Quantitative Aspects of Lagrangian Cobordism Theory and Mather-Floer Theory

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Quantitative aspects of Lagrangian cobordism theory and Mather-Floer Theory

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Careful attention to one thing often proves superior to genius

-Cicero

Til Mor
Para Helgui

Zusammenfassung

Diese Doktorarbeit besteht aus den Beiträgen des Autors zu den Gebieten der symplektischen Topologie und der dynamischen Systeme, welche dieser über die letzten 3-4 Jahre erarbeitet hat. Die Arbeit trägt einerseits zur quantitativen Theorie von Lagrange-Kobordismen bei, und anderseits versucht sie eine symplektische Mather-Floer Theorie aufzubauen. Alle Ergebnisse finden in einer symplektischen Mannigfaltigkeit (M^{2n}, ω) statt.

Der erste Teil dieser Arbeit (Kapitel 2-4) behandelt die Idee "die Grösse" eines Lagrange-Kobordismus zu messen. Insbesondere wird untersucht auf welche Weise diese "Grösse" die Topologie des Lagrange-Kobordismus einschränkt. Ein Lagrange-Kobordismus V zwischen Lagrange Untermannigfaltigkeiten in (M, ω) ist eine Lagrange Untermannigfaltigkeit in $(\mathbb{R}^2 \times M, \omega_{\mathbb{R}^2} \oplus \omega)$. "Die Grösse von V" wird durch den sogenannten *Schatten* von V (geschrieben $\mathcal{S}(V) \in [0, \infty)$) gemessen. Intuitiv misst $\mathcal{S}(V)$ die Fläche der Projektion von $V \subset \mathbb{R}^2 \times M$ in $\mathbb{R}^2$. Cornea-Shelukhin [32] (siehe auch [12]) haben vor kurzem gezeigt, dass das "minimieren von $\mathcal{S}(V)$ über passende Lagrange-Kobordismen mit fixiertem Rand" eine "Lagrange-Kobordismus Metrik" ergibt.

Das erste Kapitel (Kapitel 2) wurde als [21] veröffentlicht. Hauptergebnis dieses Kapitels ist eine Ungleichung, welche die Lagrange-Kobordismus Metrik von Cornea-Shelukhin von unten durch eine Lagrange *Spektral* Metrik abschätzt. Die Spektral Metrik ist mittels Floer-homologischen Lagrange spektral Invarianten konstruiert [58]. Eine der Hauptanwendungen dieses Kapitels ist die Folge von Ungleichungen (2.1), welche die Kobordismus Metrik zwischen der Spektral Metrik und der Chekanov-Hofer Lagrange Metrik einschliesst. Kapitel 3, welches als [20] veröffentlicht wurde, zeigt, dass die (exakte) Lagrange-Kobordismus Klasse der Null-Lagrange Untermannigfaltigkeit eines Kotangentialbündels unendlichen Durchmesser (bezüglich der Kobordismus Metrik) hat.

Das letzte Kapitel des ersten Teils der Arbeit (Kapitel 4), welches als [18] veröffentlicht wurde, untersucht Einschränkungen der Topologie eines Lagrange-Kobordismus und dessen Rand, wenn sie sehr klein (bezüglich $\mathcal{S}(V)$) ist. Das Hauptergebnis dieses Kapitels ist, dass wir die singuläre Homologie von *kleinen* Lagrange-Kobordismen bestimmen, welche auf Grund von Biran-Corneas Ergebnissen [11], Operationen auf einer Version der derivierten Fukaya Kategorie induzieren.

Der zweite Teil dieser Arbeit (Kapitel 5), welcher als [19] veröffentlicht wurde, versucht Ideen der Aubry-Mather Theorie mittels Werkzeugen der symplektischen Topologie zu verallgemeinern. Das Hauptergebnis dieses Kapitels zeigt die Existenz von Borel Wahrscheinlichkeitsmassen, welche unter dem Fluss einer Hamiltonschen Funktion invariant sind, und deren Rotationsvektoren mittels dem Subdifferential einer α-Funktion vorgegeben sind. Ergebnisse von Viterbo [100] und Monzner-Vichery-Zapolsky [62] zeigen, dass diese Funktion Mather's α-Funktion ist, falls die Hamiltonsche Funktion Tonelli auf dem Kotangentialbündel einer geschlossene Mannigfaltigkeit ist. In Kapitel 5 befassen wir uns mit der Stabilität dieser verallgemeinerter "Mather Masse". Darüber hinaus wird, im Falle wo die

Lagrange Untermannigfaltigkeit "displaceable" ist, ein Zugang entwickelt um die Existenz invarianter Masse mit vorgeschriebenen Rotationsvektoren zu zeigen

Abstract

This thesis contains the author's contributions to the field of symplectic topology and dynamical systems, obtained over the last 3-4 years. These contributions concern two distinct topics: quantitative aspects of Lagrangian cobordism theory and a symplectic Mather-Floer theory. The stage on which all results play out is a symplectic manifold. We will denote our symplectic manifold by (M^{2n}, ω).

The first part of the thesis (chapters 2-4) is concerned with the idea of measuring "the size" of a Lagrangian cobordism, and understanding what kind of restrictions its size puts on the topology of the cobordism and its boundary components. A Lagrangian cobordism V, connecting Lagrangian submanifolds in (M, ω), is a Lagrangian submanifold in $(\mathbb{R}^2 \times M, \omega_{\mathbb{R}^2} \oplus \omega)$. The main interesting quantity which one can assign to a Lagrangian cobordism $V \subset \mathbb{R}^2 \times M$ is its *shadow* (denoted by $\mathcal{S}(V) \in [0, \infty)$), which intuitively measures the area of the projection of V to the $\mathbb{R}^2$-plane. Cornea-Shelukhin [32] (see also [12]) recently showed that "minimizing $\mathcal{S}(V)$ over suitable Lagrangian cobordisms with fixed boundary components" gives rise to a "Lagrangian cobordism metric".

The first chapter (chapter 2) appeared as [21]. It studies the Lagrangian cobordism metric introduced by Cornea-Shelukhin by bounding it from below with a Lagrangian *spectral* metric on monotone Lagrangian cobordism classes, constructed using Lagrangian spectral invariants [58]. The main result of this chapter is the sequence of inequalities (2.1) which "sandwiches" the cobordism metric in between the spectral metric and the Lagrangian Chekanov-Hofer metric [28]. Chapter 3, which is based on the paper [20], checks that the (exact) Lagrangian cobordism class of the zero-section in a cotangent bundle of a closed manifold has infinite diameter with respect to the Lagrangian cobordism metric. The last chapter of the first part of the thesis (chapter 4), which appeared as [18], studies restrictions on the topology of a Lagrangian cobordism V when it is very "small" (in the sense of $\mathcal{S}(V)$). The main result of this chapter is to determine the homology of the *small* Lagrangian cobordisms which, by Biran-Cornea's results [11], are known to induce operations on a suitable version of the derived Fukaya category.

The second part of the thesis (chapter 5), which is appeared as [19], attempts to generalize ideas from Aubry Mather theory using methods coming from symplectic topology. The main result detects the existence of Borel probability measures which are invariant under a Hamiltonian flow ϕ_H and whose rotation vector is prescribed by the subdifferential of an "α-function". By results due to Viterbo [100] and Monzner-Vichery-Zapolsky [62] it is known that this function coincides with Mather's α-function when the Hamiltonian is Tonelli on the cotangent bundle of a closed manifold. Moreover, chapter 5 also studies the stability of the support of the constructed "Mather-measures" as well as an approach to constructing measures with "prescribed" rotation vector in settings where the Lagrangian submanifold can be displaced by a Hamiltonian diffeomorphism.

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Contents

1. Introduction

A symplectic manifold (M^{2n}, ω) is a $2n$-dimensional manifold M equipped with a closed non-degenerate 2-form $\omega \in \Omega^2(M)$ (i.e. $d\omega = 0$ and ω^n is a volume form). A symplectic manifold is a mathematical formalization of the concept of a *phase space* in classical mechanics. The dynamics determined by a Hamiltonian (or energy function) $H \in C^\infty(S^1 \times M)$ is the flow $\phi_H = \{\phi_H^t\}_{t\in\mathbb{R}}$ generated by the symplectic gradient X_{H_t}. X_{H_t} is characterized by the equation

$$\iota_{X_{H_t}}\omega = -dH_t.$$

Modern symplectic topology/geometry studies (among other things) topological and intersection phenomena exhibited by n-dimensional submanifolds $L \subset (M, \omega)$ satisfying $\omega|_L \equiv 0$. Such submanifolds are called *Lagrangian.* There are many good reasons why one should care about Lagrangian submanifolds. From the point of view of this thesis we point out the following two:

Example A The Lagrangian suspension construction: Let $L \subset (M, \omega)$ be a Lagrangian and fix $H \in C^\infty([0,1] \times M)$. Then the embedding

$$[0,1] \times L \ni (t,x) \mapsto (t, H_t(\phi_H^t(x)), \phi_H^t(x)) \in [0,1] \times \mathbb{R} \times M$$

is a Lagrange embedding into $[0,1] \times \mathbb{R} \times M = T^*[0,1] \times M$, which encodes both the path $t \mapsto \phi_H^t(L)$ as well as information about H along the path $t \mapsto \phi_H^t(L)$.

Example B Herman's theorem [46]: Fix a Hamiltonian $H \in C^\infty(M)$ and an energy-level $c \in \mathbb{R}$. Suppose $L \subset \{H = c\}$ is a ϕ_H-invariant embedded torus on which the dynamics is quasi-periodic and uniquely ergodic. Then L is Lagrangian.

These examples exhibit two different aspects of the way in which Lagrangian submanifolds encode information about Hamiltonian dynamics. Generalizing Example A leads naturally to the notion of a *Lagrangian cobordism.* The first part (chapters 2-4) of this thesis studies quantitative and topological aspects of Lagrangian cobordisms. Example B indicates how important Lagrangian submanifolds are for the dynamics of H and how, if one wants to detect invariant sets for Hamiltonian systems, one should study Lagrangian submanifolds. The second part (chapter 5) of this thesis studies invariant sets of Hamiltonian systems, detected using pseudo holomorphic curve techniques. In the rest of this introduction we outline the main results of the thesis. The validity of the results depend on various technical assumptions and definitions about the objects considered. For clarity and purposes of exposition, we mostly suppress such technicalities here, and refer to the chapters below for further details.

1.1 Quantitative aspects of Lagrangian cobordisms

Chapters 2-4, which are based on the papers [18], [21] and [20], explore quantitative phenomena exhibited by Lagrangian cobordisms. At its early stages, symplectic topology was mostly developed as a *qualitative* theory, whose spirit was in many ways similar to that of classical differential topology. Gromov's non-squeezing theorem [41], which gave rise to the concept of the Gromov radius of a symplectic manifold, seems to be the first indication that symplectic topology has a *quantitative* side to it. Later such phenomena were formalized by Ekeland-Hofer's [33], [34] notion of a symplectic capacity. Another aspect to this quantitative story of symplectic topology arose when Hofer [48] asked the question "How much energy is required to displace a set from itself?". Exploring answers to this question led to the notion of Hofer's metric on the group of Hamiltonian diffeomorphisms as well as the notion of displacement energy of a subset of a symplectic manifold. Deep insight, first by Polterovich [75] and later Chekanov [27], shows that the notion of displacement energy really gives rise to *quantitative* analogues of *qualitative* phenomena conjectured by Arnold'.

1.1.1 Lagrangian cobordisms and spectral numbers

Consider a symplectic manifold (M, ω) and equip the plane $\mathbb{R}^2(x, y)$ with the standard symplectic structure $\omega_{\mathbb{R}^2} = dx \wedge dy$. A Lagrangian cobordism $V : (L_i')_i \rightsquigarrow (L_j)_j$ connecting two ordered tuples $(L_j)_{j=1}^m, (L_i')_{i=1}^{m'}$ of Lagrangians in (M, ω), is a properly embedded Lagrangian submanifold

$$V \subset (\tilde{M}, \tilde{\omega}) := (\mathbb{R}^2 \times M, \omega_{\mathbb{R}^2} \oplus \omega)$$

whose positive (respectively negative) asymptotic end in the x-direction is given as a product of disjoint intervals in the plane with the L_js (respectively L_i's). The main quantity one can associate to a Lagrangian cobordism is its *shadow* $\mathcal{S}(V) \in [0, \infty)$, which can intuitively be described as the area of the subset of $\mathbb{R}^2$ consisting of all bounded connected components of $\mathbb{R}^2 \backslash \pi(V)$, with $\pi : \tilde{M} \to \mathbb{R}^2$ the canonical projection. The shadow of a Lagrangian cobordism was first introduced in [32]. An indication that this quantity contains meaningful information is seen in Example A above, where $\mathcal{S}(V)$ equals the Hofer norm of H along the path $t \mapsto \phi_H^t(L)$. The main question one can ask is:

Given two Lagrangians $L, L' \subset (M, \omega)$, how small (measured in shadow) can a Lagrangian cobordism $V : L' \rightsquigarrow L$ (satisfying a number of "good" conditions) be? (Q)

Analogous to the Lagrangian version of the Hofer metric this leads to the quantity

$$d_c(L, L') := \inf\{\mathcal{S}(V) \mid V : L' \rightsquigarrow L \text{ satisfies "good conditions"}\}. \tag{1.1}$$

One of the main results of [32] is that, if one chooses a monotonicity condition as the "good condition," then $d_s(L, L') > 0$ if $L \neq L'$, so (1.1) defines a non-degenerate metric. If L is Hamiltonian isotopic to L' then Example A above guarantees that $d_c(L, L') \leq d_H(L, L')$, where d_H denotes the Lagrangian Hofer metric [28].

Chapter 2, which appeared in [21], presents results concerning how to estimate the answer to (Q) using Lagrangian spectral invariants. The concept of Lagrangian spectral invariants goes back to Viterbo's [102] and Oh's [71], [72], but the version we use is due to Leclercq-Zapolsky [58]. Given a monotone Lagrangian $L \subset (M, \omega)$ they constructed a *spectral invariant function*

$$l_L : QH_*(L; \mathbb{Z}_2) \times \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R} \cup \{-\infty\}.$$

Here $QH_*(L;\mathbb{Z}_2)$ denotes the quantum homology of L and $\widetilde{\mathrm{Ham}}(M,\omega)$ denotes the universal cover of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ on (M,ω). Throughout this section we will only consider monotone Lagrangians with $QH_*(L;\mathbb{Z}_2) \neq 0$, so in particular the unity $[L] \in QH_*(L;\mathbb{Z}_2)$ with respect to the ring structure is $\neq 0$. By a result due to Biran-Cornea [10], a monotone Lagrangian cobordism $V : L' \rightsquigarrow L$ induces a ring isomorphism

$$\Phi_V : QH_*(L;\mathbb{Z}_2) \stackrel{\cong}{\to} QH(L';\mathbb{Z}_2).$$

Theorem 1.1.1 (Theorem 2.2.2 on page 10)**.** *Let $L, L' \subset (M,\omega)$ be two monotone Lagrangians and let $V : L' \rightsquigarrow L$ be a monotone Lagrangian cobordism. Then*

$$|l_L(\alpha,\phi) - l_{L'}(\Phi_V(\alpha),\phi)| \leq \mathcal{S}(V) \tag{1.2}$$

for all $\alpha \in QH_*(L)\backslash\{0\}$ and all $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$.

The main upshot of this result is that Lagrangian spectral invariants can be used to estimate the size of (monotone) Lagrangian cobordisms. A direct consequence of the fact that Φ_V preserves the ring structure is that

$$|l_L([L],\phi) - l_{L'}([L'],\phi)| \leq \mathcal{S}(V). \tag{1.3}$$

for all monotone $V : L' \rightsquigarrow L$ and all $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$. Define for every pair of monotone Lagrangians $L, L' \subset (M,\omega)$ the quantity

$$d_s(L,L') := \sup\{|l_L([L],\phi) - l_{L'}([L'],\phi)| \mid \phi \in \widetilde{\mathrm{Ham}}(M,\omega)\},$$

where we use the convention that the supremum over $\emptyset$ equals $+\infty$.

Proposition 1.1.2 (Proposition 2.2.6 on page 11)**.** *d_s defines a non-degenerate metric on the space of all monotone Lagrangian submanifolds of (M,ω).*

As a consequence of (1.3) we in particular have the estimate

$$d_s(L,L') \leq d_c(L,L') \leq d_H(L,L') \quad \forall\ L, L' \text{ monotone and Hamiltonian isotopic},$$

which in certain cases allows one to compute $d_c(L,L')$. Another consequence of (1.3) is that the asymptotic Lagrangian spectral invariant $\sigma_L : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$, defined by

$$\sigma_L(\phi) = \lim_{k\to\infty} \frac{l_L([L],\phi^k)}{k}$$

is an honest Lagrangian cobordism invariant, which can distinguish different cobordism classes.

Theorem 1.1.3 (Theorem 2.2.9 on page 13)**.** *Let $L, L' \subset (M,\omega)$ be monotone Lagrangians with $QH_*(L;\mathbb{Z}_2) \neq 0$. If there exists a monotone Lagrangian cobordism $V : L' \rightsquigarrow L$ then $QH_*(L';\mathbb{Z}_2) \neq 0$, so $\sigma_{L'} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ is well-defined and $\sigma_L = \sigma_{L'}$.*

This theorem implies a cute intersection result originally due to Biran-Cornea (see page 14).

1.1.2 Expanding exact Lagrangian cobordism classes

Consider an exact symplectic manifold $(M, d\lambda)$ which is convex at infinity and whose Liouville flow $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ is complete. Recall that ψ is generated by the vectorfield Z on M, characterized by

$$\iota_Z\omega = \lambda.$$

The Lagrangian $L \subset (M, d\lambda)$ is said to be *exact* if $\lambda|_L = df$ for some $f \in C^\infty(L)$. In this setting $(\tilde{M}, \tilde{\omega})$ is also an exact symplectic manifold with $d(\lambda_{\mathbb{R}^2} \oplus \lambda) = \tilde{\omega}$, so it makes sense to require that a Lagrangian cobordism $V \subset (\tilde{M}, \tilde{\omega})$ be exact (with respect to $\lambda_{\mathbb{R}^2} \oplus \lambda$). We denote by $\mathcal{L}(L)$ the space of Lagrangian submanifolds of $(M, d\lambda)$ which are exact and exact Lagrangian cobordant to L. It makes sense to consider a version of d_c (see above) on $\mathcal{L}(L)$

$$d_c(L', L'') := \inf\{\mathcal{S}(V) \mid V : L' \rightsquigarrow L \text{ is exact}\} \quad \forall\ L', L'' \in \mathcal{L}(L).$$

In this setting Cornea-Shelukhin's result [32] implies that d_c is a non-degenerate metric on $\mathcal{L}(L)$. Chapter 3, which appear as the note [20], discusses the proof of

Theorem 1.1.4 (Theorem 3.0.1 on page 29)**.** *Let $L \subset (M, d\lambda)$ be a closed exact Lagrangian submanifold. Then $\{\psi_t\}_{t\in\mathbb{R}}$ induces a flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on $(\mathcal{L}(L), d_c)$ via $\Psi_t(L') := \psi_t(L')$. More precisely, Ψ defines a continuous $\mathbb{R}$-action on $\mathcal{L}(L)$ and every $\Psi_t : \mathcal{L}(L) \to \mathcal{L}(L)$ satisfies*

$$d_c(\Psi_t(L'), \Psi_t(L'')) = e^t d_c(L', L'') \quad \forall\ L', L'' \in \mathcal{L}(L).$$

In particular the metric space $(\mathcal{L}(L), d_c)$ has infinite diameter.

1.1.3 Topology of (small) Lagrangian cobordism classes

For two Lagrangians $L, L' \subset (M, \omega)$, consider again the distance $d_c(L, L')$ defined by (1.1), with the "good condition" chosen such that d_c is a non-degenerate metric [32]. In order for this quantity to be interesting, one would hope that L and L' share similar properties if $d_c(L, L')$ is sufficiently small. Perhaps, one might even hope that they would have to be Hamiltonian isotopic. $d_c(L, L')$ being small means exactly that L and L' are the boundary components of a suitable Lagrangian cobordism whose shadow is small. Section 4, which appeared as [18], explores what one can say about the topology of so-called *small* Lagrangian cobordisms and their boundary. The main "moral" of the story is that the topology of small Lagrangian cobordisms to a large extend is determined by its boundary. Parts of these results generalize ideas due to Biran-Cornea [10] and Suarez [94].

Apart from the shadow of a Lagrangian cobordism $V : (L'_i)_i \rightsquigarrow (L_j)_j$, there is another (positive) quantity which can be associated to V, namely the *bubbling threshold* $A(\tilde{M}, V) \in (0, \infty]$. Our understanding is that this quantity, for a closed Lagrangian submanifold, first appear in Chekanov's [27]. However, one can also see hints of this quantity in Polterovich's [75]. For an $\tilde{\omega}$-compatible almost complex structure $\tilde{J}$ on $\tilde{M}$ which is *standard at* ∞[1] we denote by $A(\tilde{M}, V, \tilde{J})$ the infimum over all $\tilde{\omega}$-areas of non-constant $\tilde{J}$-holomorphic spheres in $(\tilde{M}, \tilde{\omega})$ and non-constant $\tilde{J}$-holomorphic disks in $(\tilde{M}, \tilde{\omega})$ with boundary on V. Under suitable assumptions on (M, ω), standard compactness arguments show that $A(\tilde{M}, V, \tilde{J}) > 0$, so

$$A(\tilde{M}, V) := \sup_{\tilde{J}} A(\tilde{M}, V, \tilde{J}) > 0.$$

Here the supremum runs over all $\tilde{J}$ as above. We then say that $V : (L'_i)_i \rightsquigarrow (L_j)_j$ is *small* if it satisfies the inequality

$$\mathcal{S}(V) < A(\tilde{M}, V).$$

[1] See Section 4.3 on page 42 for a precise definition of this term.

A key novelty of this condition is that it places no a priori conditions on the boundary of V. E.g. for two arbitrary closed Lagrangians $L, L' \subset (M, \omega)$ it makes sense to ask whether or not a small $V : L' \rightsquigarrow L$ exists. The following result is in some sense the cobordism analogue of the main theorem in Chekanov's [27].

Theorem 1.1.5 (Theorem 4.2.2 on page 34)**.** *Let $(L_i')_{i=1}^{m'}, (L_j)_{j=1}^{m}$ be two tuples of Lagrangians in (M, ω) such that $L_i \pitchfork L_j$ and $L_i' \pitchfork L_j'$ for all $i \neq j$. If $V : (L_i')_i \rightsquigarrow (L_j)_j$ is a small Lagrangian cobordism which is spin, then*

$$\dim_{\mathbb{F}} H_*(V, \partial_- V; \mathbb{F}) \leq \sum_{1 \leq i < j \leq m} \#(L_i \cap L_j) + \sum_{1 \leq i < j \leq m'} \#(L_i' \cap L_j') \tag{1.4}$$

for all fields $\mathbb{F}$. If V is not spin, then (1.1) holds with $\mathbb{F} = \mathbb{Z}_2$.

This result has a fairly long list of applications. Perhaps the most important applications are the following ones (for the full list we refer to Sections 4.2.1 and 4.2.2 on pages 34 and 36 respectively).

Corollary 1.1.6 (Corollary 4.2.7 on page 35)**.** *Let $L, L' \in \mathcal{L}_1$ and suppose at least one of them is spin. If there exists a small Lagrangian cobordism $V : L' \rightsquigarrow L$ then $L \cap L' \neq \emptyset$. Moreover, if $L \pitchfork L'$ then*

$$\dim_{\mathbb{F}} H_*(L; \mathbb{F}) \leq \#(L \cap L').$$

for every field $\mathbb{F}$. Of course, if neither L nor L' is spin then the same conclusion holds for $\mathbb{F} = \mathbb{Z}_2$.

Corollary 1.1.7 (Corollary 4.2.9 on page 35)**.** *Let $L, L' \in \mathcal{L}$ be simply connected and suppose there exists a small, simply connected Lagrangian cobordism $V : L' \rightsquigarrow L$. If $\dim(V) \geq 6$ then V is diffeomorphic to $[0,1] \times L$, so in particular L is diffeomorphic to L'.*

These two results lead to the natural question, whether or not it is true that all small Lagrangian cobordisms $V : L' \rightsquigarrow L$ are images under some $\psi \in \mathrm{Symp}(\widetilde{M}, \tilde{\omega})$ of a Lagrangian suspension. I.e. whether or not a small Lagrangian cobordism is really the same as an exact Lagrangian isotopy.

Corollary 1.1.8 (Corollary 4.2.15 on page 38)**.** *Let $L \subset (M, \omega)$ be a Lagrangian and let $(L_i)_{i=1}^m$ be an m-tuple of Lagrangians in (M, ω) satisfying the following conditions:*

a) All singular points of the immersion $\sqcup_i L_i \hookrightarrow M$ induced by the inclusion are transverse and double.

b) The subset $(\cup_i L_i) \subset M$ is connected.

c) The intersection graph of the $(L_i)_i$ is a graph.[2]

d) L_i is spin for every $i = 1, \ldots, m$.

Then every small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$ satisfies

$$H_*(V; \mathbb{Z}) \cong H_*(\widetilde{V}; \mathbb{Z}),$$

where $\widetilde{V} : \widetilde{\#}_i L_i \rightsquigarrow (L_i)_i$ is a Lagrangian cobordism originating from surgering the $(L_i)_i$ to $\widetilde{\#}_i L_i$ (see Example 4.2.17 on page 38). Moreover,

$$H_*(L'; \mathbb{Z}) \cong H_*(\widetilde{\#}_i L_i; \mathbb{Z}).$$

If some L_i isn't spin, then the same conclusions hold with coefficients in $\mathbb{Z}_2$.

[2] The set of vertices of the intersection graph of $(L_i)_{i=1}^m$ is by definition $\{L_1, \ldots, L_m\}$. The set of edges is defined as follows: There is a bijection from the set of (unoriented) edges connecting L_i to L_j to the set $L_i \cap L_j$.

Note that there are many examples of tuples $(L_i)_i$ which meet these conditions. The last result leads to the natural question, is it true that all small Lagrangian cobordisms of the type $V : L' \rightsquigarrow (L_i)_i$, which satisfy reasonable additional conditions, arise as surgery cobordisms.

1.2 Mather theory and symplectic rigidity

Chapter 5, which appeared in [19], studies invariant sets of (autonomous) Hamiltonian systems using tools coming from symplectic topology. The first part of the theory developed uses the Leclercq-Zapolsky Lagrangian spectral invariant discussed in Section 1.1.1 above and builds heavily on the philosophy of Viterbo's homogenization paper [100], as well as Monzner-Vichery-Zapolsky's [62] and Vichery's [99]. Given a monotone Lagrangian $L \subset (M, \omega)$ with $QH_*(L; \mathbb{Z}_2) \neq 0$, we will here denote by $l_L : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ the Leclercq-Zapolsky spectral invariant associated to the unity in $QH_*(L; \mathbb{Z}_2)$ (l_L was denoted by $l_L([L], -)$ in Section 1.1.1). Given a normalized $H \in C^\infty(M)$, l_L gives rise to a function $H^1(M; \mathbb{R}) \to \mathbb{R}$ by defining

$$\sigma_{H:L}(c) := \lim_{\mathbb{N} \ni k \to \infty} \frac{l_L(\psi_1^{-1} \tilde{\phi}_H^k \psi_1)}{k}, \quad c \in H^1(M; \mathbb{R}).$$

Here $\psi : [0, 1] \times M \to M$ is any symplectic isotopy with $\mathrm{Flux}(\psi) = c$. $\sigma_{H:L}$ is locally Lipschitz (with respect to any norm on $H^1(M; \mathbb{R})$) and thus has a well-defined set of Clarke subdifferentials

$$\partial \sigma_{H:L}(c) \subset H_1(M; \mathbb{R}) = H^1(M; \mathbb{R})^*$$

at every point $c \in H^1(M; \mathbb{R})$. The way in which $\sigma_{H:L}$ encodes information about the dynamics of ϕ_H is that its subdifferentials parametrise rotation vectors of ϕ_H-invariant measures.

Theorem 1.2.1 (Theorem 5.2.2 on page 59)**.** *Let $H \in C^\infty$ be normalized. For any $c \in H^1(M; \mathbb{R})$ and any subdifferential $h \in \partial \sigma_{H:L}(c) \subset H_1(M; \mathbb{R})$, there exists a ϕ_H-invariant Borel probability measure μ on M satisfying*

$$\rho(\mu) = h.$$

Here $\rho(\mu) \in H_1(M; \mathbb{R})$ denotes the rotation vector of μ. We refer to Section 5.5.1 for a discussion on rotation vectors and Clarke subdifferentials. It turns out that, if the homomorphism $H^1(M; \mathbb{R}) \to H^1(L; \mathbb{R})$ is surjective, then $\sigma_{H:L}$ descends to a function

$$\alpha_{H:L} : H^1(L; \mathbb{R}) \to \mathbb{R}$$

which is locally Lipschitz. As a consequence, under this assumption, one has some control over the direction of $\sigma_{H:L}$'s subdifferentials in $H_1(M; \mathbb{R})$.

Corollary 1.2.2 (Corollary 5.2.5 on page 60)**.** *Suppose the map $H^1(M; \mathbb{R}) \to H^1(L; \mathbb{R})$ induced by the inclusion $\iota : L \hookrightarrow M$ is surjective. Then, for every $c \in H^1(L; \mathbb{R})$ and every subdifferential $h \in \partial \alpha_{H:L}(c) \subset H_1(L; \mathbb{R})$, there exists a ϕ_H-invariant Borel probability measure μ on M such that*

$$\rho(\mu) = \iota_*(h) \in H_1(M; \mathbb{R}).$$

The fact that these measures arise as (convex combinations of) weak*-limits of probability measures supported on orbits of ϕ_H allows one to study their support in simple situations. In Section 5.3.1 below we give an example of a Hamiltonian on $T^*\mathbb{T}^2$, for which the measures obtained in the previous two results are extremely unstable.

Section 5.4, which is the second part of chapter 5, is an attempt to study invariant measures of certain Hamiltonian systems in a setting where Theorem 1.2.1 cannot be applied. More precisely, it studies invariant measures of Hamiltonian systems on symplectic manifolds (M, ω), where every compact subset can be displaced by a Hamiltonian diffeomorphism, using ideas due to Buhovsky-Entov-Polterovich [24]. Throughout the rest of this introduction, we will assume that our Hamiltonian $H \in C^\infty(M)$ is proper and bounded from below. For $k \in \mathbb{R}$ we define $\Sigma_k = \Sigma_k(H) := \{H < k\} \subset M$. Due to our assumptions on M and H, Σ_k can be displaced by a Hamiltonian diffeomorphism, so its displacement energy $e(\Sigma_k)$ is a non-negative real number. To every ϕ_H-invariant Borel probability measure μ, which is *compactly* supported in Σ_k, one can associate a rotation vector

$$\rho(\mu) \in H_1(\Sigma_k; \mathbb{R}).$$

Theorem 1.2.3 (Theorem 5.4.1 on page 68)**.** *Suppose (M, ω) is weakly exact and let $H \in C^\infty(M)$ be proper and bounded from below. Suppose $L \subset \Sigma_k \subset (M, \omega)$ is a Lagrangian with Abelian $\pi_1(L)$ and fix $\tilde{c} \in H^1(L; \mathbb{R})$ such that $\partial\tilde{c}|_{\pi_2(M,L)} \equiv [\omega]|_{\pi_2(M,L)}$. Then for every $a \in H^1(\Sigma_k; \mathbb{R})$ satisfying*

$$a|_L \in (e(\Sigma_k), \infty) \cdot H^1(L; \mathbb{Z}) - \tilde{c} \quad (\subset H^1(L; \mathbb{R}))$$

and every $\epsilon \in (0, k - \max_L H)$ there exists a measure $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ satisfying the two conditions

$$\langle a, \rho(\mu)\rangle \leq \max_L H - k + \epsilon$$
$$H(\mathrm{Supp}(\mu)) \subset [\max_L(H) + \tfrac{\epsilon}{2}, k - \tfrac{\epsilon}{2}].$$

The existence of $\tilde{c} \in H^1(L; \mathbb{R})$ with the described properties is guaranteed by Lemma 5.6.1 below. Theorem 1.2.3 is really a consequence of a more abstract result which relates the classical notion of the symplectic *shape* of a sublevel set Σ_k to the dynamical properties of $\phi_k|_{\Sigma_k}$. Classically, Sikorav [88], [87] and later Eliashberg [36], defined the symplectic shape of a subset $X \subset (M, \omega)$, when $\omega = d\lambda$ is an exact symplectic form. For our purposes it is convenient to introduce the following definition.

Definition 1.2.4 (Definition 5.4.2 on page 69)**.** Fix $c \in H^1(L; \mathbb{R})$. The *(Lagrangian) (L, c)-shape* of a subset $X \subset M$, denoted by $\mathrm{Sh}(X; L, c)$, is defined as the subset of $H^1(X; \mathbb{R})$ consisting of those classes a for which there exists a Lagrange isotopy $\psi : [0, 1] \times L \to M$ starting at L and ending in X in the sense that $\psi_1(L) \subset X$ and which in addition satisfies the condition

$$\mathrm{Flux}_L(\psi) - c = \psi_1^*(a|_{\psi_1(L)}) \in H^1(L; \mathbb{R}).$$

In order for the (L, c)-shape of Σ_k to say anything about the dynamics of $\phi_H|_{\Sigma_k}$, one need to choose the energy level $k \in \mathbb{R}$ and $c \in H^1(L; \mathbb{R})$ in a compatible way. To make this precise one associates a function $\kappa_{H:L} : H^1(L; \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ to the pair (H, L), given by

$$\kappa_{H:L}(c) := \inf_{\psi} \max_{\psi_1(L)}(H),$$

where the infimum runs over all *Lagrange isotopies* $\psi : [0, 1] \times L \to (M, \omega)$ with $\psi_0 : L \hookrightarrow (M, \omega)$ the inclusion and whose Lagrange flux $\mathrm{Flux}_L(\psi) \in H^1(L; \mathbb{R})$ satisfy $\mathrm{Flux}_L(\psi) = c$. $\kappa_{H:L}$ should be thought of as a generalization of Mather's α-function.

Theorem 1.2.5 (Theorem 5.4.7 on page 70)**.** *Consider an autonomous Hamiltonian $H \in C^\infty(M; \mathbb{R})$ which is proper and bounded from below. Let $c \in H^1(L; \mathbb{R})$ and suppose $k \in \mathbb{R}$ is an energy level such that $k \leq \kappa_{H:L}(c)$. Then for every*

$$a \in \mathrm{Sh}(\Sigma_k; L, c)$$

there exists a ϕ_H-invariant Borel probability measure μ which satisfies

$$\langle a, \rho(\mu) \rangle < 0. \tag{1.5}$$

In particular, μ with the property (1.5) can be chosen to be ergodic.

The main applications of this last result are to Hamiltonian systems on $\mathbb{R}^{2n}$ as well as Hamiltonian systems on twisted cotangent bundles of tori.

2. Invariants of Lagrangian cobordisms via spectral numbers

2.1 Introduction

One approach to studying Lagrangian submanifolds of a symplectic manifold (M,ω) which has attracted a lot of attention lately is by studying their Lagrangian cobordisms in $(\mathbb{R}^2 \times M, \omega_{\mathbb{R}^2} \oplus \omega)$ (see precise definitions in Section 2.4). Biran and Cornea showed in [10] and [11] that suitable Lagrangian cobordisms preserve symplectic invariants. Considering "Lagrangians up to Lagrangian cobordism" thus seems like a very natural notion for studying the *symplectic* topology of Lagrangian submanifolds. Moreover, Cornea and Shelukhin [32] recently discovered the existence of a remarkable "cobordism metric" d_c on suitable subspaces of the space of Lagrangians in (M,ω) (see precise definitions in Section 2.2.1). This metric can be viewed as a generalization of the Lagrangian version of the well-known Hofer metric introduced by Chekanov [28]. Motivated by these discoveries we produce a Lagrangian cobordism invariant by applying the Lagrangian spectral invariant package recently developed by Leclercq and Zapolsky in [58]. We also show that spectral numbers provide functions on subsets of the space of Lagrangian submanifolds in (M,ω) to $\mathbb{R}$ which are Lipschitz continuous with respect to d_c and use this to define a non-degenerate "spectral metric" which bounds d_c from below.

2.1.1 Setting and notation

Throughout this chapter we consider a connected symplectic manifold (M^{2n},ω) which is either closed or open and convex at infinity [35]. We also consider the associated symplectic manifold $(\tilde{M},\tilde{\omega})$ defined by $\tilde{M} := \mathbb{R}^2(x,y) \times M$ and $\tilde{\omega} := \omega_{\mathbb{R}^2} \oplus \omega$, where $\omega_{\mathbb{R}^2} := dx \wedge dy$. Unless otherwise stated, any Lagrangian submanifold $L^n \subset (M,\omega)$ will be assumed closed, connected and *monotone*. By this we mean that there is a positive constant $\tau_L > 0$ satisfying

$$\omega|_{\pi_2(M,L)} = \tau_L \cdot \mu|_{\pi_2(M,L)},$$

where $\omega : \pi_2(M,L) \to \mathbb{R}$ denotes integration of ω and $\mu : \pi_2(M,L) \to \mathbb{Z}$ denotes the Maslov index. We will also assume that the minimal Maslov number $N_L := \min\{\mu(\alpha) > 0 \mid \alpha \in \pi_2(M,L)\}$ associated to L satisfies $N_L \geq 2$. If $\mu|_{\pi_2(M,L)} \equiv 0$ we set $N_L = \infty$ (this is a special case of the *weakly exact* setting - see Section 2.2.3).

In the monotone setting both Floer homology $HF_*(L)$ and quantum homology $QH_*(L)$ of L with Λ-coefficients are well-defined, where $\Lambda := \mathbb{Z}_2[t,t^{-1}]$ [15], [16], [58], [104].[1] Here

[1] [15] and [16] use language/notation which is slightly different from [58] and [104]. To avoid confusion we therefore point out that $QH(L)$ (respectively $HF(L)$) with Λ-coefficients in [15] and [16] corresponds to $QH(L)$ (respectively $HF(L)$) of a suitable quotient complex with $\mathbb{Z}_2$-coefficients in [58] and [104] (see Section 2.6 in [58] for details).

we will only work with Λ-coefficients and therefore omit them from the notation.

Given $\tau > 0$ we denote by $\mathcal{L}_\tau = \mathcal{L}_\tau(M, \omega)$ the space of all Lagrangian submanifolds $L \subset (M, \omega)$ as above, satisfying the additional condition that $\tau_L = \tau$ (i.e. all Lagrangians are *uniformly monotone*). We also denote by $\mathcal{L}_\tau^* \subset \mathcal{L}_\tau$ the subspace consisting of Lagrangians L for which $QH_*(L) \neq 0$.

2.2 Main results

2.2.1 The Lagrangian cobordism metric structure

Given a Lagrangian cobordism $V \subset (\tilde{M}, \tilde{\omega})$ connecting two Lagrangians $L, L' \in \mathcal{L}_\tau$ (see Section 2.4 for precise definitions) it is natural to consider the set $\pi(V) \subset \mathbb{R}^2$, where $\pi : \tilde{M} = \mathbb{R}^2 \times M \to \mathbb{R}^2$ denotes the projection. The insight that $\pi(V)$ contains valuable information originally arose during Biran and Cornea's extensive study of Lagrangian cobordism [10], [11]. The idea was made quantitative in [32] where Cornea and Shelukhin established the existence of a remarkable natural *cobordism metric* on the space $\mathcal{L}_\tau$.[2] We say that the Lagrangian cobordism V is *elementary* if it is connected and monotone with $N_V \geq 2$ when viewed as a Lagrangian submanifold of $(\tilde{M}, \tilde{\omega})$ (see also Section 2.4).

Definition 2.2.1 ([32])**.** Given $L, L' \subset \mathcal{L}_\tau$ as well as an elementary Lagrangian cobordism $V : L' \rightsquigarrow L$ the *outline of* V, $ou(V)$, is by definition the closed subset of $\mathbb{R}^2$ given as the complement of the union of unbounded components of $\mathbb{R}^2 \backslash \pi(V)$. The *shadow of* V is defined by

$$\mathcal{S}(V) := \mathrm{Area}(ou(V)).$$

The main result of this chapter shows that $\mathcal{S}(V)$ provides a natural upper bound on the difference in spectral numbers coming from the bounding Lagrangians. The theory of spectral numbers for Lagrangian submanifolds has been developed in various settings by various authors, starting with Viterbo [102] and Oh [70], [72]. Here we will use the version for monotone Lagrangians recently developed by Leclercq and Zapolsky [58]. Associated to a given $L \in \mathcal{L}_\tau$ they defined a *spectral invariant* function

$$l_L : QH_*(L) \times \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R} \cup \{-\infty\}$$

satisfying $l(\alpha, \phi) = -\infty$ if and only if $\alpha = 0 \in QH_*(L)$ (see Section 2.3 for preliminaries on Lagrangian spectral invariants). To state our main result we recall that, if $L, L' \in \mathcal{L}_\tau$ and $V : L' \rightsquigarrow L$ is an elementary Lagrangian cobordism then, by Theorem 2.2.2 in [10], V induces a *ring isomorphism* $\Phi_V : QH_*(L) \xrightarrow{\cong} QH_*(L')$.

Theorem 2.2.2. *Let $L, L' \in \mathcal{L}_\tau$ and let $V : L' \rightsquigarrow L$ be an elementary Lagrangian cobordism. Then*

$$|l_L(\alpha, \phi) - l_{L'}(\Phi_V(\alpha), \phi)| \leq \mathcal{S}(V)$$

for all $\alpha \in QH_*(L) \backslash \{0\}$ and all $\phi \in \widetilde{\mathrm{Ham}}(M, \omega)$.

Example 2.2.8 below shows that the result is sharp in the sense that there exist cobordisms V for which the statement becomes false if $\mathcal{S}(V)$ is replaced by a smaller number. Cornea and Shelukhin further considered the following

Definition 2.2.3 ([32])**.** Define a function $d_c : \mathcal{L}_\tau \times \mathcal{L}_\tau \to [0, \infty]$ by

$$d_c(L, L') := \inf\{\mathcal{S}(V) \mid V : L' \rightsquigarrow L\}, \quad L, L' \in \mathcal{L}_\tau.$$

Here the infimum runs over all elementary Lagrangian cobordisms $V : L' \rightsquigarrow L$.

[2] In fact Cornea and Shelukhin showed that there is a cobordism metric in several different settings.

One of the main results in Cornea and Shelukhin's paper [32] is that d_c in fact is a metric on $\mathcal{L}_\tau$. Of course $d_c(L, L') = \infty$ if and only if there do not exist any elementary Lagrangian cobordisms $V : L' \rightsquigarrow L$. Recall that, given $L \in \mathcal{L}_\tau$, the fundamental class $[L] \in QH_n(L)$ is the unity with respect to the ring structure on $QH_*(L)$. It is convenient to introduce the notation $l_L^+(\phi) := l_L([L], \phi)$ for $\phi \in \widetilde{\mathrm{Ham}}(M, \omega)$. Since Φ_V preserves the ring structure, Theorem 2.2.2 implies

Corollary 2.2.4. *For every pair of Lagrangian submanifolds $L, L' \in \mathcal{L}_\tau^*$ we have*

$$|l_L^+(\phi) - l_{L'}^+(\phi)| \leq d_c(L, L') \quad \forall\ \phi \in \widetilde{\mathrm{Ham}}(M, \omega).$$

Remark 2.2.5. Corollary 2.2.4 says that, for any fixed $\phi \in \widetilde{\mathrm{Ham}}(M, \omega)$, the function $(\mathcal{L}_\tau^*, d_c) \to (\mathbb{R}, |\cdot|)$ given by $L' \mapsto l_{L'}^+(\phi)$ is 1-Lipschitz. Of course this statement is only interesting when $L' \mapsto l_{L'}^+(\phi)$ is restricted to an elementary Lagrangian cobordism class $\subset \mathcal{L}_\tau^*$.

The following definition and proposition were generously suggested to us by the anonymous referee whom we wholeheartedly thank! Define $d_s : \mathcal{L}_\tau^* \times \mathcal{L}_\tau^* \to [0, \infty]$ by

$$d_s(L, L') := \sup\{|l_L^+(\phi) - l_{L'}^+(\phi)| \mid \phi \in \widetilde{\mathrm{Ham}}(M, \omega)\}, \quad L, L' \in \mathcal{L}_\tau^*.$$

As the notation suggests d_s is a (spectral) metric. It is clear that d_s is symmetric and satisfies the triangle inequality. Hence, the only non-trivial property to check in order for d_s to be a metric, is non-degeneracy.

Proposition 2.2.6. $d_s : \mathcal{L}_\tau^* \times \mathcal{L}_\tau^* \to [0, \infty]$ *is a non-degenerate metric.*

Note that this result shows that the estimate in Theorem 2.2.2 is non-trivial whenever $L \neq L'$. As an immediate consequence of Corollary 2.2.4 we obtain

Corollary 2.2.7. *For all $L, L' \in \mathcal{L}_\tau^*$ we have $d_s(L, L') \leq d_c(L, L')$.*

This result together with Proposition 2.2.6 gives a new proof of the fact that the restriction of d_c to $\mathcal{L}_\tau^*$ is non-degenerate. Fixing $L \in \mathcal{L}_\tau^*$ one often considers the subset $\mathcal{H}(L) := \{\phi(L) \mid \phi \in \mathrm{Ham}(M, \omega)\} \subset \mathcal{L}_\tau^*$ equipped with the Hofer metric d_H [28]. It follows from Corollary 2.2.7 and the Lagrangian suspension construction [32] that

$$d_s(L', L'') \leq d_c(L', L'') \leq d_H(L', L'') \quad \forall\ L', L'' \in \mathcal{H}(L). \tag{2.1}$$

Denote by $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega}) \leq \mathrm{Symp}(\tilde{M}, \tilde{\omega})$ the subgroup of symplectomorphisms ψ which are *compactly supported relative to M* in the sense that there is a compact $K \subset \mathbb{R}^2$ such that $\psi|_{(\mathbb{R}^2 \setminus K) \times M} = (\mathrm{id}_{\mathbb{R}^2} \times \varphi)|_{(\mathbb{R}^2 \setminus K) \times M}$ for some $\varphi \in \mathrm{Symp}(M, \omega)$. Given $L, L' \in \mathcal{L}_\tau^*$, $\psi \in \mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega})$ and an elementary Lagrangian cobordism $V : L' \rightsquigarrow L$ we obtain a new cobordism $\psi(V) : \varphi(L') \rightsquigarrow \varphi(L)$. It is easy to check that d_s is $\mathrm{Symp}(M, \omega)$-invariant and therefore Corollary 2.2.7 gives a Lagrangian non-squeezing type inequality:

$$d_s(L, L') \leq \mathcal{S}(\psi(V)) \quad \forall\ \psi \in \mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega}). \tag{2.2}$$

From this point of view it would be interesting to understand for which pairs $L', L'' \in \mathcal{H}(L)$ it holds that $d_s(L', L'') = d_H(L', L'')$. For such pairs one can find Lagrangian suspensions which are (close to) "optimal" in the sense that they (almost) minimize shadow among all elementary Lagrangian cobordisms $L'' \rightsquigarrow L'$ (see also Example 2.2.8 below). Some investigations in this direction were already carried out in Remark 5.1 in [32]. Moreover, Corollary 2.2.4 above can be viewed as a generalization of a bound found in [58] for the Hofer distance on the universal cover of $\mathcal{H}(L)$ for a fixed $L \in \mathcal{L}_\tau^*$.

It is an open problem to understand the extent to which d_c differs from d_H in our setting: The main examples of elementary Lagrangian cobordisms are Lagrangian suspensions and their images under $\mathrm{Symp}_{c|M}(\tilde{M},\tilde{\omega})$-elements. On the other hand there are many explicit examples of non-monotone Lagrangian cobordisms which do not arise as Lagrangian suspensions [10], [26], [44]. In [18] we study how properties similar to (2.2) of such cobordisms are intimately linked to their topology.

Example 2.2.8. [3] Consider $T^*S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ with coordinates $(q,p) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ and equipped with the symplectic structure $dp \wedge dq$. Denote by $L \subset T^*S^1$ the 0-section. Define an autonomous Hamiltonian $H \in C^\infty(T^*S^1)$ by $H(q,p) = (\sin(2\pi q) + 1)$ and let $L' := \phi^1_H(L) \subset T^*S^1$. Choose a monotone function $\rho \in C^\infty(\mathbb{R};[0,1])$ such that for some small $\epsilon > 0$ we have $\rho = 0$ on $(-\infty, \epsilon]$ and $\rho = 1$ on $[1-\epsilon, \infty)$. Then the Lagrangian suspension construction [76] applied to the time-dependent Hamiltonian $\hat{H}_t(q,p) := \rho'(t)H(q,p)$ produces an exact Lagrangian cobordism $V : L' \rightsquigarrow L$ with shadow $\mathcal{S}(V) = 2$. Moreover, since H has Hofer norm $= 2$ we conclude that $d_H(L,L') \leq 2$. We claim that $d_s(L,L') \geq 2$, which by (2.1) implies

$$d_s(L,L') = d_c(L,L') = d_H(L,L') = 2.$$

To see this fix a small $\epsilon > 0$ and a corresponding $1 \gg \delta > 0$ such that

$$\max\{H(q,p) \mid (q,p) \in T^*S^1 \backslash B_{3\delta}(\tfrac{1}{4})\} \geq 2 - \frac{\epsilon}{3},$$

where $B_{3\delta}(\frac{1}{4})$ denotes the ball of radius 3δ centered at $\frac{1}{4} \in L$. Choose $\varphi_1 \in C^\infty(L;[0,1])$ such that

$$\varphi_1 \begin{cases} = 0, & \text{on } L \cap B_{2\delta}(\frac{1}{4}) \\ = 1, & \text{on } L \backslash B_{3\delta}(\frac{1}{4}) \end{cases} \quad \& \quad \varphi_1' \begin{cases} \leq 0, & \text{on } [\frac{1}{4} - 3\delta, \frac{1}{4} - 2\delta] \\ \geq 0, & \text{on } [\frac{1}{4} + 2\delta, \frac{1}{4} + 3\delta], \end{cases}$$

and define $H^1(q,p) := \varphi_1(q)H(q,p)$. Applying Lemma 2.5.1 below together with an easy approximation argument one sees that for all $s \in [0,1]$ we have $l^+_L(sH^1) = \max_{T^*S^1}(sH^1) \geq s(2-\frac{\epsilon}{3})$. Fix now $\varphi_2 \in C^\infty(T^*S^1;[0,1])$ such that $\varphi_2 = 1$ outside a very small neighborhood of $L' \backslash (B_\delta(\frac{1}{4}) \cup B_\delta(\frac{3}{4}))$ and $\varphi_2 = 0$ on an even smaller neighborhood of $L' \backslash (B_\delta(\frac{1}{4}) \cup B_\delta(\frac{3}{4}))$. Define $H^2(q,p) := \varphi_2(q,p)H^1(q,p)$. By continuity and the Lagrangian control property from [58] we have $|l^+_{L'}(sH^2)| \leq \frac{\epsilon}{3}$ for all $s \in [0,1]$ if δ is chosen small enough. For a $s_* \in (0,1)$ very close to 1 (depending only on the set $\{\varphi_2 \neq 1\}$) the path $\{\phi^t_{s_*H^1}(L) = \phi^t_{s_*H^2}(L)\}_{t\in[0,1]}$ is contained in the set $\{\varphi_2 = 1\}$. In particular, for every $s \in [0,s_*]$, the Hamiltonian chords of the autonomous Hamiltonian $sH^2 + (s_* - s)H^1 = (s_* + s(\varphi_2 - 1))H^1$ connecting L to itself coincide with those of the Hamiltonian s_*H^1, and are contained in the set $\{\varphi_2 = 1\}$. Hence, spectrality [58] gives

$$l^+_L(sH^2 + (s_* - s)H^1) \in \mathcal{A}_{s_*H^1:L}(\mathrm{Crit}(\mathcal{A}_{s_*H^1:L})) \quad \forall\ s \in [0,s_*].$$

Since $\mathcal{A}_{s_*H^1:L}(\mathrm{Crit}(\mathcal{A}_{s_*H^1:L})) \subset \mathbb{R}$ is nowhere dense and

$$[0,s_*] \ni s \mapsto l^+_L(sH^2 + (s_* - s)H^1) \in \mathbb{R}$$

is continuous we conclude that $l^+_L(s_*H^2) = l^+_L(s_*H^1) \geq s_*(2-\frac{\epsilon}{3}) \geq 2-\frac{2\epsilon}{3}$ if s_* is sufficiently close to 1. Hence,

$$d_s(L,L') \geq |l^+_L(s_*H^2) - l^+_{L'}(s_*H^2)| \geq |l^+_L(s_*H^2)| - |l^+_{L'}(s_*H^2)| \geq 2 - \epsilon.$$

Now the claim follows by letting $\epsilon \to 0$.

[3] We thank the anonymous referee for suggesting this example.

2.2.2 A Lagrangian cobordism invariant

Fix $L \in \mathcal{L}^*_\tau$. A nice fact about l^+_L is that it satisfies the triangle inequality

$$l^+_L(\phi\psi) \leq l^+_L(\phi) + l^+_L(\psi) \quad \forall\ \phi, \psi \in \widetilde{\mathrm{Ham}}(M, \omega).$$

This and the continuity property of l^+_L allows one to consider the asymptotic spectral invariant $\sigma_L : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ given by

$$\sigma_L(\phi) := \lim_{k \to \infty} \frac{l^+_L(\phi^k)}{k}, \quad \phi \in \widetilde{\mathrm{Ham}}(M, \omega).$$

We want to point out that, just like l_L, σ_L is known to satisfy a number of "nice" properties. In the case when L is the 0-section of a cotangent bundle many of these are documented in [62]. For the monotone setting we are considering many analogous properties follow immediately from the properties of l_L which are documented in [58].

Our next result shows that σ_L can be considered as an object associated to L's elementary Lagrangian cobordism class.

Theorem 2.2.9. *Let $L \in \mathcal{L}^*_\tau$. Then the function $\sigma_L : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ is an elementary Lagrangian cobordism invariant of L. In other words, if $L' \in \mathcal{L}_\tau$ is in the same elementary Lagrangian cobordism class as L then $\sigma_{L'} : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}$ is well-defined and $\sigma_L = \sigma_{L'}$.*

As mentioned in the introduction Biran and Cornea's Lagrangian cobordism theory [10], [11] shows that it is desirable to be able to detect whether or not two given Lagrangians are in the same elementary Lagrangian cobordism class. To our knowledge, $L \mapsto \sigma_L$ is one of very few numerical invariants known for Lagrangian cobordism. Naturally one would like to make use of the algebraic structures on $\widetilde{\mathrm{Ham}}(M, \omega)$ and properties of σ_L to derive criteria for detecting the non-existence of elementary Lagrangian cobordisms. One example of how this can be done is the following result. Recall that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$. In particular $\mathrm{Symp}(M, \omega)$ acts on $\widetilde{\mathrm{Ham}}(M, \omega)$ by conjugation

$$\begin{gathered}\mathrm{Symp}(M, \omega) \times \widetilde{\mathrm{Ham}}(M, \omega) \to \widetilde{\mathrm{Ham}}(M, \omega) \\ (\psi, \phi) \mapsto \psi\phi\psi^{-1}\end{gathered}$$

As a consequence of Theorem 2.2.9 and the symplectic invariance property from [58] we obtain

Corollary 2.2.10. *Let $L \in \mathcal{L}^*_\tau$ and $\psi \in \mathrm{Symp}(M, \omega)$. If L and $\psi(L)$ are in the same elementary Lagrangian cobordism class then σ_L is invariant under conjugation by ψ. I.e.*

$$\sigma_L(\phi) = \sigma_L(\psi\phi\psi^{-1}) \quad \forall\ \phi \in \widetilde{\mathrm{Ham}}(M, \omega).$$

By the Lagrangian suspension construction this result implies in particular that σ_L is invariant under conjugation by elements of $\mathrm{Ham}(M, \omega)$. We do not know of any examples $\psi \in \mathrm{Symp}(M, \omega) \setminus \mathrm{Ham}(M, \omega)$ such that L and $\psi(L)$ are in the same elementary Lagrangian cobordism class.

Remark 2.2.11. Consider a pair of Lagrangians $L, L' \in \mathcal{L}^*_\tau$ satisfying $L \cap L' = \emptyset$. Now choose a normalized[4] autonomous Hamiltonian $H \in C^\infty(M)$ satisfying $H|_L \equiv c$ and $H|_{L'} \equiv c'$ for constants $c \neq c'$. Then

$$\sigma_L(\phi_H) = c \neq c' = \sigma_{L'}(\phi_H),$$

[4] If (M^{2n}, ω) is a closed symplectic manifold we say that a Hamiltonian $H \in C^\infty([0,1] \times M)$ is *normalized* if $\int_M H_t \omega^n = 0$ for all $t \in [0,1]$. If (M^{2n}, ω) is non-compact we say that $H \in C^\infty([0,1] \times M)$ is *normalized* if it has compact support.

by the Lagrangian control property from [58]. In view of Theorem 2.2.9 this observation gives a new proof of the following result which can also be derived from Biran and Cornea's work [10] (see Remark 2.4.5 below).

Corollary 2.2.12. *Let $L \in \mathcal{L}_\tau^*$. If $L' \in \mathcal{L}_\tau$ is in the same elementary Lagrangian cobordism class as L then $L \cap L' \neq \emptyset$.*

In fact there is even a third proof of this fact based on the metric d_s: If $L, L' \in \mathcal{L}_\tau^*$ satisfy $L \cap L' = \emptyset$ then it is easy to check that $d_s(L, L') = \infty$. In particular Corollary 2.2.7 implies that $d_c(L, L') = \infty$ and therefore L and L' cannot be in the same elementary Lagrangian cobordism class.

Example 2.2.13. Spectral invariants coming from Floer theory are known to be very hard to compute. However, given $L \in \mathcal{L}_\tau^*$, a consequence of Theorem 2.2.9 is that there is a rather large subset of $\widetilde{\mathrm{Ham}}(M, \omega)$ on which σ_L can be computed explicitly! Consider the subgroup $\mathcal{G}_L \subset \widetilde{\mathrm{Ham}}(M, \omega)$ defined by

$$\phi \in \mathcal{G}_L \overset{Def.}{\Longleftrightarrow} (\exists \{\phi_t\}_{t \in [0,1]} \in \phi \ : \ \phi_t(L) = L \ \forall\ t \in [0, 1]).$$

In other words $\mathcal{G}_L$ consists exactly of the homotopy classes (rel. endpoints) of paths ϕ in $\mathrm{Ham}(M, \omega)$, based at the identity, which contain a path $\{\phi_t\}_{t\in[0,1]}$ satisfying $\phi_t(L) = L$ for all $t \in [0, 1]$. It is easy to check that $\phi \in \mathcal{G}_L$ if and only if $\phi = \phi_H$ for a normalized $H \in C^\infty([0, 1] \times M)$ satisfying $H_t|_L = c(t)$ for some $c \in C^\infty([0, 1])$. For such a Hamiltonian the Lagrangian control property from [58] reads

$$\sigma_L(\phi_H) = \int_0^1 c(t)dt. \tag{2.3}$$

Applying Theorem 2.2.9 this has the following interesting consequence: *$\sigma_L(\phi)$ can be computed explicitly by a formula similar to (2.3) for every*

$$\phi \in \bigcup_{L'} \mathcal{G}_{L'}. \tag{2.4}$$

Here L' runs over the entire elementary Lagrangian cobordism class of L. Note that, by the Lagrangian suspension construction, the orbit of L under the natural action $\mathrm{Ham}(M, \omega) \times \mathcal{L}_\tau \to \mathcal{L}_\tau$ is contained in the elementary Lagrangian cobordism class of L. It is therefore clear that the union in (2.4) is a rather large set in general.

Analogues of σ_L have a very prominent history in symplectic topology. In the case of spectral invariants coming from Hamiltonian Floer homology the study of the analogue of this quantity was pioneered by Entov and Polterovich in their development of *Calabi quasimorphisms* on $\widetilde{\mathrm{Ham}}(M, \omega)$ [37] (see also [79]). In case L is the zero-section of the cotangent bundle T^*N of a closed manifold N Monzner, Vichery and Zapolsky [62] showed, using ideas due to Viterbo [100], that σ_L is closely related to Mather's α-function. Moreover, for the special case $N = \mathbb{T}^n$, they showed that σ_L is closely related to Viterbo's homogenization operator.

2.2.3 What happens in the (weakly) exact case?

If one chooses to work with weakly exact Lagrangians one can obtain the results in Section 2.2 in a slightly different form. For the convenience of the reader we here point out these changes.

The case $\mu|_{\pi_2(M,L)} \equiv 0$

One alternative construction of spectral invariants which is relevant for our purposes was carried out by Leclercq [57] for closed Lagrangians $L \subset (M, \omega)$ verifying

$$\omega|_{\pi_2(M,L)} \equiv 0 \quad \& \quad \mu|_{\pi_2(M,L)} \equiv 0. \tag{2.5}$$

Note that the existence of such a Lagrangian in (M, ω) implies that M is symplectically aspherical in the sense that $\omega|_{\pi_2(M)} \equiv 0$ and $c_1(TM)|_{\pi_2(M)} \equiv 0$. We denote by $\mathcal{L}_0$ the space of all closed Lagrangians $L \subset (M, \omega)$ satisfying (2.5). Note that for any $L \in \mathcal{L}_0$, $QH_*(L)$ reduces to $H_*(L) = H_*(L; \mathbb{Z}_2)$. As already mentioned this setting is covered by our results in Section 2.2. However, it is also possible to recover some of our results using Leclercq's spectral invariants which satisfy particularly nice properties. The spectral invariant function

$$c(\cdot\ ; L, \cdot) : H_*(L) \times \mathcal{H}(L) \to \mathbb{R} \cup \{-\infty\},$$

constructed by Leclercq in [57], is associated to a fixed $L \in \mathcal{L}_0$. Among other properties he showed that $c(\alpha; L, L') = -\infty$ if and only if $\alpha = 0 \in H_*(L)$ and, for every pair $(\alpha, L') \in H_*(L) \times \mathcal{H}(L)$ for which $\alpha \neq 0$, one has $0 \leq c(\alpha; L, L') \leq d_H(L, L')$. For $L \in \mathcal{L}_0$ the results in Section 2.2.1 continue to be true, *mutatis mutandis*, if one replaces l_L by $c(\cdot\ ; L, \cdot)$. More precisely, when replacing l_L by $c(\cdot\ ; L, \cdot)$, $QH_*(L)$ is replaced by $H_*(L)$, $\widetilde{\mathrm{Ham}}(M, \omega)$ is replaced by $\mathcal{H}(L)$ and the statements hold for elementary Lagrangian cobordisms V verifying $\tilde{\omega}|_{\pi_2(\tilde{M},V)} \equiv 0 \equiv \mu|_{\pi_2(\tilde{M},V)}$. One could of course also study the asymptotic version of $c(\cdot\ ; L, \cdot)$. However, it is not clear to us that this quantity contains information about Lagrangian cobordisms.

The case $\mu|_{\pi_2(M,L)} \neq 0$

Recall that a Lagrangian $L \subset (M, \omega)$ is said to be weakly exact if $\omega|_{\pi_2(M,L)} \equiv 0$. We denote by $\mathcal{L}_{we}(M, \omega)$ the space of all closed and weakly exact Lagrangian submanifolds in (M, ω). In case $(M, \omega = d\lambda)$ is exact[5] the exact Lagrangians are special cases of weakly exact Lagrangians. The version of spectral invariants developed in [58] was initially constructed in the exact setting for the particular case of the zero-section in a cotangent bundle by Oh [70], [72]. The parts of Oh's scheme which are needed for our results can also be carried out for Lagrangians in $\mathcal{L}_{we}(M, \omega)$ (see [58], [103]). In this setting $HF(L)$ does not necessarily carry a $\mathbb{Z}$-grading but is still isomorphic to $H(L) = \oplus_{k=0}^{n} H_k(L; \mathbb{Z}_2)$. Zapolsky [103] showed that, for $L \in \mathcal{L}_{we}$, l_L in fact descends to $\mathrm{Ham}(M, d\lambda)$:

$$l_L : H(L) \times \mathrm{Ham}(M, d\lambda) \to \mathbb{R} \cup \{-\infty\}.$$

Therefore, one recovers all results from Section 2.2.2 with the one difference that $\widetilde{\mathrm{Ham}}(M, \omega)$ can be replaced by $\mathrm{Ham}(M, d\lambda)$ throughout, given that one also restricts to looking at Lagrangian cobordisms V satisfying $\tilde{\omega}|_{\pi_2(\tilde{M},V)} \equiv 0$. The same goes for the results in Section 2.2.1. For additional properties of σ_L for $L \in \mathcal{L}_{we}$ we refer to [62], where the case of a zero-section in a cotangent bundle is studied in detail.

2.3 Preliminaries on Lagrangian spectral invariants

Fix $L \in \mathcal{L}_\tau$. Given a pair (H, J), where $H \in C^\infty([0,1] \times M)$ is a (time-dependent) Hamiltonian satisfying $\phi_H^1(L) \pitchfork L$ and $J = \{J_t\}_{t \in [0,1]}$ is a generic smooth path of ω-compatible almost complex structures, one can construct the Floer homology group $HF_*(H, J : L)$.

[5] Recall that (M, ω) is said to be *exact* if $\omega = d\lambda$ for some 1-form λ on M. In this case a Lagrangian $L \subset (M, d\lambda)$ is said to be *exact* (with respect to λ) if $\lambda|_L = df$ for some $f \in C^\infty(L)$.

Recall that $HF_*(H, J : L)$ can be thought of as the Morse homology of the action functional $\mathcal{A}_{H:L}$. Here we view $\mathcal{A}_{H:L}$ as being defined on the space $\overline{\Omega}_L$ consisting of equivalence classes of pairs $\widetilde{\gamma} = [\gamma, \widehat{\gamma}]$ where $\gamma : ([0,1], \{0,1\}) \to (M, L)$ and $\widehat{\gamma} : (\dot{D}^2, \partial\dot{D}^2) \to (M, L)$ is a capping of γ ($\dot{D}^2 = D^2 \backslash \{1\} \subset \mathbb{C}$ denotes the punctured unit disc). The equivalence relation is given by identifying cappings of equal symplectic area. Following [104] we use the convention[6]

$$\mathcal{A}_{H:L}(\widetilde{\gamma} = [\gamma, \widehat{\gamma}]) = \int_0^1 H_t(\gamma(t))\ dt - \int \widehat{\gamma}^*\omega, \quad \widetilde{\gamma} \in \overline{\Omega}_L.$$

Lagrangian Floer homology was first developed by Floer [39] and later developments were carried out by Oh [67], [68]. Today Lagrangian Floer theory is a well-documented theory and some of the standard references to which we refer for further details are [84], [73] and [74]. Here and throughout the paper we will follow the conventions and notation appearing in [104], to which we also refer the interested reader. Assuming $L \in \mathcal{L}^*_\tau$ one can use $HF_*(H, J : L)$ to extract so-called *spectral invariants*. This idea was recently developed in the monotone setting by Leclercq and Zapolsky [58]. Leclercq and Zapolsky constructed a *Lagrangian spectral invariant* function

$$l_L : QH_*(L) \times \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R} \cup \{-\infty\}, \tag{2.6}$$

satisfying $l_L(\alpha, \phi) = -\infty$ if and only if $\alpha = 0 \in QH_*(L)$. This function is defined by "mimicking" classical critical point theory as follows. $HF_*(H, J : L)$ is the homology of the Floer chain complex $(CF_*(H, J : L), d)$ where $CF_*(H, J : L)$ is the $\mathbb{Z}_2$-vector space generated by critical points of $\mathcal{A}_{H:L}$ and d is defined by "counting finite energy Floer trajectories". Given $a \in \mathbb{R}$ we denote by $CF^a_*(H, J : L) \subset CF_*(H, J : L)$ the subspace generated by those $\mathrm{Crit}(\mathcal{A}_{H:L})$-points whose action is $< a$. Floer-trajectories can be interpreted as negative gradient flow lines for $\mathcal{A}_{H:L}$, so d restricts to a differential on $CF^a_*(H, J : L)$. We denote by

$$\iota^a : CF^a_*(H, J : L) \hookrightarrow CF_*(H, J : L)$$

the inclusion and by $\iota^a_* : HF^a_*(H, J : L) \to HF_*(H, J : L)$ the map induced on homology. Identifying all the groups $HF_*(H, J : L)$ for different choices of data (H, J) we obtain the Floer homology ring of L, $HF_*(L)$. After choosing a quantum datum for L, $QH_*(L)$ is well-defined and ring-isomorphic to $HF_*(L)$ via a PSS-type isomorphism

$$\mathrm{PSS} : QH_*(L) \xrightarrow{\cong} HF_*(L).$$

Given $\alpha \in QH_*(L)$ and a Floer datum (H, J) Leclercq and Zapolsky define

$$l_L(\alpha, H, J) := \inf\{a \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{Image}(\iota^a_*) \subset HF_*(H, J : L)\}.$$

They then further show that $l_L(\alpha, H, J)$ is independent of J and that l_L descends to a function (2.6) satisfying many additional properties [58].

2.4 Preliminaries on Lagrangian cobordism

Recently Biran and Cornea introduced several new methods for studying Lagrangian submanifolds via Lagrangian cobordisms [10], [11]. Here we follow their work. Recall that $\pi : \tilde{M} = \mathbb{R}^2 \times M \to \mathbb{R}^2$ denote the canonical projection. For subsets $V \subset \tilde{M}$ and $U \subset \mathbb{R}^2$ we write $V|_U = V \cap \pi^{-1}(U)$.

[6]Note that $\mathcal{A}_{H:L}$ is defined *absolutely* here. In other words, since the definition of $HF_*(H, J : L)$ in [58] and [104] does not require the choice of a base point in Ω_L there is no need to normalize spectral invariants. This will be important below.

Definition 2.4.1. We say that two families $(L_i)_{i=0}^{k_-}$ and $(L'_j)_{j=0}^{k_+}$ of closed connected Lagrangian submanifolds of (M, ω) are *Lagrangian cobordant* if for some $R > 0$ there exists a smooth compact Lagrangian submanifold $V \subset ([-R, R] \times \mathbb{R} \times M, \omega_{\mathbb{R}^2} \oplus \omega)$ with boundary $\partial V = V \cap (\{\pm R\} \times \mathbb{R} \times M)$ satisfying the condition that for some $\epsilon > 0$ we have

$$V|_{[-R,-R+\epsilon)\times\mathbb{R}} = \bigsqcup_{i=0}^{k_-}([-R, -R+\epsilon) \times \{i\}) \times L_i \tag{2.7}$$

$$V|_{(R-\epsilon,R]\times\mathbb{R}} = \bigsqcup_{j=0}^{k_+}((R-\epsilon, R] \times \{j\}) \times L'_j. \tag{2.8}$$

In particular V defines a smooth compact cobordism $(V, \bigsqcup_{i=0}^{k_-} L_i, \bigsqcup_{j=0}^{k_+} L'_j)$. We write $V : (L'_j)_j \rightsquigarrow (L_i)_i$.

Our notation will not distinguish between a Lagrangian cobordism and its obvious horizontal $\mathbb{R}$-extension. This extension is a *Lagrangian with cylindrical ends*. More generally we have

Definition 2.4.2 ([10])**.** A *Lagrangian with cylindrical ends* is a Lagrangian submanifold $V \subset (\tilde{M}, \tilde{\omega})$ without boundary satisfying the conditions that $V|_{[a,b]\times\mathbb{R}}$ is compact for all $a < b$ and that there exists $R > 0$ such that

$$V|_{(-\infty,-R]\times\mathbb{R}} = \bigsqcup_{i=0}^{k_-}((-\infty, -R] \times \{a_i^-\}) \times L_i$$

$$V|_{[R,\infty)\times\mathbb{R}} = \bigsqcup_{j=0}^{k_+}([R, \infty) \times \{a_j^+\}) \times L'_j$$

for Lagrangians $L_i, L'_j \subset (M, \omega)$ and constants $a_i^-, a_j^+ \in \mathbb{R}$ verifying $a_i^- \neq a_{i'}^-$ for $i \neq i'$ and $a_j^+ \neq a_{j'}^+$ for $j \neq j'$.

We will be interested in specific Lagrangian cobordisms and Lagrangians with cylindrical ends which allow us to compare Floer-theoretic invariants of the ends.

Definition 2.4.3. Given two families $(L_i)_{i=0}^{k_-}, (L'_j)_{j=0}^{k_+} \subset \mathcal{L}_\tau$ we say that a Lagrangian cobordism $V : (L'_j)_j \rightsquigarrow (L_i)_i$ is *admissible* if $V \subset (\tilde{M}, \tilde{\omega})$ is itself a monotone, connected Lagrangian submanifold with monotonicity constant $\tau_V = \tau$ and minimal Maslov number $N_V \geq 2$. We say that V is an *elementary Lagrangian cobordism* if V is admissible and satisfies $k_+ = k_- = 0$, i.e. if there is only one positive and one negative end.

For examples of Lagrangian cobordisms we refer to [44], [10] and [26].

Remark 2.4.4. Note that "being cobordant by an elementary Lagrangian cobordism" is an equivalence relation on $\mathcal{L}_\tau$.

Remark 2.4.5. If $L, L' \in \mathcal{L}_\tau$ are in the same elementary Lagrangian cobordism class then the Floer homology group $HF(L, L')$ with coefficients in the universal Novikov ring over the base ring $\mathbb{Z}_2$ is well-defined [10]. If $QH_*(L) \neq 0$ then results from [10] imply that $HF(L, L') \neq 0$. In particular Corollary 2.2.12 follows.

Quantum (and Floer) homology for Lagrangians with cylindrical ends was introduced by Biran and Cornea [10], [11] and further studied by Singer [89]. Since action estimates are crucial for our intentions we will make some small adaptions in the construction of Lagrangian Floer homology from [10] to make it suit our purposes.

2.5 Proofs of results

Here we develop the theory needed to prove our results. Most of our results are in fact consequences of Theorem 2.2.2 whose proof we postpone until the end. For the proof of Proposition 2.2.6 it will be convenient to view l_L as a function

$$l_L : QH_*(L) \times C_c^\infty([0,1] \times M) \to \mathbb{R} \cup \{-\infty\},$$

so that we don't have to worry about normalizing our Hamiltonians [58]. Given $L \in \mathcal{L}_\tau$ we will denote by $U = U(L) \subset M$ a Darboux-Weinstein neighborhood of $L \subset M$. In particular we have a neighborhood $W = W(L) \subset T^*L$ of $L \subset T^*L$ and a symplectic identification $U \approx W$ which restricts to the identity on L [60]. For the proof of Proposition 2.2.6 we will need

Lemma 2.5.1. *Fix $L \in \mathcal{L}_\tau^*$. Denote $b : U \to L$ the restriction of the base-point map $T^*L \to L$ to $W \approx U$. Let $h \in C^\infty(L)$ be a Morse function such that $\max_L |h| < \frac{\tau_L N_L}{2}$ and $\mathrm{Graph}(dh) \subset Y$, where Y is a precompact and fiber-wise convex neighborhood of the 0-section in $W \approx U$. Define $H \in C_c^\infty(M)$ by $H := \varphi b^* h$, where $\varphi \in C_c^\infty(U; [0,1])$ is a cutoff satisfying $\varphi|_Y \equiv 1$. Then there exists $q \in \mathrm{Crit}_n(h)$ such that*[7]

$$l_L^+(H) = h(q).$$

Proof of Proposition 2.2.6. Let $L, L' \in \mathcal{L}_\tau^*$ with $L \neq L'$. Denote by $b : U(L) \to L$ the restriction of the base-point map $T^*L \to L$ to $W(L) \approx U(L)$. Choose $q \in L \backslash L'$ and a Morse function $f \in C^\infty(L)$ such that $\mathrm{Crit}_n(f) = \{q\}$. Fix a Morse chart $B \subset L \backslash L'$ at q and a bump function $h \in C_c^\infty(B)$ attaining its unique maximum at q with $0 < h(q) < \frac{\tau_L N_L}{2}$. By perhaps rescaling h we may assume that $\mathrm{Graph}(dh|_B) \cap (L' \cap U(L)) = \emptyset$ and choose a cutoff $\varphi \in C_c^\infty(U(L))$ as in Lemma 2.5.1 such that $\varphi|_{b^{-1}(B)\cap L'} = 0$. Consider for small $\epsilon \geq 0$ the autonomous Hamiltonian $H^\epsilon := \varphi b^*(h + \epsilon f) \in C_c^\infty(M)$. Since $H^0|_{L'} \equiv 0$ the Lagrangian control property from [58] implies that $l_{L'}^+(H^0) = 0$. Moreover, for all small $\epsilon > 0$ we have

$$l_L^+(H^\epsilon) = h(q) + \epsilon f(q)$$

by the lemma and $H^\epsilon \overset{\epsilon \to 0}{\longrightarrow} H^0$ uniformly. Thus, continuity of l_L^+ implies $l_L^+(H^0) = h(q)$. Hence,

$$d_s(L, L') \geq |l_L^+(\phi_{H^0}) - l_{L'}^+(\phi_{H^0})| = |l_L^+(H^0) - l_{L'}^+(H^0)| = h(q) > 0.$$

□

Proof of Lemma 2.5.1. By the spectrality property of l_L [58] and the fact that $[L] \in QH_n(L)$ we know that $l_L^+(H) = \mathcal{A}_{H:L}([\gamma, \widehat{\gamma}])$ for some $[\gamma, \widehat{\gamma}] \in \mathrm{Crit}(\mathcal{A}_{H:L})$ whose Conley-Zehnder index equals n. By construction of H we can identify $[\gamma, \widehat{\gamma}] \approx [q, \widehat{q}]$ where $q \in \mathrm{Crit}(h)$ and $\widehat{q}$ is a topological disc in M with boundary on L. The Conley-Zehnder index of $[q, \widehat{q}]$ equals $|q|_h - \mu(\widehat{q})$, where $|q|_h$ denotes the Morse index of q and μ denotes the Maslov index [104]. We claim that we must have $|q|_h = n$. To see this, assume for contradiction that $|q|_h < n$. Then $\mu(\widehat{q}) = |q|_h - n < 0$ and thus

$$\begin{aligned}\mathcal{A}_{H:L}([\gamma, \widehat{\gamma}]) &= \int_0^1 H(\gamma(t))dt - \int \widehat{\gamma}^*\omega = h(q) - \omega(\widehat{q}) \\ &= h(q) - \tau_L \mu(\widehat{q}) \geq -\max_L |h| + \tau_L N_L > \frac{\tau_L N_L}{2}.\end{aligned}$$

[7] Here $\mathrm{Crit}_n(h)$ denotes the critical points of h whose Morse index equals n.

But by the continuity property of l_L^+ we also have

$$|\mathcal{A}_{H:L}([\gamma,\widehat{\gamma}])| = |l_L^+(H)| \leq \max_M |H| = \max_L |h| < \frac{\tau_L N_L}{2},$$

which is a contradiction. This shows that $|q|_h = n$ and therefore $\mu(\widehat{q}) = |q|_h - n = 0$. It follows that

$$l_L^+(H) = \mathcal{A}_{H:L}([\gamma,\widehat{\gamma}]) = h(q) - \tau_L\mu(\widehat{q}) = h(q).$$

□

Remark 2.5.2. In the above proof we used the $\mathbb{Z}$-grading on $HF_*(L)$. If $L \in \mathcal{L}_{we}(M,\omega)$ with $\mu|_{\pi_2(M,L)} \neq 0$ then $HF(L)$ does not necessarily carry a $\mathbb{Z}$-grading (see Section 2.2.3). However, Proposition 2.2.6 continues to hold true also in this setting. We will not need this and therefore not carry out the proof. The basic idea is that, if the condition $\max_L |h| < \frac{\tau_L N_L}{2}$ in the statement of Lemma 2.5.1 is replaced by the condition that h be C^2-small, then the Floer chain complex $CF(H,J:L)$ of the weakly exact Lagrangian L reduces to the Floer chain complex $CF(H|_U,J|_U:L)$ of L viewed as a the 0-section in $W \approx U$ (see [69]). But this chain complex carries a $\mathbb{Z}$-grading, simply given by the index of the critical points of h, so the above argument can be carried out.

Proof of Theorem 2.2.9. Note first that the existence of Φ_V guarantees that any $L' \in \mathcal{L}_\tau$ in the same elementary Lagrangian cobordism class as L is in fact an element of $\mathcal{L}_\tau^*$, so indeed $\sigma_{L'} : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ is well-defined. Moreover L' and L are in the same class if and only if $d_c(L,L') < \infty$. Assuming this is the case Corollary 2.2.4 gives

$$|\sigma_L(\phi) - \sigma_{L'}(\phi)| = \lim_{k\to\infty} \frac{|l_L^+(\phi^k) - l_{L'}^+(\phi^k)|}{k} \leq \lim_{k\to\infty} \frac{d_c(L,L')}{k} = 0$$

for all $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$. □

Proof of Corollary 2.2.10. Note first that $\psi(L) \in \mathcal{L}_\tau$. Recall from [15] and [58] that any $\psi \in \mathrm{Symp}(M,\omega)$ induces an isomorphism

$$\psi_* : QH_*(L) \to QH_*(\psi(L)).$$

Clearly ψ_* maps $[L]$ to $[\psi(L)]$. In particular it follows from the symplectic invariance property of Lagrangian spectral invariants [58] that

$$l_L^+(\phi) = l_{\psi(L)}^+(\psi\phi\psi^{-1}) \quad \forall\ \phi \in \widetilde{\mathrm{Ham}}(M,\omega).$$

Assuming the existence of an elementary Lagrangian cobordism $V : L \rightsquigarrow \psi(L)$ it therefore follows from Theorem 2.2.9 that

$$\begin{aligned}\sigma_L(\phi) &= \lim_{k\to\infty} \frac{l_L^+(\phi^k)}{k} = \lim_{k\to\infty} \frac{l_{\psi(L)}^+(\psi\phi^k\psi^{-1})}{k} \\ &= \lim_{k\to\infty} \frac{l_{\psi(L)}^+((\psi\phi\psi^{-1})^k)}{k} = \sigma_{\psi(L)}(\psi\phi\psi^{-1}) = \sigma_L(\psi\phi\psi^{-1}).\end{aligned}$$

□

2.5.1 Floer homology, PSS and spectral invariants for Lagrangians with cylindrical ends

Throughout this section we consider a connected monotone Lagrangian submanifold $V \subset (\tilde{M},\tilde{\omega})$ with cylindrical ends and minimal Maslov number $N_V \geq 2$. We denote by $(L_i)_{i=0}^{k_-}$ the family of Lagrangians in (M,ω) corresponding to negative ends of V and by $(L'_j)_{j=0}^{k_+}$ the family of Lagrangians in (M,ω) corresponding to positive ends of V.

Floer homology with Hamiltonian perturbations for Lagrangians with cylindrical ends

Our reference for Lagrangian Floer homology is [104] and we adopt the conventions used there. Since we work in the setting of Lagrangians with cylindrical ends we will apply the machinery developed in [10] and [11] to deal with compactness issues. Due to the fact that we use many different references we will here point out how to combine the different approaches.

Here, following [11], Floer homology will be based on the choice of a class of *perturbation functions* $h \in C^\infty(\mathbb{R}^2)$. Our requirements of h will differ slightly from those in [11]. We therefore point out the specific conditions which h needs to satisfy. Fix a number $R > 0$ such that V is cylindrical outside $[-R, R]^2$,

$$V|_{\mathbb{R}^2\backslash[-R,R]^2} = \left(\bigsqcup_{i=0}^{k_-} (-\infty, -R] \times \{a_i^-\} \times L_i \right) \cup \left(\bigsqcup_{j=0}^{k_+} [R, \infty) \times \{a_j^+\} \times L'_j \right). \tag{2.9}$$

We will require the following of h.

(i) Fix $\epsilon > 0$ so small that all the sets $V_j^+ = [R, \infty) \times [a_j^+ - \epsilon, a_j^+ + \epsilon]$ and $V_j^- = (-\infty, -R] \times [a_j^- - \epsilon, a_j^+ + \epsilon]$ are pairwise disjoint. We require that the support of h be contained in the union of these and $[-C, C]^2$ where $C := R + 1$.

(ii) The Hamiltonian isotopy ϕ_h^t associated to h exists for all $t \in \mathbb{R}$.

(iii) The restriction of h to each of the sets $T_j^+ = [C, \infty) \times [a_j^+ - \frac{\epsilon}{2}, a_j^+ + \frac{\epsilon}{2}]$ and $T_j^- = (-\infty, -C] \times [a_j^- - \frac{\epsilon}{2}, a_j^+ + \frac{\epsilon}{2}]$ takes the form

$$h(x, y) = \alpha_j^\pm x + \beta_j^\pm \tag{2.10}$$

where each $\alpha_j^\pm \in \mathbb{R}\backslash\{0\}$ has absolute value so small that

$$\phi_h^t([C, \infty) \times \{a_j^+\}) \subset T_j^+ \quad \forall\, t \in [-1, 1]$$

and

$$\phi_h^t((-\infty, -C] \times \{a_j^-\}) \subset T_j^- \quad \forall\, t \in [-1, 1].$$

(iv) $\phi_h^t([-C, C]^2) = [-C, C]^2$ for all $t \in [-1, 1]$.

It is easy to verify the existence of such an h and having fixed one we denote by $\mathfrak{h}$ the corresponding class of perturbation functions. This class is defined as follows: $h' \in C^\infty(\mathbb{R}^2)$ is an element of $\mathfrak{h}$ if and only if it satisfies (i)-(iv) and $h = h'$ outside $[-C, C]^2$.

Given a fixed class of perturbation functions $\mathfrak{h}$ we now specify the requirements for the data going into the definition of the Floer chain complexes we want to consider.

(i) $\tilde{\mathcal{H}}_\mathfrak{h}$ denotes the space of all Hamiltonians $\tilde{H} \in C^\infty([0, 1] \times \tilde{M})$ satisfying the condition that there is a compact subset $Y \subset (-C, C)^2$ (depending on $\tilde{H}$) such that

$$\tilde{H}_t(z, p) = h(z) + H_t(p) \quad \forall\, (t, z, p) \in [0, 1] \times (\mathbb{R}^2\backslash Y) \times M, \tag{2.11}$$

for some Hamiltonian $H \in C_c^\infty([0, 1] \times M)$ and some $h \in \mathfrak{h}$.

(ii) $\tilde{\mathcal{J}}_\mathfrak{h}$ denotes the space of time dependent $\tilde{\omega}$-compatible almost complex structures $\tilde{J} = \{\tilde{J}_t\}_{t\in[0,1]}$ on $\tilde{M}$ satisfying the additional condition that the canonical projection $\pi : \tilde{M} \to \mathbb{R}^2$ restricts to a $(\tilde{J}_t, (\phi_h^t)_* i)$-holomorphic map on $(\mathbb{R}^2\backslash[-C, C]^2) \times M$ for all $t \in [0, 1]$. Here i denotes the canonical complex structure on $\mathbb{C} \approx \mathbb{R}^2$ and h is some element of $\mathfrak{h}$.

Note that $\tilde{\mathcal{H}}_{\mathfrak{h}}$ is a convex space. Given a non-degenerate $\tilde{H} \in \tilde{\mathcal{H}}_{\mathfrak{h}}$, in the sense that $\phi^1_{\tilde{H}}(V) \pitchfork V$, and a generic $\tilde{J} \in \tilde{\mathcal{J}}_{\mathfrak{h}}$ we want to consider the Floer chain complex

$$(CF_*(\tilde{H}, \tilde{J} : V), d),$$

defined in [104]. Due to our non-compact setting we need to verify that all finite energy Floer trajectories, i.e. finite energy solutions $u : \mathbb{R} \times [0,1] \to \tilde{M}$ of Floer's equation

$$\begin{cases} \partial_s u + \tilde{J}_t(u)(\partial_t u - X_{\tilde{H}_t}(u)) = 0 \\ u(\mathbb{R} \times \{0,1\}) \subset V, \end{cases}$$

stay in a compact set. The next proposition ensures that this is the case.

Proposition 2.5.3. *Let $\tilde{H} \in \tilde{\mathcal{H}}_{\mathfrak{h}}$ be non-degenerate and let $\tilde{J} \in \tilde{\mathcal{J}}_{\mathfrak{h}}$. Then all finite energy solutions of Floer's equation are contained in $[-C, C]^2 \times M$. As a consequence the pair $(\tilde{H}, \tilde{J})$ is regular for generic $\tilde{J} \in \tilde{\mathcal{J}}_{\mathfrak{h}}$ in the sense that $(CF_*(\tilde{H}, \tilde{J} : V), d)$ is a well-defined chain complex. Moreover, for every two regular Floer data $(\tilde{H}^-, \tilde{J}^-), (\tilde{H}^+, \tilde{J}^+) \in \tilde{\mathcal{H}}_{\mathfrak{h}} \times \tilde{\mathcal{J}}_{\mathfrak{h}}$ and every regular homotopy of Floer data from $(\tilde{H}^-, \tilde{J}^-)$ to $(\tilde{H}^+, \tilde{J}^+)$, there is a continuation chain map*

$$(CF_*(\tilde{H}^-, \tilde{J}^- : V), d) \to (CF_*(\tilde{H}^+, \tilde{J}^+ : V), d),$$

which induces an isomorphisms on homology. This isomorphism is canonical in the sense that it is independent of the choice of regular homotopy of Floer data.

Proof. This follows immediately from the compactness and transversality arguments carried out in [10] and [11]. In fact the compactness argument runs analogously to the one carried out in the proof of Proposition 2.5.4 below. ☐

As usual we will identify all Floer homology groups via the canonical isomorphisms induced by continuation maps. In this way we obtain an abstract Floer homology group which we denote by $HF_*(V, \mathfrak{h})$.

As explained in Section 2.3 the main structure needed to extract spectral invariants from homology groups is an $\mathbb{R}$-filtration. Given $a \in \mathbb{R}$ and a regular Floer datum $(\tilde{H}, \tilde{J}) \in \tilde{\mathcal{H}}_{\mathfrak{h}} \times \tilde{\mathcal{J}}_{\mathfrak{h}}$, we denote by $(CF^a_*(\tilde{H}, \tilde{J} : V), d)$ the Floer chain complex generated by critical points of the action functional $\mathcal{A}_{\tilde{H}:V}$ whose action is $< a$. This is a well-defined chain complex because the Floer differential d is action decreasing. We denote by $HF^a_*(\tilde{H}, \tilde{J} : V)$ its homology and by $\iota^a_* : HF^a_*(\tilde{H}, \tilde{J} : V) \to HF_*(\tilde{H}, \tilde{J} : V)$ the map induced by the inclusion $\iota^a : (CF^a_*(\tilde{H}, \tilde{J} : V), d) \hookrightarrow (CF_*(\tilde{H}, \tilde{J} : V), d)$.

The PSS isomorphism

Suppose we are given a Lagrangian $V \subset (\tilde{M}, \tilde{\omega})$ with cylindrical ends as above together with a regular quantum datum $\mathcal{D} = (\tilde{f}, \tilde{\rho}, \tilde{J}')$ *adapted to the exit region* $S = \partial V$ in the sense of Section 3 in [89]. Here $(\tilde{f}, \tilde{\rho})$ denotes a Morse-Smale pair on V satisfying additional conditions as in [89]. In particular $\tilde{f}$ is required to be split on

$$\left(\bigsqcup_{i=0}^{k_-} (-\infty, -R + \delta] \times \{a_i^-\} \times L_i \right) \cup \left(\bigsqcup_{j=0}^{k_+} [R - \delta, \infty) \times \{a_j^+\} \times L'_j \right)$$

for some small $\delta > 0$ and $-\nabla^{\tilde{\rho}} \tilde{f}$ is required to point outwards along $\partial V|_{[-R,R]^2}$. Also, $\tilde{J}'$ denotes a generic almost complex structure on $\tilde{M}$ satisfying the condition that $\pi : \tilde{M} \to \mathbb{R}^2$ restricts to a $(\tilde{J}', i)$-holomorphic function on $(\mathbb{R}^2 \backslash [-R + \delta, R - \delta]^2) \times M$. As showed in [89] and [10] the quantum chain complex $(QC_*(\mathcal{D} : V, \partial V), d)$ is then an honest chain complex whose homology $QH_*(\mathcal{D} : V, \partial V)$ is independent of the choice of regular quantum datum $\mathcal{D}$.

We now fix a choice of perturbation function h for V and require it satisfy the following condition, which is identical to the one used in Section 5.2 of [10].

$$\begin{gathered}\text{For every } j \in \{1, \ldots, k_\pm\} \text{ the constant } \alpha_j^\pm \text{ in} \\ \text{(2.10) is required to satisfy } \pm\alpha_j^\pm < 0.\end{gathered} \tag{2.12}$$

Denote by $\mathfrak{h}$ the corresponding class of perturbation functions. It was discovered in [10] (see also Remark 3.5.1. in [11]) that this specific choice of class implies that there is a PSS-type isomorphism $QH_*(V, \partial V) \cong HF_*(V, \mathfrak{h})$. Fixing a regular Floer datum $(\tilde{H}, \tilde{J}) \in \tilde{\mathcal{H}}_\mathfrak{h} \times \tilde{\mathcal{J}}_\mathfrak{h}$ we will now point out how this isomorphism adapts to our setup. More precisely, we will define chain maps

$$\mathrm{PSS}_+ : QC_*(\mathcal{D} : V, \partial V) \to CF_*(\tilde{H}, \tilde{J} : V) \tag{2.13}$$

$$\mathrm{PSS}_- : CF_*(\tilde{H}, \tilde{J} : V) \to QC_*(\mathcal{D} : V, \partial V) \tag{2.14}$$

which, at the level of homology, are inverse to each other and induce a canonical isomorphism $QH_*(V, \partial V) \cong HF_*(V, \mathfrak{h})$. In the standard case of closed monotone Lagrangians of closed symplectic manifolds this was carried out in [15]. Moreover, the construction is described in great detail in [104].

We first introduce some notation. Define $Z := \mathbb{R} \times [0, 1]$ and $Z_\pm := \{(s, t) \in Z \mid \pm s \geq 0\}$, viewed as subsets of $\mathbb{R}^2 \approx \mathbb{C}$. We will think of $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ as a Riemann surface with boundary equipped with the conformal structure it inherits from $\mathbb{C}$. Define also $D_\pm := D^2 \backslash \{\pm 1\}$ where we view ± 1 as a positive $(+)$, repectively negative $(-)$, boundary puncture in the sense of [104] (see also [84]) and equip the punctures with the standard strip-like ends $\epsilon_\pm : Z_\pm \to D_\pm$ given by

$$\epsilon_\pm(z) = \frac{e^{\pi z} - i}{e^{\pi z} + i}, \quad z \in Z_\pm.$$

Choose once and for all two functions $a_\pm \in C^\infty(D_\pm, [0, 1])$ satisfying the following conditions:

(i) $a_\pm(z) = 0$ whenever $z \notin \mathrm{Image}(\epsilon_\pm)$ or $z = \epsilon_\pm(s, t)$ for $t = 0$ and/or $\pm s \leq 1$.

(ii) $a_\pm(\epsilon_\pm(s, t)) = t$ whenever $\pm s \geq 2$.

(iii) $\pm\partial_s(a_\pm \circ \epsilon_\pm)(s, 1) > 0$ whenever $1 < \pm s < 2$.

Following [11] we consider now a specific type of perturbation data $(\tilde{K}^\pm, \tilde{I}^\pm)$ on $D_\pm$, compatible with the Floer data $(\tilde{H}, \tilde{J})$. That is, we will consider pairs $(\tilde{K}^\pm, \tilde{I}^\pm)$ where $\tilde{K}^\pm \in \Omega^1(D_\pm, C^\infty(\tilde{M}))$ is a 1-form on $D_\pm$ with values in $C^\infty(\tilde{M})$ and $(\tilde{I}_z^\pm)_{z \in D_\pm}$ is a family of $\tilde{\omega}$-compatible almost complex structures on $\tilde{M}$. The specific requirements we make are as follows.

(i) Globally (on all of $D_\pm$) we have $\tilde{K}^\pm = da_\pm \otimes \tilde{h} + k_\pm$ where $\tilde{h} := h \circ \pi$ for some $h \in \mathfrak{h}$ and each of the other ingredients are required to satisfy

  (a) $\epsilon_\pm^* \tilde{K}^\pm = \tilde{H} dt$ on $\{(s, t) \in Z_\pm \mid \pm s \geq 2\}$.

  (b) $\tilde{K}^\pm = 0$ on $D_\pm \backslash \mathrm{Image}(\epsilon_\pm)$ and $\epsilon_\pm^* \tilde{K}^\pm = 0$ on $\{(s, t) \in Z_\pm \mid \pm s \leq 1\}$.

  (c) $k_\pm(\xi) = 0$ for all $\xi \in T\partial D_\pm$.

  (d) For $C = R+1$ as in the previous subsection we have $d\pi(X_{k_\pm}) = 0$ on $(\mathbb{R}^2 \backslash [-C, C]^2) \times M$.

(ii) $\tilde{I}^{\pm}_z = \tilde{J}'$ for all $z = \epsilon_{\pm}(s,t)$ with $\pm s < 1$.

(iii) $\tilde{I}^{\pm}_z = \tilde{J}_t$ for all $z = \epsilon_{\pm}(s,t)$ with $\pm s > 2$.

(iv) $\tilde{I}^{\pm}$ satisfies the condition, that the restriction of $\pi : \tilde{M} \to \mathbb{R}^2$ to $(\mathbb{R}^2 \backslash [-C,C]^2) \times M$ is $(I^{\pm}_z, (\phi_h^{a_{\pm}(z)})_* i)$-holomorphic for all $z \in D_{\pm}$.

We will call a perturbation datum $(\tilde{K}^{\pm}, \tilde{I}^{\pm})$ satisying these specified criteria a PSS-*admissible perturbation datum.* Having chosen $(\tilde{K}^{\pm}, \tilde{I}^{\pm})$ we consider solutions $u_{\pm} \in C^{\infty}(D_{\pm}, \mathbb{R}^2 \times M)$ of

$$\begin{cases} d_z u_{\pm} + \tilde{I}^{\pm}_z \circ d_z u_{\pm} \circ i = X_{\tilde{K}^{\pm}} + \tilde{I}^{\pm}_z \circ X_{\tilde{K}^{\pm}} \circ i \\ u_{\pm}(\partial D_{\pm}) \subset V, \end{cases} \tag{2.15}$$

where for $\xi \in T_z D_{\pm}$ the term $X_{\tilde{K}^{\pm}(\xi)}$ denotes the Hamiltonian vector field of the autonomous Hamiltonian $\tilde{K}^{\pm}(\xi) \in C^{\infty}(\tilde{M})$. The following compactness result is a small adaption of the compactness argument appearing in [11].

Proposition 2.5.4. *Let $u_+ \in C^{\infty}(D_+, \tilde{M})$ be a solution of the "$+$"-case of (2.15) and let $u_- \in C^{\infty}(D_-, \tilde{M})$ be a solution of the "$-$"-case of (2.15) satisfying the condition*

$$u_-(1) \in [-C,C]^2 \times M. \tag{2.16}$$

Moreover, suppose both $u_{\pm}$ have finite energy. Then $u_{\pm}(D_{\pm}) \subset [-C,C]^2 \times M$.

Proof. The argument is the same for the two cases, so we only consider $u := u_+ : D_+ \to \tilde{M}$. First note that, since u has finite energy, $u \circ \epsilon_+(s,t)$ converges to a Hamiltonian chord of $\tilde{H}$ connecting V to itself when $s \to \infty$. Since $\tilde{H} \in \tilde{\mathcal{H}}_{\mathfrak{h}}$ all such chords are contained in $[-C,C]^2 \times M$. Now define $\tilde{h} := h \circ \pi : \tilde{M} \to \mathbb{R}$ for some $h \in \mathfrak{h}$ and consider the map $v \in C^{\infty}(D_+, \tilde{M})$ defined by the equation $u(z) = \phi_{\tilde{h}}^{a_+(z)}(v(z)),\ z \in D_+$. Differentiation reveals that v satisfies

$$d_z v + \tilde{I}'_z \circ d_z v \circ i = Y + \tilde{I}'_z \circ Y \circ i,$$

where $X_{\tilde{K}^+} = d\phi_{\tilde{h}}^{a_+(z)}(Y) + da_+ \otimes X_{\tilde{h}}$ and $\tilde{I}^+_z = (\phi_{\tilde{h}}^{a_+(z)})_* \tilde{I}'_z$. Moreover, v satisfies the "moving boundary condition"

$$v(z) \in (\phi_{\tilde{h}}^{a_+(z)})^{-1}(V) \quad \forall\ z \in \partial D_+.$$

Note that it follows from the requirements of $\tilde{K}^{\pm}$ and $\tilde{I}^{\pm}$ that, outside the compact subset[8]

$$U := [-C,C]^2 \times M = \bigcup_{t \in [0,1]} (\phi_{\tilde{h}}^t)^{-1}([-C,C]^2 \times M) \subset \tilde{M},$$

we have $\tilde{I}'_z = i \oplus J_z$ for some almost complex structure J_z on M and $d_{v(z)}\pi(Y) = 0$ for all $z \in D_+$. In particular $\tilde{v} := \pi \circ v : D_+ \to \mathbb{C}$ restricts to a holomorphic function on $v^{-1}(\tilde{M} \backslash U)$. It then follows, using the open mapping theorem from complex analysis and the conformal properties of holomorphic maps as in the proof of Lemma 3.3.2 of [11], that the assumption $\tilde{v}(D_+) \cap (\mathbb{R}^2 \backslash [-C,C]) \neq \emptyset$ contradicts the convergence statement made in the beginning of the proof. Hence $\tilde{v}(\mathbb{R} \times [0,1]) \subset [-C,C]^2$. Since ϕ_h^t preserves $[-C,C]^2$ for all $t \in [-1,1]$ the statement follows. ☐

We note that, since $-\nabla^{\tilde{\rho}} \tilde{f}$ points outwards along $\partial V|_{[-R,R]^2}$, the only relevant solutions of the "$-$"-case of (2.15) for defining PSS are those satisfying (2.16). Transversality issues and energy estimates for moduli spaces of such solutions are dealt with in [11]. With these observations at hand we can define (2.13) and (2.14) exactly as in [15] or [104], to which

[8] Here the last condition imposed on our perturbation function h is crucial.

we refer for details. We recall that (2.13) is defined "by counting" rigid constellations of pearly trajectories and finite energy solutions u_+ of (2.15) subject to the condition that the pearly trajectory "ends" at $u_+(-1)$. (2.14) is defined similarly.

Following [104] one now checks that the homology isomorphism PSS : $QH_*(\mathcal{D} : V; \partial V) \to HF_*(\tilde{H}, \tilde{J} : V)$ is independent of the chosen data and that it respects continuation isomorphisms. Moreover in [89] it is shown that $QH_*(V, \partial V)$ is a unital algebra, and by the standard arguments we have a canonical isomorphism PSS : $QH_*(V, \partial V) \to HF_*(V, \mathfrak{h})$ of unital algebras.

Remark 2.5.5. It is important to note that the specific requirement (2.12) imposed on the elements in $\mathfrak{h}$ in order for the PSS map $QH_*(V, \partial V) \cong HF_*(V, \mathfrak{h})$ to exist is closely connected with the definition of $QH_*(V, \partial V)$. To see the connection we suggest the curious reader take a look at the proof of Proposition 5.2 in [10].

Remark 2.5.6. Note that a consequence of the above discussion is that for any two choices of perturbation functions $h^{\pm}$ satisfying (2.12), but are in distinct classes $h^- \in \mathfrak{h}^-$, $h^+ \in \mathfrak{h}^+$ there is a natural isomorphism $HF_*(V, \mathfrak{h}^-) \cong QH_*(V, \partial V) \cong HF_*(V, \mathfrak{h}^+)$ provided by PSS.

Spectral invariants for Lagrangians with cylindrical ends

We will apply the machinery developed in [58] to Lagrangians with cylindrical ends. The translation to our setup is more or less immediate and we will only need a minimum of properties developed there, so we will here only mention the details needed to carry those properties over to our setup. Let h be a choice of perturbation function satisfying (2.12) and $\mathfrak{h}$ the corresponding perturbation function class. If $(\tilde{H}, \tilde{J}) \in \tilde{\mathcal{H}}_{\mathfrak{h}} \times \tilde{\mathcal{J}}_{\mathfrak{h}}$ is a regular Floer datum and $\alpha \in QH_*(V, \partial V)$ we define

$$l(\alpha, \tilde{H}, \tilde{J}) := \inf\{a \in \mathbb{R} \mid \mathrm{PSS}(\alpha) \in \mathrm{Image}(\iota^a_*) \subset HF_*(\tilde{H}, \tilde{J} : V)\}. \tag{2.17}$$

which is an element of $\mathbb{R} \cup \{-\infty\}$. Here ι^a_* denotes the map on homology induced by the natural map $\iota^a : CF^a_*(\tilde{H}, \tilde{J} : V) \to CF_*(\tilde{H}, \tilde{J} : V)$. It is immediate that $l(0, \tilde{H}, \tilde{J}) = -\infty$. For $\alpha \neq 0$ an argument from[9] [58] shows that the existence of continuation isomorphisms implies that

$$\int_0^1 \min_{\tilde{M}}(\tilde{H}^-_t - \tilde{H}^+_t)dt \le l(\alpha, \tilde{H}^-, \tilde{J}^-) - l(\alpha, \tilde{H}^+, \tilde{J}^+) \le \int_0^1 \max_{\tilde{M}}(\tilde{H}^-_t - \tilde{H}^+_t)dt \tag{2.18}$$

for any two regular Floer data $(\tilde{H}^-, \tilde{J}^-), (\tilde{H}^+, \tilde{J}^+) \in \tilde{\mathcal{H}}_{\mathfrak{h}} \times \tilde{\mathcal{J}}_{\mathfrak{h}}$. In particular it follows that (2.17) does not depend on the specific choice of compatible almost complex structure $\tilde{J}$. We therefore write $l(\alpha, \tilde{H}) = l(\alpha, \tilde{H}, \tilde{J})$. When we want to emphasize that l is associated to the relative quantum homology $QH_*(V, \partial V)$ we write $l_{(V,\partial V)}(\alpha, \tilde{H}) = l(\alpha, \tilde{H})$. Moreover, it follows from (2.18) and genericity of non-degenerate Floer data that $l_{(V,\partial V)}$ extends by continuity to a function

$$l_{(V,\partial V)} : QH_*(V, \partial V) \times \tilde{\mathcal{H}}_{\mathfrak{h}} \to \mathbb{R} \cup \{-\infty\} \tag{2.19}$$

satisfying $l_{(V,\partial V)}(\alpha, \tilde{H}) = -\infty$ if and only if $\alpha = 0 \in QH_*(V, \partial V)$.

2.5.2 Proof of Theorem 2.2.2

For the convenience of the reader the proof is split into several steps. We make use of the notation from the statement of the theorem.

[9] A different setup is considered in the reference, but the argument carries over to our case *mutatis mutandis*.

Step 1: The definition of Φ_V. We briefly recall the definition of the canonical restriction map $j' : QH_*(V, \partial V) \to QH_{*-1}(L')$. For details we refer to Section 9 in [89]. Fix $R > 0$ such that

$$V|_{\mathbb{R}^2\setminus[-R,R]^2} = ((-\infty, -R] \times \{0\} \times L) \cup ([R, \infty) \times \{0\} \times L') .$$

The Morse function $\tilde{f} \in C^\infty(V)$ in the regular quantum datum $\mathcal{D} = (\tilde{f}, \tilde{\rho}, \tilde{J}')$ for $QH_*(V, \partial V)$ which we consider is required to satisfy the following condition.

$$\tilde{f}(t, 0, p) = f^+(p) + \sigma^+(t) \quad \forall\ (t, p) \in [R, R + 1] \times L'$$

where $\sigma^+ : [R, R + 1] \to \mathbb{R}$ has a unique maximum at $R + \frac{1}{2}$ and $f^+ \in C^\infty(L')$ is Morse. Moreover, on $[R, R+1] \times L'$ the Riemannian metric $\tilde{\rho}$ is given by $\rho \oplus \rho^+$ for some metric ρ on $[R, R+1]$ and some metric ρ^+ on L' just as well as $\tilde{J}' = i \oplus J'$ outside $[-R, R]^2 \times M$ for some generic ω-compatible almost complex structure J' on M. In this setup the quotient map

$$QC_*(\mathcal{D} : V, \partial V) \to QC_{*-1}(\mathcal{D}' : L'),$$

where $\mathcal{D}' = (f^+, \rho^+, J')$, is a chain map. The map induced on homology is exactly the map j'. Of course there is similarly a map $j : QH_*(V, \partial V) \to QH_{*-1}(L)$. By Theorem 2.2.2 in [10] V is a quantum h-cobordism. From Lemma 5.1.2. in the same paper it now follows that both j and j' are isomorphisms. That they also respect multiplication is shown in [89], Theorem 1.2. By definition $\Phi_V = j' \circ j^{-1}$. The estimate in Theorem 2.2.2 is therefore equivalent to the estimate

$$|l_L(j(\alpha), \phi) - l_{L'}(j'(\alpha), \phi)| \leq \mathcal{S}(V), \tag{2.20}$$

for all $\alpha \in QH_*(V, \partial V)\setminus\{0\}$ and all $\phi \in \widetilde{\mathrm{Ham}}(M, \omega)$.
Step 2: Adapting V. Our strategy is based on the following trick from [32] which replaces V by a new elementary Lagrangian cobordism $V' : L' \rightsquigarrow L$. Fix once and for all a small $\tilde{\epsilon} > 0$. Given $\psi \in \mathrm{Symp}(\mathbb{R}^2, \omega_{\mathbb{R}^2})$ we define $\tilde{\psi} := \psi \times \mathrm{id} \in \mathrm{Symp}(\tilde{M}, \tilde{\omega})$. We choose a ψ such that every point outside $[-R, R] \times \mathbb{R}$ is fixed and such that $V' := \tilde{\psi}(V)$ satisfies

$$\pi(V') \subset \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \beta(x)\}, \tag{2.21}$$

where $\beta \in C_c^\infty(\mathbb{R}, [0, \infty))$ satisfies $\mathrm{Supp}(\beta) \subset (-R, R)$ and

$$\int_{-\infty}^{\infty} \beta(t)\ dt \leq \mathcal{S}(V') + \tilde{\epsilon}. \tag{2.22}$$

The existence of such ψ and β is quite obvious. Moreover, the construction implies that $V' : L' \rightsquigarrow L$ is an elementary Lagrangian cobordism satisfying $\mathcal{S}(V) = \mathcal{S}(V')$ and $\Phi_V = \Phi_{V'}$.
Step 3: Constructing suitable extensions of H. We first fix a perturbation function h satisfying the following criteria (here we use the notation from the first conditions (i)-(iv) in Section 2.5.1)

- $\mathrm{Supp}(h) \subset V_0^- \cup V_0^+$ and $\partial_y h(x, y) = 0$ for all (x, y) satisfying $|y| < \frac{\epsilon}{2}$.
- h must satisfy (2.12) as well as

$$\partial_x h(x, 0) \geq 0 \quad \forall\ x \leq -R$$
$$\partial_x h(x, 0) \leq 0 \quad \forall\ x \geq R.$$

  Moreover, we require these inequalities be strict for $x < -(R + \frac{1}{3})$ and $x > R + \frac{1}{3}$ respectively.

- Lastly, we require $|h(-(R+\frac{1}{2}),0)|, |h(R+\frac{1}{2},0)| \leq \tilde{\epsilon}$.

Denote by $\mathfrak{h}$ the class corresponding to h. Fix now $0 < \delta \ll \tilde{\epsilon}$ and choose a cut-off $b \in C_c^\infty(\mathbb{R}, [0,\delta])$ such that

$$b = \begin{cases} \delta, & \text{on } [-R-\frac{1}{2}, R+\frac{1}{2}] \\ 0, & \text{on } \mathbb{R}\backslash(-C+\frac{1}{5}, C-\frac{1}{5}). \end{cases}$$

Define the constant $K := \int_{-R-1}^{R+1} (b(t)+\beta(t))\ dt$ which for small enough choice of δ satisfies $K \leq \mathcal{S}(V') + 2\tilde{\epsilon}$. We now define 3 auxiliary functions as follows. We denote by $\rho \in C_c^\infty(\mathbb{R}, [0,1])$ a cut-off satisfying

$$\rho = \begin{cases} 1, & \text{on } [-R, R] \\ 0, & \text{on } \mathbb{R}\backslash(-(R+\frac{1}{2}), R+\frac{1}{2}). \end{cases}$$

By $\eta_-, \eta_+ \in C^\infty(\mathbb{R}, [0,1])$ we denote *monotone* functions satisfying

$$\eta_- = \begin{cases} 0, & \text{on } (-\infty, -R-\frac{4}{5}] \\ -1, & \text{on } [-R-\frac{2}{3}, \infty), \end{cases} \quad \& \quad \eta_+ = \begin{cases} 1, & \text{on } (-\infty, R+\frac{2}{3}] \\ 0, & \text{on } [R+\frac{4}{5}, \infty), \end{cases}$$

as well as $\max_{\mathbb{R}} |\eta_-'|, \max_{\mathbb{R}} |\eta_+'| \leq 10$. Using this auxiliary data we can finally define two specific perturbation functions $h_-, h_+ \in \mathfrak{h}$ by

$$\begin{aligned} h_+(x,y) &= \left(\int_{-R-1}^{x} (b(t)+\beta(t))\ dt\right) \eta_+(x)\rho(y) + h(x,y) \\ h_-(x,y) &= \left(\int_{-R-1}^{x} (b(t)+\beta(t))\ dt - K\right) \eta_-(x)\rho(y) + h(x,y), \end{aligned}$$

where $(x,y) \in \mathbb{R}^2$. We have constructed $h_\pm$ such that they satisfy the following properties. Assuming our data is chosen carefully we achieve

$$\phi_{h_-}^1(\pi(V')) \cap \pi(V') = \{(-R-\tfrac{1}{2}, 0)\} \quad \& \quad \phi_{h_+}^1(\pi(V')) \cap \pi(V') = \{(R+\tfrac{1}{2}, 0)\}.$$

Moreover, $x \mapsto h_-(x,0)$ has a local non-degenerate maximum at $x = -(R+\frac{1}{2})$ and $x \mapsto h_+(x,0)$ has a local non-degenerate maximum at $x = R+\frac{1}{2}$. Given any Hamiltonian $H \in C_c^\infty([0,1]\times M)$ we define associated Hamiltonians $\tilde{H}^-, \tilde{H}^+ \in \tilde{\mathcal{H}}_{\mathfrak{h}}$ by

$$\tilde{H}_t^-(z,p) = H_t(p) + h_-(z) \quad \& \quad \tilde{H}_t^+(z,p) = H_t(p) + h_+(z),$$

for $(t,z,p) \in [0,1]\times\mathbb{R}^2\times M$. For future use we denote $c_\pm := h_\pm(\pm(R+\frac{1}{2}))$. Note that we can estimate

$$|c_- - c_+| \leq 5\tilde{\epsilon}. \tag{2.23}$$

Step 4: Relating spectral invariants of the ends to those of V'. Let $H \in C_c^\infty([0,1]\times M)$ be a normalized Hamiltonian which is non-degenerate both for L and L'. Define $\phi := \phi_H \in \widetilde{\mathrm{Ham}}(M,\omega)$. It suffices to prove the theorem for such ϕ. In the notation of [58] we have $l_L(\alpha, H) = l_L(\alpha,\phi)$ for $\alpha \in QH_*(L)$.

Lemma 2.5.7. *For all $\alpha \in QH_*(V', \partial V')$ it holds that*

$$l_{L'}(j'(\alpha), H) = l_{(V',\partial V')}(\alpha, \tilde{H}^+) - c_+ \quad \& \quad l_L(j(\alpha), H) = l_{(V',\partial V')}(\alpha, \tilde{H}^-) - c_-.$$

Proof. The proofs of the two equalities are similar, so we only prove the first one. Fix once and for all a generic path of ω-compatible almost complex structures $J = \{J_t\}_{t\in[0,1]}$ in M so that the Floer chain complex $(CF_*(H, J : L'), d)$ is well defined. We also want to consider the Floer chain complex $CF_*(\tilde{H}^+, \tilde{J} : V')$ for a specific choice of regular $\tilde{J}$. We

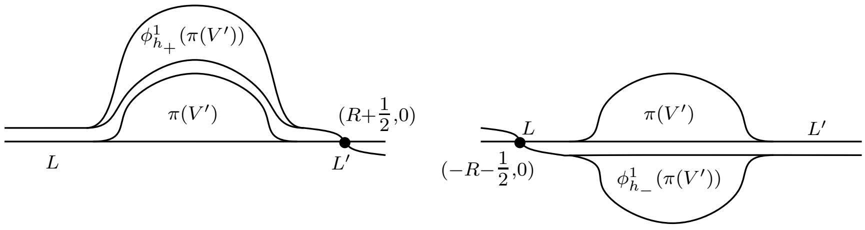

Figure 2.1: The main point about our construction of $h_\pm$ is that they are elements of $\mathfrak{h}$ and their Hamiltonian flows have the following "effect": In the left picture the image of $\pi(V')$ under $\phi^1_{h_+}$ is indicated. In the right picture the image under $\phi^1_{h_-}$.

specify this choice below. Note that, by the construction of h_+ there is a 1-1 correspondence between chords of $X_{\tilde{H}^+}$ connecting V' to itself and chords of X_H connecting L' to itself (see Figure 2.1). We define the subspace $Y_* \subset CF_*(\tilde{H}^+, \tilde{J} : V')$ as the $\mathbb{Z}_2$-vector space generated by those critical points[10] $[\gamma, \widehat{\gamma}]$ of $\mathcal{A}_{\tilde{H}^+:V'}$ for which $\widehat{\gamma}$ is *not* equivalent to any capping of γ whose image is completely contained in the fiber $\{(R+\frac{1}{2},0)\} \times M$. Since any capping of a X_H-chord sitting inside M can be viewed as a capping of the corresponding $X_{\tilde{H}^+}$-chord sitting inside $\{(R+\frac{1}{2},0)\} \times M$ there is a well-defined inclusion map[11]

$$\iota : CF_{*-1}(H, J : L') \hookrightarrow CF_*(\tilde{H}^+, \tilde{J} : V') \tag{2.24}$$

of $\mathbb{Z}_2$-vector spaces. It is clear from the definition of Y_* that the image of ι is a direct complement to Y_*. I.e. we obtain a splitting of $\mathbb{Z}_2$-vector spaces:

$$CF_*(\tilde{H}^+, \tilde{J} : V') = CF_{*-1}(H, J : L') \oplus Y_*. \tag{2.25}$$

Now choose any $\tilde{J} = \{\tilde{J}_t\}_{t\in[0,1]} \in \tilde{\mathcal{J}}_{\mathfrak{h}}$ satisfying the condition that $\tilde{J}_t = (\phi^t_{h_+})_* i \oplus J_t$ outside $[-R,R]^2 \times M$. We claim that $\tilde{J}$ is regular for $\tilde{H}^+$. To see this, note that all finite energy Floer trajectories corresponding to the data $(\tilde{H}^+, \tilde{J})$ are completely contained in the fiber $\{(R+\frac{1}{2},0)\} \times M$. This is easy to see using the same trick as in the proof of Proposition 2.5.4 but is in fact also a simple case of the *bottleneck* construction in the proof of Lemma 3.3.2 in [11]. In particular the linearized operator associated to Floer's equation splits along any finite energy Floer trajectory. It therefore follows from the automatic transversality result in Corollary 4.3.2 from [11], which is based on the theory developed in [85], that $(\tilde{H}^+, \tilde{J})$ is a regular Floer datum. Here it was crucial that $x = R + \frac{1}{2}$ is a local maximum for $\mathbb{R} \ni x \mapsto h_+(x, 0)$. Considering the definition of Y_* it is not hard to see that this implies that (2.25) is in fact a splitting of chain complexes. As a consequence the quotient map $q : (CH_*(\tilde{H}^+, \tilde{J} : V'), d) \to (CH_{*-1}(H, J : L'), d)$ which collapses the Y_* component is a chain map satisfying

$$q(CF^{a+c_+}_*(\tilde{H}^+, \tilde{J} : V')) \subset CF^a_{*-1}(H, J : L') \quad \forall\, a \in \mathbb{R}. \tag{2.26}$$

Similarly one sees that

$$\iota(CF^a_{*-1}(H, J : L')) \subset CF^{a+c_+}_*(\tilde{H}^+, \tilde{J} : V') \quad \forall\, a \in \mathbb{R}. \tag{2.27}$$

[10] Recall that a critical point $[\gamma, \widehat{\gamma}]$ of $\mathcal{A}_{\tilde{H}^+:V'}$ consists of a chord $\gamma : ([0,1], \{0,1\}) \to (\tilde{M}, V')$ satisfying $\dot{\gamma} = X_{\tilde{H}^+}(\gamma)$ and an equivalence class of cappings $\widehat{\gamma}$ of γ, where we say that two cappings of γ are equivalent if they have the same symplectic area.

[11] The fact that ι increases the degree by 1 comes from the fact that we follow the normalization convention of the Conley-Zehnder index from [104]. We point out that this convention corresponds to assigning Maslov index 1 to the loop $\mathbb{R}/\mathbb{Z} \ni t \mapsto e^{-t\pi i}\mathbb{R}$ in the Lagrangian Grassmannian of $(\mathbb{C}, dx \wedge dy) \approx (\mathbb{R}^2, \omega_{\mathbb{R}^2})$.

We now want to choose regular perturbation data for $\mathrm{PSS}_{V'}$ and $\mathrm{PSS}_{L'}$ such that the two diagrams

$$\begin{tikzcd} CF_*(\tilde{H}^+, \tilde{J} : V') & CF_{*-1}(H, J : L') \arrow[l, hook', "\iota"'] \\ QC_*(\mathcal{D} : V', \partial V') \arrow[u, "\mathrm{PSS}_{V'}"] \arrow[r, "j'"] & QC_{*-1}(\mathcal{D}' : L') \arrow[u, "\mathrm{PSS}_{L'}"'] \end{tikzcd}$$

$$\begin{tikzcd} CF_*(\tilde{H}^+, \tilde{J} : V') \arrow[r, "q"] & CF_{*-1}(H, J : L') \\ QC_*(\mathcal{D} : V', \partial V') \arrow[u, "\mathrm{PSS}_{V'}"] \arrow[r, "j'"] & QC_{*-1}(\mathcal{D}' : L') \arrow[u, "\mathrm{PSS}_{L'}"'] \end{tikzcd}$$

commute. Denote by (K^+, I^+) a regular perturbation datum for $\mathrm{PSS}_{L'} : QC_*(\mathcal{D}' : L') \to CF_*(H, J : L')$. By definition of j' we see[12] that in order to accomplish commutativity of the above diagrams it suffices to extend (K^+, I^+) to a regular perturbation datum $(\tilde{K}^+, \tilde{I}^+)$ such that the image of every finite energy solution $u_+ \in C^\infty(D_+, \tilde{M})$ of (2.15) is contained in the fiber $\{(R + \frac{1}{2}, 0)\} \times M$ and can be identified with a solution of the corresponding equation for $\mathrm{PSS}_{L'} : QC_*(\mathcal{D}' : L') \to CF_*(H, J : L')$. Once this has been carried out the claim easily follows from (2.26) and (2.27).

We now argue how to extend (K^+, I^+). First choose a PSS-admissible family of almost complex structures $\{\tilde{I}_z^+\}_{z \in D_+}$ on $\tilde{M}$ satisfying the condition that

$$\tilde{I}_z^+|_{(\mathbb{R}^2 \setminus [-R,R]^2) \times M} = (\phi_{h_+}^{a_+(z)})_* i \oplus I_z^+ \quad \forall\ z \in D_+.$$

Now define

$$\tilde{K}^+ := da_+ \otimes \tilde{h}_+ + k_+,$$

where $k_+ \in \Omega^1(D_+, C^\infty(\tilde{M}))$ is defined by $k_+(\xi) = K^+(\xi)$ for $\xi \in TD_+$. Needless to say, we here view $K^+ \in \Omega^1(D_+, C^\infty(\tilde{M}))$ in the obvious way. Moreover $\tilde{h}_+ := h_+ \circ \pi$. We note that, using this data, it follows from the same argument as above that the image of any finite energy solutions u_+ of (2.15) is contained in the fiber $\{(R + \frac{1}{2}, 0)\} \times M$. Hence, the linearization of the operator associated to (2.15) along any finite energy solution is split. One last time we use Lemma 4.3.1 in [11], for the case $k = 0$ in their terminology, to argue that the perturbation datum $(\tilde{K}^+, \tilde{I}^+)$ is regular.[13] Again it is crucial that $x = R + \frac{1}{2}$ is a local maximum for $x \mapsto h_+(x, 0)$. This finishes the proof of the claim. $\square$

Step 5: The estimate. Applying (2.18) and making use of the lemma together with the estimates obtained in Step 3 one easily computes that, for all $\alpha \in QH_*(V', \partial V') \backslash \{0\}$, we have

$$\begin{aligned} |l_L(j(\alpha), \phi) - l_{L'}(j'(\alpha), \phi)| &\leq |l_{(V', \partial V')}(\alpha, \tilde{H}^-) - l_{(V', \partial V')}(\alpha, \tilde{H}^+)| + 5\tilde{\epsilon} \\ &\leq \int_0^1 \max_{\tilde{M}} |\tilde{H}_t^- - \tilde{H}_t^+| \, dt + 5\tilde{\epsilon} \\ &= \max_{[-C,C]^2} |h_- - h_+| + 5\tilde{\epsilon} \\ &\leq \mathcal{S}(V') + 10\tilde{\epsilon} = \mathcal{S}(V) + 10\tilde{\epsilon}. \end{aligned} \tag{2.28}$$

Since $\tilde{\epsilon} > 0$ was arbitrary the proof is done.

[12] Lemma 1.3 in [89] implies that pearly trajectories which are not completely contained in the fiber $\{(R + \frac{1}{2}, 0)\} \times M$ cannot "end in it".

[13] Technically speaking the case $k = 0$ is not contained in the statement of Lemma 4.3.1 in [11]. However, applying the methods from [85] exactly as in the proof of this lemma one sees that the conclusion of the lemma also holds for the case $k = 0$.

3. Expanding exact Lagrangian cobordism classes

Consider an exact symplectic manifold $(M^{2n}, d\lambda)$ which is tame at infinity. We assume throughout that the Liouville flow $\{\psi_t\}_{t\in\mathbb{R}}$ on $(M, d\lambda)$ is complete. Given a fixed closed exact Lagrangian submanifold $L \subset (M, d\lambda)$ we denote by $\mathcal{L}(L)$ the space of all closed exact Lagrangian submanifolds of $(M, d\lambda)$, which are exact Lagrangian cobordant to L (see definition below). In [32] Cornea and Shelukhin proved the existence of a remarkable "cobordism metric" d_c on $\mathcal{L}(L)$. In this note we prove

Theorem 3.0.1. *Let $L \subset (M, d\lambda)$ be a closed exact Lagrangian submanifold. Then $\{\psi_t\}_{t\in\mathbb{R}}$ induces a flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on $(\mathcal{L}(L), d_c)$ via $\Psi_t(L') := \psi_t(L')$. More precisely, Ψ defines a continuous $\mathbb{R}$-action on $\mathcal{L}(L)$ and every $\Psi_t : \mathcal{L}(L) \to \mathcal{L}(L)$ satisfies*

$$d_c(\Psi_t(L'), \Psi_t(L'')) = e^t d_c(L', L'') \quad \forall\ L', L'' \in \mathcal{L}(L). \tag{3.1}$$

In particular the metric space $(\mathcal{L}(L), d_c)$ has infinite diameter.

Before proving Theorem 3.0.1 we note the following special case.

Corollary 3.0.2. *Suppose $L \subset (M, d\lambda)$ in Theorem 3.0.1 satisfies $\lambda|_L \equiv 0$. Then $\Psi_t(L) = L$ for all $t \in \mathbb{R}$ and L is the unique fixed point of every $\Psi_{t\neq 0}$. In particular Ψ restricts to a free $\mathbb{R}$-action on $\mathcal{L}(L)\backslash\{L\}$. Hence, if $L' \subset (M, d\lambda)$ is another closed Lagrangian satisfying $\lambda|_{L'} \equiv 0$, then $L' \notin \mathcal{L}(L)$.*

Proof. The condition $\lambda|_L \equiv 0$ is equivalent to the condition $\psi_t(L) = L$ for all $t \in \mathbb{R}$. If $L' \in \mathcal{L}(L)$ is a fixed point for some $\Psi_{t\neq 0}$ then

$$d_c(L, L') = d_c(\Psi_t(L), \Psi_t(L')) = e^t d_c(L, L').$$

Since $t \neq 0$, non-degeneracy of d_c implies $L = L'$. $\square$

Prerequisites

Recall that $\psi_t^*\lambda = e^t\lambda$ for all $t \in \mathbb{R}$. Consider $(\mathbb{R}^2(x, y), d\lambda_{\mathbb{R}^2})$, where $\lambda_{\mathbb{R}^2} = xdy$, and denote by $\{\psi_t^{\mathbb{R}^2}\}_{t\in\mathbb{R}}$ the corresponding Liouville flow. Note that the Liouville flow $\{\tilde{\psi}_t\}_{t\in\mathbb{R}}$ on the product $(\tilde{M}, d\tilde{\lambda}) := (\mathbb{R}^2 \times M, d(\lambda_{\mathbb{R}^2} \oplus \lambda))$ is equal to the product flow $\{\psi_t^{\mathbb{R}^2} \times \psi_t\}_{t\in\mathbb{R}}$. We will use the notion of Lagrangian cobordism from [10]. In short, given two closed Lagrangian submanifolds $L, L' \subset (M, d\lambda)$, a Lagrangian cobordism $V \subset (\tilde{M}, d\tilde{\lambda})$ with negative end L and positive end L' is a Lagrangian submanifold $V \subset (\tilde{M}, d\tilde{\lambda})$ which has two cylindrical ends of the type $((-\infty, -R] \times \{0\}) \times L$ and $([R, \infty) \times \{0\}) \times L'$ for some $R > 0$, and satisfies the condition that $([-R, R] \times \mathbb{R} \times M) \cap V$ is compact (we refer to [10] and [32] for further details on cobordisms). We write $V : L' \rightsquigarrow L$ and say that V is exact if it is an exact Lagrangian submanifold of $(\tilde{M}, d\tilde{\lambda})$. Given a Lagrangian cobordism $V : L' \rightsquigarrow L$ one considers its *outline* $ou(V) \subset \mathbb{R}^2$, which by definition is the complement of the union of all unbounded connected components of $\mathbb{R}^2 \backslash \pi(V)$. Here $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ denotes the projection. The shadow $\mathcal{S}(V)$ of a Lagrangian cobordism $V : L' \rightsquigarrow L$ is then by definition the area of $ou(V)$. Given a fixed closed exact Lagrangian submanifold $L \subset (M, d\lambda)$ Cornea and Shelukhin [32] showed that

$$d_c(L', L'') := \inf\{\mathcal{S}(V) \mid V : L'' \rightsquigarrow L' \text{ is an exact Lagrangian cobordism}\}$$

defines a non-degenerate metric on the space $\mathcal{L}(L)$. Note that "being exact Lagrangian cobordant" is an equivalence relation. To fix conventions we remark that, given $H \in C_c^\infty([0,1] \times M)$, we denote by $\{\phi_H^t\}_{t \in [0,1]}$ the Hamiltonian isotopy generated by the vector field X_{H_t} satisfying $i_{X_{H_t}} d\lambda = -dH_t$.

Proof of Theorem 3.0.1

The proof of Theorem 3.0.1 is inspired by a technique originally employed by Biran and Cieliebak in [9] for a different purpose. The first step is the following beautiful observation which (to our knowledge) was first noted by Gromov [41]. We will need the proof later, so we include it here.

Lemma 3.0.3 ([41])**.** *If $L \subset (M, d\lambda)$ is a closed exact Lagrangian then $\{\psi_t(L)\}_{t \in \mathbb{R}} \subset \mathcal{L}(L)$.*

Proof. Let $f \in C^\infty(L)$ satisfy $\lambda|_L = df$. Consider the Lagrangian isotopy $F : \mathbb{R} \times L \to M$ given by $F(t,x) = \psi_t(x)$. One checks that $F^*\lambda|_{(t,x)} = e^t df|_x$, so F is an exact Lagrangian isotopy. By [73, proof of Theorem 3.6.7] or [76, Exercise 6.1.A] we conclude the existence of $H \in C_c^\infty([0,1] \times M)$ such that $\psi_t(L) = \phi_H^t(L)$ for all $t \in [0,1]$ and

$$H_t(\psi_t(x)) = e^t f(x) \quad \forall\ (t,x) \in [0,1] \times L. \tag{3.2}$$

The claim now follows from the Lagrangian suspension construction [73], [76]. □

Lemma 3.0.4. *Fix $t \in \mathbb{R}$ and consider a Lagrangian cobordism $V : L' \rightsquigarrow L$. Then $\tilde{\psi}_t(V) : \psi_t(L') \rightsquigarrow \psi_t(L)$ is a Lagrangian cobordism satisfying*

$$\mathcal{S}(\tilde{\psi}_t(V)) = e^t \mathcal{S}(V). \tag{3.3}$$

Proof. The first claim is immediate from the definitions. Using the fact that $(\psi_t^{\mathbb{R}^2})^* \lambda_{\mathbb{R}^2} = e^t \lambda_{\mathbb{R}^2}$ (3.3) follows from an easy application of the substitution formula in measure and integration theory. □

Now fix two closed Lagrangian submanifolds $L, L' \subset (M, d\lambda)$. Lemma 3.0.4 shows that $\tilde{\psi}_t$ sets up a bijection between the space of Lagrangian cobordisms $L' \rightsquigarrow L$ and the space of Lagrangian cobordisms $\psi_t(L') \rightsquigarrow \psi_t(L)$. Moreover, this bijection increases the shadow (if $t > 0$) or decreases the shadow (if $t < 0$) in the sense of (3.3). If in addition L and L' are exact then so are $\psi_t(L)$ and $\psi_t(L')$ and the bijection restricts to a bijection of exact Lagrangian cobordisms. This proves

Proposition 3.0.5. *Fix $t \in \mathbb{R}$ and a closed exact Lagrangian submanifold $L \subset (M, d\lambda)$. For $L' \in \mathcal{L}(L)$ we have $\psi_t(L') \in \mathcal{L}(\psi_t(L))$ and*

$$d_c(\psi_t(L), \psi_t(L')) = e^t d_c(L, L').$$

Proof of Theorem 3.0.1. Let $L' \in \mathcal{L}(L)$. Then, by Lemma 3.0.3 and Proposition 3.0.5, $\psi_t(L') \in \mathcal{L}(\psi_t(L)) = \mathcal{L}(L)$. This proves that Ψ is well-defined. Now apply Proposition 3.0.5 again to obtain (3.1). To prove that Ψ is continuous as a map $\mathbb{R} \times \mathcal{L}(L) \to \mathcal{L}(L)$ we first show that, for a fixed $L' \in \mathcal{L}(L)$, the map $t \mapsto \Psi_t(L')$ is continuous. Without loss of generality we might assume that $L' = L$ and prove continuity at $t \in [0,1]$. To do this, fix f and H as in the proof of Proposition 3.0.3 and denote by $I \subset [0,1]$ a closed interval with endpoints s_1, s_2. Using (3.2) we compute that the shadow of the Lagrangian suspension corresponding to $\{\phi_H^t(L)\}_{t\in I}$ equals

$$\int_I \max_{\psi_t(L)}(H_t) - \min_{\psi_t(L)}(H_t) dt = (\max_L(f) - \min_L(f)) \int_I e^t dt = (\max_L(f) - \min_L(f))|e^{s_2} - e^{s_1}|.$$

In particular continuity of $t \mapsto \Psi_t(L)$ follows:

$$d_c(\Psi_{s_2}(L), \Psi_{s_1}(L)) \leq (\max_L(f) - \min_L(f))|e^{s_2} - e^{s_1}| \to 0 \quad \text{for } s_2 \to s_1.$$

To finish the proof, fix $(s, L'') \in \mathbb{R} \times \mathcal{L}(L)$. Then

$$\begin{aligned} d_c(\Psi_t(L'), \Psi_s(L'')) &\leq d_c(\Psi_t(L'), \Psi_t(L'')) + d_c(\Psi_t(L''), \Psi_s(L'')) \\ &= e^t d_c(L', L'') + d_c(\Psi_t(L''), \Psi_s(L'')) \to 0 \quad \text{for } (t, L') \to (s, L''). \end{aligned}$$

□

Relation to Hofer geometry

Using ideas from [9, Proof of Lemma 3.2] one can also obtain versions of Theorem 3.0.1 for both Lagrangian and Hamiltonian Hofer geometry. This might already be known to some extend, but we have not been able to find the explicit statement in the literature. So, since it shows a close connection between Hofer geometry and d_c, we sketch the argument. For details and specifics on the Hofer metric d_H on $\mathrm{Ham}(M, d\lambda)$ we refer to [76], whose conventions we follow here.

Fix $T \in \mathbb{R}$. A small adaption of [9, Proof of Lemma 3.2] shows that, for a given Hamiltonian $H \in C_c^\infty([0,1] \times M)$, the Hamiltonian $e^T \psi_{-T}^* H_t(x) := e^T H_t(\psi_{-T}(x))$ generates the flow $\{\psi_T \circ \phi_H^t \circ \psi_{-T}\}_{t\in[0,1]}$. Thus, given a fixed $\phi \in \mathrm{Ham}(M, d\lambda)$ we have a bijection

$$\begin{aligned} \{H \in C_c^\infty([0,1] \times M) \mid \phi_H^1 = \phi\} &\xrightarrow{\approx} \{G \in C_c^\infty([0,1] \times M) \mid \phi_G^1 = \psi_T \circ \phi \circ \psi_{-T}\} \\ H &\longmapsto e^T \psi_{-T}^* H, \end{aligned}$$

which evidently in- or decreases oscillation by a factor of e^T. In particular

$$d_H(\psi_T \circ \phi \circ \psi_{-T}, \mathrm{id}) = e^T d_H(\phi, \mathrm{id}). \tag{3.4}$$

Defining $\Psi : \mathbb{R} \times \mathrm{Ham}(M, d\lambda) \to \mathrm{Ham}(M, d\lambda)$ by $(t, \phi) \mapsto \Psi_t(\phi) := \psi_t \circ \phi \circ \psi_{-t}$, equation (3.4) reads

$$d_H(\Psi_t(\phi), \mathrm{id}) = e^t d_H(\phi, \mathrm{id}) \quad \forall\ (t, \phi) \in \mathbb{R} \times \mathrm{Ham}(M, d\lambda).$$

Using the fact that each $\Psi_t : \mathrm{Ham}(M, d\lambda) \to \mathrm{Ham}(M, d\lambda)$ is a group isomorphism and the fact that d_h is biinvariant the above equation implies

$$d_H(\Psi_t(\phi_1), \Psi_t(\phi_2)) = e^t d_H(\phi_1, \phi_2) \quad \forall\ t \in \mathbb{R},\ \phi_1, \phi_2 \in \mathrm{Ham}(M, d\lambda). \tag{3.5}$$

To check that Ψ is continuous as a map $\mathbb{R} \times \mathrm{Ham}(M, d\lambda) \to \mathrm{Ham}(M, d\lambda)$ one can apply the same argument as in the proof of Theorem 3.0.1 once it is established that, for a fixed $\phi \in \mathrm{Ham}(M, d\lambda)$, the map $\mathbb{R} \ni t \mapsto \Psi_t(\phi) \in \mathrm{Ham}(M, d\lambda)$ is continuous. But this is easily seen to be the case. More precisely, an easy exercise shows the existence of a Hamiltonian $H \in C_c^\infty([0,1] \times M)$ such that $\phi_H^t \circ \phi = \psi_t \circ \phi \circ \psi_{-t}$ for every $t \in [0,1]$. The existence of such H can also be seen as a special case of a famous result due to Banyaga [6, Proposition II.3.3]. We sum up these observations in the following lemma.[1]

Lemma 3.0.6. *$\{\psi_t\}_{t\in\mathbb{R}}$ induces a flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on $(\mathrm{Ham}(M, d\lambda), d_h)$ via $\Psi_t(\phi) := \psi_t \circ \phi \circ \psi_{-t}$. More precisely Ψ defines a continuous $\mathbb{R}$-action on $\mathrm{Ham}(M, d\lambda)$ and, for each $t \in \mathbb{R}$, $\Psi_t : \mathrm{Ham}(M, d\lambda) \to \mathrm{Ham}(M, d\lambda)$ is a group isomorphism satisfying*

$$d_H(\Psi_t(\phi_1), \Psi_t(\phi_2)) = e^t d_H(\phi_1, \phi_2) \quad \forall\ \phi_1, \phi_2 \in \mathrm{Ham}(M, d\lambda).$$

In particular id *is the unique element of* $\mathrm{Ham}(M, d\lambda)$ *which commutes with $\psi_{t\neq 0}$ and the metric space* $(\mathrm{Ham}(M, d\lambda), d_h)$ *has infinite diameter.*

The Lagrangian Hofer geometric version of Theorem 3.0.1 follows easily from arguments similar to the ones above. To state the corresponding result, recall that if $L \subset (M, d\lambda)$ is a closed Lagrangian submanifold then a result due to Chekanov [28] shows that the Lagrangian Hofer metric d_l is non-degenerate on $\mathcal{H}(L) := \{\phi_H^1(L) \mid H \in C_c^\infty([0,1] \times M)\}$.

Lemma 3.0.7. *Let $L \subset (M, d\lambda)$ be a closed exact Lagrangian submanifold. Then $\{\psi_t\}_{t\in\mathbb{R}}$ induces a flow $\{\Psi_t\}_{t\in\mathbb{R}}$ on $(\mathcal{H}(L), d_l)$ via $\Psi_t(L') := \psi_t(L')$. More precisely, Ψ defines a continuous $\mathbb{R}$-action on $\mathcal{H}(L)$ and every $\Psi_t : \mathcal{H}(L) \to \mathcal{H}(L)$ satisfies*

$$d_l(\Psi_t(L'), \Psi_t(L'')) = e^t d_l(L', L'') \quad \forall\ L', L'' \in \mathcal{H}(L).$$

In particular the metric space $(\mathcal{H}(L), d_l)$ has infinite diameter.

[1] One does not need $(M, d\lambda)$ to be tame at infinity in order for the Hofer metric to be non-degenerate on $\mathrm{Ham}(M, d\lambda)$. Therefore Lemma 3.0.6 also holds without this assumption on $(M, d\lambda)$.

4. Topology of (small) Lagrangian cobordisms

4.1 Introduction

In this chapter we study the topology of Lagrangian cobordisms connecting Lagrangian submanifolds of a symplectic manifold (M^{2n}, ω). The idea of relating Lagrangian submanifolds by Lagrangian cobordisms was first conceived by Arnol'd [2]. The idea has recently received a lot of attention, in particular due to Biran-Cornea's work [10], [11] which (among many other things) shows that Lagrangian cobordisms provide a geometric realization of operations in a suitable version of the (derived) Fukaya category. Biran-Cornea further showed that examples of such cobordisms arise as the trace of Lagrange surgery and it is therefore natural to ask if all such cobordisms come from Lagrange surgery. In general there are by now a few explicit constructions available for producing Lagrangian cobordisms. However, the topological and geometric nature imposed on a cobordism by requiring it admit a Lagrangian embedding into $\mathbb{R}^2 \times M$ remain rather mysterious. The present chapter aims at exploring this nature. Some of the questions we attempt to answer are the following: How different can the topology of Lagrangian cobordant Lagrangians be? Does Lagrange surgery of two Lagrangians always give rise to a Lagrangian "trace of surgery"-cobordism? Is there a quantitative way to detect if a cobordism "originates" from Lagrange surgery?

Setting and notation

(M^{2n}, ω) will be assumed either closed or convex at infinity [35]. Our Lagrangian cobordisms live in $\tilde{M} := \mathbb{R}^2(x, y) \times M$ equipped with the symplectic structure $\tilde{\omega} := \omega_{\mathbb{R}^2} \oplus \omega$ where $\omega_{\mathbb{R}^2} := dx \wedge dy$. We denote by $\mathcal{L} = \mathcal{L}(M, \omega)$ the space of all closed, connected Lagrangian submanifolds of (M, ω). A Lagrangian cobordism $V \subset (\tilde{M}, \tilde{\omega})$ relating two ordered tuples $(L_i)_{i=1}^m, (L'_i)_{i=1}^{m'} \subset \mathcal{L}$ will always be assumed connected and is symbolically written

$$V : (L'_i)_i \rightsquigarrow (L_j)_j.$$

Viewing V as an abstract cobordism its boundary ∂V has a positive part and a negative part: $\partial_- V \approx \sqcup_{j=1}^m L_j$, $\partial_+ V \approx \sqcup_{j=1}^{m'} L'_j$. When V is oriented the L_i and L'_j inherit an orientation via the convention $\partial V = -\partial_- V \sqcup \partial_+ V$ (see Section 4.3).

4.2 Main results

The two main results represent "two sides of the same coin". The first one being a cobordism version of the classical adjunction formula for Lagrangian submanifolds. Given oriented

$L, L' \in \mathcal{L}$ we denote by $I(L, L')$ the intersection index of (L, L') computed with respect to the orientation ω^n on M.

Theorem 4.2.1. *Let $V : (L'_i)_i^{m'} \rightsquigarrow (L_j)_j^m$ be an oriented Lagrangian cobordism between two oriented ordered tuples $(L_i)_{i=1}^m, (L'_i)_{i=1}^{m'} \subset \mathcal{L}$. Then*

$$(-1)^{\frac{(n+1)n}{2}} \chi(V, \partial_- V) = \sum_{1 \leq i < j \leq m} I(L_i, L_j) - \sum_{1 \leq i < j \leq m'} I(L'_i, L'_j). \tag{4.1}$$

In the non-oriented case this formula holds true modulo 2.

The other side of the coin is the Floer-homological version of Theorem 4.2.1. It should be thought of as a relative version of the main result in Chekanov's beautiful paper [27]. To state it, we will say that a tuple $(L_i)_{i=1}^m \subset \mathcal{L}$ is *transverse* if $L_i \pitchfork L_j$ for every $i \neq j$.

Theorem 4.2.2. *Let $(L'_i)_{i=1}^{m'}, (L_j)_{j=1}^m \subset \mathcal{L}$ be two transverse tuples and let $V : (L'_i)_i \rightsquigarrow (L_j)_j$ be a small Lagrangian cobordism. If V is spin then*

$$\dim_{\mathbb{F}} H_*(V, \partial_- V; \mathbb{F}) \leq \sum_{1 \leq i < j \leq m} \#(L_i \cap L_j) + \sum_{1 \leq i < j \leq m'} \#(L'_i \cap L'_j). \tag{4.2}$$

for every field $\mathbb{F}$. If V is not spin then (4.2) holds with $\mathbb{F} = \mathbb{Z}_2$.

Let us explain the meaning of the word *small* in the assumptions of this result. Denote by $A(\tilde{M}, V) > 0$ the *bubbling threshold* of V. $A(\tilde{M}, V)$ can intuitively be though of as the area of the smallest non-constant holomorphic disk on V. For closed Lagrangian submanifolds this quantity was introduced by Chekanov [27], but his definition immediately generalizes to Lagrangian cobordisms (see Section 4.3). Recently Cornea and Shelukhin [32] introduced another non-negative quantity associated to V - namely the so-called shadow of V, denoted by $\mathcal{S}(V)$. One can think of $\mathcal{S}(V)$ as measuring the "size" of the projection of V to the $\mathbb{R}^2$-plane (see Section 4.3).

Definition 4.2.3. We say that a Lagrangian cobordism $V : (L'_i)_{i=1}^{m'} \rightsquigarrow (L_i)_{i=1}^m$ is *small* if

$$\mathcal{S}(V) < A(\tilde{M}, V). \tag{4.3}$$

Remark 4.2.4. A main novelty of the class of *small* Lagrangian cobordisms is that they impose no topological restrictions on the bounding Lagrangians beyond being closed. In fact every $L \in \mathcal{L}$ is the end of a small Lagrangian cobordism (e.g. the trivial cobordism $\mathbb{R} \times L \subset (\tilde{M}, \tilde{\omega})$). Moreover, most currently available constructions of Lagrangian cobordisms yield small cobordisms in many examples.

Remark 4.2.5. Recall that if $(V^{n+1}, \partial_+ V, \partial_- V)$ is a compact orientable cobordism and $\mathbb{F}$ denotes a field then Poincaré -Lefschetz duality gives $\mathbb{F}$-vector space isomorphisms

$$H_k(V, \partial_+ V; \mathbb{F}) \cong H_{n+1-k}(V, \partial_- V; \mathbb{F}) \quad \forall\, k \in \mathbb{Z}.$$

Of course, whether orientable or not, this always holds with $\mathbb{F} = \mathbb{Z}_2$. In particular we see that any compact cobordism $(V^{n+1}, \partial_+ V, \partial_- V)$ satisfies $\chi(V, \partial_- V) = (-1)^{n+1} \chi(V, \partial_+ V)$.

4.2.1 Applications to elementary Lagrangian cobordisms

Theorems 4.2.1 and 4.2.2 are easiest to apply to *elementary* Lagrangian cobordisms, i.e. Lagrangian cobordisms $V : L' \rightsquigarrow L$ which have just one negative and one positive end. For such V the right-hand side of (4.2) equals 0. The following corollaries follow directly from this fact. For more detailed proofs we refer to Section 4.4.

Theorem 4.2.6. *Let $L, L' \in \mathcal{L}$ and suppose at least one of them is spin. If $V : L' \rightsquigarrow L$ is a small Lagrangian cobordism then the inclusions $L, L' \hookrightarrow V$ induce isomorphisms on singular (co)homology. In particular, if there exists a small Lagrangian cobordism $V : L' \rightsquigarrow L$, then $H_*(L;\mathbb{Z}) \cong H_*(L';\mathbb{Z})$ as graded groups and $H^*(L;\mathbb{Z}) \cong H^*(L';\mathbb{Z})$ as graded rings. If neither L nor L' is spin then the same result holds for homology with coefficients in $\mathbb{Z}_2$.*

The following result is very much in the spirit of Chekanov's original result [27]. One can interpret it as saying that one cannot (geometrically) displace a Lagrangian by a small cobordism.

Corollary 4.2.7. *Let $L, L' \in \mathcal{L}_1$ and suppose at least one of them is spin. If there exists a small Lagrangian cobordism $V : L' \rightsquigarrow L$ then $L \cap L' \neq \emptyset$. Moreover, if $L \pitchfork L'$ then*

$$\dim_{\mathbb{F}} H_*(L;\mathbb{F}) \leq \#(L \cap L'). \tag{4.4}$$

for every field $\mathbb{F}$. Of course, if neither L nor L' is spin then the same conclusion holds for $\mathbb{F} = \mathbb{Z}_2$.

Corollary 4.2.8. *No oriented $L \in \mathcal{L}$ with $\chi(L) \neq 0$ admits an oriented Lagrangian null-cobordism. Similarly, no $L \in \mathcal{L}$ admits a small Lagrangian null-cobordism.*

Of course the only case where the first conclusion in Corollary 4.2.8 is a *symplectic* phenomenon is when both n and $\chi(L)$ are even.[1] The corollary in particular implies that the only oriented Lagrange surfaces in a symplectic 4-manifold which can be Lagrangian null-cobordant are Lagrangian tori. In contrast, recall that in the smooth category *every* oriented surface is oriented null-cobordant! A final application of Theorem 4.2.6 to elementary Lagrangian cobordisms yields the following result[2]

Corollary 4.2.9. *Let $L, L' \in \mathcal{L}$ be simply connected and suppose there exists a small, simply connected Lagrangian cobordism $V : L' \rightsquigarrow L$. If $\dim(V) \geq 6$ then V is diffeomorphic to $[0,1] \times L$, so in particular L is diffeomorphic to L'.*

Remark 4.2.10. Corollary 4.2.9 concerns the case $\dim(M) \geq 10$. In the case $\dim(M) = 4$ one can apply Theorem 4.2.1 to get a conclusion in the same spirit: Suppose $V : L \rightsquigarrow L'$ is an orientable Lagrangian cobordism between two orientable Lagrange surfaces $L, L' \in \mathcal{L}(M^4, \omega)$. Then, by additivity of χ and Theorem 4.2.1, we have $\chi(L) = \chi(V) = \chi(L')$. Hence L and L' are diffeomorphic!

Example 4.2.11. [3] Consider $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the symplectic structure $\omega_{\mathbb{T}^2}$ inherited from $\mathbb{R}^2$. Denote by $L_h := \{y = \frac{1}{2}\}$ and $L_v := \{x = \frac{1}{2}\}$ the standard horizontal and vertical Lagrangians. Fix two curves $\gamma_1, \gamma_2 \subset \mathbb{T}^2$ as in Figure 4.1 and denote by $\epsilon > 0$ the sum of the areas of the little gray "triangles". Performing Lagrange surgery [75] along γ_1 we obtain the surgered Lagrangian $L_h \# L_v$. By Biran-Cornea's Lagrangian cobordism theory [10] the trace of this surgery can be realized as a Lagrangian cobordism $V_1 : L_h \# L_v \rightsquigarrow (L_v, L_h)$. Similarly we can perform Lagrange surgery along γ_2 and obtain a Lagrangian cobordism $V_2 : (L_v, L_h) \rightsquigarrow L_v \# L_h$. Concatenating V_1 and V_2 we obtain a Lagrangian cobordism $V : L_h \# L_v \rightsquigarrow L_v \# L_h$. Denote by B the bounded connected component of $\mathbb{R}^2 \backslash \pi(V)$, where $\pi : \tilde{M} \to \mathbb{R}^2$ denotes the projection. Consider now a split almost complex structure on $\tilde{\mathbb{T}}^2 := \mathbb{R}^2 \times \mathbb{T}^2$ of the type $\mathfrak{i} \oplus J$, where $\mathfrak{i}$ denotes the standard

[1] Recall that every closed odd dimensional manifold N satisfies $\chi(N) = 0$. Similarly it is well-known that the boundary of a compact manifold has even Euler characteristic, so for $\chi(L)$ odd the conclusion of the corollary follows from classical topology.

[2] I am extremely grateful to François Charette for pointing out this application to me.

[3] This example is very much inspired by the construction carried out in Section 6.2 of [10].

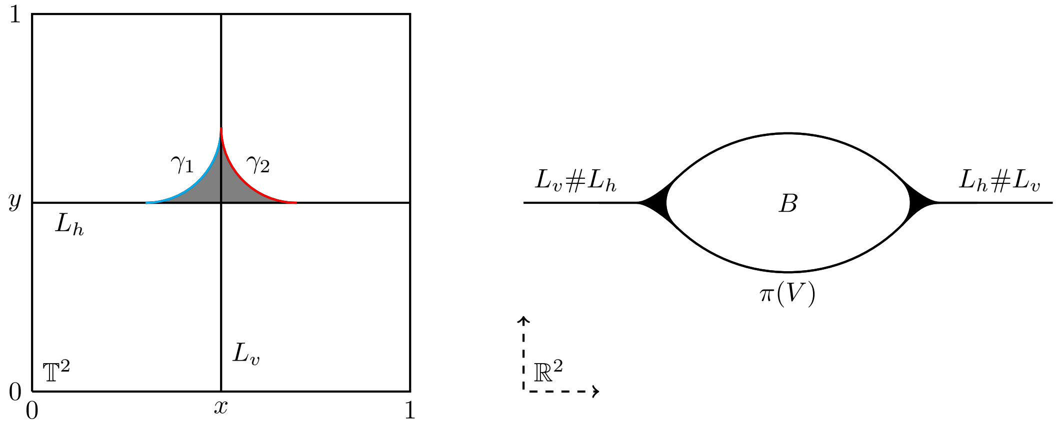

Figure 4.1: In the left figure $L_h, L_v \subset \mathbb{T}^2$ are indicated together with the curves γ_1 (blue) and γ_2 (red) along which Lagrange surgery is performed. The gray region has area ϵ. In the right figure the projection $\pi(V) \subset \mathbb{R}^2$ of V is indicated in black. As a consequence of the construction of Lagrangian cobordism via surgery [10] we have $\text{Area}(\pi(V)) = \epsilon$.

complex structure on $\mathbb{R}^2 \approx \mathbb{C}$. We then have an $\mathfrak{i} \oplus J$-holomorphic disk with boundary on V:

$$\begin{aligned} u : (\overline{B}, \partial\overline{B}) &\to (\tilde{\mathbb{T}}^2, V) \\ z &\mapsto (z, (\tfrac{1}{2}, \tfrac{1}{2})). \end{aligned}$$

Since the curve $u|_{\partial\overline{B}} \subset V$ projects to the non-trivial element of $H_1(\mathbb{R}^2 \backslash B; \mathbb{Z}_2) \cong \mathbb{Z}_2$ the inclusion $L_H\#L_V \hookrightarrow V$ does *not* induce an isomorphism in $\mathbb{Z}_2$-homology. Hence, V is *not* small. In fact it is easy to check that $[u]$ generates $\pi_2(\tilde{\mathbb{T}}^2, V)$ and therefore, if $\epsilon < \text{Area}(B)$, we conclude that the class $[u] \in \pi_2(\tilde{\mathbb{T}}^2, V)$ must contain a $\tilde{J}$-holomorphic disk for every $\tilde{J} \in \tilde{\mathcal{J}}_c$.[4] This implies that $A(\tilde{\mathbb{T}}^2, V) = \int u^*\tilde{\omega} = \text{Area}(B)$ and thus

$$\mathcal{S}(V) = A(\tilde{\mathbb{T}}^2, V) + \epsilon. \tag{4.5}$$

The construction in Example 4.2.11 can be carried out for every $\epsilon > 0$, so in particular (4.5) implies that Theorem 4.2.2 is optimal in the following sense: Its statement would cease to be true if one were to replace $A(\tilde{M}, V)$ by a larger number (or $\mathcal{S}(V)$ by a smaller number) in (4.3). We do not know, however, if Theorem 4.2.2 continues to be true if one replaces "$<$" by "$\leq$" in (4.3).

4.2.2 Applications to Lagrangian cobordisms with multiple ends

Above we saw that if we have a small Lagrangian cobordism $V : L' \rightsquigarrow L$, then the homology of L determines that of both V and L'. In this section we consider small Lagrangian cobordisms $V : L' \rightsquigarrow (L_i)_i$ from a "singleton" $L' \in \mathcal{L}$ to an ordered m-tuple $(L_i)_{i=1}^m \in \mathcal{L}$. The main interest in such cobordisms comes from the fact that, in certain situations, they are known to correspond to (possibly multiple) exact triangles in a suitable version of the derived Fukaya category [11, Theorem A]. The main questions we are interested in concern homological uniqueness: Does the data

$$\bigoplus_{i=1}^m H_*(L_i; \mathbb{Z}) \quad \& \quad \mathcal{I} := \sum_{1 \leq i < j \leq m} \#(L_i \cap L_j) \tag{4.6}$$

[4]Here $\tilde{\mathcal{J}}_c$ denotes the space $\tilde{\omega}$-compatible almost complex structures on $\tilde{\mathbb{T}}^2$ which are *standard at* ∞. -See Section 4.3 below for a precise definition.

associated to $(L_i)_i$ determine $H_*(V;\mathbb{Z})$ and $H_*(L';\mathbb{Z})$? We first note an obstruction to finding small Lagrangian cobordisms with many ends:

Corollary 4.2.12. *Let $(L_i)_{i=1}^m \subset \mathcal{L}$ be a transverse m-tuple of Lagrangians in (M,ω). If $L' \in \mathcal{L}$ and there exists a small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$ then*

$$\mathcal{I} \geq m-1. \tag{4.7}$$

Our first result in the direction outlined above is

Theorem 4.2.13. *Let $(L_i)_{i=1}^m \subset \mathcal{L}$ be a transverse m-tuple such that every L_i is spin and $\mathcal{I} = m-1$. Then every $L' \in \mathcal{L}$ for which there exists a small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$ satisfies*

$$H_l(L';\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } l = 0, n \\ \oplus_{j=1}^m H_l(L_j;\mathbb{Z}), & \text{if } l \neq 0, n \end{cases}$$

and V satisfies

$$H_k(V;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0 \\ \oplus_{j=1}^m H_k(L_j;\mathbb{Z}), & \text{if } k \neq 0, \end{cases}$$

where the isomorphism in the case $k \neq 0$ is induced by the inclusion $\sqcup_{j=1}^{m'} L_j \hookrightarrow V$. Of course, if some L_i is not spin then the same conclusion holds for homology with coefficients in $\mathbb{Z}_2$.

Let's put this result into perspective.

Definition 4.2.14. We will say that an ordered m-tuple $(L_j)_{j=1}^m \subset \mathcal{L}$ is *simple* if it is transverse and satisfies the following two conditions

a) $L_i \cap L_j \cap L_k = \emptyset$ for all distinct i, j, k

b) $(\cup_{j=1}^m L_j) \subset M$ is a connected subset.

Note that, if $(L_j)_{j=1}^m \subset \mathcal{L}$ is simple then all singular points of the Lagrange immersion $(\sqcup_j L_j) \hookrightarrow M$ are transverse and double.

Biran-Cornea [10] discovered that under certain conditions the trace of Lagrange surgery can be realized as Lagrangian cobordisms with multiple ends. In fact, as we will see below these "trace of surgery"-cobordisms are often small. Conversely, Theorem 4.2.13 suggests that if $\mathcal{I}$ is not too large, then every small $V : L' \rightsquigarrow (L_i)_i$ is (homologically) the "trace of surgery"-cobordism of the $(L_i)_i$ and L' is (homologically) a surgery of $(L_i)_i$. In order to explore this idea further we point out that in the present paper the term "Lagrange surgery" should be understood in the sense of [76]. Recall that, given a simple m-tuple $(L_j)_{j=1}^m \subset \mathcal{L}$, the operation developed in [76] allows one, after choosing an equipment at every singular point of the immersed Lagrangian $(\cup_i L_i) \subset (M,\omega)$, to paste in a Lagrange handle in order to obtain an embedded singleton $\widetilde{\#}_i L_i \in \mathcal{L}$.[5] Although $\widetilde{\#}_i L_i$ in general depends on many choices, the diffeomorphism type of $\widetilde{\#}_i L_i$ only depends on the choice of an equipment of each singular point of $\cup_i L_i$ [75]. While there are no obstructions in the choice of equipment at each intersection point from the point of view of Lagrange surgery, the equipments must be chosen consistently in order to obtain an associated Lagrangian cobordism $\widetilde{\#}_i L_i \rightsquigarrow (L_i)_i$ (see Example 4.2.17 below).

[5] We use $\widetilde{\#}$ in order to emphasize that, if the Lagrange immersion $(\cup_i L_i) \hookrightarrow (M,\omega)$ has multiple singular points, then the surgered Lagrangian $\widetilde{\#}_i L_i$ will not in general coincide with the connected sum $\#_i L_i$.

Corollary 4.2.15. *Let $(L_i)_{i=1}^m \subset \mathcal{L}$ be a simple m-tuple whose intersection graph is a tree. Suppose in addition that every L_i is spin. Then every small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$ from a singleton $L' \in \mathcal{L}$ to $(L_i)_i$ satisfies*

$$H_*(V;\mathbb{Z}) \cong H_*(\widetilde{V};\mathbb{Z}),$$

where $\widetilde{V} : \widetilde{\#}_i L_i \rightsquigarrow (L_i)_i$ is a Lagrangian "trace of surgery" cobordism, the surgery resulting in $\widetilde{\#}_i L_i$ being performed with respect to any equipment of $(L_i)_i$. Moreover,

$$H_*(L';\mathbb{Z}) \cong H_*(\widetilde{\#}_i L_i;\mathbb{Z}).$$

If some L_i isn't spin then these conclusions hold for homology with coefficients in $\mathbb{Z}_2$.

Example 4.2.16. There are many examples of symplectic manifolds (M,ω) which admit simple m-tuples $(L_j)_{j=1}^m \subset \mathcal{L}(M,\omega)$ whose intersection graphs are trees. One such example is the plumbing of m unit codisk bundles of m closed Riemannian manifolds. Other examples are (A_m)-configurations $(m > 1)$ of Lagrangian two-spheres in symplectic 4-manifolds, in the sense of [83] (see also [83, Section 8] for explicit examples of such configurations inside $\{z_1^2 + z_2^2 = z_3^{m+1} + \frac{1}{2}\} \subset \mathbb{C}^3(z_1,z_2,z_3)$). Yet another example can be found in [13] where the authors (among many other things) study Lagrangian submanifolds of $\mathbb{C}P^2$ (symplectically) blown up at two points. Here they show the existence of two Lagrangian spheres having a single transverse intersection point.

Example 4.2.17. Fix $n > 1$ and consider a simple pair $(L_1, L_2) \subset \mathcal{L}(M^{2n},\omega)$ of orientable Lagrangians. Suppose $\#(L_1 \cap L_2) = k+1$ for some $k \in \mathbb{N} \cup \{0\}$. We equip the immersed Lagrangian $L_1 \cup L_2 \subset (M,\omega)$ consistently in the sense that in a Darboux-Weinstein neighborhood of every element of $L_1 \cap L_2$, L_1 is identified with $\mathbb{R}^n$ and L_2 with $i\mathbb{R}^n$. Then fix some point $p \in L_1 \cap L_2$ and prescribe that the equipment at p is positive. This choice induces a sign to the equipment at every other element of $L_1 \cap L_2$ (see [75]). We will say that (L_1, L_2) is *positive* if the sign of the equipment of every element of $L_1 \cap L_2$ thus produced is positive, and we will say that it is *negative* otherwise. This terminology does not depend on the choice of $p \in L_1 \cap L_2$ and (L_1, L_2) is positive if and only if (L_2, L_1) is. Note in particular that if $k = 0$, then the pair (L_1, L_2) is automatically positive. If instead at least one of the Lagrangians in the simple pair $(L_1, L_2) \subset \mathcal{L}$ is non-orientable then we say that (L_1, L_2) is negative.

In case (L_1, L_2) is a positive simple pair a construction due to Biran-Cornea [10] yields a Lagrangian *trace of surgery cobordism*

$$V : \widetilde{\#}_i L_i \rightsquigarrow (L_1, L_2) \tag{4.8}$$

where $\widetilde{\#}_i L_i \approx L_1 \# L_2 \# k P^n$ as a smooth manifold, with $P^n = S^{n-1} \times S^1$. If on the other hand (L_1, L_2) is negative then the construction yields a Lagrangian trace of surgery cobordism as in (4.8) but this time $\widetilde{\#}_i L_i \approx L_1 \# L_2 \# k Q^n$ as a smooth manifold, where Q^n denotes the mapping torus of an orientation reversing involution of S^{n-1}. In either case the "trace of surgery"-cobordism V has the homotopy type of the topological subspace $L_1 \cup L_2 \subset M$ and $\tilde{\omega}(\pi_2(\tilde{M}, V)) = \omega(\pi_2(M, L_1 \cup L_2))$. It is easy to see from the construction of V in [10] that one can achieve $\mathcal{S}(V) < \delta$ for any $\delta > 0$. In particular we see that V can be made small if $\omega(\pi_2(M, L_1 \cup L_2)) = \delta \cdot \mathbb{Z}$ for some $\delta > 0$.

Counting holomorphic disks

We will use the terminology that a Lagrangian $L \in \mathcal{L}(M,\omega)$ is *monotone* if

$$\omega|_{H_2(M,L;\mathbb{Z})} \equiv \tau_L \cdot \mu_L|_{H_2(M,L;\mathbb{Z})}$$

for some constant $\tau_L \geq 0$. Here $\omega|_{H_2(M,L;\mathbb{Z})}$ denotes integration of ω and $\mu_L|_{H_2(M,L;\mathbb{Z})}$ denotes the Maslov index. Given a monotone Lagrangian $L \in \mathcal{L}(M, d\lambda)$ we denote by $\mathcal{D}_L \subset \pi_2(M, L)$ the elements which have Maslov index 2. Suppose now that L is monotone and spin, and that a spin structure for L has been fixed. For $\alpha \in \mathcal{D}_L$ and a ω-compatible almost complex structure $J \in \mathcal{J}(M, \omega)$ we consider the moduli space

$$\widetilde{\mathcal{M}}_L(\alpha, J) := \{u : (\mathbb{D}, \partial\mathbb{D}) \to (M, L) \mid \bar{\partial}_J u = 0,\ [u] = \alpha\}.$$

For generic J this is a $(n+2)$-dimensional manifold, so the quotient $\mathcal{M}_L(\alpha, J)$ by the group of conformal transformation of the disk preserving $1 \in D$ is an n-dimensional oriented compact manifold (the orientation is induced by the choice of a spin structure). For $\alpha \in \mathcal{D}_L$ we define $\eta_L(\alpha; \mathbb{Z}) \in \mathbb{Z}$ to be the degree of the evaluation map $ev : \mathcal{M}_L(\alpha, J) \to L$ given by $ev([u]) = u(1)$. By the usual cobordism argument $\eta_L(\alpha; \mathbb{Z})$ does not depend on the choice of J and it depends on the choice of a spin structure on L only up to a sign. If L is orientable, but not spin then $\mathcal{M}_L(\alpha, J)$ need not be orientable, but it is still compact so the mod 2 degree $\eta_L(\alpha; \mathbb{Z}_2) \in \mathbb{Z}_2$ of ev is well-defined. The following result first appeared in [26] with the assumption that the cobordism V is monotone rather than small. However, the proof presented there seems to contain a gap. We do not know if the result as stated in [26] holds true but here we prove it under the stronger assumption that V is small.

Corollary 4.2.18. *Let $L, L' \in \mathcal{L}(M, \omega)$ be monotone and spin and equipped with spin structures. Suppose moreover that $V : L' \rightsquigarrow L$ is a small Lagrangian cobordism. Then the isomorphism $H_*(L; \mathbb{Z}) \cong H_*(L'; \mathbb{Z})$ from Theorem 4.2.6 induces a bijection $\mathcal{D}_L \leftrightarrow \mathcal{D}_{L'}$ such that the diagram*

$$\begin{tikzcd} \mathcal{D}_L \arrow[rr, leftrightarrow] \arrow[dr, "\eta_L(\cdot;\mathbb{Z})"'] & & \mathcal{D}_{L'} \arrow[dl, "\eta_{L'}(\cdot;\mathbb{Z})"] \\ & \mathbb{Z} & \end{tikzcd}$$

commutes up to sign.

In [25] and [26] Chekanov found and studied his famous *exotic* Lagrangian tori in standard symplectic vector space. For each $k \in \{1, \dots, n\}$ Chekanov produced a monotone Lagrangian torus $T^n_k \in \mathcal{L}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ with the property that T^n_k and $T^n_{k'}$ are in different Hamiltonian isotopy classes whenever $k \neq k'$.[6] This result was proved in [26] by showing that there are exactly k elements $\alpha \in \mathcal{D}_{T^n_k}$ for which $\eta_{T^n_k}(\alpha; \mathbb{Z}_2) = 1$. A consequence of Corollary 4.2.18 and [26, Lemma 2.1] is that the same holds true for small cobordisms: T^n_k and $T^n_{k'}$ are not cobordant by a small Lagrangian cobordism if $k \neq k'$. It was of course already known that there exist no orientable monotone Lagrangian cobordism $T^n_k \rightsquigarrow T^n_{k'}$ for k odd and k' even (see e.g. [10, Remark 2.3.1.v.]).

We will now consider an explicit example using the T^n_k in order to demonstrate that Lagrange surgery not always can be realized as a (small) Lagrange cobordism. Given $a > 0$ we denote by $T^n_k(a)$ Chekanov's special tori embedded in $\mathbb{R}^{2n}$ in such a way that a Maslov 2 disk has area a. Suppose in the following that $n \geq 3$ is odd and consider $T := T^n_k(a)$ for k even and $T' := T^n_{k'}(a)$ for k' odd. Now (Hamiltonian) perturb T and T' such that $T \pitchfork T'$ and $\#(T \cap T') = 2$. Choose a compatible equipment of the tuple (T, T'). Clearly this equipment must be a negative one. We denote by $T \widetilde{\#}_- T'$ the surgery performed with respect to this compatible equipment. Note that $T \widetilde{\#}_- T'$ is non-orientable and that we have a "trace of surgery"-cobordism $T \widetilde{\#}_- T' \rightsquigarrow (T, T')$. Since we assume n is odd we can switch the sign of the equipment at one of the intersection points by simply changing the choice of which torus is identified with $\mathbb{R}^n$ and which is identified with $\mathrm{i}\mathbb{R}^n$ at that point. We denote by $T \widetilde{\#}_+ T'$ the surgery performed with respect to this changed equipment. Note that $T \widetilde{\#}_+ T'$ is orientable.

[6] In this notation T^n_n is the Lagrangian product torus consisting of the product of n circles in $\mathbb{R}^{2n} = (\mathbb{R}^2)^n$. For the definition of T^n_k we refer to [25] or [26].

Corollary 4.2.19. *Suppose in the above setting that we have a small Lagrangian cobordism $V : L \rightsquigarrow (T, T')$ for some $L \in \mathcal{L}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$. Then both L and V are non-orientable and*

$$H_*(L; \mathbb{Z}_2) \cong H_*(T \widetilde{\#}_- T'; \mathbb{Z}_2).$$

Moreover,

$$H_*(V; \mathbb{Z}_2) \cong H_*(\widetilde{V}; \mathbb{Z}_2)$$

where $\widetilde{V} : T \widetilde{\#}_- T' \rightsquigarrow (T, T')$ denotes the trace of surgery Lagrangian cobordism. In particular there does not exist a small Lagrangian cobordism $T \widetilde{\#}_+ T' \rightsquigarrow (T, T')$.

Remark 4.2.20 (Kazaryan-Vassilyev characteristic classes)**.** In general, given $(L_i)_i \subset \mathcal{L}$ with $\mathcal{I} \geq m$, one cannot expect that the data (4.6) determines the homology of every $L' \in \mathcal{L}$ for which there exists a small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$. However, in certain situations one can deduce information about $H_*(L')$ if additional information about (Lagrange) characteristic classes of L' is known. Consider a closed manifold W^n with cotangent bundle $T^*W \xrightarrow{\pi_W} W$, and denote by ω the canonical (exact) symplectic form on T^*W. Recall that a *caustic* of a Lagrangian $L \subset (T^*W, \omega)$ is a singularity of the map $\pi_W|_L : L \to W$. Arnol'd observed that there are topological obstructions to the coexistence of different types of caustics for a single Lagrangian $L \subset (T^*W, \omega)$ [98]. In his beautiful book [98] Vassilyev introduced Lagrange characteristic classes in the cohomology ring of a Lagrangian in (T^*W, ω) which "measure" these obstructions (see also [97] and [4, Chapter 6, Section 3.3-3.4]). Later Kazaryan [54], [53] found additional Lagrange characteristic classes corresponding to what he called "hidden singularities" and developed the theory of Lagrange characteristic classes in greater generality. Theorem 4.2.2 shows that the existence of a small Lagrangian cobordism imposes even further restrictions on the Lagrange characteristic classes of its ends than the ones coming from purely homological reasons. More precisely we have the following estimate (here "Lagrange characteristic classes" should be understood in the sense of either [98] or [53]):

Proposition 4.2.21. *Let $(L_i)_{i=1}^m \subset \mathcal{L}(T^*W, \omega)$ be an m-tuple of Lagrangians and suppose there are exactly $k \in \mathbb{N} \cup \{0\}$ Lagrange characteristic classes which are non-zero in some $H^*(L_i; \mathbb{Z}_2)$. Suppose $L' \in \mathcal{L}(T^*W, \omega)$ is a Lagrangian for which there exists a small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$. Then L' has at most $\mathcal{I} + k + 1 - m$ distinct non-zero Lagrange characteristic classes in $H^{\geq 1}(L'; \mathbb{Z}_2)$.*

Given a tuple $(L_i)_{i=1}^m \subset \mathcal{L}(T^*W, \omega)$, one can apply this estimate and Kazaryan-Vassilyev's theory to obtain information about the caustics of a Lagrangian $L' \in \mathcal{L}(T^*W, \omega)$ for which there exists a small Lagrangian cobordism $V : L' \rightsquigarrow (L_i)_i$. The proposition is particularly nice if every L_i is an exact perturbation of the zero-section $W \subset T^*W$, because in this case one has $k = 0$. As an example, consider $S^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid |x| = 1\}$ for $n \leq 6$ and let $f : S^n \to \mathbb{R}$ be the height function $f(x) = x_{n+1}$. Denote by L_1 the zero section in (T^*S^n, ω) and by $L_2 := \phi_F^1(L_1)$, where $F := \pi_{S^n}^* f \in C^\infty(T^*S^n)$. Then $\mathcal{I} = m = 2$ and $k = 0$ for the pair $(L_1, L_2) \subset \mathcal{L}(T^*S^n, \omega)$. If $L' \in \mathcal{L}(T^*S^n, \omega)$ is a Lagrangian for which there exists a small Lagrangian cobordism[7] $V : L' \rightsquigarrow (L_1, L_2)$ then the proposition implies that L' can have at most 1 non-trivial Lagrange $\mathbb{Z}_2$-characteristic class. Hence, if in addition it is known that L' is nonorientable Vassilyev's theory [98] implies that all singularities of the map $\pi_{S^n}|_{L'} : L' \to S^n$ of codimension > 1 are $\mathbb{Z}_2$-homologically trivial, in the sense that their associated characteristic classes vanish.

Remark 4.2.22. Consider for $\delta > 0$ the space $\mathcal{L}_\delta := \{L \in \mathcal{L} \mid \omega(\pi_2(M, L)) = \delta \cdot \mathbb{Z}\}$. In [32] Cornea and Shelukhin defined $d_c : \mathcal{L}_\delta \times \mathcal{L}_\delta \to [0, \infty]$ by

$$d_c(L, L') := \inf_V (\mathcal{S}(V)),$$

[7] By Example 4.2.17 such a Lagrangian is obtained by performing Lagrange surgery on the tuple (L_1, L_2).

where the infimum runs over all Lagrangian cobordisms $V : L' \rightsquigarrow L$ satisfying $\tilde{\omega}(\pi_2(\tilde{M}, V)) = \delta \cdot \mathbb{Z}$. They showed that d_c defines a (non-degenerate) metric on $\mathcal{L}_\delta$. From this point of view Theorem 4.2.2 says that $H_*(L; \mathbb{Z}_2) \cong H_*(L'; \mathbb{Z}_2)$ if $L, L' \in \mathcal{L}_\delta$ satisfy $d_c(L, L') < \delta$.

Remark 4.2.23. Consider the subgroup $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega}) \leq \mathrm{Symp}(\tilde{M}, \tilde{\omega})$ of symplectomorphisms which are *compactly supported relative to* M. $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega})$ consists, by definition, of the $\psi \in \mathrm{Symp}(\tilde{M}, \tilde{\omega})$ for which there exist a compact subset $C \subset \mathbb{R}^2$ and a $\psi' \in \mathrm{Symp}(M, \omega)$ such that $\psi = \mathrm{id} \times \psi'$ on $(\mathbb{R}^2 \backslash C) \times M$. As will be clear from Definition 4.3.4 below, $A(\tilde{M}, V)$ it is invariant under elements of $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega})$. Hence, in the situation of Example 4.2.11 we conclude that

$$\mathcal{S}(\psi(V)) \geq \mathrm{Area}(B) \quad \forall\, \psi \in \mathrm{Symp}_{c|M}(\tilde{\mathbb{T}}^2, \tilde{\omega}).$$

This can be viewed as a kind of non-squeezing statement. More generally we obtain the following "Lagrangian non-squeezing statement": If $V : L' \rightsquigarrow L$ is a Lagrangian cobordism satisfying $H_*(V, L; \mathbb{Z}_2) \neq 0$, then Corollary 4.2.6 implies that

$$\mathcal{S}(\psi(V)) \geq A(\tilde{M}, V) \quad \forall\, \psi \in \mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega}).$$

See also [21] for a different Lagrangian cobordism non-squeezing result.

Remark 4.2.24. The main observation in the proof of Corollary 4.2.12 can also be used to obtain the following estimate: If $V : (L_i')_{i=1}^{m'} \rightsquigarrow (L_i)_{i=1}^{m}$ is a small Lagrangian cobordism and both $(L_i')_{i=1}^{m'}$ and $(L_i)_{i=1}^{m}$ are transverse then

$$\max(m, m') - 1 \leq \sum_{1 \leq i < j \leq m} \#(L_i \cap L_j) + \sum_{1 \leq i < j \leq m'} \#(L_i' \cap L_j').$$

4.2.3 Outlook and questions

To the author's knowledge there a currently 3 known explicit constructions of elementary Lagrangian cobordisms:[8]

a) The Lagrangian suspension construction (see Section 3.1.E in [76]).

b) Lagrangian antisurgery. This construction was recently introduced by Haug [44].

c) Concatenating multi-ended Lagrangian cobordisms which are constructed as the trace of Lagrange surgery (this construction is due to Biran-Cornea [10]).

Elementary Lagrangian cobordisms of type c) are never small (see Example 4.2.11) and we do not know of any examples where a Lagrangian cobordism of type b) is small. On the other hand there are many examples of small Lagrangian suspensions.

Question 4.2.25. Is every small elementary Lagrangian cobordism $V \subset (\tilde{M}, \tilde{\omega})$ the image of a Lagrangian suspension under an element of $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega})$?

This question is very closely related to a conjecture by Biran-Cornea which states that every exact Lagrangian cobordism is Hamiltonian isotopic to a Lagrangian suspension [94]. Although this conjecture remains unsolved both Suárez [94] and Tanaka [95] have made good progress towards confirming it.

To our knowledge there is only one known explicit construction which produces Lagrangian cobordisms of the type $L' \rightsquigarrow (L_i)_i$ for an m-tuple $(L_i)_{i=1}^{m} \subset \mathcal{L}$ and $L' \in \mathcal{L}$ and that is the Biran-Cornea trace of surgery cobordism [10]. As was noted in Example 4.2.17 such cobordisms can often be made small. Motivated by Corollary 4.2.15 and 4.2.19 we ask:

[8] Up to concatenating Lagrangian cobordisms coming from these constructions and applying symplectomorphisms of course...

Question 4.2.26. Given a simple m-tuple $(L_i)_{i=1}^m \subset \mathcal{L}$ whose intersection graph is a tree as well as $L' \in \mathcal{L}$, is every small Lagrangian cobordism $L' \rightsquigarrow (L_i)_i$ the image under an element of $\mathrm{Symp}_{c|M}(\tilde{M}, \tilde{\omega})$ of a trace of surgery cobordism coming from Lagrange surgery of $(L_i)_i$?

4.3 Preliminaries on Lagrangian cobordisms

Here we collect a few facts and definitions about Biran-Cornea's Lagrangian cobordism theory [10], [11]. We also give precise definitions of the objects used above. Given subsets $V \subset \tilde{M} = \mathbb{R}^2 \times M$ and $U \subset \mathbb{R}^2$ we write $V|_U = V \cap \pi^{-1}(U)$, where $\pi : \tilde{M} \to \mathbb{R}^2$ denotes the natural projection. Given an oriented manifold K^k with boundary ∂K we use the convention that the induced boundary orientation of ∂K is given by the "outward normal first" convention. I.e. if $q \in \partial K$ then $(v_1, \ldots, v_{k-1})$ is an oriented basis for $T_q(\partial K)$ if $(n_q, v_1, \ldots, v_{k-1})$ is an oriented basis for $T_q K$, where $n_q \in T_q K$ points outward from K.

Definition 4.3.1 ([10]). We say that two ordered tuples $(L_i)_{i=1}^m, (L_i')_{i=1}^{m'} \in \mathcal{L}$ are *Lagrangian cobordant* if for some $R > 0$ there exists a smooth compact Lagrangian submanifold $V \subset ([-R, R] \times \mathbb{R} \times M, \omega_{\mathbb{R}^2} \oplus \omega)$ with boundary $\partial V = V \cap (\{\pm R\} \times \mathbb{R} \times M)$ satisfying the condition that for some $\epsilon > 0$ we have

$$V|_{[-R,-R+\epsilon)\times\mathbb{R}} = \bigsqcup_{i=1}^{m} ([-R, -R + \epsilon) \times \{i\}) \times L_i \tag{4.9}$$

$$V|_{(R-\epsilon,R]\times\mathbb{R}} = \bigsqcup_{j=1}^{m'} ((R - \epsilon, R] \times \{j\}) \times L_j'. \tag{4.10}$$

In particular V defines a smooth compact cobordism $(V, \bigsqcup_{i=1}^m L_i, \bigsqcup_{j=1}^{m'} L_j')$. We write $V : (L_j')_j \rightsquigarrow (L_i)_i$. If each L_i and each L_j' is oriented then we say that V is an oriented Lagrangian cobordism if the boundary orientation of ∂V coincides with the orientation given by $\partial V = (-\bigsqcup_{i=1}^m L_i) \bigsqcup (\bigsqcup_{j=1}^{m'} L_j')$.

As is customary in the field our notation does not distinguish between a Lagrangian cobordism and its horizontal $\mathbb{R}$-extension. This extension is a *Lagrangian with cylindrical ends*. More generally we have

Definition 4.3.2 ([10]). A *Lagrangian with cylindrical ends* is a boundaryless Lagrangian submanifold $V \subset (\tilde{M}, \tilde{\omega})$ satisfying the conditions that $V|_{[a,b]\times\mathbb{R}}$ is compact for all $a < b$ and that there exists $R > 0$ such that

$$V|_{(-\infty,-R]\times\mathbb{R}} = \bigsqcup_{i=0}^{m} ((-\infty, -R] \times \{a_i^-\}) \times L_i$$

$$V|_{[R,\infty)\times\mathbb{R}} = \bigsqcup_{j=0}^{m'} ([R, \infty) \times \{a_j^+\}) \times L_j'$$

for Lagrangians $L_i, L_j' \subset (M, \omega)$ and constants $a_i^-, a_j^+ \in \mathbb{R}$ verifying $a_i^- \neq a_{i'}^-$ for $i \neq i'$ and $a_j^+ \neq a_{j'}^+$ for $j \neq j'$.

The shadow and bubbling threshold of a Lagrangian cobordism

Given a Lagrangian with cylindrical ends $V \subset (\tilde{M}, \tilde{\omega})$ we denote by $\mathcal{B} = \mathcal{B}(V)$ the collection of gaps in V, i.e. the collection of *bounded* connected componenets of $\mathbb{R}^2 \backslash \pi(V)$. The following notions were coined by Cornea and Shelukhin in [32].

Definition 4.3.3 ([32])**.** Given a Lagrangian with cylindrical ends $V \subset (\tilde{M}, \tilde{\omega})$ we define the *outline of* V as the closed subset of $\mathbb{R}^2$

$$\mathrm{Ou}(V) := \pi(V) \cup (\cup_{B \in \mathcal{B}} B) \,.$$

The *shadow of* V is then defined as the non-negative number

$$\mathcal{S}(V) := \mathrm{Area}\,(\mathrm{Ou}(V))\,.$$

Denote now by $\mathcal{J}$ (respectively $\tilde{\mathcal{J}}$) the space of smooth almost complex structures on M (respectively $\tilde{M}$) which are compatible with the symplectic structure. We denote by $\tilde{\mathcal{J}}_c \subset \tilde{\mathcal{J}}$ the subset consisting of almost complex structures which are *standard at* ∞ in the following sense: For every $\tilde{J} \in \tilde{\mathcal{J}}_c$ there exists a compact set $C \subset \mathbb{R}^2$ such that the restriction of $\tilde{J}$ to $(\mathbb{R}^2 \backslash C) \times M$ has the form $\mathfrak{i} \oplus J$, for some $J \in \mathcal{J}$. We say that $\tilde{J}$ *is supported in* C and we denote by $\tilde{\mathcal{J}}(C) \subset \tilde{\mathcal{J}}_c$ the subset consisting of almost complex structures which are supported in C. Given $\tilde{J} \in \tilde{\mathcal{J}}$ we denote by $A_S(\tilde{M}, \tilde{J})$ the minimal symplectic area of a non-constant $\tilde{J}$-holomorphic sphere in $\tilde{M}$. Given a Lagrangian with cylindrical ends $V \subset (\tilde{M}, \tilde{\omega})$ we denote by $A_D(\tilde{M}, V, \tilde{J})$ the minimal symplectic area of a non-constant $\tilde{J}$-holomorphic disk in $\tilde{M}$ with boundary on V. Suppose $\tilde{J} \in \tilde{\mathcal{J}}(C)$ for a compact set $C \subset \mathbb{R}^2$ and let u be a $\tilde{J}$-holomorphic disk/sphere. It follows from the open mapping theorem that if u satisfies $\mathrm{Image}(\pi \circ u) \not\subset C$ then $z \mapsto \pi \circ u(z)$ is constant. With this fact at hand it is easy to adapt the usual compactness argument to show that

$$A_S(\tilde{M}, \tilde{J}), A_D(\tilde{M}, V, \tilde{J}) > 0 \quad \forall\ \tilde{J} \in \tilde{\mathcal{J}}_c.$$

Definition 4.3.4. Let $V \subset (\tilde{M}, \tilde{\omega})$ be a Lagrangian with cylindrical ends. We define the bubbling threshold $A(\tilde{M}, V)$ of V by

$$A(\tilde{M}, V) := \sup_{\tilde{J} \in \tilde{\mathcal{J}}_c} A(\tilde{M}, V, \tilde{J}),$$

where $A(\tilde{M}, V, \tilde{J}) := \min\{A_D(\tilde{M}, V, \tilde{J}), A_S(\tilde{M}, \tilde{J})\}$.

4.4 Proofs

We begin by proving our applications of Theorem 4.2.1 and 4.2.2. The following remark will be used frequently.

Remark 4.4.1. Applying Theorem 4.2.2 with coefficients in a field $\mathbb{F} \neq \mathbb{Z}_2$ requires us to know that the (small) cobordism $V : (L_i')_{i=1}^{m'} \rightsquigarrow (L_j)_{j=1}^{m}$ is spin. However, in many cases the spin condition follows from the smallness assumption if we know e.g. that every L_i is spin. The idea is the following bootstrapping argument: Suppose the intersection points in (4.2) are so few that one can can apply the $\mathbb{Z}_2$-version of the Theorem 4.2.2 to verify that the inclusion $i : (\sqcup_i L_i) \hookrightarrow V$ induces injections $0 \to H^k(V; \mathbb{Z}_2) \to H^k(\sqcup_i L_i; \mathbb{Z}_2)$ for $k = 1, 2$. Applying i^* to the Stiefel-Whitney classes we have $i^*(w_k(V)) = \sum_{i=1}^{m} w_k(L_i) = 0$ for $k = 1, 2$. Here we use the assumption that every L_i is spin. It follows that $w_k(V) = 0$ for $k = 1, 2$, so V is spin as claimed.

4.4.1 Proofs of results from Section 4.2.1

Proof of Theorem 4.2.6. By Theorem 4.2.2 every small elementary Lagrangian cobordism $V : L' \rightsquigarrow L$ satisfies $H_*(V, L; \mathbb{Z}_2) = 0 = H_*(V, L'; \mathbb{Z}_2)$. In particular the inclusions $L, L' \hookrightarrow V$ induce isomorphisms on $\mathbb{Z}_2$-(co)homology. The $\mathbb{Z}_2$-version of the theorem follows. To obtain the $\mathbb{Z}$-version we apply Remark 4.4.1 to conclude that one of L and L' being spin implies that V is spin. Now we can apply Theorem 4.2.2 to conclude that $H_*(V, L; \mathbb{F}) = 0 = H_*(V, L'; \mathbb{F})$ for every field $\mathbb{F}$. It therefore follows from the homological universal coefficients theorem [43, Corollary 3A.6.] that $H_*(V, L; \mathbb{Z}) = 0 = H_*(V, L'; \mathbb{Z})$. Hence, the inclusions $L, L' \hookrightarrow V$ induce isomorphisms on $\mathbb{Z}$-(co)homology. $\square$

Proof of Corollary 4.2.7. Given a small Lagrangian cobordism $V : L' \rightsquigarrow L$ we can "bend" its right end in order to obtain a Lagrangian null-cobordism $V' : \emptyset \rightsquigarrow (L, L')$. It is not hard to see that this bending can be done in such a way that V' again is small. If $L \pitchfork L'$ then V' has transversally intersecting ends. From the proof of Corollary 4.2.6 we know that V and hence V' are spin. Now the conclusion follows by applying (4.2) to V'. To see this, note that $H_*(V', \partial_+ V'; \mathbb{F}) = H_*(V'; \mathbb{F}) \cong H_*(V; \mathbb{F}) \cong H_*(L; \mathbb{F})$, where the last isomorphism comes from the inclusion $L \hookrightarrow V$. $\square$

Proof of Corollary 4.2.8. If $V : \emptyset \rightsquigarrow L$ is an oriented Lagrangian null-cobordism with boundary $L \in \mathcal{L}$ then elementary algebraic topology implies $[L] = 0 \in H_n(M; \mathbb{Z})$, so $\chi(L) = 0$. For the second part of the corollary, suppose for contradiction that $L \in \mathcal{L}$ admits a *small* Lagrangian null-cobordism $V : \emptyset \rightsquigarrow L$. Then $\partial_+ V = \emptyset$, so $H_0(V, \partial_+ V; \mathbb{Z}_2) = H_0(V; \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand Theorem 4.2.2 implies that $H_0(V, \partial_+ V; \mathbb{Z}_2) = 0$. This contradiction finishes the proof. $\square$

Proof of Corollary 4.2.9. L and L' being simply connected implies that they are both spin. Therefore the assumption that V is small and Theorem 4.2.6 imply that $H_*(V; L; \mathbb{Z}) = 0 = H_*(V, L'; \mathbb{Z})$. Now the conclusion follows from Smale's famous h-cobordism theorem [90], [61, Theorem 9.1]. $\square$

4.4.2 Proofs of results from Section 4.2.2

Proof of Corollary 4.2.12. The inequality (4.7) follows from Theorem 4.2.2 if only we show

$$\dim_{\mathbb{F}} H_n(V, L; \mathbb{F}) \geq m - 1 \tag{4.11}$$

with $\mathbb{F} = \mathbb{Z}_2$. To see this we consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(V, \partial V; \mathbb{F}) \arrow[r, "\partial_{\mathbb{F}}"] & H_n(\partial V, L; \mathbb{F}) \arrow[r, "i_{\mathbb{F}}"] & H_n(V, L; \mathbb{F}) \arrow[r] & \cdots \\ & & \oplus_{j=1}^m H_n(L'_j; \mathbb{F}) \arrow[u, "\cong"] \arrow[r] & H_n(V; \mathbb{F}) \arrow[u] & \end{tikzcd} \tag{4.12}$$

with $\mathbb{F} = \mathbb{Z}_2$. Here the top horizontal line is a piece of the long exact sequence associated with the triple $(V, \partial V, L)$. Note that trivially $\oplus_{j=1}^m H_n(L'_j; \mathbb{Z}_2) \cong \mathbb{Z}_2^m$ and $H_{n+1}(V, \partial V; \mathbb{Z}_2) \cong \mathbb{Z}_2$, so $i_{\mathbb{Z}_2}$ induces an embedding $\mathbb{Z}_2^{m-1} \hookrightarrow H_n(V, L; \mathbb{Z}_2)$, which proves (4.11). $\square$

Proof of Theorem 4.2.13. We first prove the $\mathbb{Z}_2$-version of the result. Since we are assuming $\mathcal{I} = m-1$ the proof above together with Theorem 4.2.2 and Poincaré-Lefschetz duality gives

$$H_k(V, L; \mathbb{F}) = 0 = H_{n+1-k}(V, \partial_+ V; \mathbb{F}) \quad \forall\ k \neq n \tag{4.13}$$

and

$$H_n(V, L; \mathbb{F}) = \mathbb{F}^{m-1} = H_1(V, \partial_+ V; \mathbb{F}) \tag{4.14}$$

with $\mathbb{F} = \mathbb{Z}_2$. Since the square in (4.12) commutes we also know that the map $H_n(V; \mathbb{Z}_2) \to H_n(V, L; \mathbb{Z}_2)$ is onto. Therefore the map $H_k(L; \mathbb{Z}_2) \to H_k(V; \mathbb{Z}_2)$ induced by the inclusion is an isomorphism for all $k < n$. A similar consideration for the long exact sequence associated with the pair $(V, \partial_+ V)$ shows that the map $H_k(\sqcup_{j=1}^m L'_j; \mathbb{Z}_2) \to H_k(V; \mathbb{Z}_2)$ is an isomorphism for all $k > 0$ and therefore

$$\oplus_{j=1}^m H_k(L'_j; \mathbb{F}) \cong H_k(\partial_+ V; \mathbb{F}) \cong H_k(V; \mathbb{F}) \cong H_k(L; \mathbb{F}), \quad 0 < k < n \tag{4.15}$$

follows for $\mathbb{F} = \mathbb{Z}_2$. This finishes the proof for $\mathbb{Z}_2$-coefficients. For the $\mathbb{Z}$-version we first claim that the assumption that every L'_i is spin implies that V is spin. To see this we first check that V is orientable. For this, note that the image of the first Stiefel-Whitney class $w_1(V) \in H^1(V; \mathbb{Z}_2)$ in $H^1(\partial_+ V; \mathbb{Z}_2)$ vanishes, so $w_1(V)$ lifts to an element $\alpha \in H^1(V, \partial_+ V; \mathbb{Z}_2)$ which is dual to $\alpha \frown [V] \in H_n(V, L; \mathbb{Z}_2)$. Exactness in (4.12) and surjectivity of $i_{\mathbb{Z}_2}$ implies that $j_{\mathbb{Z}_2} : H_n(V, L; \mathbb{Z}_2) \to H_n(V, \partial V; \mathbb{Z}_2)$ vanishes. We therefore conclude that

$$0 = j_{\mathbb{Z}_2}(\alpha \frown [V]) = w_1(V) \frown [V],$$

which by Poincaré-Lefschetz duality implies $w_1(V) = 0$, so V is orientable. Note that L too is orientable, being a boundary component of an orientable manifold. To see that also $w_2(V) = 0 \in H^2(V; \mathbb{Z}_2)$ it suffices to note that (4.13) implies that $H^2(V, \partial_+ V; \mathbb{Z}_2) = 0$, so the inlcusion $\partial_+ V \hookrightarrow V$ induces an injection $0 \to H^2(V; \mathbb{Z}_2) \to H^2(\partial_+ V; \mathbb{Z}_2)$. Hence, by Remark 4.4.1 $w_2(V) = 0$ and V is spin. We can therefore fix any field $\mathbb{F}$ and apply Theorem 4.2.2 to conclude $\dim_{\mathbb{F}} H_*(V, L; \mathbb{F}) \leq m - 1$. Since every L'_i is orientable the isomorphism in (4.12) implies $H_n(\partial V, L; \mathbb{F}) \cong \mathbb{F}^m$. Exactly as above we can apply Poincaré-Lefschetz duality to conclude (4.13) and (4.14), this time with coefficients in $\mathbb{F}$. Since these considerations hold for every field $\mathbb{F}$ it follows from the homological universal coefficients theorem [43, Corollary 3A.6.] that (4.13) and (4.14) also hold with $\mathbb{F} = \mathbb{Z}$. To finish the proof we need to check that $i_{\mathbb{Z}}$ is onto. Since all groups displayed in the top horizontal line of (4.12) with $\mathbb{F} = \mathbb{Z}$ are free, we can count ranks to conclude that if $i_{\mathbb{Z}}$ were not onto then $\mathrm{Coker}(i_{\mathbb{Z}})$ would be torsion (see e.g. [49, Chapter II, Theorem 1.6]). Thus, by exactness in (4.12) we conclude that, if $i_{\mathbb{Z}}$ were not onto, then $H_n(V, \partial V; \mathbb{Z})$ would *not* be free. However, duality implies that $H_n(V, \partial V; \mathbb{Z}) \cong H^1(V; \mathbb{Z})$ is free and therefore $i_{\mathbb{Z}}$ must be onto. As in the previous proof it follows that $H_n(V; \mathbb{Z}) \to H_n(V, L; \mathbb{Z})$ is onto so that the inclusion $L \hookrightarrow V$ induces an isomorphism $H_k(L; \mathbb{Z}) \xrightarrow{\cong} H_k(V; \mathbb{Z})$ for all $k < n$. A similar consideration shows that the map $H_1(V, \partial_+ V; \mathbb{Z}) \to H_0(\partial_+ V; \mathbb{Z})$ is injective. Comparing this to (4.13) and (4.14) with $\mathbb{F} = \mathbb{Z}$ one sees that the inclusion $\partial_+ V \hookrightarrow V$ induces an isomorphism $H_k(\partial_+ V; \mathbb{Z}) \xrightarrow{\cong} H_k(V; \mathbb{Z})$ for every $k > 0$, so (4.15) follows with $\mathbb{F} = \mathbb{Z}$. This proves the $\mathbb{Z}$-version of the theorem. □

Proof of Corollary 4.2.15. If the intersection graph of a simple m-tuple $(L'_i)_{i=1}^m \in \mathcal{L}_m$ is a tree then $\mathcal{I} = m - 1$, so the homology of L is computed in Theorem 4.2.13. Moreover, any equipment of the immersed Lagrangian $(\cup_i L'_i) \subset (M, \omega)$ will result in a surgery $\tilde{\#}_i L'_i$ which equals the connected sum $\#_i L'_i$ at the level of smooth manifolds. The result follows. □

Proof of Corollary 4.2.18. Let $L, L' \in \mathcal{L}$ be monotone and spin. Suppose $V : L' \rightsquigarrow L$ is a monotone Lagrangian cobordism. The idea of the following argument is originally due to Chekanov [26]. Chekanov's idea was that the (signed) count of holomorphic disks in a given class $\alpha \in H_2(M, L; \mathbb{Z})$ should coincide with the corresponging number associated to $i_*(\alpha) \in H_2(\tilde{M}, V; \mathbb{Z})$, where $i : (M, L) \hookrightarrow (\tilde{M}, V)$ denotes the inclusion. However, it appears that the proof of this presented in [26] contains a gap, because it seemingly requires that i_* is injective. This is the case if V is small. Hence, assume $V : L' \rightsquigarrow L$ is a small Lagrangian cobordism. By Theorem 4.2.6 we then know that the inclusions $L, L' \hookrightarrow M$ induce isomorphisms $H_*(M, L; \mathbb{Z}) \cong H_*(\tilde{M}, V; \mathbb{Z}) \cong H_*(M, L'; \mathbb{Z})$, so V is also monotone and spin. Since the Maslov index is a characteristic class we have bijections

$$\mathcal{D}_L \leftrightarrow \mathcal{D}_V \leftrightarrow \mathcal{D}_{L'}. \tag{4.16}$$

Choose $R > 0$ such that V is cylindrical outside the "box" $B := [\epsilon - R, R - \epsilon]^2$ for some small $\epsilon > 0$. As in Definition 4.3.1 we will view V as a subset of $([-R, R] \times \mathbb{R} \times M, \tilde{\omega})$ by "cutting off" its ends. For $\tilde{J} \in \tilde{\mathcal{J}}(B)$ and $\alpha_V \in \mathbb{D}_V$ we now consider the moduli space

$$\widetilde{\mathcal{M}}_V(\alpha_V, \tilde{J}) := \{u : (\mathbb{D}, \partial\mathbb{D}) \to (\tilde{M}, V) \mid \partial_{\tilde{J}} u = 0 \;\&\; [u] = \alpha_V\},$$

where $\mathbb{D} \subset \mathbb{C}$ denotes the closed unit disk. By the open mapping theorem in complex analysis the image of every $\tilde{J}$-holomorphic disk on $u \in \widetilde{\mathcal{M}}_V(\alpha_V, \tilde{J})$ passing through a point $(x, y, q) \in V \subset \tilde{M}$ with $|x| > R - \epsilon$ is contained in the fiber $\{(x, y)\} \times M$. Recall that any $\tilde{J} \in \tilde{\mathcal{J}}(B)$ satisfies $\tilde{J}|_{\mathbb{R}^2 \setminus B} = \mathfrak{i} \oplus J$ for some $J \in \mathcal{J}$. It follows that for regular $\tilde{J} \in \tilde{\mathcal{J}}(B)$ the space $\widetilde{\mathcal{M}}_V(\alpha_V, \tilde{J})$ (if $\neq \emptyset$) is a $(n+3)$-dimensional manifold with boundary $\partial\widetilde{\mathcal{M}}_V(\alpha_V, \tilde{J}) = \widetilde{\mathcal{M}}_L(\alpha_L, J) \sqcup \widetilde{\mathcal{M}}_{L'}(\alpha_{L'}, J)$, where $\alpha_L \in \mathcal{D}_L$ and $\alpha_{L'} \in \mathcal{D}_{L'}$ are the unique elements which correspond to α_V under (4.16). The action of the group of automorphisms of $\mathbb{D}$ which preserve $1 \in \partial\mathbb{D}$ respects the boundary, so the quotient $\mathcal{M}_V(\alpha_V, \tilde{J})$ is a smooth $(n+1)$-dimensional manifold with boundary $\partial\mathcal{M}_V(\alpha_V, \tilde{J}) = \mathcal{M}_L(\alpha_L, J) \sqcup \mathcal{M}_{L'}(\alpha_{L'}, J)$. Choosing a spin structure on V, we obtain an orientation of $\mathcal{M}_V(\alpha_V, \tilde{J})$ and by Gromov compactness $\mathcal{M}_V(\alpha_V, \tilde{J})$ is compact. The claim of the corollary now follows from commutativity of the diagram

$$\begin{tikzcd} H_{n+1}(\mathcal{M}_V, \partial\mathcal{M}_V; \mathbb{Z}) \arrow[r, "ev"] \arrow[d, "\partial"] & H_{n+1}(V, \partial V; \mathbb{Z}) \arrow[d, "\partial"] \\ H_n(\partial\mathcal{M}_V; \mathbb{Z}) \arrow[r, "ev"] & H_n(\partial V; \mathbb{Z}) \end{tikzcd}$$

where $\mathcal{M}_V = \mathcal{M}_V(\alpha_V, \tilde{J})$ and ev denotes the evaluation map $ev([u]) = u(1)$. □

Proof of Corollary 4.2.19. Recall the setting of Corollary 4.2.19: We consider $T = T^n_k(a), T' = T^n_{k'}(a) \in \mathcal{L}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ for some k even, k' odd and $n \geq 3$ odd, perturbed in such a way that $T \pitchfork T'$ and $\#(T \cap T') = 2$. Suppose $L \in \mathcal{L}(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ and that we have a *small* Lagrangian cobordism $V : L \rightsquigarrow (T, T')$. Fix $R > 0$ such that V is cylindrical outside $B := [\epsilon - R, R - \epsilon]$ for some small $\epsilon > 0$. We will view $V \subset [-R, R] \times \mathbb{R} \times M$. For $\alpha \in \mathcal{D}_V$ and $\tilde{J} \in \tilde{\mathcal{J}}(B)$ we consider the moduli space $\widetilde{\mathcal{M}}^*_V(\alpha; \tilde{J})$ of *simple* holomorphic disks in $\tilde{M}$ with boundary on V (see [15, Definition 3.1.1.]). For generic $\tilde{J} \in \tilde{\mathcal{J}}(B)$ the quotient $\mathcal{M}^*_V(\alpha; \tilde{J})$ of $\widetilde{\mathcal{M}}^*_V(\alpha; \tilde{J})$ by the group of automorphisms of $\mathbb{D}$ which preserve $1 \in \partial\mathbb{D}$ is a smooth $(n+1)$-dimensional manifold with boundary

$$\bigcup_{\beta} \mathcal{M}^*_L(\beta; J) \sqcup \bigcup_{\xi} \mathcal{M}^*_T(\xi; J) \sqcup \bigcup_{\zeta} \mathcal{M}^*_{T'}(\zeta; J), \tag{4.17}$$

where the unions run over all elements of $\mathcal{D}_L, \mathcal{D}_T, \mathcal{D}_{T'}$ which hit α when pushed into $H_2(\mathbb{R}^{2(n+1)}, V; \mathbb{Z})$. By [26, Lemma 2.1] there is an odd number of elements $\zeta \in \mathbb{D}_{T'}$ such that $\eta_{T'}(\zeta; \mathbb{Z}_2) = 1 \in \mathbb{Z}_2$ and an even number of elements $\xi \in \mathbb{D}_T$ such that $\eta_T(\xi; \mathbb{Z}_2) = 1 \in \mathbb{Z}_2$. It follows that by perhaps changing $\alpha \in \mathbb{D}_V$ we can arrange that the parity of the number of $\zeta \in \mathbb{D}_{T'}$ for which $\eta_{T'}(\zeta; \mathbb{Z}_2) = 1$ in (4.17) differs from the parity of the number of $\xi \in \mathbb{D}_{T'}$ for which $\eta_{T'}(\xi; \mathbb{Z}_2) = 1$ in (4.17). Given such a choice of α we choose points $q \in T$ and $q' \in T'$ which are regular for all the evaluation maps

$$\mathcal{M}^*_T(\xi; J) \to T \quad \& \quad \mathcal{M}^*_{T'}(\zeta; J) \to T'.$$

Now choose a smooth embedding $\gamma : \mathbb{R} \to V$ such that[9]

$$\gamma(t) = \begin{cases} (t, 1, q) & \text{if } t \leq \epsilon - R \\ (t, 2, q') & \text{if } t \geq R - \epsilon \end{cases}$$

[9] Here we view V as a Lagrangian with cylindrical ends.

After perhaps perturbing $\gamma|_{[\epsilon-R,R-\epsilon]}$ we obtain that the evaluation map $ev : \mathcal{M}_V^* \to V$ as well as the restriction to the boundary $ev|_{\partial\mathcal{M}_V^*} : \partial\mathcal{M}_V^* \to \partial V$ are transverse to γ. It follows that $N := ev^{-1}(\gamma) \subset \mathcal{M}_V^*$ is a smooth 1-dimensional manifold with boundary $\partial N = N \cap \partial\mathcal{M}_V^*$. The choice of α implies that N has an odd number of boundary points. Hence, we can find a sequence $(u_j)_{j\in\mathbb{N}} \subset N$ which has no convergent subsequence in N. Applying Gromov convergence to (u_j) we obtain a (Gromov) convergent subsequence, again denoted by (u_j). By construction there are two possible (Gromov) limits:

a) (u_j) converges to a genuine cusp curve. I.e. the limit consists of multiple ($\# > 1$) $\tilde{J}$-holomorphic disks in $\tilde{M}$ with boundary on V.

b) (u_j) converges to a $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (\tilde{M}, V)$ representing the class $\alpha \in \mathbb{D}_V$. Since $ev(u) \in \gamma$ we conclude that u cannot be simple.

Suppose now for contradiction that V is orientable. Under this assumption we study the limits in the two cases a) and b) above: In case a) one of the limit disks $v : (\mathbb{D}, \partial\mathbb{D}) \to (\tilde{M}, V)$ will satisfy $0 < \tilde{\omega}(v) < a$ and $\mu_V(v) \leq 1$. Moreover, since V is assumed orientable we must have $\mu_V(v)$ even, so

$$0 < \tilde{\omega}(v) < a \quad \& \quad \mu_V(v) \leq 0. \tag{4.18}$$

It follows that $[\partial v] \in H_1(V;\mathbb{Z})/H_1(\partial_- V;\mathbb{Z})$ generates an infinite cyclic subgroup so the image $\beta \in H_1(V, \partial_- V;\mathbb{Z})$ of ∂v also generates an infinite cyclic subgroup. If instead we are in case b) we apply a result due to Lazzarini [56] to extract a $\tilde{J}$-holomorphic disk $v : (\mathbb{D}, \partial\mathbb{D}) \to (\tilde{M}, V)$ which (by the same considerations as above) must satisfy (4.18). Again we conclude that the image $\beta \in H_1(V, \partial_- V;\mathbb{Z})$ of ∂v generates an infinite cyclic subgroup. We conclude that in both cases above $\mathrm{Range}(H_1(V;\mathbb{Z}) \to H_1(V, \partial_- V;\mathbb{Z}))$ contains an infinity cyclic subgroup. However, by the long exact sequence in homology we also know that $H_1(V, \partial_- V;\mathbb{Z})/H_1(V;\mathbb{Z})$ is an infinity cyclic subgroup, so we conclude that $\mathrm{rank}(H_1(V, \partial_- V;\mathbb{Z})) \geq 2$ which implies

$$\dim_{\mathbb{Z}_2} H_1(V, \partial_- V;\mathbb{Z}_2) \geq 2.$$

Since V is small this together with Theorem 4.2.2 implies

$$H_l(V, \partial_- V;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } l = 1 \\ 0 & \text{if } l \neq 1. \end{cases} \tag{4.19}$$

Hence, $\chi(V, \partial_- V) = -2$ which, by Theorem 4.2.1, contradicts the assumption that V is orientable. This contradiction shows that V is non-orientable. With this at hand one again easily computes that (4.19) must hold which by Poincaré-Lefschetz duality implies $H_1(V, L;\mathbb{Z}_2) = 0$ so that the first Stiefel-Whitney class of L is non-trivial. We conclude that L too is non-orientable. Computing the $\mathbb{Z}_2$-homology of L and V now comes down to writing out long exact sequences. This finishes the proof of Corollary 4.2.19. □

Proof of Proposition 4.2.21. The statement is a consequence of the following basic fact: A non-trivial characteristic class $0 \neq \alpha_{L'} \in H^k(L';\mathbb{Z}_2)$ in degree $k \geq 1$ which satisfies $\alpha_{L_i} = 0 \in H^k(L_i;\mathbb{Z}_2)$ for every $i = 1, \ldots, m$ gives one dimension in $H^k(V, \partial_- V;\mathbb{Z}_2)$, because $0 \neq \alpha_V \in H^k(V;\mathbb{Z}_2)$ is in the image of the map $H^k(V, \partial_- V;\mathbb{Z}_2) \to H^k(V;\mathbb{Z}_2)$. Clearly the image of $H^0(\partial_- V;\mathbb{Z}_2) \to H^1(V;\mathbb{Z}_2)$ has dimension $m - 1$. Hence, the bound in the proposition follows from Theorem 4.2.2. □

4.4.3 Proofs of Theorem 4.2.1 and 4.2.2

The proof of Theorem 4.2.1 is an easy consequence of Weinstein's Lagrangian tubular neighborhood theorem, so we will carry it out first.

Proof of Theorem 4.2.1

Consider two ordered, oriented tuples $(L_i)_{i=1}^m, (L'_j)_{j=1}^{m'} \subset \mathcal{L}$ and an oriented Lagrangian cobordism $V : (L'_j)_j \rightsquigarrow (L_i)_i$. By Remark 4.2.5 it suffices to compute $\chi(V, \partial_+ V)$. To do this we may as well assume that $L_i \pitchfork L_j$ and $L'_i \pitchfork L'_j$ for $i \neq j$. This can be achieved by attaching small Lagrangian suspensions to the ends of V [76, Section 3.1 E], which clearly does not change the topology of V. Now fix a Darboux-Weinstein neighborhood $U \subset \tilde{M}$ of V, so that we have a symplectic identification $U \approx W$ which restricts to the identity on V, where $W \subset T^*V$ is a neighborhood of $V \subset T^*V$. Denote by $B := [-R, R]^2 \subset \mathbb{R}^2$ a "box" such that V is cylindrical outside $B \times M$. We may choose U such that it is of product type outside $B \times M$. Denote by $g_{\mathbb{R}^2}$ the standard Euclidean metric on $\mathbb{R}^2$ and fix a Riemannian metric g_M on M. Now fix a Morse function $f \in C^\infty(V)$ such that $-\nabla^g f$ points *outwards* along $\partial_+ V$ and *inwards* along $\partial_- V$, where $g := g_{\mathbb{R}^2} \oplus g_M$. We further require that

$$f(x, y, p) = \left\{ \begin{array}{ll} \sigma_j^-(x), & \forall\ (x, y, p) \in (-\infty, -R) \times \{j\} \times L_j \\ \sigma_j^+(x), & \forall\ (x, y, p) \in (R, \infty) \times \{j\} \times L'_j \end{array} \right. \tag{4.20}$$

where σ_j^- and σ_j^+ have the form $\sigma_j^\pm(x) = \alpha x + \beta^\pm$ for a constant $\alpha < 0$ and constants $\beta^\pm \in \mathbb{R}$. We extend f to a (non-compactly supported and autonomous) Hamiltonian $F \in C^\infty(\tilde{M})$ by first extending it constantly along fibers in $W \approx U$ and then cutting off outside of a fiberwise convex neighborhood containing both Graph(df) and Graph$(-df)$. If $f|_{V \cap (B \times M)}$ is C^2-small then we can achieve that $V_1 := \phi_F^1(V)$ is cylindrical outside $B \times M$ (see Figure 4.2). We assume that f is chosen such that this is the case. We equip V_1 with the orientation induced by the diffeomorphism $\phi_F^1 : V \to V_1$. We now have an identification $\mathrm{Crit}(f) \approx V \cap V_1$ and it is easy to check that

$$(-1)^{|q|_f} = (-1)^{\frac{(n+1)n}{2}} I_q(V, V_1) \quad \forall\ q \in \mathrm{Crit}(f) \approx (V \cap V_1),$$

where $I_q(V, V_1)$ denotes the intersection index at q with respect to the orientation $\tilde{\omega}^{n+1}$ of $\tilde{M}$ and $|q|_f$ denotes the Morse index. Since the Morse homology of f is $H_*(V, \partial_+ V)$ we conclude that

$$\chi(V, \partial_+ V) = \sum_{q \in \mathrm{Crit}(f)} (-1)^{|q|_f} = (-1)^{\frac{(n+1)n}{2}} I(V, V_1), \tag{4.21}$$

where $I(V, V_1)$ denotes the intersection index of (V, V_1) in $\tilde{M}$. Standard arguments in differential topology [42] imply that if $\{\rho_t\}_{t \in [0,1]}$ is an isotopy of $\tilde{M}$ supported in $I \times \mathbb{R} \times M$ for some compact interval I such that $V \pitchfork \rho_1(V_1)$ then $I(V, V_1) = I(V, \rho_1(V_1))$. Denote by $\eta : [-R-2, R+2] \to \mathbb{R}$ a compactly supported bump-function such that

$$\eta'(t) \left\{ \begin{array}{ll} \geq 0 & \text{if } t \in [-R-2, -R-1] \\ \leq 0 & \text{if } t \in [R+1, R+2] \end{array} \right.$$

and $\eta = C$ on $[-R, R]$ for some large constant $C > 0$. Denote by $\{\rho_t\}_{t \in [0,1]}$ the isotopy generated by $-\eta \partial_y$. If C is large enough it is easy to see that $V \pitchfork \rho_1(V_1)$ and each intersection point $q = (x, y, p) \in V \cap \rho_1(V_1) \subset \mathbb{R}^2 \times M$ corresponds to some $p \in L_i \cap L_j$ for $i < j$ if $x < 0$ or some $p \in L'_i \cap L'_j$ for $i < j$ if $x > 0$. It suffices to compare $I_q(V, \rho_1(V_1))$ to $I_p(L_i, L_j)$ in the former case and to $I_p(L'_i, L'_j)$ in the latter case. Recall that we are using the convention that the orientation ∂V inherits as a boundary of V corresponds to the orientation of each L_i, L'_j via the convention $\partial V = (-\sqcup_i L_i) \sqcup (\sqcup_j L'_j)$. One therefore easily checks that if $q = (x, y, p) \in V \cap \rho_1(V_1)$ with $x < 0$ and $p \in L_i \cap L_j$ for $i < j$ then

$$I_q(V, \rho_1(V_1)) = (-1)^{n+1} I_p(L_i, L_j).$$

If on the other hand $q = (x, y, p) \in V \cap \rho_1(V_1)$ with $x > 0$ and $p \in L'_i \cap L'_j$ for $i < j$ then

$$I_q(V, \rho_1(V_1)) = (-1)^n I_p(L'_i, L'_j) = -(-1)^{n+1} I_p(L'_i, L'_j).$$

It follows that

$$I(V, \rho_1(V_1)) = (-1)^{n+1} \left(\sum_{1 \le i < j \le m} I(L_i, L_j) - \sum_{1 \le i < j \le m'} I(L'_i, L'_j) \right)$$

Together with (4.21) this finishes the computation of $\chi(V, \partial_+ V)$ and therefore the proof of Theorem 4.2.1.

Proof of Theorem 4.2.2

From now on we consider a *small* Lagrangian cobordism $V : (L'_1, \dots, L'_{m'}) \rightsquigarrow (L_1, \dots, L_m)$ which is spin. The $\mathbb{Z}_2$-case when V is not assumed spin follows the same line of arguments and will not be mentioned further. Fix once and for all a $\tilde{J} \in \tilde{\mathcal{J}}_c$ such that $\mathcal{S}(V) < A(\tilde{M}, V, \tilde{J})$ together with a small $\delta > 0$ such that

$$\Delta := \mathcal{S}(V) + 2\delta < A(\tilde{M}, V, \tilde{J}). \tag{4.22}$$

Fix also $R > 0$ such that $\tilde{J}$ is supported in $[-R, R]^2 \times M$ and set $B := [-R, R]^2$.

Shaping V

We will reduce to the situation where V is a Lagrangian with cylindrical ends satisfying the following conditions: There exists a $\beta \in C_c^\infty(\mathbb{R}; [0, \infty))$ such that

$$\int_{-\infty}^{\infty} \beta(s) ds < \mathcal{S}(V) + \delta, \tag{4.23}$$

and such that V is cylindrical outside the set $Y \times M$ where

$$Y := \{(x, y) \in \mathbb{R}^2 \ : \ 0 < y < \beta(x)\}.$$

I.e. we can write

$$V \cap (\mathbb{R}^2 \backslash Y \times M) = \left(\bigsqcup_{j=1}^{m} I_j^- \times \{a_j^-\} \times L_j \right) \cup \left(\bigsqcup_{j=1}^{m'} I_j^+ \times \{a_j^+\} \times L'_j \right) \tag{4.24}$$

for intervals of the type $I_j^- = (-\infty, r_j^-]$ and $I_j^+ = [r_j^+, \infty)$ and numbers $a_j^\pm \in \mathbb{R}$ satisfying $a_j^\pm \neq a_i^\pm$ for $i \neq j$. By perhaps increasing R we may as well assume that $Y \subset B$.

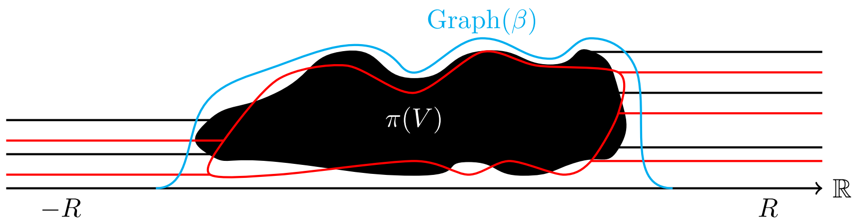

Figure 4.2: The geometric picture here indicates the situation in the proof of both Theorem 4.2.1 and 4.2.2, Graph(β) is only needed for the proof of the latter. The projection of V to $\mathbb{R}^2$ is indicated in black and the graph of β is indicated in blue. The red figure outlines the projection of $\phi_F^1(V)$ to $\mathbb{R}^2$.

Remark 4.4.2. The assumption that β satisfying (4.23) exists and (4.24) is satisfied can be made without loss of generality: If V does not satisfy these conditions then it is easy to find $\psi \in \mathrm{Symp}(\mathbb{R}^2, \omega_{\mathbb{R}^2})$ such that the Lagrangian $\tilde{V} := \psi \times \mathrm{id}_M(V) \subset (\tilde{M}, \tilde{\omega})$ satisfies them. All structures used in the proof below can then be conjugated by $\psi \times \mathrm{id}_M$ in order to transfer the results from $\tilde{V}$ to V. From now on we will therefore (without further mentioning) assume the existence of β such that (4.23) and (4.24) are satisfied.

Computing $H_*(V, \partial_\pm V; \mathbb{F})$

We will estimate the dimension of $H_*(V, \partial_+ V; \mathbb{F})$ using (a suitable adaption of) local Floer homology. To do this we fix a Morse function on V as in the proof of Theorem 4.2.1. However, this time we need to be a bit more specific about the choice we make. More precisely, instead of (4.20) f is this time required to staisfy

$$f(x,y,p) = \begin{cases} \sigma_j^-(x), & \forall\ (x,y,p) \in I_j^- \times \{a_j^-\} \times L_j \\ \sigma_j^+(x), & \forall\ (x,y,p) \in I_j^+ \times \{a_j^+\} \times L_j', \end{cases}$$

where again σ_j^- and σ_j^+ have the form $\sigma_j^\pm(x) = \alpha x + \beta^\pm$ for a constant $\alpha < 0$ and constants $\beta^\pm \in \mathbb{R}$. Here $I_j^\pm$ denote the intervals from (4.24). Exactly like before f extends to a (non-compactly supported and autonomous) Hamiltonian $F \in C^\infty(\tilde{M})$ and by choosing f such that $f|_{V \cap (Y \times M)}$ is C^2-small we achieve that $\phi_F^1(V)$ is cylindrical outside $Y \times M$. Note that $\phi_F^t(V) \pitchfork V$ for all $t \in [-1,1] \backslash \{0\}$. Consider now the strip $Z := \mathbb{R} \times [0,1]$ with coordinates $z = (s,t)$. We study solutions $u \in C^\infty(Z, \tilde{M})$ to the problem

$$\begin{cases} \partial_s u + \tilde{J}_t'(u)(\partial_t u - X_F(u)) = 0 \\ u(\mathbb{R} \times \{0,1\}) \subset V \end{cases} \tag{4.25}$$

where $\tilde{J}' = \{\tilde{J}_t'\}_{t \in [0,1]} \subset \tilde{\mathcal{J}}$ denotes a smooth path of almost complex structures and X_F denotes the symplectic gradient of F, defined by $i_{X_F}\tilde{\omega} = -dF$. Given a solution $u \in C^\infty(Z, \tilde{M})$ to (4.25) we recall that its energy (with respect to $\tilde{J}'$) is defined by

$$E_{\tilde{J}'}(u) := \int_{-\infty}^{\infty} \int_0^1 \tilde{\omega}(\partial_s u, \tilde{J}_t'(u)\partial_s u) dt ds.$$

Due to our non-compact setting we will need to impose some restrictions on $\tilde{J}'$ in order to obtain a well-defined Floer theory. In order to do so we first introduce a bit of notation. Denote by $K \Subset \mathbb{R}^2$ a subset[10] with $Y \subset K$. We will then denote by $\tilde{\mathcal{J}}_F(K)$ the space of smooth paths of $\tilde{\omega}$-compatible almost complex structures $\tilde{J}' = \{\tilde{J}_t'\}_{t \in [0,1]}$ satisfying the condition that

$$(\phi_F^t)^* \tilde{J}_t'|_{(\mathbb{R}^2 \backslash K) \times M} = (\mathfrak{i} \oplus J_t')|_{(\mathbb{R}^2 \backslash K) \times M} \tag{4.26}$$

for all $t \in [0,1]$, where $\{J_t'\}_{t \in [0,1]}$ is some smooth path of ω-compatible almost complex structures on M.

Remark 4.4.3. It was shown in [11] and [21] that a generic path in $\tilde{\mathcal{J}}_F(K)$ is regular for (4.25) in the usual sense of Floer theory. By this we mean both that transversality is achieved for generic $\tilde{J}' \in \tilde{\mathcal{J}}_F(K)$ and that there is a compact subset of $\tilde{M}$ (depending on F and K) which contains the image of every finite energy solutions to (4.25) for any path $\{\tilde{J}_t'\}_{t \in [0,1]} \in \tilde{\mathcal{J}}_F(K)$. In the following we will need to consider variations of (4.25) and therefore also variations of the almost complex structures. The precise equations (and therefore also transversality issues) we will face have been dealt with in practically identical settings before (see for instance [9], [11], [70] and references therein). The only non-standard aspect here is the compactness issue. However, in each case below compactness follows directly from the arguments in [11].

Given a path $\{\tilde{J}_t'\}_{t \in [0,1]} \in \tilde{\mathcal{J}}_F(K)$ and $t \in [0,1]$ we can define $A_S(\tilde{M}, \tilde{J}_t')$ and $A_S(\tilde{M}, V, \tilde{J}_t')$ exactly as in Section 4.3. Due to (4.26) it is an easy consequence of the of the open mapping theorem in complex analysis and the usual compactness argument that

$$A_S(\tilde{M}, \tilde{J}_t'),\ A_S(\tilde{M}, V, \tilde{J}_t') > 0.$$

[10] We use $C \Subset D$ to emphasize that C is a *compact* subset of D.

It follows that

$$A(\tilde{M}, V, \tilde{J}'_t) := \min\{A_S(\tilde{M}, \tilde{J}'_t), A_S(\tilde{M}, V, \tilde{J}'_t)\} > 0 \quad \forall\, t \in [0,1].$$

For the next lemma, note that the path $[0,1] \ni t \mapsto (\phi^t_F)_* \tilde{J}$ is an element of $\tilde{\mathcal{J}}_F(B)$.

Lemma 4.4.4. *If $F|_{B\times M}$ is sufficiently C^∞-small then*

a) $\Delta < A(\tilde{M}, V, (\phi^t_F)_*\tilde{J})$ for all $t \in [0,1]$.

Moreover, if in addition $\{\tilde{J}'_t\}_{t\in[0,1]} \in \tilde{\mathcal{J}}_F(B)$ is sufficiently C^∞-close to the path $t \mapsto (\phi^t_F)_*\tilde{J}$ then the following two conditions are satisfied:

b) Given a solution $u \in C^\infty(Z, \tilde{M})$ to (4.25) it holds that $E_{\tilde{J}'}(u) \leq \Delta$ if and only if $E_{\tilde{J}'}(u) \leq \delta$. Moreover, if $E_{\tilde{J}'}(u) < \infty$ then $u(Z) \subset U$ if and only if $E_{\tilde{J}'}(u) \leq \delta$.

c) $\Delta < A(\tilde{M}, V, \tilde{J}'_t)$ for all $t \in [0,1]$.

Proof. The proof of a) is a standard compactness argument. Note that for any $\epsilon > 0$ the path $[0,1] \ni t \mapsto (\phi^t_{\epsilon F})_*\tilde{J}$ is an element of $\tilde{\mathcal{J}}_{\epsilon F}(B)$. Therefore any $(\phi^t_{\epsilon F})_*\tilde{J}$-holomorphic sphere (respectively disk) into $\tilde{M}$ (respectively $(\tilde{M}, V)$) whose $\mathbb{R}^2$-component is non-constant is contained in $B \times M$. Hence we can apply Gromov compactness. Given a), the points b) and c) are nothing but special cases of [74, Propositions 17.1.2 and 17.1.3]. Note that in [74] the results are stated for closed Lagrangians (see also [69], [70] and [27]). However, since Gromov compactness applies, the proof in our setting is identical to the ones in [74]. □

From now on we assume that the data $\tilde{J}'$ and F is chosen according to Lemma 4.4.4 and that J' is regular for (4.25). We will now discuss the Floer chain complex which we will be using for the proof of Theorem 4.2.2. Our main reference for Lagrangian Floer homology is Zapolsky's excellent paper [104], where the orientation issues for Floer homology are worked out in every detail. For more details on the construction of the Floer chain complex we therefore refer to [104] (see also [84]). We will denote by Ω_V the space of equivalence classes of pairs $\widetilde{\gamma} = [\gamma, \widehat{\gamma}]$ where $\gamma : ([0,1], \{0,1\}) \to (\tilde{M}, V)$ is a smooth curve and $\widehat{\gamma}$ is a *capping* for γ. The equivalence relation is given by identifying two cappings if and only if they have equal symplectic area and equal Maslov index. Elements $\widetilde{\gamma} = [\gamma, \widehat{\gamma}] \in \Omega_V$ for which γ is an integral curve of X_F are exactly the critical points of the action functional $\mathcal{A}_{F:V} : \Omega_V \to \mathbb{R}$, defined by

$$\mathcal{A}_{F:V}(\widetilde{\gamma} = [\gamma, \widehat{\gamma}]) = \int_0^1 F(\gamma(t))dt - \int \widehat{\gamma}^*\tilde{\omega}.$$

We define

$$CF(F:V) := \bigoplus_{\widetilde{\gamma}\in \mathrm{Crit}(\mathcal{A}_{F:V})} C(\widetilde{\gamma}), \tag{4.27}$$

where $C(\widetilde{\gamma}) \cong \mathbb{Z}$ is generated by the two orientations of a suitable determinant line bundle of Fredholm operators defined on representatives of $\widehat{\gamma}$ as in [104]. Note that since we identify cappings which have the same symplectic area and Maslov indices, $C(\widetilde{\gamma})$ is only well-defined once we have fixed a spin structure on V so that we can identify the different rank 1 $\mathbb{Z}$-modules coming from different equivalent cappings [104, Section 7.3].[11] We will therefore fix a spin structure on V from now on. We also define $\Gamma := \pi_2(\tilde{M}, V)/\sim$ where $a \sim b$ if and only if $\tilde{\omega}(a) = \tilde{\omega}(b)$ and $\mu_V(a) = \mu_V(b)$.[12] Then

$$\tilde{\omega} \times \mu_V : \Gamma \to \mathbb{R} \times \mathbb{Z} \tag{4.28}$$

[11] In fact [104] only requires that a relative $\mathrm{Pin}^{\pm}$-structure for V has been chosen. However, for our purposes it is more convenient to assume V is spin, so we will require the choice of a spin structure.

[12] Here μ_V denotes the Maslov class of V.

is a monomorphism and $CF_*(F : V)$ is a Γ-module. In fact, by the construction of F, every $\widetilde{\gamma} = [\gamma, \widehat{\gamma}] \in \mathrm{Crit}(\mathcal{A}_{F:V})$ is naturally identified with a pair $[\gamma, \widehat{\gamma}] \approx (q, \widehat{q})$ where $q \in \mathrm{Crit}(f) \subset V$ and $\widehat{q} \in \Gamma$. We denote by $CF_0(F : V) \subset CF(F : V)$ the direct sum of the $C(\widetilde{\gamma})$ for which $\widetilde{\gamma} = [\gamma, \widehat{\gamma}] \approx (q, \widehat{q}) \in \mathrm{Crit}(\mathcal{A}_{F:V})$ for which $\widehat{q} = 0 \in \Gamma$. From this point of view it is easy to see that

$$CF_0(F : V) \otimes_{\mathbb{Z}} \Gamma \cong CF(F : V) \tag{4.29}$$

as Γ-modules. A crucial ingredient for understanding Chekanov's construction is the *length* between elements $\widetilde{\gamma}_-, \widetilde{\gamma}_+ \in \mathrm{Crit}(\mathcal{A}_{F:V})$, defined by

$$l(\widetilde{\gamma}_-, \widetilde{\gamma}_+) := \mathcal{A}_{F:V}(\widetilde{\gamma}_-) - \mathcal{A}_{F:V}(\widetilde{\gamma}_+) \in \mathbb{R}.$$

It is important to note that l is Γ bi-invariant. We denote by $\mathcal{M}(F, \tilde{J}', \widetilde{\gamma}_-, \widetilde{\gamma}_+)$ the moduli space of finite-energy and *unparametrized* solutions u of (4.25) satisfying the asymptotic conditions

$$\lim_{s \to -\infty} u_s = \widetilde{\gamma}_- \quad \& \quad \lim_{s \to \infty} u_s = \widetilde{\gamma}_+$$

in Ω_V. For such u we have the energy identity

$$0 \leq E_{\tilde{J}'}(u) = \tilde{\omega}(u) + f(q_-) - f(q_+) = l(\widetilde{\gamma}_-, \widetilde{\gamma}_+),$$

where we set $\widehat{\gamma}_\pm = (q_\pm, \widehat{q}_\pm)$. In particular, if $\widehat{q}_\pm = 0$ we see that $E_{\tilde{J}'}(u) = f(q_-) - f(q_+)$, so if the Hofer norm of F satisfies

$$\|F|_{B \times M}\| \leq \delta, \tag{4.30}$$

then automatically $u(Z) \subset U$ by Lemma 4.4.4. After perhaps scaling F we can (and will) assume that F has been chosen to satisfy (4.30) from now on. We can then define a Γ-linear operator $\partial : CF(F : V) \to CF(F : V)$ by declaring that its $(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$'th matrix element be 0 if either $l(\widetilde{\gamma}_-, \widetilde{\gamma}_+) > \delta$ or $\dim \mathcal{M}(F, \tilde{J}', \widetilde{\gamma}_-, \widetilde{\gamma}_+) \neq 0$ and

$$\sum_{u \in \mathcal{M}(F, \tilde{J}', \widetilde{\gamma}_-, \widetilde{\gamma}_+)} C(u) : C(\widetilde{\gamma}_-) \to C(\widetilde{\gamma}_+) \tag{4.31}$$

if $l(\widetilde{\gamma}_-, \widetilde{\gamma}_+) \leq \delta$ and $\dim \mathcal{M}(F, \tilde{J}', \widetilde{\gamma}_-, \widetilde{\gamma}_+) = 0$. Here $C(\widetilde{\gamma})$ denotes the $\mathbb{Z}$-linear operator defined in [104, Section 3.8.1.1]. We point out that ∂ being Γ-invariant is a non-trivial matter. This fact uses the choice of a spin structure for V [104, Section 7.3]. Note that by the remarks above $\partial(CF_0(F : V)) \subset CF_0(F : V)$, so we have an operator $\partial|_{CF_0(F:V)} : CF_0(F : V) \to CF_0(F : V)$. From the point of view of the identification (4.29) we see that

$$\partial|_{CF_0(F:V)} \otimes_{\mathbb{Z}} \mathrm{id}_\Gamma = \partial.$$

because of Γ-linearity. We will therefore denote $\partial|_{CF_0(F:V)}$ simply by ∂. Similarly, given a field $\mathbb{F}$ we continue to denote the induced operator on $CF(F : V; \mathbb{F}) := CF(F : V) \otimes_{\mathbb{Z}} \mathbb{F}$ by ∂.

Proposition 4.4.5. *$(CF_0(F : V; \mathbb{F}), \partial)$ is a chain complex (i.e. $\partial^2 = 0$) and its homology $HF_0(F : V; \mathbb{F}) := H(CF_0(F : V; \mathbb{F}), \partial)$ satisfies*

$$HF_0(F : V; \mathbb{F}) \cong H_*(V, \partial_+ V; \mathbb{F}). \tag{4.32}$$

Proof. For closed Lagrangian submanifolds this is a classical result for whose proof we refer to [74, Section 17.2]. The only non-standard aspect when checking $\partial^2 = 0$ in our situation is making sure that Floer trajectories cannot "escape" along the non-compact ends corresponding to $L_1, ..., L_m, L'_1, ..., L'_{m'}$. This is achieved by simply choosing almost complex structures which are the restriction of paths form $\tilde{\mathcal{J}}_F(B)$ to $U \subset T^* V_0$. One can check this using the arguments from [10]. Checking (4.32) can now be done using a PSS argument. This has been carried out in the setting of Lagrangian cobordisms in [10] or [21]. Those accounts easily adapt to our setting. $\square$

Chekanov's homotopy lemma

The inequality in Theorem 4.2.2 will follow from an observation due to Chekanov. We will need a slightly modified version of his beautiful result, so we cover the details we need here. Consider a subgroup $A \leq \mathbb{R} \times \mathbb{Z}$ and denote by $\lambda : A \to \mathbb{R}$ the homomorphism given by projection to the first coordinate. Given a field $\mathbb{F}$ we consider the group ring $\Lambda := \mathbb{F}[A]$. We write an element of Λ as a finite sum[13]

$$\sum_k f_k T^{a_k} \tag{4.33}$$

where $a_k \in A$ and $f_k \in \mathbb{F}$. We note that Λ is both a commutative ring with $1 \neq 0$ as well as a $\mathbb{F}$-vector space. Consider also the natural positive and negative $\mathbb{F}$-subspaces

$$\Lambda_{\pm} := \left\{ \sum_i f_i T^{a_i} \ : \ \pm\lambda(a_i) \geq 0 \ \forall \ i \right\}$$

together with their $\mathbb{F}$-linear "projections"

$$\begin{aligned} P_{\pm} : \Lambda &\to \Lambda_{\pm} \\ \sum_k f_k T^{a_k} &\mapsto \sum_{k : \pm\lambda(a_k) \geq 0} f_k T^{a_k} \end{aligned}$$

Given a finite dimensional $\mathbb{F}$-vector space W we obtain a free Λ-module $W \otimes_{\mathbb{F}} \Lambda$ with $\mathrm{rank}_{\Lambda}(W \otimes_{\mathbb{F}} \Lambda) = \dim_{\mathbb{F}}(W)$. Considering the $\mathbb{F}$-linear subspace $W_0 := W \otimes_{\mathbb{F}} \mathbb{F}[\ker \lambda] \leq W \otimes_{\mathbb{F}} \Lambda$ we have natural positive and negative $\mathbb{F}$-linear subspaces

$$W_{\pm} := \Lambda_{\pm} \cdot W^0 \subset W \otimes_{\mathbb{F}} \Lambda,$$

together with the associated $\mathbb{F}$-linear "projection" maps

$$\mathrm{id}_W \otimes_{\mathbb{F}} P_{\pm} : W \otimes_{\mathbb{F}} \Lambda \to W^{\pm},$$

which we (by abuse of notation) continue to denote by $P_{\pm}$. Suppose now that (W, ∂) is a finite dimensional differential $\mathbb{F}$-vector space. Denoting by $\partial' := \partial \otimes_{\mathbb{F}} \mathrm{id}_{\Lambda}$ the induced differential on $W \otimes_{\mathbb{F}} \Lambda$ we have a free and finitely generated Λ-differential module $(W \otimes_{\mathbb{F}} \Lambda, \partial')$. Following Chekanov [27] we say that two Λ-linear maps

$$f, g : W \otimes_{\mathbb{F}} \Lambda \to W \otimes_{\mathbb{F}} \Lambda$$

are λ-homotopic if there exists a Λ-linear map $h : W \otimes_{\mathbb{F}} \Lambda \to W \otimes_{\mathbb{F}} \Lambda$ such that

$$P_{-}(f - g - h\partial' - \partial' h)P_{+} = 0 \tag{4.34}$$

as a map $W \otimes_{\mathbb{F}} \Lambda \to W \otimes_{\mathbb{F}} \Lambda$. The version of Chekanov's homotopy lemma which we need is the following. Chekanov's original formulation seems to differ slightly from the one we use here, but his proof easily carries over to our setup.

Lemma 4.4.6 ([27]). *Denote by N a free, finitely generated Λ-module and by (W, ∂) a finite dimensional differential $\mathbb{F}$-vector space. If there exist Λ-linear maps $\Phi : W \otimes_{\mathbb{F}} \Lambda \to N$ and $\Psi : N \to W \otimes_{\mathbb{F}} \Lambda$ such that $\Psi\Phi$ is λ-homotopic to the identity then*

$$\dim_{\mathbb{F}} H(W, \partial) \leq \mathrm{rank}_{\Lambda}(N).$$

[13] Note that A is Abelian.

An $\tilde{\omega}$-homotopy

Viewed through (4.28) Γ will play the role of A above. So, $\Lambda = \mathbb{F}[\Gamma]$ and λ is simply given by $\tilde{\omega} : \Gamma \to \mathbb{R}$. We point out now that, with coefficients in a field $\mathbb{F}$, (4.29) translates into an isomorphism of Λ-modules

$$CF_0(F : V; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda \cong CF(F : V; \mathbb{F})$$

Fix now $C, \epsilon > 0$ and choose two functions $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}; [0,1])$ satisfying

$$\varphi_1(y) = \begin{cases} 1, & \text{for } |y| < C \\ 0, & \text{for } |y| \geq C + 1 \end{cases} \quad \& \quad \varphi_2(x) = \begin{cases} 1, & \text{for } x < R \\ 0, & \text{for } x \geq R + C \end{cases}$$

as well as

$$|\varphi_2'(x)| \leq \epsilon \quad \forall\, x \in \mathbb{R}.$$

Consider the Hamiltonian $H \in C_c^\infty(\tilde{M})$ defined by[14]

$$H(x, y, p) = \left(\int_{-\infty}^{x} -\beta(s) ds \right) \varphi_1(y) \varphi_2(x).$$

We then define the time-dependent and *compactly supported* Hamiltonian $\tilde{H} \in C_c^\infty([0,1] \times \tilde{M})$ by $\tilde{H}_t(z,p) = H(\phi_F^{1-t}(z,p))$ and note that $\tilde{H}$ has Hofer norm

$$\|\tilde{H}\| = \|H\| \leq \mathcal{S}(V) + \delta.$$

Consider now the time-dependent Hamiltonian $G_t(z,p) := F(z,p) + \tilde{H}_t(z,p) \in C^\infty([0,1] \times \tilde{M})$ generating the flow $\phi_G^t = \phi_F^{t-1} \phi_H^t \phi_F^1$. Since $\phi_G^1 = \phi_H^1 \phi_F^1$ it is easy to see from the choices made in Section 4.4.3 that $CF(G : V; \mathbb{F})$ is a Λ-module of rank

$$\operatorname{rank}_\Lambda CF(G : V; \mathbb{F}) = \sum_{1 \leq i < j \leq m} \#(L_i \cap L_j) + \sum_{1 \leq i < j \leq m'} \#(L_i' \cap L_j')$$

provided that $C > 0$ is chosen large enough and $\epsilon > 0$ small enough. Therefore (4.2) follows from Lemma 4.4.6 and

Proposition 4.4.7. *There exist Λ-linear maps*

$$\begin{aligned} \Phi :& CF(F : V; \mathbb{F}) \to CF(G : V; \mathbb{F}) \\ \Psi :& CF(G : V; \mathbb{F}) \to CF(F : V; \mathbb{F}) \end{aligned}$$

whose composition $\Psi\Phi$ is $\tilde{\omega}$-homotopic to the identity.

Proof of Proposition 4.4.7. The following is basically Chekanov's proof from [27] (see also [70]). Chekanov constructed Ψ, Φ together with a suitable $\tilde{\omega}$-homotopy using Floer's continuation principle. Fix two monotone functions $\rho_\pm \in C^\infty(\mathbb{R}; [0,1])$ satisfying

$$\rho_+(s) = \begin{cases} 0, & \text{if } s \leq -1 \\ 1, & \text{if } s \geq 1 \end{cases} \quad \& \quad \rho_-(s) = \begin{cases} 1, & \text{if } s \leq -1 \\ 0, & \text{if } s \geq 1 \end{cases}$$

Consider also the positive and negative parts of $\|\tilde{H}\|$

$$b_+ := \int_0^1 \max_{\tilde{M}}(\tilde{H}_t)\; dt \geq 0 \quad \& \quad b_- := \int_0^1 \min_{\tilde{M}}(\tilde{H}_t)\; dt \leq 0,$$

[14] Recall the choice of $\beta \in C_c^\infty(\mathbb{R})$ made in Section 4.4.3.

so that $\|\tilde{H}\| = b_+ - b_-$ and choose $\tilde{R} > R$ such that $\mathrm{Supp}(\tilde{H}_t) \subset \tilde{B} \times M$ for all $t \in [0,1]$, where $\tilde{B} := [-\tilde{R}, \tilde{R}]$. Now consider for $u \in C^\infty(Z; \tilde{M})$ the problem

$$\begin{cases} \partial_s u + \tilde{I}^\pm_{(s,t)}(u)(\partial_t u - X_F(u) - \rho_\pm(s) X_{\tilde{H}_t}(u)) = 0 \\ u(\mathbb{R} \times \{0,1\}) \subset V \end{cases} \tag{$P_\pm$}$$

where $\{\tilde{I}^\pm_z\}_{z\in Z}$ is a smooth Z-family of $\tilde{\omega}$-compatible almost complex structures satisfying

$$(\phi^t_F)^* \tilde{I}^\pm_{(s,t)}|_{(\mathbb{R}^2\setminus\tilde{B})\times M} = (\mathfrak{i} \oplus I^\pm_{(s,t)})|_{(\mathbb{R}^2\setminus\tilde{B})\times M} \tag{4.35}$$

for some Z-family of ω-compatible almost complex structures $\{I^\pm_z\}_{z\in Z}$ on M. We additionally require that there exists a constant $0 < C < \infty$ such that

$$\tilde{I}^\pm_{(s,t)} = \begin{cases} \tilde{J}'_t, & \text{if } \pm s < -C \text{ and/or } t \in \{0,1\} \\ \tilde{J}^\infty_t, & \text{if } \pm s > C \text{ and/or } t \in \{0,1\}, \end{cases} \tag{4.36}$$

where $\{\tilde{J}^\infty_t\}_{t\in[0,1]} \in \tilde{\mathcal{J}}_F(\tilde{B})$ satisfies the condition that $\tilde{J}^\infty_t = \tilde{J}'_t$ for $t \in \{0,1\}$. Exactly as in Lemma 4.4.4 one sees that $\{\tilde{I}^\pm_z\}_{z\in Z}$ may be chosen such that it is regular in the usual sense of Floer theory and such that

$$\Delta < A(\tilde{M}, V, \tilde{I}^\pm_z) \quad \forall\, z \in Z \tag{4.37}$$

We will therefore assume that this is the case from now on. Given $\widetilde{\gamma}_- \in \mathrm{Crit}(\mathcal{A}_{F:V})$ and $\widetilde{\gamma}_+ \in \mathrm{Crit}(\mathcal{A}_{G:V})$ we define the length

$$l_+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) := \mathcal{A}_{F:V}(\widetilde{\gamma}_-) - \mathcal{A}_{G:V}(\widetilde{\gamma}_+) \in \mathbb{R}$$

and denote by $\mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ the space of finite energy solutions u to (P_+) enjoying the asymptotic conditions $\lim_{s\to-\infty} u_s = \widetilde{\gamma}_-$ and $\lim_{s\to\infty} u_s = \widetilde{\gamma}_+$ in Ω_V. Since $\tilde{I}^+$ is regular $\mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ is a smooth manifold. Given $u \in \mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ one integrates by parts to see that

$$E_{\tilde{I}^+}(u) = l_+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) + \int_{-\infty}^{\infty} \int_0^1 \dot{\rho}_+(s) \tilde{H}_t(u)\, dtds. \tag{4.38}$$

If $l_+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) \le \delta - b_-$ it follows from this and monotonicity of ρ_+ that

$$E_{\tilde{I}^+}(u) \le \delta - b_- + \int_0^1 \max_{\tilde{M}}(\tilde{H}_t)\, dt = \|\tilde{H}\| + \delta \le \mathcal{S}(V) + 2\delta = \Delta,$$

for every $u \in \mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$. In particular, in this case, (4.37) implies that no bubbling occurs along $\mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$. If in addition $\dim \mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) = 0$ it follows from regularity of $\tilde{I}^+$ that $\mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ is compact. Hence, we can define $\Phi : CF(F : V; \mathbb{F}) \to CF(G : V; \mathbb{F})$ as the unique Λ-linear operator whose $(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$'th matrix element equals 0 if $\dim \mathcal{M}^+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) \neq 0$ or $l_+(\widetilde{\gamma}_-, \widetilde{\gamma}_+) > \delta - b_-$ and otherwise equals

$$\sum_{u\in\mathcal{M}^+(\widetilde{\gamma}_-,\widetilde{\gamma}_+)} C(u) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{F}} : C(\widetilde{\gamma}_-) \otimes_{\mathbb{Z}} \mathbb{F} \to C(\widetilde{\gamma}_+) \otimes_{\mathbb{Z}} \mathbb{F} \tag{4.39}$$

as defined in [104, Section 3.8.1]. Similarly, given $\widetilde{\gamma}_- \in \mathrm{Crit}(\mathcal{A}_{G:V})$ and $\widetilde{\gamma}_+ \in \mathrm{Crit}(\mathcal{A}_{F:V})$ we consider the quantity

$$l_-(\widetilde{\gamma}_-, \widetilde{\gamma}_+) := \mathcal{A}_{G:V}(\widetilde{\gamma}_-) - \mathcal{A}_{F:V}(\widetilde{\gamma}_+),$$

and denote by $\mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ the space of finite energy solutions u to (P_-) satisfying $\lim_{s\to-\infty} u_s = \widetilde{\gamma}_-$ and $\lim_{s\to\infty} u_s = \widetilde{\gamma}_+$ in Ω_V. Again $\mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ is a smooth manifold and integration by parts yields

$$E_{\tilde{I}^-}(u) = l_-(\widetilde{\gamma}_-, \widetilde{\gamma}_+) + \int_{-\infty}^{\infty} \int_0^1 \dot{\rho}_-(s) \tilde{H}_t(u)\, dtds. \tag{4.40}$$

for every $u \in \mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$. In particular if $l_-(\widetilde{\gamma}_-, \widetilde{\gamma}_+) \leq \delta + b_+$ then

$$E_{\tilde{I}^-}(u) \leq \delta + b_+ - \int_0^1 \min_{\tilde{M}}(\tilde{H}_t)\ dt \leq \Delta,$$

for every $u \in \mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$, so no bubbling occurs along $\mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$. We can therefore define $\Psi : CF(G : V; \mathbb{F}) \to CF(F : V; \mathbb{F})$ as the unique Λ-linear map whose $(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$'th matrix element equals 0 if $\dim \mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+) \neq 0$ or $l_-(\widetilde{\gamma}_-, \widetilde{\gamma}_+) > \delta + b_+$ and otherwise equals

$$\sum_{u \in \mathcal{M}^-(\widetilde{\gamma}_-, \widetilde{\gamma}_+)} C(u) \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{F}} : C(\widetilde{\gamma}_-) \otimes_{\mathbb{Z}} \mathbb{F} \to C(\widetilde{\gamma}_+) \otimes_{\mathbb{Z}} \mathbb{F}$$

The aim now is to construct an $\tilde{\omega}$-homotopy from $\Psi\Phi$ to the identity. I.e. we need to construct a Λ-linear map $h : CF(F : V; \mathbb{F}) \to CF(F : V; \mathbb{F})$ such that

$$P_-(\mathrm{id} - \Psi\Phi - h\partial - \partial h)P_+ = 0 \tag{4.41}$$

In order to construct h we choose a function $\rho \in C^\infty([0, \infty) \times \mathbb{R}; [0, 1])$, written $(\tau, s) \mapsto \rho_\tau(s)$, such that for every $\tau \in [0, \infty)$ the function $\rho_\tau \in C_c^\infty(\mathbb{R}; [0, 1])$ satisfies the monotonicity condition

$$\frac{d\rho_\tau}{ds}(s) \begin{cases} \geq 0, & \text{if } s \leq 0 \\ \leq 0, & \text{if } s \geq 0 \end{cases} \tag{4.42}$$

Moreover, we require the condition that $[0, \infty) \ni \tau \mapsto \rho_\tau(0)$ is a monotone function onto $[0, 1]$ as well as the condition that, for $\tau \geq 2$, we have

$$\rho_\tau(s) = \begin{cases} \rho_+(s + \tau), & \text{if } s \leq 0 \\ \rho_-(s - \tau), & \text{if } s \geq 0. \end{cases}$$

Consider for $\tau \in [0, \infty)$ the problem

$$\begin{cases} \partial_s u + \tilde{I}^\tau_{(s,t)}(u)(\partial_t u - X_F(u) - \rho_\tau(s) X_{\tilde{H}_t}(u)) = 0 \\ u(\mathbb{R} \times \{0, 1\}) \subset V, \end{cases} \tag{4.43}$$

where $\{\tilde{I}^\tau_z\}_{(\tau,z) \in [0,\infty) \times Z}$ is a family of $\tilde{\omega}$-compatible almost complex structures satisfying (4.35) (with $\pm$ replaced by τ) for a family of ω-compatible almost complex structures $\{I^\tau_z\}_{(\tau,z) \in [0,\infty) \times Z}$ on M. We require that

$$\tilde{I}^\tau_{(s,t)} = \tilde{J}'_t \quad \text{if } \tau = 0 \text{ and/or } t \in \{0, 1\}.$$

We also require the existence of a constant $0 < C < \infty$ such that for all $\tau \geq C$

$$\tilde{I}^\tau_{(s,t)} = \begin{cases} \tilde{I}^+_{(s+\tau,t)}, & \text{if } s \leq 0 \\ \tilde{I}^-_{(s-\tau,t)}, & \text{if } s \geq 0, \end{cases}$$

and for all $\tau \leq C$

$$\tilde{I}^\tau_{(s,t)} = \tilde{J}'_t \quad \text{if } s \text{ is sufficiently large.}$$

As above $\{\tilde{I}^\tau_z\}_{(\tau,z)}$ can be chosen to be regular and satisfy

$$\Delta < A(\tilde{M}, V, \tilde{I}^\tau_z) \quad \forall\ (\tau, z) \in [0, \infty) \times Z. \tag{4.44}$$

Given $\tau \in [0, \infty)$ and $\widetilde{\gamma}_-, \widetilde{\gamma}_+ \in \mathrm{Crit}(\mathcal{A}_{F:V})$ we denote by $\mathcal{M}^{\rho_\tau}(\widetilde{\gamma}_-, \widetilde{\gamma}_+)$ the space of all finite energy solutions $u \in C^\infty(Z, \tilde{M})$ to (4.43) satisfying $\lim_{\tau \to -\infty} u_s = \widetilde{\gamma}_-$ and $\lim_{\tau \to \infty} u_s = \widetilde{\gamma}_+$ in Ω_V and we define

$$\mathcal{M}^\rho(\widetilde{\gamma}_-, \widetilde{\gamma}_+) := \{(\tau, u)\ :\ \tau \in [0, \infty),\ u \in \mathcal{M}^{\rho_\tau}(\widetilde{\gamma}_-, \widetilde{\gamma}_+)\}.$$

Since $\{\tilde{I}^\tau_{(s,t)}\}_{(\tau,s,t)}$ is regular, $\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$ is a smooth manifold for every $\widetilde{\gamma}_-,\widetilde{\gamma}_+\in \mathrm{Crit}(\mathcal{A}_{F:V})$ and integration by parts yields

$$E_{\tilde{I}^\tau}(u) = l(\widetilde{\gamma}_-,\widetilde{\gamma}_+) + \int_{-\infty}^{\infty}\int_0^1 \frac{d\rho_\tau}{ds}(s)\tilde{H}_t(u)\ dtds \quad \forall\ u\in \mathcal{M}^{\rho_\tau}(\widetilde{\gamma}_-,\widetilde{\gamma}_+). \tag{4.45}$$

In particular we see that if $l(\widetilde{\gamma}_-,\widetilde{\gamma}_+)\le\delta$ then

$$\begin{aligned} E_{\tilde{I}^\tau}(u) &\le \delta + \int_{-\infty}^{0}\int_0^1 \frac{d\rho_\tau}{ds}(s)\tilde{H}_t(u)\ dtds + \int_0^{\infty}\int_0^1 \frac{d\rho_\tau}{ds}(s)\tilde{H}_t(u)\ dtds \\ &\le \delta + \rho_\tau(0)b_+ - \rho_\tau(0)b_- = \delta + \rho_\tau(0)\|\tilde{H}\| \le \Delta \quad \forall\ u\in\mathcal{M}^{\rho_\tau}(\widehat{\gamma}_-,\widehat{\gamma}_+) \end{aligned}$$

so, in this case, (4.44) implies that no bubbling occurs along $\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$. If in addition $\dim\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)=0$ then $\#\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)<\infty$, so we can define $h: CF(F:V)\to CF(F:V)$ as the unique Λ-linear map whose $(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$'th matrix element equals 0 if $\dim\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)\neq 0$ or $l(\widehat{\gamma}_-,\widehat{\gamma}_+)>\delta$ and otherwise equals

$$\sum_{(\tau,u)\in\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)} C(u)\otimes_{\mathbb{Z}}\mathrm{id}_{\mathbb{F}} : C(\widetilde{\gamma}_-)\otimes_{\mathbb{Z}}\mathbb{F}\to C(\widetilde{\gamma}_+)\otimes_{\mathbb{Z}}\mathbb{F}.$$

To finish the proof we need to check that (4.41) is satisfied. To do that we fix $\widetilde{\gamma}_- = (q_-,\widehat{q}_-), \widetilde{\gamma}_+ = (q_+,\widehat{q}_+)\in\mathrm{Crit}(\mathcal{A}_{F:V})$ with $\tilde{\omega}(\widehat{q}_-)\ge 0$ and $\tilde{\omega}(\widehat{q}_+)\le 0$ and we need to check that the $(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$'th matrix element of the operator

$$\mathrm{id} - \Psi\Phi - h\partial - \partial h \tag{4.46}$$

equals 0. This is clearly the case if the Conley-Zehnder indices of $\widetilde{\gamma}_-$ and $\widetilde{\gamma}_-$ differ, so we only consider the case when these coincide, in which case $\dim\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)=1$. Since

$$l(\widetilde{\gamma}_-,\widetilde{\gamma}_+) = f(q_-) - \tilde{\omega}(\widehat{q}_-) - f(q_+) + \tilde{\omega}(\widehat{q}_+) \le f(q_-) - f(q_+)\le\delta,$$

no bubbling occurs along $\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$, so it is compact up to Floer breaking. By the usual gluing argument every configuration counted in the $(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$'th matrix element of (4.46) occurs as a boundary point of the compactification of $\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$ and it therefore follows as in [104, Section 3.8.2] that the $(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$'th matrix element of (4.46) equals 0 if only we argue that every boundary point of the compactification of $\mathcal{M}^\rho(\widetilde{\gamma}_-,\widetilde{\gamma}_+)$ occurs in (4.46). This follows from the estimates on the last page of [27]. $\square$

5. Symplectic "Mather-Floer Theory"

5.1 Introduction

Consider a symplectic manifold (M^{2n}, ω) and a Hamiltonian $H \in C^\infty(M)$ with complete flow $\phi_H = \{\phi_H^t\}_{t\in\mathbb{R}}$. This chapter concerns the question of how to systematically determine interesting invariant sets for ϕ_H. The celebrated KAM theorem asserts (loosely speaking) that, if H is non-degenerate and sufficiently regular, an invariant Lagrangian torus, on which the dynamics of ϕ_H has a Diophantine rotation vector, will persist under small (sufficiently regular) perturbations of H. Mather [59] discovered that one can loosen the regularity assumptions in the KAM theorem and still find plenty of interesting invariant sets which persist perturbations of H, if one pays the price of replacing the non-degeneracy condition with a stronger *convexity* assumption. One of the aims of the current paper is to study, using methods coming from symplectic topology, what happens when one relaxes the convexity assumption. It is interesting to understand the connection between Aubry-Mather theory and symplectic topology. Here we hope to provide some understanding of how pseudo-holomorphic curve techniques can explain phenomena from Aubry-Mather theory. A first indication that these theories can be connected was obtained by Bernard [8]. The approach we take here builds heavily on Viterbo's symplectic homogenization paper [100] as well as the Floer-homological version studied by Monzner-Vichery-Zapolsky [62] and Vichery [99]. The second part of the paper obtains interesting invariant measures by studying a generalization of the notion of symplectic shape (due to Sikorav [88]) of sublevel sets of a Hamiltonian. It employs ideas due to Buhovsky-Entov-Polterovich [24], Entov-Polterovich [38] and Polterovich [78].

The contents of the paper are as follows: In Section 5.2 we present our results on existence of invariant measures using homogenized Lagrangian spectral invariants. Section 5.3 compares properties of these measures to Mather measures. Section 5.4 contains our results on invariant measures using the C^0-techniques developed by Buhovsky-Entov-Polterovich. Finally, Section 5.5 and 5.6 contain preliminaries and proofs.

Setting and notation: $\mathcal{M}(X)$ will denote the space of Borel probability measures which are *compactly* supported on $X \subset M$. We use the convention that the symplectic gradient X_{H_t} associated to $H \in C^\infty([0,1] \times M)$ is defined by $\iota_{X_{H_t}}\omega = -dH_t$. The flow generated by X_{H_t} is denoted by $\phi_H = \{\phi_H^t\}_{t\in\mathbb{R}}$. We denote by $\mathcal{M}(X;\phi_H) \subset \mathcal{M}(X)$ the subset of ϕ_H-invariant measures and write $\mathcal{M} = \mathcal{M}(M)$ as well as $\mathcal{M}(\phi_H) = \mathcal{M}(M;\phi_H)$.

5.2 Symplectic "Mather-Floer theory"

Throughout this section (M^{2n},ω) will denote a closed symplectic manifold.[1] We consider a closed *monotone* Lagrangian submanifold $L \subset (M,\omega)$ which is non-narrow in the sense that its quantum homology $QH_*(L;\mathbb{Z}_2)$ doesn't vanish (see Section 5.5). In this setting Leclercq-Zapolsky [58] recently developed a theory of *Lagrangian spectral invariants.* We will denote by

$$l_L : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}.$$

the Leclercq-Zapolsky spectral invariant associated to the unity in $QH_*(L;\mathbb{Z}_2)$.[2] An idea due to Viterbo [100] says that homogenizing l_L gives rise to an analogue of Mather's α-function. The approach we consider here was first studied by Monzner-Vichery-Zapolsky [62] and Vichery [99]. Denote by $\mathcal{H} := \{H \in C^\infty(M) \mid \int H\omega^n = 0\}$ the space of *normalized* Hamiltonians.

Definition 5.2.1. Associated to $H \in \mathcal{H}$ we define a function $\sigma_{H:L} : H^1(M;\mathbb{R}) \to \mathbb{R}$ by

$$\sigma_{H:L}(c) = \lim_{\mathbb{N}\ni k\to\infty} \frac{l_L(\psi_1^{-1}\tilde{\phi}_H^k\psi_1)}{k}, \tag{5.1}$$

where $\psi : [0,1]\times M \to M$ is any smooth symplectic isotopy with $\psi_0 = \mathrm{id}$ and $\mathrm{Flux}(\psi) = c \in H^1(M;\mathbb{R})$, and $\tilde{\phi}_H^k \in \widetilde{\mathrm{Ham}}(M,\omega)$ denotes the element represented by the path $[0,1] \ni t \mapsto \phi_H^{tk}$ in $\mathrm{Ham}(M,\omega)$.

For more details on the construction and properties of $\sigma_{H:L}$ we refer to Section 5.5. It turns out that $\sigma_{H:L}$ is locally Lipschitz, so at every point $c \in H^1(M;\mathbb{R})$ it has a well-defined set of *Clarke subdifferentials* (see Section 5.5.1)

$$\partial\sigma_{H:L}(c) \subset H_1(M;\mathbb{R}) = H^1(M;\mathbb{R})^*.$$

The dynamical information contained in $\sigma_{H:L}$ is that it guarantees existence of analogues of *Mather measures* (from Aubry-Mather theory) to the present setting. I.e. $\mathcal{M}(\phi_H)$-measures whose rotation vector (or asymptotic cycle, see Section 5.5.1) is prescribed by $\partial\sigma_{H:L}(c)$.

Theorem 5.2.2. *Let $H \in \mathcal{H}$. For any $c \in H^1(M;\mathbb{R})$ and any Clarke subdifferential $h \in \partial\sigma_{H:L}(c) \subset H_1(M;\mathbb{R})$ of $\sigma_{H:L}$ at c, there exists a $\mu \in \mathcal{M}(\phi_H)$ which has rotation vector*

$$\rho(\mu) = h.$$

For a more detailed discussion of how Theorem 5.2.2 relates to classical Aubry-Mather theory we refer to Section 5.3.

Remark 5.2.3. After having proved the main part of Theorem 5.2.2 we learned about Vichery's [99], where a result similar to Theorem 5.2.2 is proved in the case when L is the zero-section of a cotangent bundle as well as Viterbo's [101] where results from [100] are applied to yield a result similar to Theorem 5.2.2 for $T^*\mathbb{T}^n$. The main difference between our result and Vichery's is greater generality. Moreover, our approach detects the measures directly using pseudo-holomorphic curve techniques and avoids the use of generating functions.

Remark 5.2.4. In order to understand the support of the measures whose existence is guaranteed by Theorem 5.2.2, one needs to understand how they are produced. The $\mu \in \mathcal{M}(\phi_H)$

[1]See Section 5.2.1 for the non-compact (exact) case.
[2]In [58] our l_L goes under the name l_+.

with $\rho(\mu) \in \partial\sigma_{H:L}(0)$ produced in the proof of Theorem 5.2.2, are convex combinations of weak*-limits of sequences $(\mu_\varsigma)_{\varsigma\in\mathbb{N}} \subset \mathcal{M}$ characterized by

$$\int f \, d\mu_\varsigma = \frac{1}{k_\varsigma}\int_0^{k_\varsigma} f\phi_H^t(x_\varsigma)\, dt \quad \forall\, f \in C^0_{bd}(M). \tag{5.2}$$

Here, $(x_\varsigma)_{\varsigma\in\mathbb{N}}$ is a sequence of points with $x_\varsigma \in \psi_\varsigma(L)$ for a sequence of symplectomorphisms $(\psi_\varsigma)_{\varsigma\in\mathbb{N}} \subset \mathrm{Symp}(M,\omega)$ satisfying

$$\psi_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} \mathrm{id}$$

in the Whitney C^∞-topology, and $(k_\varsigma)_{\varsigma\in\mathbb{N}} \subset (0,\infty)$ is a sequence of times with $k_\varsigma \uparrow \infty$. In addition, $\phi_H^{k_\varsigma}(x_\varsigma) \in \psi_\varsigma(L)$ for all $\varsigma \in \mathbb{N}$ and to each orbit $\gamma_\varsigma = \{\phi_H^t(x_\varsigma)\}_{t\in[0,k_\varsigma]}$ is associated an action $\mathcal{A}_{H:\psi_\varsigma(L)}(\gamma_\varsigma) \in \mathbb{R}$ such that[3]

$$\frac{\mathcal{A}_{H:\psi_\varsigma(L)}(\gamma_\varsigma)}{k_\varsigma} \overset{\varsigma\to\infty}{\longrightarrow} \sigma_{H:L}(0).$$

We will denote by $\mathfrak{M}_{H:L} \subset \mathcal{M}(\phi_H)$ the set of measures $\mu \in \mathcal{M}(\phi_H)$ with $\rho(\mu) \in \partial\sigma_{H:L}(0)$ arising as convex combinations of weak*-limits of sequences $(\mu_\varsigma)_\varsigma$ given by (5.2), such that $(x_\varsigma)_\varsigma$ and $(k_\varsigma)_\varsigma$ satisfy the criteria mentioned above. This notation is justified by results, first due to Viterbo [100, Proposition 13.3] and later Monzner-Vichery-Zapolsky [62, Theorem 1.11], saying that $\sigma_{H:L}$ coincides with Mather's α-function when $L \subset T^*L$ is the zero-section and H is Tonelli. In particular, $\mathfrak{M}_{H:L}$-measures are Mather measures in this setting (see Remark 5.2.9 below).

Under certain circumstances, one can reduce the domain of $\sigma_{H:L}$ further, thus getting further restrictions on its subdifferentials. In the following we will denote by $r_L : H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$ and by $i_L : H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ the (co)homological maps induced by the inclusion $L \hookrightarrow M$.

Corollary 5.2.5. *Suppose r_L is surjective. Then $\sigma_{H:L}$ descends to a locally Lipschitz function*

$$\alpha_{H:L} : H^1(L;\mathbb{R}) \to \mathbb{R}.$$

In particular, for every $c \in H^1(L;\mathbb{R})$ and every subdifferential $h \in \partial\alpha_{H:L}(c) \subset H_1(L;\mathbb{R})$ there exists a $\mu \in \mathcal{M}(\phi_H)$ with rotation vector

$$\rho(\mu) = i_L(h) \in H_1(M;\mathbb{R}).$$

Of course, the notation is meant to suggest that $\alpha_{H:L}$ should play the role of Mather's α-function on a general symplectic manifold.

Polterovich [78] used Poisson bracket invariants to study invariant measures which have "large" rotation vectors. In certain situations Corollary 5.2.5 allows us to sharpen his result in the sense that we can detect that the rotation vector of the constructed measure is contained in the subspace $i_L(H_1(L;\mathbb{R})) \leq H_1(M;\mathbb{R})$. For the definition of Lagrange Flux (Flux_L) we refer to [91].

Corollary 5.2.6. *Suppose r_L is surjective. Let $\psi : [0,1]\times L \to M$ denote a Lagrange isotopy with ψ_0 the inclusion $L \hookrightarrow M$ and set*

$$c := \mathrm{Flux}_L(\psi) \in H^1(L;\mathbb{R}).$$

Then there exists $h \in H_1(L;\mathbb{R})$ with

$$\langle c, h\rangle \geq \min_{\psi_1(L)}(H) - \max_L(H)$$

such that $i_L(h) \in H_1(M;\mathbb{R})$ is realized as the rotation vector of a $\mathcal{M}(\phi_H)$-measure.

[3] See the Section 5.5 for a precise definition of $\mathcal{A}_{H:\psi_\varsigma(L)}(\gamma_\varsigma)$.

5.2.1 The non-compact setting

Here we will discuss the analogue of the above results in the setting of a non-compact (M, ω). Since the case of the cotangent bundle was covered by Vichery [99], we will focus on the general case.

Denote by $(M, \omega = d\lambda)$ an exact symplectic manifold which is convex at infinity [35] and denote by $L \subset (M, \omega)$ a closed Lagrangian submanifold with $\lambda|_L \equiv 0$. We will denote by $\mathcal{H} = \mathcal{H}(M, \omega)$ the class of autonomous Hamiltonians $H \in C^\infty(M)$ whose Hamiltonian flow $\phi_H = \{\phi_H^t\}_{t\in\mathbb{R}}$ is complete and satisfies the condition that $\mathcal{O}_+(X) \subset M$ is compact for every compact $X \subset M$. Here we make use of the notation

$$\mathcal{O}_+(X) := \overline{\bigcup_{t\geq 0} \phi_H^t(X)} \quad \& \quad \mathcal{O}_-(X) := \overline{\bigcup_{t\leq 0} \phi_H^t(X)}.$$

Note that $\mathcal{H}$ contains all proper Hamiltonians. We will denote by $H_*^{BM}(M, \mathbb{R})$ the Borel-Moore homology groups (or homology with closed support) of M with coefficients in $\mathbb{R}$ (see e.g. [23]). We view $H_k^{BM}(M, \mathbb{R})$ as the dual of $H_c^k(M; \mathbb{R})$, the compactly supported cohomology of M with coefficients in $\mathbb{R}$. Given $H \in \mathcal{H}$, every measure $\mu \in \mathcal{M}$ has a well-defined action $\mathcal{A}_{H,\lambda}(\mu) \in \mathbb{R}$ given by[4]

$$\mathcal{A}_{H,\lambda}(\mu) := \int H - \langle \lambda, X_H \rangle \; d\mu. \tag{5.3}$$

For $\mu \in \mathcal{M}(\phi_H)$, one can think of its rotation vector $\rho(\mu)$ either as an element of $H_1^{BM}(M; \mathbb{R})$ or as an element of $H_1(M; \mathbb{R})$, depending on whether or not one allows non-compactly supported 1-forms η in (5.27).

In this setup, every $H \in \mathcal{H}$ gives rise to a function

$$\sigma_{H:L} : H_c^1(M; \mathbb{R}) \to \mathbb{R}$$

by requiring that the symplectic isotopy in (5.1) be compactly supported. When $(M, d\lambda)$ is a cotangent bundle and $L \subset (M, d\lambda)$ the 0-section one can in fact define $\sigma_{H:L} : H^1(M; \mathbb{R}) \to \mathbb{R}$ as was done in [62]. The non-compact version of Theorem 3 now reads as follows:

Theorem 5.2.7. *Let $H \in \mathcal{H}(M, d\lambda)$. Then, for every $c \in H^1(M; \mathbb{R})$ and every Clarke subdifferential $h \in \partial\sigma_{H:L}(c) \subset H_1^{BM}(M; \mathbb{R})$, there exists a measure $\mu \in \mathcal{M}(\phi_H)$ whose action and rotation vector satisfy*

$$\mathcal{A}_{H,\lambda}(\mu) = \sigma_{H:L}(c) - \langle c, \rho(\mu) \rangle \quad \textit{and} \quad \rho(\mu) = h.$$

Of course, one would like to think of rotation vectors as being elements of $H_1(M; \mathbb{R})$ rather than $H_1^{BM}(M; \mathbb{R})$. In some cases this can be done effortlessly: Since L is compact, the inclusion $L \hookrightarrow M$ is proper. In particular there is an induced map $r_{L,c} : H_c^1(M; \mathbb{R}) \to H_c^1(L; \mathbb{R}) = H^1(L; \mathbb{R})$. The following is the non-compact version of Corollary 5.2.5.

Corollary 5.2.8. *Let $H \in \mathcal{H}(M, d\lambda)$ and suppose $r_{L,c}$ is surjective. Then $\sigma_{H:L}$ descends to a locally Lipschitz function*

$$\alpha_{H:L} : H^1(L; \mathbb{R}) \to \mathbb{R}.$$

In particular, for every $c \in H^1(L; \mathbb{R})$ and every Clarke subdifferential $h \in \partial\alpha_{H:L}(c)$, there exists a measure $\mu \in \mathcal{M}(\phi_H)$ satisfying

$$\mathcal{A}_{H,\lambda}(\mu) = \alpha_{H:L}(c) - \langle c, \rho(\mu) \rangle \quad \textit{and} \quad \rho(\mu) = i_L(h) \in H_1(M; \mathbb{R}). \tag{5.4}$$

[4]Recall that all $\mathcal{M}$-measures are required to have *compact* support.

Remark 5.2.9. Suppose now that $M = T^*L$ with L a closed manifold viewed as the zero-section $L \subset T^*L$ and $\omega = d\lambda$ with $\lambda = pdq$ being the canonical Liouville one-form. A Tonelli Lagrangian $l \in C^\infty(TN)$ (see [92, Definition 1.1.1]) sets up a Legendre transform

$$\mathcal{L} : TN \to T^*N,$$

so that the associated Tonelli Hamiltonian H and l satisfy the equation

$$l(\mathcal{L}^{-1}(q,p)) = \langle \lambda, X_H \rangle(q,p) - H(q,p) \quad \forall\, p \in T_q^*N.$$

In particular, in this setting one sees that the condition on the action in (5.4) can be written

$$-\alpha_{H:L}(c) = \int l \circ \mathcal{L}^{-1}\; d\mu - \langle c, \rho(\mu) \rangle. \tag{5.5}$$

As mentioned in Remark 5.2.4, it is known that $\alpha_{H:L}$ coincides with Mather's α-function in this setting so (by [92, Section 3.1]) (5.5) implies that the measures detected in Corollary 5.2.8 are Mather measures in this setting.

5.3 Comparison with Mather's theory

In this section we compare properties of the support of the measures whose existence is guaranteed by Theorem 5.2.2 and 5.2.7 to those which arise in Mather's theory and place our results in a historical context. To accomplish this, it will suffice to consider $(M, \omega) = (T^*\mathbb{T}^2, d\lambda)$. We will construct an H on $T^*\mathbb{T}^2$ for which the support of $\mathfrak{M}_{H:L}$-measures becomes extremely "wild". This phenomenon is closely related to diffusion phenomena such as Arnold' diffusion and superconductivity channels [22]. Studying $\mathfrak{M}_{H:L}$-measures in this setting was generously suggested to me by Vadim Kaloshin.

Consider $\mathbb{R}^2(p)$ as well as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the (mod $\mathbb{Z}^2$) coordinate q. Given sufficiently smooth functions $H_0 : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as well as $\epsilon \geq 0$ we consider the Hamiltonian

$$H_\epsilon(q,p) := H_0(p) + \epsilon F(q,p)$$

on $(T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2, dp \wedge dq)$. Throughout this section we make use of the canonical identification $H^1(T^*\mathbb{T}^2, \mathbb{R}) \cong \mathbb{R}^2$. The flow ϕ_{H_0} is integrable in the sense that it leaves every Lagrangian torus of the form $T(p) := \mathbb{T}^2 \times \{p\}$ invariant. KAM theory guarantees that, if H_0 is non-degenerate in the sense that its Hessian satisfies

$$\det(\mathrm{Hess}(H_0))(p) \neq 0 \quad \forall\, p \in \mathbb{R}^2,$$

then, for all small enough $\epsilon > 0$, many Lagrangian tori $T(p)$ persist. More precisely, if the rotation vector of $\phi_{H_0}|_{T(p)}$ is Diophantine and $\epsilon > 0$ is small enough, then there is a Lagrangian torus which is invariant for ϕ_{H_ϵ} close to $T(p)$. We are interested in what happens to the invariant torus $T(p)$ for $\epsilon > 0$ when $\phi_{H_0}|_{T(p)}$ does *not* have Diophantine rotation vector. Does it (at least partially) persist or does it disappear? Mather [59] studied the case

$$\det(\mathrm{Hess}(H_0))(p) > 0 \quad \forall\, p \in \mathbb{R}^2. \tag{5.6}$$

In this case all $T(p)$ *partially* persist. More precisely, Mather's theory tells us that the *p-Mather set*[5] $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(p)}) \subset \mathbb{T}^2 \times \mathbb{R}^2$ is the graph of a Lipschitz function defined on a subset of $T(p)$ (graph property). Moreover, $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(p)})$ satisfies the even stronger property that it converges to a subset of $T(p)$ in the Hausdorff distance as $\epsilon \to 0$ (localization property). Mather's proof [59, Section 5] of the localization property uses both the

[5] It is a consequence of Mather's theory (or Theorem 5.2.7) that this set is always $\neq \emptyset$.

graph property as well as the fact that Mather measures minimize a Lagrange functional. Mather's theory thus guarantees that, if (5.6) is satisfied, then the system cannot be *too* unstable. In a different direction, Arnold' conjectured [1], [3] that a generic Hamiltonian system on $T^*\mathbb{T}^n,\ n > 2$ will exhibit diffusing orbits (this phenomenon is today known as *Arnold' diffusion*). Hence, if Arnold's conjecture holds true, then all systems will exhibit *some* instability.

The question we address here is: Do the $T(p)$ persist for $\epsilon > 0$ also when

$$\det(\mathrm{Hess}(H_0))(p) < 0 \quad \forall\, p \in \mathbb{R}^2? \tag{5.7}$$

The candidate for a perturbation of $T(p)$ is again $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(p)})$, so the question is how well/badly-behaved this set is. To our knowledge Herman [45] was the first to observe that, in general, in the case (5.7) $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(p)})$ will *not* be a Lipschitz graph over $T(p)$ (see also [29] for an English exposition of Herman's example). Below we construct an example showing that not only can the graph property be violated, but so can the localization property. In fact it can be violated as dramatically as possible. The dynamics responsible for this behavior is a very fast type of diffusion arising from so-called *superconductivity channels*. We learned about this phenomenon from Bounemoura-Kaloshin's [22]. A first indication that symplectic methods can be used to study this phenomenon was found in [38]. The concept of superconductivity channels goes back to the work of Nekoroshev [64], [65] who discovered that for every $x = (q,p) \in \mathbb{T}^2 \times \mathbb{R}^2$, $\phi^t_{H_\epsilon}(x)$ remains close to $\phi^t_{H_0}(x) \in T(p)$ for $|t| \le e^{c\epsilon^{-a}}$ for some positive constants $c, a > 0$, as long as $\epsilon > 0$ is sufficiently small and H_0 is *steep*.[6] By a result due to Ilyashenko [50], the steepness condition can be phrased as requiring that the restriction of H_0 to any 1-dimensional linear subspace of $\mathbb{R}^2$ only has isolated critical points. Hence, the philosophy we follow is that, in order to find diffusion which is fast enough to "push" $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(p)})$ far away from $T(p)$, we should choose H_0 *non*-steep.

5.3.1 An example

Consider

$$H_0(p_1,p_2) = \frac{p_1^2 - p_2^2}{4}, \quad F(q_1,q_2,p_1,p_2) = \sin(2\pi(q_1 - q_2))\varphi(\tfrac{p_1-p_2}{2})$$

for some $\varphi : \mathbb{R} \to [0,1]$ with $\varphi(t) = \varphi(-t)$ and

$$\varphi(s) = \begin{cases} 1 & \text{if } |s| \le K \\ 0 & \text{if } |s| > K+1 \end{cases}, \quad \dot\varphi(s) \begin{cases} \ge 0 & \text{if } s \le 0 \\ \le 0 & \text{if } s \ge 0 \end{cases}$$

where $K > 0$ is a large constant. Note that H_0 is as far from being convex as possible in the sense that $\mathrm{Hess}(H_0)$ has one positive and one negative eigenvalue. After the symplectic change of coordinates

$$(\theta_1,\theta_2,I_1,I_2) = (q_1 - q_2, q_1 + q_2, \tfrac{p_1-p_2}{2}, \tfrac{p_1+p_2}{2}) \in \mathbb{T}^2 \times \mathbb{R}^2$$

we have

$$H_\epsilon(\theta, I) = I_1 I_2 + \epsilon\varphi(I_1)\sin(2\pi\theta_1),$$

so the resulting equations of motion are given by

$$\begin{cases} \dot I_1 = -2\pi\epsilon\varphi(I_1)\cos(2\pi\theta_1) \\ \dot I_2 = 0 \\ \dot\theta_1 = I_2 + \epsilon\dot\varphi(I_1)\sin(2\pi\theta_1) \\ \dot\theta_2 = I_1. \end{cases} \tag{5.8}$$

[6] The "closeness" in this Nekoroshev estimate is in fact quantitative. For precise details we refer to [22, Section 1.1.2]

In particular, we see that $\mathcal{O}_+(X)$ is compact for every compact X, so Theorem 5.2.7 applies.

Remark 5.3.1. The superconductivity channels $\{I_1 = 0\} \cup \{I_2 = 0\}$ give rise to diffusion: Consider initial conditions

$$(\theta_1(0), \theta_2(0), I_1(0), I_2(0))$$

such that $I_2(0) = 0$, $|I_1(0)| < K$ and $\sin(2\pi\theta_1(0)) = 0$. Then $t \mapsto \theta_1(t)$ is constant and

$$\dot{I}_1 = \pm 2\pi\epsilon\varphi(I_1),$$

so the function $t \mapsto |I_1(t) - I_1(0)|$ is strictly increasing. We denote by

$$\mathcal{D} := \{(\theta_1, \theta_2, I_1, I_2) \mid \sin(2\pi\theta_1) = 0,\ I_2 = 0,\ |I_1| < K\}$$

the set of initial conditions with $|I_1(0)| < K$ which diffuse. Poincaré recurrence implies

$$\mu(\mathcal{D}) = 0 \quad \forall\, \mu \in \mathcal{M}(\phi_H). \tag{5.9}$$

Note that if $\epsilon = 0$, then (by Remark 5.2.4 and the fact that each $T(I_1, I_2)$ is ϕ_{H_0}-invariant)

$$\mathrm{Supp}(\mathfrak{M}_{H_0:T(I_1,I_2)}) \subset T(I_1, I_2).$$

In contrast, when $\epsilon > 0$ we have

Proposition 5.3.2. *Fix $I \in (-K, K)$ and denote by $B_K \subset \mathbb{R}^2$ the open K-ball centered at 0. For every $\epsilon > 0$ we have*

$$\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(I,0)}) \cap (\mathbb{T}^2 \times B_K) = \emptyset.$$

The rest of this section will be spent proving this result.

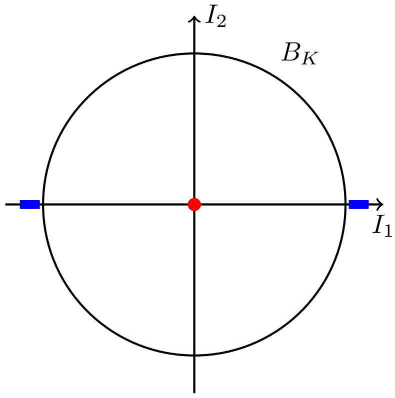

Figure 5.1: The projection of $\mathrm{Supp}(\mathfrak{M}_{H_0:T(0,0)})$ to $\mathbb{R}^2$ is contained in the red spot in the center. For all $\epsilon > 0$, the projection of $\mathrm{Supp}(\mathfrak{M}_{H_\epsilon:T(0,0)})$ to $\mathbb{R}^2$ is contained in the blue regions.

Remark 5.3.3. The conclusion of Proposition 5.3.2 resonates well with classical insight from symplectic topology. For example, it is nowadays well understood that $\dim_{\mathbb{R}} H_*(M;\mathbb{R}) \leq \mathrm{Fix}(\phi)$ for non-degenerate $\phi \in \mathrm{Ham}(M,\omega)$ (this is one of the celebrated Arnol'd conjectures). Symplectic topology can verify this inequality, but it cannot say anything about where in M the fixed points are located.

Lemma 5.3.4. *For every $\epsilon \geq 0$ we have*

$$\alpha_{H_\epsilon}(I, 0) = 0 \quad \forall\ I \in \mathbb{R}.$$

The proof of this lemma (which uses nothing but standard properties of Lagrangian spectral invariants) is presented in Section 5.6.2. The proof of Proposition 5.3.2 consists of studying carefully the solutions to (5.8). From Remark 5.2.4 we know that, in order to study the support of $\mathfrak{M}_{H_\epsilon:T(I,0)}$-measures, it suffices to study the support of $\mu \in \mathcal{M}(\phi_H)$ which arise as the weak*-limits of sequence $(\mu_\varsigma)_{\varsigma\in\mathbb{N}} \subset \mathcal{M}$ characterized by (5.2). In the current setting we can write $(x_\varsigma)_\varsigma = (\theta_1^\varsigma(0), \theta_2^\varsigma(0), I_1^\varsigma(0), I_2^\varsigma)_{\varsigma\in\mathbb{N}}$ (I_2 is an integral of motion). The conditions described in Remark 5.2.4 now amount to

$$(I_1^\varsigma(0), I_2^\varsigma) \overset{\varsigma\to\infty}{\longrightarrow} (I,0) \tag{5.10}$$
$$I_1^\varsigma(k_\varsigma) = I_1^\varsigma(0) \quad \forall\, \varsigma \in \mathbb{N}$$
$$\mathcal{A}_{H_\epsilon,\lambda}(\mu_\varsigma) \overset{\varsigma\to\infty}{\longrightarrow} \alpha_{H_\epsilon}(I,0) = 0. \tag{5.11}$$

After passing to a subsequence we may assume either that a) $I_2^\varsigma \neq 0$ for all ς or b) that $I_2^\varsigma = 0$ for all ς.

a) Note first that if $I_2(t) = I_2 \neq 0$ we can compute (using (5.8)) that

$$-\dot I_1 I_2 = \dot I_1(\dot\theta_1 - I_2) - \dot I_1\dot\theta_1 = \frac{d}{dt}(\epsilon\varphi(I_1)\sin(2\pi\theta_1)),$$

which implies that

$$I_1(t) = \frac{-\epsilon\varphi(I_1(t))\sin(2\pi\theta_1(t))}{I_2} + C \tag{5.12}$$

with

$$C = \frac{\epsilon\varphi(I_1(0))\sin(2\pi\theta_1(0))}{I_2} + I_1(0) = \frac{\epsilon\sin(2\pi\theta_1(0))}{I_2} + I_1(0),$$

whenever $|I_1(0)| < K$. Assuming $I_2^\varsigma \neq 0$ we therefore have[7]

$$I_1^\varsigma(t) = \frac{-\epsilon\varphi(I_1^\varsigma(t))\sin(2\pi\theta_1^\varsigma(t))}{I_2^\varsigma} + \frac{\epsilon\sin(2\pi\theta_1^\varsigma(0))}{I_2^\varsigma} + I_1^\varsigma(0). \tag{5.13}$$

By further passing to a subsequence we may in fact assume that either $I_2^\varsigma > 0$ or $I_2^\varsigma < 0$ for all ς. We will only discuss the first case as the second is similar. Assuming $I_2^\varsigma > 0$ we see that $I_1^\varsigma(t) \in [-K, K]$ implies $\varphi(I_1^\varsigma(t)) = 1$ and

$$\frac{(K + I_1^\varsigma(0))I_2^\varsigma}{\epsilon} \geq \sin(2\pi\theta_1^\varsigma(t)) - \sin(2\pi\theta_1^\varsigma(0)) \geq \frac{(I_1^\varsigma(0) - K)I_2^\varsigma}{\epsilon}. \tag{5.14}$$

We now claim that the sequence $(\sin(2\pi\theta_1^\varsigma(0)))_{\varsigma\in\mathbb{N}}$ cannot accumulate at ± 1. For if (after passing to a subsequence) we have $\sin(2\pi\theta_1^\varsigma(0)) \to \pm 1$, then using (5.10) and the last equation in (5.8) one deduces that the percentage of time $t \in [0, k_\varsigma]$ for which (5.14) is violated tends to 0. By looking at Figure 5.2 one sees that this implies that μ will be supported on the set

$$\{(\theta_1, \theta_2, I_1, I_2) \mid \sin(2\pi\theta_1) = \pm 1,\ |I_1| < K,\ I_2 = 0\}$$

on which H_ϵ takes the value $\pm\epsilon$. We can also estimate

$$\int \langle \lambda, X_{H_\epsilon}\rangle\, d\mu_\varsigma = \frac{I_2^\varsigma}{k_\varsigma}\int_0^{k_\varsigma} I_1^\varsigma(t)\, dt + o(1) \overset{\varsigma\to\infty}{\longrightarrow} 0.$$

Together these facts imply that

$$\lim_{\varsigma\to\infty} \mathcal{A}_{H_\epsilon,\lambda}(\mu_\varsigma) = \mathcal{A}_{H_\epsilon,\lambda}(\mu) = \pm\epsilon,$$

[7] By (5.10) we may assume that $|I_1^\varsigma(0)| < K$ for all ς.

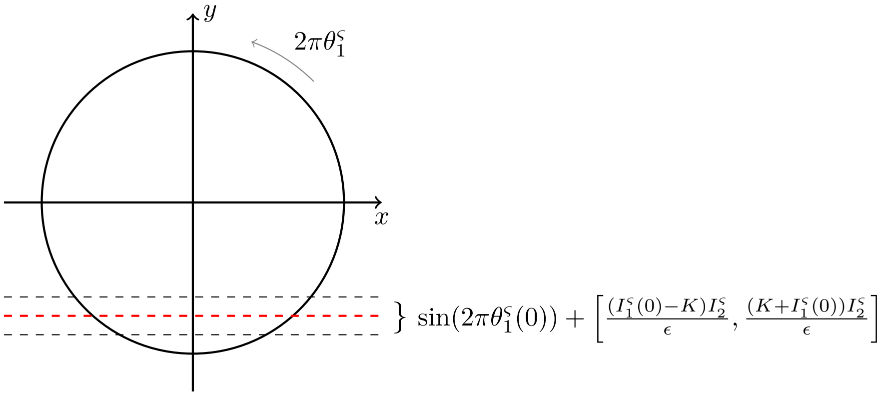

Figure 5.2: $I_1^\varsigma(t) \in [-K, K]$ implies that $2\pi\theta_1^\varsigma(t)$ is contained in the set indicated in the figure.

which contradicts (5.11). This contradiction shows that we may assume

$$|\sin(2\pi\theta_1^\varsigma(0)) \pm 1| \geq \delta > 0 \quad \forall\, \varsigma \in \mathbb{N}. \tag{5.15}$$

Now we have two subcases to deal with: Either, 1) the sequence $(\sin(2\pi\theta_1^\varsigma(0)))_{\varsigma\in\mathbb{N}}$ accumulates at 0 or 2) it doesn't.

1) In the first case we may (after passing to a subsequence) assume that

$$\sin(2\pi\theta_1^\varsigma(0)) \overset{\varsigma\to\infty}{\longrightarrow} 0.$$

This corresponds to the red dashed line in Figure 5.2 tending to $\{y = 0\}$. This implies that μ satisfies

$$\mathrm{Supp}(\mu) \cap (\mathbb{T}^2 \times B_K) \subset \mathcal{D}.$$

However, we already know from (5.9) that $\mu(\mathcal{D}) = 0$. Hence, $\mathrm{Supp}(\mu) \cap (\mathbb{T}^2 \times B_K) = \emptyset$ which confirms the statement of Proposition 5.3.2 for such measures.

2) In the second case we may assume that $|\sin(2\pi\theta_1^\varsigma(0))| \geq \delta > 0$ for all $\varsigma \in \mathbb{N}$. Together with (5.15) this implies the existence of a constant $C_1 > 0$ (uniform in ς) such that $|\sin(2\pi\theta_1^\varsigma(t))|, |\cos(2\pi\theta_1^\varsigma(t))| \geq C_1$ for all $t \in [0, k_\varsigma]$ with $I_1^\varsigma(t) = \pm K$. In turn, this implies the existence of two constants $C_-, C_+ > 0$ (uniform in ς) such that if $t \in [0, k_\varsigma]$ satisfies

$$\pm \dot{I}_1^\varsigma(t) > 0 \quad \& \quad \pm I_1^\varsigma(t) = K$$

then the smallest time $s \in (t, \infty)$ such that

$$\pm \dot{I}_1^\varsigma(s) < 0 \quad \& \quad \pm I_1^\varsigma(s) = K$$

satisfies $C_- \leq s - t \leq C_+$. I.e. once I_1^ς enters either $\{I_1 > K\}$ or $\{I_1 < -K\}$ then the time it takes before it leaves again lies in the interval $[C_-, C_+]$. Denote by J_ς the number of connected components of the set

$$\{t \in [0, k_\varsigma] \mid I_1^\varsigma(t) > K\}.$$

Again we consider two cases. In the first case $(J_\varsigma)_{\varsigma\in\mathbb{N}}$ is a bounded sequence. In this case we can estimate[8]

$$\mu_\varsigma(\{|I_1| > K\}) = \frac{|\{t \in [0, k_\varsigma] \mid |I_1^\varsigma(t)| > K\}|}{k_\varsigma} \leq \frac{(2J_\varsigma + 1)C_+}{k_\varsigma} \overset{\varsigma\to\infty}{\longrightarrow} 0,$$

[8] $|X|$ denotes the Lebesgue measure of a measurable set $X \subset \mathbb{R}$.

so $\mu_\varsigma(\{|I_1| \leq K\}) \to 1$. The same estimate as earlier thus implies that

$$\int \langle \lambda, X_{H_\epsilon} \rangle \, d\mu_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} 0,$$

and since $(\sin(2\pi\theta_1^\varsigma(0)))_{\varsigma\in\mathbb{N}}$ is bounded away from 0 we conclude that the weak*-limit of the sequence $(\mu_\varsigma)_\varsigma$ has non-zero action, which again contradicts (5.11). If instead $(J_\varsigma)_\varsigma$ is *unbounded*, then we denote by $T_\varsigma > 0$ the time it takes I_1^ς to run through the interval $[-K, K]$ once. Since $(\sin(2\pi\theta_1^\varsigma(0)))_{\varsigma\in\mathbb{N}}$ is bounded away from ± 1 one can use the fact that the derivative of $t \mapsto \arcsin(t)$ is bounded for t away from ± 1 to conclude that the sequence $(T_\varsigma)_{\varsigma\in\mathbb{N}}$ is bounded from above. By counting connected components we get the following upper bound:

$$\frac{|\{t \in [0, k_\varsigma] \mid |I_1^\varsigma(t)| \leq K\}|}{k_\varsigma} \leq \frac{(J_\varsigma + 2)T_\varsigma}{(J_\varsigma + 2)T_\varsigma + (2J_\varsigma - 1)C_-}.$$

Since $(T_\varsigma)_{\varsigma\in\mathbb{N}}$ is bounded and we may assume $J_\varsigma \to \infty$, we conclude that

$$\lim_{\varsigma\to\infty} \mu_\varsigma(\mathbb{T}^2 \times B_K) = \lim_{\varsigma\to\infty} \frac{|\{t \in [0, k_\varsigma] \mid |I_1^\varsigma(t)| \leq K\}|}{k_\varsigma} = 0.$$

Hence, the Portmanteau Theorem implies $\mathrm{Supp}(\mu) \cap (\mathbb{T}^2 \times B_K) = \emptyset$, which again confirms Proposition 5.3.2 in this case.

b) The case when $I_2^\varsigma = 0$ for all ς is similar, but in fact simpler, than the previous one. We therefore only indicate the argument. Again one considers different cases: If we can pass to a subsequence such that $\sin(2\pi\theta_1^\varsigma(0)) \to \pm 1$, then (as above) one concludes that

$$\mu_\varsigma(\{|I_1| > K\}) \overset{\varsigma\to\infty}{\longrightarrow} 0$$

and

$$\int \langle \lambda, X_H \rangle d\mu_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} 0.$$

From these conditions it follows that $\mathcal{A}_{H_\epsilon,\lambda}(\mu_\varsigma) \to \pm\epsilon$ which contradicts (5.11). If instead one can extract a subsequence with $\sin(2\pi\theta_1^\varsigma(0)) \to 0$, then it follows that

$$\mathrm{Supp}(\mu) \cap (\mathbb{T}^2 \times B_K) \subset \mathcal{D},$$

so from (5.9) it follows that $\mathrm{Supp}(\mu) \cap (\mathbb{T}^2 \times B_K) = \emptyset$, confirming Proposition 5.3.2 in this case. The last case is when (after passing to a subsequence)

$$0 < \delta \leq |\sin(2\pi\theta_1^\varsigma(0))| \leq 1 - \delta \quad \forall\, \varsigma \in \mathbb{N}.$$

In this case, the time $T_\varsigma > 0$ which it takes I_1^ς to pass through the interval $[-K, K]$ once can be computed to be

$$T_\varsigma = \frac{2K}{2\pi\epsilon|\cos(2\pi\theta_1^\varsigma(0))|}.$$

As this admits both a positive lower bound and a positive upper bound (uniformly in ς) the previous argument carries through to confirm Proposition 5.3.2 also in this case. This finishes the proof of Proposition 5.3.2.

5.4 Symplectic "Mather theory" in the absence of intersections

From the point of view of the previous sections, it is tempting to say that the existence of Mather-like measures is a consequence of symplectic intersection phenomena. In this section we study invariant measures in the "absence of intersections", e.g. when every compact subset of (M, ω) can be displaced by a Hamiltonian diffeomorphism. More precisely, we develop a C^0-approach to studying invariant measures of (autonomous) Hamiltonian systems using ideas due to Entov-Polterovich [38] and Polterovich [78]. Our main application is to the study of Hamiltonian systems on twisted cotangent bundles and $\mathbb{R}^{2n}$. The setup we consider here is slightly different from the one in the previous section. Here (M, ω) will denote a symplectic manifold which is either closed or geometrically bounded (see e.g. [5]) and $L \subset (M, \omega)$ will denote any closed (i.e. compact without boundary) connected Lagrangian submanifold. Consider an autonomous Hamiltonian $H \in C^\infty(M)$ with complete flow $\phi_H = \{\phi_H^t\}_{t \in \mathbb{R}}$. Since H is an integral of motion, sublevel sets $\Sigma_k = \Sigma_k(H) := \{H < k\}$ are invariant. As discussed previously, every $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ has a well-defined rotation vector or asymptotic cycle

$$\rho(\mu) \in H_1(\Sigma_k; \mathbb{R})$$

defined by (5.27). For the statement of the next result we denote by $e(X) \in [0, \infty]$ the *displacement energy* of a subset $X \subset M$. We recall that

$$e(X) := \inf \left\{ \|H\| : \ H \in C_c^\infty([0,1] \times M) \ \text{ s.t. } \phi_H^1(\overline{X}) \cap \overline{X} = \emptyset \right\},$$

where we use the convention that the infimum over $\emptyset$ equals ∞ and

$$\|H\| := \int_0^1 \max_M(H_t) - \min_M(H_t)\, dt$$

denotes the Hofer norm of H.

Theorem 5.4.1. *Suppose (M, ω) is weakly exact and let $H \in C^\infty(M)$ be proper and bounded from below. Fix an energy-level $k \in \mathbb{R}$ such that Σ_k is displaceable. Suppose $L \subset \Sigma_k \subset (M, \omega)$ is a Lagrangian with Abelian $\pi_1(L)$ and fix $\tilde{c} \in H^1(L; \mathbb{R})$ such that $\partial \tilde{c}|_{\pi_2(M,L)} \equiv [\omega]|_{\pi_2(M,L)}$. Then for every $a \in H^1(\Sigma_k; \mathbb{R})$ satisfying*

$$a|_L \in (e(\Sigma_k), \infty) \cdot H^1(L; \mathbb{Z}) - \tilde{c} \quad (\subset H^1(L; \mathbb{R}))$$

and every $\epsilon \in (0, k - \max_L H)$ there exists a measure $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ satisfying the two conditions

$$\langle a, \rho(\mu) \rangle \leq \max_L H - k + \epsilon$$
$$H(\mathrm{Supp}(\mu)) \subset [\max_L(H) + \tfrac{\epsilon}{2}, k - \tfrac{\epsilon}{2}].$$

In this statement, $(e(\Sigma_k), \infty) \cdot H^1(L; \mathbb{Z}) - \tilde{c}$ denotes the subset $\{t \cdot c - \tilde{c} \mid c \in H^1(L; \mathbb{Z}),\ t \in (e(\Sigma_k), \infty)\}$ of $H^1(L; \mathbb{R})$ and $\partial : H^1(L; \mathbb{R}) \to H^2(M, L; \mathbb{R})$ denotes the cohomological boundary map. Lemma 5.6.1 on page 77 below guarantees the existence of a $\tilde{c} \in H^1(L; \mathbb{R})$ such that $\partial \tilde{c}|_{\pi_2(M,L)} = [\omega]|_{\pi_2(M,L)}$ when (M, ω) is weakly exact (i.e. $\omega|_{\pi_2(M)} \equiv 0$) and $\pi_1(L)$ is Abelian. Before exhibiting an example showing the phenomenon captured by Theorem 5.4.1 we consider this result from a different point of view, which is closer to the main result of this section.

We will use the terminology that a Lagrange isotopy

$$\psi : [0,1] \times L \to (M, \omega), \quad (t, x) \mapsto \psi_t(x)$$

starts at L if ψ_0 is the inclusion $L \hookrightarrow M$. To a Lagrange isotopy ψ starting at L is associated its (Lagrangian) Flux

$$\mathrm{Flux}_L(\psi) \in H^1(L;\mathbb{R}),$$

see [91]. Here we study invariant measures of Hamiltonian systems using Flux_L.

Definition 5.4.2. Fix $c \in H^1(L;\mathbb{R})$. The *(Lagrangian) (L,c)-shape* of a subset $X \subset M$ is defined as the subset of $H^1(X;\mathbb{R})$ consisting of those classes a for which there exists a Lagrange isotopy $\psi : [0,1] \times L \to M$ starting at L and ending in X in the sense that $\psi_1(L) \subset X$ and which in addition satisfies the condition

$$\mathrm{Flux}_L(\psi) - c = \psi_1^*(a|_{\psi_1(L)}) \in H^1(L;\mathbb{R}).$$

We will denote the (L,c)-shape of X by $\mathrm{Sh}(X;L,c) \subset H^1(X;\mathbb{R})$. By the *$L$-shape of X* we simply mean the $(L,0)$-shape of X and use the shorthand notation $\mathrm{Sh}(X;L) := \mathrm{Sh}(X;L,0)$.

Remark 5.4.3. The notion of *symplectic shape* was introduced by Sikorav [88] [87] and later studied by Eliashberg [36]. The above definition is "modelled" on this classical concept, but also makes sense for closed symplectic manifolds. In case $(T^*N, \omega = d\lambda)$ is a cotangent bundle, our definition of (N,c)-shape of $X \subset T^*N$ should be thought of as corresponding to the shape of X in the sense of [36] computed with respect to the primitive $\lambda - \eta_c$ of ω, where η_c is a closed 1-form representing $c \in H^1(N;\mathbb{R}) = H^1(T^*N;\mathbb{R})$.

For the statement of the following result, recall that our Lagrangian L is said to be *rational* if the subgroup $\langle[\omega], \pi_2(M,L)\rangle \leq \mathbb{R}$ is discrete. If L is rational we denote by $\gamma(L)$ the positive generator of $\langle[\omega], \pi_2(M,L)\rangle$ in case this subgroup is non-trivial and otherwise we set $\gamma(L) := +\infty$.

Theorem 5.4.4. *Let $L \subset (M,\omega)$ be a closed rational Lagrangian with $\gamma(L) < \infty$ and let $H \in C^\infty(M)$ be proper and bounded from below. Suppose $k \in \mathbb{R}$ is an energy-level such that*

$$e(\Sigma_k) < \gamma(L).$$

Then for every $c \in \gamma(L) \cdot H^1(L;\mathbb{Z}) \subset H^1(L;\mathbb{R})$ and every $a \in \mathrm{Sh}(\Sigma_k;L,c) \subset H^1(\Sigma_k;\mathbb{R})$ there exists a $\mu \in \mathcal{M}(\Sigma_k;\phi_H)$ which satisfies

$$\langle a, \rho(\mu)\rangle < 0. \tag{5.16}$$

In particular there exists an ergodic *measure $\mu \in \mathcal{M}(\Sigma_k;\phi_H)$ which satisfies (5.16).*

Again, this is in fact a corollary of a more general result given below.

Remark 5.4.5. Theorem 5.4.4 deduces information about a Hamiltonian system using Lagrange isotopies which *cannot* be realized by globally defined symplectic isotopies. This should be compared with information about invariant measures coming from Lagrange isotopies induced by globally defined symplectic isotopies as in [78]. More precisely, the Lagrange isotopies ψ which are the source of information in Theorem 5.4.4 are those for which $\mathrm{Flux}_L(\psi)$ does *not* lie in the image of the restriction map $H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$.

Example 5.4.6. Let's illustrate the phenomenon captured in Theorem 5.4.4 by a very simple example. Denote by $h : [0,\infty) \to \mathbb{R}$ the function whose graph is illustrated in Figure 5.3 and consider the Hamiltonian $H(x,y) = h(\sqrt{x^2+y^2})$ on the plane. The symplectic structure we use on $\mathbb{R}^2(x,y)$ is $dx \wedge dy$. The sublevel set $\Sigma_{3/2}$ is an annulus containing periodic orbits of ϕ_H which represent both positive and negative classes in $H_1(\Sigma_{3/2};\mathbb{R}) \cong \mathbb{R}$. This

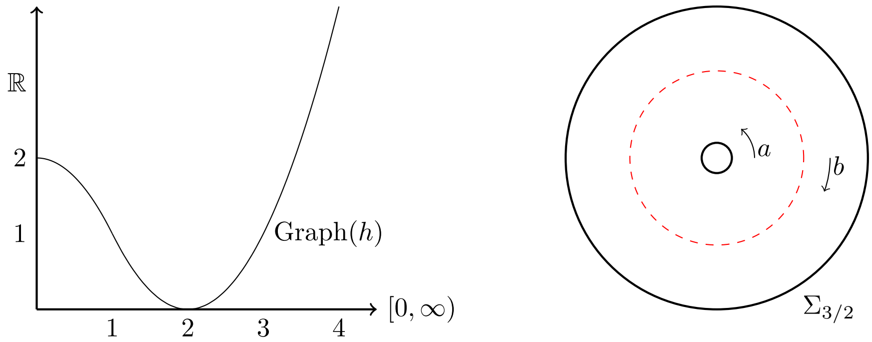

Figure 5.3: On the left Graph(h) is indicated. On the right the sublevel set $\Sigma_{3/2}$ is indicated. The red dashed line indicates the level set $\{H = 0\}$ which consists of fixed points for ϕ_H. The arrows a and b indicate the direction of ϕ_H-flowlines.

is exactly what our theory detects: We choose the Lagrangian $L := \{x^2 + y^2 = 16\}$, so that $e(\Sigma_{3/2}) < 16\pi = \gamma(L)$. Denote by $d\varphi$ the angle 1-form on L (so that $\int_l d\varphi = 2\pi$ for $l(t) = (4\cos(t), 4\sin(t))$, $t \in [0, 2\pi]$). The Lagrange isotopy ψ which "shrinks" L to the Lagrangian $\{x^2 + y^2 = 4\} = \{H = 0\} \subset \Sigma_{3/2}$ has Lagrange Flux $-2[d\varphi] \in H^1(L;\mathbb{R})$. Hence, $-2[d\varphi] \in \mathrm{Sh}(\Sigma_{3/2}; L) \subset H^1(\Sigma_{3/2};\mathbb{R})$ and Theorem 5.4.4 guarantees the existence of a $\mu \in \mathcal{M}(\Sigma_{3/2}; \phi_H)$ such that

$$-\langle 2[d\varphi], \rho(\mu)\rangle < 0 \Leftrightarrow \langle [d\varphi], \rho(\mu)\rangle > 0,$$

I.e. the motion detected by μ is that which is indicated by arrow a in Figure 5.3. Moreover, $-8[d\varphi] \in \gamma(L) \cdot H^1(L;\mathbb{Z})$ so

$$6[d\varphi] = \mathrm{Flux}_L(\psi) - (-8[d\varphi]) \in \mathrm{Sh}(\Sigma; L, -8[d\varphi]).$$

Hence, Theorem 5.4.4 also detects the existence of a $\nu \in \mathcal{M}(\Sigma_{3/2}; \phi_H)$ such that

$$\langle 6[d\varphi], \rho(\nu)\rangle < 0 \Leftrightarrow \langle [d\varphi], \rho(\nu)\rangle < 0.$$

I.e. the motion detected by ν is that which is indicated by arrow b in Figure 5.3. For more sophisticated applications of the above results, see Section 5.4.1.

The theorems above are in fact corollaries of a general theorem based on the study of a κ-function which we now define. Given a Hamiltonian $H \in C^\infty(M)$ which is bounded from below we associate to it a function $\kappa_{H:L} : H^1(L;\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\kappa_{H:L}(c) := \inf_{\psi} \max_{\psi_1(L)}(H), \quad c \in H^1(L;\mathbb{R}),$$

where the infimum runs over all Lagrange isotopies $\psi : [0,1] \times L \to M$ starting at L and satisfying $\mathrm{Flux}_L(\psi) = c$. Again we use the convention that the infimum over $\emptyset$ equals $+\infty$. The motivation for this function comes from Aubry-Mather theory where it was discovered by Contreras-Iturriaga-Paternain [31, Corollary 1] that the Mather α-function, associated to a convex Hamiltonian on the cotangent bundle of a closed manifold, can be defined in a similar way.

Theorem 5.4.7. *Consider an autonomous Hamiltonian $H \in C^\infty(M;\mathbb{R})$ which is proper and bounded from below. Let $c \in H^1(L;\mathbb{R})$ and suppose $k \in \mathbb{R}$ is an energy level such that $k \leq \kappa_{H:L}(c)$. Then for every*

$$a \in \mathrm{Sh}(\Sigma_k; L, c)$$

there exists a $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ which satisfies

$$\langle a, \rho(\mu)\rangle < 0. \tag{5.17}$$

In particular there exists an ergodic *$\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ satisfying (5.17).*

The proof of Theorem 5.4.7 (presented in Section 5.6.1) in fact provides an estimate for $\langle a, \rho(\mu)\rangle$. It relies heavily on beautiful ideas due to Buhovsky-Entov-Polterovich [24], Entov-Polterovich [38] and Polterovich [78]. Of course this result is only useful if we can find reasonable lower bounds for $\kappa_{H:L}$. The main source of such estimates comes from rigidity results in symplectic topology. Recall that $\partial : H^1(L;\mathbb{R}) \to H^2(M,L;\mathbb{R})$ denotes the cohomological boundary map. Given $c \in H^1(L;\mathbb{R})$, let's define the (additive) subgroup

$$\mathcal{G}_L(c) := \langle [\omega] - \partial c, \pi_2(M,L)\rangle \leq \mathbb{R}.$$

We are interested in $\mathcal{G}_L(c)$ for the following reason. If $\psi : [0,1] \times L \to M$ is a Lagrange isotopy starting at L with $\mathrm{Flux}_L(\psi) = c \in H^1(L;\mathbb{R})$, then the period group of $\psi_1(L) \subset (M,\omega)$ is exactly $\mathcal{G}_L(c)$:

$$\langle [\omega], \pi_2(M, \psi_1(L))\rangle = \mathcal{G}_L(c). \tag{5.18}$$

The following Proposition was the original idea employed in [38]. For more quantitative versions of this idea, see below.

Proposition 5.4.8 ([38])**.** *Suppose $c \in H^1(L;\mathbb{R})$ satisfies $\partial c = [\omega] \in H^2(M,L;\mathbb{R})$ and suppose that (M,ω) does not admit any weakly exact Lagrangian submanifolds. Then $\kappa_{H,L}(c) = +\infty$ for all Hamiltonians H.*

Proof. There exists no Lagrange isotopy ψ starting at L with $\mathrm{Flux}_L(\psi) = c$. This follows from (5.18) and our assumption. □

This proposition can be refined in several ways. Suppose (M,N,ω) is a *subcritical polarized Kähler manifold* in the sense of [14]. I.e. (M,ω) is a closed Kähler manifold and $N \subset M$ is a subset such that $(M\backslash N,\omega)$ admits the structure of a subcritical Stein manifold. In this situation we have the following estimate of $\kappa_{H:L}$.

Proposition 5.4.9. *Let (M,N,ω) be as above and suppose $H \in C^\infty(M)$. If $L \subset (M,\omega)$ is a closed Lagrangian and $c \in H^1(L;\mathbb{R})$ satisfies the condition that $\partial c = [\omega] \in H^2(M,L;\mathbb{R})$ then*

$$\kappa_{H:L}(c) \geq \min_N H.$$

Proof. Suppose $\psi : [0,1] \times L \to (M,\omega)$ is a smooth Lagrange isotopy starting at L with $\mathrm{Flux}_L(\psi) = c \in H^1(L;\mathbb{R})$. It suffices to show that $\psi_1(L) \cap N \neq \emptyset$. Suppose for contradiction that $\psi_1(L) \subset (M\backslash N,\omega)$. Since $(M\backslash N,\omega)$ is subcritical Stein it follows from [9, Lemma 3.2] that $\psi_1(L)$ can be displaced by a compactly supported Hamiltonian diffeomorphism. However, $\mathcal{G}_L(c) = \{0\}$, which means that $\psi_1(L)$ is weakly exact, so it cannot be displaced by a Hamiltonian diffeomorphism. This finishes the proof. □

The following Proposition connects Theorem 5.4.7 to the main result in [78].

Proposition 5.4.10. *Suppose that M is closed, that the restriction map $r_L : H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$ is surjective and that L is non-displaceable, i.e. $\phi(L) \cap L \neq \emptyset$ for all $\phi \in$* $\mathrm{Ham}(M,\omega)$*. Then*

$$\kappa_{H:L}(0) \geq \min_L H \tag{5.19}$$

for all $H \in C^\infty(M)$.

Proof. Let $\psi : L \times [0,1] \to M$ be a Lagrange isotopy starting at L with $\mathrm{Flux}_L(\psi) = 0 \in H^1(L;\mathbb{R})$. By [91, Lemma 6.6] there exists a symplectic isotopy $\theta : M \times [0,1] \to M$ with $\theta_0 = \mathrm{id}$ such that $\theta_t(L) = \psi_t(L)$ for all $t \in [0,1]$ and such that $\mathrm{Flux}(\theta) = 0 \in H^1(M;\mathbb{R})$. In particular $\theta_1 \in \mathrm{Ham}(M,\omega)$. Hence, $\psi_1(L) \cap L = \theta_1(L) \cap L \neq \emptyset$ which implies (5.19). □

The guiding philosophy above is that symplectic rigidity prevents a given Lagrangian from being (Lagrangian) isotoped with a given (Lagrange) flux into a fixed sublevel set of our Hamiltonian H. Perhaps the most fundamental situation where this idea can be used to estimate $\kappa_{H:L}$ from below is the following:

Proposition 5.4.11. *Let $H \in C^\infty(M;\mathbb{R})$ be bounded from below. Suppose $c \in H^1(L;\mathbb{R})$ satisfies the condition that the subgroup*

$$\mathcal{G}_L(c) \leq \mathbb{R}$$

is discrete and non-trivial. Denote by $\gamma \in (0,\infty)$ its positive generator, so $\mathcal{G}_L(c) = \gamma \cdot \mathbb{Z}$. If $k \in \mathbb{R}$ is an energy-level such that

$$e(\Sigma_k) < \gamma \tag{5.20}$$

then

$$\kappa_{H:L}(c) \geq k.$$

Proof of Proposition 5.4.11. Consider a Lagrange isotopy $\psi : [0,1] \times L \to M$ starting at L and satisfying

$$\mathrm{Flux}_L(\psi) = c. \tag{5.21}$$

We need to show that $\max_{\psi_1(L)} H \geq k$. Since $\mathcal{G}_L(c)$ is the period group of $\psi_1(L)$, a fundamental result first due to Polterovich [77] and later Chekanov [27][9] says that $\gamma \leq e(\psi_1(L))$. Hence, (5.20) implies $e(\Sigma_k) < e(\psi_1(L))$ which in turn gives

$$\psi_1(L) \not\subseteq \Sigma_k = \{H < k\}.$$

It follows that $\max_{\psi_1(L)} H \geq k$. This finishes the proof. $\square$

Remark 5.4.12. Together Theorem 5.4.7 and Proposition 5.4.11 establish a link between the fundamental notion of *displacement energy* in symplectic topology and the fundamental idea of looking for invariant measures with a prescribed rotation vector in Aubry-Mather theory. Other results linking the idea of Hofer geometry/displacement energy with Aubry-Mather theory known to the author are [86] and [93].

Proof of Theorem 5.4.1. Recall that we have an element $\tilde{c} \in H^1(L;\mathbb{R})$ such that $\partial\tilde{c}|_{\pi_2(M,L)} \equiv [\omega]|_{\pi_2(M,L)}$ and consider $a \in H^1(\Sigma_k;\mathbb{R})$ such that

$$a|_L \in (e(\Sigma_k),\infty) \cdot H^1(L;\mathbb{Z}) - \tilde{c}.$$

I.e. $a|_L = c - \tilde{c}$ for some $c \in K \cdot H^1(L;\mathbb{Z})$, where $K \in (e(\Sigma_k),\infty)$. Then we compute

$$\mathcal{G}_L(\tilde{c} - c) = \langle \partial c, \pi_2(M,L)\rangle \subset \langle c, \pi_1(L)\rangle \subset K \cdot \mathbb{Z},$$

and since $K > e(\Sigma_k)$ Proposition 5.4.11 implies that $\kappa_{H:L}(\tilde{c} - c) \geq k$. Moreover, $a \in \mathrm{Sh}(\Sigma_k; L, \tilde{c} - c)$ as one sees by considering the constant Lagrange isotopy $\psi_t(x) = x,\ (t,x) \in [0,1] \times L$. Hence, Theorem 5.4.1 follows from Theorem 5.4.7 (the estimate for $\langle a, \rho(\mu)\rangle$ follows from the proof of Theorem 5.4.7). $\square$

Proof of Theorem 5.4.4. Recall that we consider an energy-level such that $e(\Sigma_k) < \gamma(L)$. By Theorem 5.4.7 it suffices to show that $\kappa_{H:L}(c) \geq k$ for all $c \in \gamma(L) \cdot H^1(L;\mathbb{Z})$. For such c we have

$$\mathcal{G}_L(c) = \langle [\omega] - \partial c, \pi_2(M,L)\rangle \subset \langle [\omega], \pi_2(M,L)\rangle + \langle c, \pi_1(L)\rangle = \gamma(L) \cdot \mathbb{Z}.$$

Hence, $\kappa_{H:L}(c) \geq k$ follows from Proposition 5.4.11. $\square$

[9] More precisely, Polterovich proved $\gamma \leq 2e(\psi_1(L))$ and Chekanov proved $\gamma \leq e(\psi_1(L))$ under very general assumptions.

5.4.1 Examples

Example 5.4.13. Consider $\mathbb{R}^{2n}$ equipped with the standard symplectic structure $\omega := \sum_{l=1}^{n} dx_l \wedge dy_l$ and denote by $p : \mathbb{R}^{2n} \to \mathbb{R}^2$ the projection onto the first factor of $\mathbb{R}^{2n} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$. Choose a Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$ which is proper and bounded from below and fix an energy-level $k \in \mathbb{R}$ such that $\Sigma_k \neq \emptyset$. Choose $r > 0$ such that $p(\Sigma_k) \subset \mathbb{R}^2$ is contained in the open disc of radius r centered at the origin. Denote by $L \subset (\mathbb{R}^{2n}, \omega)$ the split Lagrangian $L = S^1(r) \times \cdots \times S^1(r)$ which is the product of circles in $\mathbb{R}^2$ of radius r, centered at the origin. Then $e(\Sigma_k) < r^2\pi = \gamma(L)$, so Theorem 5.4.4 gives the following statement: For every $c \in r^2\pi \cdot H^1(L;\mathbb{Z}) \subset H^1(L;\mathbb{R})$ and every $a \in \mathrm{Sh}(\Sigma_k; L, c)$ there exists $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ such that

$$\langle a, \rho(\mu)\rangle < 0.$$

Apart from $\mathbb{R}^{2n}$, the main example we have in mind is that of Hamiltonian dynamics on a twisted cotangent bundle.

Example 5.4.14. Consider the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and denote by Ω a closed 2-form on $\mathbb{T}^2$ such that $\int_{\mathbb{T}^2} \Omega = 1$. The symplectic structure on the *twisted cotangent bundle* $(T^*\mathbb{T}^2, \omega)$ is defined by

$$\omega := d\lambda_{\mathbb{T}^2} + \pi^*\Omega,$$

where $\pi : T^*\mathbb{T}^2 \to \mathbb{T}^2$ denotes the footpoint map and $\lambda_{\mathbb{T}^2}$ denotes the Liouville 1-form. Note that $(T^*\mathbb{T}^2, \omega)$ is weakly exact. Let $H \in C^\infty(T^*\mathbb{T}^2)$ be a Hamiltonian which is proper and bounded from below. From [40] it follows that every sublevel set $\Sigma_k = \{H < k\}$ is displaceable. Suppose $L \subset \Sigma_k \subset (T^*\mathbb{T}^2, \omega)$ is a Lagrangian 2-torus.[10] Since $\pi_1(L)$ is Abelian, Lemma 5.6.1 on page 77 guarantees the existence of a $\tilde{c} \in H^1(L;\mathbb{R})$ such that $\partial\tilde{c}|_{\pi_2(M,L)} \equiv [\omega]|_{\pi_2(M,L)}$. Applying Theorem 5.4.1 we then conclude that for any $a \in H^1(\Sigma_k;\mathbb{R})$ satisfying

$$a|_L \in (e(\Sigma_k), \infty) \cdot H^1(L;\mathbb{Z}) - \tilde{c}$$

and any $\epsilon > 0$ there exists a $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ such that

$$\langle a, \rho(\mu)\rangle \leq \max_L H - k + \epsilon.$$

Example 5.4.15. Consider the 2-sphere S^2 equipped with a closed 2-form Ω such that $\int_{S^2} \Omega = 1$. Denote by $\pi : T^*S^2 \to S^2$ the footpoint map. We want to apply our results above to study Hamiltonian systems on the twisted cotangent bundle (T^*S^2, ω), where

$$\omega := d\lambda_{S^2} + \pi^*\Omega.$$

Clearly (T^*S^2, ω) is *not* weakly exact. There exists a rational Lagrangian 2-torus $L \subset (T^*S^2, \omega)$ with $\gamma(L) = 1$. Assuming this fact for now, let $H \in C^\infty(T^*S^2)$ denote an autonomous Hamiltonian which is proper and bounded from below and let $k \in \mathbb{R}$ denote an energy level such that $e(\Sigma_k) < 1$. Then by Theorem 5.4.4, for every $c \in H^1(L;\mathbb{Z})$ and every $a \in \mathrm{Sh}(\Sigma_k; L, c)$ there exists a $\mu \in \mathcal{M}(\Sigma_k; \phi_H)$ such that

$$\langle a, \rho(\mu)\rangle < 0.$$

In order to see the existence of L, let's be explicit and view

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

[10] There are several ways of finding Lagrangian 2-tori in $(T^*\mathbb{T}^2, \omega)$. One can find "small" ones in Darboux charts but there are also several other approaches. See e.g. Example 5.4.15. Note that, since every closed Lagrangian in $(T^*\mathbb{T}^2, \omega)$ is displaceable, it follows from the adjunction formula, that every closed orientable Lagrangian in $(T^*\mathbb{T}^2, \omega)$ is topologically a 2-torus.

Since $\Omega|_{S^2\cap\{z>-1\}}$ is exact we can denote by ξ a 1-form on $S^2\cap\{z>-1\}$ such that $\Omega = d\xi$ and choose a smooth function $\rho : S^2 \to [0,1]$ such that

$$\rho(x,y,z) = \begin{cases} 1, & \text{if } z \geq -1/2 \\ 0, & \text{if } z \leq -2/3. \end{cases}$$

By Moser's argument the symplectic manifold $(T^*S^2, \widetilde{\omega})$, where

$$\widetilde{\omega} = \omega - d\pi^*(\rho\xi),$$

is symplectomorphic to (T^*S^2, ω). Denote by $f : B^2(2) \to S^2$ an embedding which identifies the closed 2-disc $B^2(2) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4\}$ with the upper hemisphere $S^2 \cap \{z \geq 0\}$. Since $\widetilde{\omega}|_{S^2\cap\{z\geq 0\}} = d\lambda_{S^2}|_{S^2\cap\{z\geq 0\}}$ f induces a symplectic embedding

$$F = (df^*)^{-1} : (T^*B^2(2), \lambda_{B^2(2)}) \to (T^*S^2, \widetilde{\omega})$$

Denote by $T \subset B^4(2) \subset B^2(2) \times \mathbb{R}^2 = T^2B^2(2)$ a rational Lagrangian 2-torus with $\gamma(T) = 1$. We claim that $F(T) \subset (T^*S^2, \widetilde{\omega})$ too is a rational Larangian 2-torus with $\gamma(F(T)) = 1$. In order to see this, let $u : (B^2(1), \partial B^2(1)) \to (T^*S^2, F(T))$ denote a smooth topological disc on $F(T)$. It suffices to check that $\int_u \widetilde{\omega} \in \mathbb{Z}$. Since we clearly have $\int_u d\lambda_{S^2} \in \mathbb{Z}$ it in fact suffices to check that $\int_v (\Omega - d(\rho\xi)) \in \mathbb{Z}$, where $v = \pi \circ u$. To see this we note that, since $\pi(F(T)) \subset S^2 \cap \{z \geq 0\}$ and $S^2 \cap \{z \geq 0\}$ is contractible, we can extend v to a map $\widetilde{v} : B^2(2) \to S^2$ such that $\widetilde{v}(\partial B^2(2)) = \{(0,0,1)\}$. Since the 2-form $(\Omega - d(\rho\xi))$ vanishes on $S^2 \cap \{z \geq 0\}$ we have $\int_{\widetilde{v}} (\Omega - d(\rho\xi)) = \int_v (\Omega - d(\rho\xi))$ and since $\widetilde{v}(\partial B^2(2)) = \{(0,0,1)\}$, $\widetilde{v}$ induces a map $\widehat{v} : S^2 \to S^2$. Now $\int_{\widetilde{v}} (\Omega - d(\rho\xi))$ is simply the degree of $\widehat{v}$. In particular it is an integer. This proves the claim that $F(T) \subset (T^*S^2, \widetilde{\omega})$ is a rational Lagrangian torus with $\gamma(F(T)) = 1$. Since $(T^*S^2, \widetilde{\omega}) \cong (T^*S^2, \omega)$ we conclude that there exists a rational Lagrangian torus $L \subset (T^*S^2, \omega)$ with $\gamma(L) = 1$.

Remark 5.4.16. The choice of the above example is not coincidental: One could ask if it is possible to study Hamiltonian systems on the twisted T^*S^2 from the previous example using Lagrangian Floer homology. However, this approach seems unlikely to succeed. Indeed, as pointed out to me by MathOverflow-user Nikolaki [66], if one manages to find a Lagrangian $L \subset (T^*S^2, \omega)$ for which there exists a well-defined Floer homology $HF(L)$, then probably $HF(L) = 0$. This follows from the fact that (if well-defined) $HF(L)$ should be a module over the symplectic homology $SH(T^*S^2)$ of (T^*S^2, ω). A result due to Ritter [80, Theorem 5] implies that $SH(T^*S^2) = 0$ (see also Benedetti's thesis [7] for applications of symplectic homology to Hamiltonian systems on twisted cotangent bundles). Indeed, Ritter's result says that the symplectic homology (with coefficients in $\mathbb{Z}_2$) of the total space $\mathrm{Tot}(\mathcal{O}(-2) \to \mathbb{C}P^1)$ of the holomorphic line bundle $\mathcal{O}(-2) \to \mathbb{C}P^1$ vanishes. In order to see that (T^*S^2, ω) is symplectomorphic to $\mathrm{Tot}(\mathcal{O}(-2) \to \mathbb{C}P^1)$, we note that the Chern number of the holomorphic tangent bundle $T\mathbb{C}P^1 \to \mathbb{C}P^1$ equals $\chi(\mathbb{C}P^1) = 2$. Hence, $\mathcal{O}(-2) \to \mathbb{C}P^1$ is realized by the holomorphic cotangent bundle $T^*\mathbb{C}P^1 \to \mathbb{C}P^1$. Since the canonical symplectic structure on $\mathrm{Tot}(T^*\mathbb{C}P^1 \to \mathbb{C}P^1)$ represents the cohomology class $[\omega] \in H^2(T^*S^2;\mathbb{R}) = H^2(\mathbb{C}P^1;\mathbb{R})$ one can apply Moser's trick to see that $\mathrm{Tot}(\mathcal{O}(-2) \to \mathbb{C}P^1)$ is symplectomorphic to (T^*S^2, ω).

5.5 Preliminaries

Here we discuss some of the background material used frequently in the above sections. Recall that a Lagrangian submanifold $L \subset (M, \omega)$ is said to be *monotone* if there exists a constant $\tau_L \geq 0$ such that

$$\omega|_{\pi_2(M,L)} = \tau_L \cdot \mu|_{\pi_2(M,L)},$$

where $\omega|_{\pi_2(M,L)}$ denotes integration of ω and $\mu|_{\pi_2(M,L)}$ denotes the Maslov index. If in addition the minimal Maslov number of L is ≥ 2, then the quantum homology $QH_*(L;\mathbb{Z}_2)$ and Floer homology $HF(L;\mathbb{Z}_2)$ of L are well-defined [15], [104]. Our main reference for $QH_*(L;\mathbb{Z}_2)$ and $HF_*(L;\mathbb{Z}_2)$ is Zapolsky's excellent [104]. In order to set the notation we use here we will discuss a few details about $HF_*(L;\mathbb{Z}_2)$ and the constrution of spectral invariants. Recall that, given a Hamiltonian $H \in \mathcal{H}$ such that $\phi^1_H(L) \pitchfork L$ and a generic path $\{J_t\}_{t\in[0,1]}$ of ω-compatible almost complex structures, $HF_*(L{:}H,J)$ is the homology of a Morse-type chain complex $(CF_*(L{:}H,J),d)$ generated by critical points of the action functional

$$\mathcal{A}_{H:L}(\widetilde{\gamma} = [\gamma,\widehat{\gamma}]) = \int_0^1 H_t(\gamma(t))\,dt - \int \widehat{\gamma}^*\omega.$$

Here the input $\widetilde{\gamma}$ consists of a smooth path $\gamma{:}([0,1],\{0,1\}) \to (M,L)$ as well as an equivalence class of *cappings* $\widehat{\gamma}$ of γ in the sense of [104]. The equivalence relation identifies cappings which have the same symplectic area. As a result, generators of $CF_*(L{:}H,J)$ have an asociated action and we denote by $(CF^a_*(L{:}H,J),d)$ $(a \in \mathbb{R})$ the subcomplex generated by elements whose action is $< a$. This is indeed a subcomplex as d decreases action, and its homology is denoted by $HF^a_*(L{:}H,J)$. The inclusion $i_a : CF^a_*(L{:}H,J) \to CF_*(L{:}H,J)$ induces maps $i^a_* : HF^a_*(L{:}H,J) \to HF_*(L{:}H,J)$. By standard arguments $HF_*(L{:}H,J)$ is independent of the data (H,J) and identifying all such groups via canonical continuation isomorphisms we obtain a ring $HF(L;\mathbb{Z}_2)$, which is isomorphic to $QH(L;\mathbb{Z}_2)$ via a PSS-isomorphism

$$\mathrm{PSS} : QH_*(L;\mathbb{Z}_2) \xrightarrow{\cong} HF_*(L;\mathbb{Z}_2).$$

Under the assumption $QH_*(L;\mathbb{Z}_2) \neq 0$, the Leclercq-Zapolsky spectral invariant $l_L : \widetilde{\mathrm{Ham}}(M,\omega) \to \mathbb{R}$ which we consider is defined by

$$l_L(\tilde{\phi}_H) := \inf\{a \in \mathbb{R} \mid \mathrm{PSS}([L]) \in \mathrm{Image}(i^a_*) \subset HF_*(L;\mathbb{Z}_2)\},$$

where $i^a_* : HF^a_*(L{:}H,J) \to HF_*(L{:}H,J) \cong HF_*(L;\mathbb{Z}_2)$ and $[L] \in QH(L;\mathbb{Z}_2)$ denotes the "fundamental class", i.e. the unity for the ring-structure on $QH_*(L;\mathbb{Z}_2)$. For further details on l_L and the properties satisfied by l_L we refer to [58] and [62]. An important property for us is that l_L satisfies the *triangle inequality*:

$$l_L(\tilde{\phi}_H\tilde{\phi}_K) \leq l_L(\tilde{\phi}_H) + l_L(\tilde{\phi}_K) \quad \forall\ H,K \in \mathcal{H}.$$

As a consequence of this, the limit (5.1) exists and $\sigma_{H:L}$ is well-defined. The following theorem is due to [62] in a slightly different setting. In fact, [62] presents many more properties of $\sigma_{H:L}$ in the setting of a cotangent bundle. Here we only list the properties which will be useful to us.

Theorem 5.5.1 ([62])**.** *Given any $H \in \mathcal{H}$, the function $\sigma_{H:L} : H^1(M;\mathbb{R}) \to \mathbb{R}$ from Definition 5.2.1 is well-defined and it satisfies the following properties*

a) $\sigma_{H:L}$ is locally Lipschitz with respect to any norm on $H^1(M;\mathbb{R})$. In particular $\sigma_{H:L}$ is differentiable almost everywhere.

b) Suppose $\psi = \{\psi_t\}_{t\in[0,1]}$ is a symplectic isotopy with $\psi_0 = \mathrm{id}$. Then

$$\min_{x\in\psi_1(L)} H(x) \leq \sigma_{H:L}(c) \leq \max_{x\in\psi_1(L)} H(x),$$

where $c := \mathrm{Flux}(\psi) \in H^1(M;\mathbb{R})$.

c) If $\psi = \{\psi_t\}_{t\in[0,1]} \subset \mathrm{Symp}(M,\omega)$ is a path of symplectomorphisms with $\psi_0 = \mathrm{id}$ and $c' := \mathrm{Flux}(\psi) \in H^1(M;\mathbb{R})$ then

$$\sigma_{H:\psi_1(L)}(c) = \sigma_{H:L}(c + c') \quad \forall\ c \in H^1(M;\mathbb{R}).$$

d) For every $\phi \in \mathrm{Ham}(M,\omega)$ *we have* $\sigma_{H:L} = \sigma_{H:\phi(L)}$.

In [21] we obtained an alternative version of the last property: If $L' \subset (M,\omega)$ is another monotone Lagrangian submanifold which is monotone Lagrangian cobordant to L then $\sigma_{H:L} = \sigma_{H:L'}$.

5.5.1 Subdifferentials and rotation vectors

In this section we recall a few basic facts about the Clarke subdifferential ∂f of a function f as well as about the approximate subdifferential $\partial_A f$ which is needed in the proofs below. For further references on the Clarke subdifferential we refer to [30] and for further references on the approximate subdifferential we refer to [51]. Denote by V a finite dimensional vector space over $\mathbb{R}$ and by $f : V \to \mathbb{R}$ a function which is locally Lipschitz (with respect to any norm on V). The generalized directional derivative $f^\circ(x,v)$ of f at $x \in V$ in direction $v \in V$ is by definition the number

$$f^\circ(x,v) := \limsup_{\substack{y\to x\\ \tau\downarrow 0}} \frac{f(y+\tau v)-f(y)}{\tau}.$$

$f^\circ(x,v)$ is a real number because f is locally Lipschitz. The *Clarke subdifferential* (or *generalized derivative*) of f at $x \in V$ is by definition the non-empty subset

$$\partial f(x) = \{v^* \in V^* \mid \langle v^*, v\rangle \le f^\circ(x,v)\ \forall\ v \in V\}$$

of V^*. Note that since f is Locally Lipschitz it is differentiable almost everywhere.[11] A key property of ∂f is [30, Theorem 2.5.1], according to which, for any Lebesgue 0-set $S \subset V$ containing the non-differentiability points of f we have

$$\partial f(x) = \mathrm{conv}(\{\lim_{\varsigma\to\infty} df(x_\varsigma) \mid (x_\varsigma)_{\varsigma\in\mathbb{N}} \subset V\backslash S \text{ s.t. } x_\varsigma \to x\}), \tag{5.22}$$

where $\mathrm{conv}(X) \subset V$ denotes the convex hull of the subset $X \subset V$. I.e. $\partial f(x)$ is the convex hull of the set of all limit points of sequences of differentials $df(x_\varsigma)$ of f with $(x_\varsigma)_{\varsigma\in\mathbb{N}} \subset V\backslash S$ a sequence such that $x_\varsigma \to x$. Another important property which will be used below is Lebourg's mean value theorem for ∂f [30, Theorem 2.3.7]: For every pair $x, y \in V$ there exists a $t \in (0,1)$ such that for some

$$v^* \in \partial f(ty + (1-t)x)$$

we have

$$\langle v^*, x-y\rangle = f(x) - f(y).$$

We will now discuss some of the basic properties of the approximate subdifferential $\partial_A f$ of f due to Ioffe [51]. We point out that the $\partial_A f$ makes sense also if f is not locally Lipschitz, but here we restrict ourselves to considering the case when f is locally Lipschitz. For $x, v \in V$, the *lower Dini directional derivative* of f at x in direction v is defined by

$$d^- f(x;v) := \liminf_{\tau\downarrow 0} \frac{f(x+\tau v)-f(x)}{\tau} \in \mathbb{R}$$

and the *Dini subdifferential* $\partial^- f(x) \subset V^*$ of f at x is the subset

$$\partial^- f(x) := \{v^* \in V^* \mid \langle v^*, v\rangle \le d^- f(x;v)\ \forall\ v \in V\} \tag{5.23}$$

[11] This should be understood in the following sense: If we identify $V \cong \mathbb{R}^k$ by choosing a basis, then the set of points in $\mathbb{R}^k$ at which f isn't differentiable is a Lebesgue 0-set.

of V^*. Finally the approximate subdifferential of f at x is defined as the subset

$$\partial_A f(x) := \bigcap_{\delta>0} \bigcup_{z\in U(f,x,\delta)} \partial^- f(z), \tag{5.24}$$

of V^*, where $U(f,x,\delta) := \{z \mid 0 < |x-z| < \delta \text{ and } |f(x)-f(z)| < \delta\}$. A key property of $\partial_A f$ (when f is locally Lipschitz) is [51, Theorem 2], according to which the Clarke subdifferential and the approximate subdifferential are related by

$$\partial f(x) = \overline{\operatorname{conv}(\partial_A f(x))} \quad \forall\, x \in V. \tag{5.25}$$

Our main interest in ∂_A is that it behaves well under uniform approximation: By a result due to Jourani [52, Theorem 3.2], if $(f_\varsigma)_{\varsigma\in\mathbb{N}}$ is a sequence of locally Lipschitz functions $V \to \mathbb{R}$ such that

$$f_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} f$$

uniformly, then

$$\partial_A f(x) \subset \limsup_{\substack{\varsigma\to\infty\\ y\to x}} \partial_A f_\varsigma(y) \quad \forall\, x \in V, \tag{5.26}$$

where

$$\limsup_{\substack{\varsigma\to\infty\\ y\to x}} \partial_A f_\varsigma(y) := \bigcap_{\varsigma=1}^{\infty} \bigcap_{\delta>0} \overline{\bigcup_{\substack{k\geq\varsigma\\ 0<|x-y|<\delta}} \partial_A f_k(y)}.$$

We point out that this result is a generalization of Ioffe's result [51, Theorem 3].

We are interested in the subdifferential of $\sigma_{H:L} : H^1(M;\mathbb{R}) \to \mathbb{R}$ because they parametrize rotation vectors of $\mathcal{M}(\phi_H)$-measures (Theorem 5.2.2). We recall that, under the canonical identification $H_1(M;\mathbb{R}) = H^1_{dR}(M;\mathbb{R})^*$, the rotation vector $\rho(\mu) \in H_1(M;\mathbb{R})$ of a ϕ_H-invariant measure $\mu \in \mathcal{M}(\phi_H)$ is given by

$$\langle [\eta], \rho(\mu)\rangle = \int \langle \eta, X_H\rangle \, d\mu, \quad [\eta] \in H^1_{dR}(M;\mathbb{R}). \tag{5.27}$$

For further details and in-depth explanations of rotation vectors we refer to [82] or [92].

5.6 Proofs of results

Before proving the results from Section 5.2 we present the remaining proofs of results from Section 5.4. This means proving Theorem 5.4.4. Before doing that we prove a lemma which was implicitely used in the statement of Theorem 5.4.1.

Lemma 5.6.1. *Let (M,ω) be weakly exact (i.e. $\omega|_{\pi_2(M)} \equiv 0$) and denote by $L \subset (M,\omega)$ a Lagrangian satisfying either the condition that $\pi_1(L)$ is Abelian or that the map $\pi_1(L) \to \pi_1(M)$ is trivial. Then there exists $\tilde{c} \in H^1(L;\mathbb{R})$ such that*

$$\partial\tilde{c}|_{\pi_2(M,L)} \equiv [\omega]|_{\pi_2(M,L)}.$$

Proof. Since (M,ω) is weakly exact we may view $[\omega]$ as a homomorphism

$$\pi_2(M,L)/\pi_2(M) \to \mathbb{R}.$$

By the long exact sequence for homotopy groups we may view $\pi_2(M,L)/\pi_2(M)$ as a subgroup $\pi_2(M,L)/\pi_2(M) \leq \pi_1(L)$. If $\pi_1(L) \to \pi_1(M)$ is trivial then

$$\pi_2(M,L)/\pi_2(M) \cong \pi_1(L),$$

so $[\omega]$ gives rise to a homomorphism $\pi_1(L) \to \mathbb{R}$ which is given by an element $\tilde{c} \in H^1(L;\mathbb{R})$. In case $\pi_1(L)$ is Abelian (so $\pi_1(L) = H_1(L;\mathbb{Z})$) $[\omega]$ descends to a homomorphism $G \to \mathbb{R}$, where G denotes the image of $\pi_2(M,L)/\pi_2(M)$ under the quotient map

$$\pi_1(L) \to \pi_1(L)/\text{Torsion}.$$

Any homomorphism $G \to \mathbb{R}$ extends to a homomorphism $\pi_1(L)/\text{Torsion} \to \mathbb{R}$ by elementary algebra (use e.g. [49, Chapter II, Theorem 1.6]). The composition

$$\pi_1(L) \to \pi_1(L)/\text{Torsion} \to \mathbb{R}$$

extends $[\omega]$ and corresponds to an element $\tilde{c} \in H^1(L;\mathbb{R})$. □

5.6.1 Proof of Theorem 5.4.7

The proof relies heavily on the work of Entov-Polterovich [38] as well as the work of Polterovich [78]. Given a 1-form η on M we denote by X_η the vector field defined by requiring

$$\iota_{X_\eta}\omega = \eta.$$

Given an autonomous Hamiltonian $H \in C^\infty(M)$, the Poisson bracket $\{H,\eta\}$ is the function given by

$$\{H,\eta\} := \omega(X_\eta, X_H) = \eta(X_H) = dH(X_\eta).$$

Given an open subset $U \subset M$ we consider the following quantity which is a slight modification of a quantity appearing in [78].

$$pb_{H,U}(c) := \inf_{\eta \in c} \max_{x \in U} \{H,\eta\}(x), \quad c \in H^1(U;\mathbb{R}).$$

If $X \subset U$ is a compact subset we also define

$$pb_{H,U}(c;X) := \inf_{\eta \in c} \max_{x \in X} \{H,\eta\}(x), \quad c \in H^1(U;\mathbb{R}).$$

The following theorem which is due to Polterovich is the main tool which allows us to construct invariant measures with given rotation vectors using $pb_{H,U}$. In [78] the statement appears in a slightly different form, but the version pesented here follows directly from the theory developed in [78].

Theorem 5.6.2 ([78])**.** *Let $H \in C^\infty(M)$ be an autonomous Hamiltonian which is proper and bounded from below. Given $k \in \mathbb{R}$ and $\delta > 0$ we set $X := \{H \leq k - \delta\}$. Suppose for some $c \in H^1(U;\mathbb{R})$ there exists $\epsilon > 0$ such that*

$$pb_{H,\Sigma_k}(c;X) \geq \epsilon.$$

Then there exists a $\mu \in \mathcal{M}(\Sigma_k;\phi_H)$ with $\mathrm{Supp}(\mu) \subset X$, which satisfies

$$\langle c, \rho(\mu) \rangle \geq \epsilon.$$

The following proof is a small variation of the proof of the main technical lemma in Entov-Polterovich's [38].

Proof of Theorem 5.4.7. Recall that we consider $c \in H^1(L;\mathbb{R})$ and $k \in \mathbb{R}$ such that

$$k \leq \kappa_{H:L}(c).$$

Fix now

$$a \in \mathrm{Sh}(\Sigma_k; L, c) \subset H^1(\Sigma_k;\mathbb{R}).$$

This means (by definition) that there exists a Lagrange isotopy $\psi : [0,1] \times L \to M$, starting at L, which satisfies

$$L_1 := \psi_1(L) \subset \Sigma_k \quad \text{and} \quad \mathrm{Flux}_L(\psi) - c = \psi_1^*(a|_{L_1}) \in H^1(L;\mathbb{R}). \tag{5.28}$$

Choose $\delta > 0$ so small that $\max_{L_1}(H) + \delta < k - \delta$. For every $\epsilon > 0$ we can find a smooth function $u : \mathbb{R} \to \mathbb{R}$ such that

$$u(t) = \begin{cases} \max_{L_1}(H) + \delta, & \text{if } t \leq \max_{L_1}(H) + \delta \\ k - \delta, & \text{if } t \geq k - \delta, \end{cases}$$

and $0 \leq u'(t) \leq 1 + \epsilon$. Consider now the Hamiltonian

$$F := u \circ H - (k - \delta).$$

Setting $X := \{H \leq k - \delta\}$ it suffices (by Theorem 5.6.2) to show that

$$pb_{H,\Sigma_k}(-a, X) \geq \Delta := k - \max_{L_1}(H) - 2\delta > 0.$$

Since $\{F, \eta\} \leq (1 + \epsilon)\{H, \eta\}$ for every 1-form η it suffices (by letting $\epsilon \downarrow 0$) to show that $pb_{F,\Sigma_k}(-a, X) \geq \Delta$. To see that this inequality holds we choose a closed 1-form η on Σ_k with $[\eta] = -a$. Our task is to show that $\max_X\{F, \eta\} \geq \Delta$. Suppose for contradiction that this is not the case, i.e. that

$$\max_X\{F, \eta\} < \Delta.$$

Exactly as in [38] we then have that the closed 2-form

$$\omega_s := \omega - s dF \wedge \eta$$

is symplectic for all $s \in [0, T]$, where $T := \frac{1}{\Delta}$. Note that ω_s is a well-defined 2-form on all of M because $dF = 0$ on $M \backslash X$. Hence, we can define a family of vectorfields $(Y_s)_{s \in [0,T]}$ by requiring

$$\iota_{Y_s}\omega_s = F\eta. \tag{5.29}$$

Since F is compactly supported we see that Y_s is compactly supported, so it integrates to an isotopy $(\rho_s)_{s\in[0,T]}$ which by Moser's argument [63] satisfies $\rho_s^*\omega_s = \omega$ for all $s \in [0, T]$. Since $L_1 \subset (M, \omega)$ is Lagrangian and F is constant on L_1 we have $\omega_s|_{L_1} \equiv 0$, so $\rho_s^{-1}(L_1) \subset (M, \omega)$ is also Lagrangian for all $s \in [0, T]$. Hence, we obtain a Lagrange isotopy $\tilde{\psi} : [0,1] \times L \to M$ by defining

$$\tilde{\psi}_t := \begin{cases} \psi_{2t}, & \text{if } t \in [0, \frac{1}{2}] \\ \rho^{-1}_{(2t-1)T} \circ \psi_1, & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

which starts at L and is smooth after a reparametrisation. Now note that because $\mathrm{Supp}(F) \subset X$ it follows from (5.29) that $\mathrm{Supp}(Y_s) \subset X$ for all $s \in [0, T]$. In particular $\rho_s|_{M\backslash X} = \mathrm{id}_{M \backslash X}$ for all $s \in [0, T]$. Since $L_1 \subset X$ it follows that $\tilde{\psi}_1(L) = \rho_T^{-1}(L_1) \subset X$, meaning

$$\max_{\tilde{\psi}_1(L)} H \leq k - \delta. \tag{5.30}$$

Since the family of vectorfields $(Z_s)_s$ generating $(\rho_s^{-1})_s$ is given by

$$Z_s(x) = -d\rho_s^{-1}(\rho_s(x)) \cdot Y_s(\rho_s(x))$$

it is easy to check that

$$\omega(Z_s, v) = -\rho_s^*(F\eta)(v) \quad \forall\, v \in T_x M.$$

From $F|_{L_1} \equiv -\Delta$ it now follows that

$$\mathrm{Flux}_{L_1}((\rho_s^{-1})_{s\in[0,T]}) = \int_0^T \Delta[\eta|_{L_1}]ds = -a|_{L_1} \in H^1(L_1;\mathbb{R}),$$

and thus (using (5.28))

$$\mathrm{Flux}_L(\tilde{\psi}) = \mathrm{Flux}_L(\psi) - \psi_1^*(a|_{L_1}) = c.$$

In particular we can estimate

$$\max_{\tilde{\psi}_1(L)} H \geq \kappa_{H:L}(c) \geq k.$$

But this contradicts (5.30). This contradiction shows that $\max_M\{F,\eta\} \geq \Delta$ which by Theorem 5.6.2 finishes the proof of the existence of μ. In order to see that μ can be chosen to be ergodic we consider the function

$$R : \mathcal{M}(X;\phi_H) \to \mathbb{R}, \quad \nu \mapsto R(\nu) := \langle a, \rho(\nu)\rangle.$$

Image(R) is a closed interval: Image(R) $= [A,B]$. The above proof shows that $A < 0$. By [17, Lemma 2.2.1] there exists an ergodic $\nu \in \mathcal{M}(X;\phi_H)$ such that $R(\nu) = A$. Since $\mathcal{M}(X;\phi_H) \subset \mathcal{M}(\Sigma_k;\phi_H)$ this finishes the proof of Theorem 5.4.7. $\square$

Remark 5.6.3. Combining the proof of Theorem 5.6.2 and the above proof, it is easy to deduce that the constructed measure μ satisfies $H(\mathrm{Supp}(\mu)) \subset [\max_{L_1} H + \delta, k - \delta]$.

5.6.2 Proofs of results from Section 5.2

Proof of Lemma 5.2.5. We first prove that $\sigma_{H:L}$ descends to a Lipschitz function

$$\alpha_{H:L} : H^1(L;\mathbb{R}) \to \mathbb{R}$$

under the assumption that $r_L : H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$ is onto. Suppose $c, c' \in H^1(M;\mathbb{R})$ satisfy $r_L(c) = r_L(c') \in H^1(L;\mathbb{R})$. Fix a closed 1-form η on M with $[\eta] = c - c'$ and denote by $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ the symplectic isotopy generated by the vector field X_η characterised by

$$\iota_{X_\eta}\omega = \eta.$$

By property c) of Theorem 5.5.1 we can now compute

$$\sigma_{H:L}(c) = \sigma_{H:L}(c' + (c - c')) = \sigma_{H:\psi_1(L)}(c').$$

We can view ψ as a Lagrange isotopy starting at L

$$\psi|_L : [0,1] \times L \to M, \tag{5.31}$$

with *Lagrangian Flux path* $s \mapsto \mathrm{Flux}_L(\{\psi_t|_L\}_{0\leq t\leq s}) \in H^1(L;\mathbb{R})$. We compute that

$$\mathrm{Flux}_L(\{\psi_t|_L\}_{0\leq t\leq s}) = \int_0^s r_L[\iota_{X_\eta}\omega]dt = r_L(c - c')s = 0 \in H^1(L;\mathbb{R})$$

for all $s \in [0,1]$. Hence, (5.31) is an exact Lagrange isotopy, so there exists $\phi \in \mathrm{Ham}(M,\omega)$ such that $\phi(L) = \psi_1(L)$. Applying property d) of Theorem 5.5.1 we conclude that $\sigma_{H:\psi_1(L)}(c') = \sigma_{H:\phi(L)}(c') = \sigma_{H:L}(c')$. It follows that, under the assumption that $r_L : H^1(M;\mathbb{R}) \to H^1(L;\mathbb{R})$ is onto, $\alpha_{H:L} : H^1(L;\mathbb{R}) \to \mathbb{R}$,

$$\alpha_{H:L}(r_L(c)) := \sigma_{H:L}(c), \quad c \in H^1(M;\mathbb{R})$$

is well-defined. Clearly $\alpha_{H:L}$ is Lipschitz. The second part of Corollary 5.2.5 follows from Theorem 5.2.2 if only we prove that

$$\partial\sigma_{H:L}(c) = i_L(\partial\alpha_{H:L}(r_L(c))) \quad \forall\, c \in H^1(M;\mathbb{R}).$$

This is essentially an exercise in non-smooth analysis. Denote by $S_L \subset H^1(L;\mathbb{R})$ the set of points $c \in H^1(L;\mathbb{R})$ at which $\alpha_{H:L}$ is *not* differentiable and define $S_M := r_L^{-1}(S_L) \subset H^1(M;\mathbb{R})$. By Rademacher's theorem S_L is a nullset in $H^1(L;\mathbb{R})$, so S_M is also a nullset in $H^1(M;\mathbb{R})$. By the usual chain rule $\sigma_{H:L} = \alpha_{H:L} \circ r_L$ is differentiable at every point $c \in H^1(M;\mathbb{R})\backslash S_M$ with differential

$$d\sigma_{H:L}(c) = d\alpha_{H:L}(r_L(c)) \circ r_L$$

Under the canonical identification $H_1(-;\mathbb{R}) \cong H^1(-;\mathbb{R})^*$, this identity reads

$$d\sigma_{H:L}(c) = i_L(d\alpha_{H:L}(r_L(c))) \quad \forall\, c \in H^1(M;\mathbb{R})\backslash S_M. \tag{5.32}$$

Fix $c \in H^1(M;\mathbb{R})$. Using (5.22) and the fact that i_L is both continuous and linear it follows from (5.32) that

$$\partial\sigma_{H:L}(c) \supset i_L(\partial\alpha_{H:L}(r_L(c))).$$

In order to see the other inclusion we recall that a classical result due to Carathéodory [81, Theorem 1.1.4] says that any element of the convex hull $\mathrm{conv}(A)$ of a set $A \subset \mathbb{R}^n$ is a convex combination of at most $n+1$ points from A. Thus, by (5.22) it suffices to take a sequence $(c_\varsigma)_{\varsigma\in\mathbb{N}} \subset H^1(M;\mathbb{R})\backslash S_M$ with $c_\varsigma \to c$, such that $h := \lim_{\varsigma\to\infty} d\sigma_{H:L}(c_\varsigma)$ exists and show that $h \in i_L(\partial\alpha_{H:L}(r_L(c)))$. To see this, consider the sequence $(d\alpha_{H:L}(r_L(c_\varsigma)))_{\varsigma\in\mathbb{N}}$. Since $\alpha_{H:L}$ is locally Lipschitz it follows that this sequence is bounded (see e.g. [30, Proposition 2.1.2]). Hence, after passing to a subsequence (still denoted by $(d\alpha_{H:L}(c_\varsigma))_{\varsigma\in\mathbb{N}}$) we may assume that

$$\tilde{h} := \lim_{\varsigma\to\infty} d\alpha_{H:L}(c_\varsigma) \in \partial\alpha_{H:L}(c)$$

exists. By continuity of i_L we conclude $h = i_L(\tilde{h})$ which finishes the proof. □

Proof of Corollary 5.2.6. Consider the path $\gamma : [0,1] \to H^1(L;\mathbb{R})$ given by $\gamma(t) = tc$. By Lebourg's mean value theorem (see Section 5.5.1) there exists $s \in (0,1)$ and $h \in \partial\alpha_{H:L}(\gamma(s))$ such that

$$\langle c, h\rangle = \alpha_{H:L}(c) - \alpha_{H:L}(0).$$

By a result due to Solomon [91, Lemma 6.6] there exists a symplectic isotopy $\tilde{\psi} : [0,1] \times M \to M$ such that $\tilde{\psi}_t(L) = \psi_t(L)$ and such that $r_L(\mathrm{Flux}(\tilde{\psi})) = c$. Using property b) of Theorem 5.5.1 we can therefore estimate

$$\langle c, h\rangle \geq \min_{x\in\tilde{\psi}_1(L)} H(x) - \max_{x\in L} H(x) = \min_{x\in\psi_1(L)} H(x) - \max_{x\in L} H(x).$$

The existence of $\mu \in \mathcal{M}(\phi_H)$ with $\rho(\mu) = i_L(h)$ now follows from Corollary 5.2.5. □

Proof of Lemma 5.3.4. Recall that we need to show $\alpha_{H_\epsilon,T(0,0)}(I,0) = 0$ for $I \in \mathbb{R}$, where

$$H_\epsilon(\theta, I) = H_0(I) + \epsilon F(\theta, I),$$

for

$$H_0(I) = I_1 I_2 \quad \& \quad F(\theta, I) = \varphi(I_1)\sin(2\pi\theta_1).$$

By Theorem 5.5.1 we have

$$\alpha_{H_\epsilon,T(0,0)}(I_1, I_2) = \alpha_{H_\epsilon,T(I_1,I_2)}(0,0) \quad \forall\, (I_1, I_2) \in \mathbb{R}^2.$$

Now fix $I \in \mathbb{R}$. We need to show that $\alpha_{H_\epsilon,T(I,0)}(0,0) = 0$. We will think of $\mathbb{T}^2 \times \mathbb{R}^2 = (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$ with coordinates (θ_1, I_1) on the first factor and (θ_2, I_2) on the second. Choose an embedded circle $S \subset (S^1 \times \mathbb{R})$ such that S is Hamiltonian isotopic to $\{I_1 = I\}$ and

$$S \cap \{I_1 \leq K+2\} = \{(\theta_1, I_1) \mid \theta_1 \in \{0, \tfrac{1}{2}\} \ \& \ |I_1| \leq K+2\}.$$

Then there exists $\phi \in \mathrm{Ham}(\mathbb{T}^2 \times \mathbb{R}^2, d\lambda)$ such that $\phi(T(I,0)) = S \times \{I_2 = 0\} \subset (S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$ and applying Theorem 5.5.1 gives

$$\alpha_{H_\epsilon,T(I,0)}(0,0) = \alpha_{H_\epsilon,S\times\{I_2=0\}}(0,0).$$

Now consider the action functionals $\mathcal{A}_{H_\epsilon:S\times\{I_2=0\}}$ associated to H_ϵ (see Section 5.5). By the choice of S, Remark 5.3.1 implies, that the *spectrum*[12]

$$\mathrm{Spec}(\mathcal{A}_{H_\epsilon:S\times\{I_2=0\}}) \subset \mathbb{R}$$

is independent of $\epsilon \geq 0$. Since $\mathrm{Spec}(\mathcal{A}_{H_\epsilon:S\times\{I_2=0\}})$ is nowhere dense and

$$\epsilon \mapsto l_{S\times\{I_2=0\}}(\phi^1_{H\epsilon}) \in \mathbb{R}$$

is continuous, it follows that

$$l_{S\times\{I_2=0\}}(\phi^1_{H_\epsilon}) = l_{S\times\{I_2=0\}}(\phi^1_{H_0}) \quad \forall\ \epsilon \geq 0$$

(and similarly $l_{S\times\{I_2=0\}}(\phi^k_{H_\epsilon}) = l_{S\times\{I_2=0\}}(\phi^k_{H_0})$ for all $k \in \mathbb{N}$). This implies that

$$\alpha_{H_\epsilon,S\times\{I_2=0\}}(0,0) = \alpha_{H_0,S\times\{I_2=0\}}(0,0).$$

Applying Theorem 5.5.1 one last time we conclude

$$\alpha_{H_0,S\times\{I_2=0\}}(0,0) = \alpha_{H_0,T(I,0)}(0,0) = H_0|_{T(I,0)} = 0,$$

which finishes the proof. □

The next several pages take up the proof of Theorem 5.2.2. First let's consider the setup: Set $m = \beta_1(M)$ and fix once and for all a basis $e_1, \ldots, e_m$ for $H^1(M;\mathbb{R})$. Choose closed one forms $\eta_1, \ldots, \eta_m$ on M such that $e_l = [\eta_l]$ and define vector fields $X_{\eta_1}, \ldots X_{\eta_m}$ by requiring

$$\iota_{X_{\eta_l}}\omega = \eta_l \quad \forall\ l = 1, \ldots, m.$$

Denote the symplectic isotopy generated by X_{η_l} by $\psi^l = \{\psi^l_t\}_{t\in\mathbb{R}}$. For each $k \in \mathbb{N}$ we define a function $a_k : H^1(M;\mathbb{R}) \to \mathbb{R}$ by

$$a_k(c) = \frac{l_L(\psi^m_{-\kappa_m} \cdots \psi^1_{-\kappa_1} \tilde{\phi}^k_H \psi^1_{\kappa_1} \cdots \psi^m_{\kappa_m})}{k}$$

if $c = \sum_{l=1}^m \kappa_l e_l$. The following lemma is due to Vichery [99]. In [99] the case of the zero-section in a cotangent bundle is considered. But in the general case the same proof applies.

Lemma 5.6.4 ([99]). *The sequence $(a_k)_{k\in\mathbb{N}}$ converges uniformly to $\sigma_{H:L}$ on compact subsets.*

[12]One should think of $\mathrm{Spec}(\mathcal{A}_{H_\epsilon:S\times\{I_2=0\}})$ as the set of critical values for $\mathcal{A}_{H_\epsilon:S\times\{I_2=0\}}$. For a precise definition we refer to [58].

Proposition 5.6.5. *Let $h \in H_1(M;\mathbb{R})$ be a Dini subdifferential of a_k at the point $c = \sum_{l=1}^m \kappa_l e_l \in H^1(M;\mathbb{R})$.*[13] *We associate to any $c' = \sum_{l=1}^m \kappa'_l e_l \in H^1(M;\mathbb{R})$ the symplectic isotopy*

$$\psi_\tau := \psi^1_{\kappa_1+\tau\kappa'_1} \cdots \psi^m_{\kappa_m+\tau\kappa'_m} \tag{5.33}$$

and denote by $X_\tau(x) := \frac{d}{ds}\Big|_{s=0} \psi_{s+\tau}(x)$ its infinitesimal generator. Then there exists a point $x \in \psi_0(L)$ such that $\phi^k_H(x) \in \psi_0(L)$ and

$$\langle c', h\rangle \leq \frac{1}{k}\int_0^k \langle dH, X_0\rangle \phi^t_H(x)\ dt. \tag{5.34}$$

Moreover, the curve $\gamma : ([0,k],\{0,k\}) \to (M,L)$ given by $\gamma(t) = \phi^t_H(x)$ represents the trivial element in $\pi_1(M,\psi_0(L))$ and there exists a capping $\widehat{\gamma}$ of γ such that

$$a_k(c) = \frac{1}{k}\int_0^k H(\gamma(t))\ dt - \frac{1}{k}\int \widehat{\gamma}^*\omega. \tag{5.35}$$

Proof. Let us first prove the result under the non-degeneracy assumption

$$\phi^k_{H\psi_0}(L) \pitchfork L. \tag{5.36}$$

The Hamiltonian kH generates the isotopy $t \mapsto \phi^{tk}_H$. Choose a generic path $J = \{J_t\}_{t\in[0,1]}$ of ω-compatible almost complex structures so that we have a well-defined Floer chain complex $(CF(L{:}kH\psi_0, J), d)$. Choose a function $\chi \in C^\infty(\mathbb{R};[0,1])$ such that

$$\chi(s) = \begin{cases} 0, & \text{if } s \leq 0 \\ 1, & \text{if } s \geq 1 \end{cases}$$

and $\chi' \geq 0$. Now fix $\tau > 0$ so small that $\phi^k_{H\psi_\tau}(L) \pitchfork L$ and consider the s-dependent Hamiltonian[14]

$$F^\tau_s(x) = kH\psi_0(x) + \chi(s)(kH\psi_\tau(x) - kH\psi_0(x)). \tag{5.37}$$

There exists a regular homotopy of Floer data $(K_{\tau,s}, J^{\tau,s})_{s\in\mathbb{R}}$ which is stationary for $s \notin (0,1)$ such that

$$\begin{aligned} |F^\tau - K^\tau|_{C^\infty(M\times[0,1]\times[0,1])} &\leq \tau^2 \\ |J - J^{\tau,s}|_{C^\infty(M\times[0,1])} &\leq \tau^2 \quad \forall\ s \in (0,1), \end{aligned} \tag{5.38}$$

The Floer continuation map

$$\Phi : CF_*(L{:}kH\psi_0, J) \to CF_*(L{:}kH\psi_\tau, J^{\tau,1})$$

is defined "by counting" finite-energy solutions $u \in C^\infty(\mathbb{R}\times[0,1];M)$ to the problem

$$\begin{cases} \partial_s u + J^{\tau,s}_t(u)(\partial_t u - X_{K_s}(u)) = 0 \\ u(\mathbb{R}\times\{0,1\}) \subset L, \end{cases} \tag{5.39}$$

where the *energy of* u denotes the non-negative quantity

$$E_{J^\tau}(u) = \int_{-\infty}^{\infty}\int_0^1 \omega(\partial_s u, J^\tau(u)\partial_s u)\ dtds.$$

[13] See (5.23).

[14] Here and in the rest of the proof one should *not* think of τ as a continuous variable, but as a fixed parameter.

By a very general result due to Usher [96, Theorem 1.4], there exists a cycle

$$\mathfrak{c} \in CF_*(L{:}kH\psi_0, J),$$

all of whose elements have action $\leq l_L(\psi_0^{-1}\tilde{\phi}_H^k\psi_0)$ and which satisfies $[\mathfrak{c}] = \mathrm{PSS}([L]) \in HF_*(L{:}kH\psi_0, J)$.[15] Since at least one of the Hamiltonian chords of the cycle $\Phi(\mathfrak{c})$ must have action greater than or equal to $l_L(\psi_\tau^{-1}\tilde{\phi}_H^k\psi_\tau)$, we deduce the existence of a finite-energy solution $u_s^\tau(t) = u^\tau(s,t)$ to (5.39) such that

$$\mathcal{A}_{kH\psi_0:L}(u_{-\infty}^\tau) \leq l_L(\psi_0^{-1}\tilde{\phi}_H^k\psi_0) \quad \text{and} \quad l_L(\psi_\tau^{-1}\tilde{\phi}_H^k\psi_\tau) \leq \mathcal{A}_{kH\psi_\tau:L}(u_\infty^\tau). \tag{5.40}$$

Here $u_{-\infty}^\tau$ (respectively u_∞^τ) denotes one of the elements of the chain $\mathfrak{c}$ (respectively $\Phi(\mathfrak{c})$). Integrating by parts reveals that u^τ satisfies the energy identity

$$0 \leq E_{J^\tau}(u^\tau) = \mathcal{A}_{kH\psi_0:L}(u_{-\infty}^\tau) - \mathcal{A}_{kH\psi_\tau:L}(u_\infty^\tau) + \int_{-\infty}^{\infty}\int_0^1 \partial_s K_s(u^\tau(s,t))dtds. \tag{5.41}$$

Using (5.38) and (5.40), this identity implies the estimate

$$\begin{aligned}
\frac{a_k(c+\tau c') - a_k(c)}{\tau} &= \frac{l_L(\psi_\tau^{-1}\tilde{\phi}_H^k\psi_\tau) - l_L(\psi_0^{-1}\tilde{\phi}_H^k\psi_0)}{k\tau} \\
&\leq \frac{\mathcal{A}_{kH\psi_\tau:L}(u_\infty^\tau) - \mathcal{A}_{kH\psi_0:L}(u_{-\infty}^\tau)}{k\tau} \\
&\leq \frac{1}{k\tau}\int_{-\infty}^{\infty}\int_0^1 \partial_s K_s(u^\tau(s,t))dtds \\
&\leq \frac{1}{k\tau}\int_{-\infty}^{\infty}\int_0^1 \partial_s F_s^\tau(u^\tau(s,t))dtds + \frac{\tau}{k} \\
&= \int_{-\infty}^{\infty}\int_0^1 \chi'(s)\left(\frac{H\psi_\tau(u^\tau(s,t)) - H\psi_0(u^\tau(s,t))}{\tau}\right)dtds + \frac{\tau}{k}.
\end{aligned} \tag{5.42}$$

Moreover, we can use the energy identity together with (5.40) to estimate the energy of u^τ from above:

$$\begin{aligned}
E_{J^\tau}(u^\tau) &\leq l_L(\psi_0^{-1}\tilde{\phi}_H^k\psi_0) - l_L(\psi_\tau^{-1}\tilde{\phi}_H^k\psi_\tau) + k\max_M |H\psi_\tau - H\psi_0| + \tau^2 \\
&\leq 2k\max_M |H\psi_\tau - H\psi_0| + \tau^2.
\end{aligned} \tag{5.43}$$

The last inequality makes use of the Lipschitz property of Lagrangian spectral invariants (see [58]). Choose now a sequence $(\tau_\varsigma)_{\varsigma\in\mathbb{N}} \subset (0,1)$ such that $\tau_\varsigma \downarrow 0$ and

$$\frac{a_k(c+\tau_\varsigma c') - a_k(c)}{\tau_\varsigma} \overset{\varsigma\to\infty}{\longrightarrow} \liminf_{\tau\downarrow 0}\frac{a_k(c+\tau c') - a_k(c)}{\tau} \tag{5.44}$$

For each τ_ς we find a Floer trajectory $u^\varsigma := u^{\tau_\varsigma}$ meeting the requirement (5.40) with $\tau = \tau_\varsigma$. Applying Gromov convergence to the sequence $(u^\varsigma)_{\varsigma\in\mathbb{N}}$ we obtain a subsequence (still denoted by $(u^\varsigma)_{\varsigma\in\mathbb{N}}$) which C_{loc}^∞-converges. This follows from a result due to Hofer [47, Proof of Proposition 2] saying that, in our setup, the obstruction to C_{loc}^∞-precompactness is "bubbling off" of a non-constant holomorphic disc or sphere. However, this cannot occur to $(u^\varsigma)_{\varsigma\in\mathbb{N}}$ since the estimate (5.43) implies

$$E_{J^{\tau_\varsigma}}(u^\varsigma) \overset{\varsigma\to\infty}{\longrightarrow} 0.$$

[15]See Section 5.5 for the notation used here.

By Fatou's lemma, this also implies that the pointwise limit u of $(u^\varsigma)_{\varsigma\in\mathbb{N}}$ is an s-independent map given by $u(s,t) = u(t) = \phi^t_{kH\psi_0}(y)$ for some $y \in L$. Now, using $\chi' \geq 0$ and the mean value theorem, we find for every $(s,t) \in \mathbb{R}\times[0,1]$ a $\tau'_\varsigma(s,t) \in (0,\tau_\varsigma)$ such that

$$\begin{aligned}\left|\chi'(s)\left(\frac{H\psi_{\tau_\varsigma}(u^\varsigma(s,t)) - H\psi_0(u^\varsigma(s,t))}{\tau_\varsigma}\right)\right| &= \chi'(s)|\langle dH, X_{\tau'_\varsigma(s,t)}\rangle|(\psi_{\tau'_\varsigma(s,t)}(u^\varsigma(s,t)))\\ &\leq \chi'(s)\max_{(\tau,z)\in[0,1]\times M}|\langle dH, X_\tau\rangle|(z)\end{aligned}$$

Since h is a Dini subdifferential of a_k at c and

$$\int_{-\infty}^{\infty}\int_0^1 \chi'(s)\max_{(\tau,z)\in[0,1]\times M}|\langle dH, X_\tau\rangle|(z)\;dtds = \max_{(\tau,z)\in[0,1]\times M}|\langle dH, X_\tau\rangle|(z) < \infty,$$

we can apply Lebesgue dominated convergence in the estimates (5.42) to conclude that

$$\begin{aligned}\langle c', h\rangle &\leq \lim_{\varsigma\to\infty}\int_{-\infty}^{\infty}\int_0^1 \chi'(s)\left(\frac{H\psi_{\tau_\varsigma}(u^\varsigma(s,t)) - H\psi_0(u^\varsigma(s,t))}{\tau_\varsigma}\right)dtds\\ &= \int_{-\infty}^{\infty}\int_0^1 \chi'(s)\lim_{\varsigma\to\infty}\left(\frac{H\psi_{\tau_\varsigma}(u^\varsigma(s,t)) - H\psi_0(u^\varsigma(s,t))}{\tau_\varsigma}\right)dtds\\ &= \int_{-\infty}^{\infty}\int_0^1 \chi'(s)\langle dH, X_0\rangle\psi_0(u(t))dtds = \int_0^1\langle dH, X_0\rangle\phi^{kt}_H\psi_0(y)dt.\end{aligned}$$

By changing variables to $s = kt$ one now concludes that (5.34) holds for $x := \psi_0(y) \in \psi_0(L)$. In order to find a capping $\widehat{\gamma}$ for $\gamma(t) = \phi^t_H(x),\ t\in[0,k]$ such that (5.35) holds, we first note that

$$\mathcal{A}_{kH\psi_0:L}(u^\varsigma_{-\infty}), \mathcal{A}_{kH\psi_{\tau_\varsigma}:L}(u^\varsigma_\infty) \overset{\varsigma\to\infty}{\longrightarrow} l_L(\psi_0^{-1}\tilde{\phi}^k_H\psi_0), \tag{5.45}$$

which is easily seen using the estimates (5.40) and (5.41). By C^∞_{loc}-convergence we may choose ς so large that the curve $[0,1]\ni t\mapsto u^\varsigma_s(t) = u^\varsigma(s,t)$ is C^0-close to u for every $s\in[-2,2]$. In particular we can choose ς so large that the areas of the two cappings of u, obtained by adjusting the cappings of u^ς_{-2} and u^ς_2 given by u^ς slightly, differ by less than τ_L. Hence, by monotonicity of L they have the same area and thus define a class of cappings $\widetilde{u}$ of u. Given $\epsilon > 0$ we can in addition achieve

$$|\mathcal{A}_{kH\psi_0}(u^\varsigma_{-2}) - \mathcal{A}_{kH\psi_0}(\widetilde{u})| < \epsilon \quad\text{and}\quad |\mathcal{A}_{kH\psi_{\tau_\varsigma}}(u^\varsigma_2) - \mathcal{A}_{kH\psi_{\tau_\varsigma}}(\widetilde{u})| < \epsilon.$$

Since $\mathcal{A}_{kH\psi_{\tau_\varsigma}}(\widetilde{u}) \to \mathcal{A}_{kH\psi_0}(\widetilde{u})$ for $\varsigma\to\infty$ these estimates together with

$$\begin{aligned}&0 \leq \mathcal{A}_{kH\psi_{\tau_\varsigma}:L}(u^\varsigma_2) - \mathcal{A}_{kH\psi_{\tau_\varsigma}:L}(u^\varsigma_\infty) \leq E_{J^{\tau_\varsigma}}(u^\varsigma) \overset{\varsigma\to\infty}{\longrightarrow} 0\\ &0 \leq \mathcal{A}_{kH\psi_{\tau_0}:L}(u^\varsigma_{-\infty}) - \mathcal{A}_{kH\psi_0:L}(u^\varsigma_{-2}) \leq E_{J^{\tau_\varsigma}}(u^\varsigma) \overset{\varsigma\to\infty}{\longrightarrow} 0\end{aligned}$$

and (5.45) imply that

$$\mathcal{A}_{kH\psi_0}(\widetilde{u}) = l_L(\psi_0^{-1}\tilde{\phi}^k_H\psi_0).$$

Hence, the cappings of $\widetilde{u}$ give rise to a capping $\widehat{\gamma}$ of γ such that (5.35) holds.

Let's now discuss how to deal with the case when assumption (5.36) is violated. Then we proceed as follows: Again we choose a sequence $(\tau_\varsigma)_{\varsigma\in\mathbb{N}}$ such that $\tau_\varsigma\downarrow 0$ and (5.44) is satisfied as well as a path $J = \{J_\tau\}_{\tau\in[0,1]}$ of ω-compatible almost complex structures. For each ς we choose $H^\varsigma \in C^\infty([0,1]\times M)$ satisfying

$$|H^\varsigma - kH|_{C^\infty([0,1]\times M)} \leq \tau_\varsigma^2 \tag{5.46}$$

as well as

$$\phi^1_{H^\varsigma\psi_0}(L) \pitchfork L \quad\text{and}\quad \phi^1_{H^\varsigma\psi_{\tau_\varsigma}}(L) \pitchfork L.$$

Now choose paths of compatible almost complex structures $J^\varsigma = \{J_t^\varsigma\}_{\tau\in[0,1]}$ which are regular in the sense that we have well-defined Floer chain complexes

$$(CF(L{:}H^\varsigma\psi_0, J^\varsigma), d), \quad (CF(L{:}H^\varsigma\psi_{\tau_\varsigma}, J^\varsigma), d)$$

and such that J^ς satisfies

$$|J^\varsigma - J|_{C^\infty([0,1]\times M)} \le \tau_\varsigma^2.$$

For each ς we obtain by the above procedure a Floer trajectory $u^\varsigma := u^{\tau_\varsigma}$ such that (5.40) is satisfied with $H = H^\varsigma$ and $\tau = \tau_\varsigma$. In particular (5.43) follows with $H = H^\varsigma$ and $\tau = \tau_\varsigma$. Using (5.42) and (5.46) as well as the Lipschitz property of Lagrangian spectral invariants [58], we can estimate

$$\begin{aligned}
\frac{a_k(c+\tau_\varsigma c') - a_k(c)}{\tau_\varsigma} &= \frac{l_L(\psi_\tau^{-1}\tilde\phi_H^k\psi_\tau) - l_L(\psi_0^{-1}\tilde\phi_H^k\psi_0)}{k\tau} \\
&\le \frac{l_L(\psi_\tau^{-1}\tilde\phi_{H^\varsigma}^1\psi_\tau) - l_L(\psi_0^{-1}\tilde\phi_{H^\varsigma}^1\psi_0)}{k\tau_\varsigma} + \frac{2\tau_\varsigma}{k} \\
&\le \frac{1}{k}\int_{-\infty}^{\infty}\int_0^1 \chi'(s)\left(\frac{H_t^\varsigma\psi_{\tau_\varsigma}(u^\varsigma(s,t)) - H_t^\varsigma\psi_0(u^\varsigma(s,t))}{\tau_\varsigma}\right) dtds + \frac{3\tau_\varsigma}{k} \\
&\le \int_{-\infty}^{\infty}\int_0^k \chi'(s)\left(\frac{H\psi_{\tau_\varsigma}(u^\varsigma(s,t)) - H\psi_0(u^\varsigma(s,t))}{\tau_\varsigma}\right) dtds + \frac{5\tau_\varsigma}{k}.
\end{aligned} \tag{5.47}$$

Again, by applying Hofer's compactness argument [47, Proof of Proposition 2], we may assume that $(u_\varsigma)_{\varsigma\in\mathbb{N}}$ C^∞_{loc}-converges. Exactly as above the pointwise limit u is a s-constant map given by $u(t) = \phi^{kt}_{H\psi_0}(y)$ for some $y \in L$. Via dominated convergence the estimate (5.47) now gives

$$\langle c', h\rangle \le \frac{1}{k}\int_0^k \langle dH, X_0\rangle\phi_H^t(x)\ dt$$

for $x = \psi_0(y) \in \psi_0(L)$ and the existence of a capping is obtained exactly as before. This finishes the proof. ☐

Remark 5.6.6. Suppose now that $\omega = d\lambda$ is exact with $\lambda|_L \equiv 0$. In this case we need to relate the action appearing in (5.35) to the one appearing in (5.3). Choose a closed one form η in the class $c \in H^1(M;\mathbb{R})$. An easy computation shows that $[\lambda|_{\psi_0(L)}] = [\eta|_{\psi_0(L)}] \in H^1(\psi_0(L);\mathbb{R})$, so in fact η can be chosen to satisfy $(\lambda-\eta)|_{\psi_0(L)} \equiv 0$. In particular we then have

$$\int \widehat\gamma^*\omega = \int d\widehat\gamma^*(\lambda-\eta) = \int_0^k \langle\lambda, X_H\rangle\gamma(t)dt - \int_0^k\langle\eta, X_H\rangle\gamma(t)dt,$$

so (5.35) reads

$$a_k(c) = \mathcal{A}_{H,\lambda}(\mu_\gamma) + \int\langle\eta, X_H\rangle d\mu_\gamma, \tag{5.48}$$

where μ_γ is the Borel probability measure supported on $\gamma : [0,k] \to M$.

Though we are mainly interested in the Clarke subdifferentials of $\alpha_{H:L}$, the proof of Theorem 5.2.2 goes via the so-called *approximate subdifferential* due to Ioffe [51].

Proposition 5.6.7. *Let $h \in \partial_A a_k(c) \subset H_1(M;\mathbb{R})$ be an approximate subdifferential of a_k at $c = \sum_{l=1}^m \kappa_l e_l \in H^1(M;\mathbb{R})$.*[16] *Given a fixed $c' = \sum_{l=1}^m \kappa_l' e_l \in H^1(M;\mathbb{R})$ we consider the symplectic isotopy (5.33) and denote its infinitesimal generator by X_τ. Then there exists a point $x \in L$ such that the curve $[0,k] \ni t \mapsto \gamma(t) = \phi^t_{H\psi_0}(x)$ satisfies (5.34) and (5.35).*

[16] See (5.24).

Proof. By (5.24) there exists a sequence $(c_\varsigma)_{\varsigma\in\mathbb{N}} \subset H^1(M;\mathbb{R})$ such that h is a Dini subdifferential of a_k at c_ς and

$$c_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} c \tag{5.49}$$

Writing $c_\varsigma = \sum_{l=1}^m \xi_l^\varsigma e_l$ we define the symplectic isotopy

$$\upsilon_\tau^\varsigma := \psi^1_{\xi_1^\varsigma+\tau\kappa_1'} \cdots \psi^m_{\xi_m^\varsigma+\tau\kappa_m'}$$

and denote by X_τ^ς the infinitesimal generator of $\tau \mapsto \upsilon_\tau^\varsigma$. By Proposition 5.6.5 there exists a sequence of points $(x_\varsigma)_{\varsigma\in\mathbb{N}}$ with $x_\varsigma \in \upsilon_0^\varsigma(L)$ such that the curves $[0,k] \ni t \mapsto \gamma_\varsigma(t) := \phi_H^t(x_\varsigma)$ have cappings $\widehat{\gamma}_\varsigma$ for which

$$\begin{aligned}\langle c', h\rangle &\leq \frac{1}{k}\int_0^k \langle dH, X_0^\varsigma\rangle(\gamma_\varsigma(t))dt \\ a_k(c_\varsigma) &= \frac{1}{k}\int_0^k H(\gamma_\varsigma(t))dt - \frac{1}{k}\int \widehat{\gamma}_\varsigma^*\omega.\end{aligned}$$

By compactness of M, we may assume that $x_\varsigma \to x$ for some $x \in M$, after passing to a subsequence. The claim now follows by passing to the limit, using the fact that a_k is locally Lipschitz with respect to any norm on $H^1(M;\mathbb{R})$. □

Proof of Theorem 5.2.2. Consider the set

$$K := \{\rho(\mu) \in H_1(M;\mathbb{R}) \mid \mu \in \mathcal{M}(\phi_H)\}$$

of rotation vectors realized by $\mathcal{M}(\phi_H)$-measures. Clearly $K \subset H_1(M;\mathbb{R})$ is non-empty, convex and compact. Hence, K is characterized by its *support function* [81, Section 1.7.1] $\chi_K : H^1(M;\mathbb{R}) \to \mathbb{R}$, defined by

$$\chi_K(c') := \max_{h\in K}\langle c', h\rangle, \quad c' \in H^1(M;\mathbb{R}).$$

The characterization is that $h \in K$ if and only if

$$\langle c', h\rangle \leq \chi_K(c') \quad \forall\ c' \in H^1(M;\mathbb{R}). \tag{5.50}$$

We need to show that $\partial\sigma_{H:L}(c) \subset K$ for all $c \in H^1(M;\mathbb{R})$. In fact, by property c) of Theorem 5.5.1, it suffices to show that $\partial\sigma_{H:L}(0) \subset K$. Moreover, since $\sigma_{H:L}$ is locally Lipschitz and K is convex and closed, it suffices (by (5.25)) to show that

$$\partial_A\sigma_{H:L}(0) \subset K. \tag{5.51}$$

Fix some $h \in \partial_A\sigma_{H:L}(0)$. By the above discussion we need to show that (5.50) holds, so choose some $c' = \sum_{l=1}^m \kappa_l' e_l \in H^1(M;\mathbb{R})$. By Lemma 5.6.4 $a_k \to \sigma_{H:L}$ uniformly on compact subsets. Hence, applying (5.26) we can find an increasing sequence $(k_\varsigma)_{\varsigma\in\mathbb{N}} \subset \mathbb{N}$ as well as sequences $(c_\varsigma)_{\varsigma\in\mathbb{N}} \subset H^1(M;\mathbb{R})$ and $(h_\varsigma)_{\varsigma\in\mathbb{N}} \subset H_1(M;\mathbb{R})$ such that $h_\varsigma \in \partial_A a_{k_\varsigma}(c_\varsigma)$ and

$$c_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} 0 \quad \text{and} \quad h_\varsigma \overset{\varsigma\to\infty}{\longrightarrow} h.$$

Writing $c_\varsigma = \sum_{l=1}^m \kappa_l^\varsigma e_l$ we define a symplectomorphism ψ_ς by

$$\psi_\varsigma := \psi^1_{\kappa_1^\varsigma} \cdots \psi^m_{\kappa_m^\varsigma}$$

and denote by X_τ^ς the infinitesimal generator of the symplectic isotopy

$$\tau \mapsto \psi^1_{\kappa_1^\varsigma+\tau\kappa_1'} \cdots \psi^m_{\kappa_m^\varsigma+\tau\kappa_m'}$$

Applying Proposition 5.6.7 we obtain for each ς the existence of $x_\varsigma \in \psi_\varsigma(L)$ such that

$$\langle c', h_\varsigma \rangle \leq \frac{1}{k_\varsigma} \int_0^{k_\varsigma} \langle dH, X_0^\varsigma \rangle \phi_H^t(x_\varsigma)\ dt$$

Denote by $\mu_\varsigma \in \mathcal{M}$ the unique probability measure characterized by

$$\int f\ d\mu_\varsigma = \frac{1}{k_\varsigma} \int_0^{k_\varsigma} f \phi_H^t(x_\varsigma)\ dt \quad \forall\ f \in C^0(M),$$

so that

$$\langle c', h_\varsigma \rangle \leq \int \langle dH, X_0^\varsigma \rangle\ d\mu_\varsigma. \tag{5.52}$$

By weak*-compactness of $\mathcal{M}$ we may after passing to a subsequence (still denoted by $(\mu_\varsigma)_{\varsigma \in \mathbb{N}}$) assume that

$$\mu_\varsigma \overset{w^*}{\rightharpoonup} \mu \in \mathcal{M}.$$

By the classical Krylloff-Bogoliouboff argument [55] μ is ϕ_H-invariant (see also [92, Proposition 3.1.1]). Passing to the limit in (5.52) we conclude that

$$\langle c', h \rangle \leq \int \langle dH, Y \rangle\ d\mu, \tag{5.53}$$

where

$$Y = \sum_{l=1}^m \kappa_l' X_{\eta_l}.$$

In particular (using $\langle dH, X_{\eta_l} \rangle = \langle \eta_l, X_H \rangle$) we obtain

$$\langle c', h \rangle \leq \sum_{l=1}^m \kappa_l' \int \langle \eta_l, X_H \rangle d\mu = \sum_{l=1}^m \kappa_l' \langle e_l, \rho(\mu) \rangle = \langle c', \rho(\mu) \rangle \leq \chi_K(c').$$

Since $c' \in H^1(M;\mathbb{R})$ was arbitrary we conclude that (5.50) holds, so $h \in K$ and the proof is done. □

We will now discuss the adaptions required to prove the results in the non-compact setting from Section 5.2.1.

Sketch of proof of Theorem 5.2.7 and Corollary 5.2.8. Recall that in the non-compact setting we assume that $\omega = d\lambda$ is exact, with a distinguished primitive λ satisfying $\lambda|_L \equiv 0$. One then defines each a_k exactly as above, except that $\eta_1, \dots, \eta_m$ are compactly supported closed 1-forms which represent a basis for $H^1_{dR,c}(M;\mathbb{R})$. Now Proposition 5.6.5 and 5.6.7 hold exactly as in the compact setting. Fix some $c = \sum_{l=1}^m \kappa_l [\eta_l] \in H^1_{dR,c}(M;\mathbb{R})$ and denote by $C \subset M$ the closure of the smallest ϕ_H-invariant set containing the subset

$$\bigcup_{(s_1,\dots,s_m) \in [0,1]^m} \psi^1_{\kappa_1+s_1} \cdots \psi^m_{\kappa_m+s_m}(L).$$

Since $H \in \mathcal{H}$, C is a compact subset of M. Now define

$$K := \{\rho(\mu) \in H_1^{BM}(M;\mathbb{R}) \mid \mu \in \mathcal{M}(C;\phi_H) \text{ with } \mathcal{A}_{H,\lambda}(\mu) + \langle c, \rho(\mu) \rangle = \sigma_{H:L}(c)\}.$$

K is again a convex compact subset of $H_1^{BM}(M;\mathbb{R})$, so it is characterized by its support function $\chi_K : H^1_c(M;\mathbb{R}) \to \mathbb{R}$. Given $h \in \partial_A \sigma_{H:L}(c)$ and $c' \in H^1_c(M;\mathbb{R})$, an argument similar to that of the proof of Theorem 5.2.2 shows the existence of a $\mu \in \mathcal{M}(C;\phi_H)$ satisfying

$$\langle c', h \rangle \leq \langle c', \rho(\mu) \rangle. \tag{5.54}$$

Moreover, applying the statement of Proposition 5.6.7[17] together with the weak*-convergence it follows that

$$\sigma_{H:L}(c) = \mathcal{A}_{H,\lambda}(\mu) + \langle c, \rho(\mu) \rangle,$$

which implies that $\rho(\mu) \in K$ and (5.54) now implies $\langle c', h \rangle \leq \chi_K(c')$. Since $c' \in H^1_c(M;\mathbb{R})$ was arbitrary this shows that $h \in K$, finishing the proof of Theorem 5.2.7. The proof of Corollary 5.2.8 is identical to that of Corollary 5.2.5. $\square$

Remark 5.6.8. Suppose each a_k is C^1, $\alpha_{H:L}$ is C^1 and that $(a_k)_{k\in\mathbb{N}}$ C^1-converges to $\alpha_{H:L}$. In this setting one can avoid applying Jourani's result and "upgrade" the proof of Theorem 5.2.2 to show the following: For every $c \in H^1(L;\mathbb{R})$ there exists a measure $\mu \in \mathcal{M}(\phi_H)$ with rotation vector $\rho(\mu) = d\alpha_{H:L}(c)$, whose support satisfies

$$\mathrm{Supp}(\mu) \subset \overline{\bigcup_{t\in\mathbb{R}} \phi^t_H(L)}.$$

In particular, if L is ϕ_H-invariant then $\mathrm{Supp}(\mu) \subset L$.

[17] More precisely we make use of the fact that (5.48) holds.

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