Operational approach to time and clocks in quantum theory

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Operational approach to time and clocks in quantum theory

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presented by

Sandra Stupar

MMath, University of Oxford

born on 16th July 1988
citizen of Serbia

accepted on the recommendation of

Renato Renner, examiner
Tony Short, co-examiner
Vlatko Vedral, co-examiner

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To my family

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Abstract

Throughout the existence of humankind, people have organised their lives based on the sunrise and sunset, the Moon phases and star movements. Nature has guided our daily routines, as it still does, and it would be hard to imagine what life would look like without the already known cycles imposed by nature. As the humanity progressed, we have realised that a day is not always of the same length, but our sleep cycle most often is, and that it would be extremely useful to have further references for organising our day and habits. So people started calculating time with the Sun clock, sand clock, water clock etc. Since these were all limited in their performance, mechanical and pendulum clocks were introduced later, followed by quartz clocks, and as the ultimate performing ones, in the 20th century the first atomic clocks were constructed. This revolutionised time-keeping, speeded up technology development and made very precise coordination in every-day lives possible. It is now hard to think of spending a day without an accurate watch.

But where does this notion of time and clocks physically arise from? And is it valid only in the classical limit that we experience, or is it possible to introduce very precise and well-synchronised quantum clocks as well? This is what we will try to deal with in this thesis. We start from the assumption that the only observable time is the one given by clocks, and hence to learn more about quantum time we need to build good quantum clocks and try to synchronise them. We define clocks as quantum systems that need to provide a time reference, such as a sequence of ticks. Further, we attempt to find limits on their synchronisation, since it is well known that quantum systems interact with their environment thereby getting disturbed. What we find out is that the performance of the quantum clocks we define depends crucially on their dimension (number of the distinguishable states available), and that the quantum clocks of a certain dimension could possibly perform better than the classical ones with the same state space size. The limitations to synchronising local quantum clocks suggest that global time in quantum mechanics might not be a physically grounded notion.

We also investigate what limitations would be imposed on the evolution of our Universe, if it itself was/contained a quantum clock providing ticks that we observe as

time. Our results show that the theory of inflation is consistent with the time keeping the Universe needed to perform at its early beginning, while the speed of evolution as of today would not be fast enough for the Universe to be able to store its ticks then. Although our argument is still not completely general, and does not exclude other possible theories for the evolution of the Universe, it is a novel approach to combining cosmology with quantum information and it would be very interesting to extend this research further.

Zusammenfassung

Im Laufe der gesamten Menschheitsgeschichte haben die Leute ihr Leben nach dem Sonnenaufgang und dem Sonnenuntergang, nach den Mondphasen und den Bewegungen der Sterne ausgerichtet. Die Natur bestimmte unseren Alltag und bestimmt ihn immer noch und es wäre nicht einfach, sich vorzustellen, wie unser Leben aussehen würde ohne bereits vorhandene Zyklen in der Natur. Im Laufe der Menschheitsgeschichte realisierte man, dass, obwohl ein Tag nicht immer gleich lange dauert, unser Schlafzyklus sich kaum ändert, und dass weitere Referenzpunkte für die Organisation des Alltags und unserer Routinen extrem nützlich sein können. Deshalb begann man, die Zeit zu messen, sowohl mit Sonnen- als auch mit Sand- und Wasseruhren. Da die Genauigkeit dieser Uhren stark limitiert war, wurden Mechanische- und Pendeluhren entwickelt, gefolgt von Quarzuhren und im zwanzigsten Jahrhundert von den sehr genauen Atomuhren. Diese Revolution in der Zeitmessung beschleunigte technologische Entwicklungen und ermöglichte eine präzise Koordination von Alltagstätigkeiten. Man kann sich heutzutage kaum mehr einen Tag ohne eine präzise Uhr vorstellen.

Aber wie kommt der Begriff der Zeit und der Uhren physikalisch zustande? Ist er nur gültig im klassischen Grenzfall, welchen wir wahrnehmen, oder ist es auch möglich, präzise und gut synchronisierte Quantenuhren zu entwickeln? Um diese Fragestellung geht es in der folgenden Dissertation. Wir beginnen mit der Annahme, dass die einzige messbare Zeit jene einer Uhr ist und dass wir deshalb gute Quantenuhren entwickeln und synchronisieren müssen, um mehr über die Quantenzeit zu lernen. Wir definieren Uhren als Quantensysteme, welche eine Zeitreferenz zur Verfügung stellen, wie zum Beispiel eine Sequenz von Uhren- Ticks. Ferner suchen wir Grenzen für die Synchronisierbarkeit, da bekannt ist, dass Quantensysteme gestört werden durch Interaktion mit der Umgebung. Wir zeigen, dass die Leistung der Quantenuhren, welche wir definieren, stark von ihrer Dimension (Anzahl der vorhandenen unterscheidbaren Zustände) abhängt und dass Quantenuhren von einer gewissen Dimension möglicherweise besser performen als klassische Uhren mit derselben Anzahl Zustände.

Wir untersuchen zudem, welche Limitationen an die Entwicklung des Universums gestellt werden, wenn wir es selbst als Quantenuhr betrachten oder es eine solche enthält, und wir das Ticken als Zeit registrieren. Unsere Resultate zeigen, dass die

Theorie der Inflationsphase konsistent ist mit der Zeitmessung, welche das Universum zu Beginn ausführen musste, während die heutige Geschwindigkeit der Entwicklung nicht schnell genug wäre für das Universum, so dass es seine eigenen Uhren-Ticks speichern könnte. Obwohl unser Argument noch nicht vollkommen allgemein ist und andere mögliche Evolutionstheorien des Universums nicht ausschliesst, ist es eine neue Vorgehensweise, um Kosmologie und Quanteninformationstheorie zu vereinigen, und es wäre sehr interessant, diese Forschungsrichtung weiter zu entwickeln.

Contents

"He who made eternity out of years remains beyond our reach. His ways remain inscrutable because He not only plays dice with matter but also with time."

Karel V. Kuchař [1]

Chapter 1

Introduction

An extremely frequent misconception in quantum mechanics is that time is a parameter that is external, absolute and on top of that classical. Without even wondering about this notion, we see most works using this parameter 't' as a pre-given truth on which most of the other quantities and equations rely. This is understandable - absolute parametric time makes many equations and calculations easier in the current quantum mechanics, and is deeply rooted in the way we teach and think about the theory. As John A. Wheeler said "Time is defined so that motion looks simple" [2]. But, to go further, we try to approximate this abstract 't' in the best way possible, define good clocks as the ones that approximate it well, and synchronise clocks with respect to this time [3, 4, 5], without defining where it comes from. This is thus usually done without the operational meaning and is an approach that the author believes should be overcome with operational ones. By operational, we will in particular consider time defined as the output of a quantum system, usually simply known as a clock. If the parametric time is still decided to be used, it should be properly defined via a physical clock that gives this standard of time. Note that assuming a perfect, continuous clock as a reference is unphysical (since real clocks always have constraints) and in practice unattainable. One of the approaches to avoid parametric time, the so-called "time as correlations" framework, has been introduced in [6, 7, 8, 9, 10] and developed further recently, see *e.g.* [11, 12], but the clock system currently used, a particle on a line, is also unphysical. We will consider this approach further in Section 2.5.

The parametric time has also introduced many caveats when reconciling quantum theory with other areas of physics. Most notably, there is the so-called *problem of time* arising when equations of general relativity are quantised [13, 1, 14, 15], or the issues with establishing the second law of thermodynamics [16]. Unfortunately, despite the attempts, no satisfactory time operator exists [17, 18, 19, 20], and the equations we use from day to day in quantum theory mostly rely on the parametric time.

In the so-called Salecker—Wigner—Peres clock model [21, 22], it is argued that a clock nec-

essarily possesses a limit to its precision, and if we want to use a clock to measure time we need to limit ourselves to discretised time intervals, since infinite precision is not possible. As Asher Peres wrote [22]:"The Hamiltonian approach to quantum physics carries the seeds of its own demise". Peres argues further that the usual time derivative definition of the state ψ is operationally meaningless, since we have:

$$\dot{\psi} = \lim_{\varepsilon \to 0} \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon}$$

and ε cannot be taken as small as we like but has a limitation that depends on the clock one uses. Peres concludes that the notion of a time-dependent wave-function $\psi(t)$ is operationally ill-defined and should be replaced with a "more completed dynamical formalism, maybe non-local in time" [22]. Hence, if we want to develop a proper theory of quantum time and clocks we need to change the usual approach. We believe that starting from the assumption of a global time, we skip a whole theory that explains where this time comes from and whether it is a valid assumption. The clocks we will consider in this thesis are the ones that can provide us with a time reference, hence they output a stream of ticks that define their clock time. The literature often uses other types of clocks, the so-called stopwatches, which only output time information once.

One could claim that parametric time works well in the classical limit of quantum mechanics, but on the smaller scales there is yet nothing more than intuition. To investigate this parametric time on the smaller scales, one can ask if it is possible to start from the local time as defined by local (quantum) clocks, and extend this notion to a global time arising from the synchronisation of these clocks. This is anyway how we usually think of time classically - our local clocks all synchronise well to give one global parametric time we use in everyday lives. To do this, we first need to define quantum clocks of an arbitrary dimension, and then resolve what is meant by their synchronisation. In this thesis we will describe the framework for the clock construction and synchronisation, and the results obtained lead to the conclusion that the notion of the global time constructed from systems (clocks) of smaller dimensions is not an achievable concept in a satisfactory manner (*i.e.* low-dimensional quantum clocks cannot stay synchronised, and hence define a more global time, for long). It is shown that the synchronisation of clocks (even classical ones), depends on the dimension of these clocks. This is an interesting and important result when thinking of time scales in quantum theory.

This bottom-up approach to building time scales could also turn out to be very useful to the *problem of time*. Namely, by defining time as a local concept given by clocks in quantum theory, it becomes compatible with the notion of time in relativity. In the case when we cannot synchronise clocks well, we cannot achieve the notion of a global time, which happens in general relativity, one of the reasons also being gravitational effects that disturb the clock synchronisation (see *e.g.* [23]).

Let us also mention some related work that relaxes the assumption of the global time/ordering

- research on indefinite causal structures and implementations thereof, see *e.g.* [24, 25, 26, 27, 28, 29, 30] and Sec. 2.6 for more details. In these works, the possibility of causal structures in the quantum world that do not obey the usual causal inequalities which hold classically, has been investigated (so called *indefinite causal structures*). It has been found that if no global ordering/time between the parties is assumed, but only a local one, there are indefinite causal structures that are compatible with the set up. In the last few years, practical implementations of the scenario and their physical meaning have been a highly investigated topic.

Finally, note that the question of time is not the only one that suggests changing or questioning some of the equations of physics we are so used to, like the Schrödinger equation. In the search for the best interpretations of quantum mechanics, there has been much heated debate, especially when it comes to the best resolution of the measurement problem. By the measurement problem one means the incompatibility of wave-function collapse after the measurement with the ordinary unitary evolution in quantum mechanics. One way of avoiding this problem, at least in part, is a many-world interpretation of quantum mechanics, where in each measurement the Universe branches out, leaving all possibilities to develop further in one of the parallel Universes created. There have been numerous thought experiments devised to try to decide between interpretations, one of the most famous being Wigner's friend experiment [31] and a recent extension thereof [32]. But as argued in [33], possibly the most plausible solution to the measurement problem is a modification of the Schrödinger equation.

This is something we also think could be a solution for many issues in quantum mechanics in the long term, especially the ones related to time. As John A. Wheeler said *"Time, among all concepts in the world of physics, puts up the greatest resistance to being dethroned from ideal continuum to the world of the discrete, of information, of bits... Of all obstacles to a thoroughly penetrating account of existence, none looms up more dismayingly than time. Explain time? Not without explaining existence. Explain existence? Not without explaining time. To uncover the deep and hidden connection between time and existence, to close on itself our quartet of questions, is a task for the future. [34]*

1.1 Contributions of this thesis

The following is the list of publications (pre-prints) during my thesis:

[35] *"Quantum clocks and their synchronisation - the Alternate Ticks Game"*, S. Ranković, Y-C. Liang, R.Renner, arXiv:1506.01373, jun 2015

[36] *"Was inflation necessary for the existence of time?"*, S. Stupar, V. Vedral, arXiv:1710.04260, oct 2017.

[37] *"Performance of the stochastic clocks in the Alternate Ticks Game"*, S. Stupar, C. Klumpp, N. Gisin, R. Renner, arXiv: 1806.08812, jun 2018

[38] *"Quantum clocks are more accurate than classical ones"*, M. P. Woods, R. Silva, G. Pütz, S. Stupar, R. Renner, arXiv:1806.00491, jun 2018

The work in these papers and extensions thereof will mainly be presented in Chapters 4, 5 and 6, and commented upon in other chapters. In particular, in Chapter 4 I will describe the two-part clock construction (consisting of a *clockwork* and *tick registers*) developed in [35] and further requirements we found for good quantum clocks (*e.g.* ε-continuity condition, Sec. 4.2). This continuity condition aims to capture the non - deterministic evolution of quantum systems and restricts a set of clocks to physical ones.

Our model of a clock is based on the assumption that the role of the clock system is to provide a time scale via ticks that need to be stored in the so-called "tick registers". We do not have to evolve a clock with the parametric time nor do we assume that clock ticks need to come in certain 'constant intervals' since this has no meaning without another clock that provides the interval. On the contrary, the clock itself provides time intervals, and what we can do is compare them to the time intervals of another clock. To describe this more practically, if you would be closed in a room with just one local clock and no windows to decide time by daylight, the clock you possess would be the most accurate clock for you and the only one that defines time. The notion of any parametric or global time would be useless. What you could compare your local clock to however, is you inner biological clock that dictates when you are hungry, sleepy etc. In our everyday lives this clock usually becomes synchronised to the daylight intervals very fast, and hence we have the additional feeling of a global time. How the inner clock would behave in the absence of daylight (the Sun) is a different topic of research and will not be discussed here (see [39, 40] for some research on the topic, showing that the 24h cycle extends for about an hour in the absence of sunlight, and returns to the usual one when the contact with sunlight is restored).

A way to decide if a clock is ε-continuous for certain ε, by using semidefinite programming, both analytically and numerically, is presented in Sec. 4.4. A review of the notion of a specific stochastic clock introduced in the master thesis [41] and an extension to more general stochastic clocks and their conditions for ε-continuity are shown in Sec. 4.5. Stochastic clocks are a subset of clocks that are defined in Sec. 4.1, that have classical dynamics and can be described by a probability distribution for a jump at each step.

In Chapter 5 I introduce the Alternate Ticks (AT) Game developed in [35]. This is a synchronisation game, *i.e.* it checks for how many ticks two or more clocks can stay synchronised. Its advantage over other similar games is that it checks clock synchronisation directly, instead of comparing them to some external reference. This is done by quantifying correlations between the tick registers of the clocks. The game starts from the local clocks, and their successful synchronisation leads to the possibility of a construction of a more global quantum clock, that we can

call *two-local*, that would exchange two original local clocks we started from. Continuing in this manner, if local clocks can be synchronised well enough one can reach a global notion of time as we define it in classical theory. But our results show a limitation on building such a global time scale, that crucially depends on the dimension of the clocks involved. This means that we should probably not assume global time in quantum mechanics and work from there, since there seem to be no physical way to construct such a time. In addition to dimension, power source for the clocks and other physical constraints also play a role.

Physical stochastic clocks have been shown to perform at best linearly in the AT Game. Bounds on a stochastic ladder clock were derived in [41] and here I develop the proof further, to show that more general stochastic clocks also cannot perform better, see Sec. 5.3 (their performance depends on the dimension and the probability distribution that describes their dynamics). Similar result was shown in [38] and discussed in Sec. 5.5, but for completely continuous stochastic clocks and a different measure of the clock quality.

In Sec. 5.2 I provide an information theoretic approach to obtaining an upper bound on the number of synchronised ticks two quantum clocks that are correlated at the start could produce. The argument is based on several assumptions, and one of them should include the ε-continuity parameter explicitly, but the exact dependence remains to be shown. Even though the proof would benefit from more development, under the assumptions posed the current results agree with the ones discussed in [35, 37, 38] - a potential super-linear bound in terms of the dimension on the synchronisation of quantum clocks.

Chapter 6 is based on the paper [36] where we make an assumption of the *universal clock*. This is a clock that would be provided by the Universe, for example as a variable of the Universe (its size) or the biggest clock that can be stored within the Universe. The time provided by this clock, could be the 'global' time that we notice in our everyday lives. Under this assumption, we consider the early beginnings of the Universe and its speed of evolution then. We take the approach from Chapter 4 and suppose that the universal clock needs to provide ticks and store them within the Universe. If these ticks were each to be stored in a distinguishable state (so that they could be read), we show that the Universe had to expand very fast at the beginning in order to create enough states to keep track of time. The speed of evolution required is, in particular, consistent with the theory of inflation. Hence we ask - are there other evolution laws that would provide sufficient speed at the beginning and if not, would we experience the global time as we do if there was no inflation? These are of course very philosophical and hard questions, but we show a possible starting step to approach them.

In appendices we briefly discuss some additional ideas, such as further reasons for the importance of the ε-continuity condition and an attempt to construct entropic uncertainty relations for the measurements of the clock time.

Chapter 2

Time in physics

2.1 Is time a parameter?

Parametric time used in quantum theory has been very convenient to work with, but has also introduced many issues when reconciling quantum theory with other areas of physics. Since no satisfactory time-operator exists [17, 18, 42, 43, 19, 20], implicit assumption when trying to design and synchronise clocks has usually been a background time that one tries to approximate as well as possible or use it to synchronise two clocks [3, 4, 5, 44, 38]. We think this approach is incomplete without describing how this parametric time comes about (one possibility by introducing a perfect clock that produces it — but this perfect clock, even sometimes used, would be unphysical by construction [6]). In this thesis we will introduce a framework that constructs time scales from quantum clocks and check their synchronisation without a need for an explicit external time parameter.

In 1920's Wolfgang Pauli presented an argument leading to a conclusion that time is not an observable in quantum theory [17, 18]. His argument is based on the reasoning that a time operator T should be canonically conjugate to a Hamiltonian of the system, *i.e.* $[H, T] \approx i$. Since any quantum operator is hermitian, T needs to be hermitian as well, and assuming that the spectrum of T is unbounded and continuous, this would lead to the conclusion that the same holds for the spectrum of H. However, such H is unphysical since it leads to possible unbounded negative energy. To cite Pauli's conclusion [17]: *"We, therefore, conclude that the introduction of an operator t is basically forbidden and the time t must necessarily be considered an ordinary number ('c-number') in Quantum Mechanics".*

Pauli's argument is often considered as too strong, and there have been numerous attempts to construct a time operator that would serve a specific role [42, 43, 19, 20]. As described in [42] this can be attained by either:

1. dropping the condition that T is self-adjoint

2. dropping the condition that T and H are canonically conjugate

3. attempt to quantise classical time directly for some cases, *e.g.* for the time of arrival

To add to this list, we believe that one solution is to consider time not to be a continuous variable/operator taking all real values, but described via discretised output from a quantum clock. In the work by Busch [19], author takes $T = P$ as an example easy to work with, but this allows non-physical eigenstates of the time operator. Note that a similar problem occurs when taking $T = X$, like in [6, 7], which also yields an unphysical clock system (particle on a line) defining time.

Up to date, more general and satisfactory time operator has not been found and we will not attempt to argue in its favour. What we will argue in this thesis, though, is that even if there is no time operator, one can still approach time operationally. To be more precise, here operationally will mean using a clock system as a resource and allowed quantum operations to extract time information from it. And most importantly, although time is most probably not a quantum operator, this does not mean it is a parameter like often assumed. As we will discuss more below, even if one cannot build a general time operator in quantum theory, there is no physical reason to just take parametric classical time and copy it into quantum mechanics. As has been learned during the history of physics, many theories that surpassed old ones needed a radically new way of thinking in order to be constructed (probably most famous example is the Newtonian theory and surpassing Special and General relativity [45], that radically changed our view on space-time and gravity). Hence keeping the parametric time from our old friend classical theory, and hoping to combine quantum theory that uses such time with the general relativity in order to construct so called 'Theory of everything', is probably an approach ill-posed from the start.

2.2 Measurement of time

Parmenides of Elea said that everything is 'frozen' in time and that time is infinite and change is an illusion. Zeno, a student of Parmenides, introduced a famous Zeno's paradox which states that linear space and time cannot explain what we experience. Namely, by dividing space into smaller and smaller intervals, Zeno argued that Achilles could win over the tortoise if a race would take place - "it is impossible to traverse an infinite number of things in a finite time". We could apply the same principle to time - by dividing a second into smaller and smaller intervals, we would never actually reach it. One of the solutions of the paradox is the existence of the convergent infinite series. In quantum mechanics, measuring system too often, without well-defined time intervals in between, is akin of Zeno's paradox [46]. What would happen if we do not regulate

measurements, is that we would disturb the system so much that one would lose any information about it.

The theory of Julian Barbour [15, 47], in consistency to Parmenides' view, argues in favour of the timeless quantum theory of the Universe. Note that here we assume that the timeless view corresponds to an infinite time view, since both defy an explicit evolution of the Universe. Barbour argues that time is constructed from sequences of 'now' that we experience as a flow of time. But this flow of time does not actually exist, and we can build up time as a discrete sequence, as one could build a whole, apparently continuous movie from the fast changing movie frames.

On the other hand, St. Augustine believed that Universe was made together with time. As Stephen Hawking argues in his book [48], the question what came before the Big Bang cannot be answered, since both time and space were only created then. Even if these notions existed before, any signs are inaccessible to us. Hence one can indeed think of time as being a distinct feature of our Universe. We will expand more on this idea and some of its consequences for the rate of the evolution of the Universe in Chapter 6.

Clocks have been so important during the history since only information we can obtain about time is by using clocks (*i.e. the clock time*). Note that by clock one also considers very simple clocks given by nature, like the Earth's rotation. What we observe when measuring time is a relation between the state of the system and the position of the clock pointer (even Newton emphasised that introducing a global time is only a handy tool [49]). The same is accepted in relativity. As Einstein wrote:"We have to bear in mind that all our propositions involving time are always propositions about simultaneous events. If, for example, I say that "the train arrives here at 7 o'clock," that means, more or less, "the pointing of the small hand of my clock to 7 and the arrival of the train are simultaneous events." [45]. But as he continues to argue, this is not enough for the definition of time if we talk about events that occur spatially away from the clock. If we would like to always put a clock near to the event, then the time we measure is dependent on the position of the observer holding the clock. Note that in this thesis we will not deal with the relativistic effects on the clocks we construct, but this would be a very important and useful topic for the future research.

Clock read-out is discretised but we nonetheless usually think of time as continuous. This is even more interesting when one remembers that Planck's time is most often taken as the smallest unit of time that has a physical meaning and could be measured [50]. If one thinks of time as given by clocks, no clock could measure interval smaller than Planck unit, hence these intervals of time have no operational meaning. Ofcourse, even if in classical theory time is continuous, our clocks are only able to measure it up to their precision, hence they introduce a time discretisation of their own, corresponding to the local time scale of the observer who holds that clock. One can

argue that there are ways to use the clock output to obtain a continuous notion of time. Most important examples are atomic clocks, that use frequency output of a clock to obtain the best definition of a second, and give us a feeling of constantly keeping track of time. But this output is still used to obtain an interval of time up to its precision.

What we can conclude, is that except for the common belief, there is no physical reasoning behind time being continuous, nor a way in which we could measure it to check this assumption. If one accepts that time read out can only be discretised, this raises a question of the events synchronisation. To decide if two events are synchronised, without a second clock in the picture, the best we can do is to define their synchronisation as occurrence in the same time interval as measured by our clock. In general, one can say that the purpose of time is to order events, and decide their possible synchronisation.

2.3 Classical versus quantum time

Does time in quantum theory also obeys the properties of classical time? Newtonian (absolute) time tends to explain well what we see and experience around us. We use highly synchronised clocks, updated according to the best precision atomic clocks (see Sec. 3.1). The time our clocks define seems to be global and absolute, due to the fact that we all measure it in the same (or similar) manner. Do we feel and experience it in the same way relates to our inner biological clocks and is a different question that will not be considered here.

From the discovery of quantum theory it has been assumed that classical theory is just its limiting case. Hence we should be able to obtain classical time also starting from quantum physics and taking the classical limit. It thus seems unreasonable to do the other way round, assume classical time and simply transfer this notion to the more general quantum theory. What we will present in this thesis, is a framework that constructs time scales directly in quantum theory. As a part of this model, one can take a classical limit (*i.e.* use classical maps and other quantities involved) as is shown in Section 4.5 and define classical time scales.

One of the main properties of the classical clock is its deterministic evolution that also seems to make the synchronisation of clocks an easier task. This mainly comes from the fact that observing, hence using, a classical clock does not introduce a disturbance of such, and if no other technical issues arise, clocks synchronised at some point should remain synchronised. The same applies to the usual interaction of the classical clock with its surroundings, which, except for certain purposeful forces, will usually not affect its working significantly. What has been shown and we will expand on in this work, is that the working of the classical clock is bound linearly by its dimension (number of orthogonal states). Constructing more distinguishable states in classical theory may not be such a hard task, but still has its limitations which directly impact the construction of time scales.

One can view classical parametric time as defining a total order relation between events, but it is questionable if this notion can be assumed in the non-relativistic quantum mechanics. Let us first remind ourselves of the *total order* relation.
Total order $\leq$ on the set S satisfies following properties:

1. Reflexivity: $a \leq a$, $\forall a \in S$

2. Antisymmetry: $a \leq b$ and $b \leq a \Rightarrow a = b, \forall a, b \in S$

3. Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$, $\forall a, b, c \in S$

4. Comparability: For any $a, b \in S$ either $a \leq b$ or $b \leq a$

In special relativity time can be viewed as a partial order, as there can be space-like events that cannot be ordered in time. *Partial order* satisfies the first three conditions of the total order (hence no comparability). In general relativity time is only a *preorder* (satisfying reflexivity and transitivity), since it allows the existence of causal loops, where distinct events can come both before and after each other. Hence the antisymmetry condition of a partial order is violated.

Ordering can be done by giving a 'time label' to each event. Since in relativistic theories we lose the above mentioned properties of total order for a general notion of time, what we could do is provide a local ordering using local clocks. Hence one could view a local time (as given in a local reference frame of an observer) as a function from events to the natural numbers. Then a question that would be nice to answer is — under which conditions is it possible to provide a global ordering and a global time scale in non-relativistic quantum theory using quantum clocks? The framework we will describe in this thesis will start from the local clocks (see Chapter 4), and attempts their synchronisation via the so-called Alternate Ticks Game (Chapter 5).

An experimentally backed intuition on how quantum time differs from the classical time we perceive, is the entanglement of photons in time. In paper [51] authors have shown that two photons exhibit quantum correlations even though first one was created and measured before the second one was even created, hence they never coexisted. Experiment was performed by creating entangled pair of photons 1 and 2 and subsequently measuring photon 1. Later another pair of entangled photons 3 and 4 is created and photons 2 and 3 are measured in a Bell bases. When photon 4 is later measured, quantum correlation with photon 1 is experimentally showed. How can we explain these correlations with usual clocks and parametric time that we use? It seems that we need clocks that can exhibit quantum correlations in order to explain how photons can get entangled in time. We will mostly not deal with correlated quantum clocks in this thesis (except briefly in Appendix B), but it would be an interesting application of our model to be pursued in the future.

2.4 The problem of time

What happens when one tries to quantise general relativity? One of the answers was given by Wheeler and DeWitt, who derived the so called Wheeler-DeWitt equation [13], stating:

$$H\,|\Psi\rangle = 0 \tag{2.1}$$

where H is the Hamiltonian and Ψ a wave function of the Universe (or any closed system) [52, 53]. This equation leads to the famous *'problem of time'*. *Problem of time* [1, 14, 54, 15, 55] is a term used to describe a contradiction that is reached when one quantise the equations of general relativity (GR) and obtains Eq. (2.1) that concludes that the Universe is stationary. Ofcourse, this goes against common sense in standard quantum mechanics, where not just that we can time-evolve any system with our parameter t, but it is almost unthinkable to talk of a system without describing its Schrödinger-type evolution. Hence, how to explain the validity of this equation, and more importantly, how to reconcile time in quantum theory with time in GR, where it is only a local notion? This is the core of the problem of time.

When looking at Eq. (2.1) we can ask which came first, a Hamiltonian or a wave function, or how would such quantities be defined for the Universe. The so called Hartle-Hawking [56, 52, 53] state of the Universe is a no - boundary state such that at the beginning, in the pre-Planck epoch, there was no space or time, but only a singularity. We will not go deeper into the question of a Ψ or a particular H in this thesis. Further in the thesis, in Sec. 2.5 we will discuss one possible approach for restricting H, which is using 'time as correlations' framework. In Ch. 6 we relate this more general framework for closed systems to the Universe in particular.
Note that in the so-called *box Universe* approach, where there is no state evolution, one has a similar issue as with the Wheeler-DeWitt equation. In [57], several issues with this model are mentioned, for example explaining causality and a notion of *present*, and the incompatibility with the current quantum perception of time.

One of the ideas in this thesis is that if we would start from a local time in quantum mechanics, it would make it more compatible with GR. Then we can try to build a global notion of time if at all possible. Not just that this could be a step towards the theory of quantum gravity, but it would also help us understand if the global notion of time we assume is actually justified in quantum mechanics.

2.5 Emerging time - time as correlations approach

In the attempt to explain the notion of time as we experience it, one of the most prominent approaches was the so-called *"time as correlations"* approach (sometimes known as the *PaW* - Page and Wootters - approach). Used and developed by several authors, see *e.g.* [6, 7, 8, 9, 10],

this approach aims to explain an evolution of a system by its correlations with another system used as a clock. Time is here an emergent phenomenon, that is observed because of the properties and correlations of the subsystems of the Univese.

In [6] authors start from the Wheeler-DeWitt equation for the whole Universe (or another closed system), and assume that the Universe consists of a clock and a system whose evolution we would like to determine. An important additional constraint is that the system and the clock do not interact. Hence the Hamiltonian for the total Universe is of the form

$$H = H_C \otimes \mathcal{I}_S + \mathcal{I}_C \otimes H_S \tag{2.2}$$

where H_C and H_S are clock and system Hamiltonian, respectively. Total state $|\Psi\rangle$ is defined on the Hilbert space $\mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{S}$, such that $H|\Psi\rangle = 0$. Hence we can derive states of the system conditioned on the states clock takes, and describe system's evolution. In the above mentioned paper this was done by taking a specific clock example, namely a particle on a line. Then one can write $T = X$, with T denoting a 'time operator' and X the position operator. Hamiltonian for the clock could further be expressed as proportional to the momentum operator. Note that since $[T, P] = [X, P] \approx i$, we have a well-defined time operator, albeit not a physical one.

As described in [11], eigenstates of the operator T we can label $|t\rangle$, with $T|t\rangle = t|t\rangle$, where t represents possible positions of a particle on the line, hence takes real values. Conditioned on having a clock system in $|t\rangle$ we have the state of the system: $|\psi(t)\rangle_S = \langle t|\Psi\rangle$. We can further easily obtain the Schrödinger equation,

$$\langle t|H|\Psi\rangle = 0 \Rightarrow i\frac{\partial}{\partial t}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S \tag{2.3}$$

This is a really neat way of obtaining the evolution equation we are all so used to. Another nice thing about it is the connection to cosmology, as it seems to carry a possibility for combining quantum theory with relativity. There have been first experimental realisations of this approach [58], where an entangled pair of photons was used as a universe, one representing a clock and other a system whose evolution we are tracking. An external observer would then see the total state as stationary, and internal one would see the system evolve due to the correlations with the clock.

As promising as this is, there are still outstanding issues with this framework if it was to be applied in a broader setting. Some have been approached and at least partly solved in the past years [11, 12, 59], but there are open questions still unresolved. An important one is the constraint that the system and the clock are non-interacting. This is a strong assumption, since as we know it is very hard to isolate a quantum system from the environment, and this holds for clocks as well, especially since they are usually assumed to be measured by observers. In the approach we will take in this thesis clocks are not directly measured, but they produce a stream

of ticks in the so-called tick registers that are observed, hence not influencing the working part of the clock. Using this clock could help with another issue, that the clock used should be a system with physical, bounded eigenstates. The particle on the line used in [6] is not such a system, and is therefore only useful for theoretical illustration.

In the paper [7], W. Wootters points out that we observe a global time in our Universe because all the clocks we use are, in a way, highly correlated. In Chapter 6 we will consider the possibility that the Universe provides a universal clock giving us the time we perceive in every day lives.

2.6 Causal relations in quantum mechanics

A very important field of research in the past couple of years have been causal relations. A classical causal structure is usually a set of events equipped with a partial order $\leq$, such that $A \leq B$ if A comes before B in the ordering (cf. Sec. 2.3). Famous "Reichenbach's common cause principle" in classical theory states that correlations between events come from the common cause in the past or a direct influence between them [60]. Questions mainly asked recently are how to determine causal relations between events in quantum theory, and can they violate the principles that hold for classical relations, like the Reichenbach principle. Interestingly, it has been found that there exist correlations that can violate definite causal order between two or more parties [24, 25, 61], by violating standard inequalities valid for time-like or space-like correlations or superpositions thereof. This is a very important result, and the one that could in future be related to clock synchronisation. A relevant further task is to find if such indefinite causal relations are actually physical, and observable in the lab, see *e.g.* [62]. A connection to the multi-time states (pre and post-processed in time) was investigated in [29].

In paper [24], authors consider two independent labs A and B that have their internal (local) space-time ordering well-defined, but no global time (or any causal structure) between the labs exists. This is a very different set up from the ones usually assumed, and in part of this thesis similar will be assumed for a quantum time - namely existence of a local time, but no predefined global one. Each lab receives an input once and produces one output. Output of A, labeled x, should be the guess of the input b of B (and vice versa), which means that only several orderings should be possible. Namely, the guess of A should only be correlated with the input of B if the later was in the causal past of x. But with no notion of global ordering assumed, a non-zero probability of indefinite causal order observed by experimentalists in the labs was found, against our usual experience of the events ordering. Hence, in these cases no definite global causal structure has arised from the local ones. Importantly, authors also show that in the classical limit the standard correlations always arise, and this question was further extended in [61]. It would be very interesting to compare this result to the results we will see in this thesis. Namely, we will argue that the impossibility of creating arbitrarily well synchronised quantum clocks, and

with this a notion of global quantum time, comes fundamentally from the limited dimension of quantum systems and their decoherence. In the classical world of course, these are surpassed and we can obtain clocks synchronised to a very high level.

Recent research also deals with the relation of the causal ordering in quantum theory and general relativity with the violation of Bell inequalities [27, 63]. In [27] authors consider two massive bodies in the gravitational field, that will lead to the 'entanglement' of temporal orders produced. This then leads to the violation of Bell's inequality, something that would classically be impossible. Hence, temporal order becomes non-classical, showing again that in search for quantum gravity, one needs to change its view on quantum time from the usual classical approach. Experimental demonstration of entangled temporal orders that yield an indefinite causal structure was shown in [64].

2.7 Time-energy uncertainty relations

Uncertainty relations are one of the main principles governing the quantum world, and their different forms have represented extensive topic of research since the beginnings of quantum theory [65, 66]. In the recent years the interest has especially been focused on the entropic forms of uncertainty relations [67, 68, 69]. The question of defining measurements of time and quantifying uncertainty of its measurement, has also become extremely important in quantum information society, especially in connection to the information disturbance problems (see *e.g.* [70]).

An important question both from theoretical and applied [71] point of view, has been finding the right time-energy uncertainty relation [72, 66]. In the review by Busch [72], author describes three fold role of time in quantum theory, so-called *external, intrinsic and observable time*. External time is *e.g.* a parametric time, that can be obtained when the measurements of time are done with clocks that are not a part of an experiment nor dynamically related to it, intrinsic or dynamical time is marked by a dynamical variable of a system, and observable or event time is obtained when one tries to express time as an observable (for example when time of arrival is investigated). Depending on the time one considers, different uncertainty relations can apply. We will in this thesis consider a dynamical time, in particular one measured by the quantum clocks. Heisenberg-Robertson [73, 74] time-energy uncertainty relation

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{2.4}$$

can't be derived using the operators approach as for the position-momentum one, because time is not an observable in quantum mechanics, as discussed in Sec.. 2.1. We briefly recall the general uncertainty relation here. For operators A and B that do not commute, we have that $\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|$, with ΔA denoting standard deviation of an observable A. Hence relation (2.4) is only

valid for specific cases, for example the minimum (external) time a particle needs to pass the energy gap ΔE.

Mandelstam and Tamm realised in [75] that the uncertainty relation (2.4) is valid for the intrinsic time of the system, in particular Δt should be interpreted as the time system needs to evolve from an initial to a final state [76]. In particular they derived a *quantum speed limit* relation (for the minimal time δt system requires to move to an orthogonal state), in the form:

$$\delta t \geq \frac{\pi\hbar}{2\Delta H} \tag{2.5}$$

where H is the Hamiltonian of the system and $\Delta^2 H$ its variance. Later, Margolus and Levitin further extended this relation, such that it includes the average energy of a system instead of its variance [77]:

$$\delta t \geq \frac{\pi\hbar}{2\langle H\rangle} \tag{2.6}$$

In addition, it has been shown that the maximum of the right hand sides of the two above relations, is the fastest time scale with which a quantum system can evolve. We will use relation (2.6) when determining the speed of the evolution of the Universe in Chapter 6.

With the above in mind, one possibility is to turn the attention to the entropic uncertainty relations when investigating time-measurement disturbance. The main advantage of the entropic formalism for the uncertainty relations over the more standard Heisenberg approach, is that the right hand side (RHS) of the inequality in these relations does not depend on the state of the system at the time of a measurement.

Standard entropic uncertainty relation for the system in the state ρ and the set of POVMs A_a, B_b is [68, 67]:

$$H(A|\rho) + H(B|\rho) \geq -2\log \max_{a,b} A_a^{1/2} B_b^{1/2}$$

We cannot use this approach directly for the time measurement uncertainty relations, because we would again need to introduce time as an observable. But one could use this relation in an indirect way, as we suggest in Appendix B.

As a long term goal, it would be desirable to have an uncertainty relation arising from operational view on time. By defining properties of good quantum clocks and ordering scales, and attempting theoretical constructions of such, one can eventually approach, operationally, question of the uncertainty in time measurements in quantum mechanics.

Chapter 3

Clocks in physics

Before going deeper into the definitions of clocks, let us first concentrate on the purpose for which they are introduced in science. Let us look at a few definitions in the Merriam-Webster dictionary: **'Clock'**:

1. a device other than a watch for indicating or measuring time commonly by means of hands moving on a dial; broadly :any periodic system by which time is measured
2. a registering device usually with a dial; specifically :odometer
3. a synchronising device (as in a computer) that produces pulses at regular intervals
4. biological clock - an inherent timing mechanism in a living system that is inferred to exist in order to explain the timing or periodicity of various behaviours and physiological states and processes

This of course is a clock as thought of classically, but since the classical world is considered a special case of the quantum one, all of these notions should be possible to be extended to the quantum world and there have been more or less successful attempts in this direction.

Historically, time has been measured using natural phenomena - one of the first clocks were sundials, based on the movement of the Sun and the shadows that objects produce at different times during the day. Sand clocks were also very popular and they were based on the movement of the fixed amount of sand, which served more as a stopwatch to determine a certain fixed interval. Later on, people made pendulum and quartz clocks where the frequency of a certain object was used to define a time standard. But this standard depended on the shape and properties of the object used, and during the centuries, scientists and watch makers attempted to make more general clocks and time scales, that would vary less from clock to clock.

Research on quantum clocks has been increasingly popular in the past decades, trying to answer questions of how to theoretically construct such a clock [22], how one could synchronize

quantum clocks in the best way [78], or how many clocks could be confined in a certain space (quantum geometric limits [79]). However, up to date there has been no generally accepted definition of a quantum clock. Depending on the usage there are different definitions in circulation, for example the atomic clock (see Sec. 3.1) or the Salecker-Wigner-Peres clock ([21, 22], Sec. 3.2).

Following the reasoning of Paul Busch [72]:"The issue of quantum clocks belongs in a sense to the realm of the theory of time measurements: time is being measured by means of observing the dynamical behaviour of a quantum system", we will develop a model for quantum clocks and their measurement in this thesis.

3.1 Atomic clocks

The biggest revolution in time-keeping came when atoms and their frequencies started to be used for the production of the time standard, around 1950's. In the atomic clocks development, there has been several models surpassing each other in their precision and accuracy. One of the first was the famous Cs (caesium) atom clock, the standard still used. It is now implemented as the so-called fountain clock, and recently logical clocks were developed, providing even bigger precision.

Atomic clocks have many applications, one of the biggest currently is to the GPS (global positioning system), which is very sensitive to the definition of second it uses. It is also believed that quantum information, especially theory of entanglement and its application, could improve current standards of time keeping [80].

In the first caesium atomic clock, big number of Cs ions in their ground state are sent through the Ramsey cavity, whose frequency of the electro-magnetic field is tuned in such a way that the maximum number of ions have a transition between the two hyperfine ground levels. This frequency is then considered a very good approximation of the natural frequency of Cs and is used to obtain a definition of a second, introduced in 1967:

"The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the ceasium 133 atom".

In recent years, to make clocks more precise and environment independent, scientists cool down the atoms to very low temperatures and hence slow their motion [81]. This allows for the bigger interaction time with the field. In the *fountain clock*, a collection of caesium atoms is cooled down to almost absolute zero, and sent up, first interacting with the laser beams which form them into a ball. This ball then goes through an interaction with another laser, and if the right frequency is hit, atoms emit photons. The microwave frequency of the laser is modulated until the biggest number of photons get emitted, and this frequency is the output of a clock.

Since atoms are slowed down by cooling they have more time to interact with the light and hence a bigger precision is obtained. The total length the atoms pass is around $1m$, and the trip up

and down lasts about $1s$. This clock is currently used by NIST (National Institute for Standards and Technology) in USA to provide a definition of a second. It looses around 1 second every 300 million years in precision.

Recently, a (quantum) logic clock was also developed by NIST. It is based on mercury or aluminium ion, that is trapped by electric field and ultraviolet light frequencies. These frequencies are many times (100000) higher than the microwave frequencies used in the standard Cs clock, hence obtaining much better accuracy (they can measure smaller intervals of time) [82]. Aluminium ion is coupled to the beryllium ion, and the ticks from aluminium are detected by the other atom (the laser is shined on both and if the right frequency for Al was achieved, the beryllium atom starts oscillating as well, hence giving a signal that the 'tick' has occurred). The precision of the atomic clocks is calculated by comparing its frequency output with the 'true' value of the resonating frequency of the atom used. Comparing an aluminium logic clock with another such showed a high accuracy, with the lowest uncertainty ever achieved when comparing two atomic clocks. This clock only looses a second every 3.7 billion years.
Note that the only way to decide how good these clocks are is to either compare them to the same clock, or to another atomic clock whose high precision is known (like the fountain clock). The more precise the clock is, the better results it can give when predicting the effects of general relativity, the value of Planck's constant or when used to build new GPS devices. Even though these clocks are most precise yet, there are no plans of exchanging current standard of second given by caesium fountain clock with those of a logical clock. One of the reasons is the difficulty of transmitting the ticks of this clock around the world at the moment, since optical fibres are of a relatively short range.

3.2 Salecker-Wigner-Peres clock

A very widely known clock, whose dynamics we will also use for the work in this thesis, is the clock described by Salecker-Wigner [21] and later investigated by Peres [22]. The clock was originally developed for measuring space-time events distances, and later work considered usage of the clock to time a certain system of interest. As described in [21], a quantum clock should be a microscopic object that can be read out multiply times, one option being by emitting light signals (cf. 5.1.2). Von Neumann said [83], the measurement is not completed until it is recorded by some macroscopic object. To avoid this object being a part of the clock (that is assumed to be microscopic), one way is to transmit the signal from the clock, *e.g.* by using photons.

The clock described in [22] has a simple dynamics, that in constant intervals τ moves the clock state from one orthogonal state to the next, in one direction only. In between these intervals, however, the state of the clock spreads significantly. A unitary map describing the dynamics of a

d-dimensional clock after each step τ is:

$$U(\tau) = \sum_{i=1}^{d} |i+1\rangle \langle i| = e^{-iH_C\tau} \tag{3.1}$$

with H_C being the clock Hamiltonian, $\hbar = 1$. This Hamiltonian is diagonal in the Fourier basis of its clock states $|0\rangle, ..., |d-1\rangle$.

3.3 Relativistic quantum clocks

In the work by Castro Ruiz, Giacomini and Brukner [23] two simple clocks of a certain mass in the gravitational field were considered. Due to the gravitational field influence, and since time flows slower near massive objects, one would expect that each clock will be affected by the mass of the other clock near it. This is exactly what the authors show. Namely, originally uncorrelated clocks (in a product state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle))$ become entangled due to the strength of the gravitational field. In particular, at time t:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle |\phi_0\rangle + |1\rangle |\phi_1\rangle) \tag{3.2}$$

$$\phi_0 = \frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{-it\Delta E}{\hbar}} |1\rangle) \tag{3.3}$$

$$\phi_1 = \frac{1}{\sqrt{2}}(|0\rangle + e^{\frac{-it\Delta E}{\hbar}(1-\frac{G\Delta E}{c^4 x})} |1\rangle) \tag{3.4}$$

Note that this is due to the different effect of the gravitational field on the ground state $|0\rangle$ and excited state $|1\rangle$. The clocks reach maximal entanglement after time $t_{max} = \frac{\pi\hbar c^4 x}{G(\Delta E)^2}$ where ΔE is the difference between the energy levels $|0\rangle$ and $|1\rangle$ for both clocks (assumption is that they have the same energy) [23]. Hence, it is shown that the rate of ticking of one clock will depend on the energies of the surrounding clocks.

It would be interesting to calculate this disturbance in the time flow of one clock if the clock model corresponds to the one presented in Sec. 4.1. Note that in the above described model, clock system is a simple qubit evolving with the Schrödinger equation and a parametric time t. Hence to apply this reasoning to our discretised clock model (Sec. 4.1), one would have to rewrite relevant equations to not involve explicit use of the Hamiltonians and a parameter t, but the completely positive trace preserving maps that describe the clock dynamics.

A different type of a relativistic semiclassical clock model was introduced in [84] and later reviewed and extended by Lock and Fuentes in [85]. There a simple localised quantum field is taken as a model of the clock and a clock time is defined as a phase of a single mode Gaussian state. Famous twin paradox was examined with this clock. Results presented show that proper time needs an alternative in the relativistic theory when a more realistic clock is used. Also there is a relativistic addition to the quantum uncertainty when measuring time with these kinds of

clocks. With the clock they investigated, authors showed that the change in the precision due to the relativistic motion depends highly on the state in which clock is initialised. This agrees with the results obtained in this thesis and in papers [35, 86, 38], where the precision/synchronisation of the clock depends on the starting state one chooses, and is of course even more related to the map that evolves the clock and reads the time information from it.

3.4 Thermodynamical quantum clocks

In the last few years, there has been an increasing interest in introducing suitable clocks to quantum thermodynamics [87, 86, 44]. Quantum thermal engines have been theoretically constructed, usually consisting of a quantum system, thermal bath and a work storage [88, 89]. One of the main motivations for introducing quantum clocks to these frameworks, is creating the ability to switch the unitaries affecting thermal engine on and off without an external interference. Before the clocks started to be explicitly used in theoretical models of thermodynamics, Hamiltonians describing the interaction of the external fields and engines of the systems and baths were time-dependent, which raised questions of external control and led to search for solutions [90, 89]. It is by now mostly accepted that a clock should be a part of the thermodynamical system, and should automatically control the duration of the application of the (unitary) maps.

Total Hamiltonian is usually taken to be of the form:

$$H = H_E \otimes \mathcal{I}_C + \mathcal{I}_E \otimes H_C + H_{\text{int}} \tag{3.5}$$

with H_E being the engine Hamiltonian, H_C Hamiltonian of the clock and H_{int} an interaction Hamiltonian between the two. Note that here Hamiltonians are assumed to be time-independent, as the whole system should not depend on the existence of another, external clock (that would be necessary for the time dependent Hamiltonians).

In the paper by Malabarba, Short and Kammerlander [87] simple model of the clock with $H_C = P$ is used and was shown to perform well for thermodynamic purposes. Then one can think of the time operator being $T = X_C$. Compare this to the time as correlations framework [6] described in Sec. 2.5, where the same Hamiltonian for the clock is used. The problem authors want to solve in these papers is of a different nature, but the clock system shows similar properties. This clock is nice and easy to work with and returns idealised solutions, since it provides parametric time as we are used to. But, unfortunately, its eigenstates are not physical, since they are unbounded from below, and hence this clock cannot be used in practice. In both papers [87, 6], authors comment on the natural possibility of the extension of the frameworks to more physical clocks, but unfortunately, such extensions had not yet been numerous.

As an example of more physical thermodynamical clocks, we briefly consider two papers [86, 44]. Note that both of these papers have some relation with the work in this thesis, particularly

the clock framework presented in Chapter 4. They show a nice possibility for the extensions and practical applications of our quantum clock model, and the clock from [44] can be viewed as a particular example of our two-part clock model with an addition of a power source. But in measuring quality of these clocks, and calculating their accuracy and precision, authors still use parametric time t. In this sense, the clocks are not completely autonomous, since one cannot decide good from bad ones without using an external classical clock that provides this time t.

In the paper by Woods, Silva and Oppenheim [86] a clock with a physical Hamiltonian was shown to work well for thermodynamical purposes, without much back-disturbance. The measure of how good this so-called WSO clock is, relates directly to how well can it time the maps used for working of the thermal engine, and can it be reused. The Hamiltonian of the clock is the one from Salecker-Wigner-Peres model [21, 22], and authors construct a specific interaction potential for which this clocks shows good properties. An important variable is a starting state of the clock, which in this paper is taken to be a Gaussian superposition of the usual clock states. It was shown that this starting state shows less disturbance as the interaction proceeds for large clock dimensions d. Important property of the WSO clock is that it has both a physical Hamiltonian and satisfies the properties that are asked from a thermodynamical clock. Note that in the paper [38], described in Sec. 5.5, a similar model of a quantum clock with a modified potential is used to show the possible advantage of quantum over classical clocks.

In the paper by Erker et al. [44], clock consists of a pointer represented by a two-qubit heat engine powered by two thermal baths and a d-level ladder representing a tick register. Hence this model represents a specific construction inside our framework [35], presented in Sec. 4.1. Authors show that the entropy production of their clock is connected to its accuracy and resolution in terms of the parametric time t. To ensure that clock is autonomous authors use a flow of heat between two thermal reservoirs at a different temperature. Non-equilibrium power source is necessary so that the spontaneous ticking process could continue in the right direction. Flow of heat will cause energy to climb up the d-dimensional ladder, and a tick is defined as decaying of the photon from the top (unstable level) to the ground level. As stated by authors the reverse process is also possible. To reduce the probability of this, they assume that the background temperature is much smaller than the energy of the emitted photon. Compare this to the issue with our light source model in Sec. 5.1.
The ticking process is probabilistic and to simplify analysis a reset clock model is assumed, *i.e.* after each tick the whole system is reseted to the ladder in the ground state and the engine qubits in the equilibrium with their baths (cf. Sec. 4.3). Authors note that this approximation is valid in a weakly-coupled limit.

The resolution and accuracy of this clock is measured as the number of ticks provided per unit time and the number of ticks before next tick is uncertain by an average time interval between the

ticks. In particular, they introduce a measure $R = \frac{\mu^2}{\sigma^2}$, with μ being an average ticking time and σ^2 standard deviation of the ticking time, later also used in [38] and Sec. 5.5. Authors suggest that an increase in entropy is needed for a better accuracy of thermodynamical clocks, and if this is valid more general it could be useful in the research of an arrow of time and its connection to the 2nd law of thermodynamics, which is still a highly debated topic. Note that we arrive to the similar conclusion for the universal clock in Chapter 6.

3.5 Protocols for clock synchronisation

Most clock synchronisation protocols start from two parties that have a good, stable and accurate local clock each, and aim to synchronise them in their common rest frame using given recourses [4]. Clock synchronisation protocols hence rely on different assumptions. We will review a few of them here.

1. **Eddington's slow synchronisation protocol (SSP)** doesn't require that A and B know the distance between them, but it assumes that Alice has an extra clock (called a travelling clock) that she sends to Bob with a very low speed. Then Bob can read the transit time as given on the world line of this clock, and if it was slow enough this will approximate time as elapsed on Alice's clock well enough.

   Ofcourse, there are a couple of issues with this protocol. Firstly, what does slow enough movement of the clock exactly means and what if Alice and Bob would move with respect to each other? Secondly, what this protocol explicitly assumes is that Alice synchronises her local clock with the travelling clock beforehand. The underlying argument is that it is easy to classically synchronise clocks that are on the same spacial location, but the real problem is to synchronise ones that are distant from each other.

2. **Synchronisation based on the phase estimation** Here the synchronisation problem is to find the time difference between two spatially separated clocks. In the paper by Bartlett et al. [91] quantum reference frames have been studied for different applications, one of them being clock synchronisation. They review clock synchronisation based on the phase reference alignment, which can be used if Alice an Bob have a common frequency standard. For example, one can exchange ticking qubits as a resource, and then depending on these qubits being correlated or not, different synchronisation precision can be achieved [92]. If un-entangled qubits (separable states) are used, one needs $O(2^{2k})$ classical messages to allow for an accuracy in the time offset between A and B's clock of k bits. On the other hand, there exist a quantum algorithm that uses entangled qubits and allows for this with only $O(k)$ quantum messages exchanged [92], which shows an advantage of a quantum protocol for this type of synchronisation problems.

3. **Quantum clock synchronisation (QCS)** protocol is a quantum version of the SSP developed by Jozsa et al. in [5]. Additional assumption is that there is a third party Charlie, who prepares maximally entangled states, and sends half of each pair to A and half to B. Then they both measure $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ observable on their respective qubit, and after communicating their results to each other they can infer the proper time along the world line going between A and C and then between C and B. As in the SSP, if the qubits of the original singlet were distributed slow enough, this would provide the time difference between the local times of A and B. Note the similarity of this protocol to Ekert's quantum key distribution protocol based on the entanglement distribution [93].
   One of the advantages of the QCS to SSP is that if you want to synchronise clocks many times you can send many entangled pairs in advance and keep them until you need a synchronisation. In the SSP, on the other hand, you would need to construct and provide a synchronised clock to Alice every time. However, in the QCS protocol each clock is a qubit, which yields only one bit of information after measuring X, and we need many qubits to get time with a good precision. Namely, the precision will depend on the number of qubits n as $\sqrt{n}^{-1}$[4].

   Since this protocol requires a phase synchronisation between the reference frames of Alice and Bob, one can argue that it leads to a circular argument, namely that one needs a previous kind of synchronisation to further synchronise the clocks. This has been partly overcome recently in the work [94] where the authors combined the protocol from [5] with the distillation protocol [95, 96], which then allows for the further synchronising operations to be done. In the work [91], however, it is argued that prior shared entanglement does not provide an advantage for the clock synchronisation, since one could also use shared reference frame state (that is not entangled) to obtain the same effect.

4. One of the most important classical protocols for clock synchronisation based on relativity is **Einstein's light synchronisation** (ES) protocol [45]. It is based on exchanging the light signals between the two parties, but one needs to know the distance between them in order to calculate the time in which the light is returned to a party. Another issue with this protocol, whose modification is currently used for synchronising satellites for GPS navigation, is that it doesn't take into account that the light as an EM field is in nature quantum, and the effects thereof [5]. Also, sending light to satellites is affected by refracting index of the atmosphere that changes, hence introducing and error in this synchronisation scheme. This is something that is mostly surpassed by the above mentioned QCS protocol. Recently there have been advances in building these protocols, which are based on the precise timing of light signals exchanged between parties [91]. Namely in [97] it was shown

that usage of highly entangled photons leads to an advantage. Here a problem can be that a loss of a single photon would destroy the entanglement, but there have also been techniques introduced to reduce this loss [98].

Chapter 4

Operational approach to time: two-system quantum clock definition

Part of this chapter, in particular Sec. 4.1, is based on the paper [35] and related work thereof, while Sec. 4.5 is based on [37].

Time is a central concept used for describing the world around us. We perceive it intuitively and we can measure it to a very good precision [99, 100, 101]. Nevertheless, it remains one of the biggest unknowns of modern physics [45, 73, 18, 13, 22, 72, 102, 1]. In quantum mechanics, time is not considered an observable, but plays merely a parametric role in the equation of motion. In fact, it cannot be treated like other observables, as there is no self-adjoint time operator that is canonically conjugate to a Hamiltonian having a semi-bounded spectrum [17, 18], see also Sec. 2.1. There have been various attempts, though, to establish an understanding of time that goes beyond this parametric view [66, 6, 7, 103, 104, 11, 16], *e.g.* by considering time as arising from correlations between physical systems. In this chapter we take further steps in this direction.

We will take an operational approach to time, in the sense that we study clocks, *i.e.*, physical systems that provide time information. This is motivated by earlier work [105, 7], where it was suggested to distinguish between *coordinate time* and *clock time*. While the first refers to a parameter used within a physical theory, the latter is an observable quantity. Here we are interested in the latter. We will define quantum clocks as quantum systems that produce a measurable output — *the clock time*. As any measurement of a quantum system unavoidably disturbs its state, one should expect fundamental limitations to the precision of quantum clocks, and hence the level of synchronisation attainable between them. It is one of the major motivations of the work in this thesis to understand these limitations.

It is desirable to have a notion of quantum clocks that agrees with our usual perception of classical clocks in a certain limit. Yet, for our operational approach towards time, we shall assume only some basic features that a clock must exhibit. Importantly, a clock should generate a clock

time that can be used to order events. This feature will play a central role in our treatment and will be captured by our definition of a *time scale*.

Within Newtonian theory, there do not seem to be any fundamental limitations to the accuracy to which clocks can stay synchronised (see e.g., [99, 101]). With this notion, one can define a common time for all the synchronised clocks, which we refer to as a *global* time in classical theory. A notion of global time is also assumed in the usual treatment of quantum mechanics. However, attempts to construct global causal structures from local ones, have lead to the discovery of various non-trivial features, which are a topic of ongoing research (see, e.g., [24, 106, 107, 108, 25, 61] and Sec. 2.6). Furthermore, a global notion of time is generally not achievable in relativistic theories [13, 109, 14, 54, 1, 110]. In our operational approach, we will thus treat quantum clocks as local physical systems, governed by the laws of quantum mechanics. This raises the question of synchronisation of those local clocks. In operational terms, we are interested to know whether two clocks that are separated, with no communication between them allowed, can order events consistently. If this is the case, we say that the time scales generated by the two clocks are compatible. We note that this question relates to the general idea of reference frames in quantum theory [91] and, in particular, the question of how well such reference frames can be correlated. To quantify the level of synchronisation between the clocks in an operational way, in the Chapter 5 we will consider a particular scenario, which we term the Alternate Ticks Game.

Further sections of this Chapter are structured as follows. In Section 4.1, we introduce the definition of a *quantum clock* - a quantum system consisting of a clockwork (dynamical part) and tick registers (storage units), and the associated time scale. In Sec. 4.3 we briefly consider a special class of clocks, where the state of the clock is reseted to a pre-chosen state every time a tick occurs. These so-called *reset clocks* turned out to be very useful and practical when deriving results on the clock performance. Section 4.2 is concerned with the ε-*continuity* condition, which ensures that quantum clocks can be regarded as self-contained devices, in particular that they do not implicitly rely on a time-dependent control mechanism. In Appendix A, we provide a sketch of a proof of the conjecture that in the case of ε- continuous clocks with small ε, tick registers do not need to be initialised in a particular state (*e.g.* $|0\rangle$) — the clockwork state is only negligibly disturbed by a different starting state of the registers. In Sec. 4.4 we consider one analytical and numerical method to show ε-continuity of a clock — semidefinite programming, and some results thereof. In Sec. 4.5 a particular class of clocks, so-called *stochastic clocks* are defined, together with the criterion for their ε-continuity. These have a classical dynamics and are thus analytically tractable, providing insights into clock quality and synchronisation, that will be discussed in the next chapter, Sec. 5.3 in particular. Finally, In Sec. 4.6 we briefly consider more practical approach to modelling our quantum clock, using light sources. This approach is closer to what can be achieved in a lab and we show some numerical results in the next Chapter,

Sec. 5.1.2.

4.1 Quantum clocks and time scales

The aim of this section is to motivate and define a *quantum clock* (Definition 1) and the derived concept of a *time scale* (Definition **??**). Roughly speaking, we consider a quantum clock to be a quantum physical system equipped with a mechanism that generates time information. We model this mechanism as a process involving the following two systems (see Fig. 4.1).

- *Clockwork (denoted C):* This is the dynamical part of the clock that has its own evolution and interacts with its environment (tick registers in particular), thereby outputting time information.
- *Tick registers (denoted $T_1, T_2, \ldots$):* These belong to the environment of the clockwork. Each tick register T_i is briefly acted upon by the clockwork and it records the output of the latter, in particular tick/no-tick information. After this interaction, it is separated from the clockwork and exchanged by a new register, keeping a record of the clock time that can be read out by the clock itself or an outside observer.

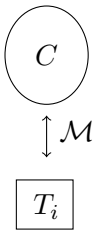

Figure 4.1: *Model of a quantum clock:* A quantum clock consists of a clockwork C that interacts with tick registers $T_1, T_2, \ldots$ via map $\mathcal{M}$. These registers are propagated away from C after their interaction, keeping a record of the clock time.

In comparison to a mechanical clock, the clockwork would be an inner mechanism, and tick registers would be similar to the hands of a clock. From our experience, we are used to assume that the evolution of a physical clock is continuous. However, as discussed in Section 2.3, the assumption of a continuous time is not well-grounded, and the notion of continuity implicitly refers to some underlying time parameter, hence we will model the evolution of a quantum clock more generally as a sequence of discrete steps. Continuity may then be approximated by imposing an additional, ε-continuity condition (see Definition 3 below). Note that even with this property imposed on good clocks, their dynamics is still discrete in nature. By explicitly letting $\varepsilon \to 0$ we can approximate completely continuous evolution (as done for clocks described in Sec. 5.5). However, we believe that the discrete evolution is more natural, since physics itself imposes conditions on

scales one can measure (*e.g.* Planck's time and length). Hence assuming continuous evolution a priory is a big assumption that might neglect important physical insights, and we want to avoid this.

Each of the discrete evolution steps of our clock consists of the interaction between the clockwork C and a fresh tick register T_i. Crucially, all steps are governed by the same dynamics, specified by a completely positive trace preserving ( ) map $\mathcal{M}$. Accordingly, all tick registers T_i are taken to be isomorphic to the given system T. In Sec. A we will argue that tick registers do not all have to be initialised in the same state for the clock to function well.
We now introduce a more formal definition of a clock.

Definition 1. A *quantum clock* is defined by a pair $(\rho_C^0, \mathcal{M}_{C\to CT})$, where ρ_C^0 is a density operator of a system C and $\mathcal{M}_{C\to CT}$ is a completely positive trace-preserving (CPTP) map from C to a composite system $C \otimes T$.

Note that our map $\mathcal{M}$ acts from a smaller space C to a larger one CT since each tick register can be seen as created by the map $\mathcal{M}$ (or one can consider it as already existing, but the only map that can create/change a state in T is the map $\mathcal{M}$). We will now introduce some further useful definitions.

Definition 2. For any map $\mathcal{M} : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A}$, with $\mathcal{H}_\mathcal{A}$ a Hilbert space, we say it is *completely positive* if $\mathcal{M} \otimes \mathcal{I}_C$ is positive for any extension space $\mathcal{H}_\mathcal{C}$. Map $S : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A}$ is positive iff $\langle \rho_A, S(\rho_A)\rangle \geq 0$ for all $\rho_A \in \mathcal{H}_\mathcal{A}$ with $\langle , \rangle$ an inner product on $\mathcal{H}_\mathcal{A}$. Note that the trace of a positive operator is positive (non-negative).
By Choi's theorem [111], map is completely positive iff it admits Kraus decomposition, *i.e.* iff it can be written as $\mathcal{M}(\rho_A) = \sum_i K_i \rho_A K_i^*$, for all $\rho_A \in \mathcal{H}_\mathcal{A}$, where K_i are Kraus operators, linear maps from $\mathcal{H}_\mathcal{A}$ to $\mathcal{H}_\mathcal{A}$ that satisfy $\sum_i K_i^* K_i = \mathcal{I}_A$.
A positive operator valued measure (POVM) is a family $\{\pi_x\}_x$ of positive hermitian operators on $\mathcal{H}$ such that $\sum_x \pi_x = \mathcal{I}_\mathcal{H}$. Note that any POVM corresponds to a non-unique physical evolution followed by a measurement. One can express it using Kraus operators $K_x = \sqrt{\pi_x}$ [112]. If one measures a state ρ with POVM $\{\pi_x\}_x$, outcome x will be obtained with probability $P(x) = \mathrm{tr}(\pi_x \rho)$.

We now return to our clock system. The clock time produced by a clock will give a rise to a *time scale*. Since, in our model, the clock time is recorded by the tick registers, a time scale corresponds to the cumulative content of these registers (see Fig. 4.2). We can denote this set as the sequence $\{\rho^N_{T_1\dots T_N}\}_{N\in\mathbb{N}}$ of density operators on $T_1 \otimes \cdots \otimes T_N$. It is easy to verify that for any

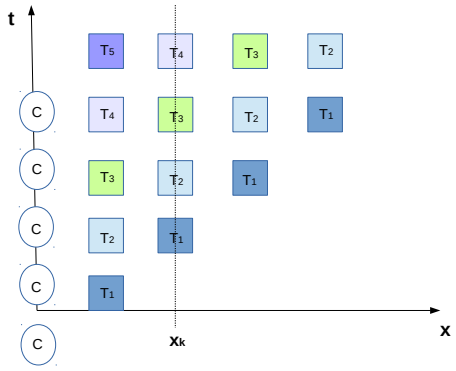

Figure 4.2: *Tick registers and the time scale.* Intuitively, one may think of the tick registers T_i as propagating through space (coordinate x), starting from the location of the clockwork C, in terms of some ordering parameter (denoted by t), that is used by an observer. An observer located at position x_k would observe them in sequential order (with respect to t). The content of the sequence of tick registers defines a time scale.

$N' > N$ we have

$$\rho^{N}_{T_1\cdots T_N} = \mathrm{tr}_{T_{N+1}\cdots T_{N'}}(\rho^{N'}_{T_1\cdots T_{N'}}) \ . \tag{4.1}$$

The sequence $\{\rho^{N}_{T_1\cdots T_N}\}_{N\in\mathbb{N}}$ is thus a way to specify the state of the system[1] $T_1 \otimes T_2 \otimes \cdots$ formed by all (infinitely many) tick registers after infinitely many invocations of the map $\mathcal{M}$. For all practical purposes we will restrict ourselves to finite number N_0 of the applications of the map, and it will be enough to consider just $\rho^{N_0}_{T_1\cdots T_{N_0}}$.

Importantly, it will often be very useful to consider a classical analogue of a time scale introduced. For this, an observer can measure a state of each tick register *e.g.* in the bases $\{|0\rangle, |1\rangle\}$ (or this measurement can be included in the clock mechanism), and note a classical result of each measurement. The occurrences of 1 will represent ticks, and the number of 0's in between will measure the interval of 'time' between ticks.

Realistic clocks consist of various physical components, such as a power source to drive the clock. Our definition of a quantum clock, i.e., the pair $(\rho^0_C, \mathcal{M}_{C\to CT})$, does not specify these explicitly. It rather provides an abstract description of how the clock generates time information. We will present and review some specific models further on.

While an abstract model is suitable for our considerations, one should keep in mind that it is quite generic. That is, any physically realisable clock admits an abstract description in terms of a pair $(\rho^0_C, \mathcal{M}_{C\to TC})$, but not any such pair corresponds to a physically realisable quantum clock.

[1] Note that the tensor product of infinitely many Hilbert spaces is not necessarily a Hilbert space.

Indeed, in Section 4.2 we will introduce an additional condition (ε-continuity) that accounts for certain physical constraints. In particular, we are interested in clock models that don't need any external time mechanism for its operation. *I.e.* consider the example of the radio alarm clock - it works receiving EM signals of certain frequency to turn the alarm on and off. A clock that works by receiving light signals, and produces a stream of ticks, will be introduced in Sec. 4.6.

Our definition of a quantum clock does not explicitly require certain features that may be expected from a clock — constant time intervals, cyclicity, and unidirectional evolution. These turn out to be unnecessary for the operational task of synchronising quantum clocks. Take, as an example, the notion of constant time intervals. Intuitively, this seems to be necessary to determine the duration of events in some well-defined units (e.g., seconds). But, if the duration of a second were to change every time the clock should tick (i.e., if it was not a constant), and if *all* clocks would perform these adjustments automatically without us noticing, then such a change would not lead to any observable consequences.

As we will see, it is often convenient to consider particular constructions of quantum clocks where a map $\mathcal{M}$ can be written as the concatenation of two CPTP maps,

$$\mathcal{M}_{C\to CT} = \mathcal{M}^{\text{meas}}_{C\to CT} \circ \mathcal{M}^{\text{int}}_{C\to C} \ . \tag{4.2}$$

$\mathcal{M}^{\text{int}}$ acts only on the clockwork C and $\mathcal{M}^{\text{meas}}$ corresponds to a measurement on C of the form

$$\mathcal{M}^{\text{meas}}_{C\to CT} : \ \rho_C \mapsto \sum_{t\in\mathcal{T}} \sqrt{\pi_t}^{*} \rho_C \sqrt{\pi_t} \otimes |t\rangle\langle t|_T \ , \tag{4.3}$$

where $\{\pi_t\}_{t\in\mathcal{T}}$ is a family of nonnegative operators on C with $\sum_t \pi_t = \mathcal{I}_C$ ($\mathcal{I}_C$ being the identity operator), Hence $\{\pi_t\}_{t\in\mathcal{T}}$ represents a POVM measurement on $\mathcal{C}$ and $\sqrt{\pi_t}$ are the Kraus operators. $\{|t\rangle\}_{t\in\mathcal{T}}$ is a family of orthonormal vectors on the Hilbert space of $\mathcal{T}$ (*i.e.* it holds $\langle s|t\rangle = 0, \forall\, |s\rangle, |t\rangle \in \mathcal{T}$). $\mathcal{M}^{\text{int}}$ can be interpreted as the internal mechanism that drives the clockwork, whereas $\mathcal{M}^{\text{meas}}$ extracts time information from it and copies it to the tick registers. Note that by construction such a $\mathcal{M}^{\text{meas}}_{C\to CT}$ is a completely positive map since it has a Kraus decomposition. Also, if one takes $\mathcal{M}^{\text{int}}_{C\to C}$ to be described by a unitary map, it is trivially also completely positive (Kraus operators equaling the unitary), and hence the map $\mathcal{M}$ is also completely positive.

For a simple example of a clock of the form (4.2), assume that the clockwork C is binary (i.e., a qubit described by the Hilbert space $\mathbb{C}^2$), with orthonormal states $|0\rangle$ and $|1\rangle$, and that the map $\mathcal{M}^{\text{int}}$ flips the state on C, corresponding to a logical NOT operation. Furthermore, $\mathcal{M}^{\text{meas}}$ could be a measurement defined by the projectors $\pi_0 = |0\rangle\langle 0|$ and $\pi_1 = |1\rangle\langle 1|$. Then, with C initially set to $|0\rangle$, the resulting time scale $\{\rho^N_{T_1\cdots T_N}\}_N$ would consist of the operators of the form

$$\rho^N_{T_1\cdots T_N} = |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes \cdots \ .$$

The corresponding classical scale would thus be $\{1, 0, 1, 0, 1, ...\}$. Because of its deterministic character such a clock would be perfectly precise, in the sense that two such clocks could stay synchronised infinitely long. In particular, they would perform arbitrarily well in the Alternate Ticks Game defined in Chapter 5. But, physically they do not correspond to proper quantum clocks, since they implicitly assume that NOT operation is done infinitely fast. This is however not possible in a quantum setup, and to capture this fact we will introduce a specific continuity condition in Sec. 4.2.

4.2 ε-continuous quantum clocks

It appears to be easy to physically realise a simple binary clock described in Section 4.1 — one merely needs to implement a NOT gate and a measurement. But in a quantum world this cannot be done instantaneously, and the evolution would have to be well-timed (*i.e.* moving to an orthogonal state is not a trivial process as in classical mechanics, see Sec. 2.7 for the discussion of a quantum speed limit). Hence, one would need an additional time-dependent control mechanism to implement such a dynamics. And if such a control mechanism is necessary, the clock can of course no longer be regarded as a self-contained device. This motivates an additional assumption, namely that the clockwork evolves almost continuously. As our model is inherently discrete, we formalise this by the requirement that the individual steps of the evolution can be made arbitrarily small.

Definition 3. A quantum clock $(\rho_C^0, \mathcal{M}_{C\to CT})$ is called *ε-continuous* if the mapping $\mathcal{M}_{C\to CT}$ is ε-close to the identity on C, i.e.,

$$\frac{1}{2}\| \operatorname{tr}_T \circ \mathcal{M}_{C\to CT} - \mathcal{I}_C\|_\diamond \leq \varepsilon \tag{4.4}$$

where $\mathcal{I}_C$ is the identity mapping on C and $\|\cdot\|_\diamond$ is the diamond norm [112, 113].

This ε-continuity, for arbitrarily small values ε, is a necessary condition for a quantum clock to be physically realisable.[1] Combined with our requirement that all evolution steps are identical it ensures that the evolution of the state of the clockwork does not implicitly rely on a time-dependent control mechanism.

We note that the clock with the NOT-based clockwork described at the end of Section 4.1 is not ε-continuous for any $0 \leq \varepsilon < 1$, as in each step the state of C is deterministically changed to an orthogonal one. In fact, constructing a continuous clock is a bit more challenging. One conceivable approach could be to split the NOT operation used in the construction into small identical steps. Specifically, the map $\mathcal{M}^{\text{int}}$ could be defined as an n^{th} root of the NOT operation,

[1] While any physically realistic clock can be modelled as an ε-continuous quantum clock, continuity alone does not necessarily guarantee that an actual physical implementation of the quantum clock exists.

for some large n, corresponding to a rotation in the state space by a small angle. For some $\varepsilon > 0$ and sufficiently large n this map would be ε-close to the identity. However, it is unclear how to choose the measurement $\mathcal{M}^{\text{meas}}$. In fact, if the described small rotations are applied in between two measurements, its outcome would no longer be deterministic. Consequently, the clock would generate a rather randomised time scale and thus not be very precise.

4.3 Reset clocks

It is of course possible to devise more precise continuous clocks, but the price to pay is that the dimension of the clockwork C is larger. As an example, consider a clockwork that mimics the propagation of a wavepacket. Specifically, let C be equipped with an orthonormal basis $\{|c\rangle\}_{c=0,\dots,m}$, for some large $m \in \mathbb{N}$ and define

$$\Pi_{c_j} = \sum_{c:\, |c-c_j|\leq\Delta} |c\rangle\langle c|$$

with $\Delta \ll m$. Furthermore, assume that the initial state ρ_C^0 is contained in the support of the projector Π_{c_0} with $c_0 = 0$, i.e.,

$$\operatorname{tr}(\Pi_{c_0}\rho_C^0) = 1 \ .$$

Intuitively, one may think of ρ_C^0 as a wavepacket of breadth Δ localised around c_0. Let $\mathcal{M}^{\text{int}}$ be a map that moves this wavepacket by some fixed distance $\nu \ll 1$, in the sense that the state ρ_C^N of the clockwork after N applications of $\mathcal{M}^{\text{int}}$ satisfies

$$\operatorname{tr}(\Pi_{c_N}\rho_C^N) \approx 1 \quad \text{for } c_N = \lfloor N\nu \rfloor \tag{4.5}$$

(provided that $N\nu + \Delta < d$). Finally, the measurement $\mathcal{M}^{\text{meas}}_{C\to CT}$ [see (4.3)] could be defined using the positive-operator-valued measure (POVM) elements

$$\pi_0 = I_C - \delta\Pi_{\bar{c}} \quad \text{and} \quad \pi_1 = \delta\Pi_{\bar{c}} \tag{4.6}$$

for $\bar{c} = d - 1$ and some $\delta \ll 1$. Intuitively, for any $\varepsilon > 0$, the clock could be made ε-continuous by choosing ν and δ sufficiently small.

Practically, this means that a certain region of the clockwork (here centered around $\bar{c}$) is a *tick region* and a tick can be produced only if the current state of the clockwork has a non-zero support in that region. Otherwise the measuring map will not change the state, and the only influence will come from the evolution map $\mathcal{M}^{\text{int}}$.

Above construction of $\mathcal{M}^{\text{int}}$, however, relies on the assumption that the (mimicked) wavepacket propagates regularly (such that condition (4.5) is satisfied). This is only possible if its breadth Δ, and hence the dimension d of the clockwork, is large enough. Note also that once the clock

has a support in the tick region defined by the operator $\Pi_{\bar{c}}$, the measuring map will influence the wavepacket and disturb its regular propagation, hence introducing more uncertainty in the sequence of ticks. The precision of the proposed construction thus depends strongly on the dimension of the clockwork, something that will be true for all clocks, as we will see later, and also on the definition of the tick region.

Under this idealised conditions, to analyse the time scale that the clock generates we first observe that, as long as $N\nu + 2\Delta < d$, the state of the clockwork ρ_C^N has only negligible overlap with π_1 and is therefore not significantly disturbed by the measurement $\mathcal{M}^{\text{meas}}$. During this phase, the clock would output a series of states $|0\rangle$ to the tick registers. Only once $N\nu$ is approximately equal to d, clock would at some point output a state $|1\rangle$. However, after producing this first tick, the clock would in each step output $|1\rangle$ with probability δ, resulting in a rather randomised pattern of ticks. This can be surpassed by making this a reset clock and reseting the state of the clock to ρ_0 after each tick.

Definition 4. *A reset clock* is any clock whose state is reseted to a fixed state after each occurrence of a tick.

Formally, an example of a reset clock is the one where the CPTP map defined by Eq. (4.3) is replaced by

$$\begin{aligned}\mathcal{M}^{\text{meas}}_{C\to CT} : \rho_C \mapsto & \sqrt{\pi_0}\,\rho_C\,\sqrt{\pi_0} \otimes |0\rangle\langle 0|_T \\ & + \operatorname{tr}(\pi_1\,\rho_C)\,\hat{\rho}_C \otimes |1\rangle\langle 1|_T\end{aligned} \tag{4.7}$$

with π_0, π_1 representing POVM on C. For simplicity we can take $\hat{\rho}_C = \rho_C^0$ as a *reset state*.

The resulting time of the clock defined in this section, would then consist of almost equally long sequences of states $|0\rangle$ separated by states $|1\rangle$. In other words, for each tick there is the same average number of applications of the map $\mathcal{M}$ until it is produced. Hence instead of considering sequence of ticks we can restrict ourselves to the properties of the first tick, which simplifies the analysis greatly.

In this thesis, we will mostly present results for the reset clocks. In particular, some numerical advances are shown in Section 5.1.1 and analytical bounds for classical clocks in 5.3. Also, in Sec. 5.5 we describe some bounds for the accuracy of classical and some quantum reset clocks as done in [38].

4.4 Determining the ε-continuity of a clock

To calculate a diamond norm needed for the calculation of the continuity parameter from Sec. 4.2, we will use the mathematical framework of semidefinite programming (SDP), as applied to the channel distance in [113]. Semidefinite programming can be used to solve convex optimisation

problems and has recently also been developed for diamond norms in particular. As shown in [113], for a quantum channel $\Phi : C \to C$, it holds that $\frac{1}{2}\|\Phi\|_\diamond \leq \varepsilon$ if and only if

$$\begin{aligned} \textbf{there exists:} \quad & Z \geq J(\Phi), \\ & Z \in \mathrm{Pos}\left(C \otimes C'\right) \\ \textbf{such that:} \quad & 2\|\operatorname{tr}_C(Z)\|_\infty \leq \varepsilon \end{aligned} \tag{4.8}$$

where we take C' isomorphic to C, Pos(S) denotes the set of positive operators on the space S and $J(\Phi) = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$ is the so-called Choi-Jamiołkowski state. Here $\Phi = \operatorname{tr}_T \circ \mathcal{M}_{C\to CT} - \mathcal{I}_C$. Note that the diamond norm of a channel distance is always bounded by 2, hence a trivial upper bound for ε is $\varepsilon \leq 1$. Also the infinity norm $\|A\|_\infty$ is defined as the maximum singular value of the map A.

For our purpose it is enough to find a good upper bound for the channel distance, since according to [113],this bound will upper bound the continuity parameter and can hence be used as ε. Any dual feasible operator Z satisfying the conditions of (4.8) can be used to compute an upper bound, and we will attempt to find Z that would provide us with a sufficiently good upper bound.

On a particular example, we will show steps needed to analytically obtain an upper bound on ε for the quantum clock.

Example 1. Let a map describing the clock dynamics be $\mathcal{M} = \mathcal{M}^{\text{meas.}} \circ \mathcal{M}^{\text{int.}}$ with $\mathcal{M}^{\text{int.}}(\rho) = e^{-iH_C\theta}\rho e^{iH_C\theta}$ an internal clockwork map for some fixed parameter θ and $\hbar = 1$. We will assume a reset model of a clock and define a clockwork Hamiltonian H_C as in the papers [21, 22] (see also Sec. 3.2). This Hamiltonian represents the dynamics of a particle on a circle, with a frequency ω, and can be written as:

$$H_C = -\frac{1}{i\tau}\log U \tag{4.9}$$

$$U(\tau) = \sum_{k=0}^{d-2} |k+1\rangle\langle k| + |0\rangle\langle d-1| \tag{4.10}$$

where a fixed parameter τ depends on ω and the dimension d. Hence the state of the clockwork jumps to the next state in constant intervals τ, but in between the state spreads a lot. This is why the starting state of such a clockwork plays an important role, which we will also see in Sec. 5.5.

For the clockwork states we will take the computational basis states $\{|0\rangle\langle 0|, ..., |d-1\rangle\langle d-1|\}$,

and it can be shown that the clockwork Hamiltonian is diagonal in the Fourier basis:

$$H_C \left|u_k\right\rangle = k\omega \left|u_k\right\rangle$$

$$\text{with } \left|m\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{-2\pi i m k}{d}} \left|u_k\right\rangle,\ m \in \{0, ..., d-1\}$$

$$\left|m\right\rangle\!\left\langle m\right| = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} e^{\frac{-2\pi i m (k-l)}{d}} \left|u_k\right\rangle \left\langle u_l\right| \tag{4.11}$$

$$\left\langle m\right| \left|n\right\rangle = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} e^{\frac{-2\pi i (mk-nl)}{d}} \left|u_k\right\rangle \left\langle u_l\right|$$

Next we define:

$$U(\theta) \left|\psi\right\rangle = e^{-iH_C\theta} \left|\psi\right\rangle \tag{4.12}$$

Then

$$U(\tau) \left|m\right\rangle = e^{-iH_C\tau} \left|m\right\rangle = e^{-iH_C\tau} \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{-2\pi i m k}{d}} \left|u_k\right\rangle = \tag{4.13}$$

$$\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{-2\pi i m k}{d}} e^{-ik\omega\tau} \left|u_k\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{-2\pi i (m+1) k}{d}} \left|u_k\right\rangle = \left|m+1\right\rangle \tag{4.14}$$

with $\tau = \frac{2\pi}{\omega d}$. For simplicity, we will take frequency $\omega = \frac{2\pi}{d}$ and $\tau = 1$.

The evolution of the clockwork $\mathcal{M}^{\text{int.}}$ on the diagonal states will be defined as:

$$\mathcal{M}^{\text{int.}}(\left|m\right\rangle\!\left\langle m\right|) = e^{-iH_C\theta} \left|m\right\rangle\!\left\langle m\right| e^{iH_C\theta} = \tag{4.15}$$

$$\frac{1}{d} \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} e^{\frac{-2\pi i \theta (k-l)}{d}} e^{\frac{-2\pi i m (k-l)}{d}} \left|u_k\right\rangle \left\langle u_l\right| \tag{4.16}$$

for some fixed (freely chosen) parameter $\theta \leq \tau$.

Tick states of the clockwork are the ones that could produce a tick (if the support of the clock state there is high enough), and we label the set of them J. We will take $J = \{d - \lceil \frac{d}{10} \rceil, ..., d-1\}$ as in Section 5.1.1. Then we can choose a measurement map to be described by the following Kraus operators (assuming that the clock is reset to the state $\left|0\right\rangle\!\left\langle 0\right|$ after the tick):

$$K_0 = (\sum_{j \notin J} \left|j\right\rangle\!\left\langle j\right| + \sqrt{1-\delta} \sum_{j \in J} \left|j\right\rangle\!\left\langle j\right|) \otimes \left|0\right\rangle_T \tag{4.17}$$

$$K_{1,j} = \sqrt{\delta} \left|0\right\rangle_C \left\langle j\right| \otimes \left|1\right\rangle_T,\ j \in J \tag{4.18}$$

We can write a CPTP map $\mathcal{M}^{\text{meas.}}$ as

$$\mathcal{M}^{\text{meas.}}(\rho_C) = K_0 \rho_C K_0^* + \sum_{j\in J} K_{1,j}\rho_C K_{1,j}^*$$

with $K_0^*K_0 + \sum_{j\in J} K_{1,j}^* K_{1,j} = \mathcal{I}$. Let $m \in \{0, ..., d-1\}$. Using Eq. (4.15), we can calculate:

$$\begin{aligned}
&\text{tr}_T(K_{1,j}\mathcal{M}^{\text{int.}}(|m\rangle\langle m|)K_{1,j}^*) = \\
&\frac{\delta}{d^3}\sum_{k1,k_2=0}^{d-1}\sum_{l1,l_2=0}^{d-1} e^{\frac{2\pi i j(l_1-k_2)}{d}} |u_{k_1}\rangle\langle u_{l_1}| \Big(\sum_{k=0}^{d-1}\sum_{l=0}^{d-1} e^{\frac{-2\pi i\theta(k-l)}{d}} e^{\frac{-2\pi i m(k-l)}{d}} |u_k\rangle\langle u_l|\Big) |u_{k_2}\rangle\langle u_{l_2}| \\
&= \frac{\delta}{d^2}\sum_{k=0}^{d-1}\sum_{l=0}^{d-1} e^{\frac{2\pi i(k-l)(j-m-\theta)}{d}} |0\rangle\langle 0|\,, \forall j \in J
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\text{tr}_T(K_{1,j}\mathcal{M}^{\text{int.}}(|m\rangle\langle n|)K_{1,j}^*) = \\
&\frac{\delta}{d^2}\sum_{k=0}^{d-1}\sum_{l=0}^{d-1} e^{\frac{2\pi i(k(j-m-\theta)-l(j-n-\theta))}{d}} |0\rangle\langle 0| = \\
&\frac{2\delta}{d^2}(1-\cos(2\pi\theta))(1-e^{2\pi i(j-m-\theta)})^{-1}(1-e^{-2\pi i(j-n-\theta)})^{-1} |0\rangle\langle 0|\,, \forall j \in J
\end{aligned}$$

Also, we have:

$$\begin{aligned}
&\text{tr}_T\left(K_0\mathcal{M}^{\text{int.}}(|m\rangle\langle n|)K_0^*\right) = \\
=&\frac{1}{d^2}\sum_{k,l=0}^{d-1} e^{-\frac{2\pi i((k-l)\theta+mk-nl)}{d}}\Big(\sum_{j_1,j_2\notin J} e^{\frac{2\pi i(j_1k-j_2l)}{d}} |j_1\rangle\langle j_2| + (1-\delta)\sum_{j_1,j_2\in J} e^{\frac{-2\pi i(j_2l-j_1k)}{d}} |j_1\rangle\langle j_2|\Big) + \\
&\frac{\sqrt{1-\delta}}{d^2}\sum_{k,l} e^{-\frac{2\pi i(\theta(k-l)+mk-nl)}{d}}\sum_{j_1\notin J}\sum_{j_2\in J}\left(e^{2\pi i(j_1k-j_2l)} |j_1\rangle\langle j_2| + e^{2\pi i(j_2k-j_1l)} |j_2\rangle\langle j_1|\right)
\end{aligned} \tag{4.20}$$

It turns out that finding a positive Z and X such that $Z - X = J(\Phi)$, wih $\Phi = \text{tr}_T\,\mathcal{M} - \mathcal{I}_C$ and the corresponding Choi-Jamilkowski state $J(\Phi)$ that would provide us with good bounds, is very hard to be done analytically. Hence we used numerics in MATLAB to find an upper bound, by optimising over positive diagonal matrices Z, and we show some of the results on Fig. 4.3. The numerics becomes very slow for higher dimensions, hence only the results for lower dimensions are presented.

4.5 Stochastic clocks

Now we will consider a specific clock construction described by the model introduced in Sec. 4.1. Inside this model, one can simplify the maps involved, reducing the dynamics to a simple classical

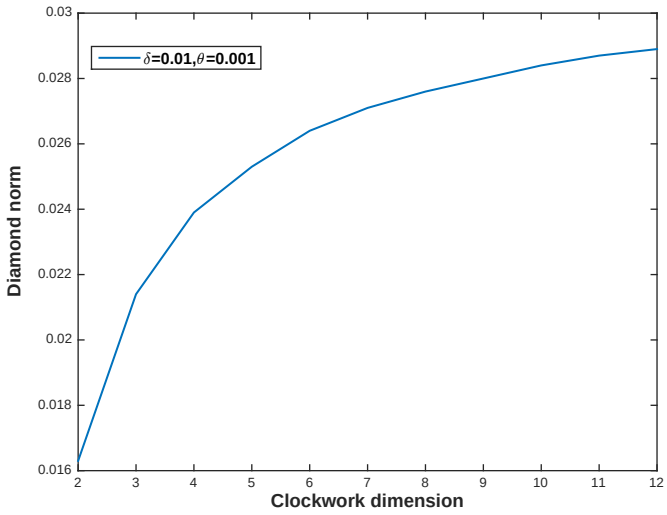

Figure 4.3: *Upper bound on the ε-continuity parameter (one half of the diamond norm), for the clockwork dimensions 2 to 12, clockwork Hamiltonian as in* (4.11) *and parameters $\delta = 0.01$, and $\theta = 0.001$*

random walk. We will call such clocks *stochastic clocks*. A specific case of a stochastic clock, that we call a *ladder clock*, was first introduced in the master thesis [41]. Here we give an extension of that clock to the more general stochastic reset clocks. Most of the results in this section were presented in the preprint [37]. Continuous stochastic clocks were investigated in [38], as we will describe in Sec. 5.5.

As we will see, stochastic clocks exhibit some simple properties which make them particularly convenient to investigate. They can be modelled by an entirely classical system, and their dynamics presents an example of a Markov chain. Even though upper bounds derived on their performance only hold for the classical dynamics, they are important steps in investigating clocks

and time scales that are described without relation to an external time. Performance of the stochastic clocks in the Alternate Ticks (AT) Game that will be introduced in Sec. 5.1, represents a specific lower bound on the performance of the best possible quantum clock that can be constructed within the given framework. This game measures the synchronisation and quality of clocks in an operational manner. Results we obtain for classical clocks also help us analyse the possible advantage of having quantum dynamics introduced.

4.5.1 Mathematical definition of a general stochastic clock

We will now describe a dynamics of a stochastic clock. Let random variables $\{S_n,\, n \geq 0\}$ represent a position on the $(d+1)$ - dimensional clockwork (depicted as a circle) after n applications of the map $\mathcal{M}$. Imagine that a clock state can jump to an arbitrary state of the clockwork $\{|0\rangle, ..., |d\rangle\}$, with a given probability distribution. We can represent this using Markov chains, such that

$$P(S_{n+1} = l \mid S_n = k) = p_{k,l}\,, n \geq 0,\, k, l \in \{0, ..., d\}$$

Hence $P(S_{n+1} = l) = \sum_{k=0}^{d} p_{k,l} P(S_n = k)$. This is the case of a homogenous Markov chain, where the probability distribution does not depend on n. We now mathematically define a general stochastic reset clock.

Definition 5. *Stochastic reset clock* on a Hilbert space $\mathcal{H}_{CT}$ is given by a pair $(\rho_C^0, \mathcal{M}^{\text{st.}})$ with $\rho_C^0 = |j\rangle\langle j|\,,\, j \in \{0, ..., d-1\}$ and the map $\mathcal{M}^{\text{st.}} : C \to CT$ taking the form:

$$\begin{aligned}\mathcal{M}^{\text{st.}}(\rho_C) &= (1-\delta)\rho_C \otimes |0\rangle\langle 0|_T \\ &+ \sum_{j=0}^{d-1}\sum_{m=j}^{d-1} \langle j|\rho_C|j\rangle (p_{jm} - \delta_{jm}(1-\delta))\, |m\rangle\langle m|_C \otimes |0\rangle\langle 0|_T + \sum_{j=1}^{d-1}\sum_{m=0}^{j-1} \langle j|\rho_C|j\rangle p_{jm}\, |0\rangle\langle 0|_C \otimes |1\rangle\langle 1|_T\end{aligned} \tag{4.21}$$

where $\delta = 1 - \min_j p_{jj}$ and δ_{jm} is a Kronecker delta symbol. Note that $\sum_{m=0}^{d-1} p_{jm} = 1,\, \forall j$.

For the diagonal pure input states, after each detected tick the clock starts anew from the state $|0\rangle\langle 0|_C$ (the tick state). Hence this is an example of a reset clock. Wlog, for these clocks, one can take $\rho_C^0 = |0\rangle\langle 0|$, and also choose any other state instead of $|0\rangle$ as a tick state for the clock. Then the parameter δ represents the maximum probability that the clock changes its state to any other state on the circle, and the values p_{jk} represent a probability that a clock jumps from the state j to k. Importantly, $p_{jk} \in [0, 1]$, $\forall j, k$ and $\sum_{k=0}^{d-1} p_{jk} = 1$, $\forall j \in \{0, ..., d-1\}$.

One of the motivations for this construction of a clock is the simplicity when investigating its properties and performance.

We will now show that above defined $\mathcal{M}^{st.}$ is a valid clock map by Def. 1.

Lemma 1. *Map $\mathcal{M}^{st.}$ from Eq. (4.21) is a completely positive trace preserving map.*

Proof. By Choi's theorem [111], it is enough to show that there is a Kraus representation for the map $\mathcal{M}^{\text{st.}}$. Define operators:

$$\begin{aligned}
M_{im} &= \sqrt{p_{im}}\,|0\rangle\langle i|_C \otimes |1\rangle_T\,,\ i \neq 0, m \in \{0,...,i-1\}\\
M_{im} &= \sqrt{p_{im} - (1-\delta)}\,|m\rangle\langle i|_C \otimes |0\rangle_T\,,\ m = i\\
M_{im} &= \sqrt{p_{im}}\,|m\rangle\langle i|_C \otimes |0\rangle_T\,,\ m \in \{i+1,...,d-1\}\\
M_0 &= \sqrt{1-\delta}\mathcal{I}_C \otimes |1\rangle_T
\end{aligned}$$

It can easily be seen that these operators are linear operators from C to CT. Also $M_{im}^\dagger M_{im} = p_{im}\,|i\rangle\langle i|\,,\ \forall i,\ m \neq i$ and $M_{ii}^\dagger M_{ii} = (p_{ii} - (1-\delta))\,|i\rangle\langle i|\,, \forall i$. Hence

$$\begin{aligned}
&\sum_{i=0}^{d-1}\sum_{m=0}^{d-1} M_{im}^\dagger M_{im} + M_0^\dagger M_0 =\\
&\sum_{i=0}^{d-1}(\sum_{m\neq i} p_{im} + (p_{ii} - (1-\delta))\,|i\rangle\langle i| + (1-\delta)\mathcal{I}_C = (1-(1-\delta))\sum_{i=0}^{d-1}|i\rangle\langle i| + (1-\delta)\mathcal{I}_C = \mathcal{I}_C
\end{aligned} \tag{4.22}$$

By definition they represent Kraus operators and we can easily see that

$$\mathcal{M}^{\text{st.}}(\rho_C) = \sum_{i=0}^{d-1}\sum_{m=0}^{d-1} M_{im}\rho_C M_{im}^\dagger + M_0\rho_C M_0^\dagger$$

is indeed a CPTP map.

□

One can think of the simplest stochastic clock model, which for the classical starting states can be described by a toss of a coin at every step. We will call it the *stochastic ladder clock*, and it was introduced in [41] (notice the usage of a similar clock in the thermodynamical setting in [44] and when investigating stochastic clocks under the different measure of performance [38]). We define it next.

Definition 6. *A stochastic ladder clock* is a clock defined by Def. 5 that in the case of pure states can jump only to the next state with the jump probability $\delta \in [0,1]$ and stay in the current state with the probability $1-\delta$. Hence for the classical states it is defined by:

$$\begin{aligned}
P(S_{n+1} = k+1 \mid S_n = k) &= p_{k,k+1} = \delta,\\
P(S_{n+1} = k \mid S_n = k) &= p_{k,k} = 1-\delta
\end{aligned} \tag{4.23}$$

independent of n and k.

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We will see later that the stochastic ladder clock is one of the best performing stochastic clocks.

Now we show how exactly ε relates to the probability distribution for the jump of the clock.

Lemma 2. *Stochastic clock* $(\rho_C^0, \mathcal{M}^{st.})$ *with* $\mathcal{M}^{\text{st.}}$ *given as in Eq. (4.21), is* ε*-continuous iff* $\delta \leq \varepsilon$

Proof. We will use the semi-definite programming described above to obtain an upper bound on $\left\|\text{tr}_T \mathcal{M}^{\text{st.}} - \mathcal{I}_C\right\|_\diamond = \left\|\Phi\right\|_\diamond$ which will also determine possible values of ε from Def. 3 (see Sec. 4.4). As a first step we need to calculate Choi-Jamiołkowski state of the $\Phi = \text{tr}_T \mathcal{M}^{\text{st.}} - \mathcal{I}_C : C \to C$ which is, assuming C' to be a copy space of C, $J(\Phi) = \sum_{i,j} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|_{C'}$. Then:

$$\Phi(|i\rangle\langle j|) = \begin{cases} -\delta\,|i\rangle\langle j| & \text{for } i \neq j \\ -\delta\,|i\rangle\langle i| + \sum_{m=i+1}^{d-1} p_{im}\,|m\rangle\langle m| + (p_{ii} - (1-\delta))\,|i\rangle\langle i| + \sum_{m=0}^{i-1} p_{im}\,|0\rangle\langle 0| & \text{for } i = j \geq 1 \\ -\delta\,|0\rangle\langle 0| + \sum_{m=1}^{d-1} p_{0m}\,|m\rangle\langle m| + (p_{00} - (1-\delta))\,|0\rangle\langle 0| & \text{for } i = j = 0 \end{cases} \tag{4.24}$$

and we have:

$$\begin{aligned} J(\Phi) = -\delta \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes |i\rangle\langle j|_{C'} + \sum_{i=0}^{d-1} \sum_{m=i+1}^{d-1} p_{im}\,|m\rangle\langle m| \otimes |i\rangle\langle i|_{C'} + \\ \sum_{i=0}^{d-1} (p_{ii} - (1-\delta))\,|i\rangle\langle i| \otimes |i\rangle\langle i|_{C'} + \sum_{i=1}^{d-1} \sum_{m=0}^{i-1} p_{im}\,|0\rangle\langle 0| \otimes |i\rangle\langle i|_{C'} \end{aligned} \tag{4.25}$$

Now we need to find a dual feasible map Z such that both Z and $X = Z - J(\phi)$ are positive maps on $\mathcal{C} \otimes \mathcal{C}'$. Let

$$X = \delta \sum_{i,j=0}^{d-1} |i\rangle\langle j|_C \otimes |i\rangle\langle j|_{C'}$$

Then it follows:

$$Z = \sum_{i=0}^{d-1} \sum_{m=i}^{d-1} p_{im}\,|m\rangle\langle m|_C \otimes |i\rangle\langle i|_{C'} + \sum_{i=1}^{d-1} \sum_{m=0}^{i-1} p_{im}\,|0\rangle\langle 0|_C \otimes |i\rangle\langle i|_{C'} + \sum_{i=0}^{d-1} (p_{ii} - (1-\delta))\,|i\rangle\langle i| \otimes |i\rangle\langle i|_{C'} \tag{4.26}$$

Now let $|\phi\rangle = \sum_{i,j} z_{i,j}\,|i\rangle_C \otimes |j\rangle_{C'} \in C \otimes C'$. Then to show X and Z are positive we need to show that $\langle\phi|X|\phi\rangle \geq 0$ for any ϕ and correspondingly for Z. Very similar map X was used in [41] and we follow the steps from there.

$$
\begin{aligned}
&\langle\phi|\, X \,|\phi\rangle \\
&= \delta \sum_{i,j,k,l,m,n} z_{ij}^* z_{mn} \left(\langle i| \otimes \langle j|\right) \left(|k\rangle \langle l| \otimes |k\rangle \langle l|\right) \left(|m\rangle \otimes |n\rangle\right) \\
&= \delta \sum_{k,l} z_{kk}^* z_{ll} \\
&= \delta \sum_{k<l} \left(z_{kk}^* z_{ll} + z_{kk} z_{ll}^*\right) + \delta \sum_k |z_{kk}|^2 \\
&= \delta \sum_{k<l} 2\mathrm{Re}\left(z_{kk}^* z_{ll}\right) + \delta \sum_k \left(\mathrm{Re}^2(z_{kk}) + \mathrm{Im}^2(z_{kk})\right)
\end{aligned}
$$

Now, notice that $\mathrm{Re}(z_{kk}^* z_{ll}) = \mathrm{Re}(z_{kk})\mathrm{Re}(z_{ll}) + \mathrm{Im}(z_{kk})\mathrm{Im}(z_{ll})$ and the sum can be written as:

$$\langle\phi|X|\phi\rangle = \delta(\sum_{i=0}^{d-1} \mathrm{Re} z_{ii})^2 + \delta(\sum_{i=0}^{d-1} \mathrm{Im} z_{ii})^2 \geq 0$$

since all of the terms are non-negative. We also calculate:

$$\langle\phi|Z|\phi\rangle = \sum_{i=0}^{d-1} \sum_{m=i+1}^{d-1} p_{im}|z_{mi}|^2 + \sum_{i=1}^{d-1} \sum_{m=0}^{i-1} p_{im}|z_{0i}|^2 + \sum_{i=0}^{d-1} (p_{ii} - (1-\delta))|z_{ii}|^2 \geq 0 \tag{4.27}$$

Further, we can see that

$$
\begin{aligned}
\mathrm{tr}_C\, Z &= \sum_{i=0}^{d-1} (\sum_{m\neq i}^{d-1} p_{im} + p_{ii} - (1-\delta))\, |i\rangle\langle i|_{C'} \\
&= \delta \mathcal{I}_{C'}
\end{aligned} \tag{4.28}
$$

Hence $\mathrm{tr}_C\, Z$ is a diagonal matrix with elements δ on the main diagonal. We can hence deduce that $||\,\mathrm{tr}_C\, Z||_\infty = \delta$. Since $||\,\mathrm{tr}_C\, Z||_\infty$ represents a bound for the $\frac{1}{2}\left\|\Phi\right\|_\diamond$ we see that our stochastic clock is ε-continuous for all $\varepsilon \geq \delta$. Also, given an ε we can choose a probability distribution for our clock jumps to be such that $\delta \leq \varepsilon$ and we know that our clock will be ε-continuous by Def. 3. □

4.6 Model of a clock with light sources

Let us now consider a practical model of the clock as introduced in Def. 1. To fully discribe a physical clock one needs a power source as well. Let us assume we have a laser available that is providing photons through an *incoming mode* to interact with the clockwork. Depending on the part of the clockwork photons interact with, they will be sent toward the tick registers through the *outgoing mode* (in the case the clockwork state had a support in the tick region) or get annihilated. Each map $\mathcal{M}^{\mathrm{meas}}$ will be performed on a different register, which with a certain

probability contains a photon. If registers are two-dimensional systems, then measurement on the register in the $\{|0\rangle, |1\rangle\}$ basis, could yield an outcome 0 meaning that there was no tick and 1 denoting that the tick has occurred. One can introduce a referee that receives the tick registers and measures them in the order he/she receives them. This is a useful set up when one wants to investigate clock synchronisation, like it will be done in Sec. 5.1.2.

Note that here we can also use one large tick register instead of a collection of the low dimensional ones. Then to obtain the number of ticks produced, we measure the register with a number operator $N = a^\dagger a$, where a represents an annihilation operator.

Essential parameters defining a quantum clock in this particular model are thus:

1. dimension of the clockwork d, the initial state of this system ρ_C^0, and a CPTP map $\mathcal{M}^{\text{int}}$ (or a Hamiltonian H^{int}) governing its evolution;

2. dimension of the tick registers of the clock;

3. specification of the energy source and dimensions of the underlying Hilbert spaces, the initial states of the incoming and outgoing modes - $|\varphi_{0,in}\rangle$, $|\varphi_{0,out}\rangle$ and the Hamiltonians H^{in}, H^{out} governing their local evolution;

4. CPTP map $\mathcal{M}^{\text{meas}}$ (or a Hamiltonian H^{meas}) governing the interaction between the clockwork and the tick registers (*i.e.* the description of the tick mechanism);

5. Positive-operator-valued measure (POVM) specifying the measurements on the tick registers by the clock or the referee.

An advantage of this model is that it fully describes the workings of the clock and hence could be implemented in the lab. It is though most easily described via Hamiltonians as we show next.

Evolution of this clock can then be described by a map

$$\mathcal{M}(\rho_C) = e^{-i(H^{\text{out}}+H^{\text{meas}}+H^{\text{int}}+H^{\text{in}})\theta}\rho_C e^{i(H^{\text{out}}+H^{\text{meas}}+H^{\text{int}}+H^{\text{in}})\theta}$$

where we supressed the identity maps (*e.g.* one should write $H^{\text{out}} \otimes \mathcal{I}_{CT} \otimes \mathcal{I}_C \otimes \mathcal{I}_{\text{in}}$ instead of just H^{out}).

As an example, we can assume that the dimensions of the tick registers are $\dim(T_i) = 2$. We can take a harmonic-oscillator-type Hamiltonians for the evolution in both the incoming and the outgoing mode of the clocks:

$$\begin{aligned} H^{\text{in}} &= a_{\text{in}}^\dagger a_{\text{in}} + \frac{1}{2}, \\ H^{\text{out}} &= a_{\text{out}}^\dagger a_{\text{out}} + \frac{1}{2}, \end{aligned} \tag{4.29}$$

where a and $a^\dagger$ are the annihilation and creator operators (for the incoming and the outgoing mode, labeled by the subscript *in* and *out*), respectively. For the interaction between the clockwork and the energy source (photons), we choose an interaction Hamiltonian of the form:

$$H^{\text{meas}} = \alpha(a_{\text{in}} \otimes |c\rangle\langle c| \otimes a^\dagger_{\text{out}} + h.c.) \tag{4.30}$$

where $|c\rangle$ is a tick state of the clockwork in a chosen clock basis, α is a real number specifying the coupling strength (it can be *e.g* a constant or a number depending on the clock dimension d), and "*h.c.*" denotes the Hermitian conjugate. Further specifications will be introduced for the Alternate Ticks Game in Sec. 5.1.2.

We can see that since Hamiltonians need to be hermitian, there will always be a probability that a photon comes back from the outgoing to the incoming mode. To minimise the probability for this to happen, one would need to add extra conditions on the power source available. In practice there is also a non-zero probability that the laser produces more than one photon immediately one after another. This would then create two closely consecutive ticks from the clock that would disturb its possibility of synchronisation with another clock. This will be discussed more in the next chapter.

4.7 Discussions and future work

In the literature, one often refers to a (quantum) clock as to a periodic system with a certain frequency. It is usually assumed to be measured directly by an observer when a time information is wanted. When two clocks are compared, this would be the comparison of how well they approximate some assumed external, parametric time (one that is used to describe the clockwork evolution). Here we extended these notions by proposing a novel two-part definition of clock (consisting of the *clockwork* - usually considered a clock by itself, and *tick registers*) that outputs a time scale. In addition, measurements done by an observer should influence the clock working as little as possible, hence these measurements are done on the tick registers and not directly on the clockwork. This is what classical clocks from our daily lives mostly do. We observe their output to keep track of time (this corresponds to reading out the time from the tick registers), but the time scale in form of *ticks* (usually in intervals denoted as seconds) is created by the clock mechanism even when it is not observed. Since classical clocks are almost negligibly disturbed by the interaction with the environment, if we observe them the tick pattern will not be influenced, unlike in the quantum case. Even though this is a well known fact, it is often overlooked, and one assumes classical parametric time in quantum theory. We find the two-part clock model we introduce very important for investigating limitations for building time scales in quantum theory, and importantly, it can give us insights into the way time measurement differs in quantum theory from the classical one.

4. OPERATIONAL APPROACH TO TIME: TWO-SYSTEM QUANTUM CLOCK DEFINITION

Evidently, our definition of a quantum clock includes physical systems that are not conventionally referred to as a clock. In particular, our quantum clock does not necessarily exhibit features that are expected from a clock — constant time intervals, cyclicality, unidirectional evolution, and passage through a succession of states. However, lots of the features usually required in the clock constructions only make sense if one assumes a global notion of time. Without the global time, or another clock as a measure to compare with, constant interval has no meaning. Also, in that case evolution of the clockwork in one direction is not important, since one is only interested in the ticks of one clock. To bring a meaning to all of these notions one can add another clock/s, to compare the original time scale with. This is what we will present in the next chapter.

To comment on possible further applications of our model, we relate to clocks in relativity. In Sec. 3.3 we have reviewed some research done on clocks that are put into the gravitational field [23]. However, clock comparison there is done only between the clockworks, tick registers are not a part of the model. Even with just the clockworks, significant lag between the originally synchronised clocks is noticed due to the gravitational field effects. It would be great to investigate how would the synchronisation between tick registers be affected in the presence of gravitation. Similar problem of interest would be to see how the tick pattern is affected if the clock would move with a relativistic speed.

Another field of research on time where our clock could be very useful, is the so-called 'time as correlations' approach, see Sec. 2.5. For now, this approach was seriously investigated only with a simple continuous model of a clock where time is measured by a position of a particle on a line. This clock is however unphysical, and it would be very insightful to see results if a physical clock is used instead. Particle on a line yields a Schrödinger equation for the evolution of a system, but it would be surprising to see the same result with a non-trivial clock used. Indeed, in a master thesis project [114], a first step was taken to include our two-part clock model into the time as correlations approach. Results obtained show that the evolution law does not look like the Schrödinger equation, but no closed form equation or rule has yet been obtained. This would be a very fruitful area for a further research.

Chapter 5

Synchronisation and accuracy of quantum clocks: the Alternate Ticks Game and other approaches

Parts of this chapter are based on the work in [35], while Sec. 5.3 is mostly based on the research described in [37] and Sec. 5.5 on [38].

So far we have defined which type of systems we consider as quantum clocks, but we have not yet provided any criteria to assess their capacity to generate precise time information. As it seems impossible to define the accuracy of a clock in an operational manner without comparing it to another one, we consider scenarios with more than one clock. We then ask how well the clocks can stay synchronised.

There are various ways to define synchronisation between two clocks. In Section 3.5 we reviewed some of the synchronisation protocols commonly used. But note that these protocols aim to construct two synchronised clocks using parametric time, while we are interested in checking the possible synchronisation (that is already achieved or could be achieved by *e.g.* choosing suitable starting states) of any two clocks in an operational manner.

In general, synchronisation of two existing clocks means that the time scales generated by them are in some sense compatible. The strongest possible criterion would be that they are identical. However, such perfect synchronisation cannot be achieved with continuous clocks of finite size, as argued in Chapter 4. Another issue with demanding identical time scales, is how to define and measure if they are actually identical? One would need a third clock which provides a 'reference time' for a comparison.

We are interested in a notion of a local *clock time* given by the physical system that fulfils our definition of a clock, Def. 1. Note that the relation defined via the time scale given by the ticks of the clock (we can call it 'time' or 'before or synchronous with' relation), satisfies the

properties of a *total order* (reflexivity, antisymmetry, transitivity and totality). The global time parameter used in classical theory and currently in quantum mechanics satisfies these properties. It is an important question to ask under what conditions can the local ordering given by clocks be extended to the global one (cf. Sec. 2.6).

In Section 5.1, we will introduce the *Alternate Ticks Game (AT Game)*, which serves as a method to investigate synchronisation and quality of quantum clocks in an operational way. Numerical results for some strategies pertinent to this game are presented here and further in Sec. 5.1.1. As we will see, performance of the clocks in the AT Game is a very important property of a clock, especially since it is defined in an operational way and without any direct relation to the external, parametric time. We provide some results for the average number of ticks in the AT Game in this chapter, and we hope to see more clock examples being investigated in future.

In Section 5.2 we attempt to analytically bound the maximum number of synchronised ticks of the initially correlated quantum clocks clocks, under few necessary assumptions. Bound obtained could be improved in some future work, by further investigating the assumptions made and including ε from the ε-continuity explicitly. In Section 5.3 we will introduce an application of our model to the classical case. This was first done in the master thesis [41] for the so-called *stochastic ladder clock*, whose properties and performance in the AT Game was investigated. We extend these results to the more general class of the *stochastic clocks* - clocks inside our framework that behave classically and can be described via classical Markov chains. We derive further results for these clocks, as presented in [37].

In Sec. 5.5 we consider a different measure of the clock accuracy $R = \frac{\mu^2}{\sigma^2}$ with μ a mean ticking time and σ standard deviation of the ticking time (measured by an external parameter). This measure was introduced for measuring clock accuracy in [44] and was also used in [38] for the particular clocks corresponding to our model and to the model introduced in the paper [86]. Interesting results for classical and quantum clocks were found, as described in this section. As a future work, for more operational results, it would be fruitful to relate this value R to the average number of alternate ticks N.

Finally, in Section 5.6 we discuss this and other possible future research directions, some of them described in [35].

5.1 The Alternate Ticks Game

In the following, we will introduce a quantitative measure for the clock synchronisation. It is formulated in terms of a game, called the *Alternate Ticks Game*. The game involves two collaborating players, A and B, each equipped with a quantum clock as defined in Sec. 4.1. Players can agree on a common strategy, but then, upon the start of the game, they are separated from each other, so that they cannot communicate. They are asked to send *tick signals* to a referee,

who checks whether these are received in an alternating order, *i.e.*, first from A, then from B, then again from A, and so on (see Fig. 5.1). The goal is to maximise the number of tick signals that are received until this criterion fails. This number can be made large if the players possess well synchronised clocks; it is thus a measure of the degree of synchronisation between the two clocks held by the players. If the game is played between two copies of the same clock, it provides a measure of the quality of this clock.

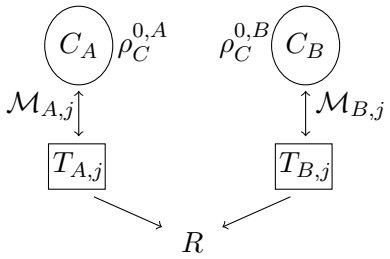

Figure 5.1: Schematic representation of the Alternate Ticks Game. Players A and B decide, respectively, on the initial states $\rho_C^{0,A}$, $\rho_C^{0,B}$ of their clockworks C_A, C_B and the local maps $\mathcal{M}_{A,j}$, $\mathcal{M}_{B,j}$ acting on their quantum clock. Output of these maps is a stream of tick registers $\{T_{A,j}, T_{B,j}\}$. They are sent, correspondingly, by A and B to the referee R.

Formally, the strategy of the players is defined by their respective choice of a quantum clock, $(\rho_C^{0,A}, \mathcal{M}^A_{C\to CT})$ and $(\rho_C^{0,B}, \mathcal{M}^B_{C\to CT})$ (cf. Def. 1). This choice may be subject to certain constraints, e.g., on the size of the clockwork. To capture the idea that the players have no access to an additional time information, we do not allow them to carry out any non-trivial operations on the individual tick registers T_j (such as operations that depend on j). Specifically, we shall assume that they simply transmit their tick registers to the referee. Note that the tick registers can be considered quantum or classical systems. Referee can monitor the incoming stream of tick registers from both players via projective measurements $\{\tau^A, \mathcal{I}_T - \tau^A\}$ and $\{\tau^B, \mathcal{I}_T - \tau^B\}$, whose outcomes are interpreted as "tick" and "no tick", respectively. *E.g.* we can have $\tau^A = \tau^B = |1\rangle\langle 1|$ for a simple case of the two-state tick registers. Let us stress that the referee's measurements do not cause any back action on the clockwork, because each fresh tick register is separated from the clockwork upon production.

For a quantitative analysis, it is convenient to introduce an operator that counts ticks generated by each of the players. Let us denote by τ_j^A the projection operator τ^A applied to player A's j^{th} tick register, and define $\nu_j^A = \mathcal{I}_{T_j} - \tau_j^A$ for the *no-tick* measurement. Furthermore, for any

$N \in \mathbb{N}$ and $0 < k_1 < k_2 < \cdots < k_s < N$, we can define the operator:

$$\begin{aligned} \Pi^N_{k_1,k_2,\ldots,k_s} = \quad & \nu_1 \cdots \nu_{k_1-1}\tau_{k_1} \\ & \nu_{k_1+1} \cdots \nu_{k_2-1}\tau_{k_2} \\ & \cdots \\ & \nu_{k_{s-1}+1} \cdots \nu_{k_s-1}\tau_{k_s} \end{aligned} \tag{5.1}$$

on $T_1 \otimes \cdots \otimes T_N$. Our measure of success in the Alternate Ticks Game can then be formalised as follows.

Definition 7. Suppose that the Alternate ticks game is played between the clocks $(\rho_C^{0,A}, \mathcal{M}^A_{C\to CT})$ and $(\rho_C^{0,B}, \mathcal{M}^B_{C\to CT})$. Then we define the *success probability* for any $n \in \mathbb{N}$ as

$$p_n = \lim_{N\to\infty} \mathrm{tr}(\rho_A^N \otimes \rho_B^N \Pi_n^N) \ , \tag{5.2}$$

where $\{\rho_A^N\}_{N\in\mathbb{N}}$ and $\{\rho_B^N\}_{N\in\mathbb{N}}$ are the time scales (joint states of the tick registers) of the two clocks and where Π_n^N is the projector defined by (as an example we take n to be even)

$$\Pi_n^N = \sum_{\substack{0<k_1<k_2<k_3<k_4 \\ <\cdots<k_{n-1}<k_n<N}} \Pi^N_{k_1,k_3,\ldots,k_{n-1}} \otimes \Pi^N_{k_2,k_4,\ldots,k_n} \ , \tag{5.3}$$

with the projectors in the sum given by (5.1).

Operationally, p_n corresponds to the probability that *at least* n ticks are produced in the correct *alternating* order. In particular, we have $p_{n'} \leq p_n$ whenever $n' \geq n$. There are different ways to condense this probabilistic statement into a single quantity. One would be to consider the *expected* (or the average) number N of ticks until the referee detects a failure. This is the figure of merit that we will adopt in our results. Alternatively, one may ask for the maximum number $n_{\max}$ of ticks such that the failure probability is below a specified *threshold* δ, i.e.,

$$n_{\max} = \max\{n : p_n \geq 1 - \delta\} \ .$$

We observe that two players equipped with the NOT-based clock described in Section 4.1 could continue the game arbitrarily, i.e., $p_n = 1$ for all $n \in \mathbb{N}$. In this sense, the NOT-based clock is perfectly precise. However, as already mentioned earlier, it is not continuous for $\varepsilon < 1$. Conversely, the ε-continuous clock described in Section 4.3 has a probabilistic behaviour, which would at some point lead to a failure of the alternating ticks condition. However, for large sizes $d \gg 1$ of the clockwork, this failure may happen only after many tick signals. In fact, if the size is unbounded, there may be strategies that achieve an arbitrarily large expected number N of ticks.

5.1.1 Implementation independent strategies for the Alternate Ticks Game

In this subsection we discuss selected numerical results and bounds on how certain quantum clocks perform in the Alternate Ticks Game. Finding the optimal evolution of the clockwork is one of the crucial parts in constructing the optimal strategy for the game, and hence good quantum clocks as well. Strategies of the players are defined by the choice of their quantum clocks. Specifically, we consider clocks of the form (4.2) characterised by the following parameters (for each clock A and B):

1. dimension d of the clockwork C and its initial state ρ_C^0;

2. a CPTP map $\mathcal{M}^{\text{int}}$ describing evolution of the clockwork (Note that we will here for simplicity use unitaries
$$\mathcal{M}^{\text{int}}(\rho) : \rho \to \mathrm{e}^{-\mathrm{i}H^{\text{int}}\theta}\,\rho\,\mathrm{e}^{\mathrm{i}H^{\text{int}}\theta} \tag{5.4}$$
specified by a Hermitian operator H^{int} as well as a fixed small parameter θ);

3. a map $\mathcal{M}^{\text{meas}}$ of the form (4.7), specified by nonnegative operators π_0 and π_1 and a reset state $\hat{\rho}_C$ for the reset clocks (see Sec. 4.3).

For our numerical simulations, we assume that both players adopt the same strategy and construct the same clockwork, but with different initial states

$$\rho_C^{0,A} = |0\rangle\langle 0|\,, \quad \rho_C^{0,B} = \left|\left\lfloor \frac{d}{2} \right\rfloor\right\rangle\!\left\langle\left\lfloor \frac{d}{2} \right\rfloor\right|, \tag{5.5}$$

where $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ is an orthonormal basis of the clock states.

We consider two strategies here, corresponding to two types of clocks. We will assume $\hbar = 1$.

1. First model is inspired by the Salecker-Wigner-Peres [21, 22] model of a simple quantum clock (see Sec. 3.2) and defined by the Hermitian operator:
$$H_P^{\text{int}} = \frac{\mathrm{i}}{t_P} \ln U_P\ , \tag{5.6a}$$
where
$$U_P = |0\rangle\langle d-1| + \sum_{k=0}^{d-2} |k+1\rangle\langle k| \tag{5.6b}$$
is a unitary matrix that cyclically permutes the basis states — bringing state $|k\rangle$ to $|k+1 \bmod d\rangle$ for all $k \in \{0, \ldots, d-1\}$.

2. Second type of the clock is defined by
$$H_S^{\text{int}} = U_P + U_P^\dagger\ . \tag{5.6c}$$

For both types of clock we let the map $\mathcal{M}^{\text{meas}}$ be defined by the state $\hat{\rho}_C = |0\rangle\langle 0|$ as well as the operators

$$\pi_0 = \mathcal{I} - \pi_1 \quad \text{and} \quad \pi_1 = \delta \sum_{j=d-d_0}^{d-1} |j\rangle\langle j|\,, \tag{5.7}$$

where in results below we chose $d_0 = \lceil \frac{d}{10} \rceil$. Hence

$$\mathcal{M}^{\text{meas}}(\rho_C) = \sqrt{\pi_0}\rho_C\sqrt{\pi}_0 \otimes |0\rangle\langle 0|_T + \text{tr}(\pi_1\rho_C)\hat{\rho}_C \otimes |1\rangle\langle 1|_T \tag{5.8}$$

Note that there were many more parameter choices we did numerics with, but we present here only selected results.

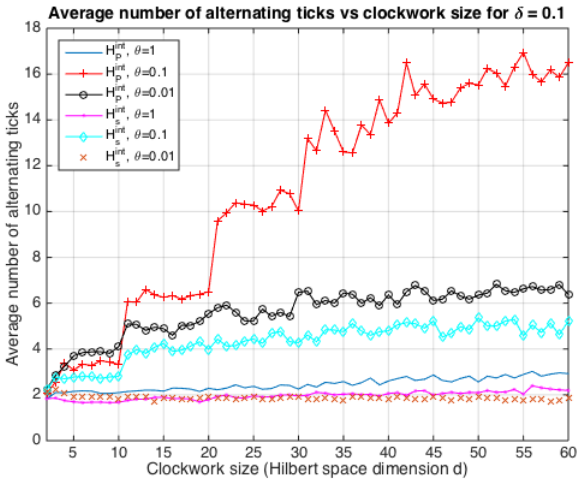

Figure 5.2: *Performance in the Alternate Ticks Game for the two Hamiltonians, with fixed* $\delta = 0.1$ *and varying steps* θ. The plot shows the average number of alternate ticks as a function of the size of the clockwork $d \in \{2, ..., 60\}$. Averaged over 500 runs.

Example numerical results showing the performance of these strategies in the Alternate Ticks Game are summarised in Fig. 5.2. Let us highlight a few features that they exhibit. Firstly, the performance obviously depends strongly on the choice of H^{int} as well as the parameters θ (which determines the step size in between two interactions with the tick register) and δ (which determines the strength of the interaction with the tick register). Nonetheless, the results clearly show that the number of achievable alternate ticks increases with the size d of the clock, at least in the regime that we explored.

While the optimal choice of θ depends on the size of the clockwork, a quantum clock that evolves according to H_P^{int} with $\theta = 1$ has a good performance for a large range of clock sizes, as shown in Fig. 5.3. This is to be expected since one should note that this clock is not ε continuous for small ε. Namely $\varepsilon = 1$ for most choices of δ, which we have also checked with the semi definite program (SDP) optimisation in Matlab (see also Sec. 4.4).

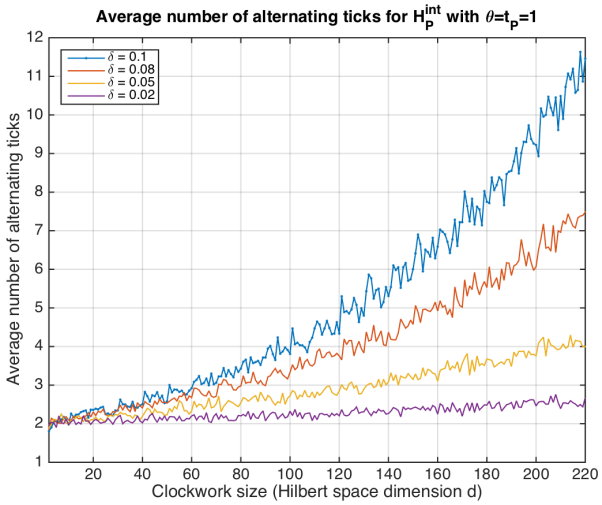

Figure 5.3: *Performance in the Alternate Ticks Game for the Peres Hamiltonian and different δ parameters.* The plot shows the average number of alternating ticks as a function of the size of the Peres clockwork, $d \in \{2, \ldots, 220\}$ and continuity parameter $\delta \in \{0.02, 0.05, 0.08, 0.1\}$ (with $\theta = 1$). Results averaged over 500 simulation runs.

Let us also remark that although particular quantum clocks we considered here are already reasonably precise (in terms of the number of alternating ticks generated), they are certainly not optimal. While the construction of good quantum clocks appears to be a non-trivial task, it is certainly an interesting and important one.

5.1.2 The Alternate Ticks Game with photons

For the numerical results in this subsection, we will consider a specific case described in Sec. 4.6 in which there is an energy source for the clock, providing photons that propagate through some incoming mode towards the clockwork. They can then interact with the clockwork and pass to the outgoing mode where they fill in one of the tick registers — systems that are located near the clockwork and are measured by the clock itself, or by an external observer. Operators and parameters required to fully describe the clock are described in Sec. 4.6.

Note that in this particular application of the game, for practical reasons we will work with system Hamiltonians.

In our numerical simulations, we will assume that each player constructs the same clock. For both models that we are considering below, in the case of the finite-dimensional clocks, we denote the dimensions of the clockworks by $\dim(C_A) = \dim(C_B) = d$, and assume dimensions of tick registers are $\dim(T_A) = \dim(T_B) = 2$. As mentioned in Sec. 4.6, Eq. (4.29) and (4.30), we can take harmonic-oscillator-type Hamiltonians for the incoming and the outgoing mode of the

particles:

$$\begin{aligned} H^A_{\text{in}} = H^B_{\text{in}} &= a^\dagger_{\text{in}} a_{\text{in}} + \frac{1}{2}, \\ H^A_{\text{out}} = H^B_{\text{out}} &= a^\dagger_{\text{out}} a_{\text{out}} + \frac{1}{2}, \end{aligned} \tag{5.9}$$

and an interaction Hamiltonian of the form:

$$H^{\text{meas}} = \alpha(a_{\text{in}} \otimes |c\rangle\langle c| \otimes a^\dagger_{\text{out}} + h.c.) \tag{5.10}$$

where $|c\rangle$ is a tick state of the clockwork in a chosen clock basis and α is a real number specifying the coupling strength.

Moreover, we will initialise the systems by setting:

$$\begin{aligned} &\rho_C^{0,A} = |0\rangle\langle 0|\,, \quad \rho_C^{0,B} = |\lceil d/2\rceil\rangle\langle\lceil d/2\rceil|\,, \\ &\left|\varphi^A_{0,\text{in}}\right\rangle = \left|\varphi^B_{0,\text{in}}\right\rangle = |n\rangle\,, \\ &\left|\varphi^A_{0,\text{out}}\right\rangle = \left|\varphi^B_{0,\text{out}}\right\rangle = |\Omega\rangle \end{aligned} \tag{5.11}$$

where, for the clockwork, $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ denotes the set of the computational basis states (that we will call *clock states*). For the incoming (outgoing) mode of the energy source, $|n\rangle$ is the state of n excitations, i.e., $a^\dagger_{\text{in}} a_{\text{in}} |n\rangle = n\,|n\rangle$ while $|\Omega\rangle$ is the vacuum state giving $a^\dagger_{out} a_{\text{out}} |\Omega\rangle = 0$, $a_{\text{out}} |\Omega\rangle = 0$. Our choice above thus corresponds to having all information carriers (e.g., n photons) in the incoming mode at the beginning.

Hence we have the total map for the clock:

$$\mathcal{M}(\rho_0) = e^{-i(H^{\text{out}}+H^{\text{meas}}+H^{\text{in}}+H^{\text{int}})\theta} \rho_0 e^{i(H^{\text{out}}+H^{\text{meas}}+H^{\text{in}}+H^{\text{int}})\theta} \tag{5.12}$$

for some constant, possibly randomly chosen small interval θ. ρ_0 is the joint starting state of the clock and the photon modes, *e.g.* $\rho_0^A = \rho^A_{0,\text{in}} \otimes \rho^A_{0,\text{out}} \otimes \rho_C^{0,A}$. Here ρ_{out} plays the role of the tick register system, and ρ_{in} is a power source of the clock. Note that the ε-continuity of this clock will depend crucially on θ.

For the measurements of the tick registers by the clock, we consider the two-outcome POVM:

$$\Pi_{\mathsf{X}} = |\Omega\rangle\langle\Omega|\,, \quad \Pi_{\checkmark} = \mathcal{I} - |\Omega\rangle\langle\Omega|\,, \tag{5.13}$$

where $\mathbf{1}$ is the identity operator acting on the Hilbert space of the register. Hence the strategies for A and B are defined by projectors $\tau_A = \tau_B = |1\rangle\langle 1|$. Moreover, for both players, with a certain uniform probability, the clock performs this measurement after each application of the map $\mathcal{M}$. In the numerics, we apply H^{int} for a time interval θ, uniformly chosen from (0,1), after which we perform a measurement on the register at the outgoing mode. Afterwards, referee receives each register carrying classical information. In our numerical simulations, we consider that a tick

occurred if a random number — sampled uniformly from the interval $(0, 1)$ — is less than or equal to the probability of getting the ✓ outcome when the register is measured. [1]

Next, we specify two clockwork Hamiltonians and present some numerical results obtained for each model.

Model 1

For the first model, we use the following Hamiltonian for the clockwork of both players:

$$H^{\text{int}} = |0\rangle\langle d-1| + \sum_{k=0}^{d-2} |k+1\rangle\langle k| + h.c.$$

where $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ are clock states, as in Eq. (5.11). Moreover, we set $n = 2$ and $|c\rangle = |1\rangle$ in our numerical simulations, cf. Eq. (5.11). Hence, if the energy source consists of photons, there are 2 photons in the incoming mode, and they only interact with the states of C that have a non-trivial support on the state $|1\rangle$. Note that here $\mathcal{M}^{\text{int}}(\rho_C) = e^{-iH^{\text{int}}\theta}\rho_C e^{iH^{\text{int}}\theta}$ for some small constant θ. After each application of this map, $\mathcal{M}^{\text{meas}} = e^{-i(H^{\text{out}}+H^{\text{meas}}+H^{\text{in}})\theta}\rho_{C,\text{in,out}}e^{i(H^{\text{out}}+H^{\text{meas}}+H^{\text{in}})\theta}$ followed by the POVM measurement by the clock on the tick register, is performed.

In Figure 5.4 we plot the average number of the alternate ticks obtained — averaged over 1000 runs of the simulation — vs the size d of the clockworks. Note that since $n = 2$, $2n = 4$ is the maximum number of alternate ticks limited by our model. From the figure, we see that for some values of n, the average number of ticks that we observed are actually comparable with this maximum.

In Figure 5.5 we show an example of how the probability of detecting a signal from the players changes as a function of the external time parameter evolution steps (specified by the uniformly sampled interval $\theta = dt$) at which the tick registers are measured. Peaks represent possible times when a signal is registered by the referee.

Model 2

An arguably more natural Hamiltonian is the one corresponding to the unitary operation considered by Salecker, Wigner and Peres [21, 22]. They modelled a quantum clock as a 'dynamical system evolving through succession of states in some constant time intervals τ'. We will denote clock states as $\{|0\rangle, |1\rangle, .., |d-1\rangle\}$. Explicitly, the corresponding clockwork Hamiltonian takes the form:

$$H^{\text{int}} = \frac{i}{\tau}\ln\left(|0\rangle\langle d-1| + \sum_{k=0}^{d-2}|k+1\rangle\langle k|\right) \tag{5.14}$$

[1] The measurement process is thus unrealistically treated as an *instantaneous* operation in the current framework. However, we expect that if the measurement process is short in the Schrödinger time compared with the evolution time of the system, the qualitative feature of the result should remain unchanged.

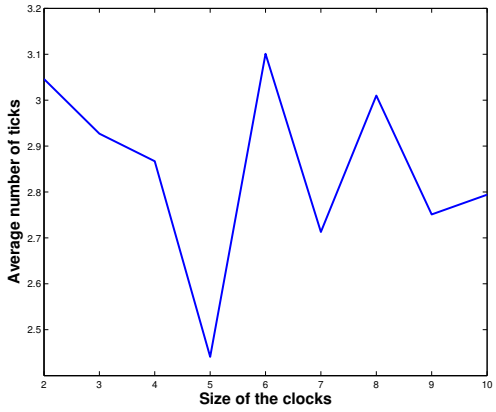

Figure 5.4: Average number of tick signals detected by a referee, depending on the size d of the clockwork. Two photons in the energy source, and only the states of C that have a non-trivial support in the state $|1\rangle$, interact with them. Averaged over 1000 runs.

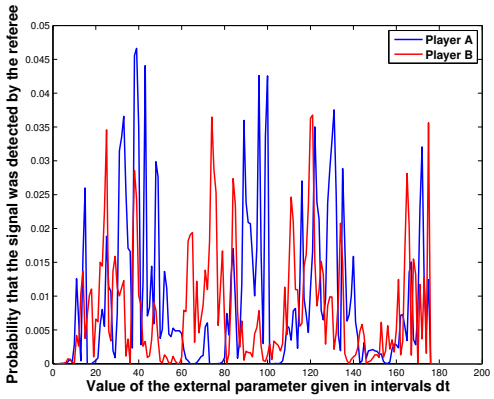

Figure 5.5: Probability of the ticks detection by the referee, for both players, in the case when $d_A = d_B = 4$. Two photons in the energy source, and only the states of C that have a non-trivial support in the state $|1\rangle$, interact with them.

This Hamiltonian is diagonal in the Fourier basis with the eigenvalues $\omega k, k \in 1, .., d$ and the frequency $\omega = \frac{2\pi}{d\tau}$, where d is the size of the clock, and τ denotes the quantum speed limit of this clock (note that we have assumed $\hbar = 1$). Then the eigenstates of the Hamiltonian will be the Fourier transforms of the clock states, $v_k = \frac{1}{\sqrt{d}} \sum_{j=0}^{k-1} |k\rangle\, e^{ikj}$, and the unitary evolution for this Hamiltonian is such that, if the system is not disturbed (*e.g.*, by an interaction with the environment), it will evolve from one computational state (clock state) to the next in intervals of the external time parameter τ. But, importantly, the state will spread significantly between being in the two clock states.

For a good strategy, players can start their clocks in the clock states differing for $d/2$ states

(or $(d+1)/2$ states if d is odd). Interaction with the light source is again described by the Hamiltonian in Eq. (5.10).

Our numerical results depended largely on the relationship between the value of τ and the maximal value of dt. In the following plots, we restrict our attention to the case of $\tau = 1$. The analog of Fig. 5.4 for this model, now with 4 photons in the source, can be found in Fig. 5.6, where we now have $|c\rangle = |1\rangle$ and $n = 4$, and the plotted results are averaged over 1000 simulation runs.

Figure 5.6: Average number of ticks detected by a referee, depending on the size of the clockwork. Four photons in the energy source and only states of C that have a non-trivial support on the state $|1\rangle$, interact with them. Averaged over 1000 runs.

Similarly, in Fig. 5.7, we show the probability of detecting a signal from the player as a function of the random steps of the external time parameter in which the measurements are performed, for a particular run of the simulation. Note that, in contrast with the first model, the peaks of the probability of the detection clearly alternate between the two players.

In Fig. 5.8 we show a comparison of the average number of ticks for two different interaction Hamiltonians (Eq. (5.10)) - one that has $\alpha = 1$ and one with $\alpha = \frac{1}{d}$, where d is the clock dimension.

As a conclusion, for any given size of the clockwork — as measured by the dimension of the respective state space — there seem to exist a non-trivial upper bound on the maximum number of alternate ticks achievable, irrespective of the strategy adopted by the players. This follows from the necessary interaction between the quantum clockwork and its environment as well as the limits of (time) measurements in quantum theory — interaction generally degrades the ideal evolution of the system while measurement unavoidably introduces disturbances. We thus believe that determining the optimal strategy, and hence the optimal quantum clocks for the Alternate Ticks Game would also lead to a better understanding of the limitations of time measurements.

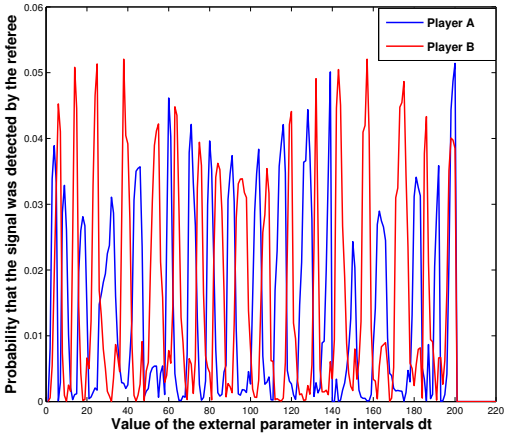

Figure 5.7: Probability of the tick signal detection by the referee, for both players, in the case when $d_A = d_B = 3$. Two photons in the energy source and only the states of C that have a non-trivial support in the state $|1\rangle$ interact with them.

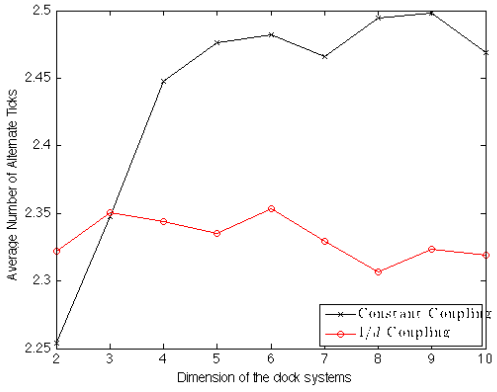

Figure 5.8: The average number of tick signals detected by a referee, depending on the size of the clockwork, for two types of the interaction Hamiltonian. Two photons in the energy source, and only the states of C that have a non-trivial support in the state $|1\rangle$ interact with them. Averaged over 1000 simulation runs.

5.2 An information-theoretic approach to finding the limitations of clock synchronisation

Let us now investigate the limitations to building global ordering scales from the two local clocks defined in Def. 1. Specifically, we consider two copies of the local quantum clock that are synchronised to a certain level (to be made precise below). We will investigate how us the number of ticks that they can produce under the synchronisation criterion bounded in terms of the dimension

of the clockworks and the level of the correlation of their tick systems. The ε-continuity of the clocks plays a big role in this argument and should ultimately be explicitly included in the upper bound, which is still a topic of research. However, the assumptions we make will exclude clocks that are continuous only for large ε. Note that the synchronisation as defined here depends on the correlation between the tick registers of the clocks.

In the steps we will show we will explicitly label the assumptions we make, and note that some of them might be quite limiting. It is an ongoing task to give a physical meaning to these assumptions and hence to clocks that are included by this reasoning, therefore we still consider this a sketch proof, nevertheless with a very important possible impact.

We will now introduce definitions and assumptions needed in our approach. Following Def. 1 we define a quantum clock as a system consisting of:

1. ***a clockwork (dynamical part)*** C on a (finite-dimensional) Hilbert space and,

2. ***a tick register*** T which stores information about the number of ticks produced by C. Dynamics of the clock is described by a completely-positive-trace-preserving (CPTP) map $\mathcal{M}_{CT\to CT}$.

Note that we can think of this map as a product $\mathcal{M}^{\text{meas}}_{CT\to CT} \circ \mathcal{M}^{\text{int}}_{C\to C}$, consisting of the internal evolution of the clockwork $\mathcal{M}^{\text{int}}_{C\to C}$ and a measuring map $\mathcal{M}^{\text{meas}}_{CT\to CT}$. We will assume that the evolution map of the clockwork is a unitary map. In addition, we will consider one tick register that can store all the ticks instead of a set of many tick registers storing one tick each (hence $\mathcal{M}_{CT\to CT}$ instead of $\mathcal{M}_{C\to CT}$). This can be done by having a ladder system for T, such that each tick raises its level to the next state. Physically, the role of the tick register is to — via its state update — keep track of the number of ticks produced by C. The clockwork C is by itself thus not a complete clock since it does not store ticks and does not provide us with a time scale. We will define a system with a trivial tick register as the one where the corresponding measuring map is the identity map, and refer to it as to a **'reference'** system, because it does not get disturbed during its course of evolution. We will label the states of such systems *ref* in subsequent discussions. In contrast, **'realistic'** clocks are those whose clockworks produce ticks by acting on the register with a non-trivial measuring map $\mathcal{M}^{\text{meas}}_{CT\to CT}$; such systems will be labeled *real*.

Let us now consider two local quantum clocks, constructed in the spatially-separated local reference frames A and B, that are correlated at the start. Using the notations introduced above, we denote by C_i, T_i and $\mathcal{M}_i$, $i \in \{A, B\}$, respectively, the clockwork, the tick system, and the associated CPTP map at site i (Fig. 5.9). Let

$$\min\{\dim(C_A), \dim(C_B)\} = d, \tag{5.15}$$

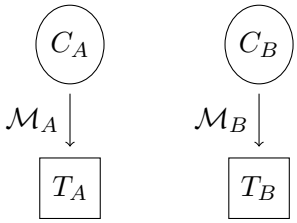

Figure 5.9: Each local quantum clock consists of a clockwork (C_A or C_B) and a tick register (T_A or T_B) on which the clockwork can act by a CPTP map ($\mathcal{M}_A$ and $\mathcal{M}_B$). Tick registers keep a record of the ticks of the clockwork through their state update by the action of $\mathcal{M}$.

and denote the maximum number of ticks produced by any of the two clocks during some interval of interest by N, *i.e.*[1]

$$\max\left(\dim\left(T_A\right), \dim\left(T_B\right)\right) = \dim\left(T_B\right) = N. \tag{5.16}$$

We will ultimately be interested in the correlations of the joint state of the registers of A, B, after $k > 0$ applications of the map $\mathcal{M}$ on both sides.

Suppose that at the start the states of the the clock systems of A and B are initialised to $\rho^i(0) = \rho^{i,C}(0) \otimes \rho^{i,T}(0)$, for $i \in \{A, B\}$. We assume that the CPTP map acts on the tick system only after the clockwork has evolved (this is a natural condition, since the first tick should only occur after the clockwork starts evolving, and also for each subsequent tick the clockwork is supposed to further evolve from the state it took at the previous tick). We label the initial joint state of the system AB $\rho^{AB}(0)$. Importantly, we require a starting correlation between the clockwork states for the subsequent analysis.

After $n \geq 1$ applications of the map $\mathcal{M}$ on both sides, the joint state becomes:

$$\rho^{AB}(n) = (\mathcal{M}^A \otimes \mathcal{M}^B)^n \rho^{AB}(0) \tag{5.17}$$

Denote by $\rho(n)^{AB}_{\text{real}}$ and $\rho(n)^{AB}_{\text{ref}}$ the joint states of the clocks A and B after n applications of the map $\mathcal{M}$, in the case when the two local quantum clocks of interest are, respectively, *realistic clocks*, and the *reference clocks*.

From the linearity of the CPTP maps and the above reasoning, we have that, for any $n \in \{1, ..., k\}$:

$$\rho(n)^{AB}_{\text{real}} = (S_{A,n} \otimes S_{B,n})\rho(n)^{AB}_{\text{ref}}. \tag{5.18}$$

with $S_{i,n} = \mathcal{M}_i^n \circ \mathcal{M}_i^{\text{int},-n}$, $i \in \{A, B\}$. Note that the inverse of the map $\mathcal{M}_i^{\text{int}}$ is well-defined, since we assumed it is a unitary map.

[1] Clearly, the tick system must consists of at least N orthogonal states in order to record N ticks from the dynamical part of the clock. Moreover, the larger clock can presumably generate more ticks and thus needs a bigger register.

5.2 An information-theoretic approach to finding the limitations of clock synchronisation

For the purpose of finding the limitations to the construction of the global ordering scales, we still need to specify our *synchronisation criterion* between the tick registers of A and B.

Assumption 1. *Synchronisation criterion*
Entropy of the state of the tick system T_B conditioned on T_A, after any number of steps $n \geq 1$, is upper bounded by $\log s$, with $s > 0$.

$$H(T_B|T_A)_{\rho_{real}(n)} \leq \log s. \tag{5.19}$$

Here $s = 1$ corresponds to the perfect synchronisation between the tick register systems since the uncertainty (entropy) of T_A given the state of T_B is $\log 1 = 0$. Likewise, if $s > 1$, the conditional entropy is non-zero and we do not have the perfect synchronisation between the registers. Generally, with greater s, we have less correlated, and hence less synchronised clocks. If we consider the case of the classical registers counting the number of ticks, the posed criterion means that, given the state of the register T_A, we have s possibilities for the state of the register T_B. Note that then also $H(T_B|T_A) \geq 0$. Hence, we can see that $s = 1$ means that knowing the state of T_A determines perfectly the state of T_B.

We now state another assumption that is needed to prove the main result of this section.

Assumption 2. *The uncertainty in the (reduced) state of the tick system, $\rho^{T_B}_{\text{real}}(n) = \text{tr}_{A,C_B}\, \rho^{AB}_{\text{real}}(n)$, $n \geq 1$ is lower bounded by a logarithmic function of the size N of the tick system:*

$$H(T_B)_{\rho_{\text{real}}(n)} \geq \log \frac{\dim T_B}{m} = \log \frac{N}{m} \tag{5.20}$$

with $1 \leq m << N$.

Note that this assumption directly relies on the ε-continuity of the clocks (see Def. 3). Namely, for the clock that has $\varepsilon = 1$ we will have a complete certainty in the state of the tick registers at any instance. We can set this clock to tick after each *e.g.* m steps and we would completely know the state of the register T after any number of the applications of the map $\mathcal{M}$ (e.g. after less then m application, there would be 0 ticks, between m and $2m-1$ applications two ticks and so on). This would yield $H(T_B)_{\rho_{\text{real}}(n)} = 0$, $\forall n$. Similarly, for ε close but not equal to 1, we would have a very small uncertainty in the state of the register. Hence to fulfil the above assumption, our clocks need to be continuous with small ε. The exact connection between the $H(T)$ and the value of ε is still unknown, but is an important task of the current and the future research.

Proposition 1. *Under the assumptions stated above, the maximum number of ticks achievable by the two local quantum clocks with correlated clockworks, satisfying the synchronisation criterion given in Eq. (5.19), is upper bounded by a quadratic function in the dimension of the clockworks, and depends also on the level of their synchronisation, expressed as the value s from Def. 1. In the case of only classical correlation between the clockworks, this bound is linear in the clockwork dimension.*

Proof. Recall that in the case of the classical correlations $H(C_A|C_B) \geq 0$, but with the quantum correlations this conditional entropy can be negative, with $H(C_A|C_B) \geq -\log \dim(C_A)$ [69]. We also know that $H(C_A) \leq \log \dim(C_A)$. Using this fact and applying a bound on the entropy of the smaller clock, we obtain the following bound on the mutual information between C_A and C_B in the state obtained after n applications of the map $\mathcal{M}$:

$$I(C_A : C_B)_{\rho_{\text{ref}}(n)} = H(C_A)_{\rho_{\text{ref}}(n)} - H(C_A|C_B)_{\rho_{\text{ref}}(n)} \leq 2 \log \dim (C_A) \tag{5.21}$$

$$\text{and } I(C_A : C_B)_{\rho_{\text{ref}}(n)} = I(C_A T_A : C_B T_B)_{\rho_{\text{ref}}(n)} \tag{5.22}$$

since the map $\mathcal{M}$ acts trivially on the registers T_A and T_B in the reference state. Note that we assume that there is a prior correlation between the clockworks (*e.g.* entanglement or classical correlations), so that the bound on the mutual information wouldn't be trivially 0. Hence the following holds:

$$I(C_A T_A : C_B T_B)_{\rho_{\text{ref}}(n)} \leq \log d^2 \tag{5.23}$$

Proposition 2. *Consider mutual information between the two clocks defined in this section, of maximum dimension d. Then the following inequality holds:*

$$\log d^2 \geq I(C_A T_A : C_B T_B)_{\rho_{\text{ref}}(n)} \geq I(C_A T_A : C_B T_B)_{\rho_{\text{real}}(n)} \tag{5.24}$$

Proof. This proposition follows from Eq. (5.18) and the data processing inequality [115]. To use the data processing we need to show that state 'real' is a *locally* processed version of the 'reference' state. As a processing map one can take $S_{A,n} \otimes S_{B,n} = (\mathcal{M}_A^n \circ \mathcal{M}_A^{\text{int},-n}) \otimes (\mathcal{M}_B^n \circ \mathcal{M}_B^{\text{int},-n})$. □

By the chain rule for the mutual information we get [115]:

$$\begin{aligned} I(C_A T_A : C_B T_B)_{\rho_{\text{real}}(n)} = {} & I(T_A : T_B)_{\rho_{\text{real}}(n)} \\ & + I(C_A : T_B|T_A)_{\rho_{\text{real}}(n)} \\ & + I(T_A : C_B|T_B)_{\rho_{\text{real}}(n)} \\ & + I(C_A : C_B|T_A T_B)_{\rho_{\text{real}}(n)}, \end{aligned}$$

which implies

$$I(C_A T_A : C_B T_B)_{\rho_{\text{real}}(n)} \geq I(T_A : T_B)_{\rho_{\text{real}}(n)} \tag{5.25}$$

since the mutual information is always non-negative.

Inequalities (5.24) and (5.25) together imply that:

$$\log d^2 \geq I(T_A : T_B)_{\rho_{\text{real}}(n)} \tag{5.26}$$

Condition (5.19) and Assumption 2 then give:

$$I(T_B : T_A)_{\rho_{\text{real}}(n)} = H(T_B)_{\rho_{\text{real}}(n)} - H(T_B|T_A)_{\rho_{\text{real}}(n)}$$

$$\Rightarrow I(T_B : T_A)_{\rho_{\text{real}}(n)} \geq \log \frac{N}{m\, s} \tag{5.27}$$

Now combining Eq. (5.26) with Eq. (5.27), and using the symmetry of the mutual information, as a final inequality we obtain:

$$\log d^2 \geq \log \frac{N}{m\, s}$$

Knowing that $d, N > 1$ and in the case of the maximum uncertainty about the state of the tick system T_B (i.e., $m = 1$), we have that, after any number of applications of the map $\mathcal{M}$, the maximum number N of synchronised ticks obeys the following inequality:

$$N \leq d^2 \cdot s \tag{5.28}$$

where d is the dimension of the smaller clockwork, and s is the measure of the clock synchronisation. This concludes the proof of Proposition 1. □

Importantly, all of the above steps also work for the classical case, replacing d^2 with d (by using $H(C_B|C_A) \geq 0$).
As a conclusion, under the validity of the assumptions we made, if one considers two synchronised clocks with the non-trivial starting correlation between their clockworks, of size (dimension of the underlying Hilbert space) d, maximal number of ticks (dimension of the underlying Hilbert space of the tick system), that any of them could produce under the posed synchronisation criterion, is $N = s \cdot d^2$ in the case of starting quantum correlation and $N = s \cdot d$ in the case of the classical correlation.

From the inequality (5.28), we can see that for better synchronisation (*i.e.* smaller s), or equivalently larger number of ticks and longer ordering scale, larger clocks (as measured by their dimension) are needed. Hence, if we want to use local ordering given by the two non-ideal clocks that are correlated at the start, as a global ordering scale valid in the reference frames of both of them, there is a bound on the length of this scale (*i.e.* on the number of ticks it provides), which depends on the size of the clocks and the extent to which these local clocks need to be synchronised. This limitation, expressed quantitatively as a bound on N may, in the future work, lead to an uncertainty relation in the measurement of time using quantum clocks, as well as to the more general bounds when constructing global clocks and ordering structures in quantum mechanics.

5.3 Performance of stochastic clocks in the Alternate Ticks Game

We now remind ourselves of the class of stochastic clocks introduced in Section 4.5. Their dynamics is defined by the map from Def. 5:

$$\mathcal{M}^{\text{st.}}(\rho_C) = (1-\delta)\rho_C \otimes |0\rangle\langle 0|_T + \sum_{j=0}^{d-1}\sum_{m=j}^{d-1}\langle j|\rho_C|j\rangle(p_{jm}-\delta_{jm}(1-\delta))\,|m\rangle\langle m|_C \otimes |0\rangle\langle 0|_T + \sum_{j=1}^{d-1}\sum_{m=0}^{j-1}\langle j|\rho_C|j\rangle p_{jm}\,|0\rangle\langle 0|_C \otimes |1\rangle\langle 1|_T \tag{5.29}$$

In [41] a specific stochastic ladder clock was shown to be ε-continuous for arbitrary small $0 < \varepsilon < 1$, by choosing the right parameters ($\delta \leq \varepsilon$), and the average number of alternate ticks obtained was linear in the dimension of the clock d, and also depended on the value of ε. Based on our numerical results from the previous sections, this shows the possible superiority of a purely quantum model, where better than linear behaviour is expected, compared to the classical one. We will now describe further the performance of a more general stochastic clock in the AT Game.

Assume now that the stochastic clock starts from one of the clock states $|j\rangle\langle j|$, $j \in \{0, ..., d\}$. We will also assume that probability for the clock to stay in the current state p_{jj} is constant for any j and also that $p_{jk} = p_{0,(k-j)}$, $\forall j, k$. Hence we have $1 - \delta = p_{jj}$.

We will describe the number of jumps (applications of the map $\mathcal{M}$) needed for a clock to produce the tick i after the tick $i-1$ by a random variable (RV) Y_i. Note that all Y_i are *iid* (independent, identically distributed) RVs since we consider the reset clocks — after the tick they always start anew from the state $|0\rangle\langle 0|$ and move with the same probability distribution. We will see that in the circular board picture, where the pieces move on the circle in one direction, the Alternate Ticks game can be lost only if one player's peace overtakes the other for at least d positions. This is however not sufficient as the other player can catch up with this fallout. This was proved in [41] and we comment on it in more details below.

We will assume that clocks can only move forward, but note that moving forward for $d-1$ steps is equivalent to moving backward for one step, since we can describe the dynamics on a circle. Hence we are still dealing with the most general case.

To be able to analyse this further using probability theory, we will next define classical description of the stochastic clocks and the game dynamics, as first introduced in [41].

Definition 8. Let $(\rho_{\text{C}}^0, \mathcal{M}_{\text{C}\to\text{CT}}^{\text{st.}})$ be a d-dimensional stochastic clock, with $\rho_C^0 = |j\rangle\langle j|$, for some $j \in \{0, ..., d\}$. We call the triple $\left(d, P^0, \{P^n\}_{n\in\mathbb{N}}\right)$ its classical description where $P^0 = j$ for $\rho_{\text{C}}^0 = |j\rangle\langle j|$, $j \in \{0, \ldots, d\}$ and the random variable P^n denotes the number of states for which the clock has moved forward (has jumped over) after n applications of the map $\mathcal{M}_{\text{C}\to\text{CT}}^{\text{st.}}$.

Note that the k^{th} tick happens after $\min\{n : P^n = dk-1\}$ steps. Now to quantify the success in the AT Game we need to look at the RV describing the distance on the circular board between

the players A and B. Suppose $P_{\mathrm{A}}^0 \geq P_{\mathrm{B}}^0$, hence the player A is in front of the player B on the board.

Definition 9. [41] One can define a random variable

$$Q^n := Q^0 + (P_{\mathrm{A}}^n - P_{\mathrm{B}}^n)\,, \tag{5.30}$$

where $Q^0 = P_{\mathrm{A}}^0 - P_{\mathrm{B}}^0$ and call $\left(d, Q^0, \{Q^n\}_{n\in\mathbb{N}}\right)$ the classical game description after n applications of $\mathcal{M}_{\mathrm{C}\to\mathrm{CT}}^{\mathrm{st.}}$. We also introduce RVs $\Delta^n = Q^{n+1} - Q^n = P_A^{n+1} - P_A^n - (P_B^{n+1} - P_B^n)$, $n \in \{0, 1, ...\}$.

We now state the boundary conditions for losing the AT Game.

Lemma 3. *[41] For the Alternate ticks game played with the stochastic clocks to be lost, it is necessary that one player is at least d steps ahead of the other, i.e. $Q^n > d$ or $Q^n < 0$. Sufficient criterion for losing is that one of the players is $2d$ steps in front of the other, i.e. $Q^n > 2d$ or $Q^n < -d$.*

The expected number of ticks in the AT Game is hence lower bounded by the number of ticks until one player is exactly d steps in front of the other, and upper bounded by the expected number of ticks until one is exactly $2d$ steps in front of the other.

If we assume that each clock can move up to $m \leq d$ states in one jump, then the distance between the positions of the clocks A and B can change for any value between $\{0, ..., m\}$ in one jump, *i.e.* $\Delta^n = Q^{n+1} - Q^n \in \{0, ..., m\}$. We are interested in the minimal number of steps n before the RV Q^n reaches one of the boundaries (one player being one or two circles in front of the other). This will lead to a higher-order differential equation, in the general case of order d. We will assume player A was in front of the player B at the start of the game, and $Q^0 = z$.

Denote with T a RV counting the number of ticks, Y_i a RV counting number of jumps between two ticks, and with $\mathbb{E}(S) = D_z$ expected number of jumps until the game is lost for initial distance between the clock states $Q^0 = z$. Note that $Y = Y_i$ and T are independent RVs. Then the following holds:

$$\mathbb{E}(S) = D_z = \sum_{i=1}^{T} Y_i = \mathbb{E}(Y \cdot T) = \mathbb{E}(Y) \cdot \mathbb{E}(T) = \mathbb{E}(Y) \cdot N$$
$$N = \frac{D_z}{\mathbb{E}(Y)}$$

In general, for the expected number of jumps S until the end of the game (number of applications of the map $\mathcal{M}$), we have:

$$\begin{aligned}\mathbb{E}(S|Q^0 = z) = D_z = p_1(D_{z+1} + 1) + p_2(D_{z+2} + 1) \\ + ... + p_m(D_{z+m} + 1) + p_0(D_z + 1) + p_{-1}(D_{z-1} + 1) \\ + p_{-2}(D_{z-2} + 1) + ... + p_{-m}(D_{z-m} + 1)\end{aligned} \tag{5.31}$$

where $m < d$ is the maximal number of states for which the positions of A and B can differ in one jump, and p_k denotes the probability that the difference between positions of players A and B on the circle changed for exactly k in one jump. We can have maximum $m = d$ which is the case that one clock jumps for d steps and the other stays where it was.
Note that the value of m will be important in determining the ε-continuity of a clock, since the bigger possible m we have, the less possibilities we have for small ε. In other words, the probability for the clock to move for many states in one jump has to remain low for an ε-continuous clock with small ε.

5.3.1 Identical stochastic clocks in the Alternate ticks game

If the two clocks are identical, the probability to change the distance between A and B for $+k$ or $-k$ states in one step will be the same. This means that variable Q^n changes for $+k$ or for $-k$ with the same probability, leading to $p_k = p_{-k}\,, \forall k \in \{1, ..., m\}$. Note also that

$$\sum_{k=-m}^{m} p_k = 2\sum_{k=1}^{m} p_k + p_0 = 1$$

Hence the recurrence relation (5.31) becomes:

$$\begin{aligned} (2\sum_{k=1}^{m} p_k)D_z = \\ p_1 D_{z+1} + p_2 D_{z+2} + ... + p_m D_{z+m} + \\ p_1 D_{z-1} + p_2 D_{z-2} + ... + p_m D_{z-m} + 1 \end{aligned} \tag{5.32}$$

Solving this relation and inputing optimal $z = \frac{d}{2}$, we obtain:

$$\frac{d^2/4 + d + 1}{2\sum\limits_{k=1}^{m} k^2 p_k} \leq D_{\frac{d}{2}} \leq \frac{9d^2/4 + 3d + 1}{2\sum\limits_{k=1}^{m} k^2 p_k} \tag{5.33}$$

and we know $N = \frac{D_z}{\mathbb{E}(Y)}$. Note that for for $\varepsilon = 0$ we have a clock described completely by the identity map, that will not produce a tick. Hence $\mathbb{E}(Y) = \infty$ for this clock and it can produce zero expected alternate ticks. Note also that by the definition of the diamond norm $\varepsilon \leq 1$. Clock with $\varepsilon = 1$ will be described in the next section.

We can then show the following Theorem.

Theorem 5.3.1. *Suppose that the AT Game is played between two identical ε-continuous d-dimensional stochastic clocks, with $\varepsilon \in (0, 1)$. Then the following bounds hold on the expected number of alternate ticks N:*

$$\frac{(d + 4 + \frac{4}{d})p_{i,i+1}}{4m^2\varepsilon(2 - \varepsilon)} \leq N \leq \frac{\frac{9d}{4} + 3 + \frac{1}{d}}{2(1 - \varepsilon)mp_{0,m}}$$

where m is the furthest state the clock can jump to from the state 0 with the probability $p_{0,m} \neq 0$, and $p_{i,i+1}$ is the probability for the clock to jump from state i to $i+1$.

Proof. We can consider homogenous equation from Eq. (5.32) (which excludes the constant term 1 on the RHS). One can notice that $A+Bz$ will be the solution. Namely inserting $D_z^{\text{hom}} = A+Bz$ to the homogenous part of Eq. (5.32), we obtain:

$$\begin{aligned}
&(2\sum_{k=1}^{m} p_k)(A+Bz) = \\
&p_1(A+Bz+B) + ... + p_m(A+Bz+mB) + \\
&p_1(A+Bz-B) + ... + p_m(A+Bz-mB) \\
&= (2\sum_{k=1}^{m} p_k)A + (2\sum_{k=1}^{m} p_k)Bz
\end{aligned} \tag{5.34}$$

Note that this is the case because of the symmetry of the particular problem. Hence we are left to find a particular solution. After considering a quadratic function in z^2, we obtain following solution:

$$D_z^{part} = -\frac{z^2}{2\sum_{k=1}^{m} k^2 p_k}$$

Hence the total solution is $D_z = D_z^{\text{hom}} + D_z^{\text{part}} = A + Bz - \frac{z^2}{2\sum_{k=1}^{m} k^2 p_k}$.

We can now impose boundary conditions (recall: $D_{-1} = D_{d+1} = 0$ for the lower bound, and $D_{-d-1} = D_{2d+1} = 0$ for the upper bound on the game). For the lower bound we have

$$A = \frac{d+1}{2\sum_{k=1}^{m} k^2 p_k},\ B = \frac{d}{2\sum_{k=1}^{m} k^2 p_k} \tag{5.35}$$

This leads to

$$D_z^{\text{lower}} = \frac{d+1+dz-z^2}{2\sum_{k=1}^{m} k^2 p_k} \tag{5.36}$$

For the upper bound we obtain

$$A = \frac{(d+1)(2d+1)}{2\sum_{k=1}^{m} k^2 p_k},\ B = \frac{d}{2\sum_{k=1}^{m} k^2 p_k} \tag{5.37}$$

and:

$$D_z^{\text{upper}} = \frac{2d^2+3d+1+dz-z^2}{2\sum_{k=1}^{m} k^2 p_k} \tag{5.38}$$

Now recall that $\mathbb{E}(S) = \mathbb{E}(Y \cdot T) = \mathbb{E}(Y) \cdot \mathbb{E}(T)$, where T is the number of ticks and $\mathbb{E}(S) = D_z$ for initial distance between clock states $Q^0 = z$. We will have D_z maximised for $z = d/2$ as

expected for identical clocks (they start from opposite states on the circle). Note that $\mathbb{E}(Y)$ is not trivial to calculate as it depends on the probability distribution of the particular clock. Inputing $z = \frac{d}{2}$, we have:

$$\frac{d^2/4 + d + 1}{2\sum_{k=1}^{m} k^2 p_k} \leq D_{\frac{d}{2}} \leq \frac{9d^2/4 + 3d + 1}{2\sum_{k=1}^{m} k^2 p_k} \tag{5.39}$$

Now write $\mathbb{E}(Y) = \sum yP(Y = y)$. We can restrict to the case when clock can not jump more than $\lfloor \frac{d}{s} \rfloor = m \geq 1$ places in one step. This is the same m as the maximum number of place differences players can achieve in one step. Note also that the average number of steps each clock needs until the tick is reached is then lower bounded by s, $\mathbb{E}(Y) \geq s$. Hence, expected number of ticks can be upper bounded as follows (note that $p_m, p_{0,m} > 0$ by assumption):

$$N = \mathbb{E}(T) \leq \frac{D^{\text{upper}}_{d/2}}{\mathbb{E}(Y)} \leq \frac{\frac{9d^2}{4} + 3d + 1}{2s\sum_{k=1}^{m} k^2 p_k} \tag{5.40}$$

$$\leq \frac{\frac{9d^2}{4} + 3d + 1}{2sm^2 p_m} = \frac{\frac{9d}{4} + 3 + \frac{1}{d}}{2mp_m} \leq \frac{\frac{9d}{4} + 3 + \frac{1}{d}}{2(1-\varepsilon)mp_{0,m}} \tag{5.41}$$

using

$$1 - p_{ii} \leq \varepsilon,\ p_m \geq p^A_{ii} p^B_{0,m} \geq (1-\varepsilon)p_{0,m},\ sm = d,\ m \geq 1 \tag{5.42}$$

Note that p_{ii} is a constant for any i and same for both clocks. Now the average number of steps until the tick of one clock, $\mathbb{E}(Y)$ is upper bounded by $\frac{d}{p_{i,i+1}}$, where $p_{i,i+1}$ is also a constant in i. Hence we have:

$$N = \mathbb{E}(T) \geq \frac{(d + 1 + \frac{d^2}{4})p_{i,i+1}}{2d\sum_{k=1}^{m} k^2 p_k} = \tag{5.43}$$

$$d\frac{p_{i,i+1}}{8\sum_{k=1}^{m} k^2 p_k} + \frac{p_{i,i+1}}{2\sum_{k=1}^{m} k^2 p_k} + \frac{p_{i,i+1}}{2d\sum_{k=1}^{m} k^2 p_k} \tag{5.44}$$

$$\geq \frac{(d + 4 + \frac{4}{d})p_{i,i+1}}{8m^2\sum_{k=1}^{m} p_k} \geq \frac{(d + 4 + \frac{4}{d})p_{i,i+1}}{4m^2(1-p_0)} \tag{5.45}$$

$$\geq \frac{(d + 4 + \frac{4}{d})p_{i,i+1}}{4m^2\varepsilon(2-\varepsilon)} \tag{5.46}$$

using the fact that for two identical clocks playing the game,

$$2\sum_{k=1}^{m} p_k = 1 - p_0$$

and

$$1 - p_0 \leq 1 - p^A_{ii} p^B_{kk} \leq 1 - (1-\varepsilon)^2 = \varepsilon(2-\varepsilon)$$

where i and k are positions of players A and B. □

From the above result we can conclude that for $mp_{0,m}$ small, we have a greater linear upper bound on the number of ticks, and for $mp_{0,m}$ approaching d we have a very bad constant bound. Recall that for the ladder clock $p_{ii} = 1 - \delta$, $\forall i \in \{0, ..., d\}$ and $p_{i,i+1} = \delta$. This suggest that the *ladder* clock is one of the best stochastic clocks one can build, since $mp_{0,m} = \delta \leq \varepsilon$. To maximise lower bound one needs $m^2\varepsilon$ to be small, which again is minimised for a ladder clock (where $m = 1$) with small ε.

This is in consistency with the results from [41], where an upper and lower bound linear in d was obtained for the performance of the two identical ladder clocks in the AT Game.

5.3.2 Perfect stochastic clocks in the Alternate Ticks Game

We will now introduce a clock that we will call *the perfect clock*. When perfect clock plays the AT Game with an identical clock, it always produces infinite number of ticks. But this clock is, however, not ε-continuous for $\varepsilon < 1$. As described in Sec. 4.1 and results thereof, this clock is hence not a physically acceptable one.

Definition 10. *Perfect stochastic clock* is a clock that always produces a tick after the exactly same number of applications of the map $\mathcal{M}$.

Note that $p_k = 0$, $k \geq 1$ for this clock, and from Eq. (5.33) the expected number of ticks is $\mathbb{E}(E) = \infty$. In the ladder clock model, this clock will always produce a tick after $D - 1$ steps, where D is the dimension of the clockwork system. Since we have taken that $D = d + 1$, the tick is produced after exactly d steps. Hence $\mathbb{E}(Y) = d$ for the perfect ladder clock. Again note that this clock is not ε-continuous for $\varepsilon < 1$ (since $1 - p_{ii} = 1$).

Let us now label perfect ladder clock A and any stochastic clock B and assume that clock A is at least one position in front of the clock B at the start. The difference equation obtained when perfect clock plays the AT Game against ordinary stochastic clock B, is of the same form as Eq. (5.31), but with different probability distribution. Namely, one obtains:

$$\begin{aligned}
&\mathbb{E}(S|Q^0 = z) = D_z = p_1(D_{z+1} + 1) + p_2(D_{z+2} + 1) \\
&+ ... + p_m(D_{z+m} + 1) + p_0(D_z + 1) + p_{-1}(D_{z-1} + 1) \\
&+ p_{-2}(D_{z-2} + 1) + ... + p_{-m}(D_{z-m} + 1) = \\
&q_0(D_z + 1) + q_1(D_{z+1} + 1) \\
&+ ... + q_{-m}(D_{z+m+1} + 1) + q_2(D_{z-1} + 1) \\
&+ ... + q_m(D_{z-m+1} + 1)
\end{aligned} \tag{5.47}$$

where q_k, $k \in \{-m, ..., m\}$ is the probability that clock B moves for k states in one step.

However, note that in the case of the non-identical clocks playing, obtained difference equation is not symmetric. Hence one can check that the particular solution as before (quadratic function

in z) will not work this time. The homogenous solution is still $D_z^{hom} = A + Bz$ (due to the fact that $\sum_{k=i}^{i+m+1} p_{k,i} = 1$).

Once we have a probability distribution of a specific clock we can easily obtain the average number of alternate ticks, since $\mathbb{E}(Y) = d$ in this case (due to the property of the perfect ladder clock). If the perfect ladder clock would play against an ordinary ladder clock, the solution of the differential equation becomes $D_z = D_0 - \frac{z}{1-\delta}$, $z \geq 0$. Note that since we assume that the perfect clock has an advantage at the beginning, it is not possible that the ladder clock overtakes it, hence $D_z = 0$ for $z < 0$ always. Boundary conditions for the lower bound then become $D_{d+1} = 0$, which leads to $D_z^{\text{lower}} = \frac{d+1-z}{1-\delta}$ and for the upper bound $D_{2d+1} = 0$, hence $D_z^{\text{upper}} = \frac{2d+1-z}{1-\delta}$. Note that results are minimised for starting condition $z = 0$. Hence

$$\frac{d+1}{1-\delta} \leq D_z^{\text{ideal vs ladder}} \leq \frac{2d+1}{1-\delta} \tag{5.48}$$

Then $N = D_0/d$ gives a bound on the number of alternate ticks, which will be very bad for small δ, as expected. Note that if a perfect clock plays against a perfect clock we have $\delta = 1$ and they can play the AT Game forever.

Results proved above show that stochastic clocks have maximally a linear in the clockwork dimension success rate in the AT Game, and the success also crucially depends on the probability distribution for the clock jumps (and hence on the ε continuity as well). We can take the performance of the clock when playing the AT Game with an identical clock to be a measure of how good a clock is, and hence we have bounded the quality of this class of stochastic (classical) clocks.

Lower bounds obtained also provide us with the minimum average number of alternate ticks achievable with the best possible quantum clocks. It is an open question whether it is possible to achieve a stronger-than-linear relationship, but current numerical results and work on different clock accuracy definition presented in the next section, confirm such a superiority of well-defined quantum to any classical clocks for certain measures used.

5.4 Possible extensions of the game

The AT Game we introduced can be generalised to the one involving multiple players, *i.e.* multiple local clocks. For instance, even when the game involves more than two players, one can again impose the synchronisation criterion that no two successive ticks should come from the same player. This can be achieved in several ways, *e.g.* by asking that ticks come from any different clocks in succession, or demanding a clear order of clocks that should provide ticks one after the other. Conceivably, by increasing the number of players, one can obtain combined time scale with smaller intervals (as measured by some external time parameter) between consecutive ticks and

hence achieve a more refined but still synchronised ordering scale. Suppose we have n players in the game, for some finite integer n. Label the players $\{1, 2, ..., n\}$, and assume that the payer $i \in \{1, ..., n\}$ starts the game. Then we can define following generalisations of the game:

1. The goal is that each n successive ticks (starting from any clock chosen) come from n different players, hence they always play in the same order. This means that in the joint string s (consisting of characters labelling the player's number) any succession of n characters will be the same permutation of the n-string $123...n$.

2. The goal is that n successive ticks come from different players, but after finishing one round of n ticks, the next round can be performed in a different order.

3. The goal is only that no two successive ticks come from the same player.

Investigating these scenarios is one of the topics for the future work.

5.5 Other measures of the clock accuracy

We can compare results obtained for the Alternate Ticks Game to the ones presented in the paper [38], where different quantity is used to measure the precision of the clock, so-called R-value. This value is defined as $R = \frac{\mu^2}{\sigma^2}$, with μ the average ticking time and σ standard deviation of the ticking time (note that this quantity was also used in work [44]). For the reset clocks, measure R describes how long can a clock run before the time uncertainty between its ticks becomes of the order of a mean of the tick time μ. Average and standard deviation are expressed in terms of the parametric time which is hence assumed to be measured perfectly. At the moment this measure is only developed for completely continuous clocks. This could be overcome by counting time in terms of the applications of the fixed maps defined as in our model.
In [38] it was shown that R value grows linearly with the clockwork dimension for classical clocks, and almost quadratically for purely quantum clocks, but at the moment only in the limit of an infinite dimension. This already shows an improvement of quantum vs classical clocks for large dimensions that could maybe also be used for practical purposes. These results are in consistency with what we have seen for the maximum number of alternate ticks in Sec. 5.2, and with the analytical results for the stochastic clocks shown in Section 5.3. This suggest that the measure R could be related to the average or maximum number of alternate ticks. If this is true, it could especially be useful when trying to find upper bounds for the performance of the general quantum clocks in the AT Game, the task that has shown to be very hard analytically.

On Fig. 5.10 we provide numerical results for the value of the measure R of the first tick, for the Salecker-Wigner-Peres (SWP) clockwork Hamiltonian, Eq. (5.14), with the interaction optimised over a choosen set of interaction potentials. Results are shown for two different starting states of

the clock, usual clock state and a Gaussian superposition of those. One can see that for the choices of potentials we optimised over, the later has a better precision. This is because Hamiltonian is such that the Gaussian packet will be less disturbed during the evolution, while a highly peaked state would spread and then peak again, leading to a worse precision. This confirms what we have seen in the numerics for the AT Game as well, high dependence of the clock quality on the choice of the starting state.

In [38] particular quantum clock with the SWP clockwork Hamiltonian was presented, that starts in the Gaussian superposition of the clock states with standard deviation σ. The dynamics of this clock is similar to the clock from [86], with a specifically chosen potential. For $d \to \infty$ it has a value of $R \geq \frac{d^{2-\mu}}{\sigma^2}$, $\mu \in (0, 1]$. Note that the proof only works for $\sigma \in (d^{\mu/2}, d)$ and other limiting parameters. This clock is thus impractical, as it demands infinite state space and infinite standard deviation of the starting state. But the importance of the result is that it suggest that there may be more physical quantum clocks that can have almost a quadratic improvement in the precision compared to classical ones. We hope such clock examples will also be found in the future.

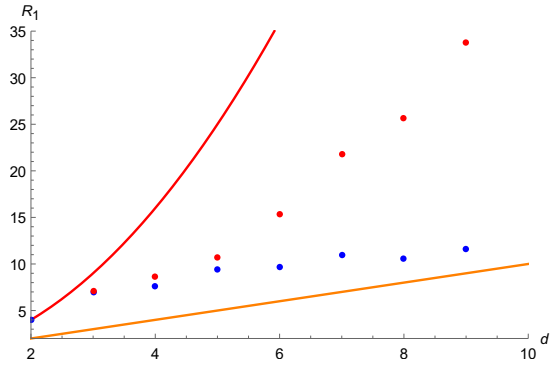

Figure 5.10: **Numerical comparison for the R value of the first tick depending on the clockwork dimension**, for the Salecker-Wigner-Peres clockwork with different starting states (a particular clock state and a Gaussian superposition of those). R value was optimised over a chosen set of interaction potentials.

5.6 Discussions and future work

In this chapter we have introduced an operational measure of the clock synchronisation, that we call the Alternate Ticks Game. This game checks if the two clocks are synchronised by demanding a specific correlation from the ticks of the clocks - namely that they come in an alternate order. By extending this game to more than two players we can have stronger conditions of synchronisation and clock quality. To compare two different clcoks, we could have them both play against a same

clock that would represent a reference for the time scale. Another option for comparing clocks of the same dimension and continuity, is that each plays the AT Game against multiple copies of itself and then compare its performance to the performance of the other clock in the same setting.

Usual approach taken when determining the clock quality and synchronisation is to quantify how well a clock can approximate 'seconds' or other intervals as defined by some external time parameter. There are several issues with this approach, important one being that there is always a need of another clock (probably perfect classical clock) that actually produces this external time - otherwise there is no operational meaning in these definitions. But this clock is rarely explicitly defined. One could think of *e.g.* atomic clocks [101, 99, 81] (see Sec. 3.1), or some universal clock [36] (Chapter 6), that would provide this standard of time. On the other hand, the Alternate ticks game provides a natural way of measuring clock quality and synchronisation, by observing correlations between the time scales of the clocks. By playing this game, one can calculate the level of synchronisation (number of alternate ticks) of any clock compared to the clock providing a time standard, or define the quality of a clock without the need for any other clock or time parameter (by playing the game with an identical clock).

In general, one can think of more ways to measure the clock synchronisation, by demanding different correlation criteria between the tick registers. It is a task of the future research to constructe other games and scenarios for the operational clock synchronisation, and compare them to the AT Game. Performance of the clocks in a certain synchronisation setting was investigated using mutual information and entropy in Sec. 5.2. Under few necessary assumptions, results suggest that the maximum synchronisation of the ε-continuous quantum clocks with small ε is upper bounded quadratically in the dimension of their clockwork spaces.

In Sec. 5.3 we have investigated the performance of stochastic clocks, introduced in Sec. 4.5. Specific stochastic ladder clock was already investigated in [41], and we have now extended results to the more general stochastic clocks and showed how their performance depends on the clockwork dimension and the clock dynamics.

We have thus also reviewed results obtained for a different measure of the clock accuracy, so-called R-value, Sec. 5.5. For the fully continuous reset stochastic clocks this value also grows linearly with the clockwork size, and a particular fully continuous quantum clock has been found that has almost a quadratic in d value of R when $d \to \infty$. Even though measure R is not satisfactory enough since it is not operational and is only developed for completely continuous clocks, this result is important, since it suggest a possible superiority of quantum to classical clocks (although note that the particular quantum clock presented in [38] is not practically achievable since it has an infinite dimension and infinite spread in its starting state). Relating R value to the expected number of alternate ticks would be a useful result, since we could obtain bounds for

the operational measure of synchronisation (the AT Game) using more analytical measure that depends on the mean ticking time and its standard deviation.

Optimal clocks and game scenarios in the future, as well as extension to multiple players, could provide us with bounds on how well quantum clocks can perform, and also with a step towards the limits of time measurements in quantum mechanics. In this sense, the framework that we have introduced serves as a basis for a future consideration of a more global, yet operational notion of time in quantum mechanics.

Going beyond non-relativistic quantum theory, it would be fruitful to look at the cases when clocks are playing the AT Game in the presence of the relativistic effects, like the ones described in *e.g.* [23, 85] (see Sec. 3.3). As mentioned in the previous chapter, only a clockwork model is considered in these papers , and the delay between previously synchronised clockworks is noticed due to the gravitational or relativistic effects. It would be interesting to see how they would further influence the AT Game, since we showed that already with the quantum effects tick registers are due to get out of synchronisation after some while (that depends on the clockwork dimension). Now we would have further disturbance due to the relativistic effects, probably yielding the game to finish earlier. How exactly this would effect the game is left for some future research.

Over the past years there has been renewed interest in the question of how quantum mechanics can be reconciled with general relativity (GR) — one of the biggest problems of modern physics. Here the "problem of time" (see Sec. 2.4) arises as witnessed by the seeming impossibility to combine the time parameter used to describe the evolution of quantum systems with the structure of GR, where time is defined locally. This has led to interesting novel approaches to define time. For example, since the Wheeler-DeWitt equation [13] corresponds to a timeless Universe, a possible way to think about time is via the correlations among systems, which leads to a notion of an internal time defined between subsystems of the Universe [6, 7, 8, 9, 10, 11, 12]. We reviewed this approach in Sec. 2.5. Our approach has various similarities with this "time as correlations" idea. In particular, the time scales are ultimately a stationary concept and the question of synchronisation may be reformulated as the question whether the correlations between two time scales have a certain desired structure. Our interest lies on deriving fundamental limitations to how well we can use separated, local quantum clocks, that are not a priori correlated (synchronised), to define a notion of time and ordering on a more global level. Note that first steps to using our clock model in the time as correlations framework were done in the master thesis [114].

Chapter 6

Universal quantum clock

What if the clock described in Section 4.1 would be the whole Universe? According to the definition, it would consist of a clockwork and a storage for the ticks - tick registers. Time this clock would define should be a kind of the universal time, perhaps exactly the time we feel in our everyday lives, since the ticks it provides would be unique for the whole Universe. I will expand on this topic in this Chapter.

In past decades the so called problem of time came into focus. How can one relate parametric time used in quantum theory to the time in general relativity and the fact that Wheeler-DeWitt equation suggests that the Universe as a closed system might be stationary? On the way towards answering this question it is fruitful to look into related problems first. One of them is the physical interpretation of the time we experience every day and the relation between evolution of the Universe and the passage of time. Since we all seem to experience the same parametric time, is it somehow embedded into factory of the Universe - does the Universe posses a 'universal clock' that produces this global time? Could we use radius of the Universe as a 'clock variable' that measures time?

As postulated in previous chapters, to talk about time operationally, we need to define a quantum clock that also records its output - ticks that it creates. Hence if a clock is stored within the whole Universe the ticks of the clock also need to be stored within the Universe (see Sec. 4.1). This implies that the Universe has to posses enough storage/memory for all the ticks that this clock has produced by now and to create enough memory for all the ones that will be produced until the 'end of time'.

Number of ticks that have been produced up to time t would equal $\frac{t}{\Delta t}$ in the case that there is an approximately constant ticking interval Δt. One obvious value for this interval is the Planck time $t_p = \sqrt{\frac{\hbar G}{c^5}} = 5 \cdot 10^{-44} s$, which is usually considered the minimal time necessary for a physically observable process to occur [50]. Planck's time equals the time a photon takes to travel a unit of Planck's length $l_p = 10^{-35} m$. We can alternatively define the ticking interval by

demanding that each tick needs an orthogonal state to be stored in. Then the minimal ticking time is a minimal time that a system needs to pass to the orthogonal state, the so-called 'quantum speed limit' [116]. There have been many approaches in calculating this time and here we will use the Margolus-Levitin bound [77]. The advantage of this bound is that it depends on the average energy of the system and not on the particular energy difference. We will see that with both Planck's time and the Margolus-Levitin bound we get a very similar result.
To upper bound the number of ticks, we require that there are at least as many distinguishable states in the Universe as ticks produced by an universal clock. To estimate the number of distinguishable (orthogonal) states, we can use the Boltzmann formula [117], that equates this number with 2^H, H being the system entropy. Furthermore, we also know that the entropy of any system is upper bounded by the entropy of the Black hole of the same size (Hawking-Bekenstein formula [118, 119, 120]). And in particular, by the so-called area law, the entropy of the Black Hole depends only on its area and on the Boltzmann's constant [118, 119, 120]. This important results will allow us to obtain an upper bound on the number of ticks of the universal clock, that depends only on the radius of the Universe.
Next discussion is based on the pre-print [36].

6.1 The theory of inflation

The theory of inflation came out of the need to describe the large scale features of the Universe as we observe today. In contrast to the theory of Big Bang, inflation offers us relatively simple explanations for the *flatness problem* (the fact that the cosmological constant today has a value extremely close to 1 — $\Omega \approx 1.012$), the *horizon problem* (distant regions of the Universe should have been at the causal contact at the past), the *monopole problem* (Big bang theory without inflation predicts too many magnetic monopoles to have been created - albeit these have not been observed) and others (the interested reader is encouraged to have a look at [121, 122]). Another argument in support of inflation comes from the assumption that the observable and the unobservable Universe might be entangled [104].

We will not restrict ourselves to a specific inflationary scenario (such as *e.g.* eternal inflation), but will consider general theory that assumes exponential expansion in the very early Universe, usually taken to have happened between around $10^{-37}s$ and $10^{-32}s$ [121, 123, 124]. The most important variable in this theory is the number of so called e-folds the Universe had to go through during inflation, *i.e.* the factor of exponential expansion that we will label C, and which we will lower bound in our approach.

Our argument provides additional explanation in support of the theory of inflation - namely, the existence of time and its passage as we record it. We term this the *'universal clock problem'*. Our results tell us that the inflation at the beginning of the Universe was needed for the notion

of time to be physically meaningful. They also lead us to propose an intriguing starting point of the Universe: the Universe comes into being at the point when its smallest size is large enough to record its own temporal evolution. Our calculations are based on the well-established quantum metrological and general relativistic bounds, linking the entropy and energy of the Universe to its capacity to store and process information.

Physical quantum clocks designed to keep track of time need to produce ticks, that can be described in the form of the so-called *tick registers* (see *e.g.* [35]). The Universe itself also needs to be able to store this information about the ticks/time it provides. The ticks produced should be stored in distinguishable quantum states. We will therefore make the following assumption: *In order for the common notion of time to be possible, the Universe itself has to contain (represent) a universal clock with a large enough memory that can store its own 'ticks'. In particular, the Universe needs to have at least as many possible distinguishable states as ticks it has produced as a clock.*

We can also relate our approach to more specific thermodynamical quantum clocks, where it was shown that the accuracy and ability of the clock to store time directly relates to its increase in entropy [44]. In our case, the increase of entropy of the Universe as a thermodynamical clock simply means the increase in the number of possible distinguishable states. This allows the Universe to keep counting ticks and measuring the universal time. We will show how the ability to store time information is directly related to the speed of the evolution of the Universe which then relates to its entropy increase via the Bekenstein formula.

Our approach fits well within the Page-Wootters picture [6]. Here one starts by considering the Wheeler-DeWitt equation [13] which states that the Universe as a closed system is stationary. A number of authors [6, 7, 125, 8, 10, 9] have shown that one can define 'time evolution' of a system via its correlations with the rest of the Universe that acts as a clock. One important assumption that has to be made, is that the clock system is independent from the rest of the Universe whose 'time' it attempts to measure (*i.e.* there is no interaction term in the total Hamiltonian). Some of the obstacles in this approach have recently been overcome [11, 12]. Even though we will not use the Page-Wootters picture explicitely, one can consider a cosmological clock that is represented by the size or some other specific variable of the Universe, which is to a good degree decoupled from the observers.

Note that we are not speculating about the fundamental origins of what time we experience really is or how best to measure it. These aspects belong to another fundamental area of research. Our argument here is simply that for an experience of time we use in physics to be possible (be it an illusion, a comfortable assumption or a fundamental entity) our Universe needed to go through an exponential expansion at the very beginning.

Below we will present calculations for the case with and without inflation, first taking the

Planck time as a fundamental ticking interval of the Universe as a clock. In our second approach we will rely on the Margolus-Levitin bound to obtain minimal time interval needed for a Universe to change to an orthogonal state and hence for a tick to happen. We will see that in both approaches we obtain similar lower bounds on the factor of exponential expansion of the Universe at the beginning. We note that our calculations are based on the FLRW model of the Universe [126].

6.2 Bekenstein bound and the number of ticks of the universal clock

It is known that the entropy of a black hole is the maximum entropy that can exist within a certain area, hence we can use the Hawking-Bekenstein bound to put an upper bound on the entropy of the Universe at the very beginning [118, 119, 120]. We use the spherically symmetric model of the Universe and denote the entropy of the black hole by S_{BH}, the entropy of the Universe S_U and the smallest interval in which universal clock could tick by Δt. The following then holds:

$$2^{\frac{S_{BH}}{k_B}} \geq 2^{\frac{S_U}{k_B}} \geq \frac{t}{\Delta t} \tag{6.1}$$

which simply encapsulates the fact that the number of distinguishable states in the universe must be at least as large as the number of ticks $\frac{t}{\Delta t}$, where t is the total time of evolution of the Universe. Since $S_{BH} = \frac{k_B A}{4l_p^2}$ and $A(t) = 4\pi l(t)^2$, we have that:

$$2^{\frac{\pi l(t)^2}{l_p^2}} \geq \frac{t}{\Delta t} \tag{6.2}$$

where $l(t)$ is the radius of the Universe at time t and $A(t)$ its area, k_B Boltzmann constant and $2^{\frac{S_{BH}}{k_B}}$ is the maximum number of distinguishable states of the Universe. Note that Eq. (6.2) rests on the assumption that each tick of the Universe needs to be stored in a distinguishable state. Hence at each moment, the Universe needs to have at least as many distinguishable states as ticks it has produced.

We are interested in the time after the Planck time $t_p \approx 10^{-44}s$ until the end of the inflation, t_2. We will take $t_2 = 10^{-32}s$ [127]. In the following calculations we will often omit the relevant units for simplicity. We will proceed to show that, without inflation, the rate of the expansion would be too slow for the Universe to be able to create enough memory to keep track of its own time.

We will consider several scenarios, first with Planck unit of tick time and second using minimal orthogonal time approach to calculate the ticking interval.

6.2.1 Ticking in the Planck units

No inflation —

Consider the expansion of the Universe after Planck's time t_p, without the inflation occurring. The following formula for the radius l of the Universe at the later times should hold [127]:

$$l(t) = K\sqrt{t} \tag{6.3}$$

$$K = l^p/\sqrt{t_p} \tag{6.4}$$

where $l^p = l(t_p)$ is the radius of the Universe at Planck's time, and depends on the horizon of the Universe one is considering (e.g. particle horizon, event horizon, Hubble horizon,...). For example, in the paper by Atkatz [128] $l^p = 10^{-55}m$ while *e.g.* the Planck length is $l_p = 10^{-35}m$. We will take this value $l^p = 10^{-55}m$ to be the radius of the observable Universe at the Planck time. Hence $K \sim 10^{-33}$. Note that we use scaling with time of the cosmological scale factor to express the scaling of the radius of the Universe.

We will assume the Planck unit of time to be the smallest interval of time that is observable in the Universe, hence we set $\Delta t = t_p$. This stems from the accepted view that the times smaller than the Planck time are not detectable or even, possibly, not physically meaningful. In this scenario the Universe needs to 'remember' and store its own state each Planck unit of time.

From Eq. (6.2) it follows that:

$$2^{\frac{K^2\pi t}{l_p^2}} \geq \frac{t}{t_p} \tag{6.5}$$

$$2^{10^{70}10^{-66}\pi t} \geq 10^{44}t \tag{6.6}$$

The above inequality is not satisfied for approximately $10^{-44} < t < 10^{-2}$. The Universe could not have expanded as given in (6.3) at the beginning if we assume that it needs to be able to keep track of its own time.

If we would, however, insert the current values of the size and age of the Universe, the above inequality would be satisfied. This inequality therefore describes the current Universe well, but for it to be able to keep track of time since the beginning it would need a more rapid evolution at the start.

Planck units — inflation

One of the implications of the previous analysis is that after the time we denote as t_1 (close to Planck's time), there had to be a different evolution equation for the size of the Universe, in order for it to be able to continue measuring its time. We therefore assume that there was an inflationary epoch ending around $t_2 = 10^{-32}s$ [127] during which the Universe grew exponentially:

$$l_2 = l(t_2) = l_1 e^C \tag{6.7}$$

where C is the factor of expansion during the inflation and $l_1 = l(t_1)$ radius of the Universe at the start of the inflation. We know $l_1 > l^p \geq 10^{-55}m$ since at that moment Universe was at least as large as it was at Planck's time (usual start of the inflation is taken around $t_1 = 10^{-37}s$ or $10^{-36}s$ [127, 129]). There is no consensus on the value of these parameters (l_1, $t_1$and C), nor is there any strong experimental evidence.

Going back to Eq. (6.2), we would like the following to hold:

$$2^{\pi l_1^2 e^{2C}/l_p^2} \geq \frac{t_2}{t_p} \tag{6.8}$$

Hence, we obtain an inequality between C and l_1:

$$l_1 \geq e^{-78-C} \tag{6.9}$$

If *e.g.* the initial size of the Universe is $l_1 = 10^{-51}m$ the rate of expansion would be $C \geq 38$. If, on the other hand $l_1 = 10^{-55}m$ then $C \geq 48$.

We can combine above inequality in Eq. (6.9) with the fact that $l_1 > l^p \geq 10^{-55}m$, and conclude that Eq. (6.8) is certainly fulfilled for: $10^{-55} \geq e^{-78-C}$, $C \geq 48$. Note that this conclusion is stronger than necessarily needed, but, if it holds, then our inequality (6.2) is certainly correct and we know that Universe was able to still count its time at least until t_2. This, as previously shown, does not hold when using the model of expansion without inflation.

Result obtained agrees with most cosmological models that state that exponential factor $C \approx 50$ or larger (C number of e - folds), depending on the problem the inflation is aimed to solve [127, 124, 129] and the starting moment of the inflation. Our argument suggests that in order for inflation to solve the 'universal clock' problem we need at least 48 - fold inflation period to occur at the beginning of the Universe.

6.2.2 The Margolus-Levitin bound

One should note that a similar approach to the one shown, can be used for a Universe that does not tick in Planck's units. For this approach we need to be able to approximate the average energy of the Universe at the time of our interest. Then one can obtain a minimal time needed for the Universe to change to a distinguishable (orthogonal) state using the Margolus-Levitin bound [77] and follow the same reasoning as above. In this bound, unlike in the Heisenberg-Robertson uncertainty relation for time and energy [73, 74], one does not consider change in energy between two states, but uses the average energy instead.

The Margolus-Levitin bound is given by:

$$\Delta t \geq \frac{\pi\hbar}{2\overline{E}} \tag{6.10}$$

where $\overline{E}$ is the average energy of the system and Δt is the minimal time needed for a system to pass to an orthogonal (completely distinguishable) state.

We can bound the average energy by demanding that the radius of the Universe is not smaller than the Schwarzschild radius [130] of a system with the total energy of $E = Mc^2$:

$$l \geq R_S = \frac{2GE}{c^4},\ E \leq \frac{lc^4}{2G} \tag{6.11}$$

$$\Rightarrow \overline{E} \leq \frac{lc^4}{2G} \tag{6.12}$$

Inserting $l_p^2 = \frac{\hbar G}{c^3}$ and using Eq. (6.10) we have:

$$\Delta t \geq \frac{\pi l_p^2}{lc},\ l_p = ct_p \tag{6.13}$$

Hence the following holds for the number of ticks:

$$\frac{t}{\Delta t} \leq \frac{lt}{\pi l_p t_p} \tag{6.14}$$

To satisfy Eq. (6.2), knowing that above inequality holds, we can assume:

$$2^{\frac{\pi l^2}{l_p^2}} \geq \frac{lt}{\pi l_p t_p}\,. \tag{6.15}$$

Note that, as in the chapter with the Planck unit of time, this condition is not necessary but is sufficient for Eq. (6.2) to hold and our Universe to create enough orthogonal states to store all of its ticks.

Let us label with $l(t)$ size of the Universe at time t, to obtain an inequality in terms of $y = \frac{l(t)}{l_p}$:

$$\pi 2^{\pi y^2} \geq yt10^{44} \tag{6.16}$$

No inflation —

In the case of a non-inflationary model, we have that $l(t) \sim K\sqrt{t}$, where $K = l^p/\sqrt{t_p} \sim 10^{-33}$. Let us consider time $t = 10^{-30}s$. Then $y = 10^{-13}$ and the inequality (6.16) is not satisfied. This shows that, in this case, the Universe could not have kept track of time until or after $10^{-30}s$.

Inflation —

Now consider scenario in which inflation takes place. We have that, at the end of inflation, $t_2 \approx 10^{-32}s$, and as before (labelling by C the factor of exponential expansion):

$$l_2 = l(t_2) = l(t_1)e^C \tag{6.17}$$

where t_1 labels the beginning of exponential expansion. Again we now know that $l_1 = l(t_1) > l^p = 10^{-55}m$.

Imposing inequality (6.16) to work for all l_1 we have

$$l_1 e^C \geq 10^{-34} \tag{6.18}$$

$$\Rightarrow C \geq 48 \tag{6.19}$$

Depending on the exact value of l_1 smaller C could be sufficient, but this bound works for any given $l_1 > 10^{-55}m$. Note that one would get a similar result with a bit different C if we consider later times, $t > 10^{-32}$. Hence our result is not dependent of taking this exact time t_2 for the end of inflationary period.

Interestingly, we have obtained the same bound as when considering Planck time as a unit time step of the universal clock.

6.3 Discussions and future work

Presented argument shows that the inflationary theory allows for the existence of time that the Universe has had enough capacity to store. Our reasoning indicates that without the inflation, the notion of time as we know it today might not have existed. Our approach works well with Planck's unit of time but also when using the Margolus-Levitin bound for the calculation of the smallest time interval of a tick of the universal clock. It would be interesting to investigate this result when using other possible values for the ticking time interval.
Note also, that the value of the size of the Universe at the Planck time depends on the horizon one considers, and further arguments in this direction would be valuable.

As a related problem, when considering the ticking of time, one naturally makes connections with the arrow of time. It is usually argued that the arrow of time arises from the 2^{nd} law of Thermodynamics, a law stating that the entropy of the closed system cannot decrease. As our Universe is considered to be a closed system (and may well be the only one such known to us), the 2^{nd} law should hold. Hence, the time flow that we notice follows the direction of the increasing entropy, and could not go 'backwards'. Our argument provides another explanation for this, when viewed as a consequence of the fact that in order to time itself, the Universe needs to be able to

provide more and more ticks, hence it needs to have more distinguishable states. From Eq (6.1) one concludes that the entropy of the Universe needs to increase (or at least not decrease) during the evolution. The flow of time again has one direction in this scenario, which is the direction of the increase in the number of ticks, *i.e.* the increase in the number of possible states and with this the entropy of the Universe.

Another way to think of the implications of the second law, is that the end of the Universe would come when it runs out of its clock time by reaching the maximum entropy. Hence the clock states of the Universe would be so mixed that there would be no more capacity to discriminate between different times. This notion is akin to the so called *heat death*, first postulated by Lord Kelvin.

Our aim was to show that the inflation was important for the Universe to be able to keep track of its time. The reason for this is that the evolutionary speed of the Universe without the inflation would simply not be large enough to create sufficient number of orthogonal states that would store the time ticks. We have shown that the exponential evolution of the size of the Universe would be enough at the early beginning, in consistency with the theory of inflation. We do not claim that this means that without the inflation Universe would not have had enough storage for its clock ticks, but we certainly do feel it is a worth while question to be asked and provide a starting step towards its answer.

The result shown is a novel stepping stone for the further investigation of the connection of the inflationary theories and the notion of time and causality we experience. Theory of inflation can already explain some of the specific features of our Universe that makes it seem quite special, and here we have argued that the existence of time is another one of them.

Chapter 7

Conclusions

In this thesis, we have presented several results related to time and clocks in quantum theory. The first and most important result is the novel framework for constructing quantum clocks and the Alternate Ticks (AT) Game, that are presented in Chapters 4 and 5, respectively. To call a system a quantum clock, we demand from it to output time information (ticks) that can be read out without disturbing the working part of the clock (the so-called clockwork) directly. We call the systems that store time information *tick registers*. To compare the synchronization of clocks, one compares their tick registers, by *e.g.* examining the correlation between their states. The AT Game is used to check the clock synchronization in an operational manner by directly comparing the clock ticks, without explicitly using any background time reference. In particular, ticks should come in an alternate order or the game is lost. We provide a few particular clock models and numerical results for the number of alternate ticks they can achieve. We also relate these models to other clocks and synchronisation measures in the literature.

We would very much like to see our framework applied in practice soon. Our model with light that interacts with the clockwork, described in Sec. 4.6 and 5.1.2, is probably the closest to the practical achievability as a first step. The issue is in constructing good photon sources (preferably a one-photon source), which are at the moment still only achievable up to an error. Errors in emitting exactly one photon could lead to the detection of two ticks from the same clock in a row. The theoretical source of an error, on the other hand, is a hermitian Hamiltonian that can with a non-zero probability return a photon to the source instead of sending it further to the detector. These are all issues that could be overcome to a certain degree satisfactory enough for the first practical trial.

Ofcourse, practically realising our more general clock model (Sec. 4.1) and the Alternate Ticks (AT) Game based on the CPTP maps (Sec. 5.1.1), would ultimately be even more desirable. Among other things, what needs to be defined if the AT Game was to be played in the lab, is the mechanism by which evolution maps $\mathcal{M}_A$ and $\mathcal{M}_B$ are applied on both sides. This mechanism is

at the moment implicitly assumed to be the same black box on both sides, that applies maps $\mathcal{M}_A$ and $\mathcal{M}_B$ in the identical manner, but no practical realisations are discussed. This is important since this mechanism should not interfere with the game itself, and should not add a lag in any of the ticks on either side. In practice these black boxes could contain a copy of the classical clock each, which would be used to implement the maps in the same way. The copies of a classical clock can be easily made, and they will not interfere with or influence the time scales we obtain from the two quantum clocks in any way. Namely, what the frequency of this classical clock is, or how its tick are defined, is not important as long as on both sides A and B we have the identical definition. Note that defining this black box mechanism is akin to using two identical light sources in the model 4.6.

In Section 5.2, an information-theoretic attempt to obtaining a bound on the level of synchronisation of two clocks with the same clockwork is shown. This approach should include the AT Game but also different synchronisation scenarios, and aims to determine how long two clocks that already obey a certain synchronisation criterion (expressed as an upper bound on the conditional entropy of their tick registers), can remain synchronised. There are a few strong assumptions that had to be made, and the criterion of ε-continuity still needs to be explicitly included (at the moment we only provide an intuitive argument). But the results obtained agree with other results on the same topic presented in Chapter 5, namely they show a linear in the dimension bound on the number of synchronised ticks in the case of the classical correlations, and quadratic for the quantum ones. This approach uses entropy and mutual information as measures, and if the assumptions made can be shown physically justified, it is a very nice and natural way of analytically approaching the synchronisation problem in a more general setting.

In Sections 4.5 and 5.3 we described a special subset of clocks, the so-called *stochastic clocks*. These clocks are relatively easy to investigate since they can be described by classical dynamics (Markov chains in particular). First analysis and analytical bounds for some stochastic clocks were obtained in [41] and we generalised them in [37]. Results for these clocks show their linear in dimension performance in the AT Game. This is an important result since it provides a lower bound on the best quantum clocks performance and gives a starting point for the comparison of the performance of clocks with classical and quantum dynamics.

In Section 5.5 we described the research done in [38], where a more tractable value R is considered. In the case of the reset clocks this measure corresponds to the number of ticks until a tick of the clock is off by the average ticking time. It was shown that there exist a reset model of a quantum clock, corresponding to the fully continuous case of the framework 4.1, for which, in the limit $d \to \infty$, R is lower bounded by an almost quadratic function in d ($d^{2-\eta}$, $\eta \in (0,1)$). It was also shown that any classical reset clock has an upper bound on R that is linear in d. This is an interesting and important result since it shows the possible advantage of carefully constructed

quantum clocks over classical ones. But, since R uses explicitly parametric time t to determine the ticking time and other variables, it is not an operational quantity, like the number of alternate ticks N. What would be nice though, and some steps in this direction are being taken, is if we could lower bound N with R and then use R as an analytically computable quantity that would provide us with an insight into N.

In Chapter 6 we provide a novel step towards bridging the gap between quantum theory and cosmology. In search of a common way of describing time in both theories, we show that under the assumption of the existence of an *universal clock* that represents a common time for the Universe, there needed to be an exponential expansion of the Universe at the early beginning. This approach has many directions for the future work. It would be fruitful if experts in general relativity would try to formulate our inequalities using some of their results. For example, a better approximation of the entropy and mass of the Universe at the beginning would give tighter bounds, as well as considering different time steps for the ticks of the universal clock. It would also be very interesting to extend this result to the current times, and provide some insight into the possible endings of the Universe, and different theories of the possible Universe - like open, closed, or stationary.

Finally, I briefly discuss a possible extension of the *"time as correlations"*, or PaW approach described in Sec. 2.5. One of the main issues with this approach, is that the clock system used in the model description (particle on a line) is oversimplified and unphysical. This clock system goes along the lines of John A. Wheeler quote *'Time is chosen to make equations simpler'*. The simplicity of this model and the fact that one equates the time operator with the position operator in this case, yields a solution with the Schrödinger equation describing the evolution of the system in question. One can then consider if the Schrödinger equation is obtained exactly because of this simplicity, since also in its basics it assumes time as a real unbounded classical parameter t, which exactly corresponds to the parameter describing the position of the particle on the line, and hence the considered 'clock time'. To investigate what would happen if the clock were a more realistic physical system, we had a master project [114] with the clock system described by our quantum clock model from Sec. 4.1. The PaW framework then had to be adjusted and to be done with the discrete instead of continuous evolution. The result, as expected, was not a Schrödinger type evolution, but a discretised evolution of the system that could still not be written as a closed form equation. It would be very interesting to investigate how different clock models fit into the PaW model, and if one can obtain a closed form evolution equation for those clock systems. This could give us insights into possible reformulations of the evolution equations in case it is confirmed that continuous parametric time indeed is not a satisfactory solution for describing time in physics.

Another issue with the PaW mechanism is that it demands that the clock system and the

system whose time we measure are completely non-interacting. It would be nice to see how the result would change by *e.g.* adding small interaction terms to the total Hamiltonian. Also, since in our approach to clocks the main result has no reference to Hamiltonians, but to the fixed CPTP maps instead, it would be very interesting to formulate a "time as correlations" framework in this manner.

What would also be interesting to be seen, is the comparison of our clock model 4.1 with the atomic clocks currently used [99]. Namely, could one model an atomic clock inside our framework? If yes, how well could this clock play the Alternate Ticks Game? Since atomic clocks are extremely precise and accurate [80, 82], it is natural to assume that they can play the AT Game successfully for a very long time. But note that a caesium atomic clock produces a frequency as an output, which is then used to define a constant interval of a second, which is a different set up than the one assumed for the AT Game. A logical clock, on the other hand, contains another atom (e.g. beryllium) that detects its ticks, and could possibly be used directly in the game. It would be interesting to see the atomic clocks in operation in a completely quantum regime as well, and to investigate how they could be used to obtain insights into the future quantum gravity theory.

Not to forget an important feature of quantum systems, it would be fruitful to investigate what would happen if we allowed quantum correlations between the clocks playing the Alternate ticks game? In particular, how much would clocks benefit from the entanglement, and would the bound on their synchronisation change significantly? One would expect that these clocks would perform better in the game, but also note that after a few ticks the clocks would lose much of the correlation, which could again lead to the similar bounds as the ones suggested in this work (linear for classical maps and better than linear, possibly quadratic for the quantum ones). Possibly the best way to investigate this would be to use Quantum clock synchronisation (see Sec. 3.5) or a similar synchronisation protocol to synchronise clockworks of the two clocks in our framework and then play the AT Game with them.

Appendix A

Importance of the ε-continuity condition

In this section we will present an argument for how is additionally the ε-continuity important condition for the quality of a quantum clock - we conjecture that it makes it more robust to the change of the state of each new tick register. Namely, we propose an argument showing that for small ε, if new tick registers do not always arrive in the state 0 after a tick, this will only negligibly affect the clockwork state, linearly in the order of ε. We provide a sketch of a proof that misses one step we intuitively know should hold, but are still attempting its rigorous proof. If the conjecture is shown correct, it provides an important reason for clocks to have the continuity property introduced in Sec. 4.2 - if the clocks are continuous only for large ε, their state will be significantly affected by the changes in the tick registers and the way they are initialised, which is certainly not what we would expect from good physical clocks.

Let $\{\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\}_{\varepsilon\in\mathbb{R}}$ and ρ^0_C specify a family of ε-continuous clocks satisfying

$$\lim_{\varepsilon\to 0}\left(\mathrm{tr}_T\,\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\right)^{\circ\lfloor\frac{1}{\varepsilon}\rfloor}(\rho^0_C)=\tilde{\rho}_C$$

where $\tilde{\rho}_C$ is some (arbitrary but fixed) density matrix acting on the clockwork Hilbert space.

We will also introduce a family of CPTP maps $\mathcal{M}^{\varepsilon}_{CT\to CT}$, such that the following holds:

$$\mathrm{tr}_T\,\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}(\rho_C)\approx^{\varepsilon^2}\mathrm{tr}_T\,\mathcal{M}_{CT\to CT}(\rho_C\otimes\rho_T) \tag{A.1}$$

for some $\rho_T\in\mathcal{H}_{\mathcal{T}}$. To this end, let us first define the Stinespring dilation $\tilde{U_{CTE}}$ of map $\tilde{\mathcal{M}}_{C\to CT}$ such that $\exists\,|\psi\rangle_{TE}$ with:

$$\mathrm{tr}_E\,\tilde{U}_{CTE}(\rho_C\otimes|\psi\rangle_{TE}\langle\psi|)\tilde{U}^*_{CTE}=\tilde{\mathcal{M}}(\rho_C) \tag{A.2}$$

Next we introduce maps $R_{TE}:|0\rangle_T|\phi\rangle_E\to|\psi\rangle_{TE}$ and $\tilde{\tilde{U}}_{CTE}=\tilde{U}_{CTE}\circ R_{TE}$ and define

$$U_{CTE}=\tilde{\tilde{U}}_{CTE}(\mathcal{I}_C\otimes|0\rangle\langle 0|_T\otimes\mathcal{I}_E)+\mathcal{I}_C\otimes(\mathcal{I}_T-|0\rangle\langle 0|_T)\otimes\mathcal{I}_E \tag{A.3}$$

with $\mathcal{I}$ denoting the identity map. Finally let

$$\mathcal{M}_{CT\to CT} : \rho_{CT} \to \mathrm{tr}_E\, U(\rho_{CT} \otimes |\phi\rangle\langle\phi|_E)U^* \tag{A.4}$$

From the definition of $\mathcal{M}$ we have that

$$\mathcal{M}(\rho_C \otimes |0\rangle\langle 0|) = \tilde{\mathcal{M}}(\rho_C)$$

$$\mathcal{M}(\rho_C \otimes |1\rangle\langle 1|) = \rho_C \otimes |1\rangle\langle 1|$$

$\mathcal{M}$ is a linear map since all the maps involved in its construction are also linear. To show that the distance between the maps $\mathcal{M}$ and $\tilde{\mathcal{M}}$ is at most ε^2, we will need Assumption 3 to hold.

We would like to show the following conjecture.

Conjecture 1. *For any sequence $s \in \{0,1\}^n$ with*

$$\lim_{n\to\infty} \mathrm{freq}(s_{1,\dots,n}) = q < 1 \tag{A.5}$$

the following holds:

$$\lim_{\varepsilon\to 0} \mathrm{tr}_T \prod_{i=1}^{\lfloor\frac{1}{\varepsilon}\rfloor} (\mathcal{T}^{(s_i)}_{T\to T} \circ \mathcal{M}_{CT\to CT})(\rho^0_C \otimes |0\rangle\langle 0|_T) = \lim_{\varepsilon\to 0} \left(\mathrm{tr}_T\, \mathcal{T}^{(0)} \tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\right)^{\circ\lfloor\frac{1}{\varepsilon}\rfloor}(\rho^0_C) = \tilde{\rho}_C \tag{A.6}$$

where $\mathcal{T}^{(b)}$ for $b \in \{0,1\}$ is defined as

$$\mathcal{T}^{(b)}(\rho_T) = \begin{cases} |0\rangle\langle 0|_T & \text{if } b = 0 \\ |0\rangle\langle 0|\,\rho_T\,|0\rangle\langle 0| + |1\rangle\langle 1|\,\rho_T\,|1\rangle\langle 1| & \text{if } b = 1 \end{cases}$$

Proof. We will usually write $\tilde{\mathcal{M}}$ for $\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}$, $\mathcal{T}^{(i)}$ for $\mathcal{T}^{(i)}_{T\to T}$ etc. and often suppress writing tensor product with identity.

Assumption 3. *Let $(\tilde{\mathcal{M}}, \rho^0_C)$ describe an ε-continuous clock for ε close to 0. Parameter p in $\mathcal{T}^1\tilde{\mathcal{M}}(\rho_C) = (1-p)\rho' \otimes |0\rangle\langle 0| + p\sigma \otimes |1\rangle\langle 1|$ is bounded by ε for all ρ_C.*

Proof. From the continuity of the maps $\tilde{\mathcal{M}}$ and $\mathcal{M}$ one can show that p is very small. Namely, assume otherwise, that p and $1-p$ are of similar value. $\mathrm{tr}_T(\mathcal{T}^1\tilde{\mathcal{M}})(\rho_C) = (1-p)\rho' + p\sigma$, with ρ' and σ orthogonal states, would then be very close to the maximally mixed state. Note that, since $\mathcal{T}^{(1)}$ only affects the T space, $\mathrm{tr}_T\,\tilde{\mathcal{M}}(\rho_C) = (1-p)\rho' + p\sigma$. Then from the continuity of $\mathrm{tr}_T\,\tilde{\mathcal{M}}$ we have that $\frac{1}{2}\left\|\mathrm{tr}_T\,\tilde{\mathcal{M}}(\rho_C) - \rho_C\right\|_1 = \frac{1}{2}\left\|(1-p)\rho' + p\sigma - \rho_C\right\|_1 \leq \varepsilon$. But, wlog taking ρ_C to be a pure state (e.g. $|0\rangle\langle 0|$) we get a contradiction (since mixed state cannot be ε-close to the pure state for small ε). Hence p and $1-p$ have to be very different, one small and one big, and since by assumption ticks are much less frequent events than no-ticks, we would get that p needs to be very small. In particular we will conjecture that it is smaller than ε, but note that this is something that still remains to be shown. □

Proposition 3. *There exist n_0 such that in each sequence of length n_0 in $s_{1,\dots,n}$, $n > n_0$, there is at least one $s_i = 0$.*

Proof. Let $\mu > 0$. Then there exists n_0 such that $\text{freq}(s_{1,\dots,n}) \overset{\mu}{\approx} q$ for all $n \geq n_0$. But this means that also the frequency of other subsequences is bounded. In particular, one can show that, for any $j \in \{n_0 + 1, \dots, kn_0 + 1\}$, $k \geq 1$

$$\text{freq}(s_{j,\dots,j+n_0-1}) \overset{(2k+1)\mu}{\approx} q \ .$$

To prove this note that by definition

$$q - \mu \leq \text{freq}(s_{1,\dots,n}) \leq q + \mu \tag{A.7}$$

for all $n \geq n_0$, hence in particular inequality (A.7) holds for $n = n_0$ and $n = n_0 + j - 1$.

We write $\text{freq}(s_{1,\dots,n}) = \frac{f^n}{n}$, where by f^n we denote number of occurrences of 1 in the n-bit sequence $s_{1,\dots,n}$.

From Eq. (A.7) and the definition of $\text{freq}(s_{j,\dots,j+n_0-1})$, it follows that

$$\text{freq}(s_{j,\dots,j+n_0-1}) = \frac{f^{n_0+j-1} - f^{j-1}}{n_0} \leq \frac{(q+\mu)(n_0+j-1) - (q-\mu)(j-1)}{n_0} \tag{A.8}$$

$$= q + \mu + \frac{(j-1)2\mu}{n_0} \leq (2k+1)\mu + q \tag{A.9}$$

In a similar way we have:

$$\text{freq}(s_{j,\dots,j+n_0-1}) = \frac{f^{n_0+j-1} - f^{j-1}}{n_0} \geq \frac{(q-\mu)(n_0+j-1) - (q+\mu)(j-1)}{n_0} \tag{A.10}$$

$$= q - \mu - \frac{(j-1)2\mu}{n_0} \geq -(2k-1)\mu + q \geq q - (2k+1)\mu \tag{A.11}$$

We will now choose k s.t. $k \leq \lfloor \frac{1-q-\mu}{2\mu} \rfloor$ so that $\text{freq}(s_{j,\dots,j+n_0-1}) < 1$. Note that we have a freedom to choose μ small enough such that k is arbitrarily large.

We also have $\text{freq}(s_{1,\dots,n_0}) < q+\mu < 1$ by choosing μ small enough. Hence $\text{freq}(s_{j,\dots,j+n_0-1}) < 1$ for all $j \in \{1, \dots, kn_0 + 1\}$.

It follows that for any k there is n_0 such that frequency of appearance of $\mathcal{T}^{(1)}$ in $s_{1,\dots,kn_0+1}$ is smaller than 1 in any subsequence of length n_0, hence map $\mathcal{T}^{(0)}$ is applied at least once every n_0 steps. □

Assumption 4. *In the following, we will make the worst-case assumption that $\mathcal{T}^{(0)}$ is applied exactly once after every n_0^{th} step in the LHS Eq. (A.6), whereas $\mathcal{T}^{(1)}$ is applied in all other steps. Note that other cases follow directly form this one, by splitting each sequence of n_0 steps into subsequences that start with $\mathcal{T}^{(0)}$ and have all other $\mathcal{T}^{(1)}$ applications. In the proof that follows, this would just interchange some of the n_0 by $n_1 < n_0$ and the bound will remain the same.*

A. IMPORTANCE OF THE ε-CONTINUITY CONDITION

Lemma 4. *Let $\tilde{\mathcal{M}}, \rho_C$ describe an ε-continuous clock for ε close to 0. Following holds for the maps $\mathcal{M}, \tilde{\mathcal{M}}, \mathcal{T}$:*

$$\begin{aligned}&\left\|\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\mathcal{M}((\mathcal{I}_C \otimes \mathcal{T}^{(1)}_{T\to T})\mathcal{M})^{n_0-1}(\rho_C \otimes |0\rangle\langle 0|) - (\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT})^{n_0}(\rho_C)\right\|_1 \\ &\leq (n_0-1)\varepsilon^2 \end{aligned} \tag{A.12}$$

1. Base step, $n_0 = 1$:

$$\begin{aligned}&\left\|\mathrm{tr}_T\, \mathcal{T}^0\mathcal{M}(\rho_C \otimes |0\rangle\langle 0|) - \mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C)\right\|_1 = \\ &\left\|\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C) - \mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C)\right\|_1 = 0 \leq (n_0-1)\varepsilon^2 = 0\end{aligned}$$

   For illustration (although not needed for the induction), we also show the case $n_0 = 2$: Denote $\mathcal{T}^1\tilde{\mathcal{M}}(\rho_C) = p_1\rho_C^{11} \otimes |1\rangle\langle 1| + (1-p_1)\rho_C^{10} \otimes |0\rangle\langle 0|$ and note that $\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C) = \mathrm{tr}_T\, \mathcal{T}^1\tilde{\mathcal{M}}(\rho_C) = p_1\rho_C^{11} + (1-p_1)\rho_C^{10}$. Then:

$$\begin{aligned}&\left\|\mathrm{tr}_T(\mathcal{T}^0\mathcal{M})(\mathcal{T}^1\mathcal{M})(\rho_C \otimes |0\rangle\langle 0|) - \mathrm{tr}_T(\mathcal{T}^0\tilde{\mathcal{M}})(\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}})(\rho_C)\right\|_1 = \\ &\left\|\mathrm{tr}_T(\mathcal{T}^0\mathcal{M})(\mathcal{T}^1\tilde{\mathcal{M}})(\rho_C) - \mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(p_1\rho_C^{11} + (1-p_1)\rho_C^{10})\right\|_1 = \\ &\left\|\mathrm{tr}_T(\mathcal{T}^0\mathcal{M})(p_1\rho_C^{11} \otimes |1\rangle\langle 1| + (1-p_1)\rho_C^{10} \otimes |0\rangle\langle 0|) - \right. \\ &\left. p_1\,\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C^{11}) - (1-p_1)\,\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C^{10})\right\|_1 = \\ &\left\|p_1\rho_C^{11} + (1-p_1)\,\mathrm{tr}_T(\mathcal{T}^0\tilde{\mathcal{M}})(\rho_C^{10}) - p_1\,\mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C^{11}) - (1-p_1)\,\mathrm{tr}_T(\mathcal{T}^0\tilde{\mathcal{M}})(\rho_C^{10})\right\|_1 \leq \\ &p_0\left\|\rho_C^{11} - \mathrm{tr}_T\, \mathcal{T}^0\tilde{\mathcal{M}}(\rho_C^{11})\right\|_1 \leq p_1\varepsilon \leq \varepsilon^2 = (n_0-1)\varepsilon^2\end{aligned} \tag{A.13}$$

   using $p_1 \leq \varepsilon$ by Assumption. 3.

2. Induction step:
   Assume following holds:

$$\begin{aligned}&\left\|\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\mathcal{M}((\mathcal{I}_C \otimes \mathcal{T}^{(1)}_{T\to T})\mathcal{M})^{n_0-1}(\rho_C \otimes |0\rangle\langle 0|) - \right. \\ &\left.(\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT})^{n_0}(\rho_C)\right\|_1 \\ &\leq (n_0-1)\varepsilon^2\end{aligned} \tag{A.14}$$

   Let us denote $((\mathcal{I}_C \otimes \mathcal{T}^{(1)}_{T\to T})\mathcal{M})^{n_0}(\rho_C \otimes |0\rangle\langle 0|) = p\rho_C^{n_0} \otimes |0\rangle\langle 0| + (1-p)\,\overline{\rho}_C^{n_0} \otimes |1\rangle\langle 1|$ and $(\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}})^{n_0}(\rho_C) = \tilde{p}\sigma_C^{n_0} + (1-\tilde{p})\,\overline{\sigma}_C^{n_0}$. Note that from (A.14) it follows that also

$$\begin{aligned}&\left\|\mathrm{tr}_T((\mathcal{I}_C \otimes \mathcal{T}^{(1)}_{T\to T})\mathcal{M})^{n_0}(\rho_C \otimes |0\rangle\langle 0|) - (\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT})^{n_0}(\rho_C)\right\|_1 \\ &= \left\|p\rho_C^{n_0} + (1-p)\overline{\rho}_C^{n_0} - \tilde{p}\sigma_C^{n_0} - (1-\tilde{p})\overline{\sigma}_C^{n_0}\right\|_1 \\ &\leq (n_0-1)\varepsilon^2\end{aligned}$$

since the system T is traced out in the end.

Then we have, using properties of the map M, induction hypothesis, continuity of $\tilde{\mathcal{M}}$ and the fact that $\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}$ is a CPTP map and hence does not increase the 1-norm:

$$
\begin{aligned}
&\big\|\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\mathcal{M}((\mathcal{I}_C \otimes \mathcal{T}^{(1)}_{T\to T})\mathcal{M})^{n_0}(\rho_C \otimes |0\rangle\langle 0|)\\
&- (\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT})^{n_0+1}(\rho_C)\big\|_1 =\\
&\big\|\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\mathcal{M}((1-p)\rho_C^{n_0} \otimes |0\rangle\langle 0| + p\,\overline{\rho}_C^{n_0} \otimes |1\rangle\langle 1|)\\
&- (\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)})\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT})(\tilde{p}\sigma_C^{n_0} + (1-\tilde{p})\,\overline{\sigma}_C^{n_0})\big\|_1 =\\
&\big\|(1-p)\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\rho_C^{n_0}) + p\overline{\rho}_C^{n_0} -\\
&\tilde{p}\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\sigma_C^{n_0}) - (1-\tilde{p})\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\overline{\sigma}_C^{n_0})\big\|_1 \leq\\
&\big\|(1-p)\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\rho_C^{n_0}) + p\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\overline{\rho}_C^{n_0}) -\\
&\tilde{p}\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\sigma_C^{n_0}) - (1-\tilde{p})\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\overline{\sigma}_C^{n_0})\big\|_1\\
&+ \big\|p\overline{\rho}_C^{n_0} - p\,\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\overline{\rho}_C^{n_0})\big\|_1 \leq\\
&\big\|[\mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}]((1-p)\rho_C^{n_0} + p\overline{\rho}_C^{n_0} - \tilde{p}\sigma_C^{n_0} - (1-\tilde{p})\overline{\sigma}_C^{n_0})\big\|_1 +\\
&p\big\|\overline{\rho}_C^{n_0} - \mathrm{tr}_T(\mathcal{I}_C \otimes \mathcal{T}^{(0)}_{T\to T})\tilde{\mathcal{M}}(\overline{\rho}_C^{n_0})\big\|_1 \leq\\
&\big\|(1-p)\rho_C^{n_0} + p\overline{\rho}_C^{n_0} - \tilde{p}\sigma_C^{n_0} - (1-\tilde{p})\overline{\sigma}_C^{n_0}\big\|_1 + \varepsilon^2 \leq\\
&(n_0-1)\varepsilon^2 + \varepsilon^2 \leq n_0\varepsilon^2
\end{aligned}
$$

using induction hypothesis $\left(\big\|q\rho_C^{n_0} + p\overline{\rho}_C^{n_0} - \tilde{p}\sigma_C^{n_0} - \tilde{q}\overline{\sigma}_C^{n_0}\big\|_1 \leq (n_0-1)\varepsilon^2\right)$ and $p \leq \varepsilon$ by Ass. 3.

Hence proof by induction is finished.

To show that Eq. (A.6) holds, we could again use mathematical induction.

For the clocks and notation introduced in this section, following holds:

$$
\Big\|\mathrm{tr}_T \prod_{i=1}^{kn_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})(\rho_C^0 \otimes |0\rangle\langle 0|_T) - \left(\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\right)^{kn_0}(\rho_C^0)\Big\|_1 \leq k(n_0-1)\varepsilon^2 \tag{A.15}
$$

Hence, the reset of the ticks registers does not have to occur at every step, and the state of the clockwork will be only negligibly affected. This also shows that the clocks that are ε-continuous with small ε are more robust - changes in tick registers affect the clockwork state less.

1. Base of induction: $k = 1$ - this follows directly from Lemma 4 and Assumption 4.

2. Induction step: Assume Eq. (A.15) holds for k. Label $\rho_C^n = (\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}})^n(\rho_C)$. Then

$$\Big\|\mathrm{tr}_T[\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}\prod_{i=1}^{kn_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})(\rho^0_C\otimes|0\rangle\langle 0|_T)] \tag{A.16}$$
$$-\ \mathrm{tr}_T\big(\mathcal{T}^{(0)}\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\big)^{(k+1)n_0}(\rho^0_C)\Big\|_1\leq$$
$$\Big\|\mathrm{tr}_T\,\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}\prod_{i=1}^{(k-1)n_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})[\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}(\rho^0_C\otimes|0\rangle\langle 0|_T)]-$$
$$\mathrm{tr}_T\,\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}\prod_{i=1}^{(k-1)n_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})[(\mathcal{T}^{(0)}\tilde{\mathcal{M}})(\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}})^{n_0-1}(\rho^0_C)]\Big\|_1$$
$$+\Big\|\mathrm{tr}_T\,\mathcal{T}^{(0)}\mathcal{M}(T^{(1)}\mathcal{M})^{n_0-1}\prod_{i=1}^{(k-1)n_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})[(\mathcal{T}^{(0)}\tilde{\mathcal{M}})(\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}})^{n_0-1}(\rho^0_C)]-$$
$$\mathrm{tr}_T\big(\mathcal{T}^{(0)}\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\big)^{kn_0}(\rho^{n_0}_C)\Big\|_1\leq$$
$$\big\|\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}(\rho^0_C\otimes|0\rangle\langle 0|_T)-(\mathcal{T}^{(0)}\tilde{\mathcal{M}})^{n_0}(\rho^0_C)\big\|_1+(n_0-1)k\varepsilon^2\leq$$
$$(n_0-1)\varepsilon^2+k(n_0-1)\varepsilon^2=(k+1)(n_0-1)\varepsilon^2$$

using property of CPTP maps ($\mathrm{tr}_T\,\mathcal{T}^{(0)}\mathcal{M}(\mathcal{T}^{(1)}\mathcal{M})^{n_0-1}\prod_{i=1}^{(k-1)n_0}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M})$ in particular), basis of induction (Lemma 4) and induction hypothesis with $\rho^0_C:=\rho^{n_0}_C$. Hence proof by induction is concluded.

Now to obtain Eq. (A.6) we need to take $k=\frac{1}{\varepsilon n_0}$ with $\varepsilon>0$:

$$\Big\|\mathrm{tr}_T\prod_{i=1}^{\lfloor\frac{1}{\varepsilon}\rfloor}(\mathcal{T}^{(s_i)}_{T\to T}\circ\mathcal{M}_{CT\to CT})(\rho^0_C\otimes|0\rangle\langle 0|_T)-\big(\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\big)^{\lfloor\frac{1}{\varepsilon}\rfloor}(\rho^0_C)\Big\|_1\leq\varepsilon$$

Letting $\varepsilon\to 0$ we get that the 1-norm on the LHS becomes 0. Finally, using the properties of the norm we obtain:

$$\lim_{\varepsilon\to 0}\mathrm{tr}_T\prod_{i=1}^{\lfloor\frac{1}{\varepsilon}\rfloor}(\mathcal{T}^{(s_i)}_{T\to T}\mathcal{M}_{CT\to CT})(\rho^0_C\otimes|0\rangle\langle 0|_T)=\lim_{\varepsilon\to 0}\big(\mathrm{tr}_T\,\mathcal{T}^{(0)}\tilde{\mathcal{M}}^{\varepsilon}_{C\to CT}\big)^{\lfloor\frac{1}{\varepsilon}\rfloor}(\rho^0_C)=\tilde{\rho}_C$$

which finishes the proof.

□

Appendix B

Entropic uncertainty relations for the time measurements

In this Chapter provide an idea for the investigation of the limits of the time measurements in quantum mechanics by looking at the entropic lower bounds for the measurements performed on the quantum clockworks. In the following, we will refer to the clockwork from Def 1 as a clock.

Assume that we can construct a quantum clock that we wish to use for measuring time with a certain precision, that we will call a *local clock* C_L. From the postulates of quantum mechanics it is known that after looking at the clock, *i.e.* measuring time, the state of the clock will be disturbed. In order to know how accurate our clock can be, we introduce another, reference clock, that we can compare the time on our local clock to. We will not measure the reference clock, so it remains undisturbed during the process. It is only used as a kind of a 'time counter' on which time flows so we can compare the time measured on other clock to this reference (compare this set up to the one in Section 5.2). We are interested in how much the measurements on the local clock affect its accuracy, and in particular, how accurate can the clock be after the second measurement, considering disturbance on the clock caused by the first measurement. To quantify this, we attempt to calculate the entropies of the reference clock conditioned on the outcomes of the two measurements we perform on our local clock. We would expect a very low entropy for the local clocks that approximate time as shown on the reference clock well, but because of the properties of quantum clocks we would expect the sum to be greater than zero. A lower bound on the sum of these entropies could represent a form of the time uncertainty principle for the measurements on quantum clocks of a similar construction. Note that one can also use mutual information, and demand from it to be high for well synchronised clocks.

One can the first measurement outcome to be $k = 1$ and calculate required entropies depending on the outcome l of the second measurement (this is because the total entropy will depend on $l - k$ hence we can fix k and vary l).

B. ENTROPIC UNCERTAINTY RELATIONS FOR THE TIME MEASUREMENTS

As a final result we would like to obtain a following type of equation:

$$H(C_R|C_1) + H(C_R|C2) \geq c(d) \tag{B.1}$$

where C_R is the observable describing the reference clock , C_1 and C_2 are classical results of measurements on the local clock and d is the dimension of both clock systems. $c(d)$ is the unknown function of the system dimension that we would like to find a constant lower bound on (note that this bound should be independent of the external time, that one can average over in order to eliminate it from the equations).

We could use the following Hamiltonian:

$$H = \frac{\log(\sum_{i=1}^{d} |i+1\rangle \langle i|)}{-i\tau} \tag{B.2}$$

with τ representing the time in which clock passes from one clock state to an orthogonal one. We will take $\tau = 1$ wlog.

As the joint state of both systems at the start one could take a maximally entangled state $\psi = \frac{1}{\sqrt{d}} \sum_{1}^{d} |i\rangle \otimes |i\rangle$. We have made this choice because we want clocks to be synchronised prior to the first measurement, and we assume that entanglement is enough to guarantee their starting synchronisation. Note that another choice is to have a product starting state of clocks that were synchronised using QCS protocol [5] described in Sec. 3.5.

We perform a measurement C_1 on the local clock at some starting time (taken to be 0 without loss of generality), and then we measure again, with an observable C_2, after a certain time interval (that we will average over as it represents an external time we are not interested in). One of the basic examples is measuring in the clock basis (in our case also computational basis) $\{|1\rangle, |2\rangle, ..., |d\rangle\}$. Then the first measurement on the local clock collapses the joint state, and the entropy $H(C_R|C_1) = 0$. For a more general first measurement, we consider the POVM:

$$\lambda_j = \sum_{i=1}^{d} \omega_i^j |i\rangle \langle i|, j \in \{1, ..., d\}$$

where ω_i^j are weights related to the possible clock states. We need λ_j's to sum up to the identity, $\sum_{j=1}^{d} \sqrt{\lambda_j}^* \sqrt{\lambda_j} = \mathcal{I}$. One obtains

$$\sum_i (\sum_j |\omega_i^j| - 1) |i\rangle\langle i| = 0$$

and hence we require $\sum_j |\omega_i^j| = 1, \forall i$.

As an example, one can consider Gaussian and Poisson distribution for the weights ω_i^j. In particular, the distribution for ω^j is either normalised version of $\mathcal{N}(j, \sigma)$, normally distributed

with mean j and variance σ (that we vary depending on how peaked distribution we want), or normalised version of $Poiss(j)$ - Poisson distribution with parameter j.

The first measurement process on the local clock gives us outcome k with probability $p_k = \mathrm{tr}\left((1 \otimes \lambda_k)\rho_0\right)$ and a new total state after the measurement ('collapsed state') becomes:

$$\rho'_k = (1 \otimes \sqrt{\lambda_k})\rho_0(1 \otimes \sqrt{\lambda_k}^*)$$

The normalised state is

$$\rho'_k = \frac{(1 \otimes \sqrt{\lambda_k})\rho_0(1 \otimes \sqrt{\lambda_k}^*)}{\mathrm{tr}\left((1 \otimes \sqrt{\lambda_k})\rho_0(1 \otimes \sqrt{\lambda_k}^*)\right)}$$

where $\rho_0 = |\psi\rangle\langle\psi|$ is the joint state before the measurement.

The second measurement is performed in the computational basis, after evolving the state. The probability that after the second measurement we obtain outcome j on the reference clock and outcome l on the local clock, conditioned on having obtained k on the local clock after the first measurement, with the Schrödinger time t elapsed between the measurements, is:

$$p(j,l;t) = \langle j,l| (U_t^R \otimes U_t^L)\rho'_k(U_t^{R,*} \otimes U_t^{L,*}) |j,l\rangle$$

Here $U_t = e^{-it(H_R \otimes \mathcal{I}_L + \mathcal{I}_R \otimes H_L)} = U_t^R \otimes U_t^L$, and ψ is the starting entangled joint state of the clocks. Note that $H_R = H_L = H$.

One should take large enough number of states, so that the periodicity of the clock is greater then the 'external' time we average our results over.

To calculate the entropy $H(C_R|C_2)$ independent from some 'external' time t showing as a parameter in the time evolution, we can average the probability p over many different times t from the Schrödinger equation. Then we calculate the conditional probability, conditioned on the outcome l of the second measurement on the master clock, and finally calculate the Shannon entropy of the obtained conditional probability over all possible outcomes j on the reference clock (all given fixed first measurement outcome k).

We will not present any further results here, since there is more analysis needed to obtain concrete conclusions. But what we could see from the numerics done, is that the sum of the entropies seem to be increasing with the clock dimension, and it also highly depends on the difference of the states around which the two measurements were peaked.

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