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Effectivity of uniqueness of the maximal entropy measure on p-adic homogeneous spaces

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Abstract. We consider the dynamical system given by an Ad-diagonalizable element a of the $\mathbb{Q}_p$-points G of a unimodular linear algebraic group acting by translation on a finite volume quotient X. Assuming that this action is exponentially mixing (e.g. if G is simple) we give an effective version (in terms of K-finite vectors of the regular representation) of the following statement: If μ is an a-invariant probability measure with measure-theoretical entropy close to the topological entropy of a, then μ is close to the unique G-invariant probability measure of X.

1. *Statement of the result*

Let $\mathbb{G}$ be an algebraic subgroup of SL_m defined over $\mathbb{Q}$ such that $G = \mathbb{G}(\mathbb{Q}_p)$ is a d-dimensional unimodular group. Suppose further that Γ is a lattice of G, and let us consider the (necessarily compact) quotient $X = \Gamma\backslash G$ with unique right-G-invariant probability measure m_X which we call the Haar measure on X. We normalize the Haar measure m_G such that it is compatible with m_X. Let $a \in G$ be Ad-diagonalizable over $\mathbb{Q}_p$ such that at least one eigenvalue is not of p-adic norm one, and consider the corresponding map on X, $x \mapsto a{\bullet}x = xa^{-1}$. Denote by $h_\mu(a)$ the measure-theoretic entropy of an a-invariant Borel probability measure μ on X; $h_{m_X}(a)$ equals the topological entropy $h_{\mathrm{top}}(a)$ of a.

Assume that the group $\langle G_a^+, G_a^- \rangle$ generated by the horospherical groups G_a^+ and G_a^- acts uniquely ergodically on X; that is, the Haar measure m_X is the unique measure invariant under $\langle G_a^+, G_a^- \rangle$. Then m_X is also the unique measure of maximal entropy with respect to translation by a. This goes back to [**AW67**] for toral automorphisms and see e.g. [**MT94**, **EL10**] for the present case. Say $f : X \to \mathbb{R}$ is smooth if it is locally constant and attach the integer

$$l_f = \min\{l \in \mathbb{N} \mid f(xg) = f(x)\ \forall x \in X \text{ and } \forall g \in G \text{ s.t. } g \equiv e \ (\mathrm{mod}\ p^l)\}$$

to it. Equivalently, f is a $K = \mathrm{SL}_m(\mathbb{Z}_p) \cap G$-finite vector of the right-regular representation $L^2(X, m_X)$ of G ramified of level l_f. We want to consider an effective analogue of unique ergodicity of $\langle G_a^+, G_a^- \rangle$, namely, the following mixing assumption, which we impose: The action by a is exponentially fast mixing; that is, there exist strictly positive constants c, α, δ such that for any two smooth vectors $f, g \in L_0^2(X)$ of vanishing integral and ramified of level l we have

$$|\langle f \circ a^n, g\rangle \leq cp^{l\alpha} \|f\|_{L^2(X)} \|g\|_{L^2(X)} \|a\|^{-\delta n} \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

We note that this is always the case if G is simple.

THEOREM 1.1. *Let $a \in G$ act on X with the assumption made above. Then there exists a constant $\kappa = \kappa(a, \alpha, \delta, c, X)$ such that for any a-invariant Borel probability measure μ and smooth function f, it holds that*

$$\left| \int f \, dm_X - \int f \, d\mu \right| \leq \kappa p^{(2\alpha + (d/2)) l_f} \|f\|_{L^2(X, m_X)} (h_{m_X}(a) - h_\mu(a))^{1/2}.$$

The constant κ is explicitly calculated to be

$$\kappa = \sqrt{2} c p^{2\alpha} m_G (K_2^G)^{-1/2} (1 - \|a\|^\delta)^{-1} \exp((3\alpha + d) h_{m_X}(a))$$

where K_2^G denotes the ball of radius p^{-2} in G and α, δ and c are the constants from (1).

Analogous results have been obtained for toral automorphisms [**Pol11**] and for hyperbolic maps on Riemann manifolds [**Kad14**]. We comment in the next chapter on the adoption for real quotients.

2. *Outline of the proof*

Denote by G_a^- and G_a^+ the stable and unstable horospherical subgroups of G with respect to a:

$$G_a^- = \{g \in G \mid a^n g a^{-n} \to e \text{ as } n \to \infty\},$$
$$G_a^+ = \{g \in G \mid a^n g a^{-n} \to e \text{ as } n \to -\infty\}.$$

The assumption on a of having an eigenvalue not of absolute value one implies that there is at least one contracting and one expanding direction in the Lie algebra so that G_a^- is non-trivial. Let

$$\mathrm{mod}(a, G_a^-) = |\det \mathrm{Ad}_a|_{\mathrm{Lie}(G_a^-)}|$$

denote the modular character corresponding to the conjugation action of a restricted to G_a^-. We recall that, in this notation,

$$h_{\mathrm{top}}(a) = \log \mathrm{mod}(a, G_a^-),$$

whose calculation we provide in Proposition 4.1. One can formulate the following variational principle.

THEOREM 2.1. [**EL10**, Theorems 7.6 and 7.9] *For any $a \in G$ and any a-invariant probability measure μ on X, the entropy of μ is bounded by*

$$h_\mu(a) \leq \log \mathrm{mod}(a, G_a^-)$$

and equality holds if and only if μ is G_a^--invariant.

Applying this result also to a^{-1}, one deduces that $h_\mu(a) = h_{\mathrm{top}}(a)$ if and only if μ is invariant under the group generated by G_a^- and G_a^+, and, by the unique ergodicity assumption on G, m_X is the unique measure of that property. It follows that $\mu = m_X$. We want to remark at this point on the connection to our mixing assumption. The Lie algebra generated by the Lie algebras $\mathfrak{g}^-$ and $\mathfrak{g}^+$ of G_a^- and G_a^+ is a Lie ideal $\mathfrak{f}$ of $\mathfrak{g} = \mathrm{Lie}(G)$, called the *Auslander ideal*. If a is mixing on $L^2(X)$ then G_a^- is already uniquely ergodic. On the other hand, the Howe–Moore theorem guarantees mixing in many natural cases (i.e. any ergodic action of a simple group is mixing).

We quickly compare the proof of Theorems 1.1 and 2.1. The if and only if part of the latter comes from the strict convexity of the map $x \mapsto x \log x$. A second-order approximation of this function shall enable us to deduce Theorem 1.1. More precisely, to compare μ and m_X we relate their conditional measures with respect to a G_a^-*-subordinate* σ-algebra

$$\mathcal{A} = \bigvee_{n=0}^{\infty} a^{-n}\xi$$

on X. By G_a^--subordinate we mean that the atoms of $\mathcal{A}$ consist of G_a^-*-plaques*: sets of the form $[x]_{\mathcal{A}} = xV$ for some $V \subset G_a^-$ containing a neighborhood of the identity. Here ξ is a countable measurable partition of X that generates the Borel σ-algebra $\mathcal{B}$ in the sense that $\bigvee_{n=-\infty}^{\infty} a^{-n}\bullet\xi = \mathcal{B}$ modulo μ. In our situation V will not depend on $x \in X$. The atoms of $[x]_{a^{-l}\bullet\mathcal{A}}$ will then support more and more of the G_a^--orbit at x as $l \to \infty$. By the generating assumption,

$$h_\mu(a) = H_\mu(\mathcal{A}|a^{-1}\bullet\mathcal{A}) \stackrel{\mathrm{Def}}{=} \int -\log \mu_x^{a^{-1}\bullet\mathcal{A}}([x]_{\mathcal{A}})\, d\mu.$$

As for the Haar measure, the inner information function is constant (equal to $h_{m_X}(a)$) and this shall allow us to rewrite the entropy difference $h_{m_X}(a) - h_\mu(a)$ as one integral over μ of the form

$$h_{m_X}(a) - h_\mu(a) = \int_X \sum \log \frac{q_i}{p_i} q_i \, d\mu$$

where p_i and q_i abbreviate the conditional measures $(m_X)_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}})$ and $\mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}})$, respectively, and we sum over x_i such that $\bigsqcup [x_i]_{\mathcal{A}} = [x]_{a^{-1}\bullet\mathcal{A}}$. By convexity, the integrand vanishes if and only if $p_i = q_i$ for all i. An iteration shows that all the conditional measures $\mu_x^{a^{-l}\bullet\mathcal{A}}$ are Haar, from which one can deduce that μ is necessarily G_a^--invariant.

To quantify this we will instead relate the above sum to $\frac{1}{2}(\sum |p_i - q_i|)^2$ by means of Pinsker's inequality. Pushing invariance from the conditional measures to the actual measure will require an exponential rate of equidistribution of the G_a^--plaques when expanded by conjugation with a^{-l}, which explains the restriction to K-finite vectors. We note that in this overview we have made use of a stronger property of $\mathcal{A}$ than the one that we will actually use: Instead of a σ-algebra whose atoms are parts of G_a^--orbits, we will use one whose atoms are contained in the manifold of a larger orbit set of the subgroup $\{g \in G \mid a^n g a^{-n} \text{ stays bounded as } n \to \infty\}$.

Apart from how equidistribution of the test function is achieved, the proof of Theorem 1.1 will be along the lines of [**Pol11**, **EL10**], the latter being present in a rather

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trivialized fashion as we do not have to deal with the boundary of the partition (and therefore handle different null sets with respect to μ, $\mu_y^{\mathcal{A}}$ and m_X) nor use the fact that $a^l \bullet \mathcal{A}$ actually generates $\mathcal{B}$ (for which we pay the price by having l_f appear in high degree in the final bound of Theorem 1.1).

We expect that the result also holds for real quotients. It requires, however, a more careful study of the corresponding equidistribution theorem. Indeed, in the case discussed here, the profinite structure of G provides us with a decomposition of the space X into orbits of compact groups, which then are no longer given. We note that equidistribution is in fact only needed in L^2 and not pointwise (as used here), which will give some leeway to a possible extension.

3. *Notation of and facts about linear p-adic Lie groups*

We quickly introduce the necessary notation and collect some facts about linear p-adic Lie groups along the way. Denote by $\mathbb{Z}_p$ the p-adic integers of $\mathbb{Q}_p$ and $|\cdot|_p$ the p-adic norm defined by $d(ax) = |a|_p\, dx$ if dx is a Haar measure of $\mathbb{Q}_p$. Equip the vector space $\mathfrak{m} = \mathrm{Mat}_d\,(\mathbb{Q}_p)$ with the maximum norm $\|X\| := \max_{i,j} |X_{ij}|_p$ which is again non-Archimedean so that $\|X + Y\| \leq \max\{\|X\|, \|Y\|\}$ and $\|XY\| \leq \|X\|\|Y\|$. The group of invertible matrices $\mathrm{GL}_d\,(\mathbb{Z}_p)$ with integral entries is endowed with the bi-invariant metric $d(g, h) = \|g - h\|$ that can be extended to a left-invariant metric on $\mathrm{GL}_d\,(\mathbb{Q}_p)$ in the following way: decompose $\mathrm{GL}_d\,(\mathbb{Q}_p) = \bigsqcup_{n>0} g_n\, \mathrm{GL}_d\,(\mathbb{Z}_p)$ with $g_1 = e$ and define

$$f(g) = \begin{cases} d(g, e) & \text{if } g \in \mathrm{GL}_d\,(\mathbb{Z}_p), \\ n & \text{if } g \in g_n\, \mathrm{GL}_d\,(\mathbb{Z}_p) \text{ for } n > 1, \end{cases}$$

and

$$\tilde{f}(g) = \inf\left\{\sum f(h_i) : g = h_1^{\varepsilon_1} \cdots h_k^{\varepsilon_k} \text{ with } \epsilon_i \in \{\pm 1\},\, h_i \in \mathrm{GL}_d\,(\mathbb{Q}_p)\right\}.$$

Then $d'(g_1, g_2) = \tilde{f}(g_1^{-1} g_2)$ defines a proper left-invariant metric that extends d. We will denote

$$K_k^{\mathfrak{m}} = p^k\, \mathrm{Mat}_d\,(\mathbb{Z}_p)$$

and for a p-adic Lie group $H = \mathbb{H}(\mathbb{Q}_p)$ where $\mathbb{H} < \mathrm{GL}_d$ is algebraic, the ball of radius p^{-k} in H by

$$K_k^H = (e + K_k^{\mathfrak{m}}) \cap H$$

and for the Lie algebra $\mathfrak{h}$ of H (defined as the set of tangents of analytic curves in H at the identity) we let $K_k^{\mathfrak{h}} = K_k^{\mathfrak{m}} \cap \mathfrak{h}$. The exponential function $X \mapsto \exp X$ defines a locally analytic isomorphism between the Lie algebra $\mathfrak{h}$ and H for which we refer to [**PR94**, p. 116] and the references therein. In the non-Archimedean setting exp turns out to be isometric.

LEMMA 3.1. *The exponential map* $\exp X = e + X + (X^2/2!) + \cdots$ *is isometric and maps Lie algebras to subgroups. More precisely, for any* $k \geq 2$ *one has that*

$$\exp K_k^{\mathfrak{h}} = K_k^H.$$

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Proof. The matrix exponential map and logarithm $\log(g) = (g-e) - (g-e)^2/2 + \cdots$ are mutually inverse and isometric whenever they are defined. Indeed, one calculates that $|n!|_p = p^{-\sum_{j\geq 1}\lfloor n/p^j \rfloor} \geq p^{-n/(p-1)}$ and thus by Hadamard's formula the radius of convergence of exp is at least $p^{-1/(p-1)}$ so that exp converges absolutely on $K_2^{\mathfrak{m}}$ (the special case here is $p=2$ where exp converges only on $K_2^{\mathfrak{m}}$, for higher p, the domain of convergence is actually $K_1^{\mathfrak{m}}$). In particular, for $x \in K_2^{\mathfrak{m}}$ we have $\|X^n/n!\| \leq p^n\|X\|^n \leq p^{-n}$ and thus $\|\exp(X) - e\| = \max_n \|X^n/n!\| = \|X\|$. Similarly, log converges on $e + K_1^{\mathfrak{m}}$.

As already mentioned, exp is a local isomorphism between $\mathfrak{h}$ and H. First note that this implies $\exp(K_2^{\mathfrak{h}}) \subset H$. Indeed, assume that exp and log are isomorphisms of $K_l^{\mathfrak{h}}$ and K_l^H for some sufficiently large $l > 0$. Then, for any polynomial f that defines $\mathbb{H}$, and any $X \in K_2^{\mathfrak{h}}$, we have $f(\exp tX) = 0$ for all $t \in p^l\mathbb{Z}_p$. As $f(\exp tX)$ is a power series in t, this implies $f(\exp tX) = 0$ for all $t \in \mathbb{Z}_p$. On the other hand, if $h \in K_2^H$, then $\exp p^l \log h = h^{p^l} \in K_l^H$ so that $\log h \in p^{-l}\mathfrak{h} = \mathfrak{h}$ and thus $\log h \in K_2^H$, because the logarithm map is isometric. This shows that exp is also surjective on K_2^H. □

Remark. We want to call a $\mathbb{Z}_p$-submodule $\mathfrak{h} < K_2^{\mathfrak{m}}$ closed under taking commutators a $\mathbb{Z}_p$-Lie algebra. Dynkin's form of the Baker–Campbell–Hausdorff formula for $z = \log(\exp x \exp y)$ is

$$z = \sum z_n, \quad z_n = \frac{1}{n}\sum_{i+j=n}(x_{i,j} + y_{i,j})$$

with

$$x_{i,j} = \sum \frac{(-1)^{m+1}}{m}\frac{\mathrm{ad}(x)^{i_1}\,\mathrm{ad}(y)^{j_1}\cdots\mathrm{ad}(x)^{i_m}(y)}{i_1!\,j_1!\cdots i_m!}$$

summing over $i_k + j_l \geq 1$ such that $\sum_{k\leq m} i_k = i$, $\sum_{l<m} j_l = j-1$, $i_m \geq 1$ and a similar formula holds for $y_{i,j}$ (see [**Ser92**, p. 29]). Bounding the factorials such as in the previous Lemma implies that $z \in \mathfrak{h}$ so that $\exp\mathfrak{h}$ is a p-adic Lie group if $\mathfrak{h}$ is a $\mathbb{Z}_p$-Lie algebra.

Denote by $\pi : G \to X$ the projection map to the quotient space and let us scale the Haar measure m_G on G to be compatible with the probability measure m_X, $m_X(\pi(B)) = m_G(B)$ for any sufficiently small ball B in G. We mentioned in the beginning that for a p-adic homogeneous space the existence of a finite measure already implies compactness. We give a quick argument (taken from [**Rat98**]) since it also provides an injectivity radius independent of Γ.

PROPOSITION 3.1. *The space $\Gamma\backslash G$ is compact with uniform injectivity radius p^{-2}.*

Proof. We first claim that for any $\Gamma < G$ the intersection $\Gamma \cap K_2^G$ must be trivial. As Γ is discrete and K_2^G compact, it suffices to show that $\Gamma \cap K_2^G$ does not contain any finite non-trivial subgroups. As for any $g \in \Gamma \cap K_2^G$ we have by denseness of $\mathbb{Z}$ in $\mathbb{Z}_p$ that $\overline{\{g^n\}} = \exp(\mathbb{Z}_p \log g) \subset \Gamma \cap K_2^G$. Hence any non-trivial subgroup must contain a copy of $\mathbb{Z}_p$. We may deduce the proven fact also for the conjugated variant, $g^{-1}\Gamma g \cap K_2^G$, which readily implies that the projection map π is injective on the neighborhood gK_2^G for any $g \in G$ so that p^{-2} is a uniform injectivity radius. By definition of m_X,

$$m_X(\pi(gK_2^G)) = m_G(gK_2^G) \equiv \text{const}.$$

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But m_X is also finite, so $\Gamma\backslash G$ must be covered by only finitely many translates of K_2^G. $\square$

4. *Bowen balls and topological entropy $h_{m_X}(a)$*

Let $\mathfrak{g} = \mathrm{Lie}(G) < \mathfrak{sl}_d(\mathbb{Q}_p)$ denote the Lie algebra of G. Define the stable horospherical subgroup

$$G_a^- = \{g \in G \mid a^n g a^{-n} \to e \text{ as } n \to \infty\}$$

and denote its Lie algebra by $\mathfrak{g}^-$. Analogously let

$$G_a^+ = \{g \in G \mid a^n g a^{-n} \to e \text{ as } n \to -\infty\}$$

define the unstable horospherical subgroup with Lie algebra $\mathfrak{g}^+$. Further denote by G_a^0 the group consisting of elements $g \in G$ for which $a^n g a^{-n}$ stays bounded for both $n \to \pm\infty$ with Lie algebra $\mathfrak{g}^0$, and, finally, put

$$P = \{g \in G \mid a^n g a^{-n} \text{ stays bounded as } n \to \infty\}$$

with Lie algebra $\mathfrak{g}^0 + \mathfrak{g}^-$ which we will call the thickened stable horospherical subgroup. The Lie algebras $\mathfrak{g}^-$, $\mathfrak{g}^+$ and $\mathfrak{g}^0$ correspond simply to the eigenspaces of Ad_a of eigenvalues in absolute value smaller than one, bigger than one or equal to one, respectively, and thus $\mathfrak{g} = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$. For each eigenspace E_λ of Ad_a there exists a basis of the $\mathbb{Z}_p$-module $E_\lambda \cap \mathrm{Mat}_d(\mathbb{Z}_p)$. Let $\{X_i\}$ be the union of the resulting basis vectors. We introduce a new family of balls ${}_aK_k^G$ that are adapted to this eigenbasis:

$${}_aK_k^G = \exp \sum_i p^k \mathbb{Z}_p X_i,$$

where we also denote the inner sum by

$${}_aK_k^{\mathfrak{g}} = \sum_i p^k \mathbb{Z}_p X_i.$$

Remarks.

(i) By choice of the $\mathbb{Z}_p$-eigenbasis, the set ${}_aK_k^{\mathfrak{g}}$ is a Lie algebra over $\mathbb{Z}_p$ and, by the remark below Proposition 3.1, ${}_aK_k^G$ defines a group.

(ii) The smoothness parameter l_f of a function f has been defined regarding the original balls K_k^G. We thus may replace l_f by $l_f + |\nu|$ to ensure that f is ${}_aK_{l_f}$-invariant.

(iii) As we will work on with the balls adjusted to a defined above, we will drop the subscript a from ${}_aK$.

Denote by $\{u_i^-\} \in \mathfrak{g}^-$ those eigenvectors X_i that are associated with eigenvalues of absolute values $p^{-\nu_i}$ less than one. We also abuse our previous notation by introducing

$$K_{k+\nu}^{\mathfrak{g}^-} = \sum p^{k+\nu_i} \mathbb{Z}_p u_i^- \tag{2}$$

for the vector $\nu = (\nu_1, \ldots, \nu_{\dim(\mathfrak{g}^-)})$ with norm $|\nu| = \sum \nu_i$. These boxes coincide with the image of $K_k^{\mathfrak{g}^-}$ under Ad_a. Similarly define $K_k^{\mathfrak{g}^+}$ and $K_{k,l}^P = \exp(K_k^{\mathfrak{g}^0} + K_l^{\mathfrak{g}^-})$ with image under Ad_a equal to $K_{k,l+\nu}^P$ using the eigenvectors from above. We introduced the new basis of the Lie algebra so that we are able to split the ball K_k^G into a stable component and a thickened unstable component.

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LEMMA 4.1. *The decomposition* $K_k^{G_a^+} K_{k,k}^P = K_k^G$ *holds for all* $k > 1$.

Proof. We only have to address the inclusion $K_k^G \subset K_k^{G_a^+} K_{k,k}^P$. For this, let

$$g = \exp(v_0 + w_0) \in K_k^G$$

where $v_0 \in \mathfrak{g}^-$ and $w_0 \in \mathfrak{g}^+ + \mathfrak{g}^0$, both of norm $\leq p^{-k}$. Define $f_0 = \exp v_0$ and $h_0 = \exp w_0$; then $f_0^{-1} g h_0^{-1} = \exp(v_1 + w_1)$ for some $v_1 \in \mathfrak{g}^-$ and $w_1 \in \mathfrak{g}^+ + \mathfrak{g}^0$ with norm at most p^{-2k} by application of the Baker–Campbell–Hausdorff formula mentioned in the remark below Proposition 3.1. Continuing this procedure, we find sequences $f_i \in K_k^{G_a^+}$ and $h_i \in K_{k,k}^P$ such that

$$f_i^{-1} \cdots f_0^{-1} g h_0^{-1} \cdots h_i^{-1} \to e.$$

On the other hand, $F_i = f_0 \cdots f_i$ and $H_i = h_0 \cdots h_i$ all lie in the compact set K_k^G, so we find a converging subsequence $F_i \to f \in K_k^{G_a^+}$ and $H_i \to h \in K_{k,k}^P$ for which $f^{-1} g h^{-1} = e$ also holds. □

Let $\mathcal{P}_2^X$ denote the partition of X into $m_G(K_2^G)^{-1}$ balls and $\mathcal{P}_k^X$ for $k > 2$ the corresponding refinement in which each atom of $\mathcal{P}_2^X$ is split into $p^{(k-2)d}$ smaller balls, where d is the dimension of G. This is indeed well formulated: $K_k^{\mathfrak{g}}$ is the disjoint union of p^d translates of $K_{k+1}^{\mathfrak{g}}$ (with distance p^{-k} from each other) so after applying the exponential map, K_k^G splits into p^d copies of K_{k+1}^G and after projecting, being inside the injectivity radius, the property that any two balls either coincide or are disjoint is passed onto atoms of $\mathcal{P}_k^X$ and thus gives a unique refinement.

Define the σ-algebra generated by $\mathcal{P}_k^X$ under a,

$$\mathcal{A} = (\mathcal{P}_k^X)_0^\infty = \bigvee_{l=0}^{\infty} a^{-l} \bullet \mathcal{P}_k^X,$$

to be the smallest σ-algebra containing all the partitions $a^{-l} \bullet \mathcal{P}_k^X$. If $[x]_{\mathcal{P}_k^X} = x K_k^G$ denotes the atom of $\mathcal{P}_k^X$ then

$$[x]_{a^{-l} \bullet \mathcal{P}_k^X} = a^{-l} \bullet [a^l \bullet x]_{\mathcal{P}_k^X} = x a^{-l} K_k^G a^l. \tag{3}$$

In order to understand the form of an atom $[x]_{\mathcal{A}}$ we need the following lemma.

LEMMA 4.2. *For* $k \in \mathbb{Z}_{>1}$ *such that* $k - \max \nu_i > 1$, *it holds for all* $n \in \mathbb{Z}_{\geq 0}$ *that*

$$[x]_{(\mathcal{P}_k^X)_0^n} = x \exp(K_{k+n\nu}^{\mathfrak{g}^+} + K_k^{\mathfrak{g}^0} + K_k^{\mathfrak{g}^-}).$$

Proof. We start by noting that

$$\exp(K_{k+n\nu}^{\mathfrak{g}^+} + K_k^{\mathfrak{g}^0} + K_k^{\mathfrak{g}^-})$$

is the intersection of $a^{-l} K_k^G a^l$ for all $0 \leq l \leq n$. Indeed, by our notation introduced in (2), we have

$$\begin{aligned}\bigcap_{l \leq n} a^{-l}(K_k^{\mathfrak{g}^+} + K_k^{\mathfrak{g}^0} + K_k^{\mathfrak{g}^-}) a^l &= \bigcap_{l \leq n} (K_{k+l\nu}^{\mathfrak{g}^+} + K_k^{\mathfrak{g}^0} + K_{k-l\nu}^{\mathfrak{g}^-}) \\ &= K_{k+n\nu}^{\mathfrak{g}^+} + K_k^{\mathfrak{g}^0} + K_k^{\mathfrak{g}^-}\end{aligned}$$

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so that

$$\bigcap_{l\le n} a^{-l}\exp(K_k^{\mathfrak{g}^+}+K_k^{\mathfrak{g}^0}+K_k^{\mathfrak{g}^-})a^l=\exp(K_{k+n\nu}^{\mathfrak{g}^+}+K_k^{\mathfrak{g}^0}+K_k^{\mathfrak{g}^-}).$$

We only have to take care of the inclusion $\subset$ in the statement of the lemma. For this, let $y\in[x]_{(\mathcal{P}_k^X)_0^n}$, which means by definition that for all $l\le n$ the points $a^l\bullet x$, $a^l\bullet y$ lie in the same partition element of $\mathcal{P}_k^X$. In particular, if $y=xg_0$ for $\|g_0\|\le p^{-k}$, then also $a\bullet y$ and $a\bullet x$ are p^{-k}-close, i.e. there exists $g_1\in K_k^G$ such that $a\bullet y=xg_0a^{-1}=xa^{-1}g_1$, or equivalently,

$$xg_0=xa^{-1}g_1a.$$

We chose k such that $a^{-1}g_1a\in K_2^G$. Using the bound on the injectivity radius from Proposition 3.1, the above equation lifts to the group level $g_0=a^{-1}g_1a$ from which we conclude that

$$g_0\in K_k^G\cap a^{-1}K_k^Ga.$$

Repeating this step for the points $y_1=x_1g_1$ and $x_1=xa^{-1}$, we find $g_2\in K_k^G$ such that $y_1a^{-1}=x_1a^{-1}g_2$, or, again equivalently, that $g_1\in a^{-1}K_k^Ga$, and thus

$$g_0\in K_k^G\cap a^{-1}K_k^Ga\cap a^{-2}K_k^Ga^2.$$

Continuing this argument, we find successively $g_0,\ldots,g_n$, all elements of K_k^G with relation $g_i=a^{-1}g_{i+1}a$, so that g_0 indeed lies in $\exp(K_{k+n\nu}^{\mathfrak{g}^+}+K_k^{\mathfrak{g}^0}+K_k^{\mathfrak{g}^-})$. $\square$

Letting $n\to\infty$ we see that the atoms

$$[x]_{\mathcal{A}}=x\exp(K_k^{\mathfrak{g}^0}+K_k^{\mathfrak{g}^-})=xK_k^P$$

are balls in the thickened horospherical direction and, for k large enough (depending on ν),

$$[x]_{a^{-1}\bullet\mathcal{A}}=x\exp\left(K_k^{\mathfrak{g}^0}+\sum p^{k-\nu_i}\mathbb{Z}_pu_i\right)=xK_{k,k-\nu}^P=\bigsqcup_{j=1}^{p^{|\nu|}}xg_jK_{k,k}^P \tag{4}$$

for some $p^{|\nu|}$ elements $g_j\in G_a^-$.

We use Bowen's formalism for homogeneous measures [**Bow71**] to calculate that

$$h_{m_X}(a)=|\nu|\log p$$

where $p^{|\nu|}$ is the product of the absolute values of all eigenvalues of Ad_a (with multiplicities) that are greater than one.

PROPOSITION 4.1. *We have $h_{m_X}(a)=h_{m_{\mathfrak{g}}}(\mathrm{Ad}_a)=|\nu|\log p$.*

Proof. As the space X is compact and since the projection map π is locally isometric by the construction of the metric on X, $d_X(\Gamma g,\Gamma h)=\inf_{\gamma\in\Gamma}d(g,\gamma h)$, we might as well calculate the entropy of a acting on G. If $D_n(e,k,a)=\bigcap_{l=0}^{n-1}a^{-l}K_k^Ga^l$ denotes a family of Bowen balls in G, the topological entropy is

$$\lim_{k\to\infty}\limsup_{n\to\infty}-\frac{1}{n}\log m_G(D_n(e,k,a)).$$

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From above, we have $D_n(e,k,a)=\exp(K^{\mathfrak{g}^+}_{k+(n-1)\nu}+K^{\mathfrak{g}^0}_k+K^{\mathfrak{g}^-}_k)$. Thus, $D_0(e,k,a)$ is the disjoint union of $p^{(n-1)|\nu|}$ translates of the nth Bowen ball so that $m_G(D_n(e,k,a))=p^{-(n-1)|\nu|}m_G(D_0(e,k,a))$, and we conclude that

$$\begin{aligned}\lim_{n\to\infty}-\frac{1}{n}\log m_G(D_n(e,k,a)) &= \lim_{n\to\infty}\left(\frac{n-1}{n}|\nu|\log p-\frac{1}{n}\log m_G(D_0(e,k,a))\right)\\ &= |\nu|\log p-0.\end{aligned}$$ □

This calculation shows that the modular function $\operatorname{mod}(a,G_a^-)$ of a with respect to the inner action on G_a^- is $p^{|\nu|}$ so that

$$h_{\mathrm{top}}(a)=h_{m_X}(a)=\log\operatorname{mod}(a,G_a^-)$$

as claimed earlier.

5. *Entropy generating partition*

To calculate the entropy, one usually finds a suitable generating partition. From the preceding section we see that atoms of the σ-algebra $\bigvee_{n=-\infty}^{\infty}a^{-n}\bullet\mathcal{P}_k^X$ are *plaques* of the form $xK_k^{G_a^0}$. In particular, $\mathcal{P}_k^X$ does not generate the Borel σ-algebra under the action of a. However, the following holds.

PROPOSITION 5.1. *A fine enough partition $\mathcal{P}_k^X$ will still be entropy-generating for any a-invariant measure μ, in the sense that*

$$h_\mu(a)=h_\mu(a,\mathcal{P}_k^X)=H_\mu\left(\mathcal{P}_k^X\,\middle|\,\bigvee_{n=1}^{\infty}a^{-n}\bullet\mathcal{P}_k^X\right).$$

Any $k\geq|\nu|+2$ works and, thanks to the concrete description of the plaques, this is shown rather painlessly (compare to the real analogue, [**EKL06**, Proposition 9.2]).

Proof. First, take the increasing sequence of σ-algebras $\sigma(\mathcal{P}_l^X)$ that converges to the Borel σ-algebra of X as $l\to\infty$ so that $h_\mu(a)=\lim_{l\to\infty}h_\mu(a,\mathcal{P}_l^X)$. On the other hand,

$$h_\mu(a,\mathcal{P}_l^X)\leq h_\mu(a,\mathcal{P}_l^X\vee\mathcal{P}_k^X)=h_\mu(a,\mathcal{P}_k^X)+h_\mu\left(a,\mathcal{P}_l^X\,\middle|\,\bigvee_{n=-\infty}^{\infty}a^{-n}\bullet\mathcal{P}_k^X\right)$$

and we claim the latter term vanishes, which then, by definition of the h-entropy, implies that $h_\mu(a)=h_\mu(a,\mathcal{P}_k^X)$. Indeed,

$$h_\mu\left(a,\mathcal{P}_l^X\,\middle|\,\bigvee_{n=-\infty}^{\infty}a^{-n}\bullet\mathcal{P}_k^X\right)=H_\mu\left(\mathcal{P}_l^X\,\middle|\,\bigvee_{m=1}^{\infty}a^{-m}\bullet\mathcal{P}_l^X\vee\bigvee_{n=-\infty}^{\infty}a^{-n}\bullet\mathcal{P}_k^X\right)$$

is zero if the partition in the first argument of H_μ is contained in the σ-algebra of the second argument. But this is an easy calculation for our explicit description of the atoms:

$$[x]_{\bigvee_{m=1}^{\infty}a^{-m}\bullet\mathcal{P}_l^X}\cap[x]_{\bigvee_{n=-\infty}^{\infty}a^{-n}\bullet\mathcal{P}_k^X}=xK^P_{l,l-\nu}\cap xK_k^{G_a^0}=xK_l^{G_a^0}\subset[x]_{\mathcal{P}_l^X}$$

for any $l\geq k$ and $l\geq|\nu|+2$ by equation (4). □

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6. *Proof*

We now begin the proof of Theorem 1.1. We will first rewrite the integral of f over the Haar measure of X as the limit of the conditional expectations (whose properties we recall below) with respect to the family of σ-algebras $a^{-n}\bullet\mathcal{A}$,

$$f_n(x) := E_{m_X}(f|a^{-n}\bullet\mathcal{A})(x).$$

Note also that we will ultimately assume that $m_X(f) = 0$.

6.1. *Contracting to (almost) G_a^--orbit integrals.*

We will now bound the expression $|m_X(f) - \mu(f)|$ by looking at how f integrated over thickened G_a^--orbits behaves with respect to μ on average. For that, fix $k = 2 + |\nu| + l_f$ forcing f to be $\mathcal{A}$-measurable and the exponential map to be defined on atoms of $\mathcal{A} = (\mathcal{P}_k^X)_0^\infty$. We recall that for a measure ν and a σ-algebra $\mathcal{C}$ on a measure space Y, the family of conditional measures $\{\nu_y^{\mathcal{C}}\}_{y\in Y}$ is uniquely determined by the properties of the conditional expectation $E_\nu(f|\mathcal{C})$ defined by

$$y \mapsto \int f\, d\nu_y^{\mathcal{C}}$$

to exist ν-almost everywhere and to be $\mathcal{C}$-measurable and integrable for every ν-integrable f, and that the integral equation

$$\int_C \int f\, d\nu_y^{\mathcal{C}}\, d\nu(y) = \int_C f\, d\nu$$

holds for every $C \in \mathcal{C}$.

LEMMA 6.1. *For an $\mathcal{A}$-integrable function f and an a-invariant measure μ it holds that*

$$|m_X(f) - \mu(f)| \le \sum_{n\ge 0} \int_X |f_{n+1}(x) - f_n(x)|\, d\mu$$

with f_n as defined above.

Proof. We have seen that the atoms $[x]_{\mathcal{A}}$ of $\mathcal{A}$ are xK_k^P. Because K_k^P is a subgroup, this implies that the conditional measure $(m_X)_x^{\mathcal{A}}$ is just the push forward of the map $g \mapsto g\bullet x$ from P to X of the Haar measure m_P restricted to K_k^P,

$$(m_X)_x^{\mathcal{A}} = \frac{1}{m_P(K_k^P)} m_P\bullet x.$$

Indeed, the function $(1/m_P(K_k^P)) \int_{K_k^P} f(g\bullet x)\, dm_P(g)$ is constant under the action of K_k^P and thus $\mathcal{A}$-measurable, and by an application of Fubini and invariance of m_G for every measurable $A\bullet y = BK_k^P\bullet y \in \mathcal{A}$, $B \subset K_k^{G_a^+}$ (which generate $\mathcal{A}$),

$$\begin{aligned}
&\frac{1}{m_P(K_k^P)} \int_{A\bullet y} \int_{K_k^P} f(g\bullet x)\, dm_P(g)\, dm_X(x) \\
&= \frac{1}{m_P(K_k^P)} \int_{K_k^P} \int_A f(yh^{-1}g^{-1})\, dm_G(h)\, dm_P(g) \\
&= \int_A f(h\bullet y)\, dm_G(h)
\end{aligned}$$

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so that $m_P \bullet x$ satisfies the two properties that uniquely characterize the conditional measure. Note that m_P is unimodular when restricted to the compact plaques $a^{-n} K_k^P a^n$. Now $(m_X)_x^{a^{-l}\bullet\mathcal{A}} = a^{-l}\bullet(m_X)_{a^l\bullet x}^{\mathcal{A}}$ and we similarly check that

$$f_n(x) = E_{m_X}(f|a^{-n}\bullet\mathcal{A})(x) = \frac{1}{m_P(K_k^P)} \int_{K_k^P} f(a^{-n}\bullet(g\bullet(a^n\bullet x)))\,dm_P$$
$$= \frac{1}{m_P(a^{-n}K_k^P a^n)} \int_{a^{-n}K_k^P a^n} f(g\bullet x)\,dm_P.$$

We will understand the last expression as a thickened horospherical flow and show in Theorem 6.1 that it equidistributes uniformly, i.e. $f_n(x) \to m_X(f)$ as $n \to \infty$ independent of x. Since f is $\mathcal{A}$-measurable, $f_0 = f$. Write

$$|m_X(f) - \mu(f)| = |\mu(m_X(f) - f)| = \lim_{k\to\infty} |\mu(f_k - f)|$$

and since

$$f_k - f = \sum_{n=0}^{k-1}(f_{n+1} - f_n) + f_0 - f = \sum_{n=0}^{k-1}(f_{n+1} - f_n),$$

we see that we indeed moved the problem to understanding the sum of the differences $\int_X |f_{n+1}(x) - f_n(x)|\,d\mu$. □

6.2. *Bounding the term $f_{n+1} - f_n$.* We make use of the following relation.

LEMMA 6.2. *If $f \circ a^n$ denotes the map $x \mapsto f(a^{-n}\bullet x)$ then the recursion formula*

$$f_{n+1}(x) = E_{m_X}(f_n \circ a^n | a^{-1}\bullet\mathcal{A})(a^n\bullet x)$$

holds for all $n \geq 0$.

Proof. The right-hand side is equal to

$$\int_{a^{-1}K_k^P a} f_n(a^{-n}\bullet(g\bullet(a^n\bullet x)))\,dm_P = \int_{a^{-(n+1)}K_k^P a^{n+1}} f_n(g\bullet x)\,dm_P$$
$$= E_{m_X}(f_n|a^{(n+1)}\bullet\mathcal{A})(x) = E_{m_X}(f|a^{(n+1)}\bullet\mathcal{A})(x). \qquad \square$$

The following classical lemma in information theory plays the crucial role in getting a quantitative estimate from the convexity of the entropy function.

LEMMA 6.3. (Pinsker's inequality) ***[CT91**, Lemma 12.6.1] Let*

$$\phi_{\underline{p}}(\underline{q}) = -\sum \log \frac{p_i}{q_i} q_i$$

be defined on the $n-1$-dimensional simplex of probability vectors $\underline{q} = (q_1, \ldots, q_n)$ such that the q_i are positive and sum up to one. Then for all probability vectors $\underline{q}$ and $\underline{p}$ it holds that

$$\|\underline{p} - \underline{q}\|_1^2 \leq 2\phi_{\underline{p}}(\underline{q}).$$

We will apply this to the probability vectors defined by the coordinates

$$p_i(x) = (m_X)_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}}) \quad \text{and} \quad q_i(x) = \mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}}).$$

LEMMA 6.4. *With assumptions on f and μ as before and ϕ_p defined as in Lemma 6.3, it holds for all $n \geq 0$ that*

$$\int_X |f_{n+1}(x) - f_n(x)|\,d\mu \leq \sqrt{2}\|f_n\|_\infty \bigg(\int_X \phi_{\underline{p}(x)}(\underline{q}(x))\,dm_X\bigg)^{1/2}.$$

Proof. With the new expression of f_{n+1}, we write

$$\begin{aligned}\int_X (f_{n+1}(x) - f_n(x))\,d\mu &= \int_X (f_{n+1}(a^{-n}\bullet x) - f_n(a^{-n}\bullet x))\,d\mu \\ &= \int_X (E_{m_X}(f_n \circ a^n | a^{-1}\bullet\mathcal{A})(x) - E_\mu(f_n \circ a^n | a^{-1}\bullet\mathcal{A})(x))\,d\mu.\end{aligned}$$

In order to compare the conditional expectations of the $\mathcal{A}$-measurable function $f_n \circ a^n$ with respect to μ and m_X, we decompose

$$E_{m_X}(f_n \circ a^n | a^{-1}\bullet\mathcal{A})(x) = \sum_j f_n(a^{-n}\bullet x_j)(m_X)_x^{a^{-1}\bullet\mathcal{A}}([x_j]_\mathcal{A})$$

and

$$E_\mu(f_n \circ a^n | a^{-1}\bullet\mathcal{A})(x) = \sum_j f_n(a^{-n}\bullet x_j)\mu_x^{a^{-1}\bullet\mathcal{A}}([x_j]_\mathcal{A})$$

where $x_j = xg_j \in \Gamma\backslash G$ are the $p^{|v|}$ points from (4) to represent the atoms $[x_j]_\mathcal{A}$ of the partitioning $[x]_{a^{-1}\bullet\mathcal{A}}$ in $\mathcal{A}$. By Lemma 6.3,

$$\begin{aligned}&\int_X |f_{n+1}(x) - f_n(x)|\,d\mu \\ &\quad\leq \int_X \sum_j |f_n(a^{-n}\bullet x_j)||(m_X)_x^{a^{-1}\bullet\mathcal{A}}([x_j]_\mathcal{A}) - \mu_x^{a^{-1}\bullet\mathcal{A}}([x_j]_\mathcal{A})|\,dm_X \\ &\quad\leq \sqrt{2}\|f_n\|_\infty \int_X \sqrt{\phi_{\underline{p}(x)}(\underline{q}(x))}\,dm_X. \qquad (5)\end{aligned}$$

Finally, $\int_X (\phi_{\underline{p}(x)}(\underline{q}(x)))^{1/2}\,dm_X \leq (\int_X \phi_{\underline{p}(x)}(\underline{q}(x))\,dm_X)^{1/2}$ by the Cauchy–Schwarz inequality and $m_X(X) = 1$. □

6.3. *Relating $f_{n+1} - f_n$ to the entropy difference.* The relative entropy $\phi_{\underline{p}}(\underline{q})$ of the two distributions $\underline{p}$ and $\underline{q}$ relates to their entropies as follows.

LEMMA 6.5. *For the σ-algebra $\mathcal{A} = (\mathcal{P}_k^X)_0^\infty$ constructed from the entropy-generating partition $\mathcal{P}_k^X$ (see §5), the following equality holds:*

$$h_{m_X}(a) - h_\mu(a) = \int \phi_{\underline{p}(x)}(\underline{q}(x))\,d\mu(x).$$

Proof. By assumption on the entropy generation we can write the entropy $h_\mu(a)$ as

$$\begin{aligned}H_\mu(\mathcal{A}|a^{-1}\bullet\mathcal{A}) &= \int -\log \mu_x^{a^{-1}\bullet\mathcal{A}}([x]_\mathcal{A})\,d\mu(x) \\ &= \int\int -\log \mu_y^{a^{-1}\bullet\mathcal{A}}([y]_\mathcal{A})\,d\mu_x^{a^{-1}\bullet\mathcal{A}}(y)\,d\mu(x)\end{aligned}$$

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$$= \int\int -\log \mu_x^{a^{-1}\bullet\mathcal{A}}([y]_{\mathcal{A}})\, d\mu_x^{a^{-1}\bullet\mathcal{A}}(y)\, d\mu(x)$$

$$= \int \sum_{i=1}^{p^{|\nu|}} (-\log \mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}}) \mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}}))\, d\mu(x),$$

where the first equality sign is just the definition of H, the third equality follows from the fact that the conditional measures $\mu_x^{a^{-1}\bullet\mathcal{A}}$ have support on $[x]_{a^{-1}\bullet\mathcal{A}}$, and for almost every point there is only one conditional measure on a particular atom, i.e. $\mu_x^{a^{-1}\bullet\mathcal{A}} = \mu_y^{a^{-1}\bullet\mathcal{A}}$ if $[x]_{a^{-1}\bullet\mathcal{A}} = [y]_{a^{-1}\bullet\mathcal{A}}$ for μ almost everywhere, $x, y \in X$. The final equality now follows from the fact that $-\log \mu_x^{a^{-1}\bullet\mathcal{A}}([y]_{\mathcal{A}})$ is constant on $\mathcal{A}$ atoms and $[x]_{a^{-1}\bullet\mathcal{A}}$ decomposes into exactly $p^{|\nu|}$ of those.

We repeat the last calculation for the Haar measure m_X and note that by §6.1 the information function with respect to the Haar measure $\log\,(m_X)_x^{a^{-1}\bullet\mathcal{A}}([y]_{\mathcal{A}})$ is constant (and equal to $|\nu| \log p$) so that we may integrate this information function against μ instead of m_X. We may therefore write the difference of the entropies in terms of ϕ:

$$h_{m_X}(a) - h_\mu(a) = \int -\log\,(m_X)_x^{a^{-1}\bullet\mathcal{A}}([x]_{\mathcal{A}}) + \log \mu_x^{a^{-1}\bullet\mathcal{A}}([x]_{\mathcal{A}})\, d\mu(x)$$

$$= \int \sum_i -\log \frac{(m_X)_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}})} \mu_x^{a^{-1}\bullet\mathcal{A}}([x_i]_{\mathcal{A}})\, d\mu(x)$$

$$= \int \phi_{\underline{p}(x)}(\underline{q}(x))\, d\mu(x). \qquad \square$$

We therefore obtain the bound

$$|m_X(f) - \mu(f)| \le \sum_{n=0}^{\infty} |\mu(f_{n+1} - f_n)| \le \sqrt{2} \sum_{n=0}^{\infty} \|f_n\|_\infty (h_{m_X}(a) - h_\mu(a))^{1/2} \tag{6}$$

which is only useful if one can show that

$$f_n(x) = \frac{1}{m_P(a^{-n} K_{k,k}^P a^n)} \int_{a^{-n} K_{k,k}^P a^n} f(g \bullet x)\, dm_P(g)$$

decays fast enough to zero (namely, so that $\sum_{n=0}^{\infty} \|f_n\|_\infty < \infty$). Since we may assume that $m_X(f) = 0$, as Theorem 1.1 is trivial for constant functions, it remains to prove effective equidistribution along the thickened horospherical orbits for the functions $f \in L_0^2(X)$ of vanishing integral. As we will see, the rate of convergence will depend on Γ.

6.4. *Effective equidistribution.* It remains therefore to prove the following equidistribution theorem.

THEOREM 6.1. *Assume that the action of G on $L^2(X)$ is exponentially fast mixing; that is, there exist strictly positive constants c, α and δ such that for any locally constant $f, h \in L_0^2(X)$ with degree of smoothness l_f (respectively, l_h) and for all $n \in \mathbb{Z}$ one has*

$$|\langle f \circ a^n, h\rangle| \le c p^{(l_f + l_h)\alpha} \|f\|_{L^2(X)} \|h\|_{L^2(X)} \|a\|^{-\delta n}.$$

Then, with notation as before, for $k = l_f + |\nu| + 2$ we have effective equidistribution of the family of sets $a^{-n} K^P_{k,k} a^n$,

$$\left\| \frac{1}{m_P(a^{-n} K^P_{k,k} a^n)} \int_{a^{-n} K^P_{k,k} a^n} f(g \bullet x)\, dm_P(g) \right\|_\infty \leq \frac{c}{\sqrt{m_G(K^G_2)}} p^{(\alpha+(d/2))|\nu|+2\alpha} p^{l_f(2\alpha+(d/2))} \|f\|_{L^2(X)} \|a\|^{-\delta n}.$$

We apply Margulis' trick used to prove equidistribution of the horospherical flow via mixing. A presentation of this method can be found in [**EW11**, Ch. 11].

PROPOSITION 6.1. *The left-hand side of the claimed inequality in Theorem 6.1 is equal to the sup-norm of the matrix coefficient $\langle f \circ a^n, h_x \rangle_{L^2(X, m_X)}$, where h_x is a smooth function depending on $a^n \bullet x$.*

The dependencies on h_x shall not disconcert us further, as the force of the mixing assumption lies in the fact that the bounds given depend only on the norm of f and h_x.

Proof of Proposition 6.1. We may distort the function by g^+ for any $g^+ \in a^{-n} K^{G_a^+}_k a^n$; as k is assumed to be larger than the degree of smoothness of f, and conjugation by a^{-n} only abates the amount of distortion by G_a^+ for positive n, the conditional expectation f_n at x equals

$$\frac{1}{m_{G_a^+}(a^{-n} K^{G_a^+}_k a^n) m_P(a^{-n} K^P_{k,k} a^n)} \int_{a^{-n} K^P_{k,k} a^n} \int_{a^{-n} K^{G_a^+}_k a^n} f(g^+ g \bullet x)\, dm_{G_a^+}(g^+)\, dm_P(g).$$

We now apply the following lemma (found e.g. as [**EW11**, Lemma 11.31]) to rewrite the product $dm_{G_a^+} dm_P$ as dm_G.

LEMMA 6.6. *For any closed subgroups S and T in G such that $S \cap T = \{e\}$ and ST contains a neighborhood of e, the Haar measure m_G restricted to ST is proportional to the push forward of $m^l_S \times m^r_T$ under the product map $(s, t) \mapsto st$, where m^l_S is the left Haar measure on S and m^r_T is the right Haar measure.*

We set $S = G_a^+$ which is unimodular and $T = P$ for which we had chosen the right Haar measure in §6.1, and thus

$$f_n(x) = \frac{1}{m_G(a^{-n} K^{G_a^+}_k K^P_{k,k} a^n)} \int_{a^{-n} K^{G_a^+}_k K^P_{k,k} a^n} f(g \bullet x)\, dm_G(g).$$

By Lemma 4.1 we have that $K^{G_a^+}_k K^P_{k,k} = K^G_k$. We find that f_n is equal to

$$\frac{1}{m_G(K^G_k)} \int_X f(a^{-n} \bullet y) \mathbb{1}_{K^G_k a^n \bullet x}(y)\, dm_X(y).$$

This is a matrix coefficient under the regular representation at a^{-n} with smooth vectors f and h_x defined by $h_x(y) = (1/m_G(K^G_k)) \mathbb{1}_{K^G_k a^n \bullet x}(y)$. □

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Proof of Theorem 6.1. Recall that we assume that m_G is compatible with m_X, i.e. $m_G(K_k^G) = m_X(K_k^G \bullet x)$. Further, $h_x - 1 \in L_0^2$ and is ramified of level k , $\langle f \circ a^n, h_x \rangle_{L^2(X,m_X)} = \langle f \circ a^n, h_x - 1\rangle_{L^2(X,m_X)}$, and since $m_G(K_k^G) = p^{-d(l_f+|\nu|)} m_G(K_2^G)$ we find that

$$\|h_x - 1\|_{L^2(X)} = \sqrt{\frac{1}{m_G(K_k^G)} - 1} \leq \frac{1}{\sqrt{m_G(K_2^G)}} p^{d(l_f+|\nu|)/2}$$

and consequentially,

$$|f_n(x)| \leq \frac{c}{\sqrt{m_G(K_2^G)}} p^{(\alpha+(d/2))|\nu|+2\alpha} p^{l_f(2\alpha+(d/2))} \|f\|_{L^2(X)} \|a\|^{-\delta n},$$

which proves Theorem 6.1. □

Proof of Theorem 1.1. In particular, we can conclude Theorem 1.1 since we found a bound for the sum appearing in (6), namely,

$$\sum \|f_n\| \leq \frac{c}{\sqrt{m_G(K_2^G)}} \frac{1}{1 - \|a\|^\delta} p^{(\alpha+(d/2))|\nu|+2\alpha} p^{l_f(2\alpha+(d/2))} \|f\|_{L^2(X)}.$$

As mentioned in the remark of §4 we adjusted the balls K_k^G to a. To reflect this change we replaced (implicitly) l_f by $l_f + |\nu|$. This, and the fact that $|\nu| \log p = h_{m_X}$, gives the expression for κ as in the statement of the theorem. □

7. *Effective decay of matrix coefficients for* $\mathrm{SL}_m(\mathbb{Q}_p)$

We finish by giving some references for the exponentially fast mixing property of unitary representations of simple groups and particularly consider $\mathrm{SL}_m(\mathbb{Q}_p)$. In Howe and Moore [**HM79**] it is shown that the matrix coefficients of a unitary representation of a simple algebraic group (over any local field) without non-zero invariant vectors vanish. For non-Archimedean fields [**BW00**, Lemma X.3.4] is used: For any irreducible smooth and admissible representation ρ, there exists $t > 0$ such that, for any vectors v and w,

$$|\langle \rho(g)v, w\rangle| \ll \|v\| \|w\| \Xi(g)^t$$

for any g. The assumptions that ρ is smooth and admissible mean that every vector v is smooth, and the dimension of $K_l v$ is finite for any l. The bounding function Ξ is the Harish-Chandra function, defined by

$$\Xi(g) = \int_K \delta_P^{1/2}(gk)\, dk$$

where in case of $G = \mathrm{SL}_m(\mathbb{Q}_p)$, the function δ_P is the modular character of the group of upper triangle matrices P. By [**Sil79**, Theorem 4.2.1] it holds that, for any $\alpha > -\frac{1}{2}$,

$$\Xi(g) \ll_\alpha \|g\|^\alpha.$$

In fact, the regular representation of any locally compact group G with co-compact lattice has a spectral gap in this sense [**Mar91**, Ch. III.1]. Note that in the form of the statement, the implicit constant will depend on the rate of smoothness of v and w. This can, however, be made explicit, for example if the representation is the Koopman representation as in our setup; see [**Sha00**, Proposition 2.5] or, more generally, as in the argument for [**CHH88**, Theorem 2]. In order to get hold of α we want to cite the following result for higher-rank groups $\mathrm{SL}_m(\mathbb{Q}_p)$, $m > 2$ which enjoy property (T) of Kazhdan so that one can give not only an explicit but also a uniform bound (with respect to various Γ).

THEOREM 7.1. *Let $m > 2$. For any smooth functions f and g in $L^2_0(X)$ both fixed by K_l^G, it holds for any non-negative integer n that*

$$|\langle f \circ a^n, g\rangle \ll p^{l(m^2-1)} \|f\|_{L^2(X)} \|g\|_{L^2(X)} \|a\|^{-\delta n}$$

for some (explicit) $\delta > 0$, where the implicit constant and δ are independent of Γ.

Proof. A bound for matrix coefficients valid uniformly among unitary representations for higher-rank (real) semi-simple groups is well known and is grounded on property (T) and observed, for example, by Cowling [**Cow79**]. This has been generalized to reductive groups over arbitrary local fields by Oh [**Oh02**]: For any unitary representation ρ of $\mathrm{SL}_m(\mathbb{Q}_p)$ without invariant vector, and any K-finite unit vectors v and w of ρ,

$$|\langle \rho(g)v, w\rangle| \leq (\dim Kv)^{1/2} (\dim Kw)^{1/2} \prod_{i=1}^{\lfloor m/2 \rfloor} \Xi\left(\frac{a_i}{a_{m+1-i}}\right)$$

where $g = kak'$ is the Cartan decomposition of g and $a = \mathrm{diag}(a_1, \ldots, a_m)$ is such that $|a_i| \geq |a_j|$ for $i > j$ and $a_i = p^{k_i}$. The function Ξ is the Harish-Chandra function of $\mathrm{PGL}_2(\mathbb{Q}_p)$, where we write $\Xi(x) = \Xi(\mathrm{diag}(x, 1))$, and is explicitly calculated to be

$$\Xi(p^k) = \frac{1}{p^{k/2}} \frac{k(p-1)+p+1}{p+1}.$$

The theorem now follows immediately after we sacrifice part of the exponent to get rid of the linear term appearing in Ξ. □

Let us mention that it is also possible to give an elementary proof similar to [**HT92**] where the real group $\mathrm{SL}_m(\mathbb{R})$ is treated. For the rank-one group $\mathrm{SL}_2(\mathbb{Q}_p)$ one can use the fact that G/K defines geometrically a $p+1$-regular tree $\mathcal{T}$. The eigenvalues of the associated Laplacian parameterize the irreducible unitary representations of G. In particular, X/K is a finite graph and thus has only finitely many representations appearing. We paraphrase in the following way.

THEOREM 7.2. *There holds an analogous statement for $\mathrm{SL}_2(\mathbb{Q}_p)$ as in Theorem 7.1 but with decay rate now depending on Γ.*

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