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Basso, Giuliano

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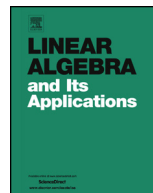


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# False-twin-free graphs with a fixed number of negative eigenvalues

Giuliano Basso<sup>a,b,\*</sup><sup>a</sup> Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland<sup>b</sup> Department of Mathematics, University of Fribourg, Chemin du Musée 23, 1700 Fribourg, Switzerland

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## ABSTRACT

We prove a quantitative version of a result of Torgašev concerning graphs with a fixed number of negative eigenvalues. We also establish a structural result stating that if for a hereditary family of graphs every graph of order  $N + 1$  and  $N + 2$  has false twins, then every graph from this family of order greater than  $N$  has false twins.

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## 1. Introduction

A finite simple graph is *false-twin-free* if the rows of its 0-1 adjacency matrix are pairwise distinct. This terminology is originally due to Burlet and Uhry [3]. Among other things, false twins naturally occur in the study of distance-hereditary graphs [1]. In the 1980s, Torgašev [11, Theorem 2] proved that for every fixed integer  $\ell \geq 0$  there

\* Correspondence to: Department of Mathematics, University of Fribourg, Chemin du Musée 23, 1700 Fribourg, Switzerland.

E-mail address: [giuliano.basso@unifr.ch](mailto:giuliano.basso@unifr.ch).

exist only finitely many false-twin-free graphs that have exactly  $\ell$  negative eigenvalues (counted with multiplicity). Bounding the number of negative eigenvalues of a graph has attracted the interest of many researchers. Recent results include [8] and [6]. The goal of this note is to establish a quantitative generalization of Torgašev’s result.

Let  $\sigma = (a, b, c)$  be a triple of real numbers, and  $G = (V, E)$  a finite simple graph with  $n \geq 1$  vertices and vertex set  $V = \{1, \dots, n\}$ . The matrix  $A_\sigma(G) = (a_{ij})_{1 \leq i, j \leq n}$  with

$$\begin{cases} a_{ij} = a & \text{if } \{i, j\} \in E, \\ a_{ij} = b & \text{if } \{i, j\} \notin E \text{ and } i \neq j, \\ a_{ij} = c & \text{otherwise} \end{cases}$$

is called  $(a, b, c)$ -adjacency matrix of  $G$ . The triple  $c := (1, 0, 0)$  yields the 0-1 adjacency matrix of a graph. Another common choice is  $s := (-1, 1, 0)$  that leads to the Seidel adjacency matrix of a graph. We denote by  $n_\sigma^+(G)$  and  $n_\sigma^-(G)$  the number of positive and the number of negative eigenvalues (counted with multiplicities) of the  $(a, b, c)$ -adjacency matrix of  $G$ . We say that  $G$  is  $\sigma$ -twin-free if the rows of the matrix  $A_\sigma(G)$  are pairwise distinct. Otherwise  $G$  is said to have  $\sigma$ -twins. Our main result reads as follows:

**Theorem 1.1.** *Let  $\sigma = (a, b, c)$  be a triple of real numbers such that  $c \leq \min\{a, b\}$ , and  $G = (V, E)$  a finite simple graph. If  $n_\sigma^-(G) \leq \ell - 2$ , for  $\ell \geq 2$ , then one of the following two possibilities must hold true:*

1.  $G$  has  $\sigma$ -twins,
2.  $G$  is  $\sigma$ -twin-free and

$$|V| \leq |\sigma|^{R(\ell)},$$

where  $R(\ell) = R(\ell, \ell)$  denotes the  $\ell$ -th Ramsey number and  $|\sigma|$  the cardinality of the set  $\{a, b, c\}$ .

By considering  $-\sigma$ , we see that if  $c \geq \max\{a, b\}$ , then the statement of Theorem 1.1 holds if we replace  $n_\sigma^-(G)$  by  $n_\sigma^+(G)$ . Hence, Theorem 1.1 tells us for instance that there are only finitely many false-twin-free graphs with a fixed multiplicity of Seidel eigenvalues strictly greater than  $-1$ .

A finite simple graph  $G = (V, E)$  is a *blow-up* of  $G_0 = (V_0, E_0)$  if  $G$  is the graph obtained from  $G_0$  by replacing each vertex  $v$  of  $G_0$  with  $n(v) \geq 1$  distinct copies and a copy of  $v \in V_0$  is adjacent to a copy of  $u \in V_0$  if and only if the vertices  $u$  and  $v$  are adjacent. Clearly, every graph with  $\sigma$ -twins is a blow-up.

**Corollary 1.2.** *Let  $\sigma = (a, b, c)$  be a triple of real numbers such that  $c \leq \min\{a, b\}$ . Then for each integer  $\ell \geq 0$  there exists a finite minimal family  $P(\ell)^\sigma$  of graphs such that every finite simple graph  $G$  with  $n_\sigma^-(G) = \ell$  is a blow-up of a graph  $G_0 \in P(\ell)^\sigma$ .*

Corollary 1.2 is a direct consequence of Theorem 1.1. Due to Theorem 1.1, to characterize the family  $P(\ell) := P(\ell)^c$  it suffices (in theory) to determine all such graphs with at most  $2^{R(\ell+2)}$  vertices. The classes  $P(2)$  and  $P(3)$  have been characterized via a different technique by Torgašev [10,9]. In view of these classifications it appears that the bound in Theorem 1.1 is not sharp. The following proposition provides a strategy to establish a sharp bound in specific cases.

**Proposition 1.3.** *Let  $\mathcal{F}$  be a hereditary family of graphs,  $\sigma = (a, b, c)$  a triple of real numbers, and  $N \geq 1$  an integer. Then the following statements are equivalent:*

1. every graph  $G \in \mathcal{F}$  of order  $N + 1$  has  $\sigma$ -twins and for each graph  $G \in \mathcal{F}$  of order  $N + 2$  the determinant of  $A_\sigma(G)$  is equal to zero.
2. every graph  $G \in \mathcal{F}$  of order  $N + 1$  and  $N + 2$  has  $\sigma$ -twins.
3. every graph  $G \in \mathcal{F}$  of order greater than  $N$  has  $\sigma$ -twins.

Here a family of graphs is *hereditary* if every induced subgraph of a graph in  $\mathcal{F}$  belongs to  $\mathcal{F}$ . For example all induced subgraphs of a fixed graph build a hereditary family of graphs. Moreover, each family  $P(0) \cup \dots \cup P(\ell)$  is hereditary. This follows directly from Cauchy’s interlacing theorem, see Theorem 2.1. Thus, as stated above, for small  $\ell$  one may use Proposition 1.3 to improve the bound in Theorem 1.1. The main tool used in the proof of Proposition 1.3 is Dodgson’s rule which we recall in Section 2.

## 2. Preliminaries

We recall Cauchy’s interlacing theorem and Dodgson’s rule. Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_m)$  with  $n > m \geq 1$  be two tuples of real numbers sorted in descending order. We say that  $\mu$  *interlaces*  $\lambda$  if

$$\lambda_i \geq \mu_i \geq \lambda_{i+n-m},$$

for all  $1 \leq i \leq m$ . Suppose that  $A$  is a symmetric  $n \times n$  matrix over the real numbers. We use the notation

$$\text{spec}(A) := (\lambda_1(A), \dots, \lambda_n(A))$$

where  $\lambda_1(A) \geq \dots \geq \lambda_n(A)$  are the eigenvalues of  $A$  (counted with multiplicity). Let  $I, J \subset \{1, \dots, n\}$  denote two subsets that have the same cardinality  $m \geq 1$ . We write  $A[I, J]$  ( $A(I, J)$ ) to denote the  $m \times m$  matrix ( $(n - m) \times (n - m)$  matrix) that is obtained from  $A$  by keeping (deleting) the rows of  $A$  with index in  $I$  and the columns of  $A$  with index in  $J$ . A matrix  $B$  is said to be a *principal submatrix* of  $A$  if there exists a non-empty subset  $I \subset \{1, \dots, n\}$  such that  $B = A[I, I]$ . The following classical result refers to Cauchy.

**Theorem 2.1** (*Interlacing theorem*). *Let  $A$  denote a symmetric  $n \times n$  matrix over the real numbers and  $B$  a principal submatrix of  $A$ . Then  $\text{spec}(B)$  interlaces  $\text{spec}(A)$ .*

An elementary proof of Cauchy’s interlacing theorem may be found in Hwang’s article [7]. We shall need the following determinantal identity first described by Dodgson [4, p. 153].

**Theorem 2.2** (*Dodgson rule*). *Let  $A$  be an  $n \times n$  matrix over the real numbers. Then*

$$\det(A) \det(A(K, K)) = \det(A(i, i)) \det(A(j, j)) - \det(A(i, j)) \det(A(j, i))$$

for all two-element subsets  $K := \{i, j\} \subset \{1, \dots, n\}$ .

In [12], Zeilberger has given a combinatorial proof of Dodgson’s rule. Berliner and Brualdi [2] used Zeilberger’s method to obtain a generalized version of Dodgson’s rule for arbitrary subsets  $K \subset \{1, \dots, n\}$ .

### 3. Proofs of the main results

In order to derive Theorem 1.1 we establish two auxiliary results, which combined with Cauchy’s interlacing theorem will directly imply Theorem 1.1.

**Lemma 3.1.** *Let  $G = (V, E)$  be a finite simple graph,  $\ell \geq 0$  an integer, and  $\sigma = (a, b, c)$  a triple of real numbers. If  $|V| > |\sigma|^\ell$ , then the rank of  $A_\sigma(G)$  is strictly greater than  $\ell$  or  $G$  has  $\sigma$ -twins.*

**Proof.** Let  $M$  denote a symmetric  $d \times d$  matrix with entries in  $\sigma$ . Suppose that  $M$  has rank  $\ell$ . Let  $M'$  denote the rectangular matrix obtained from  $M$  by keeping the first  $\ell$  columns of  $M$ , and  $M''$  the square matrix obtained from  $M'$  by keeping the first  $\ell$  rows of  $M'$ . As  $M$  is symmetric, without loss of generality we may assume that  $M'$  and  $M''$  have full rank.

It is not hard to check that for each vector  $b \in \mathbb{R}^d$  it holds that if  $M'x = b$ , then  $x = (M'')^{-1}b'$ , where  $b' \in \mathbb{R}^\ell$  is obtained from  $b$  by keeping the first  $\ell$  coordinates of  $b$ . Indeed, if  $M'x = b'$ , then  $M''x = b'$  and thereby  $x = (M'')^{-1}b'$ , since the square matrix  $M''$  has full rank.

Note that for every column  $b$  of  $M$  there is a vector  $x \in \mathbb{R}^\ell$  such that  $M'x = b$ . Thus, by the above, if there are two columns of  $M$  such that their first  $\ell$  coordinates coincide, then these two columns are identical. In other words, the first  $\ell$  coordinates of a column completely determine the column. Therefore, if  $|V| > |\sigma|^\ell$  and  $A_\sigma(G)$  has rank  $\ell$ , then there are two identical columns of  $A_\sigma(G)$ . This completes the proof.  $\square$

Due to a result of Ellingham [5, p. 253] the bound in Lemma 3.1 cannot be improved to a polynomial bound.

**Lemma 3.2.** *Let  $G = (V, E)$  be a finite simple graph, and  $\sigma = (a, b, c)$  a triple of real numbers. We write  $\lambda_1 \geq \dots \geq \lambda_n$  for the eigenvalues of  $A_\sigma(G)$ . Suppose that  $\ell \geq 1$  is an integer. If  $|V| \geq R(\ell)$  and  $c \leq \min\{a, b\}$ , then*

$$\lambda_{n-\ell+k} \leq 0$$

for all  $2 \leq k \leq \ell$ .

**Proof.** Let  $K_\ell$  denote the complete graph on  $\ell$  vertices and  $\overline{K}_\ell$  the complement graph of  $K_\ell$ . It is not hard to check that  $a(\ell - 1) + c$  is an eigenvalue of  $A_\sigma(K_\ell)$  with multiplicity one and  $c - a$  is an eigenvalue of  $A_\sigma(K_\ell)$  with multiplicity  $\ell - 1$ . Moreover,  $b(\ell - 1) + c$  is an eigenvalue of  $A_\sigma(\overline{K}_\ell)$  with multiplicity one and  $c - b$  is an eigenvalue of  $A_\sigma(\overline{K}_\ell)$  with multiplicity  $\ell - 1$ .

Since  $|V| \geq R(\ell)$ , it follows that the matrix  $A_\sigma(G)$  has  $A_\sigma(K_\ell)$  or  $A_\sigma(\overline{K}_\ell)$  as a principal submatrix. By Theorem 2.1, the eigenvalues of a principal submatrix of a symmetric matrix interlace the eigenvalues of the given matrix; thus, we obtain the desired result.  $\square$

We proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** It suffices to show that if  $|V| > |\sigma|^{R(\ell)}$ , then  $G$  has  $\sigma$ -twins. We argue by contradiction. To this end, suppose that  $|V| > |\sigma|^{R(\ell)}$  and  $G$  is  $\sigma$ -twin-free. Due to Lemma 3.1, we obtain that the rank of the matrix  $A := A_\sigma(G)$  is greater than  $R(\ell)$ . Therefore, there exists a non-singular principal  $R(\ell) \times R(\ell)$  submatrix  $A'$  of  $A$ . This is well-known, see for instance [5, Theorem 1.1]. Now, by applying Lemma 3.2 to  $A'$  we conclude that  $n_\sigma^-(G) \geq \ell - 1$ , since  $A'$  is non-singular and the eigenvalues of  $A'$  interlace the eigenvalues of  $A$ , see Theorem 2.1. But this is impossible, as  $n_\sigma^-(G) \leq \ell - 2$ . Hence, we have shown that if  $|V| > |\sigma|^{R(\ell)}$ , then  $G$  has  $\sigma$ -twins. This completes the proof.  $\square$

We conclude this section with the proof of Proposition 1.3.

**Proof of Proposition 1.3.** We show that  $1. \implies 2. \implies 3. \implies 1.$

$1. \implies 2.:$  Let  $G \in \mathcal{F}$  be a graph of order  $N + 2$ . We set  $A := A_\sigma(G)$ . Let  $B$  be a principal submatrix of  $A$  of order  $N + 1$ . Clearly,  $\det(B) = 0$ . Consequently, since additionally  $\det(A) = 0$ , the rank of  $A$ , denoted by  $r$ , satisfies  $r \leq N$ . There exists a principal submatrix  $R$  of  $A$  of dimension  $r$  such that  $R$  is invertible, see for example [5, Theorem 1.1]. Without loss of generality we may assume that  $R$  is obtained from  $A$  by keeping the first  $r$  columns and the first  $r$  rows of  $A$ . Therefore, the same reasoning as in the proof of Lemma 3.1 applies. We obtain that if there are two columns of  $A$  such that their first  $r$  coordinates coincide, then these two columns are equal to each other.

Let  $A'$  denote the matrix that is obtained from  $A$  by deleting the last row and column. By our assumptions, two columns of  $A'$  coincide. In particular, their first  $r$  coordinates

coincide. Hence, we conclude that the corresponding columns of  $A$  are not distinct, as was to be shown.

2.  $\implies$  3.: Let  $G \in \mathcal{F}$  be a graph of order  $N + 3$ . We set  $A := A_\sigma(G)$ . We claim that  $\det(A) = 0$ . By assumption, all principal  $(N + 2) \times (N + 2)$  minors and all principal  $(N + 1) \times (N + 1)$ -minors of  $A$  vanish. Using that  $A$  is symmetric and Dodgson's rule (Theorem 2.2), we see that all  $(N + 2) \times (N + 2)$  minors of  $A$  vanish. Thus, from a Laplace expansion, we obtain that the determinant of  $A$  vanishes as well. Now, we may use the implication 3.  $\implies$  2. to deduce that  $G$  has  $\sigma$ -twins. Therefore, the implication 2.  $\implies$  1. follows via induction.

3.  $\implies$  1.: This is trivial.  $\square$

### Declaration of competing interest

None declared.

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