A discourse on the Pivot rules random edge and random facet

A Discourse on the Pivot Rules Random Edge and Random Facet [1]

Technical Report

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Chapter 1

Introduction

This technical report is a summary of most of the topics I was working on during my first year at the ETH. Hopefully, I hope to expand on some of them during my dissertation. It is, therefore, to be regarded as work in progress.

The largest section is about finding new nontrivial bounds for Random Edge. It opens, however, with a few facts we observed when looking at orientations on the hypercube and their impact on the behaviour of randomized pivot rules.

The final section is dedicated to some ideas connected to Balinski's Theorem and other path properties on oriented polytopes.

Chapter 2

Randomized pivot rules on the hypercube

2.1 The Matoušek class

The subexponential bounds established for randomized pivot rules are valid for more general settings like the so-called LP-type problems – not just for LP as such. Most interestingly, Matoušek could construct a class of problems [4] whose LP instances, as shown by Gärtner [1], can be solved by Random Facet in polynomial time while some of its non-LP instances are proof that the subexponential bounds are tight.

We studied how Random Edge is doing on the Matoušek examples, and claim now the following:

In dimension d

the $\left\{ \begin{array}{c} 2k^{th} \\ 2k+1^{th} \end{array} \right\}$ row is of the form $\left\{ \begin{array}{c} 0\,1\,0\,1\,0\,1 \ldots 0\,1\,1\,0 \ldots 0 \\ 1\,1\,1\,1\,1\,1 \ldots 1\,1\,1\,0 \ldots 0 \end{array} \right\}$,

i.e. the matrix is of the form

$$\begin{pmatrix} 1 & & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 0 & 1 & 1 & & & & & & & \\ 1 & 1 & 1 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & & & & \\ & \vdots & & & & & \ddots & & & \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{if } d = 2n,$$

$$\begin{pmatrix} 1 &&&&&&&& \\ 1 & 1 &&&&&&& \\ 0 & 1 & 1 &&&&&& \\ 1 & 1 & 1 & 1 &&&&& \\ 0 & 1 & 0 & 1 & 1 &&&& \\ 1 & 1 & 1 & 1 & 1 & 1 &&& \\ & \vdots &&&&& \ddots && \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \quad \text{sonst.}$$

Taking the average expected value over all the vertices the following matrix seems to be the worst case:

$$\begin{pmatrix} 1 &&&&&&&&& \\ 1 & 1 &&&&&&&& \\ 0 & 1 & 1 &&&&&&& \\ 1 & 0 & 1 & 1 &&&&&& \\ 1 & 1 & 0 & 1 & 1 &&&&& \\ 1 & 1 & 1 & 0 & 1 & 1 &&&& \\ & \vdots &&&&& \ddots &&& \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

All these worst-case-examples are non-realisable.

Using Random Facet the picture is not quite so clear. The main problem being, that computing the exact expected number of pivot steps becomes to time-consuming already in dimension $d = 6$. Let us therefore just give the worst cases in dimension 3,4 and 5:

dim	key	WCaverage	WCmax	WCsourse
3	7	2	3.66667	3.66667
	8	2.25	3.66667	3
4	60	3.5	5.66667	4.5
	62	3.5	5.33333	5.16667
	63	3.25	5.66667	5.33333
	64	3.5	5.5	4
5	951	4.75	8.13333	7.33333
	956	5.1	8.13333	6.5
	958	5.11667	7.6	7.2
	984	5.10833	7.86667	6
	991	4.75	7.9	7.9
	992	5.14167	7.9	5.56667
	1019	4.75	7.9	7.9
	1020	5.11667	8.13333	6.4
	1023	4.75	8.13333	7

We notice some pattern here (see the following table), but rather not dare to claim to be able to recognise the decisive one.

dim	*WCaverage*	*WCmax*	*WCsourse*
3	$8\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$	$7\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$	$7\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$
4	$60\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$	$60\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$	$47\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&0&1&1\end{bmatrix}$
	$62\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\1&1&1&1\end{bmatrix}$	$63\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$	$59\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\0&1&1&1\end{bmatrix}$
	$64\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$		
5	$958\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\0&1&1&0&0\\1&1&1&1&0\\1&0&1&1&1\end{bmatrix}$	$951\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\0&0&1&1&1\end{bmatrix}$	$991\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\1&0&1&1&0\\1&1&1&1&1\end{bmatrix}$
	$984\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\1&0&1&1&0\\0&1&1&1&1\end{bmatrix}$	$956\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\1&0&1&1&1\end{bmatrix}$	$1019\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\1&1&1&1&1\end{bmatrix}$
	$992\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\1&0&1&1&0\\1&1&1&1&1\end{bmatrix}$	$1020\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\1&1&1&1&1\end{bmatrix}$	
	$1020\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\1&1&1&1&1\end{bmatrix}$	$1023\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}$	

2.2 Sparse Matrices

As briefly mentioned above, Gärtner managed to show, that RANDOM FACET takes an expected $O(d^2)$ number of pivot steps on a subclass of Matoušek's AOF, characterized by the sparsity condition A^{-1} having no more than two 1-entries per row (incl. the diagonal entry). (This class contains all realizable instances but possibly more.)

In this context two questions arised:
What is the expected number of pivot steps

1. if A^{-1} has no more than k 1-entries per row? (Possibly $O(d^{k+1})$?)

2. if some general sparsity condition holds (i.e. A^{-1} has no more than kn 1-entries)? Does still $O(d^{k+1})$ hold?

To answer the latter one, we identify the matrix $A(S)$ with $A^*(S)$ which is just A with all nondiagonal column and row entries whose indices are not in S set to zero. This again is an invertible matrix, and $\mathrm{inv}(A^*(S))$ can also be derived by setting some elements of $\mathrm{inv}(A(S))$ to zero.

In case $S = [d] \setminus \{k\}$, we can write $A^*(S)$ as the matrix product $L\,A\,M$, where L (M) are unit matrices whose k^{th} row (column) was substituted by the k^{th} row (column) of $\mathrm{inv}(A)$. Further, $\mathrm{inv}(A^*(S)) = \mathrm{inv}(M)\mathrm{inv}(A)\mathrm{inv}(L)$ where $\mathrm{inv}(M)$ und $\mathrm{inv}(L)$ have the same structure as M and L before.

$$A = (a_{ij}), \text{ where } a_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i > j; \\ 0 \text{ or } 1 & \text{o/w.} \end{cases}$$

$$B = A^{-1} = (b_{i\,j})$$

$$\begin{aligned} A^*(S) &= L\,A\,M \\ &= \begin{bmatrix} P_1 & & \\ 0 \cdots 0 & 1 & \\ & 0 & \\ Q & \vdots & P_2 \\ & 0 & \end{bmatrix} \end{aligned}$$

where all the diagonal entries are 1, P_1, P_2 are lowerdiagonal matrices, Q is just some matrix with entries 0 or 1 and L, M are just as described above.

$$\begin{aligned} (A^*(S))^{-1} &= M^{-1}\,A^{-1}\,L^{-1} \\ &= \begin{bmatrix} P_1^{-1} & & \\ 0 \cdots 0 & 1 & \\ & 0 & \\ Q + (b_{k\,1} \cdots b_{k\,k-1}) \begin{pmatrix} b_{k+1\,k} \\ \vdots \\ b_{n\,k} \end{pmatrix} & \vdots & P_2^{-1} \\ & 0 & \end{bmatrix} \end{aligned}$$

This tells us that the number of 1-entries in A^{-1} can remain constant, get smaller, or can get bigger by at most $(n-i)i - |n-1|$.

In other words, it seems impossible to answer the second question in any satisfying way.

2.2.1 Example

Given

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$$

We can write

$$A^{-1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{21} & 1 & 0 & 0 \\ a_{21}\,a_{32} - a_{31} & -a_{32} & 1 & 0 \\ -a_{21}\,a_{32}\,a_{43} + a_{21}\,a_{42} + a_{31}\,a_{43} - a_{41} & a_{32}\,a_{43} - a_{42} & -a_{43} & 1 \end{bmatrix}$$

And with

$$L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{21}\,a_{32} - a_{31} & -a_{32} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_{43} & 1 \end{bmatrix}$$

$A^*(S)$ is just

$$L\,A\,M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{bmatrix}$$

and

$$(A^*(S))^{-1} = M^{-1}A^{-1}L^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{21} & 1 & 0 & 0 \\ \begin{matrix} a_{43}(a_{21}\,a_{32} - a_{31}) - a_{21}\,a_{32}\,a_{43} \\ +a_{21}\,a_{42} + a_{31}\,a_{43} - a_{41} \end{matrix} & -a_{42} & 0 & 1 \end{bmatrix}$$

Maybe, the first sparsity condition is more promising?

Another idea from Bernd is the following: The real inverses of all matrices A not containing the 'forbidden' submatrices seem to have only 0 and $+/-1$-entries. Maybe this can be proved. And, maybe, it will be prove useful to focus the attention on the class of all matrices with this property.

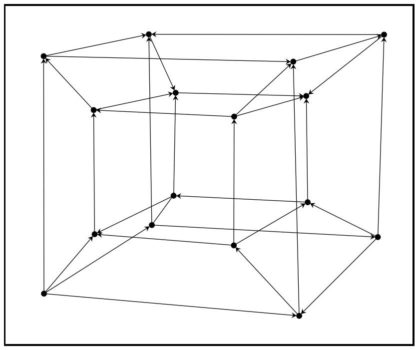
2.3 More about Random Facet on the hypercube

We examined the behaviour of Random Facet on the hypercube. In particular we wanted to know whether this pivot rule is provably faster on realizable instances compared to non-realizable ones.

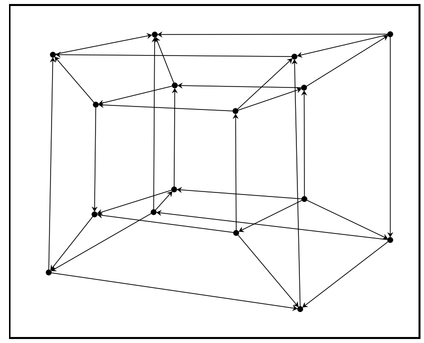
As proved by Holt and Klee, one necessary condition for realizability is the existence of d vertice-disjunctive paths from source to sink. So we calculated the expected number of steps for each possible orientation:

distance between source & sink	satisfying	path condition	not satisfying
1	1390	#	978
	3.61	$\emptyset$	3.73
	3.17	min	3.17
	4.5	max[source]	4.33
	5.75	max[all]	5.83
2	1696	#	2436
	4.23	$\emptyset$	4.37
	3.17	min	3.17
	5.5	max[source]	5.42
	5.83	max[all]	5.83
3	2494	#	2559
	4.78	$\emptyset$	4.90
	3.5	min	3.5
	6	max[source]	6.08
	6	max[all]	6.08
4	433	#	754
	5.24	$\emptyset$	5.31
	4	min	4.33
	6	max[source]	6.08
	6	max[all]	6.08
total	6113	#	6527

We observed that neither RandomFacet nor Random Edge was consistently faster than the other or vice versa.



source <—-> sink = 3



source <—-> sink = 4

We looked hard at all the 'bad' orientations on the hypercube which lead the randomized pivot rules to a high expected number of pivot steps. Unfortunately, it was impossible find some regularity in their structure which would be sufficient to show something valid in general dimension.

2.4 Even more about RANDOM FACET on the hypercube

When we run RANDOM FACET we find the optimum (v_i', say) of a randomly chosen facet i to which the starting vertex is incident. If this is not already the optimum we repeat the process from the neighbouring vertex (v_i'') not element of this original facet. Let d_i be the dimension of the smallest face containing v_i'' as well as the global optimum.

Claim: $\sum_i d_i$(realisible cubes) $<$ $\sum_i d_i$(non-realisible cubes).

Lemma 2.4.1 $\sum_i d_i'' \leq \binom{d}{2}$

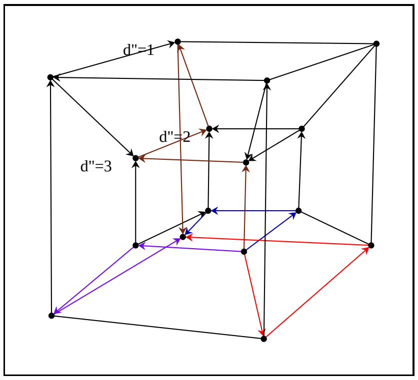
Proof: Let $F_1, \ldots, F_d$ denote the facets to which the starting vertex is incident s.th. wlog. the following holds:

$$\begin{aligned} \text{opt}(F_1) &\leq \ldots \leq \text{opt}(F_d) \\ v_i'' &< \text{opt}(F_i), \ldots, \text{opt}(F_d) \\ \text{opt},\ v_i'' &\in F_i'' \cap \ldots \cap F_d'' \\ \Rightarrow \qquad \sum_i d_i'' &\leq 0 + \ldots + (d-1) = \binom{d}{2} \end{aligned}$$

□

For instance, if we can construct a 4dimensional hypercube which satisfies the path condition but for which $\sum_i d_i'' = \binom{4}{2} = 6$ our claim is shown to be not valid.

Unfortunately, this is the case, see the following example:

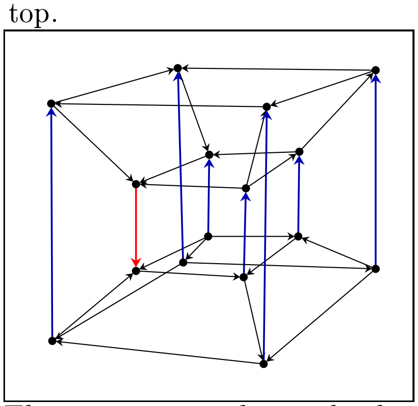


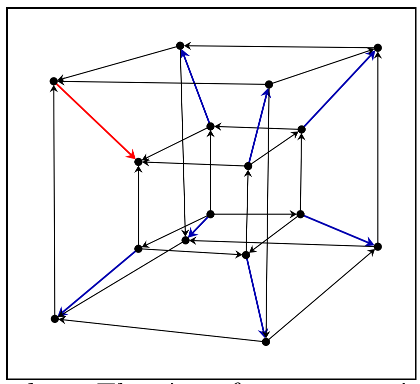
There are d vertice-disjunctive paths from source to sink in each d-dimensional subcube in this example, nevertheless, the sum of the d_i'' equals 6.

2.5 More about Random Edge on the hypercube

If we don't make any further assumptions what are the 'worst' orientations on the hypercube, which ones trigger the highest expected number of pivot steps for Random Edge?

As it turned out, (although we could check this only up to dimension 4,) the 'bad' cubes are of the following form: We have a $d-1$-dimensional 'top'-cube and a $d-1$-dimensional 'bottom'-cube. Of the 2^d edges from one to the other exactly *one* is going from the top to the bottom, namely the one connecting the two sinks; all the other edges point from the bottom to the top.





The structure on these subcubes is less clear. That is unfortunate, as it plays an important role if one wants to prove some bounds for this class of hypercubes. On the other hand, we had no idea how a construction rule for the orientation of a d-dimensional hypercube should look like such that it is provably 'bad'.

2.6 Encoding Hypercubes as trees

The Matoušek class is just a small subclass of all possible oriented hypercubes.

Here is a suggestion for a bigger one: We consider all hypercubes C with the following property:

1. The hypercube can be 'split up into' two $d-1$-dimensional hypercubes C_1 and C_2 and 2^{d-1} edges going from C_2 to C_1.

2. Each face (some hypercube of dimension k) which could be 'separated' by such a splitting process can again be split into two $k-1$-dimensional subfaces F_1, F_2 such that all edges point from one of subfaces (F_2, say) to the other (F_1).

It is clear, that given a hypercube with this property we can construct a tree according to the following recipe:

1. The hypercube itself is the root.

2. Each face which is split is a node, its two children are the two subfaces it splits into.

3. Each node has the additional information (d, i), where k is the dimension which is for each vertex of a subface constant but in which the two subfaces differ, and i is the direction, i.e. it is1 for a subface if it contains source of the face, it is 0 if it contains the sink.

(All this becomes clearer when looking at an example.)

There are some invariant operations on these trees worth mentioning:

- If the nodes on one tree level all have the same dimension we can swap that dimension with the one at the root. [This is equivalent to a rotation of the cube.]

- If on one level of the subtree dimension and direction are throughout identical we can exchange both with the entries at the root of the subtree. [This is equivalent to changing the order of the construction.]

- And, finally, we can of course always exchange the two subtrees of a node. This only needs some updating on the directions.

What is this all good for?

The main obstacles of making use of this transformation is that, although possible, it is hard to check whether two trees are encoding isomorphic hypercubes. Even more importantly, it seems even harder than on the actual orientated hypercubes to find some structure or pattern which distinguishes the 'good' from the 'bad'.

Chapter 3

Towards untrivial bounds of Random Edge

Of all the randomized pivot rules for linear programming Random Edge is somewhat the easiest: *At any nonoptimal vertex x of P, follow one of the decreasing edges leaving x with equal probability.*

Notwithstanding its simplicity, it proved to be extremely difficult to analyse its runtime-behaviour. Gärtner, Henk and Ziegler [2] managed to show the following: *The expected number E_n of steps that the* Random Edge *rule will take, starting at random vertex on the n-dimensional Klee-Minty cube, is bounded by*

$$\frac{n^2}{4(H_{n+1}-1)} \leq E_n \leq \binom{n+1}{2}.$$

This implies that there is a vertex x with $E_n(x) = \Omega(n^2/\log n)$ – but this lower bound could not be shown to hold for a specific vertex.

In the previous section we tried (more or less unsuccessfully) to find orientations on hypercubes on which the Random Edge has also a provably 'bad' behaviour.

Now we will follow a different approach. We no longer consider hypercubes but (simple) polytopes with only $d+k$ facets, where k is small.

3.1 Polytopes with $d+1$ facets

Naturally, these polytopes are just the well-studied simplices.

Theorem 1 *Given a simplex in d dimensions the expected number of steps that the* Random Edge *rule will take, starting at the source vertex (that is, the one with outdegree d), equals H_{d+1}.*

ℙroof: Note that a simplex with an orientation induced by some linear function has exactly 1 vertex with outdegree i, for each i, $0 \leq i < d+1$.

Let a_i denote the expected number of steps starting at a vertex with outdegree i. Then we have

$$\begin{aligned} a_n &= 1 + \frac{1}{n}\sum_{i=0}^{n-1} a_i \\ na_n + n &= n + \sum_{i=0}^{n-1} a_i \\ na_n + (n-1)a_{n-1} &= 1 + a_{n-1} \\ n(a_n - a_{n-1}) &= 1 \\ a_n - a_{n-1} &= \frac{1}{n} \\ \Rightarrow \qquad a_n &= H_n \end{aligned}$$

□

So in this simple case we can actually determine the expected number of steps for any starting vertex. The asymptotic behaviour is both for the average (by A.4) and for the worst case $\Omega(\log d)$.

3.2 Useful Transformations and j-facets

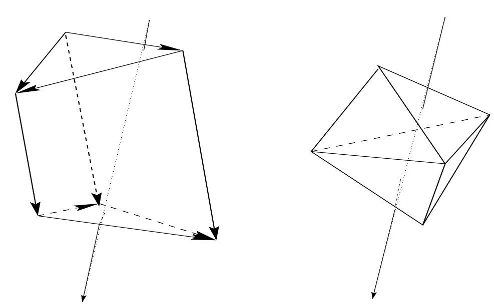
As it is to expect things are getting complicated for $k = 2$ already. Before we discuss this case in detail we therefore briefly introduce a couple of very useful transformations.

3.2.1 The Polar Of A Polytope

Let P be a polytope in $\mathbb{R}^d$. W.l.o.g. assume $\mathbf{0} \in \mathrm{int}(P)$.

Definition 3.2.1 *For any subset $P \subset \mathbb{R}^d$, the* polar *is defined by*

$$P^{\Delta} := \{\mathbf{c} \in (\mathbb{R}^d)^* : \mathbf{cx} \leq 1 \quad \forall \mathbf{x} \in P\} \subseteq (\mathbb{R}^d)^* \tag{3.1}$$



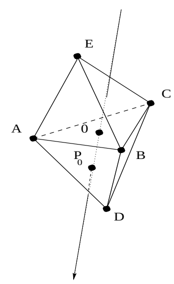
Lemma 3.2.1 *Let $l, \mathbf{0} \in l$ be a directed line representing a linear function and P as above. Then the linear order of the vertices of the polygon is identical to the linear order induced by l on the facets of the polar P^{Δ}.*

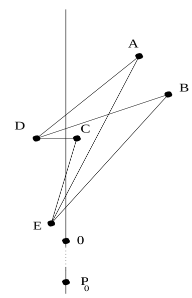
Proof: W.l.o.g. we can assume $l = x_n$, where x_n is the n^{th} axis. So given $p, q \in P$, $p < q \Leftrightarrow p_n < q_n$. We want to show that the intersection point p_0^{Δ} of l with p^{Δ} is *earlier* than q_0^{Δ}. Now $p^{\Delta} = \{(c_1, \ldots, c_n) | \sum_i c_i p_i = 1\}$, implying $p_0^{\Delta} = (0, \ldots, \frac{1}{p_n})$, $\quad q_0^{\Delta} = (0, \ldots, \frac{1}{q_n})$. But $p_n < q_n$ implies $\frac{1}{p_n} > \frac{1}{q_n}$. $\square$

3.2.2 The Gale Transformation

Definition 3.2.2 *Let $d < n$. Given $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{n \times n-d-1}$ (sequence of n points in $\mathbb{R}^d$ and $\mathbb{R}^{n-d-1}$), with $A^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$ and $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$ (origin is centroid) and both matrices of full rank (not all points on a common hyperplane), then we call B the* orthogonal dual of A*, $A \perp B$, if $B^T A = \mathbf{0}$.*

$$\left\{\begin{array}{rrr|r|rr} 0 & 0 & -1 & 1 & 80 & -96 \\ -2 & -2 & 0 & 1 & -27 & 47 \\ 2 & -2 & 0 & 1 & -19 & 39 \\ 0 & 6 & 0 & 1 & -18 & 26 \\ 1 & -1 & -5 & 1 & -16 & 0 \\ -1 & -1 & 6 & 1 & 0 & -16 \end{array}\right\}$$





When we apply this transformation on the n vertices of a simplicial polytope together with an interior point P_0 lying on some given line l that goes through the origin, we get $n+1$ points in dimension $n-d$, some line g and the following lemmas hold:

Lemma 3.2.2 *Consider a partition of $P, P = Q \cup R, Q \cap R = \emptyset$. Then Q forms a face of $conv(P) \iff conv(R \cup P_0)^*$ contains the origin. In particular, if $|Q| = d, |R| = n-d$, Q forms a facet of $conv(P) \iff conv(R \cup P_0)^*$ is a simplex containing the origin.*

Since P_0 is not contained in any of the $Q-$faces of P, P_0^* is vertice of all the $(R \cup P_0)^*$ simplices.

Lemma 3.2.3 *Let $l, \mathbf{0} \in g$ be a directed line representing a linear function and P^* as above. Then the order of the facets induced by the visibility in respect to l is identical to the linear order induced by the intersection points of the according simplices with the line $g = l^*$.*

How can we calculate the dual of a point set?
Get $\left(\frac{A^T}{1}\right)$ into the form $(I_{d+1}|M)$, then $B = \left(\frac{M}{-I_{d+1}}\right)$ will do the job.

3.2.3 j-facets

Given a set S of n points in $\mathbb{R}^d$ (no $d+1$ on a common hyperplane). A directed line l enters a $j-$facet, if it intersects the relative interior of the spanned simplex, form the positive to the negative side.

Definition 3.2.3 *An oriented simplex spanned by d points of S is called $j-$facet, if there are exactly j points of S on its positive side.*

Note: 0-facets are the facets of conv$S$.

3.3 Polytopes with $d+2$ facets

We are going to try to look at the combinatorial structure of the polytopes with $d+2$ facets by applying the duality mappings which we have just discussed in the previous section consecutively.

This we will give us the following:

Primal: That is our starting point. Given is a simple polytope P with $n+2$ facets. W.l.o.g. $\mathbf{0} \in P$. Each face $F = F_1 \cap \cdots \cap F_k$ has a unique sink $v(\in F_1 \cap \cdots \cap F_k)$; the orientation of the edges is induced by some line l_0, $\mathbf{0} \in l_0$.

Polar: The polar of P^Δ is simplicial, and each face $F^\Delta = \text{conv}(F_1^\Delta, \ldots, F_k^\Delta)$ is incident to a unique minimal facet v^Δ with $F_1^\Delta, \ldots, F_k^\Delta \subseteq v^\Delta$. The order of the facets is induced by $l_1 = l_0^\Delta$. We add an auxiliary point P_0, $P_0 \in l_1 \cap P \setminus \mathbf{0}$.

Dual of the Polar: Finally, after applying the Gale-transformation, we have a set of $n+3$ points in the 2dimensional plane and the lemma 3.2.2 holds.

3.3.1 The refined index

Definition 3.3.1 *Given a planar point set S, a line l, $S \cap l = \emptyset$, and a j-facet f. For $a :=\#$ of points on positive side of l and f, and $b :=\#$ of points on negative side of l and positive side of f, let (a,b) be the* refined index *of f.*

Lemma 3.3.1 *Let n_{pos}, n_{neg} be the number of points in P on positive, negative side of S, respectively.*

1. *All refined indices lie in $[0 \ldots n_{pos}-1] \times [0 \ldots n_{neg}-1]$.*
2. *No two $*$-facets have the same refined index!*
3. *l enters $n_{pos}n_{neg}$ $*$-facets.*

Corollary 3.3.1 *Every pair in* $[0 \ldots n_{pos}-1] \times [0 \ldots n_{neg}-1]$ *occurs exactly once as a refined index as there are:* $(0,j),(1,j-1),\ldots,(j-1,1),(j,0)$.

Theorem 2 *In the plane, line* l *enters* $\min\{j+1, n-j-1, n_{pos}, n_{neg}\}$ $j-$*facets.*

The $\vec{h}$-vector tells us the numbers of j-facets ($= h_j$). $j < n-j-1$:
$\vec{h} = (1,2,\ldots,j,\underbrace{j+1,\ldots,j+1}_{\text{all } j+1},j,j-1,\ldots,2,1)$

From this we can derive the $\vec{f}$-vector, for $f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i}$.

But it is also possible to compute it directly:

$$f_k = \sum_{i=1}^{\min\{n_{pos},n-k-1\}} \binom{n_{pos}}{i}\binom{n_{neg}}{n-k-i}$$

where w.l.o.g. $n_{pos} \leq n_{neg}$.

3.3.2 How AOFs come in

Taking the risk that this slight detour prolongs this already long chapter even more we will shortly apply yet another transformation – this time to the 2dimensional point set we received after applying GALE – and return in a way (surprise!) just where we started!

But before we do so we briefly introduce some necessary background just to have it all together..

Let L, R be two finite sets of cardinality l and r, resp., and let $<$ be a total order on $L \times R$. We define a directed graph $G = G(L, R, <)$ on $L \times R$ by introducing directed edges

$$\begin{cases} (a,b) \to (a',b), & \text{whenever } (a,b) > (a',b) \\ (a,b) \to (a,b'), & \text{whenever } (a,b) > (a,b') \end{cases}$$

Definition 3.3.2 *The pair* $A = (L \times R, <)$ *is an* abstract objective function *(AOF) iff*

- $G(L,R,<)$ *is acyclic, and*
- *the induced subgraph* $(G', R', <)$ *has a unique sink for all* $L' \subseteq L, R' \subseteq R$ *with* $L', R' \neq \emptyset$.

Lemma 3.3.2 *Let* $A = (L \times R, <)$ *be an AOF and consider* $x, a \in L, y, b, \in R$.

- *if* $(x,b),(a,y) \leq (a,b)$*, then also* $(x,y) \leq (a,b)$*,*
- *if* $(x,b),(a,y) \geq (a,b)$*, then also* $(x,y) \geq (a,b)$*.*

With this at hand we can now define an "abstract" version of the refined index:

Definition 3.3.3 *For* $(a, b) \in L \times R$ *we define*

$$\gamma(a, b) := (|L(a, b)|, |R(a, b)|),$$

where

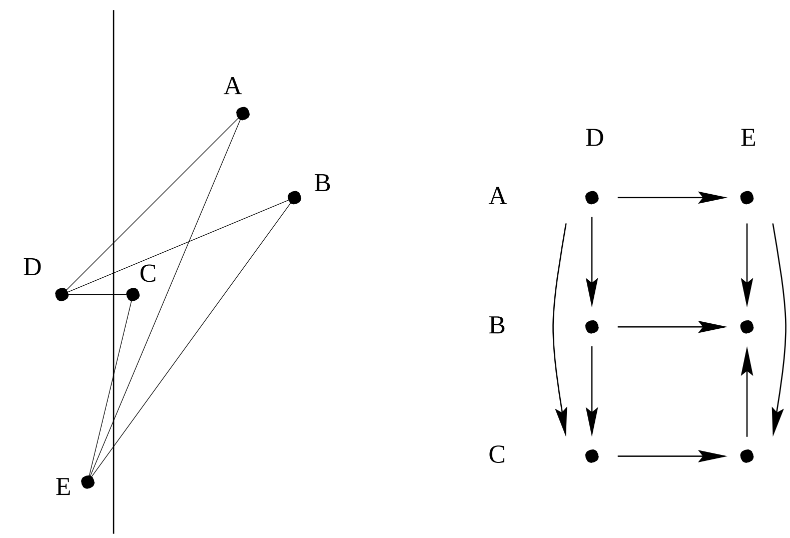
$$L(a, b) := \{x \in L | (x, b) < (a, b)\},$$

$$R(a, b) := \{y \in R | (a, y) < (a, b)\}.$$

We call $\gamma(a, b)$ *the* refined index *of* (a, b).

Theorem 3 *The refined index* $\gamma : L \times R \rightarrow \{0, \ldots, l - 1\} \times \{0, \ldots, r - 1\}$ *is a bijection.*

Back to our 2dimensional point set transversed by some line l. We number all points on the left by $1, \ldots, L$ and all points on the right by $1, \ldots, R$. Each facet is now represented by a pair $(a, b) \in L \times R$ and has a unique refined index $\gamma(a, b)$.



Two immediate questions arise.

- Which AOFs of the form $A = (L \times R, <)$ are just the image of a polytope with $d + 2$ facets in d dimensions?

or more precise:

- What distinguishes the AOFs which can be considered as images of such polytopes from the ones which aren't?

- Suppose we are given an AOF with its preimage, a $d-$polytope. What is the equivalent to a face of the polytope in the AOF?

The last question is the easiest and we will therefore answer it right now. For the very interesting other two the reader is referred to the final chapter.

Theorem 4 *Let P be a d-dimensional simple polytope with $d+2$ facets and $A = (L \times R, <)$ the equivalent abstract objective function according to the combinatoric bijections discussed above. Then there is a natural bijection between each induced subgraph $G(L', R', <)$ and the $(L'+R'-2)$-dimensional faces of P.*

Proof: Given a polytope $P(n, d)$ with facets $F_1, \dots, F_{d+2}$.
T.F.A.E.

1. $F = F_1 \cap \dots \cap F_{d-k}$ is a $k-$dimensional face of P.
2. $F^\Delta = \text{conv}(F_1^\Delta, \dots, F_{d-k}^\Delta)$ is a $d-k-1$-dimensional face of P^Δ.
3. $\text{conv}(P^{\Delta^*} - F^{\Delta^*}, P_0^*) = \text{conv}({F_{d-k+1}^\Delta}^*, \dots, {F_{d+2}^\Delta}^*, P_0^*) = S$ contains the origin, and $|P^{\Delta^*} - F^{\Delta^*}, P_0^*| = |({F_{d-k+1}^\Delta}^*, \dots, {F_{d+2}^\Delta}^*, P_0^*)| = k+3$.
4. There are L' and R' points of S on the left and right of $l_2 = l_1^*$, resp, where $\min(L', R') \geq 1$ and $L' + R' - 2 = k$.
5. The image of the points $P^{\Delta^*} - F^{\Delta^*}$ form a $L' \times R'$-subgraph G' of G.

□

Corollary 3.3.2 *The directed graph $G(P)$ of a simple polytope $P(n, d)$ with $d+2$ facets is isomorphic to the 'AOF-gridgraph' of P.*

This nice corollary shall mark a temporary endpoint under this detour, it shall get more prominence in the final chapter.

3.3.3 The FlipFlap-Game

Back to our 2-dimensional set-up. It is easy to see, that starting from a polynom of $d+2$ points, where d is the actual dimension, Random Edge gets translated into a 2-dimensional game. The n vertices translate into n points, and this point set is transversed by the dual of the line $l_1\, l_1^* = P_0^* \mathbf{0} = l_2$, say. None of the n points is incident to l_2.

We consider all pairs of points (A, B), where A and B are points on the positive and negative side of l, respectively.

FlipFlap

> Pick any such pair.
> While there exist points *below* (A, B) pick one of them at random, C say. If C lies on the positive/negative side of l substitute A/B with C.

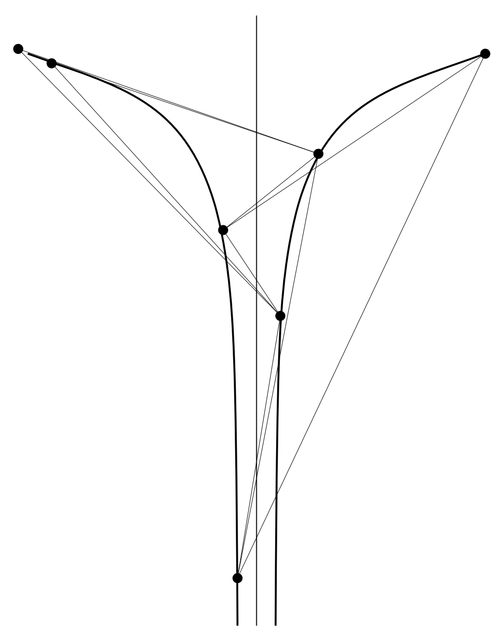
In the following we want to derive some properties of the game and ignore to some extend prior knowledge we have. For instance, the lemma below would follow directly from the fact that Random Edge is finite.

Lemma 3.3.3 FlipFlap *is a finite game.*

Proof: Through every pair (A, B) we can define a line AB which intersects l in some point, L_{AB} say. I claim that the game ends as soon as we obtained those points which induce the lowest intersection point. For suppose otherwise, i.e. there exists a pair (C, D) such that L_{CD} lies *below* L_{AB} but C, D lie *above* on AB. This is clearly a contradiction. As we deal only with a finite number of points (and hence a finite number of pairs of points) the game therefore must be final. □

3.3.4 A lower bound for the FlipFlap

We construct the following set-up: All points lie on two parabolas p_n, p_m in the following way. Place n_1, n_2, m_1 on p_n and p_m, cp. picture. Now m_i shall be below the intersection of the $m-$parabola with $n_{i-1}n_{i-2}$ on the parabola, n_j on the $n-$parabola below its intersection with the line $m_{i-1}m_{i-2}$.



Observe the following properties:

$$\begin{aligned} RI(m_i m_j) &= (j, j+n_2-i), \text{if } i \geq j \\ &= (i+n_2-j, i), \text{if } i < j \end{aligned}$$

We shall devide the game into several phases. To be precise, we say that we stay in phase $i = \min(a,b)$ as long $\min(a,b)$ does not change.

$$\begin{aligned} \mathbb{E}[\text{running time}] &= \mathbb{E}[\#\text{of flips}] \\ &= \sum_{a'<n} \mathbb{E}[\#\text{of flips in phase } a']\mathbb{P}(a' \text{ `happens'}) \end{aligned}$$

We consider the following events:

$$\begin{aligned} X &= \# \text{ of total flips} \\ X_i &= \# \text{ of flips in phase } i \\ Y_i &= \# \text{ of flips in phase } i + \# \text{ of flips of the consecutive phase.} \end{aligned}$$

Then the above translates into

$$2\mathbb{E}[X] = \mathbb{E}[Y_n] + \mathbb{E}[X_n] + \sum_{0<a'<n} \frac{\mathbb{E}[Y_{a'}]}{a'+1} + \mathbb{E}[X_0] \tag{3.2}$$

Let $t_{a,b}$ denote the expected number of flips starting at position (a,b) until we enter a new phase.

2 cases have to be considered:

$$\begin{aligned} &(A) \quad \min(a,b) = b; \qquad (a \geq b) \\ &t_{a,b} = 1 + \tfrac{1}{2b+n-a} \textstyle\sum_{a'>a} t_{a',b} \end{aligned}$$

$$\begin{aligned} (2b+n-a)t_{a,b} - (2b+n-a-1)t_{a+1,b} &= 1 + t_{a+1,b} \\ (2b+n-a)(t_{a,b} - ta+1,b) &= 1 \\ \Rightarrow t_{a,b} = \frac{1}{2b+n-a} + t_{a+1,b} & \end{aligned}$$

$$\text{Now, since} \qquad t_{n,b} = \begin{cases} 0, & \text{if} \quad b = 0, \\ 1, & \text{if} \quad b > 0, \end{cases}$$

$$\text{we have} \qquad t_{a,b} = \begin{cases} 1 + \sum_{a'=a}^{n-1} \frac{1}{2b+n-a'}, & \text{if} \quad b \neq 0 \\ \sum_{a'=a}^{n-1} \frac{1}{n-a'}, & \text{if} \quad b = 0 \end{cases}$$

$$\text{i.e.} \qquad t_{a,b} = \begin{cases} 1 + H_{2b+n-a} - H_{2b}, & \text{if} \quad b \neq 0 \\ H_{n-a}, & \text{if} \quad b = 0 \end{cases}$$

$$(B) \qquad \min(a,b) = a; \qquad (a < b)$$

$$t_{a,b} = 1 + \tfrac{1}{2a+n-b+1} \textstyle\sum_{b'>b} t_{a,b'}$$

$$\begin{aligned} (2a+n-b+1)t_{a,b} - (2a+n-b)t_{a,b+1} &= 1 + t_{a,b+1} \\ \Rightarrow t_{a,b} &= \frac{1}{2a+n-b+1} + t_{a,b+1} \end{aligned}$$

$$\begin{aligned} \text{Now, since} \quad t_{a,n} &= 1 \\ \text{we have} \quad t_{a,b} &= \frac{1}{2a+n-b+1} + \ldots + \frac{1}{2a+2} + 1 \\ t_{a,b} &= 1 + H_{2a+n-b+1} - H_{2a+1} \end{aligned}$$

Of course, these equations already imply that $\mathbb{E}[X_i] = \Theta(\log(n))$, so in particular this is true for $\mathbb{E}[Y_n], \mathbb{E}[X_n], \mathbb{E}[X_0]$. The only interesting term of (3.2) is therefore $\sum_{0<a'<n} \frac{\mathbb{E}[Y_{a'}]}{a'+1}$.

We are now considering the expected length of two consecutive phases. Again, we distinguish between two cases:

$$(A) \qquad \min(a,b) = b; \qquad (a \geq b)$$

$$\begin{aligned} \mathbb{E}[Y]_b &\geq H_{2b+n-a} - H_{2b} + 1 \\ &\quad + \frac{1}{2b} \left[\underbrace{\sum_{b'new} 1}_{=b} + \sum_{\substack{0 \leq a' < b \\ a' new}} (1 + H_{2a'+n-b+1} - H_{2a'+1}) \right] \\ &= H_{2b+n-a} - H_{2b} + 2 + \frac{1}{2b} \sum_{0 \leq a' < b} (H_{2a'+n-b+1} - H_{2a'+1}) \end{aligned}$$

We achieve a minimum for $a = n$, i.e.

$$\mathbb{E}[Y]_b \geq 2 + \frac{1}{2b} \sum_{a'<b} (H_{2a'+n-b+1} - H_{2a'+1}) \tag{3.3}$$

$$(B) \qquad \min(a,b) = a; \qquad (a < b)$$

$$\begin{aligned} \mathbb{E}[Y]_a &\geq H_{2a+1+n-b} - H_{2a} + 1 \\ &\quad + \frac{1}{2a} \left[\sum_{0 \leq a' < a} 1 + \sum_{0 < b' < a} (1 + H_{2b'+n-a} - H_{2b'}) + H_{n-a} \right] \\ &\geq H_{2a+1+n-b} - H_{2a} + 2 \\ &\quad + \frac{1}{2a} \left[\sum_{0 < b' < a} (H_{2b'+n-a} - H_{2b'}) + H_{n-a} \right] \end{aligned}$$

By (A.5) we have $H_{2b'+n-a+1} - H_{2b'+1} < H_{n-a}$, which gives us:

$$\begin{aligned}\mathbb{E}[Y]_a &\geq H_{2a+1+n-b} - H_{2a} + 2 \\ &\quad + \frac{1}{2a}\left[\sum_{0<b'<a} (H_{2b'+n-a+1} - H_{2b'+1})\right]\end{aligned}$$

We achieve a minimum for $b = n$, i.e.:

$$\mathbb{E}[Y]_a \geq 2 + \frac{1}{2a}\left[\sum_{0<b'<a} (1 + H_{2b'+n-a} - H_{2b'})\right] \tag{3.4}$$

So, in general, with $m = \min(a, b)$, we are interested to get a closed form of the sum:

$$\frac{1}{2m}\left[\sum_{0<m'<m} (H_{2m'+1+n-m} - H_{2m'+1})\right] \tag{3.5}$$

Making heavy use of (A.1)...(A.4) we are able to derive the following:

$$\begin{aligned}
&\frac{1}{2m}\left[\sum_{\substack{0<m''<2m+1 \\ m'' odd}} (H_{m''+n-m} - H_{m''})\right] \\
\geq\ &\frac{1}{2m}\left[\sum_{\substack{m''<m+n+1 \\ m'' odd}} H_{m''} - \sum_{\substack{m''<n+1-m \\ m'' odd}} H_{m''} - \sum_{\substack{m''<2m+1 \\ m'' odd}} H_{m''}\right] \\
\geq\ &\frac{1}{4m}\left[\sum_{m''<m+n+1} H_{m''} + 1 - H_{m+n+1} - \sum_{m''<n+1-m} H_{m''} - H_{n-m}\right. \\
&\left. - \sum_{m''<2m-1} H_{m''} - H_{2m}\right] \\
=\ &\frac{1}{4m}[(m+n)(H_{m+n+1} - 1) + (n-m+1)(H_{n-m+1} - 1) \\
&-H_{n-m} - (2m+1)(H_{2m+1} - 1) - H_{2m}] \\
=\ &\frac{1}{4m}\left[(m+n)H_{m+n} + (m+n)\frac{1}{m+n-1} - (m+n)\right. \\
&-(n-m)H_{n-m} - (n-m)\frac{1}{n-m+1} + (n-m) - H_{n-m+1} + 1 \\
&-(2m)H_{2m} - 2m\frac{1}{2m+1} + 2m - H_{2m+1} + 1 \\
&\left.-H_{n-m} - H_{2m}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4m}\left[(m+n)H_{m+n} - (n-m)H_{n-m} - (2m)H_{2m}\right. \\
&\quad \left. 1 - \frac{1}{m+n+1} - 2H_{2m} - 2H_{n-m}\right]
\end{aligned}
$$

$$
\mathbb{E}[Y]_m \geq 2 + \frac{1}{4m}\left[(m+n)H_{m+n} - (n-m)H_{n-m} - (2m)H_{2m} - 2H_{2m} - 2H_{n-m}\right) \tag{3.6}
$$

Now we are ready to find a lower bound for the actual game.

$$
\begin{aligned}
\frac{1}{2}\sum_{0<m<n}\frac{\mathbb{E}[Y]_m}{m+1} &\geq \frac{1}{2}\sum_{0<m<n}\frac{1}{m+1}\left[2 + \frac{1}{4m}((m+n)H_{m+n} - (n-m)H_{n-m}\right. \\
&\quad \left. -(2m)H_{2m} - 2H_{2m} - 2H_{n-m}\right] \\
&\geq \sum_{0<m<n}\frac{1}{m+1} \\
&\quad + \frac{1}{8}\sum_{0<m<n}\left[\left(\frac{1}{m} - \frac{1}{m+1}\right)\left(n\left(H_{m+n} - H_{n-m}\right) - 2\left(H_{2m} + H_{n-m}\right)\right)\right. \\
&\quad \left. + \frac{1}{m+1}(H_{m+n} + H_{n-m} - 2H_{2m})\right] \\
&= H_n - 1 - \frac{1}{4}\sum_{0<m<n}\frac{H_{2m}}{m} \\
&\quad + \frac{n+3}{8}\sum_{0<m<n}\frac{H_{n-m}}{m+1} - \frac{n+2}{8}\sum_{0<m<n}\frac{H_{n-m}}{m} \\
&\quad - \frac{n-1}{8}\sum_{0<m<n}\frac{H_{m+n}}{m+1} + \frac{n}{8}\sum_{0<m<n}\frac{H_{m+n}}{m} \\
&= H_n - 1 - \frac{1}{4}\sum_{0<m<n}\frac{H_{2m}}{m} \\
&\quad + \frac{n+3}{8}\left(\sum_{0<m<n}\frac{H_{n-m+1}}{m} - H_n + \frac{1}{n}\right) \\
&\quad - \frac{n+2}{8}\sum_{0<m<n}\frac{H_{n-m}}{m} \\
&\quad - \frac{n-1}{8}\left(\sum_{0<m<n}\frac{H_{m+n-1}}{m} - H_n + \frac{H_{2n-1}}{n}\right) \\
&\quad + \frac{n}{8}\sum_{0<m<n}\frac{H_{m+n}}{m} \\
&= \frac{n+2}{8}\sum_{0<m<n}\frac{1}{m(n-m+1)} + \frac{n-1}{8}\sum_{0<m<n}\frac{1}{m(n-m+1)}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{8}\sum_{0<m<n}\frac{H_{m+n}+H_{n-m+1}}{m}-\frac{1}{4}\sum_{0<m<n}\frac{H_{2m}}{m}+O(\log(n)) \\
= & \frac{1}{8}\sum_{0<m<n}\frac{H_{m+n}+H_{n-m+1}}{m}-\frac{1}{4}\sum_{0<m<n}\frac{H_{2m}}{m}+O(\log(n))
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{8}\left(2\sum_{m<n}\frac{H_n}{m}+\sum_{m<n}\frac{1}{m}\left(\frac{1}{n+1}+\cdots+\frac{1}{n-m}-\frac{1}{n-m+2}-\cdots-\frac{1}{n}\right)\right) \\
& -\frac{1}{4}\sum_{0<m<n}\frac{H_{2m}}{m}+O(\log(n)) \\
= & \frac{1}{8}\log^2(n)+O(\log(n))
\end{aligned}
$$

We can plug this into (3.2) and get the result:

$$
\begin{aligned}
\mathbb{E}[X] &= \frac{1}{2}\left(\mathbb{E}[Y_n]+\mathbb{E}[X_n]+\sum_{0<a'<n}\frac{\mathbb{E}[Y_{a'}]}{a'+1}+\mathbb{E}[X_0]\right) \\
\mathbb{E}[X] &= \frac{1}{8}\log^2(n)+O(\log(n)) \qquad (3.7)
\end{aligned}
$$

Lemma 3.3.4 *In the worst case* FLIPFLAP *takes an expected number of* $\Omega(\log^2(n))$ *steps, where* n *is the number of points.*

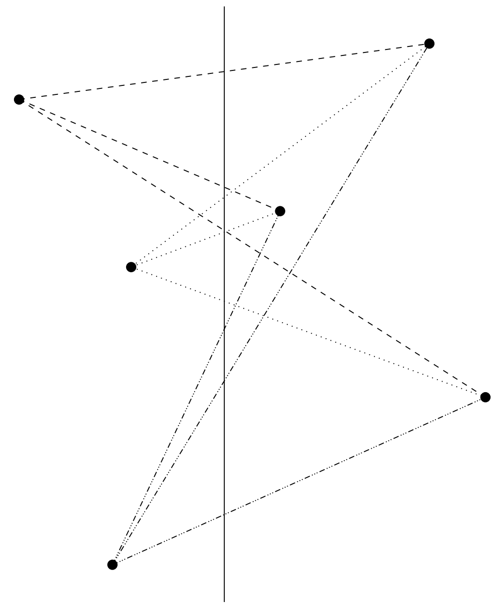
Remark: We could show this lower bound for another, non-realisable example, the 'twisted lexicographic order':

$$
\begin{aligned}
& (a,b) < (a',b') \\
& \iff (a<a') \vee \left(a=a' \wedge \begin{cases} b<b' & a \text{ even} \\ b>b' & a \text{ odd} \end{cases}\right)
\end{aligned}
$$

3.3.5 An upper bound for the FLIPFLAP

Lemma 3.3.5 FLIPFLAP *takes an expected number of* $O(n\log(n))$ *steps, where* n *is the number of points.*

Proof: Each point from the left can 'couple up' with each point on the right. When we run the game we count the expected number of occurances for each left point. Then this number will be $O(\log n)$, having $O(n)$ points on the left, the result follows. □



Claim: The actual running time of FlipFlap is $\Theta(\log^2(n))$.

This claim is supported by the following observations: Calculating all the AOFs up to size 4×4, the WCs were just the ones for which we proved the lower bound – with only one row altered. (Since #Orientierungen= $n!^{2(n-1)}$ for $n \times n$ field, the calculation is impossible for higher dimensions.) Also, the observed runtime is close to this lower bound.

A temporary list of observations which might help:

- Given a point S there are two points A and B on the other side of l such that all points on the other side lie in the cone ASB. The number of times we can flip away from S with the chance to return is bounded above by the number of points in ASB and on the same side as S.
- We aim to distinguish certain phases in the game with the property that there are $O(\log n)$ phases each taking $O(\log n)$ time. There was once hope that you could add points in pairs such that all the facets incident to these points are counting as a phase. The indegree of all the old faces would increase by exactly 2, the outdegree would remain constant. Unfortunately, counterexamples could be constructed. But a more clever way of defining 'phases' might do the trick.

3.4 Polytopes with $d+3$ facets

As we saw above, in 2D there is a natural way of defining a bijection between each $k-$facet and all pairs (i,j) in a certain range s.t. $i+j=k$ holds.

In this section, we want to discuss what properties a generalization into 3 dimensions of such an index should have and what implications this has.

(In the following, we only discuss the 3dimensional case without explicitly referring to it each time.)

Any three points whose convex hull is entered by l is defining a $k-$facet, where k is the number of points 'below'. (l shall enter the facet from below to above, say.)

Suppose we can identify each $k-$facet with a triple (x, y, z). To be the generalization of the 2dim. refined index we are looking for it definitely should fulfill the following constraints:

- $x + y + z = k$
- Points 'above' the actual facet shall have no influence on the actual triple entries. I.e. we can add as many points as we wish above the considered facet without changing its refined index.

Of course, we need to be more precise in saying, how we really define this index. One natural way would be:

- Given a facet ABC, the refined index (x, y, z) shall denote the number of choices for $A'(= x)$, $B'(= y)$ and $C'(= z)$.

Of course, this is already a bit ambigous because ABC, BCA and CAB all denote the same facet and therefore should have the same refined index... (Identifying (x, y, z) with all its even permutations would be one way to handle this, having to reassess the condition of uniqueness. Or we just find some clever way of getting a unique representative of the up to three potential choises.)

Nethertheless this definition is quite appealing. You could well imagine to extrapolate to even higher dimensions along these lines.

Let's have a look at some other properties we could ask for, now, as we have opted for a definition of the refined index:

- The index of the top-facet determines the possible range of the other indices to the extent that we can deduce exactly which indices exist.

Suppose we are given a top-facet with refined index (x, y, z) it would be great if we could find a natural bijection between all possible refined indices of our point set and the set $\{0, \ldots, x\} \times \{0, \ldots, y\} \times \{0, \ldots, z\}$ or something similar. Unfortunately, this does not seem feasible. *Just by moving the points which form the top-facet* we are able to change its refined index while leaving all the indices of facets not incident to the top-facet constant.

The main problem seems to be that we don't have 'left' or 'right' anymore. And it is far from clear what notion should be an adequate replacement!

Chapter 4

The Path-condition

Theorem 5 *Given a d-dimensional (simple) polytope with $d+2$ vertices it is realisable if there exist exactly k vertex-disjoint paths from sink to source for each k-dimensional subface.*

Proof: Of course, this is just a very special case of BALINSKI's Theorem, which says that the graph of *any* d-polytope is d-connected.

Nevertheless, we want to give a proof for this special case anyway, not just because it is a nice application of the methods from the previous chapter, but also because we conjecture that the converse can be proved as well, using again these methods.

As we saw above the oriented graph of the polytope P is just the 'AOF-gridgraph' of P and it will have the following properties:

- Each $m' \times n'$-subgraph has unique source and sink.
- There are no cycles.
- The number of vertices with a certain outdegree is equivalent to the $h-$vector.

Recall further that the image of an $m+n-2$ dimensional polytop with $m+n$ facets and mn vertices is an $m \times n$ grid-graph, that satisfies all these conditions.

We now want to show that there are $m+n-2$ knot-disjoint paths from source to sink. (Note: It has to be shown that these paths are representing vertice-disjoint paths on the original polytope from source to sink.)

We first concentrate on the source and the sink. There are three cases to consider: They are either in the same row, in the same column or neither of the two. The first two cases we can discuss at once. W.l.o.g suppose they are in the same row. Certainly there is exactly one way of getting from source to sink, namely the direct one. We now could add an additional point either on the left (i.e. we get an additional row) or on the right (which means the number of columns increases by one). In either case we will also get an additional path from source to sink, as the first condition listed above has to be fulfilled, that is uniqueness of source and sink.

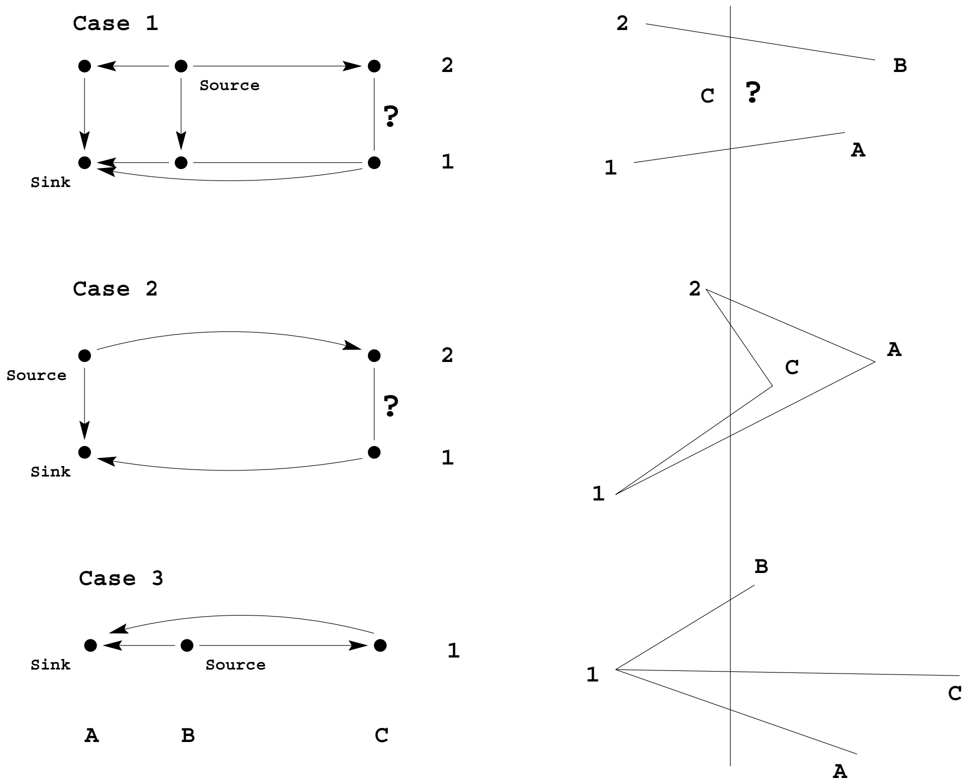


Figure 4.1:

The latter case is a bit more tricky: Look at Figure 4.1: Is it possible, that there exists a directed edge $(C, 1) \to (C, 2)$. Well, the intersection point of the line $(C, 1)$ with l is above the one of $(C, 2)$ with l iff C is on the left side of l - which is impossible. I.e. such a graph represents a non-realizable configuration. But the directed edge $(C, 2) \to (C, 1)$ garanties an additional path from source to sink.

By an induction argument we therefore conclude that the number of knot-disjoint paths from source to sink, i.e. the number of vertice-disjoint paths from source to sink on the original polytop equals the $n + m - 2 = d$, the dimension. $\square$

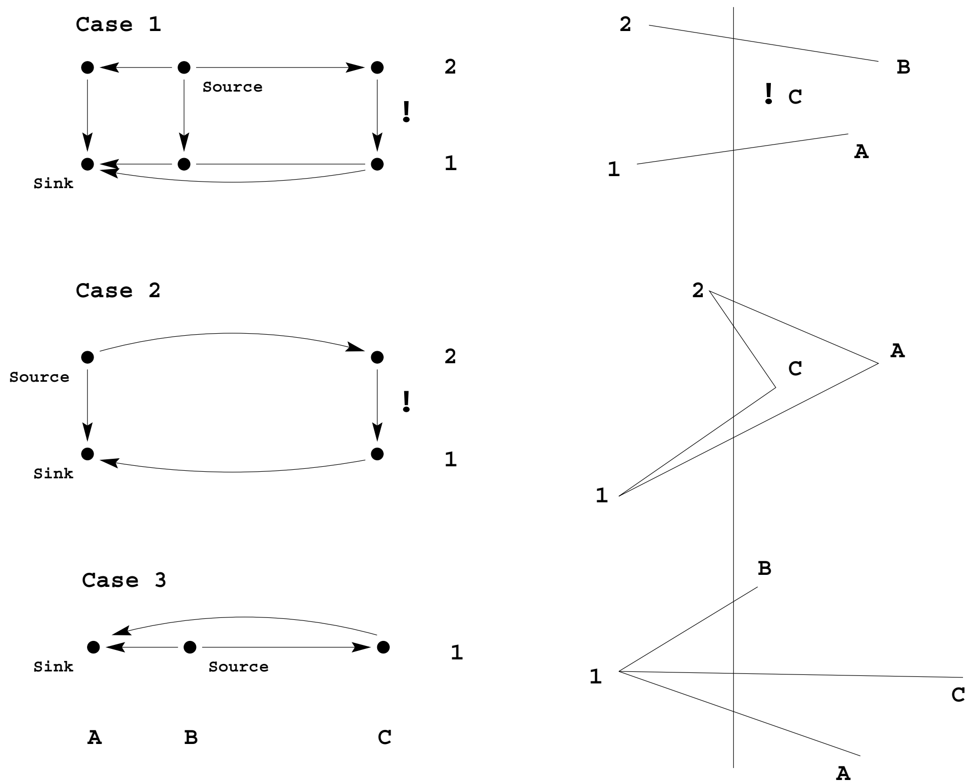

Case 1
Source
Sink
2
1
!
2
B
! C
A
1
Case 2
Source
Sink
2
1
!
2
A
C
1
Case 3
Sink
Source
1
A
B
C
B
1
C
A

Figure 4.2:

We recall two lemmas which we proved already in the previous chapter:

Lemma 4.0.1 *The grid-graph is just the oriented vertice graph of the given polytope, if it is realisable.*

Lemma 4.0.2 *Any $p \times q$ subgrid is the image of some facet of the polytope.*

It is now not far fetched to ask the following:
Open Question: *Given an $m \times n$-grid-graph G such that each $m' \times n'$-subgridgraph has:*

- *unique source and sink and*
- *$n' + m' - 2$ knot-disjunctive paths from source to sink.*

Does there exist a $d = n + m - 2$-dimensional simple polytope with $d + 2$ vertices with vertex-graph G?

The most direct way to answer this would be to (dis)prove what we state next:
Claim: *Given a grid-graph with the aforementioned properties we can construct the 2dimensional image dual to the polar of a polytope P with vertex-graph G.*

But as it seems at the moment this is probably not the case!

Appendix A

Appendix

A.1 Some useful inequalities

$$\sum_{0 \le a' < a} H_{a'+x} = \sum_{0 \le a' < a+x} H_{a'} - \sum_{0 \le a' < x} H_{a'} \tag{A.1}$$

$$\frac{1}{2}\left(\sum_{a'<a} H_{a'} + H_{a-1}\right) > \sum_{\substack{a'<a \\ a' even}} H_{a'} > \frac{1}{2}\left(\sum_{a'<a} H_{a'} - H_a\right) \tag{A.2}$$

$$\frac{1}{2}\left(\sum_{a'<a} H_{a'} + H_{a-1}\right) > \sum_{\substack{a'<a \\ a' odd}} H_{a'} > \frac{1}{2}\left(\sum_{a'<a} H_{a'} + 1 - H_a\right) \tag{A.3}$$

$$\sum_{a'<a} H_{a'} = a(H_a - 1) \tag{A.4}$$

$$\sum_{a'<a} H_{a'} + \sum_{a'<b} H_{a'} > \sum_{a'<a+b} H_{a'} \tag{A.5}$$

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