Existence of SRB measures in chaotic random motions: a graph transform approach

Author(s):
Gerber, Damian

Publication date:
2022

Permanent link:
https://doi.org/https://doi.org/10.3929/ethz-b-000600113

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Imperial College London

SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH AND
IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

Existence of SRB measures in chaotic random motions

a graph transform approach

Author:
Damian Gerber

Supervisors:
Jeroen Lamb
Manfred Einsiedler

Submitted in partial fulfillment of the requirements for the Bachelor of Science at ETH Zurich in collaboration with Imperial College London, year 2021-2022.

CID: 02175362

Declaration of originality

I hereby confirm that I am the sole author of the written work here enclosed and that I have compiled it in my own words. Parts excepted are corrections of form and content by the supervisors. Where ever any material can be constructed as the work of others, it is fully cited and referenced to the respective source and/or appropriate acknowledgement given.

Damian Gerber

Acknowledgments

My deepest thanks go out to my supervisor Jeroen Lamb for introducing me to the world of random dynamical systems and Manfred Einsiedler for supervising this project. I thank Jeroen Lamb for his guidance while writing this paper, the motivation he gave me and the many helpful suggestions. Additionally, I thank Giuseppe Tenaglia, without whom this project would not be what it is today. I thank him for his patience, the amazing help as well as his mentorship throughout the whole the process. He pushed me to understand the many non-trivial concepts of random dynamical systems and helped me realize my potential.
Furthermore, I want to express my gratitude to Sacha for his great suggestions and emotional support when ever I needed it. Regardless of the distance, he helped me stay focused and motivated from start to finish.
Lastly, my many thanks go out to my friends and family for all their support and especially Xenia, Lukáš, Timo, Melanie, Tino, Julia and Naomi for the helpful corrections and inputs.

Abstract

SRB measures present a useful class of invariant measures that help understand the chaotic nature of random dynamical systems. Inspired by a paper from Alex Blumenthal and Lai-Sang Young from 2018 titled ”Equivalence of physical and SRB measures in random dynamical systems” this BSc project discusses the existence of SRB measures in chaotic random motions. We look at an innovative geometric construction of SRB measures via a graph transform approach that uses adapted one-sided charts. By pushing forward the stationary measures and approximating the unstable manifolds with pushed forward leaves we show that the sample measures are SRB.

Contents

Abstract **iii**

1 Introduction **1**
1.1 Brief overview 1
1.2 Objective 1
1.3 Construction of the SRB property 2
1.3.1 Geometric construction 2
1.4 Structure 5
1.5 Historical evolution 5

2 Setting, definitions and concepts concerning RDS **7**
2.1 Setting and the random dynamical system 7
2.2 Definitions and concepts concerning RDS 9
2.3 Results 13
2.3.1 Main Theorem 13
2.3.2 Main Proposition and proof of the results 14

3 New chart systems and iterated graph transforms **17**
3.1 Measurable selection of the past 18
3.2 Adapted single graph transform 19
3.3 Iteration of graph transforms 23

4 Source and target sets and construction of u-graphs **27**
4.1 Uniformity sets of $\mu^n_{\underline{\omega}}$-typical points 27
4.2 Accumulating $\mu^n_{\underline{\omega}}$-mass 28
4.3 Disintegration of μ in the "source set" onto u-graphs 32

5 Geometry of pushed forward u-graphs **35**
5.1 Construction of the stack of W^n-leaves 35
5.2 Limiting properties of W^n-leaves 36

6 Proof of the SRB properties **41**
6.1 Construction of partitions respecting unstable manifolds 41
6.2 Pushed forward measures and their conditional densities 45
6.3 Passing to the weak limit as $n \to \infty$ and completing the proof 48
6.3.1 Concluding the Main Proposition 51

7 Conclusion **53**
7.1 Summary and overview 53
7.2 Reflection and achievements 54
7.3 Future work 54

A Selection of significant original contributions **55**

Bibliography **58**

Figures

1.1 On the left, construction of the local unstable laves (W^u-leaves) on Λ. On the right, the partition β_m of a fraction of the unstable leaves. . . 3

1.2 Pushing forward the foliation $\mathcal{F}_-^n$ with $f^n_{\theta^{-n}\underline{\omega}}$ from time $-n$ to time 0 in order to create the W^n-leaves. 3

1.3 Overlaying the W^u-leaves (blue), W^n-leaves (green) and the partition β_m (yellow) on the left. A deeper view of the element $\beta_{m,k}$ of the partition β_m overlayed with part $\beta_{\tilde{m},\tilde{k}}$ of the finer partition $\beta_{\tilde{m}}$ is shown on the right. Note that while a fraction of W^n-leave is fully inside single elements of the partition on the left, there is a W^n-leave that is only partly contained in $\beta_{\tilde{m},\tilde{k}}$ on the right. 4

1.4 At the top, we have an iteration from past n to $\tilde{n}$ with $\tilde{n} > n$ to create a new fraction of $W^{\tilde{n}}$-leaves that are contained in $\beta_{\tilde{m},\tilde{k}}$. At the bottom, we have the alignment of the ξ^n-leave and ξ-leave as $\tilde{m}, \tilde{n} \to \infty$. The W^n-leave ξ^n represents here the leave that with each iteration of n and m ends up in the same element of the respective partition as ξ the W^u-leave. 4

Tables

A.1 A selection of original work done in this project. 56

Chapter 1

Introduction

Invariant measures are a good way to investigate the chaotic nature of dynamical systems. Since a system can have infinitely many invariant measures, some present themselves as more helpful than others. A useful class of measures with interesting geometric and dynamical structures are the Sinai-Ruelle-Bowen measures (SRB measures). They are well studied in the deterministic setting and were first introduced by Sinai, Ruelle and Bowen in the 1970s in [10, 8, 3] to study the behaviour of chaotic topological attractors. The notion of SRB measures can be extended to the random setting but the existence of such measures is however not always guaranteed. Based on that observation, this paper is dedicated to discussing the existence of random SRB measures in chaotic random system as well as their construction.

1.1 Brief overview

Motivated by the paper [2] written by Alex Blumenthal and Lai-Sang Young in 2018 with the title "Equivalence of physical and SRB measures in random dynamical systems", one way of showing the existence of random SRB measures is to find physical measures of the system and show that they have random SRB properties. In this context, physical measures are invariant measures where the time average converges to the space average. A remarkable result from Alex Blumenthal and Lai-Sang Young is that SRB measures exist in chaotic ergodic C^2 random dynamical systems. It turns out that the sample measures $\mu_{\underline{\omega}}$ are under these assumptions SRB.

1.2 Objective

The general goal of this bachelor project is to become familiar with the core concepts of random dynamical systems, a topic which is usually not covered in undergraduate courses. More specifically, the objective is to work through the paper [2] in order to understand the relation between physical and SRB measures as well as the construction to get there. Additionally, we present complementary proofs and explanations for selected statements.

We note that Alex Blumenthal and Lai-Sang Young are not the first to look at a relation between physical and SRB measures as the paper [5] written by François Ledrappier and Lai-Sang Young in 1988 discusses a similar result. What makes the paper [2] unique, is that it presents a geometric construction of SRB measures by pushing forward the stationary measures and introduces a novel approach of working with finite pasts using adapted one-sided charts. We think that this gives a good geometric idea of the underlying mechanics and makes the result approachable for someone new to RDS.
To make it easier to distinguish between any original content and the established contributions of Alex Blumenthal and Lai-Sang Young, we refer to Table A.1 in the appendix for a selection of the major new contributions. We want to emphasize that this list is only a selection of some of the original work. A lot of additional smaller explanations, comments and proofs were added which cannot be clearly separated from the content in [2].

1.3 Construction of the SRB property

The construction of the SRB property is in itself straightforward but requires a lot of technical work. Using the fact that the sample measures can be seen as the weak limit of pushing forward the stationary measure from time $-\infty$ to time 0 we will show that they are SRB. We will see in Chapter 6 that the main idea for showing the SRB properties is that we can push forward a foliation of a small localized region (later called the "source" set) from $-\infty$ to 0 such that a positive fraction of graphs (elements of the pushed forward foliation) get aligned with local unstable leaves at time 0 in a small region (later called the "target" set). The idea is to partition parts of the stack of unstable leaves in finer and finer partitions so that the pushed forward leaves are fully contained in the elements of the partitions. By choosing a partition that eventually only contains one single unstable leave, we show that a positive fraction of pushed forward leaves gets aligned with a part of the local unstable manifolds.
How does this partitioning help us show the SRB property? We will construct at time $-\infty$ a measure that looks similar to the Lebesgue measure but is supported on a small region. We will then push this measure forward and support it on our partition that eventually contains single leaves. By choosing the source and target sets small enough, we can show that we have absolute continuity for the conditional measures with respect to the Lebsegue measure for a positive fraction of the unstable leaves as well as bounded densities from above and below which will conclude the SRB property.

1.3.1 Geometric construction

Below, there is a rough graphical explanation of the introduced construction. It intents to motivate the steps in Chapters 2-6 and present the overall strategy. This section already presupposes knowledge about RDS and a proper introduction will be done in Chapter 2. Denote by $\{f_\omega\}$ the random dynamical system and fix $\underline{\omega} \in \Omega^{\mathbb{Z}}$

with Ω being a Polish space. We write $f^n_{\theta^{-n}\underline{\omega}}$ for the push forward function from time $-n$ to time 0 where θ is the shift operator on $\Omega^{\mathbb{Z}}$ and $f_{\theta^{-n}\underline{\omega}}$ a C^2 diffeomorphism on a Riemannian Manifold M onto itself. Since we only work with small regions in M we can confuse a set $\Lambda \subset M$ with a subset in $T_{x^*}M$ where $x^* \in \Lambda$. Assume that the below steps are well defined. The construction goes as followed.

1. Following Figure 1.1, take at time 0 the group of local unstable leaves (here the W^u-leaves) of Λ and bundle most of them into a partition of open sets $\beta_m = \{\beta_{m,k}\}$ for some $m \in \mathbb{N}$. Assume that the partitions β_m are nested, i.e., each $\beta_{m,k} \subset \beta_{m-1,k'}$ for some k'.

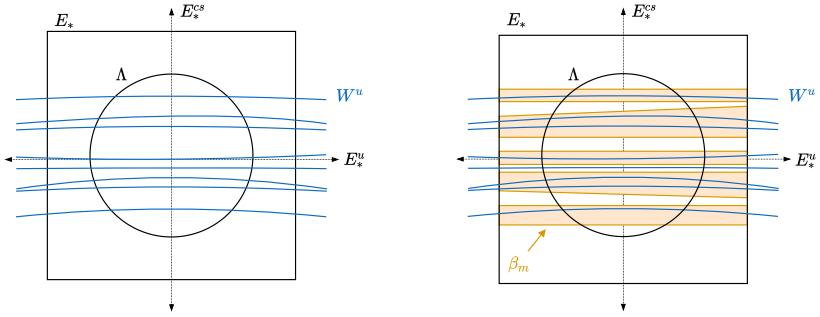

Figure 1.1: On the left, construction of the local unstable laves (W^u-leaves) on Λ. On the right, the partition β_m of a fraction of the unstable leaves.

2. Next choose a big enough n such that this and the next step hold. We take at time $-n$ a subset B such that the set gets pushed forward into Λ via $f^n_{\theta^{-n}\underline{\omega}}$. We construct a foliation $\mathcal{F}^n_-$ on B using parallel planes at time $-n$ (the parallel planes will later end up being the u-graphs and are parallel to the unstable set of a reference point in B). We push forward $\mathcal{F}^n_-$ to time 0 and end up by the construction with a partition on part of Λ. We call the elements in this new partition the W^n-leaves. The process is shown in Figure 1.2.

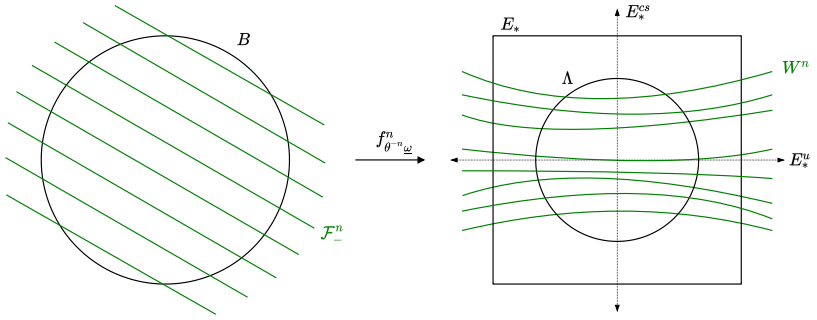

Figure 1.2: Pushing forward the foliation $\mathcal{F}^n_-$ with $f^n_{\theta^{-n}\underline{\omega}}$ from time $-n$ to time 0 in order to create the W^n-leaves.

3. We overlay the W^n-leaves with the W^u-leaves. By the choice of n, a positive fraction of W^n-leaves ends up fully inside the elements of β_m (see Figure 1.3

on the left). We choose next a finer partition $\beta_{\tilde{m}}$, $\tilde{m} > m$ and see that the fraction of W^n-leaves from before is not necessarily anymore contained in the single $\beta_{\tilde{m},\tilde{k}}$ (see Figure 1.3 on the right).

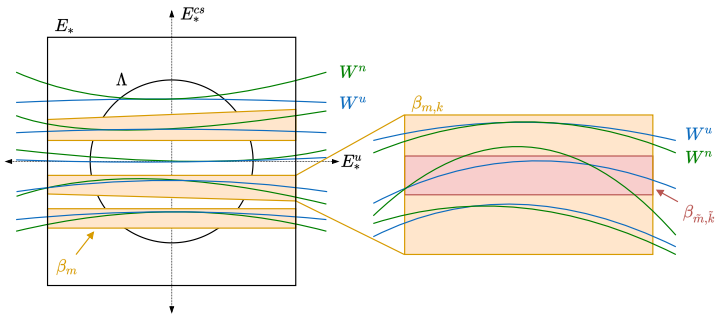

Figure 1.3: Overlaying the W^u-leaves (blue), W^n-leaves (green) and the partition β_m (yellow) on the left. A deeper view of the element $\beta_{m,k}$ of the partition β_m overlayed with part $\beta_{\tilde{m},\tilde{k}}$ of the finer partition $\beta_{\tilde{m}}$ is shown on the right. Note that while a fraction of W^n-leave is fully inside single elements of the partition on the left, there is a W^n-leave that is only partly contained in $\beta_{\tilde{m},\tilde{k}}$ on the right.

4. Remember that we started at time $-n$ and went to time 0. By choosing a new $\tilde{n} \in \mathbb{N}$, $\tilde{n} > n$, we can find a new foliation $W^{\tilde{n}}$ that is again fully contained in the $\beta_{\tilde{m},\tilde{k}}$ (see Figure 1.4 at the top). Iteratively, we choose again a finer partition (increasing $\tilde{m}$) and as a response go further back in the past (increasing $\tilde{n}$). We will show in the remainder of this paper that letting $\tilde{m}, \tilde{n} \to \infty$ results in that we get a positive fraction of single W^n-leaves ξ^n, that get aligned with their corresponding unstable leaves ξ (see Figure 1.4 at the bottom).

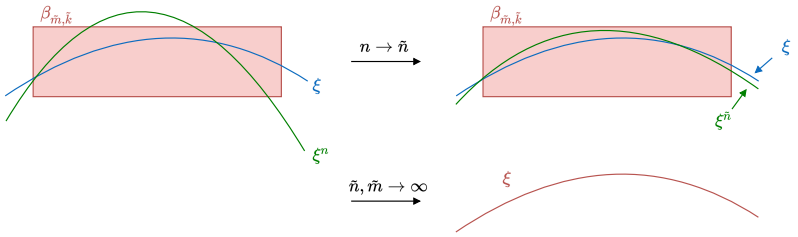

Figure 1.4: At the top, we have an iteration from past n to $\tilde{n}$ with $\tilde{n} > n$ to create a new fraction of $W^{\tilde{n}}$-leaves that are contained in $\beta_{\tilde{m},\tilde{k}}$. At the bottom, we have the alignment of the ξ^n-leave and ξ-leave as $\tilde{m}, \tilde{n} \to \infty$. The W^n-leave ξ^n represents here the leave that with each iteration of n and m ends up in the same element of the respective partition as ξ the W^u-leave.

The above geometric construction will be done in more detail in Chapter 6. Chapters 2, 3, 4 and 5 will occupy the technicalities.

1.4 Structure

This thesis follows closely the paper [2] and is organised similarly in order to make it easier to follow the content and retrace the steps.

1. Chapter 2 presents the overall setting and background information needed for the paper. It covers a brief introduction on random dynamical systems and important findings which will later be useful.

2. Section 2.3 concludes Chapter 2 by presenting the main results as well as a proof.

3. Chapters 3-6 are dedicated to proofing the proposition which was assumed for the main result.

4. Chapter 3 discusses the construction of graph transforms when considering adapted one-sided charts. It covers the problems which appear when working with adapted graph transforms and how they can be resolved.

5. Chapter 4 constructs the so called "source sets" and "target sets" and disintegrate the stationary measure μ in the "source set" onto the so called u-graphs.

6. Chapter 5 constructs a foliation of pushed forward u-graphs from time $-n$ to time 0 (also called W^n-leaves). We discover that for big enough n the stack of W^n-leaves starts to resemble the stack of local unstable leaves.

7. Chapter 6 is dedicated to the geometric construction. It proves the SRB properties by looking at finite partitions with compact subsets on the stack of unstable leaves. We introduce a new measure and show the desired properties using the results from Chapters 4 and 5.

1.5 Historical evolution

We briefly mentioned that the paper from Alex Blumenthal and Lai-Sang Young is not the first to show an equivalence between physical and random SRB measures. A proof for a random dynamical system constructed with diffeomorphisms first appeared in [5] written by François Ledrappier and Lai-Sang Young in 1988. Using an expression for the entropy of the RDS they showed the random SRB properties for sample measures. 14 years later, in 2002, Pei-Dong Liu, Min Qian and Fu-Xi Zhang generalised these results for endomorphisms in [7]. Using the Pesin Formula they showed the random SRB properties as a consequence of the entropy of the system. Alex Blumenthal and Lai-Sang Young take in [2] a more modern and geometric approach by constructing the sample measures directly with no need for a detour into the entropy. We think that this makes the result more approachable and presents a visual idea of what happens.

Chapter 2

Setting, definitions and concepts concerning RDS

2.1 Setting and the random dynamical system

Before we dive into the interesting field of random dynamical systems, we need to do a bit of preliminary work. It is important to first get familiar with the overall setting as well as a few basic definitions.
First up we have the notion of a Polish space.

Definition 2.1 (Polish space). A *Polish space* is a space that is homeomorphic to a complete metric space that has a countable dense subset.

Remark 2.2. The group of Polish spaces is extremely big and diverse as the following two examples show.

1. $\mathbb{R}^n$ is a Polish space for $n \in \mathbb{N}$, as well as any open or closed subset of $\mathbb{R}^n$.
2. Every locally compact Hausdorff space with a countable basis is a Polish space. (See [11] for a proof)

With this definition out of the way, take now a Polish space Ω, a Borel probability measure P and a compact Riemannian Manifold M. We consider the Borel measurable mapping $\omega \to f_\omega$ going from Ω to $\mathrm{Diff}^2(M)$, where $\mathrm{Diff}^2(M)$ is the space of all C^2 diffeomorphisms from M onto itself equipped with the C^2 norm. With the help of this mapping, we can construct so called random dynamical systems, commonly abbreviated as RDS. The following definition is taken from [1].

Definition 2.3 (RDS). A *measurable random dynamical system* on the measurable space $(M, \mathcal{B})$ over (or covering, or extending) a metric dynamical system $(\mathbb{T}, \tilde{\Omega}, \mathbb{P}, \theta)$ with time $\mathbb{T}$ is a mapping

$$\varphi : \mathbb{T} \times \Omega \times M \to M, \qquad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

with the following properties:

(i) Measurability: ϕ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$-measurable.

(ii) Cocycle property: The mappings $\phi(t,\omega) \coloneqq \phi(t,\omega,\cdot) : M \to M$ form a cocylce over $\theta(\cdot)$, i.e. they satisfy

$$\begin{aligned} \varphi(0,\omega) &= \mathrm{id}_M && \text{for all} \quad \omega \in \Omega \qquad (\text{if } 0 \in \mathbb{T}) \\ \varphi(t+s,\omega) &= \varphi(t,\theta(s)\omega) \circ \varphi(s,\omega) && \text{for all} \quad s,t \in \mathbb{T}, \quad \omega \in \Omega. \end{aligned}$$

Note that the below definition (taken from Chapter 1 in [2]) looks slightly different from Definition 2.3.

Definition 2.4 (RDS according to Alex Blumenthal and Lai-Sang Young). Let Ω be a Polish space and let P be a Borel probability measure on Ω. Let M be a compact Riemannian manifold and consider a Borel-measurable mapping $\omega \mapsto f_\omega$ from $\Omega \to \mathrm{Diff}^2(M)$, the space of C^2 diffeomorphisms from M onto itself equipped with the C^2-metric. A *random dynamical system* consists of compositions of sequences of maps from $\{f_\omega, \omega \in \mathbb{R}\}$ chosen i.i.d. with law P. For $\underline{\omega} = (\omega_n)_{n\in\mathbb{Z}} \in \Omega^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we write

$$f^n_{\underline{\omega}} = \begin{cases} f_{\omega_n} \circ \cdots \circ f_{\omega_1} & n > 0 \\ \mathrm{Id} & n = 0 \\ f^{-1}_{\omega_{n+1}} \circ \cdots \circ f^{-1}_{\omega_0} & n < 0 \end{cases}$$

The following lemma shows that this issue is of no concern to us as Definition 2.4 implies Definition 2.3.

Lemma 2.5. *Definition 2.4 constructs a C^2 RDS according to Definition 2.3.*

Proof. Set $\tilde{\Omega} \coloneqq \Omega^{\mathbb{Z}}$ and $\mathbb{P} \coloneqq \otimes^{\mathbb{Z}} P$ as the corresponding product measure. Furthermore, define $\theta(n) : \tilde{\Omega} \to \tilde{\Omega}$ as the n-times left-shift operator for $n \in \mathbb{N}$ and set $\mathbb{T} \coloneqq \mathbb{Z}$. It can be shown that $(\mathbb{T}, \tilde{\Omega}, \mathfrak{B}, \theta)$ is an ergodic continuous dynamical system. Define the mapping φ as

$$\varphi : \mathbb{T} \times \tilde{\Omega} \times M \to M, (t,\underline{\omega},x) \to \varphi(t,\underline{\omega},x) \coloneqq \begin{cases} f_{\omega_t} \circ \cdots \circ f_{\omega_1}(x) & t > 0 \\ x & t = 0 \\ f^{-1}_{\omega_{t+1}} \circ \cdots \circ f^{-1}_{\omega_0}(x) & t < 0 \end{cases}.$$

The next step is to prove that φ is a random dynamical system. We see that φ is generated by the time-one mapping $\varphi(1,\underline{\omega}) =: \psi(\underline{\omega})$ with

$$\psi(\underline{\omega}) \coloneqq f_{\omega_1} : M \to M$$

and

$$\varphi(t,\underline{\omega}) = f^t_{\underline{\omega}}.$$

It follows from the definition of f_ω for $\omega \in \Omega$ that $f_\omega \in \mathrm{Diff}^2(M)$ and $\omega \to f_\omega$ is Borel measurable. Because $\varphi(t,\underline{\omega})$ is a composition of maps in $\mathrm{Diff}^2(M)$ by construction, it is itself in $\mathrm{Diff}^2(M)$. The cocycle property is satisfied because we have for all $s,t \in \mathbb{T}$ and all $\underline{\omega} \in \tilde{\Omega}$ that

$$\varphi(t+s,\omega) = f^{t+s}_{\underline{\omega}} = f^t_{\theta(s)\underline{\omega}} \circ f^s_{\underline{\omega}} = \varphi(t,\theta(s)\omega) \circ \varphi(s,\omega).$$

This concludes that φ is a C^2 RDS. □

2.2 Definitions and concepts concerning RDS

We briefly mentioned in Chapter 1 that this thesis intends to push forward the so called stationary measure using the RDS to get the sample measures. Before we do that, it is a good idea to introduce these two measures.

Definition 2.6 (Stationary measure). Let (X_n) be a Markov chain on M associated with a RDS. The chain has fixed $X_0 \in M$ and $X_{n+1} = f_{\omega_{n+1}}(X_n)$. Let μ be a Borel probability measure on M. The measure μ is said to be *stationary* if for all Borel sets $E \subset M$,

$$\mu(E) = \int \mathcal{P}(E|x)\mu(dx).$$

Here, $\mathcal{P}(E|x)$ is the so called *transition probability* of the chain and is defined by

$$\mathcal{P}(E|x) = P\{\omega : f_\omega x \in E\}.$$

Definition 2.7 (Sample measures). Suppose μ^* is a probability measure on $M \times \Omega^{\mathbb{Z}}$ with marginal $\mathbb{P}$ on $\Omega^{\mathbb{Z}}$, e.g. $\pi_{\Omega^{\mathbb{Z}}}\mu^* = \mathbb{P}$ with $\pi_{\Omega^{\mathbb{Z}}}(x, \underline{\omega}) = \underline{\omega}$ being the projection onto $\Omega^{\mathbb{Z}}$. We call the family of probability measures $\{\mu_{\underline{\omega}},\ \underline{\omega} \in \Omega^{\mathbb{Z}}\}$ the *sample measures* or *disintegration* of μ^* on M-fibers if:

1. for all $B \in \mathcal{B}(M)$, $\underline{\omega} \mapsto \mu_{\underline{\omega}}(B)$ is measurable
2. for all continuous functions $\phi : M \times \Omega^{\mathbb{Z}} \to \mathbb{R}$, we have

$$\int \phi(x, \underline{\omega}) d\mu^*(x, \underline{\omega}) = \int \left(\int \phi(x, \underline{\omega}) d\mu_{\underline{\omega}}(x) \right) d\mathbb{P}(\underline{\omega}).$$

One may ask why these two types of measures are important to us. The following well known lemma shows that the sample measures can be interpreted as the weak limit of pushing forward the stationary measure, i.e. $(f^n_{\theta^{-n}\underline{\omega}})_*\mu \to \mu_{\underline{\omega}}$ as $n \to \infty$. This important relation will later help us show that sample measures are SRB measures without having to compute them. We refer to Part I. in [1] for the well known proof.

Lemma 2.8.

(a) *A Borel probability measure μ on M is a stationary measure if and only if $\mu \times \mathbb{P}^+$ is an invariant measure of $\tau^+ : M \times \Omega^{\mathbb{Z}^+} \to M \times \Omega^{\mathbb{Z}^+}$.*

(b) *Given μ as in (a), there is a unique τ-invariant probability measure μ^* on $M \times \Omega^Z$ that projects onto $\mu \times \mathbb{P}^+$. Here, the function τ denotes the skew-product map $\tau : M \times \Omega^{\mathbb{Z}} \to M \times \Omega^{\mathbb{Z}}$ with $\tau(x, \underline{\omega}) = (f_{\omega_1}x, \theta\underline{\omega})$.*

(c) *The measures $\mu_{\underline{\omega}}$ are invariant in the sense that for each $\underline{\omega} = (\omega_n) \in \Omega^{\mathbb{Z}}$,*

$$(f_{\omega_1})_*\mu_{\underline{\omega}} = \mu_{\theta\underline{\omega}}.$$

(d) *For $\mathbb{P}$-a.e. $\underline{\omega} \in \Omega^{\mathbb{Z}}$,*

$$(f^n_{\theta^{-n}\underline{\omega}})_*\mu \to \mu_{\underline{\omega}} \text{ weakly as } n \to \infty.$$

It follows that $\mu_{\underline{\omega}}$ depends only on ω_n for $n \leq 0$.

Going back to the construction of the RDS we have

$$f^n_{\theta^{-n}\underline{\omega}} = f_{\omega_0} \circ f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-(n-1)}}$$

as well as

$$(f^n_{\theta^{-n}\underline{\omega}})^{-1} = f^{-1}_{\omega_{-(n-1)}} \circ \cdots \circ f^{-1}_{\omega_0} = f^{-n}_{\underline{\omega}}$$

and it follows

$$(f^n_{\theta^{-n}\underline{\omega}})_* \mu(A) = \mu(f^{-n}_{\underline{\omega}}(A))$$

for all $A \in \mathcal{B}(M)$. Item (c) in Lemma 2.8 thus shows that $\mu_{\underline{\omega}}$ can be interpreted as the push-forward measure from $-\infty$ to 0 along $\underline{\omega} \in \Omega^{\mathbb{Z}}$.
Before we look at the concept of random SRB measures we need the notion of the Lyapunov exponent.

Definition 2.9 (Lyapunov exponent). Let $\{f_\omega\}$ be our RDS. Given $(x, \underline{\omega}) \in M \times \Omega^{\mathbb{Z}}$ the *Lyapunov exponent* of $(x, \underline{\omega})$ is defined as

$$\lambda(x, \underline{\omega}) = \lim_{n\to\infty} \frac{\log ||Df^n_{\underline{\omega}}(x)||}{n}.$$

Lyapunov exponents are very useful as they help us understand the dynamics of the RDS. If the system inhibits certain properties (see [1] for further information), the total number of different Lyapunov exponents of the system is smaller than the dimension of M. For almost every $(x, \underline{\omega}) \in M \times \Omega^{\mathbb{Z}}$ the tangent spaces T_xM can then be decomposed using the so called *Oseledets-splitting* to study where the system exhibits contraction and where expansion. These findings are summarized in the *Multiplicative Ergodic Theorem (MET)*. We refer to Chapter 4 in [1] for the MET on Manifolds and Proposition 4 in [2] for the specific case used in this thesis. The next proposition introduces a lot of important notation.

Remark 2.10. Following Chapter 4 in [2] we

$a)$ denote by Γ the set of full measure generated by the MET,

$b)$ write $E^u_{(x,\underline{\omega})}$ for the *unstable set* at $(x, \underline{\omega}) \in \Gamma$ and

$c)$ write $E^{cs}_{(x,\underline{\omega})}$ for the *center-stable* set at $(x, \underline{\omega}) \in \Gamma$.

Proposition 2.11. *Assume the RDS has a strictly positive Lyapunov exponent λ_0. Fix $\delta_0,\ \delta_1,\ \delta_2 > 0$ with $\delta_0,\ \delta_2 \ll \lambda_0$ and δ_1 sufficiently small, and let $\lambda = \lambda_0 - \delta_0$. Shrinking Γ by a set of μ^*-measure 0 (and continuing to call it Γ), there are defined on Γ*

(i) *a Borel measurable family of invertible linear maps*

$$L_{(x,\underline{\omega})} : \mathbb{R}^u \times \mathbb{R}^{cs} \to T_xM,$$

with $\mathbb{R}^u = \mathbb{R}^{\dim E^u}$, $\mathbb{R}^{cs} = \mathbb{R}^{\dim E^{cs}}$, $L_{(x,\underline{\omega})}\mathbb{R}^u = E^u_{(x,\underline{\omega})}$ and $L_{(x,\underline{\omega})}\mathbb{R}^{cs} = E^{cs}_{(x,\underline{\omega})}$, and

(ii) *a measurable function $l : \Gamma \to [1, \infty)$ satisfying $e^{-\delta_2} \geq \frac{l\circ\tau}{l} \leq e^{\delta_2}$,*

with respect to which the following hold. Let the chart *at* $(x, \underline{\omega})$ *be given by*

$$\Phi_{(x,\underline{\omega})} : B(\delta_1 l(x,\underline{\omega})^{-1}) \to M \qquad \textit{with} \qquad \Phi_{(x,\underline{\omega})} = \exp_x \circ L_{(x,\underline{\omega})},$$

and define the connecting maps between charts *to be*

$$\tilde{f}_{(x,\underline{\omega})} = \Phi^{-1}_{\tau(x,\underline{\omega})} \circ f_{\underline{\omega}} \circ \Phi_{(x,\underline{\omega})} : B(\delta_1 l(x,\underline{\omega})^{-1}) \to \mathbb{R}^u \times \mathbb{R}^{cs}.$$

Then

(a) *for any* $y,\ y' \in \Phi_{(x,\underline{\omega})} B(\delta_1 l(x,\underline{\omega})^{-1})$, *we have*

$$d(y,y') \le |\Phi^{-1}_{(x,\underline{\omega})} y - \Phi^{-1}_{(x,\underline{\omega})} y'| \le l(x,\underline{\omega})\ d(y,y')$$

and

(b) $\tilde{f}_{(x,\underline{\omega})}$ *satisfies*

(b1) $|(d\tilde{f}_{(x,\underline{\omega})})_0 u| \ge e^{\lambda}|u|$ *for* $u \in \mathbb{R}^u$ *and* $|(d\tilde{f}_{(x,\underline{\omega})})_0 v| \ge e^{\delta_0}|v|$ *for* $v \in \mathbb{R}^{cs}$,

(b2) $\mathrm{Lip}(\tilde{f}_{(x,\underline{\omega})} - (d\tilde{f}_{(x,\underline{\omega})})_0|_{B(\delta l(x,\underline{\omega})^{-1})}) \le \delta$ *for all* $\delta \in (0,\delta_1)$, *and*

(b3) $\mathrm{Lip}(d\tilde{f}_{(x,\underline{\omega})}) \le l(x,\underline{\omega})$.

Definition 2.12 (Uniformity sets). We define or fixed $l_0 \ge 1$ the *uniformity set* at l_0 as

$$\Gamma_{l_0} = \{(x,\underline{\omega}) \in \Gamma : l(x,\underline{\omega}) \le l_0\}.$$

Remark 2.13. For the remainder of this thesis, we let $B^u(r) = \{u \in \mathbb{R}^u : |u| \le r\}$, $B^{cs}(r) = \{v \in \mathbb{R}^{cs} : |v| \le r\}$ and write

$$B(r) = B^u(r) + B^{cs}(r),$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^u$ and $\mathbb{R}^{cs}$. For $w = u + v \in \mathbb{R}^u \times \mathbb{R}^{cs}$ we define the norm $|w| = \max\{|u|,|v|\}$.

We briefly mentioned the notion of unstable manifolds in Chapter 1. They are an important aid for studying the properties of the chaotic system and are defined through the *Unstable Manifold Theorem*.

Definition 2.14 (Big-Lip function). Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. The *Big-Lip function* Lip is defined as

$$\mathrm{Lip}(f) = \limsup_{r \to 0^+} \frac{\sup_{\|x-y\|_{\mathbb{R}^m} \le r} \|f(x) - f(y)\|_{\mathbb{R}^n}}{r}.$$

Theorem 2.15 (Unstable Manifold Theorem). *Assume the setting of Proposition 2.11, i.e. let* Γ *be the invariant set of full measure, generated by the MET and shrunk by a set of* μ^**-measure* 0. *Let* $\delta > 0$ *be sufficiently small. Then there exists a unique family of measurable-varying maps* $\{g_{(x,\underline{\omega})} : B^u(\delta l(x,\underline{\omega})^{-1}) \to \mathbb{R}^{cs}\}_{(x,\underline{\omega}) \in \Gamma}$ *and a constant* $C > 0$ *such that*

$$g_{(x,\underline{\omega})}(0) = 0 \quad \textit{and} \quad \tilde{f}_{(x,\underline{\omega})}(\mathrm{graph}(g_{(x,\underline{\omega})})) \supset \mathrm{graph}(g_{\tau(x,\underline{\omega})}),$$

for every $(x,\underline{\omega}) \in \Gamma$. *Moreover,*

1. $g_{(x,\underline{\omega})}$ *is* $C^{1+\mathrm{Lip}}$, *and* $(dg_{(x,\underline{\omega})})_0 = 0$,
2. $\mathrm{Lip}(g_{(x,\underline{\omega})}) \leq 1/10$, $\mathrm{Lip}(dg_{(x,\underline{\omega})}) \leq Cl(x,\underline{\omega})$ *and*
3. *if* $z_1, z_2 \in \tilde{f}^{-1}_{(x,\underline{\omega})}(\mathrm{graph}(g_{\tau(x,\underline{\omega})}))$, *then*

$$|\tilde{f}_{(x,\underline{\omega})}z_1 - \tilde{f}_{(x,\underline{\omega})}z_1| \geq (e^{\lambda} - \delta)|z_1 - z_2|.$$

The local unstable manifolds *at* $(x,\underline{\omega})$ *are defined as*

$$W^u_{(x,\underline{\omega}),\delta} = \Phi_{(x,\underline{\omega})}(\mathrm{graph}(g_{(x,\underline{\omega})})).$$

The global unstable manifold *is defined as*

$$W^u_{(x,\underline{\omega})} = \bigcup_{n\geq 0} f^n_{\theta^{-n}\underline{\omega}} W^u_{\tau^{-n}(x,\underline{\omega}),\delta}.$$

Lastly we need the notion of the disintegration and of conditional measures on an arbitrary partition $\mathcal{P}$ of a measure space M. Similarly to sample measures, the conditional measures discuss the original measure restricted to an element in the partition. The following definition is taken from [9].

Definition 2.16 (Disintegration and conditional measures). Let μ be a measure on M and $\mathcal{P}$ a partition of M into measurable sets. Denote by

$$\pi : M \to \mathcal{P} \qquad \text{with} \qquad \pi(x) = P \in \mathcal{P} \text{ if } x \in P$$

the natural projection into the partition. We turn $\mathcal{P}$ into a measure space $(\mathcal{P}, \hat{\mathcal{B}}, \hat{\mu})$, by letting

$$Q \in \hat{\mathcal{B}} \Leftrightarrow \pi^{-1}(Q) \in \mathcal{B},$$

and

$$\hat{\mu}(Q) = \mu(\pi^{-1}(Q)).$$

A *disintegration of* μ *with respect to* $\mathcal{P}$ is a family of probability measures $\{\mu_P : P \in \mathcal{P}\}$ such that for every $E \in \mathcal{B}$ one has

1. $\mu_p(P) = 1$ for $\hat{\mu}$ for almost every $P \in \mathcal{P}$.
2. $P \in \mathcal{P} \mapsto \mu_P(E) \in \mathbb{R}$ is $\hat{\mathcal{B}}$-measurable.
3. $\mu(E) = \int \mu_P(E) d\hat{\mu}(P)$.

Each measure μ_P is called a *conditional measure on the partition* $\mathcal{P}$.

Definition 2.17 (Random SRB measures). Let $\{f_\omega\}$ and μ be given. We say the $\mu_{\underline{\omega}}$ are *random SRB measures* if

1. $f^n_{\underline{\omega}}$ has a positive Lyapunov exponent μ^*-a.e. and
2. for $\mathbb{P}$-a.e. $\underline{\omega} \in \Omega^{\mathbb{Z}}$, the sample measure $\mu_{\underline{\omega}}$ has absolutely continuous conditional measures on unstable manifolds with respect to the Lebesgue measure.

The following definition for physical measures is taken from [14].

Definition 2.18 (Physical measure). Let $f : M \to M$ be an arbitrary map and μ an f-invariant probability measure. We call μ a *physical measure* if there is a positive Lebesgue measure set $V \subset M$ such that for every continuous observable $\varphi : M \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho(f^i x) = \int \varphi \, d\mu$$

for every $x \in V$.

It follows directly from Birkhoff's Ergodic Theorem that every ergodic invariant measure is a physical measure.
We refer to [2] for the construction of *stacks of unstable leaves* $\mathcal{S}$ (see Proposition 11 in [2]) and any further concepts that are not covered in this chapter but are used in the rest of this thesis.

2.3 Results

The goal of this section is to present the main results of this thesis and explain how they can be deduced from a proposition that will be proven in Chapters 3-6.

2.3.1 Main Theorem

The main result is the sample measures are random SRB measures in the context of an ergodic chaotic RDS.

Theorem 2.19 (Main Theorem). *Let $\{f_\omega\}$ be a RDS satisfying*

$$\int \log^+ \|f_\omega\|_{C^2} \, dP(\omega), \ \int \log^+ \|f_\omega^{-1}\|_{C^2} \, dP(\omega) < \infty,$$

and let μ be an ergodic stationary measure. We assume that

1. *$\mu \ll \mathrm{Leb}$ with a continuous density, and*
2. *$\{f^n_{\underline{\omega}^+}\}$ has a positive Lyapunov exponent $(\mu \times \mathbb{P}^+)$-a.e.*

Then the $\mu_{\underline{\omega}}$ are random SRB measures.

By the above assumptions, our chaotic system $\{f^n_{\underline{\omega}^+}\}$ has already a positive Lyapunov exponent $(\mu\times\mathbb{P}^+)$-a.e.. We further know that an arbitrary $\underline{\omega}^+ \in \Omega^{\mathbb{Z}^+}$ can be extended to a $\underline{\omega} \in \Omega^{\mathbb{Z}}$ by choosing a past $\underline{\omega}^- \in \Omega^{\mathbb{Z}^-}$ and recall that the measure μ^* projects onto $\mu \times \mathbb{P}^+$. Consequently, item (1) in Definition 2.17 follows directly. It is left to prove that for $\mathbb{P}$-a.e. $\underline{\omega} \in \Omega^{\mathbb{Z}}$ the sample measures $\mu_{\underline{\omega}}$ have absolutely continuous conditional measures on unstable manifolds. For the rest of this thesis, we are in the setting of the Main Theorem.

2.3.2 Main Proposition and proof of the results

To prove the Main Theorem, it is enough to show the following proposition.

Proposition 2.20 (Main Proposition). *For all sufficiently large $l_0 > 1$, there is a positive $\mathbb{P}$-measure set of $\underline{\omega}$ and a small constant $c > 0$ for which the following hold. On each $\Gamma_{l_0,\underline{\omega}}$ with*

$$\Gamma_{l_0,\underline{\omega}} = \{x \in M : (x, \underline{\omega}) \text{ and } l(x, \underline{\omega}) \leq l_0\}$$

there is a stack $\mathcal{S}_{\underline{\omega}}$ of local unstable manifolds with the following properties:

- (a) *A fraction $\geq c$ of $\mu_{\underline{\omega}}$ is supported on $\mathcal{S}_{\underline{\omega}}$, i.e. $\mu_{\underline{\omega}} = \nu_1 + \nu_2$ where ν_1, ν_2 are both positive measures, and ν_1 is supported on $\mathcal{S}_{\underline{\omega}}$ with $\nu_1(\mathcal{S}_{\underline{\omega}}) \geq c$.*
- (b) *Let Ξ be the partition of $\mathcal{S}_{\underline{\omega}}$ into unstable leaves. Then the conditional probabilities of ν_1 on elements of Ξ are absolutely continuous with respect to the Lebesgue measure, with densities uniformly bounded from above and below.*

The proof of the Main Proposition will occupy the rest of the paper and can be found in Chapters 3-6. Assuming that this proposition holds we will now prove the Main Theorem.

Definition 2.21 (Subordinate to unstable manifolds). Assume the setting of our RDS. We say a partition η of $M \times \Omega^{\mathbb{Z}}$ is *subordinate to unstable manifolds* if for μ^*-a.e. $(x, \underline{\omega})$, $\eta(x, \underline{\omega})$ is a relatively compact subset of the global unstable manifold $W^u_{(x,\underline{\omega})}$. Here, $\eta(x, \underline{\omega})$ is the element of the partition containing $(x, \underline{\omega})$.

It can be shown that the definition of SRB measures is equivalent to the measure μ^* having absolutely continuous conditional measures on elements of a partition η that is subordinate to unstable manifolds (see [6] for further details).

Proof of the Main Theorem 2.19 assuming Proposition 2.20 holds. Following the proof of the Main Theorem in [2], the set-up is the following. Let η be a partition of $M \times \Omega^{\mathbb{Z}}$ subordinate to unstable manifolds. Let a function f_η be defined by

$$f_\eta : M \times \Omega^{\mathbb{Z}} \to (M \times \Omega^{\mathbb{Z}})/\eta \,, \qquad f_\eta(x, \underline{\omega}) \longmapsto \eta(x, \underline{\omega})$$

where $(M \times \Omega^{\mathbb{Z}})/\eta$ is equipped with the induced quotient topology on the partition. Using the function f_η we denote by μ^*_T the quotient measure of μ^* on $(M \times \Omega^{\mathbb{Z}})/\eta$. For μ^*_T-a.e. element α of η, let m_α be the Riemannian measure on α. Using the quotient measure, we define a measure ν on $M \times \Omega^{\mathbb{Z}}$ by letting

$$\nu(A) = \int m_\alpha(A \cap \alpha) \, d\mu^*_T(\alpha)$$

for every Borel set $A \subset M \times \Omega^{\mathbb{Z}}$. It can be shown that ν is sigma-finite. Using Lebesgue's decomposition theorem for sigma-finite measures μ^* and ν (see [12] for more information and a proof) we find new measures μ^*_{ac} and $\mu^*_\perp$ such that

$a)$ $\mu^* = \mu^*_{ac} + \mu^*_\perp$,

b) $\mu^*_{ac} \ll \nu$ and

c) $\mu^*_\perp$ and ν are singular.

The measure μ^*_{ac} is preserved by τ since μ^* is τ-invariant. We have $\mu^*_{ac} \ll \mu^*$ and (τ, μ^*) is ergodic. This implies that μ^*_{ac} is ergodic and since $M \times \Omega^{\mathbb{Z}}$ is τ-invariant we get that $\mu^*_{ac}(M \times \Omega^{\mathbb{Z}}) \in \{0, 1\}$. Assume that $\mu^*_{ac}(M \times \Omega^{\mathbb{Z}}) = 1$, then

$$0 \leq \mu^*_\perp(A) \leq \mu^*_\perp(M \times \Omega^{\mathbb{Z}}) = 0$$

for every Borel set $A \subset M \times \Omega^{\mathbb{Z}}$ and it follows that $\mu^* = \mu^*_{ac}$. By construction of ν this means that μ^* has absolute continuous conditional measures on almost every element of η and consequently, μ^* is SRB.
Assume via contradiction that this is not true and $\mu^* = \mu^*_\perp$. Since $\mu^*_\perp$ and ν are singular, there exists a Borel set $A \subset M \times \Omega^{\mathbb{Z}}$ with $\nu(A^c) = 0$ and $\mu^*_\perp(A) = 0$. By the definition of both measures and using the relation $\mu^* = \mu^*_\perp$ we get that $m_\alpha(A^c \cap \alpha) = 0$ and $\mu^*_\alpha(A \cap \alpha) = 0$ for μ^*_T-a.e. $\alpha \in \eta$. This implies that for μ^*_T-a.e., $\alpha \in \eta$ m_α and μ^*_α are mutually singular.
Note that we can choose by Proposition 2.20 a set $B \subset M \times \Omega^{\mathbb{Z}}$ with $\mu^*(B) > 0$ such that for every $(x, \underline{\omega}) \in B$, the unstable leave corresponding to $(x, \underline{\omega})$ has a nontrivial absolute continuous part. Hence, we can find a μ^*_T-positive measure set of $\alpha \in \eta$ where the μ^*_α have a positive absolute continuous component. This contradicts that m_α and μ^*_α are mutually singular and we conclude that μ^* is an SRB measure.

□

Chapter 3

New chart systems and iterated graph transforms

Recall from Chapter 2 that the measure μ^* projects onto $\mu \times \mathbb{P}^+$. While the sample measure for μ^* are $\mu_{\underline{\omega}}$, the sample measures for $\mu \times \mathbb{P}^+$ are given by μ. This is true since we have by Fubini for a continuous function $\phi : M \times \Omega^{\mathbb{Z}^+} \to \mathbb{R}$ the relation

$$\int \phi(x, \underline{\omega}^+)\, d(\mu \times \mathbb{P}^+) \overset{\text{Fubini}}{=} \int_{\Omega^{\mathbb{Z}^+}} \int_M \phi(x, \underline{\omega})\, d\mu\, d\mathbb{P}^+.$$

We will see in Chapter 4 that a good way to study $\mu^*(\Theta)$ for a subset $\Theta \in M \times \Omega^{\mathbb{Z}}$ is to project it to $M \times \Omega^{\mathbb{Z}^+}$ where the sample measures have a much simpler form.
Motivated by above, we are interested in using charts defined at $(\mu \times \mathbb{P}^+)$-a.e. $(x, \underline{\omega}^+)$ to push forward graphs from time $-n$ to time 0. We will look as preparation at pushing forward graphs starting at time 0 using functions of the form $f^n_{\underline{\omega}^+}$ for $n \in \mathbb{N}$. However, before working with $(x, \underline{\omega}^+)$ instead of $(x, \underline{\omega})$ we need to address a problem. Even though we have for $x \in M$ and $\underline{\omega} \in \Omega^{\mathbb{Z}}$ with future $\underline{\omega}^+ \in \Omega^{\mathbb{Z}^+}$ that

$$f^n_{\underline{\omega}^+}(x) = f^n_{\underline{\omega}}(x),$$

there is no intrinsic notion for the unstable set since it depends on the past (which we do not know). The paper [2] introduces in their Chapter 3 a way around by presenting the approach of adapting one-sided charts to the two-sided case. The construction roughly goes as follows. Denote by $\Gamma^+ \in M \times \Omega^{\mathbb{Z}^+}$ the projected set of Γ onto $M \times \Omega^{\mathbb{Z}^+}$.

1) Start with a point $(x, \underline{\omega}^+) \in \Gamma^+$.

2) Select a proper past $\underline{\omega}^- \in \Omega^{\mathbb{Z}^-}$ such that $(x, \underline{\omega}) \coloneqq (x, (\underline{\omega}^-, \underline{\omega}^+)) \in \Gamma$.

We discuss in Section 3.1 how this allocation of the past is best chosen since there are possible many suitable pasts.
Using the constructed $(x, \underline{\omega}) \in \Gamma$ we can use in a next step results from Chapter 2 to look at the push forward mechanism and graph transforms. We have briefly mentioned in Section 1.3 that we are going to push forward leaves of a partition of a localized set. Assume that the following steps are all well defined.

Take a small ball B around x and a foliation $\mathcal{P}$ of B. Denote by $\xi \in \mathcal{P}$ the leave of the foliation that goes trough ξ and assume that there is a Lipschitz continuous function $g : B^u(\delta l(x, \underline{\omega})^{-1}) \to \mathbb{R}^{cs}$ that describes the leave, in the sense that

$$\Phi_{(x,\underline{\omega})}(\text{graph}(g)) \subset \xi.$$

Note that it is much easier to study the function g than the set ξ since we do not know how ξ exactly looks like. In a next step we construct the so called *graph transform* $\mathcal{T}_{(x,\underline{\omega})}g$ of g. The graph transform is the mapping $\mathcal{T}_{(x,\underline{\omega})}g : B^u(\delta l(\tau(x, \underline{\omega}))^{-1}) \to \mathbb{R}^{cs}$ with

$$\tilde{f}_{(x,\underline{\omega})}\text{graph}(g) \supset \text{graph}(\mathcal{T}_{(x,\underline{\omega})}g).$$

In other words, it is the function that describes the one-time pushed forward ξ and the process can be iterated. By looking at the properties of the k-times graph transforms we can directly make assumptions about the k-times pushed forward leave ξ for $k \in \mathbb{N}$. Note that if we find for example that the graph transform aligns with the function that describes the local unstable leave, we can conclude that the pushed forward leave also resembles the unstable leave (which is exactly what we want to show).
We will see in this chapter that if we assume that our initial g is properly bounded, we can find a big enough k such that if we focus g on a smaller region, the k-times graph transform is also properly bounded.

3.1 Measurable selection of the past

We already briefly mentioned that for a given $(x, \underline{\omega}^+) \in \Gamma^+$ there are potentially many pasts that suffice the above construction. Which one do we choose? Since the allocation is not unique, the idea is to find a measurable function that allocates a past for us.
The next lemma shows that such a function exists since $\Gamma \in M \times \Omega^{\mathbb{Z}}$ is here a countable union of compact sets by construction. We will prove this lemma using an application of the Kuratowski-Ryll-Nardzewski measurable selection theorem. See Lemma 14 in [2] for the application and [13] for the measurable selection theorem.

Lemma 3.1. *Given any Borel set $\Theta \subset M \times \Omega^{\mathbb{Z}}$ that is a countable union of compact subsets, there is a Borel measurable function*

$$\hat{\underline{\omega}}^- : \Theta^+ \to \Omega^{\mathbb{Z}^-}$$

with the property that for any $(x, \underline{\omega}^+) \in \Theta^+$, we have

$$(x, \hat{\underline{\omega}}(x, \underline{\omega}^+)) \in \Theta, \qquad \textit{where} \qquad \hat{\underline{\omega}}(x, \underline{\omega}^+) := (\hat{\underline{\omega}}^-(x, \underline{\omega}^+), \underline{\omega}^+) \in \Omega^{\mathbb{Z}}.$$

Proof. Let us define $X = M \times \Omega^{\mathbb{Z}^+}, Y = \Omega^{\mathbb{Z}^-}$. Take any Borel set $\Theta \subset X \times Y$ which is a countable union of compact sets $\{G^i\}_{i\in\mathbb{N}}$, i.e.

$$\Theta = \bigcup_{i\in\mathbb{N}} G^i.$$

If Θ^+ is the projection from $X \times Y$ to X then

$$\Theta^+ = \bigcup_{i\in\mathbb{N}} G^i_X,$$

where G^i_X denotes the projection of G^i onto X. By applying Lemma 14 in [2] we get that for every $i \in \mathbb{N}$ we have a Borel measurable mapping $\hat{\underline{\omega}}^-_i : G^i_X \to Y$ with the property that $(\tilde{x}, \hat{\underline{\omega}}^-_i(\tilde{x})) \in G^i$ for all $\tilde{x} \in G^i_X$. Let us define for $i = 0$ the subset $H^0 = G^0$. For $i \in \mathbb{N}_{>0}$ we define $H^i = G^i \setminus \bigcup_{j=0}^{i-1} G^j$. It is clear that the $\{H^i\}_{i\in\mathbb{N}}$ are disjoint Borel subsets because the G^i are compact and hence Borel. We define the mapping

$$\hat{\underline{\omega}}^- : \Theta^+ \to \Omega^{\mathbb{Z}^-}, \qquad \hat{\underline{\omega}}^-(\tilde{x}) = \hat{\underline{\omega}}^-_i(\tilde{x}) \qquad \text{if } \tilde{x} \in H^i_X.$$

The function $\hat{\underline{\omega}}^-$ is well defined because the H^i_X are disjoint and the union covers Θ^+. Borel measurability follows from the fact that $\hat{\underline{\omega}}^-_i|_{H^i_X}$ is Borel measurable for every $i \in \mathbb{N}$. Indeed, for all $B \subset Y$ and B measurable we get

$$(\hat{\underline{\omega}}^-)^{-1}(B) = \bigcup_{i\in\mathbb{N}} (\hat{\underline{\omega}}^-_i|_{H^i_X})^{-1}(B) \in \mathcal{B}(\Theta^+),$$

where we used the fact that $(\hat{\underline{\omega}}^-_i|_{H^i_X})^{-1}(B) \in \mathcal{B}(\Theta^+)$ for all $i \in \mathbb{N}$. Hence the function $\hat{\underline{\omega}}^-$ is Borel measurable and by setting $\tilde{x} = (x, \underline{\omega}^+) \in \Theta^+$, we get

$$(x, \hat{\underline{\omega}}(x, \underline{\omega}^+)) \in \Theta, \qquad \text{where} \qquad \hat{\underline{\omega}}(x, \underline{\omega}^+) := (\hat{\underline{\omega}}^-(x, \underline{\omega}^+), \underline{\omega}^+) \in \Omega^{\mathbb{Z}}.$$

□

Remark 3.2. We must be careful when working with one-sided charts. The above constructed past is somewhat random (even though measurable) and there is no guarantee that $\hat{\underline{\omega}}(\tau^+(x, \underline{\omega}^+)) = \theta(\hat{\underline{\omega}})$ holds. Hence before pushing forward/backwards, we always have to first allocate a past to retain measurability and then do the push. Proposition 6 in [2] shows that the continuity of the unstable sets depends on the past. Even though our allocation from above is measurable it is not necessarily continuous. As a consequence, the continuity of the unstable sets is not guaranteed which makes it harder to bound the slopes of our graphs. We have seen in Remark 16 in [2] that $\hat{E}^u_{(x,\underline{\omega}^+)}$ and $E^{cs}_{(x',\underline{\omega}^+)}$ are uniformly separated when $(x, \underline{\omega}^+)$ and $(x', \underline{\omega}^+)$ are sufficiently close. Uniform separation means in this case that the angle between both subspaces can be bounded from below. Since this is not the case here, our graphs can be difficult to bound and the graph transforms can possibly have large slopes.

3.2 Adapted single graph transform

We are now going to look at a single graph-transform from time k to $k + 1$. The computations follow the same structure as in Section 2.3 in [2], with the difference of having possibly large slopes. For the remainder of this chapter, we define

$$\hat{\Phi}^{(k)} := \hat{\Phi}^{(k)}_{(x,\underline{\omega}^+)}, \qquad \hat{f}^{(k)} := \hat{f}^{(k)}_{(x,\underline{\omega}^+)}, \qquad \hat{l}^{(k)} := \hat{l}^{(k)}_{(x,\underline{\omega}^+)}.$$

The main observation in this section is stated in the next lemma. It shows that for an initially bounded graph g it can be difficult to control $\mathrm{Lip}(d\mathcal{T}^{(k)}g)$ as it is not only bounded by $\hat{l}^{(k)}$ but also $\mathrm{Lip}(dg)$ needs to be considered.
Based on the results in this paper we propose an alteration for item (ii') in Lemma 3.3 to the original version (Lemma 17 in [2]). Since g can be seen as the function at time k and the graph transform $\mathcal{T}^{(k)}g$ as the transformed function at time $k+1$, we changed item (ii') to $\mathrm{graph}(\mathcal{T}^{(k)}g) \subset B(\delta(\hat{l}^{(k+1)})^{-1}$ (item (ii') was originally given by $\mathrm{graph}(\mathcal{T}^{(k)}g) \subset B(\delta(\hat{l}^{(k)})^{-1}$). This alteration holds in the scope of the below proof and the rest of the paper.

Lemma 3.3. *For any $K_0 > 0$, there exist constants $r_1 = r_1(\lambda, \delta_0, K_0) \in (0,1), C_1 = C_1(\lambda, \delta_0, K_0), C_2 = C_2(\lambda, \delta_0, K_0)$ such that the following holds when $\delta < r_1$. Let $k \geq 0, \rho \in (0, K_0^{-1}]$, and let*

$$g : B^u(\rho\delta(\hat{l}^{(k)})^{-1}) \to \mathbb{R}^{cs}$$

be a $C^{1+\mathrm{Lip}}$ map from which (i) $g(0) = 0$, (ii) $\mathrm{graph}(g) \subset B(\delta(\hat{l}^{(k)})^{-1})$ and (iii) $\mathrm{Lip}(g) \leq K_0$. Then the graph transform

$$\mathcal{T}^{(k)}g : B^u(\rho'\delta(\hat{l}^{(k+1)})^{-1}) \to \mathbb{R}^{cs} \qquad \textit{with} \qquad \rho' = \min\{\rho e^{\frac{\lambda}{2}}, 1\}$$

is defined, and is a $C^{1+\mathrm{Lip}}$ map for which

$$\begin{aligned}
&(i') \;\; \mathcal{T}^k g(0) = 0,\\
&(ii') \;\; \mathrm{graph}(\mathcal{T}^{(k)}g) \subset B(\delta(\hat{l}^{(k+1)})^{-1}),\\
&(iii') \;\; \mathrm{Lip}(\mathcal{T}^{(k)}g) \leq K_0 e^{-\frac{\lambda}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&(iv')\,(a) \;\; |(d\mathcal{T}^{(k)}g)_0| \leq e^{-\frac{\lambda}{2}}|(dg)_0|,\\
&(iv')\,(b) \;\; \mathrm{Lip}(d\mathcal{T}^{(k)}g) \leq C_1\hat{l}^{(k)} + C_2\mathrm{Lip}(dg).
\end{aligned}$$

The proof for $(iv')(b)$ is given in [2] and will not be repeated here. We will instead focus on items (i'), (ii') and (iii'). The following lemma will present itself useful.

Lemma 3.4. *The map $\hat{f}_{(x,\underline{\omega})}$ can be written in the form*

$$\hat{f}_{(x,\underline{\omega})}(u,v) = (A_{(x,\underline{\omega})}u + g^{(1)}_{(x,\underline{\omega})}(u,v), B_{(x,\underline{\omega})}v + g^{(2)}_{(x,\underline{\omega})}(u,v))$$

where $g^{(1)}_{(x,\underline{\omega})}(0,0) = dg^{(1)}_{(x,\underline{\omega})}(0,0) = g^{(2)}_{(x,\underline{\omega})}(0,0) = dg^{(2)}_{(x,\underline{\omega})}(0,0) = 0$, $(u,v) \in \mathbb{R}^u \oplus \mathbb{R}^{cs}$ and the following holds

$$\begin{aligned}
\|A_{(x,\underline{\omega})}\| &\geq e^{\lambda}\\
\|B_{(x,\underline{\omega})}\| &\leq e^{\delta_0}.
\end{aligned}$$

Furthermore, it holds

$$\|(dg^{(1)}_{(x,\underline{\omega})}, dg^{(2)}_{(x,\underline{\omega})})_{z_1} - (dg^{(1)}_{(x,\underline{\omega})}, dg^{(2)}_{(x,\underline{\omega})})_{z_2}\| \leq \delta\|z_1 - z_2\|.$$

It can be proven that the graph transform is given by

$$\hat{g} = \mathcal{T}^{(k)} g \coloneqq \pi^{cs} \circ \hat{f}^{(k)} \circ (Id \times g) \circ \phi^{-1}$$

with

$$\phi : B^u(\rho\delta\hat{l}^{-1}) \to \mathbb{R}^u, \qquad \phi(u) = \pi^u \circ \hat{f}^{(k)}(u, g(u)).$$

For the remainder of this section, we are going to write $\hat{f} = \hat{f}^{(k)}$ and $\hat{l} = \hat{l}^{(k)}$. The first step is to find a suitable choice for r_1.
We claim that the constant $r_1 = r_1(\lambda, \delta_0, K_0)$ can be taken small enough such that for every $\delta \in (0, r_1)$ we have for all $z \in B(\delta\hat{l}^{-1})$ that

$$d\hat{f}_z \mathcal{C}(K_0) \subset \mathcal{C}(K_0 e^{-\frac{\lambda}{2}}),$$

with $\mathcal{C}$ being the invariant cone defined as

$$\mathcal{C} = \mathcal{C}(K_0) = \{u_1 + u_2 : u_1 \in \mathbb{R}^u, u_2 \in \mathbb{R}^{cs}, \text{and } |v| \leq K_0|u|\}.$$

Proof of claim. Take $u = u_1 + u_2 \in \mathcal{C} \subset \mathbb{R}^u \oplus \mathbb{R}^{cs}$ and $d\hat{f}_z(u) = w_1 + w_2 \in \mathbb{R}^u \oplus \mathbb{R}^{cs}$ for $z \in B(\delta\hat{l}^{-1})$. Then

$$\begin{aligned}
|w_1| &= |\pi^u \circ d\hat{f}_z(u)| = |\pi^u \circ d\hat{f}_z(u) + \pi^u \circ d\hat{f}_0(u) - \pi^u \circ d\hat{f}_0(u)| \\
&\geq |\pi^u \circ d\hat{f}_0(u)| - |\pi^u \circ (d\hat{f}_0(u) - d\hat{f}_z(u))| \\
&= |Au_1| - |\pi^u \circ (d\hat{f}_0(u) - d\hat{f}_z(u))| \\
&\geq e^{\lambda}|u_1| - \delta|u_1 + u_2| \\
&= e^{\lambda}|u_1| - \delta \max\{|u_1|, |u_2|\} \geq (e^{\frac{\lambda}{2}} - \delta \max\{1, K_0\})|u_1|
\end{aligned}$$

by the definition of the norm $|u_1 + u_2| = \max\{|u_1|, |u_2|\}$ and because $|d\hat{f}_0 - d\hat{f}_z| \leq |z|\hat{l} \leq \delta$ for $z \in B(\delta\hat{l}^{-1})$. Similarly, we get for w_2

$$\begin{aligned}
|w_2| &= |\pi^{cs} \circ d\hat{f}_z(u)| \leq |\pi^{cs} \circ d\hat{f}_0(u)| + |\pi^{cs} \circ (d\hat{f}_0(u) - d\hat{f}_z(u))| \\
&= |Bu_2| + |\pi^{cs} \circ (d\hat{f}_0(u) - d\hat{f}_z(u))| \\
&\leq e^{\delta_0}|u_2| + \delta \max\{|u_1|, |u_2|\} \leq e^{\delta_0}|u_2| + \delta \max\{1, K_0\}|u_1|.
\end{aligned}$$

This leads to the estimate

$$\begin{aligned}
|w_2| &\leq e^{\delta_0}|u_2| + \delta \max\{1, K_0\}|u_1| \leq (e^{\delta_0} K_0 + \delta \max\{1, K_0\})|u_1| \\
&\leq \frac{e^{\delta_0} K_0 + \delta \max\{1, K_0\}}{e^{\frac{\lambda}{2}} - \delta \max\{1, K_0\}}|w_1|.
\end{aligned}$$

Because $\delta_0 \ll \lambda$ we can choose $r_1 > 0$ small enough such that for $\delta \in (0, r_1)$

$$\frac{e^{\delta_0} K_0 + \delta \max\{1, K_0\}}{e^{\frac{\lambda}{2}} - \delta \max\{1, K_0\}} \leq e^{-\frac{\lambda}{2}} K_0 \tag{3.1}$$

and $w_1 + w_2 \in \mathcal{C}(e^{-\frac{\lambda}{2}} K_0)$. □

Let us now show that $\hat{g}$ exists and is well defined.

Proof of existence and well definedness. Note that from the construction of $\hat{g}$, showing the existence and well definedness of $\hat{g}$ is the same as showing that ϕ is invertible on $\phi(B^u(\rho\delta\hat{l}^{-1}))$. This is the same as saying that ϕ is injective.
Take $u, \tilde{u} \in B^u(\rho\delta\hat{l}^{-1})$ with $\phi(u_1) = \phi(u_2)$. Then

$$\pi^u \circ \hat{f}(u, g(u)) = \pi^u \circ \hat{f}(\tilde{u}, g(\tilde{u})),$$

which is equivalent to the following, using Lemma 3.4.

$$\begin{aligned} 0 &= |Au - A\tilde{u} + g^{(1)}(u, g(u)) - g^{(1)}(\tilde{u}, g(\tilde{u}))| \\ &\geq |Au - A\tilde{u}| - \delta\, |(u, g(u)) - (\tilde{u}, g(\tilde{u}))| \\ &\geq (e^{\lambda} - \delta K_0)|u - \tilde{u}| \\ &\geq |u - \tilde{u}| \end{aligned}$$

This implies $|u - \tilde{u}| = 0$ and $u = \tilde{u}$. We conclude injectivity and the existence of an inverse of $\phi : B^u(\rho\delta\hat{l}^{-1}) \to \phi(B^u(\rho\delta\hat{l}^{(-1)}))$. □

We continue with showing the properties of the graph transform.

Proof of (i'). We have the relations $g(0) = 0$, $\hat{f}(0) = 0$ as well as $\phi(0) = \pi^u(0) = 0$ which is the same as $\phi(0)^{-1} = 0$. We conclude

$$\hat{g}(0) = \pi^{cs} \circ \hat{f} \circ (Id \times g) \circ \phi(0)^{-1} = \pi^{cs} \circ \hat{f} \circ (Id \times g)(0) = \pi^{cs}(0) = 0.$$

□

As (ii') is a direct consequence of (iii'), we first prove (iii').

Proof of (iii'). Let $\hat{u}_2, \hat{u}_2 \in B^u(\rho\delta(\hat{l}^{(k+1)})^{-1})$ and define $u_1 \coloneqq \phi^{-1}(\hat{u}_1), u_2 \coloneqq \phi^{-1}(\hat{u}_2)$. We get

$$\begin{aligned} |\hat{g}(\hat{u}_1) - \hat{g}(\hat{u}_2)| &= |\pi^{cs} \circ \hat{f} \circ (Id \times g)(u_1) - \circ\hat{f} \circ (Id \times g)(u_2)| \\ &= |\pi^{cs} \circ (\hat{f}(u_1 + g(u_1)) - \hat{f}(u_2 + g(u_2)))| \end{aligned}$$

for $u_i \in \mathbb{R}^u, g(u_i) \in \mathbb{R}^{cs}$ and $|g(u_i)| \leq K_0|u_i|, i = 1, 2$. Applying the mean value theorem we find $\xi \in B(\delta\hat{l}^{-1})$ such that

$$|\hat{g}(\hat{u}_1) - \hat{g}(\hat{u}_2)| \leq |\pi^{cs} \circ d\hat{f}_{\xi}(u_1 - u_2 + g(u_1) - g(u_2))|.$$

From $\mathrm{Lip}(g) \leq K_0$ we get $|g(u_1) - g(u_2)| \leq K_0|u_1 - u_2|$ and so $u_1 - u_2 + g(u_1) - g(u_2) \in \mathcal{C}(K_0)$. We rewrite $d\hat{f}_{\xi}(u_1 - u_2 + g(u_1) - g(u_2)) = w_1 + w_2$ with $w_1 \in \mathbb{R}^u, w_2 \in \mathbb{R}^{cs}$ and see that $(w_1 + w_2) \in \mathcal{C}(e^{-\frac{\lambda}{2}}K_0)$ by construction. Then

$$\begin{aligned} |\hat{g}(\hat{u}_1) - \hat{g}(\hat{u}_2)| &= |\pi^{cs} \circ (w_1 + w_2)| = |w_2| \\ &\leq e^{\delta_0}|g(u_1) - g(u_2)| + \delta \max\{1, K_0\}|u_1 - u_2| \\ &\leq e^{\delta_0}K_0|u_1 - u_2| + \delta \max\{1, K_0\}|u_1 - u_2| \end{aligned}$$

$$\leq (e^{\delta_0}e^{-\frac{\lambda}{2}}K_0 + \delta e^{-\frac{\lambda}{2}}\max\{1, K_0\})|\hat{u}_1 - \hat{u}_2|$$

In the last line, we used that $|d\phi_{\hat{u}}^{-1}| \leq e^{-\frac{\lambda}{2}}$ for all $\hat{u} \in B^u(\rho\delta(\hat{l}^{(k+1)})^{-1})$. We conclude

$$\begin{aligned}\mathrm{Lip}(\mathcal{T}^k g) &\leq e^{\delta_0}e^{-\frac{\lambda}{2}}K_0 + \delta e^{-\frac{\lambda}{2}}\max\{1, K_0\}\\ &= e^{-\frac{\lambda}{2}}(e^{\delta_0}K_0 + \delta\max\{1, K_0\})\\ &\leq e^{-\frac{\lambda}{2}}(e^{-\frac{\lambda}{2}}K_0(e^{\frac{\lambda}{2}} - \delta\max\{1, K_0\}))\\ &\leq K_0 e^{-\frac{\lambda}{2}},\end{aligned}$$

where we used in the second inequality the estimate (6.1). □

Proof of (ii'). From (iii') we get the estimate $\mathrm{Lip}(\mathcal{T}^{(k)}g) \leq K_0 e^{-\frac{\lambda}{2}}$. Using $\hat{g}(0) = 0$ we have for $u \in B^u(\rho'\delta(\hat{l}^{(k+1)})^{-1})$

$$|\hat{g}(u)| = |\hat{g}(u) - \hat{g}(0)| \leq K_0 e^{-\frac{\lambda}{2}}|u - 0| = K_0 e^{-\frac{\lambda}{2}}|u|.$$

This implies

$$|\hat{g}(u)| \leq K_0 e^{-\frac{\lambda}{2}}K_0^{-1}e^{\frac{\lambda}{2}}\delta(\hat{l}^{(k+1)})^{-1} \leq \delta(\hat{l}^{(k+1)})^{-1}.$$

□

Remark 3.5. The constants C_1 and C_2 are given by

$$\begin{aligned}C_1 &= e^{-\lambda}\max\{1, K_0\}^2 + (e^{\delta_0}K_0 + \delta\max\{1, K_0\})\max\{1, K_0\}^2 e^{-\frac{3\lambda}{2}}\\ C_2 &= e^{-\lambda}(e^{\delta_0} + \delta) + (e^{\delta_0}K_0 + \delta\max\{1, K_0\})(e^{\delta_0} + \delta)e^{-\frac{3\lambda}{2}}.\end{aligned}$$

In general, $C_2 > 0$ as K_0 can be arbitrarily big. This makes it hard to bound $\mathrm{Lip}(d\mathcal{T}^{(k)}g)$ when we wish to iterate the graph transform again. We see however that we can fix K_0 small enough, denoted by $\bar{K}_0$, such that $\bar{K}_0 < \frac{1}{10}$ and $C_2(\bar{K}_0)e^{\delta_2} < 1$ and write

$$\bar{C}_i = C_i(\bar{K}_0).$$

We further take r_1 small enough, denoted by $\bar{r}_1$, such that the cones $\mathcal{C}(\bar{K}_0)$ are invariant under $d\hat{f}_z$ on $B(\bar{r}_1(\hat{l}^{(k)})^{-1})$.

3.3 Iteration of graph transforms

Having done one iteration, we are interested in what happens when we do it k times. The next proposition shows that given a graph g we can find a big enough $\bar{k}$ and a small enough $\bar{r}_1$ such that the $\bar{k}$-times graph transforms $g_{\bar{k}}$ can be bounded nicely for all $\bar{k} \geq k$ on a small ball dependent on $\bar{r}_1$.

Proposition 3.6. *Given K_0, λ_0, δ_0, there exists $\bar{C} \geq 1$ (independent of K_0), $m_0 = m_0(K_0)$ and $\bar{r}_0 = \bar{r}_0(K_0, m_0) > 0$ for which the following hold. Let $r_0 \leq \bar{r}_0$. Then there exists $m_1 \in \mathbb{Z}^+$ depending on $\mathrm{Lip}(dg)$ in addition to the constants above with the following properties. Let*

$$g : B^u(r_0(\hat{l}^{(0)})^{-1}) \to \mathbb{R}^{cs}$$

be a $C^{1+\mathrm{Lip}}$ *map with* (i) $g(0)=0$ *and* (ii) $\mathrm{Lip}(g)\le K_0$. *For* $k>0$ *let*

$$g_k=\mathcal{T}^{(k-1)}\circ\cdots\circ\mathcal{T}^{(0)}g$$

be the graph transforms of g *on* $B(d_k(\hat{l}^{(k)})^{-1})$ *with* $d_k=r_0$ *for* $k\le m_0$ *and* $d_k=\bar{r}_1$ *for* $k>m_0$. *Then for all* $k\ge m_0+m_1$,

$$g_k : B^u(\bar{r}_1(\hat{l})^{-1})\to\mathbb{R}^{cs}$$

is defined and satisfies

$$\mathrm{Lip}(g_k)\le\bar{K}_0,\qquad |(dg_k)_0|\le e^{-\frac{k\lambda}{2}}|(dg)_0|,\qquad and\qquad \mathrm{Lip}(dg_k)\le\bar{C}\hat{l}^{(k)}.$$

Proof. The proof consists of two parts. In the first part, we construct an m_0 and $\bar{r}_0$ such that $\mathrm{Lip}(g_k)\le\bar{K}_0$ and $|(dg_k)_0|\le e^{-\frac{k\lambda}{2}}|(dg_k)_0|$ for $k\ge m_0$. In the second part, we construct an m_1 such that $\mathrm{Lip}(dg_k)\le\bar{C}\hat{l}^{(k)}$ for $k\ge m_0+m_1$.

1) If $K_0\le\bar{K}_0$ we set $m_0=1$. We now assume $K_0>\bar{K}_0$ and set m_0 such that $K_0e^{-\frac{m_0\lambda}{2}}<\bar{K}_0$. Define

$$K_0^{(k)}:=K_0e^{-\frac{k\lambda}{2}}.$$

Using the same procedure as in the proof of Lemma 3.3 and Remark 3.5, we find for every $0\le k\le m_0$ an $\bar{r}_0^{(k)}>0$ (dependant on K_0) such that for $z\in B(\bar{r}_0^{(k)}(\hat{l}^{(k)})^{-1})$ we have $(d\hat{f}^{(k)})_z\mathcal{C}(K_0^{(k)})\subset\mathcal{C}(K_0e^{-\frac{\lambda}{2}})$. Set

$$\bar{r}_0=\min\{\ \bar{r}_0^{(k)}\mid 0\le k\le m_0\}.$$

Then $\bar{r}_0>0$ and for $0\le k\le m_0$ and $z\in B(\bar{r}_0(\hat{l}^{(k)})^{-1})$ we have that

$$(d\hat{f}^{(k)})_z\mathcal{C}(K_0e^{-\frac{k\lambda}{2}})\subset\mathcal{C}(K_0e^{-\frac{(k+1)\lambda}{2}}).$$

Since $g : B^u(r_0(\hat{l}^{(0)})^{-1})\to\mathbb{R}^{cs}$, it follows from Lemma 3.3 that for $r_0\le\bar{r}_0$,

$$g_1 : B^u(\min\{e^{\frac{\lambda}{2}},1\}r_0(\hat{l}^{(k+1)})^{-1})\to\mathbb{R}^{cs}.$$

Because λ is the positive lyapunov exponent we have $e^{\frac{\lambda}{2}}\ge 1$ and hence

$$g_1 : B^u(r_0\hat{l}^{(1)})^{-1})\to\mathbb{R}^{cs}$$

and

$$g_1(0)=0\qquad \mathrm{graph}(g_1)\subset B(r_0(\hat{l}^{(1)})^{-1})\qquad \mathrm{Lip}(g_1)\le K_0e^{-\frac{\lambda}{2}}.$$

By construction of $\bar{r}_0$, we can reapply Lemma 3.3 to g_1 to get g_2. By induction, it follows for all $0\le k\le m_0$,

$$g_k : B^u(r_0(\hat{l}^{(k)})^{-1})\to\mathbb{R}^{cs}.$$

Especially for m_0 we get $\mathrm{Lip}(g_{m_0})\le K_0e^{-\frac{m_0\lambda}{2}}<\bar{K}_0$. By construction, the invariant cones $\mathcal{C}(\bar{K}_0)$ are preserved on $B(\bar{r}_1(\hat{l}^{(k)})^{-1})$, which implies $\mathrm{Lip}(g_k)\le\bar{K}_0$ for $k>m_0$.

By the same argumentation, we get also $|(dg_k)_0| \leq e^{-\frac{k\lambda}{2}}|(dg)_0|$ for all k.

2) Since there is an expansion in $\mathbb{R}^u$ under $\hat{f}$ we define $m_1^{'}$ big enough such that $g_{m_0+m_1^{'}}$ is defined on all of $B^u(\bar{r}_1(\hat{l}^{m_0+m_1^{'}})^{-1})$. Furthermore, $\bar{K}_0$ is defined such that $\bar{C}_2 e^{\delta_2} < 1$ by Remark 3.5. The $\mathrm{Lip}(dg_k)$ of the first $m_0 - 1$ transforms are not necessarily bounded by $\bar{K}_0$ and $\mathrm{Lip}(dg_{m_0})$ can be arbitrarily big. We start our iteration in m_0 and set $a = \mathrm{Lip}(dg_{m_0})$. Applying repeatedly Lemma 3.3 gives the inequality

$$\mathrm{Lip}(dg_{m_0+i}) \leq \bar{C}_1\big((\hat{l}^{m_0+i}) + \bar{C}_2(\hat{l}^{m_0+i-1}) + \cdots + \bar{C}_2^i a\big).$$

Using the relation $e^{-\delta_2} \leq \frac{l\circ\tau}{l} \leq e^{\delta_2}$ (recall from Proposition 2.11),

$$\mathrm{Lip}(dg_{m_0+i}) \leq \bar{C}_1\big((\hat{l}^{m_0+i}) + \bar{C}_2 e^{\delta_2}(\hat{l}^{m_0+i}) + \cdots + \bar{C}_2^{i-1}(e^{\delta_2})^{i-1}(\hat{l}^{m_0+i}) + \bar{C}_2^i a\big).$$

Because $C_2 < 1$ by construction we define the converging series

$$\bar{C} = 2\bar{C}_1 \sum_{i=0}^{\infty} (\bar{C}_2 e^{\delta_2})^i.$$

Using $l : \Gamma \to [1, \infty)$ we get

$$\mathrm{Lip}(dg_{m_0+i}) \leq \frac{1}{2}\bar{C}(\hat{l}^{(m_0+i)}) + \bar{C}_1\bar{C}_2^i a(\hat{l}^{(m_0+i)}) \leq \big(\frac{1}{2}\bar{C} + \bar{C}_1\bar{C}_2^i a\big)(\hat{l}^{(m_0+i)}).$$

Choosing m_1 big enough such that $\bar{C}_1\bar{C}_2^{m_1} a < \frac{1}{2}\bar{C}$ and $m_1 \geq m_1'$ implies

$$\mathrm{Lip}(dg_k) \leq \bar{C}\hat{l}^{(k)}, \qquad \text{for all} \qquad k \geq m_0 + m_1.$$

□

Next, we present the distortion estimate for positive times. The lemma corresponds to Lemma 19 in [2].

Lemma 3.7. *Write $a_0 = \mathrm{Lip}(dg)$. Then for any $k \geq m_0 + m_1$, there exists a constant $D = D(K_0, a_0, r_0, \hat{l}^{(0)}, \hat{l}^{(k)})$ with the following property. Write $\gamma_j = \hat{\Phi}^{(j)}(\mathrm{graph}(g_j))$ for $0 \leq j \leq k$ and let $p_1, p_2 \in \gamma_k$. Then,*

$$\left|\log \frac{\det(df^k_{\underline{\omega}^+}|_{T\gamma_0})((f_{\underline{\omega}^+})^{-1}p_1)}{\det(df^k_{\underline{\omega}^+}|_{T\gamma_0})((f_{\underline{\omega}^+})^{-1}p_2)}\right| \leq D.$$

Note that D does not depend on k, except through the value of $\hat{l}^{(k)}$.

The proof in [2] is straightforward and uses a similar distortion estimate to Lemma 9 in [2] considering positive time. We will motivate here an explanation of the main inequality in the proof in [2]. Set $p_k^i = p_i$ and $p_0^i = (f^k_{\underline{\omega}^+})^{-1}(p_k^i)$ and for $0 \leq j < k$ define $p_j^i = f^j_{\underline{\omega}^+}p_0^i \in \gamma_j$. We have

$$\left|\log\frac{\det(df^k_{\underline{\omega}^+}|_{T\gamma_0})(p^1_0)}{\det(df^k_{\underline{\omega}^+}|_{T\gamma_0})(p^2_0)}\right| \le \left|\log\frac{\det(df^{m_0+m_1}_{\underline{\omega}^+}|_{T\gamma_0})(p^1_0)}{\det(df^{m_0+m_1}_{\underline{\omega}^+}|_{T\gamma_0})(p^2_0)}\right| + \left|\log\frac{\det(df^{k-(m_0+m_1)}_{\underline{\omega}^+}|_{T\gamma_{m_0+m_1}})(p^1_{m_0+m_1})}{\det(df^{k-(m_0+m_1)}_{\underline{\omega}^+}|_{T\gamma_{m_0+m_1}})(p^2_{m_0+m_1})}\right|.$$

Rewriting $df^k_{\underline{\omega}^+}$ and using the chain rule we get

$$\begin{aligned} df^k_{\underline{\omega}^+}(p^i_0) &= d(f^{k-(m_0+m_1)}_{(\theta^{m_0+m_1})^+(\underline{\omega}^+)} \circ f^{m_0+m_1}_{\underline{\omega}^+})(p^i_0) \\ &= df^{k-(m_0+m_1)}_{(\theta^{m_0+m_1})^+(\underline{\omega}^+)}(p^i_{m_0+m_1}) \circ df^{m_0+m_1}_{\underline{\omega}^+}(p^i_0). \end{aligned}$$

Because $p^i_j \in \gamma_j$ it follows that $\det(df^{k-(m_0+m_1)}_{\underline{\omega}^+}|_{T\gamma_{m_0+m_1}})(p^i_{m_0+m_1})$ is well defined. Using the standard determinant and logarithmic rules, the inequality follows.

We finish this chapter by stating sufficient conditions for switching axes. Let the mapping $\check{g}_k : \mathrm{Dom}(\check{g}_k) \subset \hat{E}^{u,(k)} \to E^{cs,(k)}$ be given by $\check{g}_k = \hat{L}^{(k)}_{x,\underline{\omega}^+} \circ g_k \circ (\hat{L}^{(k)}_{(x,\underline{\omega}^+)})^{-1}$.

Lemma 3.8. *For any $\hat{l} > 1$ there exists $r_3 = r_3(\hat{l})$ with the following properties. Let $k \ge m_0 + m_1$ and let $r_3 = r_3(\hat{l}^{(k)})$. Then*

- (a) $\mathrm{Dom}(\check{g}_k)$ *contains* $\hat{E}^{u,(k)}(r_3)$, *and*
- (b) *if* $|(dg_k)_0| \le (20\check{l}^{(k)})^{-1}$ *holds, then we have* $\mathrm{Lip}(\check{g}_k) \le 1/10$ *on* $\hat{E}^{u,(k)}(r_3)$.

Chapter 4

Source and target sets and construction of u-graphs

In light of Proposition 2.20 we only need to push forward a localized section of μ to show the SRB property. This chapter introduces the so called "source set" and "target set" as well as a disintegration of μ in the "source set" onto the so called u-graphs. The u-graphs will be later used to approximate the unstable leaves by pushing them forward. We will set in Chapter 6 that to find an upper and lower bound for the densities from Proposition 2.20, we need to control the distortion along the graph transforms of the u-graphs. The idea is to take small enough source and target sets such that we can control this distortion but also to take them big enough such that the set of points that go from the source set to the target set has positive measure.

4.1 Uniformity sets of $\mu^n_{\underline{\omega}}$-typical points

Take $l_0 > 1$ and fix a compact subset $\Theta_0 \subset \Gamma_{l_0}$ as well as $n \in \mathbb{Z}^+$. We define

$$\Theta_n = \Theta_0 \cap \tau^{-n}\Theta_0$$

to be the set of all points $(x, \underline{\omega})$ which are in the uniformity set Θ_0 at times 0 and n. The skew product map τ is a diffeomorphism so τ^{-1} is continuous. Because Θ_0 is compact, the set $\tau^{-n}\Theta_0$ is also compact and we can use Lemma 3.1 in Chapter 3 to define a measurable selection

$$\hat{\underline{\omega}}^n : \Theta_n^+ \to \Omega^{\mathbb{Z}^-}$$

with $(x, (\hat{\underline{\omega}}^n(x, \underline{\omega}), \underline{\omega}^+)) \in \Theta_n$ for every $(x, \underline{\omega}^+) \in \Theta_n^+$.
Take $\underline{\omega} \in \Omega^{\mathbb{Z}}$. We define $x_{-n} = f^{-n}_{\underline{\omega}} x$ as the pushed back point x at time $-n$. This lets us define a subset of $\mu^n_{\underline{\omega}}$-typical points

$$M^n_{\underline{\omega}} = \{x \in M : (x_{-n}, (\theta^{-n}\underline{\omega})^+) \in \Theta_n^+\}$$

and

$$M_{\underline{\omega}} = \{x \in M : (x, \underline{\omega}) \in \Theta_0\}.$$

The next lemma shows that we have in some sense a convergence of $M^n_{\underline{\omega}}$ to $M_{\underline{\omega}}$ as $n \to \infty$.

Lemma 4.1. *For any $\delta > 0$, there exists a constant $N_0 = N_0(\delta) \in \mathbb{N}$ such that for any $n \geq N_0$, we have that $M^n_{\underline{\omega}} \subset \mathcal{N}_\delta(M_{\underline{\omega}})$. In particular,*

$$\lim_{n\to\infty} \sup_{x\in M^n_{\underline{\omega}}} \operatorname{dist}(x, M_{\underline{\omega}}) = 0.$$

Before we can prove the lemma we need the following auxiliary lemma.

Lemma 4.2. *For any sequence $\{x^n\} \subset M$ converging to a point $x \in M$ for which $x^n \in M^n_{\underline{\omega}}$ for all n, we have that $x \in M_{\underline{\omega}}$.*

Proof. Let us take a sequence $\{x^n\} \subset M$ converging to a point $x \in M$ for which $x^n \in M^n_{\underline{\omega}}$ for all n. For $n \geq 1$ write $x^n_{-n} = f^{-n}_{\underline{\omega}} x^n$ and define $\check{\underline{\omega}}_n \in \Omega^{\mathbb{Z}}$ as

$$\check{\underline{\omega}}_n = \theta^n\big(\hat{\underline{\omega}}^n(x^n_{-n}, (\theta^{-n}\underline{\omega})^+), (\theta^{-n}\underline{\omega})^+\big).$$

We have by construction of $\hat{\underline{\omega}}$ that $(x^n_{-n}, \theta^{-n}(\check{\underline{\omega}}^n)) \in \Theta_n$. This implies that

$$(x^n, \check{\underline{\omega}}_n) = \tau^n(x^n_{-n}, \theta^{-n}(\check{\underline{\omega}}^n)) \in \tau^n\Theta_n \subset \Theta_0.$$

We further see that $\check{\underline{\omega}}_n \to \underline{\omega}$ as $n \to \infty$. Hence, $(x^n, \check{\underline{\omega}}_n) \in \Theta_0$ for all $n \in \mathbb{N}$ and $(x^n, \check{\underline{\omega}}_n) \to (x, \underline{\omega})$ as $n \to \infty$. Because Θ_0 is compact it follows that $(x, \underline{\omega}) \in \Theta_0$ and we conclude that $x \in M_{\underline{\omega}}$. □

Proof of Lemma 4.1. $M_{\underline{\omega}}$ can be seen as the projection of the fiber $M \times \{\underline{\omega}\}$ restricted to Θ_0 onto M. Because $M \times \{\underline{\omega}\}|_{\Theta_0}$ is compact and the projection continuous, $M_{\underline{\omega}}$ is compact. Assume by contradiction that there exists a $\delta > 0$ and there does not exist an $N \in \mathbb{N}$ such that we have for any $n \geq N_0$ that $M^n_{\underline{\omega}} \subset \mathcal{N}_\delta(M_{\underline{\omega}})$. This lets us construct an infinite sequence $\{x^{n(i)}\}_{i\in\mathbb{N}} \subset M$ with $x^{n(i)} \in M^{n(i)}_{\underline{\omega}} \setminus \mathcal{N}_\delta(M_{\underline{\omega}})$, for all $i \in \mathbb{N}$. Because M is compact, $\{x^{n(i)}\}$ has a convergent subsequence $\{x^{\tilde{n}(i)}\}_{i\in\mathbb{N}}$ with limit $x \in M$. By Lemma 4.2 holds $x \in M_{\underline{\omega}}$ but

$$0 = \operatorname{dist}(x, M_{\underline{\omega}}) = \lim_{i\to\infty} \operatorname{dist}(x^{\tilde{n}(i)}, M_{\underline{\omega}}) \geq \delta > 0$$

gives a contradiction and the proof follows. □

4.2 Accumulating $\mu^n_{\underline{\omega}}$-mass

We are interested in this chapter in finding a set of $\underline{\omega} \in \Omega^{\mathbb{Z}}$ such that $M^n_{\underline{\omega}}$ has sufficiently large $\mu^n_{\underline{\omega}}$-mass for infinitely many n. These $\underline{\omega}$ are then used to construct the target and source set. The set-up for this section is the following.

Set $\beta_0 > 0$ to be very small and fix $l_0 > 1$ sufficiently large such that $\mu^*\Gamma_{l_0} \geq 1 - \beta_0/3$. Because every measurable set can be approximated by compact sets, we can fix a compact set $\Theta_0 \in \Gamma_{l_0}$ with $\mu^*(\Theta_0) \geq 1 - \beta_0/2$ so that $\mu^*(\Theta_n) \geq 1 - \beta_0$ for each $n \geq 0$. The last inequality holds because μ^* is a τ-invariant probability measure and

$$1 \geq \mu^*(\Theta_0 \cup \tau^{-n}\Theta_0) = \mu^*(\Theta_0) + \mu^*(\tau^{-n}\Theta_0) - \mu^*(\Theta_0 \cap \tau^{-n}\Theta_0)$$

$$\begin{aligned}&\geq (1-\beta_0/2) + /(1-\beta_0/2) - \mu^*(\Theta_n)\\ &= 2 - \beta_0 - \mu^*(\Theta_n).\end{aligned}$$

Hence, we get

$$\mu^*(\Theta_n) \geq 1 - \beta_0.$$

Lemma 4.3. *Let $\mathfrak{c} > 1$. For each $n \geq 1$ define*

$$\mathcal{G}^{(n)} = \{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu^n_{\underline{\omega}}(M^n_{\underline{\omega}}) \geq 1 - \mathfrak{c}\beta_0\},$$

and set $\mathcal{G} = \lim_{n\to\infty} \mathcal{G}^{(n)} = \cap_{N\geq 1} \cup_{n\geq N} \mathcal{G}^{(n)}$. Then we have

$$\mathbb{P}(\mathcal{G}) \geq \frac{\mathfrak{c}-1}{\mathfrak{c}}.$$

Proof. The main trick in this proof is that we claim

$$\theta^{-n}\mathcal{G}^{(n)} = \{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu\{y \in M : (y, \underline{\omega}^+) \in \Theta^+_n\} \geq 1 - \mathfrak{c}\beta_0\}.$$

Proof of the equality. We find

$$M^n_{\theta^n\underline{\omega}} = \{x \in M : (f^{-n}_{\theta^n\underline{\omega}}x, \underline{\omega}^+) \in \Theta^+_n\} = \{x \in M : ((f^n_{\underline{\omega}})^{-1}x, \underline{\omega}^+) \in \Theta^+_n\}.$$

Because $f^n_{\underline{\omega}}$ is a diffeomorphism we have that for $x \in M_{\theta^{-n}\underline{\omega}}$ there exists a $y = (f^n_{\underline{\omega}})^{-1}x \in M$ such that $f^n_{\underline{\omega}}y = x$. Hence,

$$M^n_{\theta^n\underline{\omega}} = \{x \in M : ((f^n_{\underline{\omega}})^{-1}x, \underline{\omega}^+) \in \Theta^+_n\} = f^n_{\underline{\omega}}\{y \in M : (y, \underline{\omega}^+) \in \Theta^+_n\}.$$

Using $\mu^n_{\theta^n\underline{\omega}} = (f^n_{\underline{\omega}})_*\mu$ we get

$$\begin{aligned}\theta^{-n}\mathcal{G}^{(n)} &= \theta^{-n}\{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu^n_{\underline{\omega}}(M^n_{\underline{\omega}}) \geq 1 - \mathfrak{c}\beta_0\}\\ &= \{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu^n_{\theta^n\underline{\omega}}(M^n_{\theta^n\underline{\omega}}) \geq 1 - \mathfrak{c}\beta_0\}\\ &= \{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu^n_{\theta^n\underline{\omega}}(f^n_{\underline{\omega}}\{y \in M : (y, \underline{\omega}^+) \in \Theta^+_n\}) \geq 1 - \mathfrak{c}\beta_0\}\\ &= \{\underline{\omega} \in \Omega^{\mathbb{Z}} : (f^n_{\underline{\omega}})_*\mu(f^n_{\underline{\omega}}\{y \in M : (y, \underline{\omega}^+) \in \Theta^+_n\}) \geq 1 - \mathfrak{c}\beta_0\}\\ &= \{\underline{\omega} \in \Omega^{\mathbb{Z}} : \mu\{y \in M : (y, \underline{\omega}^+) \in \Theta^+_n\} \geq 1 - \mathfrak{c}\beta_0\}.\end{aligned}$$

□

Because $\theta^{-n}\mathcal{G}^{(n)}$ depends only on the positive times, and θ^n is $\mathbb{P}$ invariant, we have

$$\mathbb{P}^+((\theta^{-n}\mathcal{G}^{(n)})^+) = \mathbb{P}(\theta^{-n}\mathcal{G}^{(n)}) = \mathbb{P}(\mathcal{G}^{(n)}).$$

By construction μ^* projects onto $\mu \times \mathbb{P}^+$ and we get

$$\begin{aligned}1 - \beta_0 \leq \mu^*(\Theta_n) &= (\mu \times \mathbb{P}^+)(\Theta^+_n)\\ &= \int_{\Theta^+_n} d(\mu \times \mathbb{P}^+) \overset{\text{Fubini}}{=} \int_{\Omega^{\mathbb{Z}^+}} \int_M 1_{\Theta^+_n}\, d\mu\, d\mathbb{P}^+\end{aligned}$$

$$= \int_{\Omega^{\mathbb{Z}^+}} \mu\{y \in M : (y, \underline{\omega}^+) \in \Theta_n^+\} \, \mathbb{P}^+(d\underline{\omega}^+)$$
$$= \left(\int_{(\theta^{-n}\mathcal{G}^{(n)})^+} + \int_{\Omega^{\mathbb{Z}^+} \setminus (\theta^{-n}\mathcal{G}^{(n)})^+} \right) \mu\{y \in M : (y, \underline{\omega}^+) \in \Theta_n^+\} \, \mathbb{P}^+(d\underline{\omega}^+).$$

The estimates

$$\begin{aligned} &\mu(A) \leq 1, && \text{for all } \ A \subset M, \\ &\mu\{y \in M : (y, \underline{\omega}^+) \in \Theta_n^+\} < 1 - \mathfrak{c}\beta_0, && \text{for all } \ \underline{\omega}^+ \in \Omega^{\mathbb{Z}^+} \setminus (\theta^{-n}\mathcal{G}^{(n)})^+ \end{aligned}$$

lead to

$$1 - \beta_0 \leq \mathbb{P}^+((\theta^{-n}\mathcal{G}^{(n)})^+) + (1 - \mathfrak{c}\beta_0)(1 - \mathbb{P}^+((\theta^{-n}\mathcal{G}^{(n)})^+)).$$

Rewriting this inequality results in

$$\frac{\mathfrak{c} - 1}{\mathfrak{c}} \leq \mathbb{P}^+((\theta^{-n}\mathcal{G}^{(n)})^+) = \mathbb{P}(\mathcal{G}^{(n)}).$$

Taking the limit over n concludes the proof,

$$\mathbb{P}(\mathcal{G}) \geq \frac{\mathfrak{c} - 1}{\mathfrak{c}}.$$

□

One may ask how this can help in finding the source and target sets we talked about at the beginning of this chapter. The next lemma shows that we can choose such sets in a way that traveling from the source to the target set has a positive $\mu_{\underline{\omega}}^n$-mass. By the initial assumptions holds $\mu \ll \mathrm{Leb}$. We write the Radon-Nikodym derivative as $\psi := \frac{d\mu}{d\mathrm{Leb}}$ and define the constant $\alpha_0 = \beta_0/\mathrm{Leb}(M)$. We get

$$\begin{aligned} \mu\{\psi \geq \alpha_0\} &= \int_{\{\psi \geq \alpha_0\}} \psi \, d\mathrm{Leb} \\ &= \int_M \psi \, d\mathrm{Leb} - \int_{M \setminus \{\psi \geq \alpha_0\}} \psi \, d\mathrm{Leb} \\ &\geq 1 - \int_M \alpha_0 \, d\mathrm{Leb} \\ &= 1 - \beta_0. \end{aligned}$$

Lemma 4.4. *For any $\epsilon > 0$, there exists a constant $c = c(\epsilon) > 0$ such that for any $\underline{\omega} \in \mathcal{G}$, we have the following. There are points $\hat{p}_- \in \{\psi \geq \alpha_0\}, \hat{p} \in M$, and a sequence $n_i \to \infty$ for which*

$$\mu_{\underline{\omega}}^n(M_{\underline{\omega}}^n \cap B(\hat{p}, \epsilon) \cap f_{\theta^{-n}\underline{\omega}}^n B(\hat{p}_-, \epsilon)) \geq c \qquad \text{for all } n = n_i.$$

Proof. Take $\underline{\omega} \in \mathcal{G}$ arbitrarily. Because $\mathcal{G} = \cap_{N \geq 1} \cup_{n \geq N} \mathcal{G}^{(n)}$, we can fix a sequence $n_i \to \infty$ with $\underline{\omega} \in \mathcal{G}^{(n)}$ for all $n = n_i$. By the initial assumptions, ψ is continuous. Then $\{\psi \geq \alpha_0\} \subset M$ is a closed subset of the compact set M, hence also compact.

We can choose a finite cover of $\{\psi \geq \alpha_0\}$ using open balls with radius ϵ and centers $p_j \in \{\psi \geq \alpha_0\}, 1 \leq j \leq J$. For each $n = n_i$, we estimate:

$$\begin{aligned} c_- &:= \frac{1}{J}(1 - (\mathfrak{c}+1)\beta_0) \\ &\leq \frac{1}{J}\mu(f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}} \cap \{\psi \geq \alpha_0\}) = \frac{1}{J}\mu\big(\bigcup_{j=1}^{J}(f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}} \cap B(p_i, \epsilon) \cap \{\psi \geq \alpha_0\})\big) \\ &\leq \frac{1}{J}\sum_{j=1}^{J} \mu(B(p_j, \epsilon) \cap f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}}). \end{aligned}$$

The first inequality follows from the following computation. Take $A, B \in \mathcal{B}(M)$. We estimate

$$1 \geq \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

which gives

$$\mu(A \cap B) \geq \mu(A) + \mu(B) - 1$$

and the inequalities

$$\begin{aligned} \mu(f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}}) = \mu^n_{\underline{\omega}}(M^n_{\underline{\omega}}) \geq 1 - \mathfrak{c}\beta_0, \\ \mu(\{\psi \geq \alpha_0\}) \geq 1 - \beta_0. \end{aligned}$$

Setting $A = f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}}$ and $B = \{\psi \geq \alpha_0\}$ results in

$$\begin{aligned} \mu(f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}} \cap \{\psi \geq \alpha_0\}) &\geq \mu(f^{-n}_{\underline{\omega}} M^n_{\underline{\omega}}) + \mu(\{\psi \geq \alpha_0\}) - 1 \\ &\geq (1 - \mathfrak{c}\beta_0) + (1 - \beta_0) - 1 \\ &= 1 - (\mathfrak{c}+1)\beta_0. \end{aligned}$$

By going back to the estimate of c_-, we see that for each i there must exist a $j = j(i)$ with

$$\mu^{n_i}_{\underline{\omega}}(f^{n_i}_{\theta^{-n_i}} B(p_j) \cap M^{n_i}_{\underline{\omega}}) = \mu(B(p_j, \epsilon) \cap f^{-n_i}_{\underline{\omega}} M^{n_i}_{\underline{\omega}}) \geq c_-.$$

Because J is a finite number and our set $\{n_i : i \in \mathbb{N}\}$ is infinite, according to the pidgeonhole principle there must exist a $\hat{j}_- \in \{1, \ldots, J\}$ such that $p_{j(i)} = p_{\hat{j}_-}$ for infinitely many i. We refine (n_i) such that

$$\mu^{n_i}_{\underline{\omega}}(f^{n_i}_{\theta^{-n_i}} B(\hat{p}_-, \epsilon) \cap M^{n_i}_{\underline{\omega}}) \geq c_-.$$

for $\hat{p}_- = p_{\hat{j}_-}$. Because M is a compact set, we can fix a finite cover of M with open balls of radius ϵ and centers $p'_j \in M, 1 \geq j \geq J'$. For each $n = n_i$, we estimate:

$$\begin{aligned} c := \frac{c_-}{J'} &\leq \frac{1}{J'}\mu^n_{\underline{\omega}}(f^n_{\theta^{-n}} B(\hat{p}_-, \epsilon) \cap M^n_{\underline{\omega}}) = \frac{1}{J'}\mu^n_{\underline{\omega}}\big(\bigcup_{j=1}^{J'}(B(p'_j, \epsilon) \cap f^n_{\theta^{-n}} B(\hat{p}_-, \epsilon) \cap M^n_{\underline{\omega}})\big) \\ &\leq \frac{1}{J'}\sum_{j=1}^{J'} \mu^n_{\underline{\omega}}(B(p'_j, \epsilon) \cap f^n_{\theta^{-n}} B(\hat{p}_-, \epsilon) \cap M^n_{\underline{\omega}}). \end{aligned}$$

Using the same pidgeonhole principle argumentation as before, we can refine (n_i) one last time to get

$$\mu_{\underline{\omega}}^{n_i}(B(\hat{p},\epsilon) \cap f_{\theta^{-n_i}}^{n_i} B(\hat{p}_-,\epsilon) \cap M_{\underline{\omega}}^{n_i}) \geq c$$

for a chosen $\hat{p} = p'_{\hat{j}}$ and $\hat{j} \in \{1,\dots,J'\}$. We find $c > 0$ with $c(\epsilon) = c$. □

In the context of Lemma 4.4, the set $B(\hat{p}_-,\epsilon)$ can be interpreted as the 'source set' and $B(\hat{p},\epsilon)$ as the 'target set'.

4.3 Disintegration of μ in the "source set" onto u-graphs

We fix $\underline{\omega} \in \mathcal{G}$ and a constant $\epsilon > 0$. Assume that Lemma 4.4 has been applied to obtain $\hat{p}_-, \hat{p} \in M$ as well as a sequence (n_i) and $c = c(\epsilon)$. For $n = n_i$, define the sets

$$\Lambda^n = M_{\underline{\omega}}^n \cap \overline{B(\hat{p},\epsilon)} \cap f_{\theta^{-n}\underline{\omega}}^n \overline{B(\hat{p}_-,\epsilon)} \qquad \text{and} \qquad \Lambda_-^n = f_{\underline{\omega}}^{-n}(\Lambda^n).$$

In Chapter 2.4 in [2], the stack of unstable leaves $\mathcal{S}_{\underline{\omega}}$ through $\overline{B(x_*,\epsilon)} \cap \Gamma_{l_0,\underline{\omega}}$ is constructed using the switching of axes onto local graphs. Here we do a similar construction. Our goal is to disintegrate μ onto the leaves of a smooth foliation $\mathcal{F}_-^n$ such that the leaves of $\mathcal{F}_-^n$ (restricted to a small enough set) can be seen as graphs corresponding to $x \in \Lambda_-^n$ going from open subsets of $\hat{E}^{u,(0)}_{(x,(\theta^{-n}\underline{\omega})^+)}$ to $E^{cs}_{(x,(\theta^{-n}\underline{\omega})^+)}$. These graphs are the so called u-graphs.
The foliation $\mathcal{F}_-^n$ is defined in the following way. We first fix a point $p_-^n \in \Lambda_-^n$. As the discussion is entirely local and the exponential map is a local diffeomorphism we can confuse a neighbourhood of q_-^n in M with a subset of $T_{q_-^n}M$ via $\exp_{q_-^n}$. The foliation $\mathcal{F}_-^n$ is defined to be the collection of the $(\dim E^u)$-dimensional hyperplanes in $T_{q_-^n}$ parallel to $\hat{E}^{u,(0)}_{(q_-^n,(\theta^{-n}\underline{\omega})^+)}$. The following Lemma states that such u-graphs exist and can be constructed.

Lemma 4.5. *For all sufficiently small $\epsilon > 0$ depending on l_0, ψ and α_0, there exist constants $K_- = K_-(l_0), r_- = r_-(l_0,\epsilon)$ for which the following hold for all $\underline{\omega} \in \mathcal{G}$. Assume Lemma 4.4 has been applied. Let $n = n_i$, and let $\mathcal{F}_-^n$ be as above. Then for every $x \in \Lambda_-^n$,*

(a) there is a function $h_x^- = h^-_{(x,(\theta^{-n}\underline{\omega})^+)}$,

$$h_x^- : B^u(r_-(\hat{l}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)})^{-1}) \to \mathbb{R}^{cs} \qquad \textit{with} \qquad \mathrm{Lip}(h_x^-), \mathrm{Lip}(dh_x^-) < K_-,$$

such that if $\mathcal{F}_-^n(x)$ is the leaf of $\mathcal{F}_-^n$ containing x, then

$$\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)}(\mathrm{graph}(h_x^-)) \subset \mathcal{F}_-^n(x);$$

(b) $\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)} B(r_-(\hat{l}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)})^{-1}) \subset \{\psi \geq \alpha_0/2\}$.

Proof. Because ψ is continuous by assumption, we can choose $\epsilon > 0$ small enough such that $\mathcal{N}_{2\epsilon}\{\psi \geq \alpha_0\} \subset \{\psi \geq \alpha_0/2\}$. Because $\hat{p}_- \in \{\psi \geq \alpha\}$ by construction it follows that $B(\hat{p}_-, 2\epsilon) \subset \{\psi \geq \alpha_0/2\}$.
We fix $n = n_i$. Because Λ_-^n is sufficiently small, uniform separation between $\hat{E}_{(q_-^n,(\theta^{-n}\underline{\omega})^+)}$ and $E^{cs}_{(x,(\theta^{-n}\underline{\omega})^+)}$ for $x \in \Lambda_-^n$ follows from from Remark 16(a) in [2]. This can be interpreted as that the angle between both subspaces is bounded from below for q_-^n and x close enough. Because the $\mathcal{F}_-^n(x)$ are all parallel to $\hat{E}_{(q_-^n,(\theta^{-n}\underline{\omega})^+)}$ we conclude uniform separation between $\mathcal{F}_-^n(x)$ and $E^{cs}_{(x,(\theta^{-n}\underline{\omega})^+)}$ for $x \in \Lambda_-^n$. Since $\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)}$ preserves angles we have a uniform separation between $(\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)})^{-1}\mathcal{F}_-^n$ and $\mathbb{R}^{cs}$. The angle between both subspaces is bounded which implies that we can find a $K_- < \infty$ with the following property: choose a function h_x^- that satisfies $\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)}(\mathrm{graph}(h_x^-)) \subset \mathcal{F}_-^n(x)$, then we have $\mathrm{Lip}(h_x^-) < K_-$ and $\mathrm{Lip}(dh_x^-) < K_-$.
Because $\mathrm{Lip}(h_x^-) < K_-$, we shrink r_- small enough to ensure that $\hat{\Phi}^{(0)}_{(x,(\theta^{-n}\underline{\omega})^+)}(\mathrm{graph}(h_x^-)) \subset B(\hat{p}_-, 2\epsilon)$. Using $B(\hat{p}_-, 2\epsilon) \subset \{\psi \geq \alpha_0/2\}$ we conclude (b).
In a last step, we need to find constants which work for all n. We observe from the above argument that the constants depend on the modulus of continuity of $x \to E^{cs}_{(x,(\theta^{-n}\underline{\omega})^+)}$. Because the subset $\Theta_n^+ \subset \Theta_0^+$ is compact, such a modulus of continuity exists. Taking the supremum over all these moduli of continuity results in a new modulus of continuity for all n and we can construct constants by the same procedure as above but independently of n. □

Chapter 5

Geometry of pushed forward u-graphs

This chapter discusses what happens when the u-graphs get pushed forward by $f^n_{\theta^{-n}\underline{\omega}}$. Using the findings from the previous chapters we get that for $n \to \infty$ the pushed forward u-graphs resemble the local unstable manifolds.

5.1 Construction of the stack of W^n-leaves

The set-up of this chapter is the following. Fix a small $\beta_0 > 0$ and an $l_0 > 1$ such that $\mu^*\{l \leq l_0\} \geq 1 - \beta_0/3$. Take $\Theta_0, \Theta_n \in \Gamma$ as in Section 4.2 and choose the constant $\mathfrak{c} = 2$ such that $\mathbb{P}(\mathcal{G}) \geq 1/2$, according to Lemma 4.3. Fix $\underline{\omega} \in \mathcal{G}$.
We wish to perform the following steps.

(1) Find suitable 'source' and 'target' sets $B(\hat{p}_-, \epsilon_*)$ and $B(\hat{p}, \epsilon_*)$ for ϵ_* chosen small enough and a sequence $\{n_i\}$ corresponding to Lemma 4.4 such that the following is well defined.

(2) Find a reference point $x_* \in M_{\underline{\omega}} \cap B(\hat{p}, \epsilon_*)$ and a box $E_*(r_*) = E^u_*(r_*) \times E^{cs}_*(r_*)$, $E^{u/cs}_*(r_*) \coloneqq E_{x_*,\underline{\omega}}$ to construct W^u-leaves according to Theorem 2.15, for r_* chosen small enough. Find a big enough n to perform a switch of axes on the n times pushed forward u-graphs (called the W^n-leaves).

We refer to Chapter 5.1 in [2] to see how these parameters need to be chosen and that they are well defined. Assume the set-up above and define $h_x \coloneqq h_{(x,\underline{\omega})} = \mathcal{T}^{(n-1)} \circ \cdots \circ \mathcal{T}^{(0)} h^-_{x_{-n}}$ for n big enough and $h^-_{x_{-n}}$ as in Lemma 4.5 with $x_{-n} \in \Lambda^-_n$. We define

$$W^n_{(x,\underline{\omega})} \coloneqq \hat{\Phi}^{(n)}_{(x_{-n},(\theta^{-n}\underline{\omega})^+)} \mathrm{graph}(h_x).$$

Using Chapter 2.4 in [2], let $\mathcal{S} \coloneqq \mathcal{S}_{\underline{\omega}} = \bigcup_{x\in\Lambda} \xi(x)$ be the stack of unstable leaves and Ξ the partition of $\mathcal{S}$ whose elements are the leaves $\xi(x)$ along $x \in \Lambda$, where

$$\Lambda \coloneqq \overline{B(\hat{p}, \epsilon_*)} \cap M_{\underline{\omega}}.$$

Here, $\xi(x)$ is the W^u-leaf through $x \in \Lambda$ and has the form $\xi(x) = \exp_{x_*} \mathrm{graph}(\gamma_x)$ for some $\gamma_x : E^u_*(r_*) \to E^{cs}_*(r_*)$ with $\mathrm{Lip}(\gamma_x) \leq 1$.
Similarly, we wish to construct a stack of W^n-leaves, denoted by $\mathcal{S}^n$, using the following lemma from [2].

Lemma 5.1. *There exists $N_* \in \mathbb{N}$, depending on all the parameters mentioned in the construction above, such that the following holds for all $n = n_i \geq N_*$.*

(a) *For each $x \in \Lambda^n$, we have that the connected component of $W^n_{(x,\underline{\omega})} \cap \exp_{x_*} E_*(r_*)$ containing x coincides with $\exp_{x_*}\mathrm{graph}(\gamma^n_x)$, where $\gamma^n_x : E^u_*(r_*) \to E^{cs}_*(r_*)$ is a $C^{1+\mathrm{Lip}}$-mapping with $\mathrm{Lip}(\gamma^n_x) \leq 1$.*

(b) *The partition Ξ^n of $\mathcal{S}^n = \bigcup_{x\in\Lambda^n} \xi^n(x)$ into leaves $\xi^n(x) \coloneqq \exp_{x_*}\mathrm{graph}(\gamma^n_x)$ is measurable.*

5.2 Limiting properties of W^n-leaves

We wish to study the behaviour of $\mathcal{S}^n$ as $n \to \infty$. For that, we have a look at what happens to γ^n_x and γ_y if x, y are close. The proposition below states that for x, y close, we can expect γ^n_x and γ_y to be close for big enough n. The proposition is important because it implies directly that $\limsup_{n\to\infty} \mathcal{S}^n \subset \mathcal{S}$.

Proposition 5.2. *For any $\epsilon > 0$, there exists $\tilde{n}_0 = \tilde{n}_0(\epsilon) \geq N_*$ and $\tilde{\eta} = \tilde{\eta}(\epsilon) > 0$ with the following property. For any $n = n_i \geq \tilde{n}_0$ and any $x \in \Lambda^n, y \in \Lambda$ with $d(x, y) < \tilde{\eta}$, we have that $\|\gamma^n_x - \gamma_y\| < \epsilon$ where $\|\cdot\|$ refers to the uniform norm on $C(E^u_*(r_*), E^{cs}_*(r_*))$.*

The idea of the proof is to choose the simplest function we can think of and show that it can approximate γ^n_x and γ_y by being pushed forward. This function turns out to be the zero function.

Proof. We use for simplicity the notation of a two-sided chart. Assume $(x, \underline{\omega}')$ and $(y, \underline{\omega})$ are such that $\omega_i = \omega'_i$ for all $i > -n$ and n as in the proposition. Denote the zero function as $\mathbf{0} : \mathbb{R}^u \to \mathbb{R}^{cs}$. First, consider $\mathbf{0}$ as a function in the chart at $\tau^{-k}(y, \underline{\omega})$ for some $k \ll \tilde{n}_0$ and define $\mathbf{0}^k_y : B^u(\delta l_0^{-1}) \to \mathbb{R}^{cs}$ to be given by

$$\mathbf{0}^k_y \coloneqq \mathcal{T}_{\tau^{-1}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})}\mathbf{0}.$$

Secondly, consider $\mathbf{0}$ as a function in the chart at $\tau^{-k}(x, \underline{\omega}')$, with $\mathbf{0}^{k,n}_x : B^u(\delta l_0^{-1}) \to \mathbb{R}^{cs}$ be given by

$$\mathbf{0}^{k,n}_x \coloneqq \mathcal{T}_{\tau^{-1}(x,\underline{\omega}')} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(x,\underline{\omega}')}\mathbf{0}.$$

Performing a switching of axes results in functions $\tilde{\gamma}^k_y, \tilde{\gamma}^{k,n}_x : E^u_*(r_*) \to E^{cs}_*(r_*)$ for $\mathbf{0}^k_y$ and $\mathbf{0}^{k,n}_x$ respectively. Next, we use the triangle inequality to obtain

$$\|\gamma^n_x - \gamma_y\| \leq \|\gamma^n_x - \tilde{\gamma}^{k,n}_x\| + \|\tilde{\gamma}^{k,n}_x - \tilde{\gamma}^k_y\| + \|\tilde{\gamma}^k_y - \gamma_y\|. \tag{5.1}$$

We see that the norm depends for a given ϵ on the choices of $k = k(\epsilon, l_0)$ and $\tilde{n}_0 = \tilde{n}_0(k, \epsilon, l_0)$. The following Lemma will present itself as useful.

Lemma 5.3. *Let $g_1, g_2 : B^u(\delta l(y, \underline{\omega})^{-1}) \to \mathbb{R}^{cs}$ be Lipschitz graphing maps in the chart at $(y, \underline{\omega})$. For $i = 1, 2$, let $\check{g}_i \coloneqq (L_{(y,\underline{\omega})} \circ \cdots \circ L^{-1}_{(y,\underline{\omega})})|_{E^u_{(y,\underline{\omega})}(2r_*)}$, and assume that $\mathrm{graph}(\check{g}_i) \subset E^u_{(y,\underline{\omega})}(2r_*) \times E^{cs}_{(y,\underline{\omega})}(\frac{1}{2}r_*)$ with $\mathrm{Lip}(\check{g}_i) \leq 1/10$. Let $\gamma_i : E^u_*(r_*) \to E^{cs}_*(r_*)$ be such that $\exp_{x_*}\mathrm{graph}(\gamma_i) = \exp_{x_*} E_*(r_*) \cap \exp_y \mathrm{graph}(\check{g}_i)$. Then $\mathrm{Lip}(\gamma_i) \leq 1/5$, and*

$$\|\gamma_1 - \gamma_2\| \leq \tilde{C}|g_1 - g_2|,$$

where $\tilde{C} = \tilde{C}(l_0) > 0$ *and* $|\cdot|$ *being the uniform norm on* $C(B^u(\delta l(y,\underline{\omega})^{-1}, \mathbb{R}^{cs})$. *The same holds when* $(y,\underline{\omega})$ *is replaced by* $(x,\underline{\omega}')$.

(i) *For the first and third term of the RHS in* (5.1)*:*
The idea is to use the contraction estimate from Lemma 8 in [2] for $|\mathbf{0}_y^k - g_{(y,\underline{\omega})}|$ and $g_{(y,\underline{\omega})}$ being the graphing map of the unstable manifold in the chart at $(y,\underline{\omega})$. Note that it follows from the Unstable Manifold Theorem (Theorem 2.15) that $g_{(y,\underline{\omega})} = \mathcal{T}_{\tau^{-1}(y,\underline{\omega})} g_{\tau^{-1}(y,\underline{\omega})}$. With a simple induction we get the iteration

$$g_{(y,\underline{\omega})} = \mathcal{T}_{\tau^{-1}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})} g_{\tau^{-k}(y,\underline{\omega})}.$$

By construction, the inequality $\mathrm{Lip}(g_{\tau^{-m}(y,\underline{\omega})}) \leq 1/10$ holds for $0 \leq m \leq k$. In order to apply Lemma 8 in [2] we additionally need that $\mathrm{Lip}(\mathcal{T}_{\tau^{-(m+1)}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})} \mathbf{0}) \leq 1/10$ for $m \leq k$. We have $\mathrm{Lip}(\mathbf{0}) = 0 \leq 1/10$ and using the first part of Lemma 8 in [2] results in the Lip of all graph transforms of $\mathbf{0}$ being bounded by $1/10$. Finally, we are allowed to use the construction estimate from Lemma 8 in [2] on $|\mathbf{0}_y^k - g_{(y,\underline{\omega})}|$ and get the bound

$$\begin{aligned} |\mathbf{0}_y^k - g_{(y,\underline{\omega})}|' &= |\mathcal{T}_{\tau^{-1}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})} \mathbf{0} - \mathcal{T}_{\tau^{-1}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})} g_{\tau^{-k}(y,\underline{\omega})}|' \\ &\leq c^k |\mathbf{0} - g_{\tau^{-k}(y,\underline{\omega})}|' = c^k |g_{\tau^{-k}(y,\underline{\omega})}|' \\ &\leq c^k. \end{aligned}$$

For the moment, take a continuous map $h : B^u(\delta l(y,\underline{\omega})^{-1}) \to \mathbb{R}^{cs}$. Recall that the norm $|\cdot|'$ is defined as

$$|h|' \coloneqq \sup_{u \in B^u(\delta l(x,\underline{\omega})^{-1}), u \neq 0} \frac{|h(u)|}{|u|}.$$

Note that we have

$$|h(u)| = \frac{|h(u)|}{|u|}|u| \leq |h|'|u| \leq |h|' \frac{\delta}{l(y,\underline{\omega})^{-1}} \leq |h|'$$

for all $u \in B^u(\delta l(y,\underline{\omega})^{-1})$ and so

$$|h| \leq |h|'.$$

Applying Lemma 5.3 results in the estimate

$$\begin{aligned} \|\tilde{\gamma}_y^k - \gamma_y\| &\leq \tilde{C} |\mathbf{0}_y^k - g_{(y,\underline{\omega})}| \\ &\leq \tilde{C} |\mathbf{0}_y^k - g_{(y,\underline{\omega})}|' \\ &\leq \tilde{C} c^k. \end{aligned}$$

Because $c \in (0,1)$, we choose k big enough that $\tilde{C} c^k < \epsilon/3$. Similarly, we want to use Lemma 8 in [2] on $|\mathbf{0}_x^{k,n} - h_x|$. For the lemma to be k-times applicable, we need $\mathrm{Lip}(\mathcal{T}_{\tau^{-(m+1)}(x,\underline{\omega}')} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(x,\underline{\omega}')} \mathbf{0}) \leq 1/10$ and $\mathrm{Lip}(\mathcal{T}_{\tau^{-(m+1)}(x,\underline{\omega}')} \circ \cdots \circ \mathcal{T}_{\tau^{-n}(x,\underline{\omega}')} h^-_{x_{-n}}) \leq 1/10$ for all $0 \leq m \leq k$. We do not need to worry about the graph transforms of $\mathbf{0}_x^{k,n}$ as the claim follows by the same argumentation as above provided that we start in the chart at $(x,\underline{\omega})'$ instead of $(y,\underline{\omega})$. For the graph transforms of $h^-_{x_{-n}}$ to be bounded by $1/10$, we need to choose $\tilde{n}_0$ big enough such that $\tilde{n}_0 - k \geq m_0 + m_1$, where m_0, m_1

are as in Proposition 3.6. Applying first Lemma 5.3 and then the contraction estimate from Lemma 8 in [2] k-times gives the suitable estimate

$$\begin{aligned}\|\gamma_x^n - \tilde{\gamma}_x^{k,n}\| &\leq \tilde{C}|\mathbf{0}_x^{k,n} - h_x|' \\ &\leq \tilde{C}c^k|\mathbf{0} - \mathcal{T}_{\tau^{-(k+1)}(x,\underline{\omega}')} \circ \cdots \circ \mathcal{T}_{\tau^{-n}(x,\underline{\omega}')} h^-_{x_{-n}}|' \\ &\leq \tilde{C}c^k \\ &\leq \epsilon/3.\end{aligned}$$

(ii) *For the second term of the RHS in* (5.1)*:*
The first step is to show the following property. Given $0 < \bar{\epsilon} \ll 1$ to be determined, we claim that for $\tilde{\eta}$ small enough and $\tilde{n}_0$ large enough depending on l_o, k and $\bar{\epsilon}$, the following hold for x, y with $d(x, y) < \tilde{\eta}$:

(a) $d(f_{\underline{\omega}}^{-k}x, f_{\underline{\omega}}^{-k}y), d_H(E^{u/cs}_{\tau^{-k}(x,\underline{\omega}')}, E^{u/cs}_{\tau^{-k}(y,\underline{\omega})}) < \bar{\epsilon}$, and

(b) $f_{\underline{\omega}'}^{-i}x \in \Phi_{\tau^{-i}(y,\underline{\omega})} B(\frac{1}{2} r_* l(\tau^{-i}(y, \underline{\omega}))^{-1})$ for all $0 \leq i \leq k$.

We recall the estimates $e^{-\delta_2} l \leq l \circ \tau \leq e^{\delta_2} l$ and $\|df_{\omega_n}^{-1}\| \leq l_1(\underline{\omega}) e^{|n|\delta_2} \leq l_0 e^{|n|\delta_2}$. For item (b) and $d(f_{\underline{\omega}}^{-k}x, f_{\underline{\omega}}^{-k}y) < \bar{\epsilon}$ we have

$$l(\tau^{-i}(y, \underline{\omega})) \leq l_0 e^{i\delta_2} \leq l_0 e^{k\delta_2}$$

because e^x is monotonous increasing. Using the norm properties of $\|\cdot\|$ and the chain rule gives

$$\begin{aligned}\mathrm{Lip}(f_{\underline{\omega}}^{-i}) \leq \|df_{\underline{\omega}}^{-i}\| &= \|d(f^{-1}_{\omega_{-(i-1)}} \circ \cdots \circ f^{-1}_{\omega_0})\| \\ &\leq \prod_{j=1}^{i} \|df^{-1}_{\omega_{-(j-1)}}\| \\ &\leq \prod_{j=1}^{i} l_0 e^{j\delta_2} = l_0^i e^{\frac{i(i+1)}{2}\delta_2} \\ &\leq l_0^k e^{\frac{k(k+1)}{2}\delta_2},\end{aligned}$$

where we used the monotonicity of e^x and $l_0 > 1$ in the last step. For $d(x, y) < \tilde{\eta}$ we find a $\delta > 0$ with

$$d(f_{\underline{\omega}}^{-k}x, f_{\underline{\omega}}^{-k}y) \leq \frac{\sup_{d(x,y)\leq\tilde{\eta}} d(f_{\underline{\omega}}^{-i}x, f_{\underline{\omega}}^{-i}y)}{\tilde{\eta}} \tilde{\eta} = (\mathrm{Lip}(f_{\underline{\omega}}^{-i}) + \delta)\tilde{\eta}$$

and $\delta \to 0$ as $\tilde{\eta} \to 0$ due to continuity. It follows that

$$d(f_{\underline{\omega}}^{-k}x, f_{\underline{\omega}}^{-k}y) \leq (l_0^k e^{\frac{k(k+1)}{2}\delta_2} + \delta)\tilde{\eta}.$$

Because δ is small for $\tilde{\eta}$ small, we can choose $\tilde{\eta}$ small enough (depending on $l_0, k, \bar{\eta}$) such that $(l_0^k e^{\frac{k(k+1)}{2}\delta_2} + \delta)\tilde{\eta} < \bar{\epsilon}$ and

$$d(f_{\underline{\omega}}^{-k}x, f_{\underline{\omega}}^{-k}y) < \bar{\epsilon}.$$

We see that for item (b) we have $l(\tau^{-i}(y,\underline{\omega})) \leq l_0 e^{k\delta_2}$. Note that for $0 \leq i \leq k$ we have $f^{-i}_{\underline{\omega}'}x = f^{-i}_{\underline{\omega}}x$. Showing the relation $f^{-i}_{\underline{\omega}}x \in \Phi_{\tau^{-i}(y,\underline{\omega})}B(\frac{1}{2}r_*(l_0e^{k\delta_2})^{-1})$ for all $0 \leq i \leq k$ is enough to show item (b). This is equivalent to proving that $\Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}x \in B(\frac{1}{2}r_*(l_0e^{k\delta_2})^{-1})$ which is the same as showing that

$$d(\Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}x, 0) = d(\Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}x, \Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}y) \leq \frac{1}{2}r_*(l_0e^{k\delta_2})^{-1}.$$

Recall from the construction that r_* depends solely on l_0. The RHS depends thus only on l_0 and k. By the continuity of $\Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}$ we can choose $\bar{\epsilon} > 0$ small enough and dependent on l_0, k such that for $f^{-k}_{\underline{\omega}}x, f^{-k}_{\underline{\omega}}y$ satisfying $d(f^{-k}_{\underline{\omega}}x, f^{-k}_{\underline{\omega}}y) < \bar{\epsilon}$ we have $d(\Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}x, \Phi^{-1}_{\tau^{-i}(y,\underline{\omega})}f^{-i}_{\underline{\omega}}y) \leq \frac{1}{2}r_*(l_0e^{k\delta_2})^{-1}$. Hence, we can choose $\bar{\epsilon}$ sufficiently small such that

$$f^{-i}_{\underline{\omega}'}x \in \Phi_{\tau^{-i}(y,\underline{\omega})}B(\frac{1}{2}r_*l(\tau^{-i}(y,\underline{\omega}))^{-1}) \text{ for all } 0 \leq i \leq k,$$

which concludes item (b).
To control $d_H(E^{u/cs}_{\tau^{-k}(x,\underline{\omega}')}, E^{u/cs}_{\tau^{-k}(y,\underline{\omega})})$ we use the continuity of the unstable- and center-stable sets in the uniformity sets. By construction of $\underline{\omega}$ and $\underline{\omega}'$, it holds $\omega_i = \omega_i'$ for all $0 \leq i \leq \tilde{n}_0 - k$. We apply Proposition 6 to $\tau^{-k}(y,\underline{\omega}), \tau^{-k}(y,\underline{\omega}) \in \{l \leq l_0e^{k\delta_2}\}$ and find an $n_0 = n_0(\bar{\epsilon}, l_0e^{k\delta_2})$. For $d_H(E^{u/cs}_{\tau^{-k}(x,\underline{\omega}')}, E^{u/cs}_{\tau^{-k}(y,\underline{\omega})}) < \bar{\epsilon}$ to hold, we require $\tilde{n}_0 \geq n_0 + k$. Now let $\mathbf{0}^y_x : B^u(\delta l(\tau^{-k}(y,\underline{\omega}))^{-1}) \to \mathbb{R}^{cs}$ be the function whose graph is the component of $(\Phi_{\tau^{-k}(y,\underline{\omega})})^{-1}\exp_{f^{-k}_{\underline{\omega}}x}E^u_{\tau^{-k}(x,\underline{\omega}')}$ in $B^u(\delta l(\tau^{-k}(y,\underline{\omega}))^{-1})$ containing the point $(\Phi_{\tau^{-k}(y,\underline{\omega})})^{-1}f^{-k}_{\underline{\omega}}x$. By item (a) above and using the continuity of unstable directions in the uniform sets we can choose $\bar{\epsilon}$ sufficiently small such that $\mathrm{Lip}(\mathbf{0}^y_x) < 1/10, \mathrm{Lip} \leq 1$, and $|\mathbf{0} - \mathbf{0}^y_x|$ as small as we want. Because x and y are really close, projecting $E^u_{\tau^{-k}(x,\underline{\omega}')}$ into the chart of $\tau^{-k}(y,\underline{\omega})$ results in almost an alignment of the graph with $B^u(\delta l(\tau^{-k}(y,\underline{\omega}))^{-1})$, which can be interpreted as the graph of $\mathbf{0}$. These properties, together with item (b), ensure according to Lemma 8(a) in [2] that the graph transform

$$\mathbf{0}^{y,k,n}_x := \mathcal{T}_{\tau^{-1}(y,\underline{\omega})} \circ \cdots \circ \mathcal{T}_{\tau^{-k}(y,\underline{\omega})}\mathbf{0}^y_x : B^u(\delta l(\tau^{-k}(y,\underline{\omega}))^{-1}) \to \mathbb{R}^{cs}$$

is well defined. Because $|\mathbf{0} - \mathbf{0}^y_x|$ can be as small as we wish for, choosing $\bar{\epsilon}$ sufficiently small we can use the contraction estimate to guarantee $|\mathbf{0}^{y,k,n}_x - \mathbf{0}^k_y| \leq \frac{\epsilon}{3\bar{C}}$. The last important thing to note is that if we look at $\exp_{x_*}E_*(r_*) \cap \exp_y \mathrm{graph}(\mathbf{0}^{k,y,n}_x)$, it is by construction precisely $\exp_{x_*}E_*(r_*) \cap \exp_x \mathrm{graph}(\mathbf{0}^{k,n}_x) = \exp_{\mathrm{x}_*}\mathrm{graph}(\tilde{\gamma}^{n,k}_x)$. This lets us apply Lemma 5.3 once more to get $\|\tilde{\gamma}^{k,n}_x - \tilde{\gamma}^k_y\| \leq \epsilon/3$ and we conclude the proof with

$$\|\gamma^n_x - \gamma_y\| \leq \epsilon.$$

□

Chapter 6

Proof of the SRB properties

This chapter is dedicated to proving Proposition 2.20. Using the results from Chapters 4 and 5, we can properly do the geometric construction we outlined in Section 1.3. By finding a new measure using partitions of the stack of unstable leaves (corresponding to ν_1 in Proposition 2.20) we complete the proof of the Main Proposition.

6.1 Construction of partitions respecting unstable manifolds

Assume the setting of Chapter 5. Our goal is to find a measure ν_1 with support on $\mathcal{S}$ that suffices the condition of Proposition 2.20 to show the SRB properties. For that, we need to show that the conditional probabilities of ν_1 on Ξ are absolutely continuous with respect to the Lebesgue measure. As we will see below, our measure of interest is the limit of measures that are supported on $\mathcal{S}^n$ rather than $\mathcal{S}$. To study the conditional measures we will group the unstable leaves in $\mathcal{S}$ into finer and finer substacks so that they still contain intact W^u-leaves.
In the rest of Chapter 6 we will use the notation of Chapter 5. First, define the measure

$$\tilde{\mu}^n_{\underline{\omega}} \coloneqq \mu^n_{\underline{\omega}}|_{\Lambda^n}.$$

Refining the sequence $\{n_i\}$, we can assume by Lemma 2.8 that $\tilde{\mu}^n_{\underline{\omega}}$ converges weakly to a measure $\tilde{\mu}_{\underline{\omega}}$ as $n \to \infty$. Because we have $\tilde{\mu}^n_{\underline{\omega}} \leq \mu^n_{\underline{\omega}}$, we note that $\tilde{\mu}_{\underline{\omega}} \leq \mu_{\underline{\omega}}$. By Lemma 4.1 it follows that also $\tilde{\mu}_{\underline{\omega}}$ is supported on Λ because the $\tilde{\mu}^n_{\underline{\omega}}$ are clearly supported on Λ^n and $\Lambda^n \in \Lambda$ for all $n = n_i$. From Lemma 4.4 we get

$$\tilde{\mu}_{\underline{\omega}}(\Lambda) \geq c_* > 0,$$

where $c_* \coloneqq c(\epsilon_*)$ as in Lemma 4.4. Next, we write for $S \in \exp_{x_*} E_*(r_*)$

$$\operatorname{diam}^{cs}(S) = \sup_{u \in E^u_*(r_*)} \operatorname{diam}(S \cap \exp_{x_*}(u + E^{cs}_*(r_*))).$$

Lemma 6.1. *There is a decreasing sequence of compact subsets $\Delta_1 \supset \Delta_2 \supset \cdots$ of Λ with the following properties:*

(i) *Each* Δ_m *is partitioned into disjoint compact subsets* $\{\Delta_{m,k}, 1 \le k \le K_m\}$ *and the* $\{\Delta_{m,k}\}$ *are nested in the sense, that each* $\Delta_{m+1,k} \subset \Delta_{m,k'}$ *for some* k'.

(ii) *The sets* $\mathcal{S}_{m,k} := \bigcup_{x\in\Delta_{m,k}} \xi(x)$ *are compact and pairwise disjoint among* $1 \le k \le K_m$.

(iii) *We have*

$$\lim_{m\to\infty} \max_{1\le k\le K_m} \operatorname{diam}^{cs}(\mathcal{S}_{m,k}) = 0.$$

(iv) *Defining* $\Delta_\infty = \bigcap_{m\ge1} \Delta_m$, *we have* $\tilde{\mu}_{\underline{\omega}}(\Delta_\infty) \ge \frac{1}{2}c_*$.

The idea of the proof is to find a subset which is transversal to the Ξ-leaves. We project the stack of unstable leaves $\mathcal{S}$ onto this set and then find suitable compact subsets with a finite collection of disjoint subsets and big enough measure. Projecting these sets back to $\mathcal{S}$ gives the desired sequence of compact subsets as in the lemma. We will focus in this proof on showing properties $(i) - (iv)$ using the construction in the proof of Lemma 28 in [2].

Proof. For ease of notation, we write in the following proof $\tilde{\mu} = \tilde{\mu}_{\underline{\omega}}$. Define $\Sigma = \exp_{x_3} E^{cs}_*(r_*)$ and choose $x \in \Lambda$ arbitrarily. By construction, $\operatorname{Lip}(\gamma_x) \le 1$ for $x \in \Lambda$ and $\gamma_x : E^u_*(r_*) \to E^{cs}_*(r_*)$ and therefore we can assume that $\operatorname{graph}(\gamma_x) \cap E^{cs}_*(r_*) = \{p\}$ for $p \in \operatorname{graph}(\gamma_x)$. Because this holds for all $x \in \Lambda$, and $\exp_{x_*}$ is a diffeomorphism, we can assume that Σ is transversal to the Ξ-leaves comprising $\mathcal{S}$. Set $\hat{\Sigma} = \Sigma \cap \mathcal{S}$ and define $\pi : \mathcal{S} \to \hat{\Sigma}$ as the projection along the Ξ-leaves. This lets us define the transverse measure $\tilde{\mu}^T = \tilde{\mu}^T_{\underline{\omega}}$ on $\hat{\Sigma}$ which can be seen as the push forward measure of $\tilde{\mu}$ using π. For each $m \ge 1$ let $\mathcal{Q}_m$ be a partition of Σ into cubes of lengths $\approx 1/2^m$ with the following properties:

(1) The sequence $\mathcal{Q}_m, m \ge 1$ is increasing, i.e., $\mathcal{Q}_{m+1} \ge \mathcal{Q}_m$ for each $m \ge 1$.

(2) We have $\vee_{m=1}^\infty \mathcal{Q}_m \stackrel{\circ}{=} \epsilon$, the partition of $\hat{\Sigma}$ into points $\tilde{\mu}$-mod 0; and

(3) For each $C \in \mathcal{Q}$, we have $\tilde{\mu}^T(\partial C) = 0$.

With $\mathcal{Q}$ fixed, the next step is to find finite collections $\check{\mathcal{Q}}_m, m \ge 1$, of disjoint compact sets. We do this via the following inductive procedure.
We state that we can fix an increasing sequence $c_1 \le c_2 \le \cdots < 1$ for which $\prod_{m=1}^\infty c_m = \frac{1}{2}$. Is this choice well defined? Assume it is true, then

$$\prod_{m=1}^{\infty} c_m = \prod_{m=1}^{\infty} e^{\log(c_m)} = e^{\sum_{m=1}^\infty \log(c_m)} = \frac{1}{2},$$

which is equivalent to

$$\sum_{m=1}^{\infty} \log(c_m) = \log(\frac{1}{2}).$$

Let $d_m = \log(c_m) < 0$. The above statement is the same as finding $d_1 \le d_2 \le \cdots < 0$ such that

$$\sum_{m=1}^{\infty} d_m = \log(\frac{1}{2})$$

which is possible as we can set for example $d_m = \frac{1}{2^m}\log(\frac{1}{2})$.
To start, for each $C \in \mathcal{Q}_1$ fix a compact subset $\check{C} \subset C \cap \pi(\Lambda)$ for which $\operatorname{dist}(\check{C}, \partial C) > 0$ and $\tilde{\mu}^T(\check{C}) \geq c_1 \tilde{\mu}^T(C \cap \pi(\Lambda))$. Set $\check{\mathcal{Q}}_1 = \{\check{C} : C \in \mathcal{Q}_1\}$ so that

$$\begin{aligned}\tilde{\mu}^T(\bigcup_{C\in\mathcal{Q}_1} \check{C}) &= \sum_{C\in\mathcal{Q}_1} \tilde{\mu}^T(\check{C}) \\ &\geq c_1 \sum_{C\in\mathcal{Q}_1} \tilde{\mu}^T(C \cap \pi(\Lambda)) \\ &= c_1 \tilde{\mu}^T(\check{\Sigma}) \geq c_1 c_*.\end{aligned}$$

Next, we construct successively $\check{\mathcal{Q}}_1, \check{\mathcal{Q}}_2, \cdots$ of disjoint compact subsets with the rule that

$(i)'$ for each $\check{C} \in \check{\mathcal{Q}}_m$, we have that $\check{C} \subset \check{C}'$ for some $\check{C}' \in \check{\mathcal{Q}}_{m-1}$, and

$(ii)'$ for each $\check{C}' \in \check{\mathcal{Q}}_{m-1}$, we have $\tilde{\mu}^T(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m, \check{C}\in\check{C}'} \check{C}) \geq c_m \tilde{\mu}^T(\check{C}')$.

Finally, with the $\{\check{\mathcal{Q}}_m\}$ constructed, we define the array of compact sets $\Delta_{m,k}$ as follows: for each $m \geq 1$ and $1 \leq k \leq K_m := |\check{\mathcal{Q}}_m|$ we define $\Delta_{m,1}, \ldots, \Delta_{m,K_m}$ to be the collection of sets of the form $\Lambda \cap \pi^{-1}(\check{C})$ as $\check{C}$ ranges over $\check{\mathcal{Q}}_m$. Λ and $\pi^{-1}(\check{C})$ are compact by construction, so are the Δ_m and $\Delta_{m,k}$ for $m \geq 1, 1 \leq k \leq K_m < \infty$. Items (i) and (ii) from the lemma follow directly.

Recall that $\mathcal{Q}_m$ is a finite collection of cubes of lengths $\approx 1/2^m$. Following the construction of the $\check{\mathcal{Q}}_m$ we expect the lengths of the different $\check{C} \in \check{\mathcal{Q}}_m$ to go to zero for $m \to \infty$ and conclude item (iii) from the lemma.

We get for $\Delta_m, m \geq 1$,

$$\begin{aligned}\tilde{\mu}(\Delta_m) &= \tilde{\mu}\big(\bigcup_{k=1}^{K_m} \Delta_{m,k}\big) \\ &= \tilde{\mu}\big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m} \Lambda \cap \pi^{-1}(\check{C})\big) \\ &= \tilde{\mu}\big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m} \pi^{-1}(\check{C})\big) = \tilde{\mu}^T\big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m} \check{C}\big),\end{aligned}$$

where we used in the last line that $\tilde{\mu}$ is supported on Λ. For $m \geq 2$ holds, using property $(i)'$ in the construction,

$$\bigcup_{\check{C}\in\check{\mathcal{Q}}_m} \check{C} = \bigcup_{\check{C}'\in\check{\mathcal{Q}}_{m-1}} \Big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m, \check{C}\in\check{C}'} \check{C}\Big)$$

and we get, using property $(ii)'$ in the construction

$$\tilde{\mu}(\Delta_\infty) = \tilde{\mu}\big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m} \pi^{-1}(\check{C})\big) = \tilde{\mu}\Big(\bigcup_{\check{C}'\in\check{\mathcal{Q}}_{m-1}} \Big(\bigcup_{\check{C}\in\check{\mathcal{Q}}_m, \check{C}\in\check{C}'} \pi^{-1}(\check{C})\Big)\Big)$$

$$
\begin{aligned}
&= \tilde{\mu}^T\Big(\bigcup_{\check{C}' \in \check{\mathcal{Q}}_{m-1}} \big(\bigcup_{\check{C} \in \check{\mathcal{Q}}_m, \check{C} \in \check{C}'} \check{C} \big) \Big) \\
&\geq c_m \tilde{\mu}^T\big(\bigcup_{\check{C}' \in \check{\mathcal{Q}}_{m-1}} \check{C}' \big) \\
&= c_m \tilde{\mu}(\Delta_{m-1}).
\end{aligned}
$$

It follows via a simple induction that $\tilde{\mu}(\Delta_m) \geq c_* \prod_{l=1}^{m} c_l$. Because the Δ_m are $\mathcal{B}$-measurable for $m \geq 1$, we can use continuity from above and get

$$
\tilde{\mu}(\Delta_\infty) = \tilde{\mu}\big(\bigcap_{m \geq 1} \Delta_m \big) = \lim_{m \to \infty} \tilde{\mu}(\Delta_m).
$$

For item (iv) we conclude

$$
\begin{aligned}
\tilde{\mu}(\Delta_\infty) &= \lim_{m \to \infty} \tilde{\mu}(\Delta_m) \\
&\geq c_* \prod_{l=1}^{\infty} c_l \\
&= \frac{1}{2} c_*.
\end{aligned}
$$

□

Given Δ_∞ as above, define:

$$
\mathcal{S}_\infty := \bigcup_{x \in \Delta_\infty} \xi(x).
$$

Next, we wish to slightly enlarge the partitions from above so that they contain intact W^n-leaves but still converge to $\mathcal{S}_\infty$.

Corollary 6.2. *There is a decreasing sequence of open sets*

$$
U_1 \supset U_2 \supset \dots \qquad \text{with} \qquad \bigcap_{m=1}^{\infty} U_m = \mathcal{S}_\infty
$$

and a sequence of partitions $\beta_m = \{\beta_{m,k}\}$ of U_m into finitely many disjoint open sets, with the properties, that

- (i) *the partition β_m are nested, i.e., each $\beta_{m,k} \subset \beta_{m-1,k'}$ for some k' ;*
- (ii) *each $\beta_{m,k}$ contains intact leaves of the compact substack $\mathcal{S}_{m,k}$, and*
- (iii) *for each ξ in $\mathcal{S}_\infty$, if $\beta_{m,k}(\xi)$ is the element of β_m containing ξ, then $\beta_{m,k}(\xi) \downarrow \xi$ as $m \to \infty$.*

Proof. Assume that the construction of Lemma 6.1 has been applied to find such a suitable sequence of compact subsets $\Delta_1 \supset \Delta_2 \supset \cdots$ of Λ. Note that for $m \geq 1$, the sets $\mathcal{S}_{m,k}$ are compact and pairwise disjoint among $1 \leq k \leq K_m$. Denote by

$$
\epsilon_m^{(k,k')} = \frac{1}{3} \mathrm{dist}(\mathcal{S}_{m,k}, \mathcal{S}_{m,k'}), \qquad k, k' \in \{1, 2, \dots, K_m\}, \qquad k \neq k',
$$

the pairwise distance between $\mathcal{S}_{m,k}$ and $\mathcal{S}_{m,k'}$. The $\mathcal{S}_{m,k}$ are compact and it follows $\epsilon_m^{(k,k')} > 0$ for $k \neq k'$. Because $|K_m| < \infty$, set

$$\tilde{\epsilon}_m = \min\{\epsilon_m^{(k,k')} : k, k' \in \{1, 2, \ldots, K_m\} \text{ and } k \neq k'\} > 0$$

and

$$\epsilon_m = \min\{\tilde{\epsilon}_m, \frac{1}{2^m}\} > 0.$$

Next we construct partitions $\beta_m = \{\beta_{m,k}\}$ for $m \geq 1$. Set

$$\beta_{m,k} \coloneqq \mathcal{N}_{\epsilon_m}(\mathcal{S}_{m,k}), \qquad 1 \leq k \leq K_m$$

and

$$U_m = \bigcup_{1 \leq k \leq K_m} \beta_{m,k}.$$

By construction, the $\beta_{m,k}$ are open and disjoint for $1 \leq k \leq K_m$ and U_m is open for $m \geq 1$. Note that $\epsilon_m \leq \epsilon_{m-1}$ for $m \geq 2$ and $\epsilon_m \to 0$ for $m \to \infty$. We conclude

$$U_1 \supset U_2 \supset \ldots$$

and

$$\begin{aligned}
\bigcap_{m=1}^{\infty} U_m &= \lim_{m \to \infty} U_m \\
&= \lim_{m \to \infty} \bigcup_{k=1}^{K_m} \mathcal{N}_{\epsilon_m}(\mathcal{S}_{m,k}) \\
&= \lim_{m \to \infty} \bigcup_{k=1}^{K_m} \mathcal{N}_{\epsilon_m}(\bigcup_{x \in \Delta_{m,k}} \xi(x)) \\
&= \mathcal{S}_\infty
\end{aligned}$$

Items (i) and (ii) follow directly by construction and Lemma 6.1(i). Take ξ in $\mathcal{S}_\infty$ and denote by $\beta_{m,k}(\xi)$ the element of β_m containing ξ. We have

$$\lim_{m \to \infty} \beta_{m,k}(\xi) = \lim_{m \to \infty} \mathcal{N}_{\delta_m}(\mathcal{S}_{m,k})(\xi)$$

using the same notation. From Lemma 6.1(iii) item (iii) follows immediately and we conclude the proof. □

6.2 Pushed forward measures and their conditional densities

In Corollary 6.2 we constructed sequences of open disjoint sets so that their union converges to $\mathcal{S}_\infty$. The following lemma states that we can always find a big enough n such that parts of the W^n-leaves are contained in the elements of the partition. This is important since we know from Proposition 5.2 that for big enough n, the W^n-leaves start to resemble the W^u-leaves.

Lemma 6.3. *For each $m \geq 1$, there exist $\eta_m > 0$ and $N_m \in \mathbb{N}$ with the property that the following hold for all $n = n_i \geq N_m$.*

(a) *Define*

$$\Lambda^n_{m,k} := \Lambda^n \cap \overline{\mathcal{N}_{\eta_m}(\Delta_{m,k})}, \quad \mathcal{S}^n_{m,k} := \bigcup_{x \in \Lambda^n_{m,k}} \xi^n(x).$$

Then $\mathcal{S}^n_{m,k} \subset \beta_{m,k}$.

(b) *Letting $\Lambda^n_m = \bigcup_{k=1}^{K_m} \Lambda^n_{m,k}$, we have*

$$\tilde{\mu}^n_{\underline{\omega}}(\Lambda^n_{m,k}) \geq \frac{2}{3}\tilde{\mu}_{\underline{\omega}}(\Lambda_{m,k}), \qquad \textit{hence} \qquad \tilde{\mu}^n_{\underline{\omega}}(\Lambda^n_m) \geq \frac{1}{3}c_*.$$

Proof. For each $m \geq 1$, let $\epsilon_m > 0$ be as in the proof of Corollary 6.2. Because $\exp_{x_*}$ is continuous, we find δ_m such that for all $x, y \in E_*(r_*)$ it holds

$$d(x,y)_{T_{x_*}M} < \delta_m \Rightarrow d(\exp_{x_*}(x), \exp_{x_*}(y))_M < \epsilon_m.$$

Applying Proposition 5.2 for δ_m gives constants $N_m \in \mathbb{N}$ and $\tilde{\eta}_m > 0$ such that for any $n = n_i \geq N_m$ and any $x \in \Lambda^n, y \in \Lambda$ with $d(x,y) < \tilde{\eta}_m$, we have that $\|\gamma^n_x - \gamma_y\| < \delta_m$ where $\|\cdot\|$ refers to the uniform norm on $C(E^u_*(r_*), E^{cs}_*(r_*))$.
The functions γ^n_x and γ_y are defined on $E^u_*(r_*)$ and so

$$d((v_1, \gamma^n_x(v_1)), (v_1, \gamma_y(v_1)))_{T_{x_*}M} < \delta_m$$

for all $v_1 \in E^u_*(r_*)$. It follows that

$$d(\exp_{x_*}(v_1, \gamma^n_x(v_1)), \exp_{x_*}(v_1, \gamma_y(v_1)))_M < \epsilon_m$$

for all $v_1 \in E^u_*$ and

$$\operatorname{dist}(\xi^n(x), \xi(y)) < \epsilon_m.$$

We conclude $\mathcal{S}^n_{m,k} \subset \beta_{m,k}$ for $\eta_m = \frac{1}{2}\tilde{\eta}_m$ because we consider the closure of $\mathcal{N}_{\eta_m}(\Delta_{m,k})$.
For (b) we assume that $\tilde{\mu}^n_{\underline{\omega}}$ converges to $\tilde{\mu}$. Choose N_m big enough such that (a) holds and additionally,

$$\tilde{\mu}^n_{\underline{\omega}}(\Delta_{m,k}) \geq \frac{2}{3}\tilde{\mu}(\Delta_{m,k})$$

for all $1 \leq k \leq K_m$ and $n = n_i \geq N_m$. We find

$$\begin{aligned}\tilde{\mu}^n_{\underline{\omega}}(\Lambda^n_{m,k}) = \tilde{\mu}^n_{\underline{\omega}}(\overline{\mathcal{N}_{\eta_m}(\Delta_{m,k})}) \geq \tilde{\mu}^n_{\underline{\omega}}(\Delta_{m,k}) \\ \geq \frac{2}{3}\tilde{\mu}(\Delta_{m,k}).\end{aligned}$$

Note that the sets $\Lambda^n_{m,k}$ are disjoint in k and so

$$\tilde{\mu}^n_{\underline{\omega}}(\Lambda^n_m) = \tilde{\mu}^n_{\underline{\omega}}(\bigcup_{k=1}^{K_m} \Lambda^n_{m,k}) = \sum_{k=1}^{K_m} \tilde{\mu}^n_{\underline{\omega}}(\Lambda^n_{m,k})$$

$$\geq \frac{2}{3}\sum_{k=1}^{K_m}\tilde{\mu}(\Delta_{m,k}) = \frac{2}{3}\tilde{\mu}(\Delta_m)$$
$$\geq \frac{1}{3}c_*,$$

where we used Lemma 6.1(b) in the last line. □

Until now, we have worked with $\mu_{\underline{\omega}}|_{\Lambda_m^n}$ to ensure that we are catching a definite fraction of $\mu_{\underline{\omega}}^n$. To prove Proposition 2.20, we are interested in $\mu_{\underline{\omega}}^n|_{\mathcal{S}_m^n}$, where $\mathcal{S}_m^n = \bigcup_{k=1}^{K_m}\mathcal{S}_{m,k}^n$. The important step is finding a part of $\mu_{\underline{\omega}}^n|_{\mathcal{S}_m^n}$ which has controlled conditional densities. Recall from Section 4.3 that we disintegrated μ on the leaves of $\mathcal{F}_-^n$, and then pushed them forward via $f_{\theta^{-n}\underline{\omega}}^n$. The foliation $\mathcal{F}_-^n$ is defined on a ball $B \subset \{\psi \geq \alpha_0/2\}$, where $\psi = \frac{d\mu}{d\mathrm{Leb}}$ is the Radon-Nikodym derivative. Define

$$\nu_-^n = \frac{\alpha_0}{2}\mathrm{Leb}|_B \qquad \text{and} \qquad \nu^n = (f_{\theta^{-n}\underline{\omega}}^n\nu_-^n)|_{\mathcal{S}^n}.$$

Since $B \subset \{\psi \geq \alpha_0/2\}$, we have $\nu_-^n \leq \mu$ and $\nu^n \leq \mu_{\underline{\omega}}^n$ for all n.
Furthermore, denote by $(\nu_\xi^n)_{\xi\in\Xi^n}$ the disintegration measures of ν^n along Ξ^n with transversal measure ν_T^n on $\mathcal{S}^n/\Xi^n$. For $\xi \in \Xi^n$, let $\phi(\xi)$ denote the leaf of $\mathcal{F}_-^n$ containing $f_{\underline{\omega}}^n\xi$. By construction of the W^n-leaves, the measure ν_ξ^n is $(f_{\theta^{-n}\underline{\omega}}^n\mathrm{Leb}_{\phi(\xi)})|_\xi$ normalized.
The next Lemma presents a lower bound of ν^n. It states that the disintegration measures ν_ξ^n are absolutely continuous and have controlled densities.

Lemma 6.4.

(a) *For measurable* $C \in \Lambda^n, \nu^n(C) \geq \frac{\alpha_0}{2\|\psi\|_\infty}\cdot\mu_{\underline{\omega}}^n(C)$

(b) *For almost every* $\xi \in \Xi^n, \nu_\xi^n$ *is absolutely continuous with density* $\rho_\xi^n : \xi \to (0,\infty)$ *satisfying the distortion estimate*

$$\left|\log\frac{\rho_\xi^n(p_1)}{\rho_\xi^n(p_2)}\right| \leq \bar{D}$$

for any $p_1, p_2 \in \xi$. *Here* $\bar{D} = \bar{D}(l_0, K_-, r_-) > 0$, *where* K_-, r_- *are as in Lemma 4.5;* $\bar{D}$ *is independent of* ξ *and* n.

In the scope of this project, we will present the proof for item (a) and the absolute continuity of the ν_ξ^n.

Proof. For the estimate in (a) we have

$$\mu_{\underline{\omega}}^n(C) = \mu(f_{\underline{\omega}}^{-n}C) = \int_{f_{\underline{\omega}}^{-n}C}\psi(x)\,dx$$
$$\leq \|\psi\|_\infty\mathrm{Leb}(f_{\underline{\omega}}^{-n}C)$$
$$= \|\psi\|_\infty\frac{2}{\alpha_0}\nu^n(C).$$

Rearranging this inequality yiuelds $\nu^n(C) \geq \frac{\alpha_0}{2\|\psi\|\|_\infty} \cdot \mu^n_{\underline{\omega}}(C)$.
For (b) we show that ν^n is absolutely continuous. Take a set $C \subset M$ with $\mathrm{Leb}(C) = 0$ and compute

$$\nu^n(C) = (f^n_{\theta^{-n}\underline{\omega}}\nu^n_-)|_{\mathcal{S}^n}(C) = \nu^n_-(f^{-n}_{\underline{\omega}}(C \cap \mathcal{S}^n)) = \frac{\alpha_0}{2}\int_{f^{-n}_{\underline{\omega}}(C\cap\mathcal{S}^n)\cap B} dx$$
$$= \frac{\alpha_0}{2}\int_{f^{-n}_{\underline{\omega}}(C\cap\mathcal{S}^n)} 1_B \, dx.$$

A substitute with $y = f^n_{\underline{\omega}}(x)$ results in

$$\nu^n(C) = \frac{\alpha_0}{2}\int_{C\cap\mathcal{S}^n} 1_B(f^n_{\theta^{-n}\underline{\omega}}(y))|\det(Df^n_{\theta^{-n}\underline{\omega}})(y)| \, dy = 0$$

due to $\mathrm{Leb}(C) \geq \mathrm{Leb}(C\cap\mathcal{S}^n) = 0$. Because C was chosen arbitrarily and measurable, ν^n is absolutely continuous for all n. Recall from Definition 2.16 that we have for $E \in \mathcal{B}$ the representation

$$\nu^n(E) = \int \nu^n_\xi(E) \, d\hat{\nu}^n(\xi).$$

Take $\tilde{C} \in M$, ν^n measurable with $\nu^n(\tilde{C}) = 0$. Then

$$\nu^n(\tilde{C}) = \int \nu^n_\xi(\tilde{C}) d\hat{\nu}^n \, (\xi) = 0.$$

Because the disintegration measures are always non-negative, it follows that $\nu^n_\xi(\tilde{C}) = 0$ for almost every $\xi \in \Xi^n$. We conclude that for almost every $\xi \in \Xi^n$, ν^n_ξ is absolutely continuous with densities $\rho^n_\xi : \xi \to (0, \infty)$.

□

6.3 Passing to the weak limit as $n \to \infty$ and completing the proof

At last, we will conclude the Main Proposition in this section. Let $\nu^n_m := \nu^n|_{\mathcal{S}^n_m}$. Using Lemma 6.4 we have that for every m,

$$\nu^n_m(\mathcal{S}^n_m) \geq \frac{\alpha_0}{6\|\psi\|_\infty}\mu^n_{\underline{\omega}}(\mathcal{S}^n_m) \geq \frac{\alpha_0}{6\|\psi\|_\infty}\mu^n_{\underline{\omega}}(\Lambda^n_m)$$

and from Lemma 6.3(b)

$$\nu^n_m(\mathcal{S}^n_m) \geq \frac{\alpha_0}{6\|\psi\|_\infty}c_* \tag{6.1}$$

for every $n = n_i \geq N_m$. We fix such a sequence $n(m)$ for each m. By Lemma 6.3(a) follows that $n(m) \to \infty$ as $m \to \infty$. Denote by ν^* any limit point of the sequence $\nu^{n(m)}_m$ as $m \to \infty$. Note that the set $\mathcal{S}^n_m$ converges to $\mathcal{S}_\infty$ for $m \to \infty$ and ν^* is

supported on $\mathcal{S}_\infty$. Moreover, $\nu^* \leq \mu_{\underline{\omega}}$ follows from $\nu_m^n \leq \nu^n \leq \mu_{\underline{\omega}}^n$ and that $\mu_{\underline{\omega}}^n$ converges weakly to $\mu_{\underline{\omega}}$ for $m \to \infty$. Note that the lower bound (6.1) passes to $\nu^*(\mathcal{S}_\infty)$ because it is a limit of $\nu_m^n(\mathcal{S}_m^n)$ as $m \to \infty$. Denote by ν_ξ^* the conditional measures of ν^* on the leaves $\xi \in \Xi$. In a last step we wish to deduce properties of the conditional measures of ν^* on leaves of Ξ from those of ν_m^n on Ξ^n which we already discussed.
Let $C^u \subset E_*^u(r_*)$ be a cube. We let $\hat{\mathrm{Leb}}(C^u) = \mathrm{Leb}(C^u)/\mathrm{Leb}(E_*^u(r_*))$, and define

$$V_{C^u} \coloneqq \exp_{x_*}(C^u + E_*^{cs}(r_*)).$$

Recall that $\mathcal{S}_m^n = \bigcup_{k=1}^{K_m} \mathcal{S}_{m,k}^n$ with $\mathcal{S}_{m,k}^n$ being disjoint for $1 \leq k \leq K_m$ and $\mathcal{S}_{m,k}^n \subset \beta_{m,k}$. Thus we have by construction $\mathcal{S}_m^n \cap \beta_{m,k} = \mathcal{S}_{m,k}^n$ and for $n = n(m)$ we have $\nu_m^n|_{\mathcal{S}_{m,k}^n} = \nu_m^n|_{\beta_{m,k}}$. The next Lemma deduces properties of disintegration of ν_* on the leaves of Ξ using the property of the disintegration of ν_m^n on leaves of Ξ^n.

Lemma 6.5. *There exists $A > 1$ such that for any $C^u \subset E_*^u(r_*)$, we have, for all large enough m and $n = n(m)$:*

$$\frac{1}{A} \cdot \hat{\mathrm{Leb}}(C^u) \leq \frac{\nu_m^n(\beta_{m,k} \cap V_{C^u})}{\nu_m^n(\beta_{m,k})} \leq A \cdot \hat{\mathrm{Leb}}(C^u). \tag{6.2}$$

It follows that for almost every $\xi \in \Xi$, we have $\pi_*^u \nu_\xi^* \ll \hat{\mathrm{Leb}}$ with $\frac{d(\pi_*^u \nu_\xi^*)}{d\mathrm{Leb}} \in [\frac{1}{A}, A]$, where π^u is the projection onto $E_*^u(r_*)$ along E_*^{cs}.

Proof. By absolute continuity of ν_ξ^n, there exists a $p_0 \in \xi$ such that $\frac{1}{D} < \rho_\xi^n(\xi)(p_0) < D$ for some $D \in [1, \infty)$. Using Lemma 6.4(b) we get the following estimates. For every $p \in \xi$ it holds

$$|\log \frac{\rho_\xi^n(p)}{\rho_\xi^n(p_0)}| \leq \bar{D}$$

which is equivalent to

$$\frac{1}{e^{\bar{D}}} \rho_\xi^n(p_0) \leq \rho_\xi^n(p) \leq \bar{D} \rho_\xi^n(p_0)$$

and we get

$$\frac{1}{e^{\bar{D}} D} \leq \rho_\xi^n(p) \leq e^{\bar{D}} D.$$

Set $A \coloneqq e^{\bar{D}} D$ and note that $0 < A < \infty$. It follows directly that (6.2) holds for $\pi_*^u \nu_\xi^n$ instead of ν_m^n but then by Definition 2.16(3) it must also hold for ν_m^n.
Let m and k be fixed. By construction of ν_m^n we have

$$\nu_{m'}^{n(m')}(\beta_{m,k} \cap V_{C^u}) = \nu^n|_{\mathcal{S}_{m'}^{n(m')} \cap \beta_{m,k}}(V_{C^u}) = \nu^n|_{\mathcal{S}_{m'}^{n(m')} \cap \mathcal{S}_{m,k}^n}(V_{C^u})$$

and

$$\nu_{m'}^{n(m')}(\beta_{m,k} \cap V_{C^u}) = \sum_{k' : \beta_{m',k'} \subset \beta_{m,k}} \nu_{m'}^{n(m')}(\beta_{m',k'} \cap V_{C^u})$$

for all $m' \geq m$ and all k' for $n(m')$ big enough (which works because $n(m') \to \infty$ for $m \to \infty$). Using (6.2) leads to

$$\sum_{k':\beta_{m',k'}\subset\beta_{m,k}} \nu_{m'}^{n(m')}(\beta_{m',k'} \cap V_{C^u}) \leq A \cdot \hat{\mathrm{Leb}}(C^u)\big(\sum_{k':\beta_{m',k'}\subset\beta_{m,k}} \nu_{m'}^{n(m')}(\beta_{m',k'} \cap V_{C^u})\big)$$
$$= A \cdot \hat{\mathrm{Leb}}(C^u) \cdot \nu_{m'}^{n(m')}(\beta_{m,k})$$

and

$$\sum_{k':\beta_{m',k'}\subset\beta_{m,k}} \nu_{m'}^{n(m')}(\beta_{m',k'}) \geq \frac{1}{A} \cdot \hat{\mathrm{Leb}}(C^u)\big(\sum_{k':\beta_{m',k'}\subset\beta_{m,k}} \nu_{m'}^{n(m')}(\beta_{m',k'})\big)$$
$$= \frac{1}{A} \cdot \hat{\mathrm{Leb}}(C^u) \cdot \nu_{m'}^{n(m')}(\beta_{m,k}).$$

We conclude that for $\beta_{m,k}$ fixed, (6.2) holds with ν_m^n replaced by $\nu_{m'}^{n(m')}$. Letting $m' \to \infty$ we get that (6.2) holds for ν_* in place of ν_m^n.
Next, we let $m \to \infty$ and run through all $\beta_{m,k}$ in $\beta_{m,k}$. Take $\xi \in \Xi$ with $\xi \in \mathrm{S}_\infty$. It follows from Corollary 6.2 that

$$\lim_{m\to\infty} \frac{\nu^*(\beta_{m,k} \cap V_{C^u})}{\nu^*(\beta_{m,k})} = \nu_\xi^*(V_{C^u}) = \pi_*^u \nu_\xi^*(C^u).$$

Recall that ν^* is supported on $\mathcal{S}_\infty$ and we can conclude that the above inequality holds for almost every $\xi \in \Xi$. Take $C \in E_*^u(r_*)$ with $\hat{\mathrm{Leb}}(C) = 0$. By the same steps as above we have for almost every $\xi \in \Xi$ that

$$\pi_*^u \nu_\xi^*(C) \leq A \cdot \hat{\mathrm{Leb}}(C) = 0$$

and $\pi_*^u \nu_\xi^* \ll \hat{\mathrm{Leb}}$. Let $\hat{\rho}_\xi$ be a Radon-Nikodym derivative of $\pi_*^u \nu_\xi^*$. Set

$$D^+ = \{x \in E_*^u(r_*) : \hat{\rho}_\xi(x) > A\}, \qquad D^- = \{x \in E_*^u(r_*) : \hat{\rho}_\xi(x) < \frac{1}{A}\}$$

and assume $\hat{\mathrm{Leb}}(D^+),\ \hat{\mathrm{Leb}}(D^-) > 0$. By the properties of the Lebesgue measure there exists a cube $C^+ \in E_*^u(r_*)$ with $\hat{\mathrm{Leb}}(C^+) > 0$. This results in

$$\pi_*^u \nu_\xi^*(C^+) = \int_{C^+} \hat{\rho}_\xi(x)\ \hat{\mathrm{Leb}}(dx) > A \cdot \hat{\mathrm{Leb}}(C^+)$$

which contradicts (6.2) and so $\hat{\mathrm{Leb}}(D^+) = 0$. Similarly, we find $\hat{\mathrm{Leb}}(D^-) = 0$. Since the Radon-Nikodym derivative is by definition defined up to a set of Lebesgue measure 0, we find $\frac{d(\pi_*^u \nu_\xi^*)}{d\hat{\mathrm{Leb}}} \in [\frac{1}{A}, A]$. □

It follows directly from Lemma 6.5 that the conditional probabilities of ν^* on elements of Ξ are absolutely continuous with respect to the Lebesgue measure and that the densities are uniformly bounded from above and below.

6.3.1 Concluding the Main Proposition

While the paper [2] concludes the proof at this point, we want to give a short overview of why the Proposition 2.20 is now indeed proven.

1. In Chapter 4, Lemma 4.3, we constructed a $\mathbb{P}$-measure set $\mathcal{G}$ with $\mathbb{P}(\mathcal{G}) \geq \frac{\mathfrak{c}-1}{\mathfrak{c}}$. At the beginning of Chapter 5 we set $\mathfrak{c} = 2$ and get the positive $\mathbb{P}$-measure set $\mathcal{G}$ of $\underline{\omega}$ with $\mathbb{P}(\mathcal{G}) \geq \frac{1}{2}$.

2. Fix $\underline{\omega} \in \mathcal{G}$ and denote by $\mathcal{S}_{\underline{\omega}}$ the stack of unstable leaves on Λ as in Chapter 5. Let $\nu_1 = \nu_*$ and $c = \frac{\alpha_0}{6\|\psi\|_\infty} c_*$ as in Equation (6.1). It follows from the definition of ν^* and the discussion in Section 6.3 that $\nu^*(\mathcal{S}_{\underline{\omega}}) \geq \nu^*(\mathcal{S}_\infty) \geq c$.

3. It follows directly from Lemma 6.5 that the conditional probabilities of ν_1 on elements of Σ are absolutely continuous with respect to the Lebesgue measure and have densities uniformly bounded from above and below.

At last, we have finally shown the Main Proposition and can conclude the Main Theorem.

Chapter 7

Conclusion

7.1 Summary and overview

In this thesis we showed that SRB measures exist in a C^2 ergodic chaotic random setting. SRB measures do not only exist but they are the sample measures and can be constructed form the stationary measure by pushing it forward from time $-\infty$ to time 0 along an $\underline{\omega} \in \Omega^{\mathbb{Z}}$. To prove the SRB properties of the sample measures, we saw that it is enough to look at a local stack of unstable leaves $\mathcal{S}_{\underline{\omega}}$ with partition Ξ on which a part of $\mu_{\underline{\omega}}$, denoted by ν_1, is supported and has positive measure. Further, if if we prove that the conditional probabilities of ν_1 on elements of Ξ are absolutely continuous with respect to the Lebesgue measure and have densities uniformly bounded from above and below, then $\mu_{\underline{\omega}}$ is SRB.
In Chapter 2 we introduced the necessary definitions and concepts concerning RDS. Using adapted one-sided charts, we showed that by choosing a small enough source set and target set, the distortion along the graph transforms can be controlled and k-times graph transforms are properly bounded if we do enough iterations of an initially bounded map g enough times on the source set (Chapter 3). The controlled distortion estimates are later used in Chapter 6 to present upper and lower bounds for the densities on the disintegration. In Chapter 4 we then used that the sample measures of $\mu \times \mathbb{P}^+$ are μ and used the adapted one-sided charts to project μ^* onto $\mu \times \mathbb{P}^+$ to construct the source and target sets. We found a sequence of $\{n_i\}$ such that the set points which get pushed forward from the source set to the target set at time $n = n_i$ has positive $\mu^n_{\underline{\omega}}$ measure. The bounded graph transforms and the source and target sets are then used in Chapter 5. There, we constructed concrete u-graphs that go from the source set into the target set and pushed them forward to create the W^n-leaves. Next we performed a switch of axes to compare them to the unstable leaves and showed that the stack of W^n-leaves $\mathcal{S}^n$ starts to resemble the stack of unstable leaves $\mathcal{S}$ (specifically $\limsup_{n\to\infty} \mathcal{S}^n \subset \mathcal{S}$).
At last, we proved in Chapter 6 the SRB properties via a geometric construction which we first introduced in Section 1.3. By pushing forward ν^n_-, a scaled Lebesgue measure that is supported on a small set, we can construct new measures ν^n that are supported on $\mathcal{S}^n$. Using the distortion estimate, we show that conditional densities of ν^n can be bounded from above and below. By partitioning parts of the stack of

unstable leaves into nested partitions of open sets which eventually only contain single unstable leaves, we saw that we can choose small enough source and target sets such that positive parts of the u-graphs get mapped into the partitions. Using the controlled densities on the W^n-leaves and the fact that we can approximate a portion of the unstable leaves with W^n-leaves as $n \to \infty$, we found a measure ν_1 that is supported on S with $\nu_1(\mathcal{S}) > c$ for some $c > 0$ and has absolute continuity of the conditional probabilities which concludes the proof.

7.2 Reflection and achievements

The paper [2] by Alex Blumenthal and Lai-Sang Young serves as a good source to become familiar with random dynamical systems as well as the disintegration of measures. Its non-trivial nature, the fact that a lot of experience is presupposed and that it leaves out a few steps which are probably obvious to the advanced reader challenged me to fully understand the topic and to do my own research on unfamiliar concepts. I fully achieved my original goals by providing explanations for almost every little detail from scratch and thoroughly explain the main result in [2].
I have learnt a lot about the process of showing absolute continuity of the conditional probabilities. It was totally new to me that by pushing forward a scaled version of the Lebesgue measure and supporting it on stacks of leaves which eventually end up being the single unstable leave we can make assumptions about the conditional measure on the unstable manifolds. I learnt the benefit of using one-sided charts for finding the source and target set and the process of bounding the distortion (something which is normally not always possible) by choosing a small enough source set which in turn leads to the bound of the conditional probabilities. In general, I think its fascinating how we were able to show the SRB by pushing forward u-graphs which, at the beginning, seem to have nothing to do with the unstable leaves but eventually get aligned with them due to the chaotic properties of the system.

7.3 Future work

For future work it would be interesting to look at a concrete example where we can see how the u-graphs evolve over time and how we can choose the source sets to control the distortion as well as observe the alignment with the unstable leaves. Since SRB measures are an interesting class of invariant measures it would be fascinating to look at a real world application and see how they can concretely help in analysing properties of a random dynamical system. As an example, the paper [4] studies climate dynamics and discusses SRB measures as time-dependent invariant measures for random attractors.

Appendix A

Selection of significant original contributions

Selection of significant original contributions

Chapter	*Location*	*Details*
1	All	The whole chapter is original researched content, especially Section 1.3.
2	Lemma 2.1	The fact that both RDS definitions give rise to the same RDS.
	Proof of Theorem 2.19	Detailed explanation using Lebesgue's decomposition theorem and Proposition 2.20.
3	Lemma 3.1	The proof of the measurable selection of the past using the Kuratowski-Ryll-Nardzewski measurable selection theorem.
	Lemma 3.3	We prove (i'), (ii') and (iii') by introducing Pesin theory.
	Proposition 3.6	Detailed construction of m_0 and m_1 so that the graph transforms are properly bounded.
4	Lemma 4.1	We show why it is enough to prove the auxiliary Lemma 4.2.
	Lemma 4.3	Thorough and detailed explanation of the important steps in the proof.
	Lemma 4.4	Additional explanations on why the inequalities in the proof hold.
5	Proposition 5.2	A lot of additional technical explanations get provided. Detailed discussion on why we choose the initial zero functions and the process of bounding the norms.
6	Lemma 6.1	The technicalities of the construction of the partition and explanation on why items $(i) - (iv)$ follow from the partitions on Σ.
	Corollary 6.2	Full proof of the corollary.
	Lemma 6.3	Full proof of the lemma.
	Lemma 6.4	Full proof of item (a) and the absolute continuity of ν_ξ^n.
	Lemma 6.5	Detailed technicalities of the lemma using our acquired knowledge from writing this project. We present clear explanations on why the limits can be taken to construct the bounds for the conditional measures on the unstable leaves.

Table A.1: A selection of original work done in this project.

Bibliography

[1] Ludwig Arnold. Random dynamical systems. In *Dynamical systems*, pages 1–43. Springer, 2003. pages 7, 9, 10

[2] Alex Blumenthal and Lai-Sang Young. Equivalence of physical and srb measures in random dynamical systems. *Nonlinearity*, 32(4):1494–1524, Mar 2019. ISSN 1361-6544. doi: 10.1088/1361-6544/aafaa8. URL http://dx.doi.org/10.1088/1361-6544/aafaa8. pages 1, 2, 5, 8, 10, 13, 14, 17, 18, 19, 20, 25, 32, 33, 35, 37, 38, 39, 42, 51, 54

[3] Rufus Bowen. Equilibrium states and the ergodic theory of anosov diffeomorphisms. *Springer Lecture Notes in Math*, 470:78–104, 1975. pages 1

[4] Mickaël D Chekroun, Eric Simonnet, and Michael Ghil. Stochastic climate dynamics: Random attractors and time-dependent invariant measures. *Physica D: Nonlinear Phenomena*, 240(21):1685–1700, 2011. pages 54

[5] F. Ledrappier and L. S. Young. Entropy formula for random transformations. *Probability Theory and Related Fields*, 80(2):217–240, December 1988. ISSN 0178-8051. doi: 10.1007/BF00356103. pages 2, 5

[6] François Ledrappier and L-S Young. The metric entropy of diffeomorphisms: Part i: Characterization of measures satisfying pesin's entropy formula. *Annals of Mathematics*, pages 509–539, 1985. pages 14

[7] Pei-Dong Liu, Min Qian, and Fu-Xi Zhang. Entropy formula of pesin type for one-sided stationary random maps. *Ergodic Theory and Dynamical Systems*, 22 (6):1831–1844, 2002. doi: 10.1017/S0143385702000767. pages 5

[8] David Ruelle. A measure associated with axiom-a attractors. *American Journal of Mathematics*, pages 619–654, 1976. pages 1

[9] Bruno Santiago. Measurable partitions, disintegration and conditional measures. 2017. pages 12

[10] Yakov G Sinai. Gibbs measures in ergodic theory. *Russian Mathematical Surveys*, 27(4):21, 1972. pages 1

[11] Daniel W Stroock. *A concise introduction to the theory of integration*. Springer Science & Business Media, 1998. pages 7

[12] Tamás Titkos. A simple proof of the lebesgue decomposition theorem. *The American Mathematical Monthly*, 122(8):793–794, 2015. pages 14

[13] Daniel H Wagner. Survey of measurable selection theorems. *SIAM Journal on Control and Optimization*, 15(5):859–903, 1977. pages 18

[14] Lai-Sang Young. What are srb measures, and which dynamical systems have them? *Journal of statistical physics*, 108(5):733–754, 2002. pages 13

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Declaration of originality

The signed declaration of originality is a component of every semester paper, Bachelor's thesis, Master's thesis and any other degree paper undertaken during the course of studies, including the respective electronic versions.

Lecturers may also require a declaration of originality for other written papers compiled for their courses.

I hereby confirm that I am the sole author of the written work here enclosed and that I have compiled it in my own words. Parts excepted are corrections of form and content by the supervisor.

Title of work (in block letters):

The existence of SRB measures in chaotic random motions: a graph transform approach

Authored by (in block letters):

For papers written by groups the names of all authors are required.

Name(s):	**First name(s):**
Gerber	Damian

With my signature I confirm that

- I have committed none of the forms of plagiarism described in the 'Citation etiquette' information sheet.
- I have documented all methods, data and processes truthfully.
- I have not manipulated any data.
- I have mentioned all persons who were significant facilitators of the work.

I am aware that the work may be screened electronically for plagiarism.

Place, date	**Signature(s)**
London, 10 June 2022	

For papers written by groups the names of all authors are required. Their signatures collectively guarantee the entire content of the written paper.