered.* In this case, equation (6.1) is equivalent to

$$q(s_{11}) + q(s_{22}) = q(s_{12}) + q(s_{21}).$$

This can be seen in the same way as the identity in case (a) by comparing the algebra elements s_{ij} to the algebra element $s_\star$ corresponding to the path of minimal length from b_1 to b_2,

$$q(s_{12}) - q(s_{11}) = q(s_\star) = q(s_{22}) - q(s_{21}).$$

This case is illustrated in Figure 14 (b). □

Often, it is useful to lift immersed curves to a covering space of $S^2_{4,*}$, namely the plane $\mathbb{R}^2$ minus the integer lattice $\mathbb{Z}^2$. We may regard $\mathbb{R}^2$ as the universal cover of the torus, and the torus as the two-fold branched cover of the sphere branched at four marked points; then the integer lattice $\mathbb{Z}^2$ is the preimage of the branch set. We choose the covering map such that the left arc of the parametrisation of $S^2_{4,*}$ (the one corresponding to $\bullet$ in Figure 12 (b)) is lifted to straight vertical lines and the other arc ($\circ$) is lifted to straight horizontal lines. This covering space along with the lifts of $\widetilde{\mathrm{BN}}_a(T)$ of a selection of tangles T is illustrated in Figure 15 (a).

Recall the following "normal form" for curves in $S^2_{4,*}$ from [16, Section 2.3]; see also [8, Section 7.1].

Definition 6.3. Consider the standard Riemannian metric on $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$, which induces a Riemannian metric on $S^2_{4,*}$. Fix some ε with $0 < \varepsilon < \frac{1}{2}$. An *$\varepsilon$-peg-board representative* of a non-compact curve γ in $S^2_{4,*}$ is a representative of the homotopy class of γ which has minimal length among all representatives that (away from the two ends) have distance ε to all four punctures in $S^2_{4,*}$.

The intuition behind this definition is to think of the four punctures of $S^2_{4,*}$ as pegs of radii ε and then to imagine pulling the curve γ "tight", like a rubber band. This is illustrated in Figure 15 (b).

Definition 6.4. Given $\gamma \in C_p$, consider the lifts of ε-peg-board representatives of γ, one for each ε with $0 < \varepsilon < \frac{1}{2}$, all with respect to the same lift of the basepoint p. We define a *singular peg-board representative of* γ as the piecewise linear curve that is the limit of these lifts as $\varepsilon \to 0$. Since singular peg-board representatives of γ are unique up to deck transformations, this allows us to define the *slope* $\sigma_p(\gamma)$ *of* γ at p as the slope of the singular peg-board representative of γ near the end p. The cyclic order on C_p (Definition 6.1) induces a strict total order on the curves of the same slope at p; we denote this order by $<_p$.

Definition 6.5. For any slope $\frac{p}{q}$, we write $\mathbf{a}_{\frac{p}{q}}$ for the curve $\widetilde{\mathrm{BN}}(Q_{\frac{p}{q}}; \mathbf{k})$.

For $\varepsilon > 0$ sufficiently small, the lift of any ε-peg-board representative of $\mathbf{a}_{\frac{p}{q}}$ is a straight line segment of slope $\frac{p}{q}$ connecting two non-special punctures. In particular, the curve $\mathbf{a}_{\frac{p}{q}}$ is independent of the field $\mathbf{k}$. For $\frac{p}{q} \in \{0, \infty\}$, the absolute bigrading is well-defined. Otherwise, whenever necessary, we will specify the absolute bigrading on $\mathbf{a}_{\frac{p}{q}}$ by specifying the orientation of $Q_{\frac{p}{q}}$.

We now relate the invariants ϑ_c from Definitions 2.2 and 2.15 to the slopes of the multicurve invariants $\widetilde{\mathrm{BN}}_a$.

Proposition 6.6. *Let P be a pattern of wrapping number 2 and T_P a pattern tangle for P (see Definition 3.8). Let σ be the slope of $\gamma := \widetilde{\mathrm{BN}}_a(T_P; \mathbb{F}_c)$ near*

$$\begin{cases} \textit{the lower right tangle end} & \textit{if } P \textit{ has winding number } 0 \ (\textit{i.e. } \mathrm{x}(T_P) =)(\,), \\ \textit{the lower left tangle end} & \textit{if } P \textit{ has winding number } \pm 2 \ (\textit{i.e. } \mathrm{x}(T_P) = \times). \end{cases}$$

Then $\vartheta_c(P) = \infty$ if and only if $\sigma = \infty$. Moreover,

$$\vartheta_c(P) = \begin{cases} \lceil \frac{\sigma}{2} \rceil & \textit{if } \sigma \in \mathbb{Q} \smallsetminus 2\mathbb{Z}, \\ \frac{\sigma}{2} & \textit{if } \sigma \in 2\mathbb{Z} \textit{ and } \gamma <_p \mathbf{a}_\sigma, \\ \frac{\sigma}{2} + 1 & \textit{if } \sigma \in 2\mathbb{Z} \textit{ and } \gamma >_p \mathbf{a}_\sigma. \end{cases} \tag{6.2}$$

Proof. Let us first consider the winding number 0 case. For $n, m \in \mathbb{Z}$, let

$$\mathbf{a}_{2n} = \widetilde{\mathrm{BN}}(Q_{2n}) \quad \text{and} \quad \mathbf{a}_{2m} = \widetilde{\mathrm{BN}}(Q_{2m}),$$

where Q_{2n} and Q_{2m} are oriented compatibly with the ∞-rational filling. Let p be the lower right tangle end. Then, by the Four Curves Lemma,

$$\begin{aligned} s(\mathbf{a}_{2n}, \gamma) + s(\mathbf{a}_{2m}, \mathbf{a}_\infty) - 2 \cdot i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_\infty) \\ = s(\mathbf{a}_{2n}, \mathbf{a}_\infty) + s(\mathbf{a}_{2m}, \gamma) - 2 \cdot i_p(\mathbf{a}_{2n}, \mathbf{a}_\infty, \mathbf{a}_{2m}, \gamma). \end{aligned}$$

By Proposition 5.20, $s_c(\mathbf{a}_{2n}, \mathbf{a}_\infty) = s_c(U) - 1 = s_c(\mathbf{a}_{2m}, \mathbf{a}_\infty)$. Similarly,

$$s_c(\mathbf{a}_{2n}, \gamma) = s_c(T_P(2n)) - 1 \quad \text{and} \quad s_c(\mathbf{a}_{2m}, \gamma) = s_c(T_P(2m)) - 1.$$

Recall from Lemma 3.12 (b) that $T_P(2n) = P_n(U)$ and $T_P(2m) = P_m(U)$. Thus,

$$s_c(P_n(U)) = s_c(P_m(U)) + 2 \cdot i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_\infty) - 2 \cdot i_p(\mathbf{a}_{2n}, \mathbf{a}_\infty, \mathbf{a}_{2m}, \gamma).$$

If $\sigma = \infty$, then clearly $i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_\infty) = i_p(\mathbf{a}_{2n}, \mathbf{a}_\infty, \mathbf{a}_{2m}, \gamma) = 0$, and so $\vartheta_c(P) = \infty$. If $\sigma \neq \infty$, let m be equal to the right-hand side of (6.2). Then $i_p(\mathbf{a}_{2n}, \mathbf{a}_\infty, \mathbf{a}_{2m}, \gamma) = 0$ for all n. Moreover, $i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_\infty) = 1$ if and only if $n < m$, so

$$s_c(P_n(U)) = \begin{cases} s_c(P_m(U)) & \text{if } n \geq m, \\ s_c(P_m(U)) + 2 & \text{if } n < m, \end{cases}$$

and hence $\vartheta_c(P) = m$.

Let us now consider the winding number ± 2 case. As before, let $n, m \in \mathbb{Z}$ and

$$\gamma = \widetilde{\mathrm{BN}}_a(T_P; \mathbb{F}_c), \quad \mathbf{a}_{2n} = \widetilde{\mathrm{BN}}(Q_{2n}), \quad \text{and} \quad \mathbf{a}_{2m} = \widetilde{\mathrm{BN}}(Q_{2m}),$$

but where now Q_{2n} and Q_{2m} are oriented compatibly with the $+1$-rational filling. Furthermore, let $N \gg 0$ be some large odd integer and $\mathbf{a}_N = \widetilde{\mathrm{BN}}(Q_N)$, where Q_N is oriented compatibly with the 0-rational filling. Let p be the lower left tangle end. Then, by the Four Curves Lemma,

$$\begin{aligned} s(\mathbf{a}_{2n}, \gamma) + s(\mathbf{a}_{2m}, \mathbf{a}_N) - 2 \cdot i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_N) \\ = s(\mathbf{a}_{2n}, \mathbf{a}_N) + s(\mathbf{a}_{2m}, \gamma) - 2 \cdot i_p(\mathbf{a}_{2n}, \mathbf{a}_N, \mathbf{a}_{2m}, \gamma). \end{aligned}$$

By Proposition 5.20, $s_c(\mathbf{a}_{2n}, \mathbf{a}_N) = s_c(Q_N(2n)) - 1 = s_c(T_{2,2n-N}) - 1$ and, similarly,

$$s_c(\mathbf{a}_{2m}, \mathbf{a}_N) = s_c(T_{2,2m-N}) - 1.$$

Since $N \gg 0$, we may assume that $2n - N$ and $2m - N$ are both negative. Therefore,

$$s_c(T_{2,2n-N}) - s_c(T_{2,2m-N}) = 2(n - m).$$

Also, $s_c(\mathbf{a}_{2n}, \gamma) = s_c(P_n(U)) - 1$ and $s_c(\mathbf{a}_{2m}, \gamma) = s_c(P_m(U)) - 1$. Thus,

$$s_c(P_n(U)) = s_c(P_m(U)) + 2(n - m) + 2 \cdot i_p(\mathbf{a}_{2n}, \gamma, \mathbf{a}_{2m}, \mathbf{a}_N) - 2 \cdot i_p(\mathbf{a}_{2n}, \mathbf{a}_N, \mathbf{a}_{2m}, \gamma).$$

With $\mathbf{a}_N$ playing the role of $\mathbf{a}_\infty$, we may now argue in the same way as in the winding number 0 case. □

Companion tangles T_K have rather restricted multicurve invariants since they are cap-trivial.

Lemma 6.7 (cf. [17, Section 3]). *Let T be a cap-trivial tangle with $\mathrm{x}(T) =$. Then the slope of $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$ near the lower left tangle end is either infinite or an even integer $n \in 2\mathbb{Z}$, in which case $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}(Q_n;\mathbb{F}_c)$ up to some grading shift. The slope of $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$ near the lower right tangle end is always an integer. Moreover, if $c = 2$, this integer is even.*

Proof. By assumption, $T(\infty)$ is the unknot U. So, by the Gluing Theorem,

$$\mathrm{HF}(\mathbf{a}_\infty, \widetilde{\mathrm{BN}}(T;\mathbb{F}_c)) \cong \widetilde{\mathrm{BN}}(U) \cong \mathbb{F}_c[H].$$

Therefore, $\widetilde{\mathrm{BN}}(T;\mathbb{F}_c)$ (and in particular $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$) can be homotoped such that it avoids the vertical arc $\mathbf{a}_\infty$ altogether. Together with the assumption about the connectivity of T and Theorem 5.9, this implies the first two statements. The final statement follows from the fact that the curve $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_2)$ does not wrap around the special puncture [18, Theorem 4.1]. □

6.1. Patterns with winding number 0. The following proposition is the basis for all results in this subsection.

Proposition 6.8. *Let T be a Conway tangle with $\mathrm{x}(T) =$ and let T' be a cap-trivial tangle. Let p be the lower right tangle end and $n \in \mathbb{Z}$. Let $\gamma = \widetilde{\mathrm{BN}}_a(-T;\mathbb{F}_c)$ and $\gamma' = \widetilde{\mathrm{BN}}_a(T';\mathbb{F}_c)$. Then, for all characteristics c,*

$$s_c(T \cup T') = s_c(T(-2n)) + \begin{cases} 2 & \textit{if } [\mathbf{a}_{2n}, \gamma, \gamma', \mathbf{a}_\infty] \textit{ is } p\textit{-ordered}, \\ -2 & \textit{if } [\gamma', \gamma, \mathbf{a}_{2n}, \mathbf{a}_\infty] \textit{ is } p\textit{-ordered}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Let $\mathbf{a}_{2n} = \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c)$, where Q_{2n} is oriented compatibly with the ∞-rational closure. Then, by the Four Curves Lemma,

$$\begin{aligned} s(\gamma,\gamma') + s(\mathbf{a}_\infty, \mathbf{a}_{2n}) - 2\cdot i_p(\gamma,\gamma',\mathbf{a}_\infty,\mathbf{a}_{2n}) \\ = s(\gamma,\mathbf{a}_{2n}) + s(\mathbf{a}_\infty,\gamma') - 2\cdot i_p(\gamma,\mathbf{a}_{2n},\mathbf{a}_\infty,\gamma'). \end{aligned}$$

By Proposition 5.20,

$$s(\mathbf{a}_\infty, \mathbf{a}_{2n}) = s_c(Q_{2n}(\infty)) - 1 = s_c(U) - 1 = -1$$

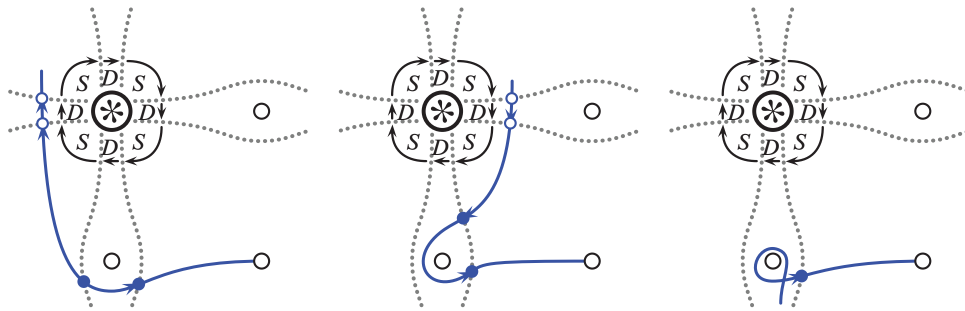
and, similarly, $s(\mathbf{a}_\infty, \gamma') = -1$, since T' is a cap-trivial tangle. Moreover,

$$s(\gamma,\gamma') = s_c(T \cup T') - 1 \quad \text{and} \quad s(\gamma, \mathbf{a}_{2n}) = s_c(T(-2n)) - 1.$$

Therefore,

$$s_c(T \cup T') = s_c(T(-2n)) + 2\cdot i_p(\gamma,\gamma',\mathbf{a}_\infty,\mathbf{a}_{2n}) - 2\cdot i_p(\gamma,\mathbf{a}_{2n},\mathbf{a}_\infty,\gamma').$$

This is equivalent to the desired identity. □



(a) (b) (c)

Figure 16. An illustration for the proof of Lemma 6.11: lifts of curves γ' to $\mathbb{R}^2 \smallsetminus \mathbb{Z}^2$.

Corollary 6.9. *Let T be a Conway tangle with $\mathrm{x}(T) =$ such that $\vartheta_c(P^T) = \infty$ for the associated pattern P^T. Then, for any cap-trivial tangle T',*

$$s_c(T \cup T') = s_c(T(2t)) \quad \text{for all } t \in \mathbb{Z}.$$

Proof. By Proposition 6.6, the slope of $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$ near the lower right tangle end is ∞. The same is true for

$$\widetilde{\mathrm{BN}}_a(-T;\mathbb{F}_c) = -\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c).$$

On the other hand, the slope of $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$ is finite by Lemma 6.7. So the claim follows from Proposition 6.8. □

Corollary 6.10. *Let T be a Conway tangle with $\mathrm{x}(T) =$ and let T' be a cap-trivial tangle. Let σ_2' be the slope of $\widetilde{\mathrm{BN}}_a(T';\mathbb{F}_2)$ near the lower right tangle end. Then*

$$s_2(T \cup T') = s_2(T(-\sigma_2')).$$

Proof. By Lemma 6.7, σ_2' is an even integer. Setting $n = \frac{\sigma_2'}{2}$ in Proposition 6.8, we obtain the desired identity, unless $[\mathbf{a}_{\sigma_2'}, \gamma, \gamma', \mathbf{a}_\infty]$ or $[\gamma', \gamma, \mathbf{a}_{\sigma_2'}, \mathbf{a}_\infty]$ are p-ordered. The following lemma says that this is impossible. □

Lemma 6.11. *Let $\gamma = \widetilde{\mathrm{BN}}_a(-T;\mathbb{F}_2)$ for some tangle T and $\gamma' = \widetilde{\mathrm{BN}}_a(T';\mathbb{F}_2)$ for some cap-trivial tangle T'. Suppose γ and γ' have the same slope $\sigma \in 2\mathbb{Z}$ near the lower right tangle end p. Suppose $\mathbf{a}_\sigma <_p \gamma <_p \gamma'$ or $\gamma' <_p \gamma <_p \mathbf{a}_\sigma$. Then $\mathrm{x}(T) =$.*

Proof. Since, by Theorem 5.6, the multicurve invariant $\widetilde{\mathrm{BN}}$ is natural with respect to twisting, we may assume that $\sigma = 0$ after replacing T by $T + Q_{-\sigma}$. Note that this preserves the connectivity of T. Then $\mathbf{a}_\sigma = \mathbf{a}_0$ is represented by the chain complex consisting of a single object $\bullet$. Up to mirroring, we may also assume $\mathbf{a}_0 <_p \gamma <_p \gamma'$. Then, in particular, $\mathbf{a}_0 <_p \gamma'$, so the complex representing γ' near p has the following form:

$$\bullet \xleftarrow{D_\bullet} \bullet \xrightarrow{S} \cdots \quad \text{or} \quad \bullet \xleftarrow{D_\bullet^j} \bullet \xleftrightarrow{S^i} \cdots$$

for some integers $i \geq 1$ and $j \geq 2$; see Figure 16. Since $\mathbf{a}_0 <_p \gamma <_p \gamma'$, the same is true for the chain complex X representing γ. Taking the mapping cone with $H \cdot \mathrm{id}_X$ produces the following complex X' given by the solid arrows:

$$\begin{tikzcd}
\bullet \arrow[d, "H"] & \bullet \arrow[l, "D_\bullet"'] \arrow[r, "S"] \arrow[d, "H"] & \cdots \arrow[ll, dotted, bend right, "S"'] \\
\bullet & \bullet \arrow[l, "D_\bullet"] \arrow[r, "S"] \arrow[ul, dotted, "\iota_\bullet"] & \cdots \arrow[ll, dotted, bend left, "S"]
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet \arrow[d, "H"] & \bullet \arrow[l, "D_\bullet^j"'] \arrow[r, leftrightarrow, "S^i"] \arrow[d, "H"] & \cdots \\
\bullet & \bullet \arrow[l, "D_\bullet^j"] \arrow[r, leftrightarrow, "S^i"] \arrow[ul, dotted, "D_\bullet^{j-1}"'] & \cdots
\end{tikzcd}$$

Applying the Clean-Up Lemma [16, Lemma 2.17] to the dotted arrows changes the complex as follows:

$$\begin{tikzcd}
\bullet \arrow[d, "H"] & \bullet \arrow[r, "S"] \arrow[d, "H"] & \cdots \\
\bullet & \bullet \arrow[r, "S"] & \cdots
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\bullet \arrow[d, "H"] & \bullet \arrow[r, leftrightarrow, "S^i"] \arrow[d, "H"] & \cdots \\
\bullet & \bullet \arrow[r, leftrightarrow, "S^i"] & \cdots
\end{tikzcd}$$

(The rest of the complex remains unchanged.) In particular, we have split off a component of the form

$$\bullet \xrightarrow{H} \bullet$$

which represents a rational curve $\mathbf{r}_1(0)$ of length 1 and slope 0 (see [18, Section 2]). Since X' is a component of $\widetilde{\mathrm{Kh}}(T)$, so is the component $\mathbf{r}_1(0)$. Rational curves of odd length detect connectivity by [18, Theorem 5.6], so $\mathrm{x}(T) =$. $\square$

We expect the proof of Lemma 6.11 to generalise to arbitrary fields of coefficients. The deep result about the invariant $\widetilde{\mathrm{BN}}(T';\mathbb{F}_2)$ that is contained in the hypothesis of the lemma is the fact that, for $c = 2$, the slope σ' is even. Computations suggest these slopes to be always even; see also [17, Questions 3.4].

The proof of Lemma 6.11 can be easily adapted to show the following general fact about rational tangle replacements in tangle sums.

Proposition 6.12. *Let T be a cap-trivial tangle with connectivity . Then there exists a unique integer $\sigma \in 2\mathbb{Z}$ with the following property: for any oriented tangle R with connectivity ,*

$$\widetilde{\mathrm{BN}}_a(T + R;\mathbb{F}_2) = \widetilde{\mathrm{BN}}_a(Q_\sigma + R;\mathbb{F}_2)$$

as bigraded curves, where R is oriented compatibly with the tangle sum $T + R$ and the orientation of $Q_\sigma + R$ is compatible with the orientation of R. $\square$

As with Corollary 6.10, we expect Proposition 6.12 to hold over arbitrary coefficients.

Remark 6.13. Proposition 6.12 implies that, for $R = Q_{\frac{p}{q}}$ with p odd and q even,

$$\widetilde{\mathrm{BN}}_a(T + R;\mathbb{F}_2) = \widetilde{\mathrm{BN}}_a(Q_{\sigma+\frac{p}{q}};\mathbb{F}_2) = \mathbf{a}_{\sigma+\frac{p}{q}}.$$

This explains the phenomenon that the authors already exploited in [25] for showing that the Rasmussen invariant s_3 is distinct from other s_c: the curves $\widetilde{\mathrm{BN}}_a(T_{8_{19}} + Q_{\frac{-1}{2}};\mathbb{F}_c)$ are equal to $\mathbf{a}_{\sigma_c-\frac{1}{2}}$ and, interestingly, σ_3 is different from σ_c for other coefficients c.

The integer invariant $\vartheta_c(K)$ from Definition 2.2 admits the following geometric interpretation.

Corollary 6.14. *Given a knot $K \subset S^3$, let σ_c be the slope of $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ near the bottom right tangle end. Then $\vartheta_c(K) = \lceil \frac{\sigma_c}{2} \rceil$. In particular, $\vartheta_2(K) = \frac{\sigma_2}{2}$.*

Proof. By definition,

$$W_t^+(K) = T_K\Big(2t + \frac{1}{2}\Big) = T \cup T'$$

with $T = Q_{-2t-\frac{1}{2}}$ and $T' = T_K$. By Lemma 6.7, the slope $\sigma' = \sigma_c$ of $\widetilde{\mathrm{BN}}_a(T')$ is an integer. So, by Proposition 6.8, for any fixed $n = t$, we obtain

$$s_c(W_t^+(K)) = s_c(Q_{-2t-\frac{1}{2}}(-2t)) + \begin{cases} 2 & \text{if } 2t + \frac{1}{2} < \sigma_c, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $Q_{-2t-\frac{1}{2}}(-2t) = Q_{-\frac{1}{2}}(0)$ is the unknot, so the first summand above vanishes. Thus, for $t = \lceil \frac{\sigma_c}{2} \rceil$,

$$s_c(W_t^+(K)) = 0 \quad \text{and} \quad s_c(W_{t-1}^+(K)) = 2.$$

So $t = \lceil \frac{\sigma_c}{2} \rceil$ agrees with $\vartheta_2(K)$. □

We can finally prove Theorem 1.1, which we restate here for clarity.

Theorem 1.1. *For every knot $K \subset S^3$, there exists a unique integer $\vartheta_2(K)$ such that*

$$s_2(P(K)) = s_2(P_{-\vartheta_2(K)}(U))$$

for all patterns P with wrapping number two and winding number zero.

Proof. The condition on the winding number means that we can write $P(K) = T_P \cup T_K$ for some pattern tangle T_P with connectivity $\mathrm{x}(T_P) =$. We now combine Corollaries 6.10 and 6.14. □

6.2. Patterns with winding number ±2. The following two results play the same role for winding number ±2 patterns as Proposition 6.8 does for winding number 0 patterns.

Proposition 6.15. *Let T be a Conway tangle with $\mathrm{x}(T) =$ and let T' be a cap-trivial tangle with $\mathrm{lk}(T') = 0$. Suppose $\widetilde{\mathrm{BN}}_a(T'; \mathbb{F}_c)$ is equal to $\mathbf{a}_{2n} = \widetilde{\mathrm{BN}}(Q_{2n}; \mathbb{F}_c)$ for some integer n up to some grading shift. Then*

$$s_c(T \cup T') = s_c(T(-2n)) + 6n.$$

Proposition 6.16. *Let T be a Conway tangle with $\mathrm{x}(T) =$ and let T' be a cap-trivial tangle. Let $\gamma = \widetilde{\mathrm{BN}}_a(-T; \mathbb{F}_c)$ and $\gamma' = \widetilde{\mathrm{BN}}_a(T'; \mathbb{F}_c)$. Suppose the slope of γ' near the lower left tangle end p is equal to ∞. Then, for all coefficients c and $n \in \mathbb{Z}$,*

$$s_c(T \cup T') = s_c(T(-2n)) + s_c(T'(2n+1)) - \begin{cases} 2 & \text{if } [\gamma, \mathbf{a}_{2n}, \gamma'] \text{ is } p\text{-ordered}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 6.15. Let us equip T' and Q_{2n} with an orientation compatible with the 1-rational closure. The key ingredient in this proof is that we can work out the absolute grading shift between $\gamma' := \widetilde{\mathrm{BN}}_a(T';\mathbb{F}_c)$ and $\mathbf{a}_{2n}$. For this, let $\widetilde{T}'$ be the tangle obtained from T' by reversing the orientation of one of the two strands. By assumption, $\mathrm{lk}(T') = 0$, so $\widetilde{\mathrm{BN}}(T';\mathbb{F}_c) = \widetilde{\mathrm{BN}}(\widetilde{T}';\mathbb{F}_c)$ as bigraded multicurves by [16, Proposition 4.8]. By the same principle, if $\widetilde{Q}_{2n}$ denotes the oriented tangle obtained from Q_{2n} by reversing the orientation of one of the two strands, then

$$\widetilde{\mathrm{BN}}(\widetilde{Q}_{2n};\mathbb{F}_c) = q^{-6\,\mathrm{lk}(Q_{2n})} \cdot \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c) = q^{6n} \cdot \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c).$$

Together with $\widetilde{T}'(\infty) = U$, these considerations imply that

$$\begin{aligned} s(\widetilde{\mathrm{BN}}(\,;\mathbb{F}_c), \gamma') &= s_c(\widetilde{T}'(\infty)) = s_c(U) - 1 = s_c(\widetilde{Q}_{2n}(\infty)) \\ &= s(\widetilde{\mathrm{BN}}(\,;\mathbb{F}_c), \widetilde{\mathrm{BN}}(\widetilde{Q}_{2n};\mathbb{F}_c)) = s(\widetilde{\mathrm{BN}}(\,;\mathbb{F}_c), q^{6n} \cdot \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c)). \end{aligned}$$

Hence $\gamma' = q^{6n} \cdot \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c)$ and

$$\begin{aligned} s_c(T \cup T') &= s(\widetilde{\mathrm{BN}}_a(-T;\mathbb{F}_c), \gamma') + 1 = s(\widetilde{\mathrm{BN}}_a(-T;\mathbb{F}_c), q^{6n} \cdot \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c)) + 1 \\ &= s_c(T \cup Q_{2n}) + 6n = s_c(T(-2n)) + 6n. \end{aligned}$$

□

Proof of Proposition 6.16. Let

$$\mathbf{a}_{2n+1} = \widetilde{\mathrm{BN}}(Q_{2n+1};\mathbb{F}_c) \quad \text{and} \quad \mathbf{a}_{2n} = \widetilde{\mathrm{BN}}(Q_{2n};\mathbb{F}_c),$$

where Q_{2n+1} is oriented compatibly with the ∞-rational filling and Q_{2n} is oriented compatibly with the 1-rational filling. Then, by the Four Curves Lemma,

$$\begin{aligned} s(\gamma, \gamma') &+ s(\mathbf{a}_{2n+1}, \mathbf{a}_{2n}) - 2 \cdot i_p(\gamma, \gamma', \mathbf{a}_{2n+1}, \mathbf{a}_{2n}) \\ &= s(\gamma, \mathbf{a}_{2n}) + s(\mathbf{a}_{2n+1}, \gamma') - 2 \cdot i_p(\gamma, \mathbf{a}_{2n}, \mathbf{a}_{2n+1}, \gamma'). \end{aligned}$$

By Proposition 5.20,

$$\begin{aligned} s(\mathbf{a}_{2n+1}, \mathbf{a}_{2n}) &= s_c(Q_{2n}(2n+1)) - 1 = s_c(U) - 1 = -1, \\ s(\gamma, \gamma') &= s_c(T \cup T') - 1, \\ s(\gamma, \mathbf{a}_{2n}) &= s_c(T \cup Q_{2n}) - 1 = s_c(T(-2n)) - 1, \\ s(\mathbf{a}_{2n+1}, \gamma') &= s_c(Q_{-2n-1} \cup T') - 1 = s_c(T'(2n+1)) - 1. \end{aligned}$$

Moreover, since the slope of γ' near p is ∞, $i_p(\gamma, \gamma', \mathbf{a}_{2n+1}, \mathbf{a}_{2n}) = 0$ for all n. We obtain

$$s_c(T \cup T') = s_c(T(-2n)) + s_c(T'(2n+1)) - 2 \cdot i_p(\gamma, \mathbf{a}_{2n}, \mathbf{a}_{2n+1}, \gamma')$$

which is equivalent to the statement of the proposition. □

Next, we explore the relationship between ϑ_c-rationality (see Definition 2.16) and the multicurve invariants, which will explain our choice of terminology.

Lemma 6.17. *Let K be a knot and σ the slope of $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)$ near the lower left tangle end p. Then $\sigma = \infty$ if and only if $\vartheta_c(P^K) = \infty$. Moreover, if $\sigma < \infty$, then $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}(Q_{2\vartheta_c(K)};\mathbb{F}_c)$ up to some grading shift and $\vartheta_c(P^K) = \vartheta_c(K) = \frac{\sigma}{2}$.*

Proof. Let us choose $T_{P^K} = T_K + Q_{-1}$ as the pattern tangle for the pattern P^K. Then, by Proposition 5.7, the slope of $\widetilde{\mathrm{BN}}_a(T_{P^K}; \mathbb{F}_c)$ near p is equal to $\sigma - 1$ if $\sigma < \infty$ and ∞ if $\sigma = \infty$. So the first statement follows from Proposition 6.6.

Now suppose $\sigma < \infty$. Lemma 6.7 implies that σ is an even integer and $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}(Q_\sigma; \mathbb{F}_c)$ up to some grading shift. So, by Corollary 6.14, $\vartheta_c(K) = \frac{\sigma}{2}$. The identity $\vartheta_c(P^K) = \vartheta_c(K)$ now follows from Proposition 6.6 and our observation from the beginning of the proof that the slope of $\widetilde{\mathrm{BN}}_a(T_{P^K}; \mathbb{F}_c)$ near p is equal to $\sigma - 1 = 2\vartheta_c(K) - 1$. □

Proposition 6.18. *A knot K is ϑ_c-rational if and only if $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}(Q_{\frac{p}{q}}; \mathbb{F}_c)$ for some slope $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1$ up to some grading shift. Moreover, if K is ϑ_c-rational, then $\frac{p}{q} = 2\vartheta_c(K) = 2\vartheta_c'(K)$.*

Proof. By Lemma 3.14, $\vartheta_c'(K) = \vartheta_c(P^{K^r}) = \vartheta_c(P^K)$, so K is ϑ_c-rational if and only if $\vartheta_c(P^K) < \infty$. By the first half of Lemma 6.17, this is equivalent to the slope σ of $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ near the lower left tangle end being finite.

So to prove the first statement of the proposition, it remains to see that $\sigma < \infty$ if and only if $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ is equal to $\widetilde{\mathrm{BN}}(Q_{\frac{p}{q}}; \mathbb{F}_c)$ for some slope $\frac{p}{q} \in \mathbb{Q}\mathrm{P}^1$ up to some grading shift. The if direction is easy to see, since the connectivity of T_K is  and the curve $\widetilde{\mathrm{BN}}(Q_{\frac{p}{q}}; \mathbb{F}_c)$ has constant slope. The converse follows from the second half of Lemma 6.17.

We have just seen that if K is ϑ_c-rational, the slope σ is finite. So, by the second half of Lemma 6.17, $\sigma = 2\vartheta_c(K) = 2\vartheta_c(P^K)$. We now conclude with the observation that $\vartheta_c'(K) = \vartheta_c(P^K)$ from the beginning of the proof. □

The above gives rise to the following simple test to determine if K is ϑ_c-rational without computing the entire curve $\widetilde{\mathrm{BN}}_a(T_K)$.

Corollary 6.19. *A knot K is ϑ_c-rational if and only if*

$$s_c(P^K_{\vartheta_c(K)-1}(U)) = s_c(P^K_{\vartheta_c(K)}(U)).$$

Proof. If K is not ϑ_c-rational, then the identity clearly does not hold. If K is ϑ_c-rational, then $\vartheta_c(K) = \vartheta_c'(K)$ by Proposition 6.18, which implies the desired identity. □

We can now prove Theorem 2.23, which we restate here for convenience.

Theorem 2.23. *Let K be a ϑ_c-rational knot. Then $\vartheta_c(K) = \vartheta_c'(K)$, and for any pattern P with wrapping number* 2 *and winding number* ± 2*,*

$$s_c(P(K)) = s_c(P_{-\vartheta_c(K)}(U)) + 6\vartheta_c(K).$$

Proof. The first statement was already shown in Proposition 6.18. The formula for $s_c(P(K))$ is obtained by applying Proposition 6.15 to $T = T_P$ some pattern tangle for P, $T' = T_K$, and $n = \vartheta_c(K)$ and using the identity

$$P_{-\vartheta_c(K)}(U) = T(-2\vartheta_c(K))$$

provided by Lemma 3.12 (b). □

Corollary 6.20. *Let K be a ϑ_c-rational knot. Then, for all $t \in \mathbb{Z}$,*

$$s_c(C_{2,2t+1}(K)) = 4\vartheta_c(K) + 2t + \begin{cases} 0 & \text{if } \vartheta_c(K) \le t, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. This follows from Theorem 2.23 applied to $P = C_{2,2t+1}$, noting that

$$P_{-\vartheta_c(K)}(U) = C_{2,2t+1-2\vartheta_c(K)}(U) = T_{2,2(t-\vartheta_c(K))+1}$$

by definition and

$$s_c(P_{-\vartheta_c(K)}(U)) = 2(t - \vartheta_c(K)) + 1 + \begin{cases} -1 & \text{if } 2(t-\vartheta_c(K)) + 1 > 0, \\ 1 & \text{if } 2(t-\vartheta_c(K)) + 1 < 0 \end{cases}$$

by Example 3.19. □

Theorem 2.27. *Let K be a knot that is not ϑ_c-rational. Then, for any pattern P with wrapping number 2 and winding number ± 2 with $\vartheta_c(P) \neq \infty$,*

$$s_c(P(K)) = s_c(P(U)) + s_c(C_{2,1}(K)) - \begin{cases} 2 & \text{if } \vartheta_c(P) > 0, \\ 0 & \text{if } \vartheta_c(P) \le 0. \end{cases}$$

Proof. Let $P(K) = T_P \cup T_K$ for some pattern tangle T_P. We apply Proposition 6.16 to $T = T_P$, $T' = T_K$, and $n = 0$. We obtain

$$s_c(P(K)) = s_c(T_P(0)) + s_c(T_K(1)) - \begin{cases} 2 & \text{if } [\gamma, \mathbf{a}_0, \gamma'] \text{ is } p\text{-ordered}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\gamma = \widetilde{\mathrm{BN}}_a(-T_P; \mathbb{F}_c)$ and $\gamma' = \widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ and p is the lower left puncture. Since $T_P(0) = P(U)$ and $T_K(1) = C_{2,1}(K)$, it suffices to show that $[\gamma, \mathbf{a}_0, \gamma']$ is p-ordered if and only if $\vartheta_c(P) > 0$. To see this, let σ be the slope of $\widetilde{\mathrm{BN}}_a(T_P; \mathbb{F}_c) = -\widetilde{\mathrm{BN}}_a(-T_P; \mathbb{F}_c) = -\gamma$ near p. So, by Proposition 6.6, $\vartheta_c(P) > 0$ if and only if $\sigma > 0$ or ($\sigma = 0$ and $-\gamma >_p \mathbf{a}_0$). This is indeed equivalent to $[\gamma, \mathbf{a}_0, \gamma']$ being p-ordered, since the slope of γ' at p is ∞. □

Theorem 6.21. *Let K be a knot that is not ϑ_c-rational. Let P be a pattern with wrapping number 2 and winding number ± 2 such that $\vartheta_c(P) = \infty$. Then*

$$s_c(P(K)) = s_c(P(U)) + s_c(C_{2,1}(K)) - \begin{cases} 2 & \text{if } \gamma_K <_p \gamma_P, \\ 0 & \text{otherwise}, \end{cases}$$

where p is the lower left tangle end, $\gamma_K := \widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$, and $\gamma_P := \widetilde{\mathrm{BN}}_a(-T_P; \mathbb{F}_c)$.

Proof. Similar to Theorem 2.27, this follows from Propositions 6.6 and 6.16. □

Example 6.22. Let K be the positive trefoil knot and the patterns P^+ and P^- defined by the pattern tangles $T_{P^+} = T'_e = T_{-K} + Q_{+1}$ and $T_{P^-} = T'_d = T_{-K} + Q_{-1}$, respectively. Then the knots $P^{\pm}(K) = C_{2,\mp 1}(K \# -K)$ are obtained by plugging $T_{P^{\pm}}$ into the grey disc in Figure 1 (a). The knot $K \# -K$ is concordant to the unknot U, so $P^{\pm}(K)$ is concordant to

$C_{2,\mp 1}(U) = U$, and hence $s_2(P^{\pm}(K)) = 0$. Moreover,

$$P^{\pm}(U) = C_{2,\mp 1}(-K) = -C_{2,\pm 1}(K).$$

Since K is not ϑ_2-rational, $s_2(C_{2,-1}(K)) = s_2(C_{2,1}(K)) - 2$, so we obtain the following identities:

$$s_2(P^{+}(K)) = s_2(P^{+}(U)) + s_2(C_{2,1}(K)),$$
$$s_2(P^{-}(K)) = s_2(P^{-}(U)) + s_2(C_{2,1}(K)) - 2.$$

Theorem 6.21 for $c = 2$ gives an alternative proof of these identities. Indeed, with the same notation as in Theorem 6.21,

$$\gamma_{P^{\pm}} = \widetilde{\mathrm{BN}}_a(-T_{P^{\pm}}; \mathbb{F}_2) = \widetilde{\mathrm{BN}}_a(T_K + Q_{\mp 1}; \mathbb{F}_2),$$

so consequently $\gamma^{+} <_p \gamma_K <_p \gamma^{-}$.

7. Computations, observations, and varying characteristics

The invariant $\vartheta_c(K)$ can be read off directly from the curve $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ using Corollary 6.14. Here is an alternative way of computing $\vartheta_c(K)$.

Proposition 7.1. *The following algorithm computes $\vartheta_c(K)$ for a given knot K and a given characteristic c that is* 0 *or a prime.*

Step 1. *Calculate $s_c(W^{+}(K))$.*

Step 2. (a) *If $s_c(W^{+}(K)) = 0$, then calculate*

$$s_c(W_t^{+}(K)) \quad \textit{for } t = -1, -2, \ldots$$

until $s_c(W_t^{+}(K)) = 2$. Return $\vartheta_c(K) = t + 1$.

(b) *If $s_c(W^{+}(K)) \neq 0$, then calculate*

$$s_c(W_t^{+}(K)) \quad \textit{for } t = 1, 2, \ldots$$

until $s_c(W_t^{+}(K)) = 0$. Return $\vartheta_c(K) = t$.

Proof. That this algorithm always stops follows from Proposition 2.3. That it returns the correct result is immediate from the definition of $\vartheta_c(K)$; see Definition 2.2. □

Note that Proposition 7.1 builds on the well-known computability of s_c. In practice, there exist a number of computer programs that calculate the Rasmussen invariants $s_c(K)$ for a given knot K and characteristic c, such as `khoca` [23], `SKnotJob` [44] and `kht++` [51]. With the aid of these programs, the above algorithms allow us to compute $\vartheta_c(K)$ for sufficiently small knots K. For example, we have computed $\vartheta_c(K)$ for various c for all prime knots up to 8 crossings, roughly half of all prime knots up to 10 crossings, and various individual knots with up to 16 crossings. For these computations, we use the naïve way to construct a diagram D' of $W_t^{+}(K)$ from a diagram D of K, which entails that D' has twice the width of D (in the sense of [32]), and at least four times as many crossings as D. So computer calculations for more complicated knots, such as iterated Whitehead doubles, seem in general unfeasible with current technology. For a complete list of the values of $\vartheta_c(K)$ that we have computed, we refer the reader to an online table [26], which we will continue to update.

We now make a number of observations and conjectures based on our calculations.

7.1. Linear independence. We find that $\vartheta_2(K)$ often differs from $\vartheta_c(K)$ for $c \neq 2$, even for many small knots. For example we have

$$\vartheta_2(T_{2,3}) = 4, \quad \vartheta_c(T_{2,3}) = 3 \tag{7.1}$$

for all $c \in \{0, 3, 5, \dots, 97\}$. Note that one direct consequence is that

$$s_2(W_3^+(T_{2,3})) = 2, \quad s_0(W_3^+(T_{2,3})) = 0.$$

This provides a knot with $s_2 \neq s_0$ (different from the first such knot, which was found by Seed [28, Remark 6.1]), similar to the example of a knot with $\tau \neq s_0$ given in (2.3). Let us consider another computational result:

$$\vartheta_3(T_{3,4}) = 9, \quad \vartheta_c(T_{3,4}) = 8,$$

for all $c \in \{0, 2, 5, \dots, 97\}$. This lead to the first known example $K = W_8^+(T_{3,4})$ of a knot K with $s_3(K) \neq s_0(K)$, and thus to the following theorem.

Theorem 7.2 ([25]). *The concordance homomorphisms s_0, s_2, s_3 are linearly independent.*

This result provided the authors with the initial impetus for the paper at hand. Can Theorem 7.2 be extended to an independence result for all the s_c? As shown by Schütz [45], for every knot K,

$$s_p(K) = s_0(K) \quad \text{holds for all but finitely many primes } p. \tag{7.2}$$

One may conjecture that this is the only restriction on the values of the s_c, i.e. that the homomorphism $(\frac{s_p - s_0}{2})_p : \mathcal{C} \to \bigoplus_p \mathbb{Z}$, where p ranges over all primes, is surjective [28, Question 6.2]; maybe even its restriction to the subgroup of $\mathcal{C}$ spanned by $\{T_{p,p+1} \mid p \text{ prime}\}$ is surjective (compare [45, Conjecture 6.2]). Unfortunately, the computation of $\vartheta_c(T_{5,6})$ seems to be just out of reach of the software and hardware currently available to us. Let us postpone further discussion of torus knots to Section 7.3.

We can show the following independence results that include τ and the invariants ϑ_c.

Proposition 7.3. *The following statements hold.*

(a) *The image of the homomorphism*

$$\left(\tau, \frac{s_0}{2}, \frac{s_2}{2}, \frac{s_3}{2}, \vartheta_2\right)$$

from the smooth concordance group $\mathcal{C}$ to $\mathbb{Z}^5$ contains $\mathbb{Z}^4 \times 4\mathbb{Z}$.

(b) *Under the assumption that ϑ_0 and ϑ_3 are homomorphisms $\mathcal{C} \to \mathbb{Z}$, the homomorphism*

$$\left(\tau, \frac{s_0}{2}, \frac{s_2}{2}, \frac{s_3}{2}, \vartheta_0, \vartheta_2, \vartheta_3\right)$$

from $\mathcal{C}$ to $\mathbb{Z}^7$ contains $\mathbb{Z}^5 \times 4\mathbb{Z} \times \mathbb{Z}$ in its image.

Knot	τ	$\frac{s_0}{2}$	$\frac{s_2}{2}$	$\frac{s_3}{2}$	ϑ_0	ϑ_2	ϑ_3
$T_{2,3} = 3_1$	1	1	1	1	3	4	3
$P(3,3,-2) = T_{3,4} = 8_{19}$	3	3	3	3	8	8	9
$P(-5,3,-2) = 10_{125}$	1	1	1	1	2	4	2
$P(5,5,-4) = 14n24552$	5	5	5	5	15	16	15
$W_2^+(T_{2,3}) = 15n115646$	0	1	1	1			
$W_3^+(T_{2,3}) = 14n22180$	0	0	1	0		unknown	
$W_8^+(T_{3,4})$ (34 crossings)	0	0	0	1			

Table 4. A table of τ, s_c, and ϑ_c for $c \in \{0, 2, 3\}$ for various knots. The values of τ may be computed using Hedden's result that $\vartheta_\tau = 2\tau$ and the computer program [39]; the values of s_c may be computed using any of the programs [23, 44, 51]; and the values of ϑ_c may be computed as described in Corollary 6.14 and Proposition 7.1.

Proof. One checks that the seven row vectors of Table 4 span a subgroup of $\mathbb{Z}^7$ containing $\mathbb{Z}^5 \times 4\mathbb{Z} \times \mathbb{Z}$. This shows part (b), and thus also part (a), of the statement. □

As a direct consequence of (7.2), we have the following.

Corollary 7.4. *For each knot K, $\vartheta_0(K) = \vartheta_c(K)$ for all but finitely many primes c.* □

One may now ask whether the conjectured divisibility of ϑ_2 by 4 (see Conjecture 2.38), (7.2), and Corollary 7.4 are the only restrictions on the values of s_c and ϑ_c.

Question 7.5. Let the infinite vectors $\underline{s}, \underline{\vartheta} \in \mathbb{Z}^\infty$ be given such that, for each of the two vectors, all but finitely many entries equal the first entry. Assume moreover that the second entry of $\underline{\vartheta}$ is divisible by four. Does there exist a knot K such that

$$(s_0(K), s_2(K), s_3(K), s_5(K), \dots) = 2\underline{s} \quad \text{and} \quad (\vartheta_0(K), \vartheta_2(K), \vartheta_3(K), \vartheta_5(K), \dots) = \underline{\vartheta}?$$

Let us note that all values of ϑ_2 that we could compute are divisible by 4, in accordance with Conjecture 2.38. In contrast, for $c \in \{0, 3, 5, \dots, 97\}$, and conjecturally for all $c \neq 2$, there does not exist $n \geq 2$ such that n divides $\vartheta_c(K)$ for all K.

7.2. Alternating knots. Recall that, for all alternating knots K (in fact, even for all quasi-alternating K), $s_c(K) = -\sigma(K)$ holds for all characteristics c (see [42, Theorem 3]).

Conjecture 7.6. *For all alternating knots K, we have*

$$\vartheta_2(K) = 2s_2(K), \tag{7.3}$$

$$\vartheta_c(K) = \frac{3}{2}s_c(K) \quad \textit{for all } c \neq 2. \tag{7.4}$$

Conjecture 7.6 is backed up by computations for around 70 prime alternating knots for $c \in \{0, 2, 3, 5, 7\}$. Interestingly, there exist quasi-alternating knots, such as the $(5, -3, 2)$-pretzel knot (10_{125} in the tables) that do not satisfy (7.4); see Table 5.

Knot	s_c for $c \leq 7$	$\vartheta_0 = \vartheta_5 = \vartheta_7$	ϑ_2	ϑ_3
10_{125}	-2	$\mathbf{-2}$	-4	$\mathbf{-2}$
$8_{19}, 10_{154}, 10_{161}$	6	**8**	**8**	9
$10_{124}, 10_{139}, 10_{152}$	8	**11**	**12**	12

Table 5. The seven prime knots with crossing number 10 or less that do not satisfy (7.3) or (7.4). The offending values are printed in boldface. Positive braid knots are printed in italics.

Knot $J = W_t^+(K)$	$\tau(J)$	$s_0(J)$	$s_2(J)$
$t = -\sigma(K) - 1$	1	2	2
$t = -\frac{3}{2}\sigma(K) - 1$	0	2	2
$t = -2\sigma(K) - 1$	0	0	2

Table 6. An alternating knot K with $\sigma(K) < 0$ and satisfying Conjecture 7.6 has three twisted positive Whitehead doubles that are linearly independent in $\mathcal{C}$.

One notes the similarity of (7.3) and (7.4) to (2.2), $\vartheta_\tau(K) = 2\tau(K)$, proven by Hedden for all knots K. For alternating (even quasi-alternating) K, (2.2) implies $\vartheta_\tau(K) = -\sigma(K)$. So, if Conjecture 7.6 holds, then every alternating knot K with $\sigma(K) \neq 0$ has three twisted Whitehead doubles generating a $\mathbb{Z}^3$ summand of the smooth concordance group, restricted to which τ, s_0, s_2 are linearly independent, as Table 6 shows in the case $\sigma(K) < 0$; the argument for $\sigma(K) > 0$ is similar.

Equation (7.4) for $c = 0$ was also mentioned in [10] as a reasonable guess for two-stranded torus knots (cf. also [41]). It does not hold for all knots, however; see Table 5. This table demonstrates that the behaviour of ϑ_c on non-alternating positive braid knots is rather different than its behaviour on alternating knots. We make the following conjecture, which we have checked on a total of 14 prime positive braid knots and $c \in \{2, 3, 5\}$.

Conjecture 7.7. *If K is a non-alternating knot that is the closure of a positive braid, then $\vartheta_2(K) < 2s_2(K)$ and $\vartheta_0(K) < \frac{3}{2}s_0(K)$. In particular,* (7.3) *and* (7.4) *do not hold.*

Note that the only knots that are both alternating and closure of a positive braid are the $T_{2,2n+1}$ torus knots with $n \geq 0$ (see [1, 37]). Beyond Conjecture 7.7, no clear pattern seems to emerge from our computations for the values of ϑ_c on positive braid knots.

7.3. Torus knots. Positive torus knots $T_{p,q}$ are arguably the simplest positive braid knots. Table 7 contains the values of $\vartheta_c(T_{p,q})$ that we could compute. Those values are compatible with the following conjectures.

Conjecture 7.8. *For all characteristics c, there exists a function $r_c\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ such that $\vartheta_c(T_{p+q,q}) = \vartheta_c(T_{p,q}) + r_c(q)$ for all coprime positive integers p and q.*

For $c = 0$ and $c = 2$, we make the following guess for r_c.

Knot	s_c	ϑ_2	ϑ_3	ϑ_5	ϑ_0
$T_{2,2n+1}$ for $0 \leq n \leq 4$	$2n$	$4n$	$3n$	$3n$	$3n$
$T_{3,4}$	6	8	9	8	8
$T_{3,5}$	8	12	12	11	11
$T_{3,7}$	12	16	18	16	
$T_{3,8}$	14	20	21	19	
$T_{3,10}$	18	24	27		
$T_{3,11}$	20	28			
$T_{4,5}$	12	16	15	15	

Table 7. Computed values for $\vartheta_c(T_{p,q})$.

Conjecture 7.9. *For all coprime positive integers p and q,*

$$\vartheta_0(T_{p+q,q}) = \vartheta_0(T_{p,q}) + q^2 - 1,$$
$$\vartheta_2(T_{p+q,q}) = \vartheta_2(T_{p,q}) + \begin{cases} q^2 & \text{if } q \text{ even,} \\ q^2 - 1 & \text{if } q \text{ odd.} \end{cases}$$

Our conjectures are in accordance with conjectures by Schütz [45, Conjectures 6.2, 6.3] on the values of s_c of certain Whitehead doubles of torus knots. Note that if Conjecture 7.8 holds, then $\vartheta_c(T_{p,q})$ is for all p, q determined by the function r_c, using Euclid's algorithm and the fact that $\vartheta_c(T_{0,1}) = 0$ since $T_{0,1}$ is the unknot. The knot $T_{p+q,q}$ arises from $T_{p,q}$ by a q-homologous right-handed twist on q strands (see Definition 4.5). So it follows from Proposition 4.8 that, if Conjecture 7.8 holds, then $0 \leq r_c(q) \leq q^2$ holds for $c \neq 2$ (and for $c = 2$ under the assumption that s_2 satisfies the generalised crossing change inequality). In order to prove Conjecture 7.8 and Conjecture 7.9, one would have to gain an understanding that goes beyond the inequalities in Proposition 4.8 of how ϑ_c changes under twists of the companion.

Interestingly, the classical knot signature σ satisfies the following recursive formula for torus knots [6, Theorem 5.2]:

$$\sigma(T_{p+2q,q}) = \sigma(T_{p,q}) + \begin{cases} q^2 & \text{if } q \text{ even,} \\ q^2 - 1 & \text{if } q \text{ odd.} \end{cases}$$

The difference to the formula given for ϑ_2 in Conjecture 7.9 is that the left-hand side pertains to $T_{p+2q,q}$ instead of $T_{p+q,q}$.

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