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INFORMATION FRACTAL DIMENSION OF MASS FUNCTION

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Abstract

Fractals play an important role in nonlinear science. The most important parameter when modeling a fractal is the fractal dimension. Existing information dimension can calculate the dimension of probability distribution. However, calculating the fractal dimension given a mass function, which is the generalization of probability, is still an open problem of immense interest. The main contribution of this work is to propose an information fractal dimension of mass function. Numerical examples are given to show the effectiveness of our proposed dimension. We discover an important property in that the dimension of mass function with the maximum Deng entropy is $\frac{\ln 3}{\ln 2} \approx 1.585$, which is the well-known fractal dimension of Sierpiski triangle. The application in complexity analysis of time series illustrates the effectiveness of our method.

Keywords: Fractal; Information Dimension; Probability Distribution; Mass Function; Shannon Entropy; Deng Entropy; Sierpiski Triangle; Time Series.

1. INTRODUCTION

Fractals are very common in nature.[1] Fractals were first proposed to measure the characteristics of the length of the coast.[2] As the field progressed, it became widely used across many different areas. In the analysis of time series, the fractal characteristics of wind speed curves[3] and interbreath interval signals[4] were examined. The properties of fractals and self-similarity are reflected in many real complex networks.[5,6] The relationship between vulnerability and fractal dimension was studied.[7] How to detect the Wada boundary[8,9] has attracted much attention. Recently, the Wada index was proposed to determine fractal basin boundaries.[10] The fractal dimension plays a vital role in fractal theory. To date, a lot of measures[11] have been proposed to determine the fractal dimension. Hausdorff dimension is a classical dimension.[12,13] However, it is not an easy task to obtain the accurate value. Thus, many other fractal dimensions were developed like information dimension,[14,15] correlation dimension,[16] and box dimension.[17,18]

The uncertainty measurement is a topic of immense interest because it can be applied to the uncertain environments.[19,20] Many models have been proposed, including probability theory,[21] Dempster–Shafer evidence theory.[22,23] Probability is a well-known representation of information. The information volume of a probability can be measured with Shannon entropy.[24,25] In addition, mass function in evidence theory is a generalization of probability set to describe uncertainty, which is also called basic probability assignment (BPA). The uncertainty of mass function can be measured by Deng entropy.[26] In contrast to probability distribution, mass function has been widely applied due to its ability in dealing with uncertain information.[27,28] Many measures and parameters have been developed to study properties of mass function such as entropy[29] and information volume.[30]

Recently, the academic community has been concerned with fractals of mass function.[14,31] However, determining the fractal dimension of mass function is still an open problem. In this paper, we have proposed information dimension of mass function based on Deng entropy and fractal theory. Importantly, we discover that the dimension of mass function with the maximum Deng entropy is 1.585, which is the same as the fractal dimension of Sierpinski triangle. Many other interesting points are discussed. The complexity of time series is analyzed using the proposed dimension.

The following are some Preliminaries.

Entropy is very important in complex systems.[32,33] There are many different kinds of entropy function such as Tsallis entropy[34] and Renyi entropy.[35] In information theory, Shannon entropy plays an important role.[36,37] As part of our literature review, we will introduce these concepts briefly.

1.1. Shannon Entropy

Given a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$, Shannon entropy is defined as follows[25]:

$$H_S = -\sum_i p_i \log(p_i). \tag{1}$$

If and only if $p_i = \frac{1}{n}$, Shannon entropy reaches maximum

$$H_{\max S} = \log(n). \tag{2}$$

1.2. Fractal and Information Dimension

Fractals have been widely studied[38] and the fractal dimension is also attracting a lot of focus.[8,39] For example, the fractal dimension of Sierpinski triangle is $\frac{\ln 3}{\ln 2}$,[40] the fractal dimension of Koch curve is $\frac{\ln 4}{\ln 3}$,[41] and the fractal dimension of Cantor sets is $\frac{\ln 2}{\ln 3}$.[42] Information dimension, as a kind of fractal dimension, also plays a critical role in dealing with probability distribution. For a discrete probability $P = \{p_1, \ldots, p_n\}$, its information dimension is defined as follows[15]:

$$\lim_{n\to\infty} \frac{-\frac{1}{q-1}\log\sum_i p_i^q}{\log(n)}. \tag{3}$$

Note that the numerator is Renyi entropy. When $q = 1$, Renyi entropy degenerates to Shannon entropy. This paper only focuses on the first-order information dimension that when $q = 1$.

$$D_r = \lim_{n\to\infty} \frac{-\sum_i p_i \log(p_i)}{\log(n)}. \tag{4}$$

1.3. Mass Function

Mass function is an important aspect of evidence theory, which is an extension of probability theory. Θ denotes the framework of discernment in evidence theory[22,23]

$$\Theta = \{\omega_1, \ldots, \omega_N\}, \tag{5}$$

the power set of Θ is 2^Θ,

$$\begin{aligned} 2^\Theta &= \{A_1,\ A_2, \ldots, A_{2^N}\} \\ &= \{\emptyset, \{\omega_1\}, \ldots, \{\omega_N\}, \{\omega_1, \omega_2\}, \ldots, \{\Theta\}\}. \end{aligned} \tag{6}$$

The mass function is defined as follows[22,23]:

$$m : 2^\Theta \to [0, 1]. \tag{7}$$

This mapping satisfies

$$m(\emptyset) = 0, \tag{8}$$

$$\sum_{A_i \subseteq 2^\Theta} m(A_i) = 1, \tag{9}$$

where A_i is called focal element when $m(A_i) > 0$. Mass function can be seen as a generalization of probability distribution, which is more efficient in dealing with uncertainty.[43,44]

1.4. Deng Entropy and Maximum Deng Entropy

Recently, a new entropy called Deng entropy has been proposed to measure uncertainty of mass function.[26,45] Given a mass function: $(\{A_1, \ldots, A_{2^N}\} : \{m(A_i), i = 1, \ldots, 2^N\})$, its Deng entropy is obtained as

$$H_D = -\sum_{A_i \subseteq 2^\Theta} m(A_i) \log\left(\frac{m(A_i)}{2^{|A_i|} - 1}\right), \tag{10}$$

where $|A_i|$ is the cardinal of focal element A_i.

When the mass function is a Bayesian structure,[23] Deng entropy degenerates to Shannon entropy.

When the mass function satisfies the condition

$$m(A_i) = \frac{2^{|A_i|} - 1}{\sum_{A_i \subseteq 2^\Theta} 2^{|A_i|} - 1}. \tag{11}$$

Deng entropy reaches the maximum, which is shown as follows[26]:

$$\begin{aligned} H_{\max D} &= -\sum_{A_i \subseteq 2^\Theta} m(A_i) \log\left(\frac{m(A_i)}{2^{|A_i|} - 1}\right) \\ &= \log \sum_{A_i \subseteq 2^\Theta} (2^{|A_i|} - 1). \end{aligned} \tag{12}$$

In probability theory, Shannon entropy reaches maximum when all results have equal probability. However, in evidence theory, Deng entropy postulates that when the uncertainty of mass function reaches the maximum, multiple subsets should have more assignment. In addition, some other extensions of Deng entropy are proposed.[46]

The rest of the paper is organized as follows. Section 2 presents the information dimension of

mass function. Some numerical examples are given in Sec. 3. An application in time series is given in Sec. 4. This paper ends with a brief conclusion.

2. A NEW PROPOSAL FOR INFORMATION DIMENSION OF MASS FUNCTION

In this section, the new dimension is proposed and we begin with some fundamental definitions. For the convenience of the reader, we also provide the proof to the properties that are being discussed.

Definition 1. For a framework of discernment Θ, the power set is $2^\Theta = \{A_1, A_2, \ldots, A_{2^N}\}$, a mass function is m. Its information dimension is defined as follows:

$$D_m = \frac{H_D}{\log \sum_{A_i \subseteq 2^\Theta} (2^{|A_i|} - 1)^{m(A_i)}}, \tag{13}$$

where H_D is Deng entropy, $(2^{|A_i|} - 1)$ is the size of power set of the focal element A_i.

Property 1. When $m(A_i) = 1, |A_i| = 1$, both 0 in numerator and denominator of Eq. (13), we defined D_m as follows:

$$D_m = 0 (m(A_i) = 1, |A_i| = 1). \tag{14}$$

Proof. Given a framework of discernment $\Theta = \{\omega_1, \ldots, \omega_N\}$, a mass function is $m(A_i) = a, m(A_1) = \cdots = m(A_{i-1}) = m(A_{i+1}) = \cdots = m(A_{2^N-1}) = b$, where $|A_i| = 1$.

When $a \to 1, b \to 0$, the mass function degenerates into $m(A_i) = 1, |A_i| = 1$. According to Eq. (10), we have

$$\lim_{a \to 1, b \to 0} H_D = \left(-a \cdot \log(a) + \sum_{j \neq i} -b \cdot \log \times \left(\frac{b}{2^{|A_j|} - 1} \right) \right) \to 0. \tag{15}$$

Thus,

$$D_m \to 0. \tag{16}$$

The special case with the condition $(m(A_i) = 1, |A_i| = 1)$ in Property 1 indicates that the information is deterministic, so its information dimension is 0.

Property 2. When mass function degenerates into probability, the proposed dimension degenerates to Renyi information dimension.

$$\begin{aligned} D_m &= \frac{H_D}{\log \sum_{i=1}^N (2^{|A_i|} - 1)^{m(A_i)}} \\ &= \frac{\sum_{i=1}^N -p_i \log(p_i)}{\log \sum (2^1 - 1)^{p_i}} \\ &= \frac{H_S}{\log(N)} = D_r, \end{aligned} \tag{17}$$

where N is the number of focal element, H_S is Shannon entropy. For denominator, the elementary event is no longer split in probability theory, which is represented by the value of any exponent of 1.

In the case of one mass function or probability distribution, only a number is obtained by Eq. (13). However, in the next section, when the mass function changes in a certain regularity, the number either stays the same or converges gradually to a constant, indicating that there is scale invariance between Deng entropy and the splitting of the mass function. Therefore, the number is used to represent this property and named as the "information fractal dimension".

3. NUMERICAL EXAMPLES

In this section, we provide some numerical examples to better illustrate the definition of the proposed dimension. In order to verify the results easily, all examples below use $\log 2$ for the purpose of calculation. Nevertheless, base two or base e (or others) does not affect the result. In Eq. (13), the numerator and the denominator will be different for different bases. However, we have proposed the dimension as a ratio reflecting scale invariance. As long as the logarithm of the numerator and denominator has the same base, the result will not change. For convenience, we just use log in the following.

Example 1. A framework of discernment is $\Theta = \{\omega_1, \omega_2\}$, a mass function is

$$m(\omega_1) = \frac{5}{6}, \quad m(\omega_1, \omega_2) = \frac{1}{6}.$$

Its Deng entropy and information fractal dimension are

$$\begin{aligned} H_D &= -\frac{5}{6}\log\left(\frac{5}{6}\right) - \frac{1}{6}\log\left(\frac{\frac{1}{6}}{2^2-1}\right) \\ &= 0.9142, \\ D_m &= \frac{0.9142}{\log[(2^1-1)^{\frac{5}{6}} + (2^2-1)^{\frac{1}{6}}]} \\ &= \frac{0.9142}{1.1381} = 0.8033. \end{aligned}$$

Example 2. Given a framework of discernment Θ, $(|\Theta| = 1, 2, \ldots, 20)$. A mass function is $m(\Theta) = 1$. The result is shown in Table 1. In Fig. 1, x-axis is Deng entropy and y-axis is $\log(2^{|\Theta|}-1)^1$.

As can been seen from Table 1, the dimension of $|\Theta| = 1$ is 0. It means the complexity of a definite information is 0. Figure 1 indicates a linear relationship between Deng entropy and the size of split of mass function when $|\Theta| = 2, 3, \ldots, 20$. The value of the slope is 1 and this means that the information dimension of the total uncertainty case is 1.

Table 1 The Convergence Process of Example 2 ($m(\Theta) = 1$).

$\lvert\Theta\rvert$	H_D	$\log\sum_i(2^{\lvert A_i\rvert}-1)^{m(A_i)}$	D_m
1	0	0	0
2	1.5850	1.5850	1
3	2.8074	2.8074	1
4	3.9069	3.9069	1
5	4.9542	4.9542	1
6	5.9773	5.9773	1
7	6.9887	6.9887	1
8	7.9944	7.9944	1
...	...	...	...
19	18.9999	18.9999	1
20	20.0000	20.0000	1

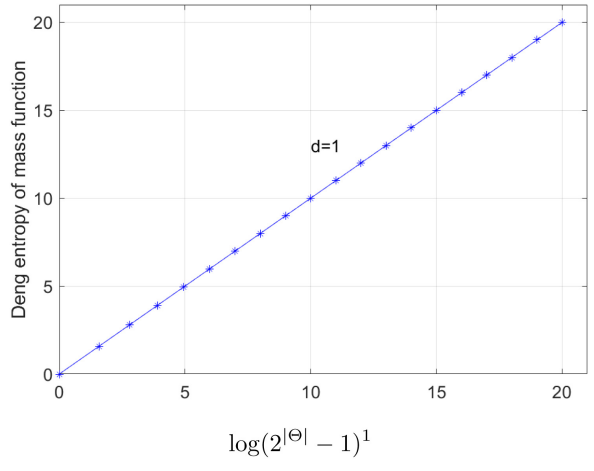

Fig. 1 The result of Example 2, where $|\Theta| = 1, 2, \ldots, 20$.

Table 2 The Convergence Process of Example 3 ($m(A) = \frac{1}{N}$).

$\lvert\Theta\rvert$	H_S	$\log\sum_i(2^{\lvert A_i\rvert}-1)^{P_i}$	D_r
1	0	0	0
2	1	1	1
3	1.5850	1.5850	1
4	2	2	1
5	2.3219	2.3219	1
6	2.5850	2.5850	1
7	2.8074	2.8074	1
8	3	3	1
9	3.1699	3.1699	1
10	3.3219	3.3219	1

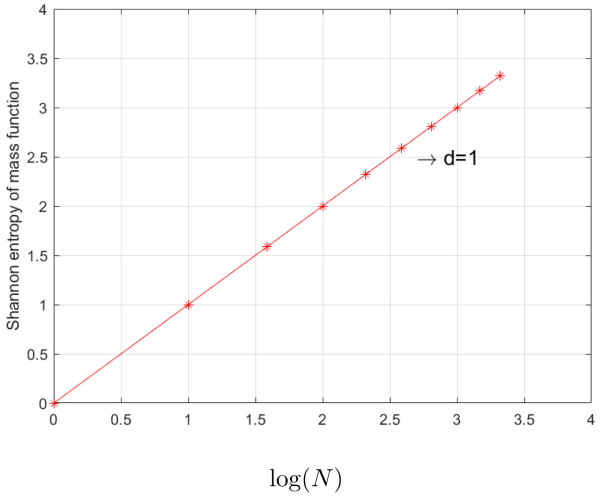

Fig. 2 The result of Example 3, where $N = |\Theta| = 1, 2, \ldots, 10$.

Example 3. Consider a framework of discernment Θ, $(|\Theta| = 1, 2, \ldots, 10)$. A mass function is $(\{A_1, A_2, \ldots, A_N\},\ m(A_i) = \frac{1}{N}))$, where $|A_i| = 1$. The information dimension of this mass function is 1. The results of D_r are listed in Table 2 and Fig. 2.

Compared with Examples 2 and 3, $m(\Theta) = 1$ means total uncertainty in evidence theory and average distribution in probability theory $m(A_i) = \frac{1}{N}$ has equal information dimension. The calculation

Table 3 The Convergence Process of Example 4 ($m(A_i) = \frac{1}{2^N-1}$).

$\lvert\Theta\rvert$	H_D	$\log\sum_i(2^{\lvert A_i\rvert}-1)^{m(A_i)}$	D_m
1	0	0	0
2	2.1133	1.7834	1.1850
3	3.8877	2.9691	1.3094
4	5.5500	4.0186	1.3811
5	7.1610	5.0260	1.4248
6	8.7428	6.0214	1.4520
7	10.3048	7.0418	1.4690
8	11.8523	8.0095	1.4798
9	13.3886	9.0058	1.4867
10	14.9162	10.0034	1.4911
...	...	...	...
21	31.4965	21.0000	1.4998
22	32.9974	22.0000	1.4999
23	34.4981	23.0000	1.4999
24	35.9985	24.0000	1.4999
25	37.4989	25.0000	1.5000

of Example 3 is

$$D_r = \frac{\sum_{i=1}^{N} -p_i \log(p_i)}{\log\sum_{i=1}^{N}(2^{|A_i|}-1)^{P_i}} = \frac{-\frac{1}{N}\log(\frac{1}{N}) \times N}{\log(1 \times N)} = 1. \tag{18}$$

The calculation of Example 2 is

$$D_m = \frac{-1 \times \log(\frac{1}{2^N-1})}{\log(2^N-1)^1} = 1. \tag{19}$$

Equation (19) can be rewritten as

$$D_m = \frac{-\frac{1}{2^N-1}\log(\frac{1}{2^N-1}) \times (2^N-1)}{\log(1 \times (2^N-1))}. \tag{20}$$

From the above equations, the case of $m(\Theta) = 1$, $|\Theta| = N$ and the case of average distribution in probability, where the number of elementary events is 2^N-1, are equivalent in expressing the complexity of information.

Example 4. Given a framework of discernment Θ, $|\Theta| = N = 1, 2, \ldots, 25$. Its power set is $2^\Theta = \{\emptyset, A_1, A_2, \ldots, A_{2^N-1}\}$. A mass function with average assignment in power set is $m(A_i) = \frac{1}{2^N-1}$. As can be seen from Table 3 and Fig. 3, with the increase of the size of Θ, D_m is changed but eventually goes to 1.5. Different from Examples 2 and 3, which is a constant from the beginning, we assume that for this example the convergent value is the final information fractal dimension of mass function with average distribution in power set.

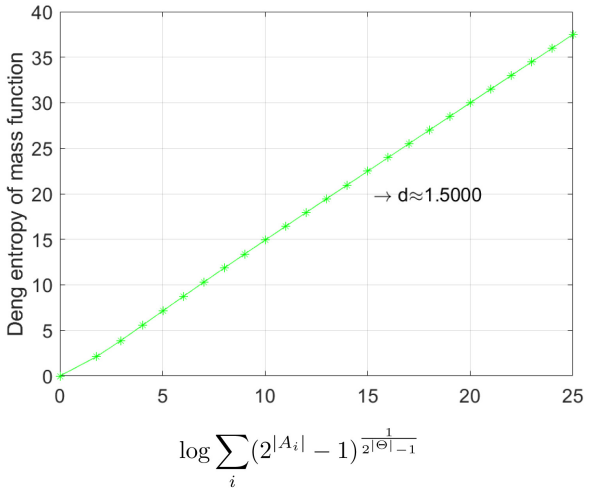

Fig. 3 The result of Example 4, where $|\Theta| = 1, 2, \ldots, 25$.

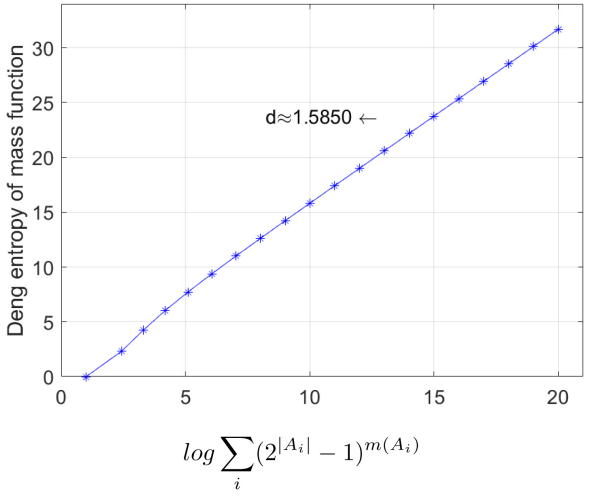

Fig. 4 The result of Example 5, where $|\Theta| = 1, 2, \ldots, 20$.

Example 5. Given a framework of discernment Θ, $|\Theta| = 1, 2, \ldots, 20$ and a mass function with maximum Deng entropy: $m(A_i) = \frac{2^{|A_i|}-1}{\sum_{A_i \subseteq 2^\Theta}(2^{|A_i|}-1)}$. Figure 4 shows the result and Fig. 5 is a Sierpinski triangle.

From Table 4, with the size of Θ increasing, D_m is a convergent sequence. The dimension of mass function with the maximum Deng entropy is 1.585, which is the well-known fractal dimension of Sierpiski triangle.

Finally, we conclude the results from Examples 2, 4 and 5 with a picture. Figure 6 shows that there is a linear relationship between Deng entropy and the size of split of mass function. However, given any mass function like Example 1, is there a special distribution form of mass function that has the

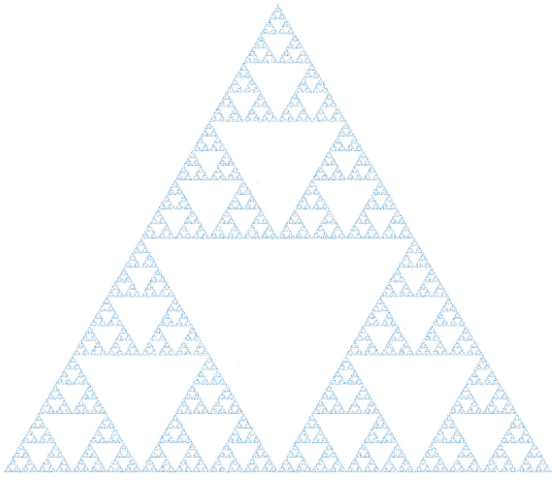

Fig. 5 Sierpinski triangle, whose fractal dimension is $\frac{\ln 3}{\ln 2} \approx 1.585$. Interested readers can refer to Ref. 47 for the full construction steps.

Table 4 The Convergence Process of Example 5 ($m(A) = \frac{2^{|A|}-1}{\sum(2^{|A_i|}-1)}$).

$\lvert\Theta\rvert$	$H_{\max D}$	$\log\sum_i(2^{\lvert A_i\rvert}-1)^{m(A_i)}$	D_m
1	0	0	0
2	2.3219	1.9757	1.1752
3	4.2479	3.1071	1.3672
4	6.0224	4.0970	1.4699
5	7.7211	5.0679	1.5235
6	9.3772	6.0434	1.5516
7	11.0077	7.0265	1.5666
8	12.6223	8.0157	1.5747
9	14.2266	9.0091	1.5791
10	15.8244	10.0052	1.5816
11	17.4178	11.0029	1.5830
12	19.0084	12.0016	1.5838
13	20.5971	13.0009	1.5843
14	22.1845	14.0005	1.5846
15	23.7711	15.0003	1.5847
16	25.3572	16.0001	1.5848
17	26.9429	17.0001	1.5849
18	28.5283	18.0000	1.5849
19	30.1136	19.0000	1.5849
20	31.6988	20.0000	1.5849

same dimension? What does the calculated dimension actually mean?

There is still no common agreement on the interpretation of dimension, but one plausible explanation is related to the degree of freedom. In Euclidean Space, one-dimensional space means that a particle can only move in one direction. In two-dimensional space, particle can move in two orthogonal directions, and for three-dimensional, particle can move in three orthogonal directions. The higher the dimension, the more directions a particle can move; more variables are needed to measure it. However, we postulate here to express information fractal dimension as complexity. That is to say, the larger the information dimension is, the more complex the information represented by a mass function will be.

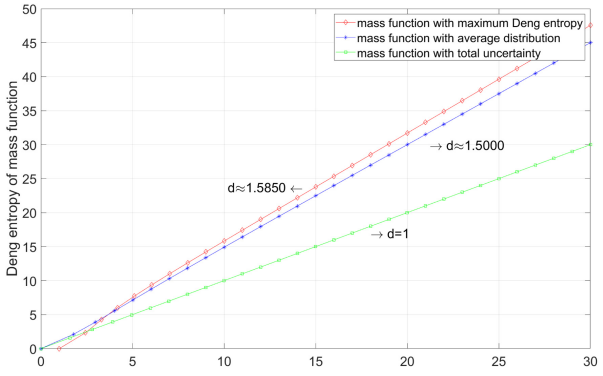

Fig. 6 Information dimension of Examples 2, 4 and 5.

4. APPLICATION IN THE COMPLEX ANALYSIS OF TIME SERIES

In the area of signal processing,[48,49] the complexity of the time series analysis has always been a hot topic.[39,50] In the physiologic area, complexity represents the adaptability of a system in a dynamic environment. Thus, the signals from a healthy system will be more complex than a pathological system.[51] Entropy is an important tool when handling uncertainty, many entropy-based methods have been proposed to analyze the complexity of signal from biological systems.[52–54] Among them, the method named EoE has a good performance because it considers the casual information by using Shannon entropy twice.[52] Recently, a generalized method BEoE based on belief entropy was developed, in which more hidden information were extracted from time series.[55] Inspired by BEoE, we proposed the BDoD model to analyze the complexity of time series. Specifically, we use information fractal dimension to replace the belief entropy. The result illustrates the effectiveness of the proposed method.

4.1. Belief Dimension of Dimension Method

First segment:

Step 1: Input a time series $x_i = \{x_1, x_2, \ldots, x_m\}$, the width of window τ.

Step 2: Find the range of x_i is $[x_{\min}, x_{\max}]$, divide $[x_{\min}, x_{\max}]$ into k_1 equal slices: $[B_s, B_{s+1}]$, where $s = 1, 2, \ldots, k_1$.

Step 3: Divide x_i into n windows: $w_j^\tau, j = 1, 2, \ldots, n$, where $n = m/\tau$.

Step 4: Determine the mass function of each window according to:

$m(B_s) = \frac{p}{\tau}$, where p is the number of x_i in w_j^τ satisfying $x_i = B_s$

$m(B_s, B_{s+1}) = \frac{P}{\tau}$, where P is the number of x_i in w_j^τ satisfying $B_s < x_i < B_{s+1}$.

Step 5: Calculate information fractal dimension of each window, the result is a sequence $\{E_j^\tau\}$.

When the information dimension sequences are calculated of all-time series, we have

Second segment:

Step 6: Find all values in information dimension sequences. The value with the highest number of occurrences in the first k_2 is used as the interval $[l_r, l_{r+1}]$, where $r = 1, 2, \ldots, k_2$.

Step 7: Determine the mass function of each information dimension sequence again according to

$m(l_r) = \frac{q}{n}$, where q is the number of E_j^τ satisfying $x_i = B_s$

$m(l_r, l_{r+1}) = \frac{Q}{n}$, where Q is the number of x_i in w_j^τ satisfying $l_r < x_i < l_{r+1}$.

Step 8: Calculate information fractal dimension of dimension BDoD, output this result.

4.2. Data Description and Parameters Setting

All data is taken from the following databases on PhysioNet[56] (a) BIDMC Congestive Heart Failure Database, (b) Long-Term AF Database and (c) MIT-BIH Normal Sinus Rhythm Database. The subjects of each database are 15, 84 and 18, respectively. For one sample sequence of subjects, we choose 70 points as time series of each database to analyze their complexity. Three time series are shown in Fig. 7. For convenience, we use BIDMC, AF, NSR to represent these three time series. The subjects from MIT-BIH Normal Sinus Rhythm Database are healthy, the other two are pathology. Thus, the final information dimension of NSR should be higher than BIDMC and AF.

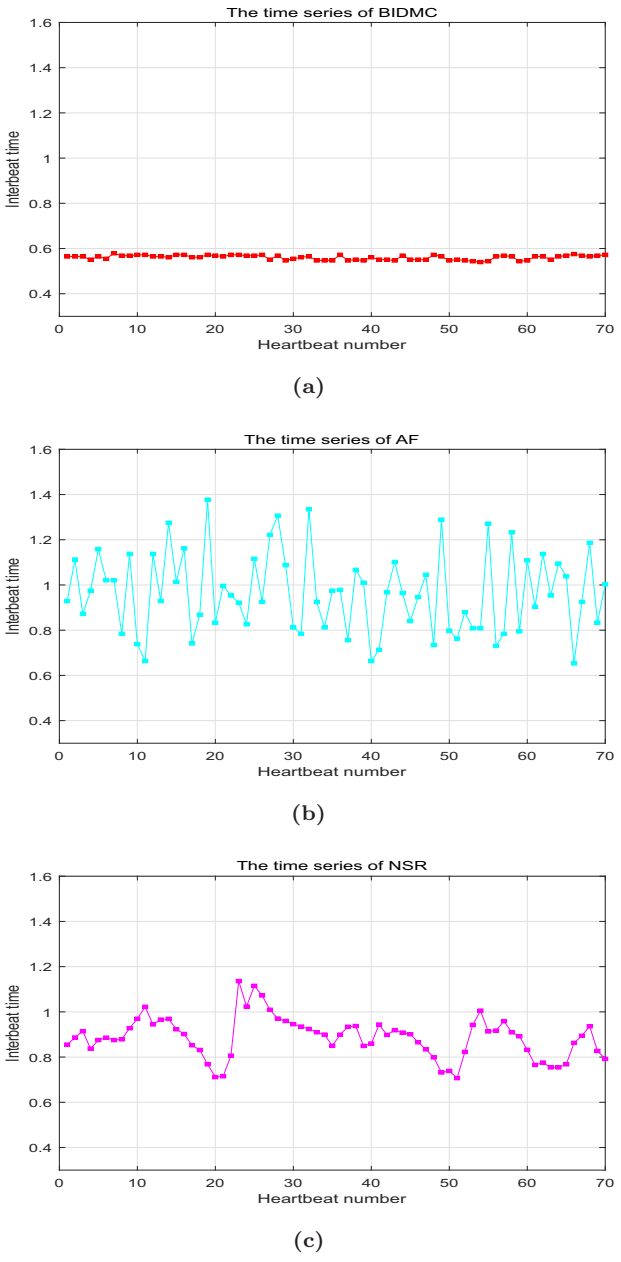

Fig. 7 The interbeat interval time series (70 datas) of three representative subjects.

After the necessary noise processing, the proposed BDoD method is applied to these time series. Clearly, the length of every signal m is 70. The width of window τ is 5 and k_2 is 7. Readers can refer to Ref. 52 for more details about the parameter setting. We will discuss these results in the following section.

4.3. Experimental Results

The results of BDoD and BEoE are listed in Tables 5, 6 and Fig. 8. It is worth noting that the

Table 5 The BEoE Result of BIDMC, AF, NSR.

Subject	Mass Function	BEoE Value
BIDMC	$m(B_1) = m(B_3) = \frac{1}{14}, m(B_2) = \frac{12}{14}$	0.5091
AF	$m(B_5, B_6) = m(B_6, B_7) = \frac{1}{14}, m(B_6) = \frac{4}{14}, m(B_7) = \frac{8}{14}$	1.2117
NSR	$m(B_2) = m(B_5, B_6) = \frac{1}{14}, m(B_4) = m(B_5) = \frac{2}{14}, m(B_6) = m(B_7) = \frac{4}{14}$	1.7273

Table 6 The BDoD Result of BIDMC, AF, NSR

Subject	Mass Function	BDoD Value
BIDMC	$m(l_1) = m(l_3) = \frac{1}{14}, m(l_2) = \frac{12}{14}$	0.4634
AF	$m(l_5, l_6) = m(l_6, l_7) = \frac{1}{14}, m(l_6) = \frac{4}{14}, m(l_7) = \frac{8}{14}$	0.8495
NSR	$m(l_2) = m(l_5, l_6) = \frac{1}{14}, m(l_4) = m(l_5) = \frac{2}{14}, , m(l_6) = m(l_7) = \frac{4}{14}$	0.9568

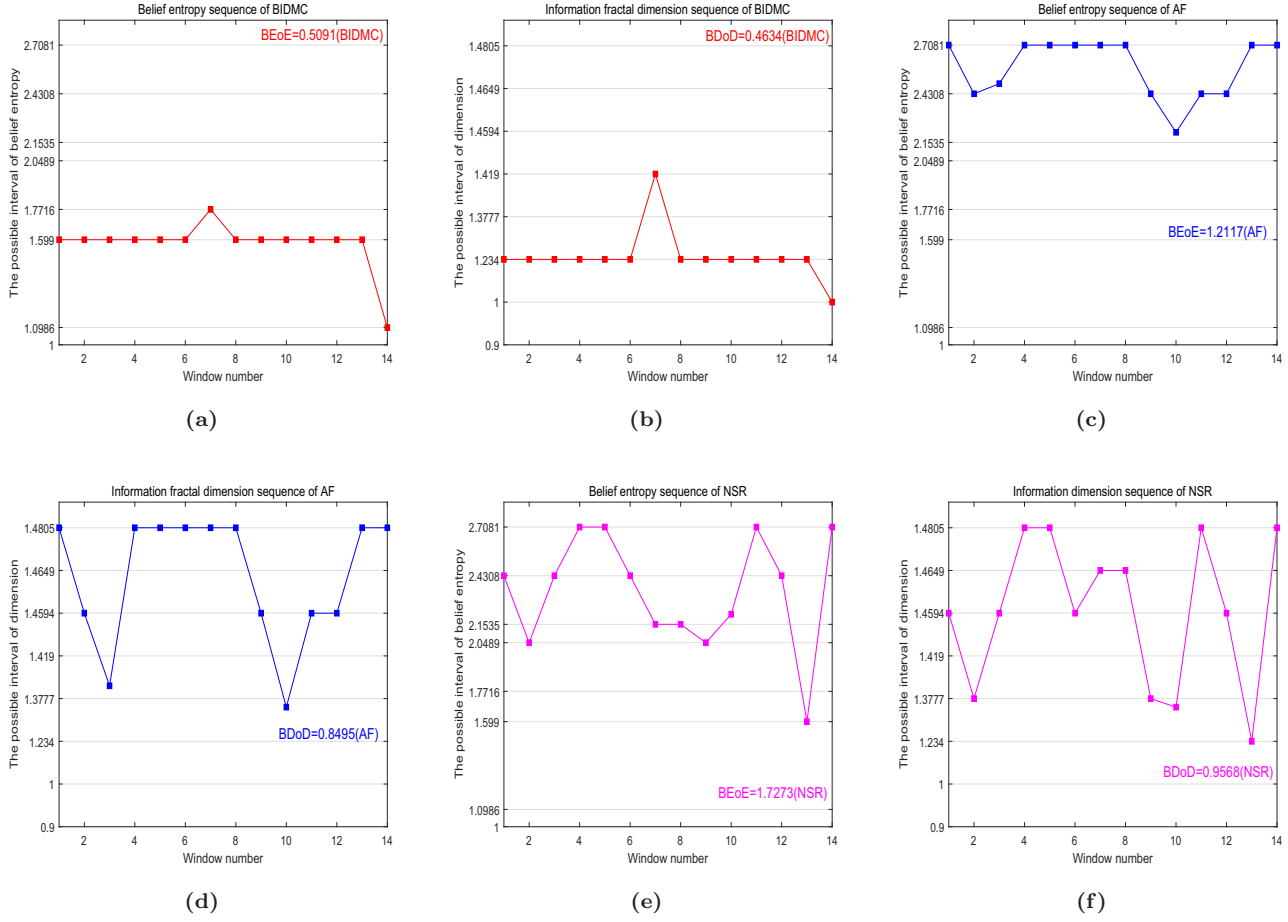

Fig. 8 The results from our experiments. The left column is the result by using BEoE method, the right column is BDoD. The result of BDoD is consistent with BEoE that BIDMC < AF < NSR. As expected, the complexity of NSR is highest.

entropy has seven levels

$$\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\} = \{1.0986, 1.5990, 1.7716, 2.0489, 2.1535, 2.4308, 2.7081\}.$$

The seven levels of information fractal dimension are

$$\{l_1, l_2, l_3, l_4, l_5, l_6, l_7\} = \{1, 1.2340, 1.3777, 1.4190, 1.4594, 1.4696, 1.4805\}.$$

From Fig. 8, the curves of NSR fluctuate wildly both in two methods. On the contrary, the curves of BIDMC are very flat. As expected, the BEoE result of the healthy subject from NSR is the highest, the value of the pathology from BIDMC is the lowest. As for AF, because it is a pathological signal, the final information dimension is 0.8495 less than the value of NSR. In addition, comparing BIDMC and AF, the BDoD method has $\mathrm{BDoD}_{\mathrm{AF}} > \mathrm{BDoD}_{\mathrm{BiDMC}}$ showing that the time series of AF is more complex, which is same as the result of BEoE. Above all, the information fractal dimension can effectively find the difference between health and pathological signals, and can clearly distinguish different pathological signals.

5. CONCLUSION

How to determine the fractal dimension of uncertain information is still an open problem. In this paper, an information fractal dimension of mass function is proposed. The proposed method not only can calculate the dimension of probability distribution in probability theory, but also the mass function in evidence theory as well. Some interesting properties have been discussed. Importantly, we discover that the dimension of mass function with the maximum Deng entropy is 1.585, which is the same as the fractal dimension of Sierpinski triangle. Our conclusion is well supported by our experiments using three biological signals analyzed by using the BDoD method. The results reveal that the information fractal dimension is a good measure of complexity.

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