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Journal Article

Author(s):
Gemünden, Thomas; Keller, Christoph A.

Publication date:
2021-11-01

Permanent link:
https://doi.org/https://doi.org/10.3929/ethz-b-000493946

Originally published in:
Journal of Algebra 585, https://doi.org/10.1016/j.jalgebra.2021.06.023

Journal of Algebra 585 (2021) 656–696

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Non-abelian orbifolds of lattice vertex operator algebras

Thomas Gemünden [a,*], Christoph A. Keller [b]

[a] *Department of Mathematics, ETH Zurich, CH-8092 Zurich, Switzerland*
[b] *Department of Mathematics, University of Arizona, Tucson, AZ 85721-0089, USA*

A R T I C L E I N F O

Article history:
Received 22 February 2021
Available online 30 June 2021
Communicated by Jonathan Kujawa

MSC:
17B69
81T40
18M20

Keywords:
Vertex operator algebras
Orbifold theory
Conformal packing
Conformal field theory

A B S T R A C T

We construct orbifolds of holomorphic lattice vertex operator algebras for non-abelian finite automorphism groups G. To this end, we construct twisted modules for automorphisms g together with the projective representation of the centralizer of g on the twisted module. This allows us to extract the irreducible modules of the fixed-point VOA V^G, and to compute their characters and modular transformation properties. We then construct holomorphic VOAs by adjoining such modules to V^G. Applying these methods to extremal lattices in $d = 48$ and $d = 72$, we construct more than fifty new holomorphic VOAs of central charge 48 and 72, many of which have a very small number of light states.

1. Introduction

In this article, we continue our exploration of holomorphic vertex operator algebras (VOA) of central charge $c = 48$ and $c = 72$. On the one hand, we are motivated by the fact that not too many examples of such VOAs have been constructed. On the other

* Corresponding author.
E-mail addresses: thomas.gemuenden@math.ethz.ch (T. Gemünden), christoph.keller@math.arizona.edu (C.A. Keller).

https://doi.org/10.1016/j.jalgebra.2021.06.023
0021-8693/

hand, we are motivated by the question of the existence of extremal VOAs [1–3], and more generally by the physics of holography, which requires VOAs with 'a small number of light states' [4]. This means that for a VOA

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} \ , \tag{1.1}$$

we want the $V_{(n)}$ to have a small dimension for the first few integers n. At $c = 48$ for instance, an extremal VOA has the following dimensions:

$$\dim V_{(1)} = 0 \ , \qquad \dim V_{(2)} = 1 \ . \tag{1.2}$$

At $c = 72$ an extremal VOA in addition satisfies $\dim V_{(3)} = 1$. These are the smallest dimensions compatible with the existence of the conformal vector and modular invariance of the character of the vacuum module, as is necessary for holomorphic VOAs. It is still an open question if such VOAs exist for central charge 48 or higher. More generally it is difficult to construct holomorphic VOAs with $\dim V_{(n)}$ small: most constructions give relatively large dimensions, or more precisely, dimensions that grow very quickly when the central charge increases. For lattice VOAs for instance, the dimensions grow at least polynomially in the rank of the lattice. We, therefore, need a way to decrease the dimensions of the $V_{(n)}$, for instance by projecting out a large number of vectors. In the present work, we achieve this by orbifolding.

Starting with a VOA V, the first step in orbifolding is to pick an automorphism group $G < \mathrm{Aut}(V)$, and restrict to the sub-VOA V^G which is invariant under G. By the combined results of [5–7], if V is strongly rational and holomorphic, and G is a finite solvable group, then V^G is again strongly rational. It is however no longer holomorphic: it will, in general, have finitely many irreducible modules W_i, which can, in fact, be obtained from the twisted modules of V [8–11]. The next step is thus to try to find a new holomorphic VOA by considering holomorphic extensions of V^G: that is, to adjoin an appropriate set of V^G-modules to V^G to recover a holomorphic VOA given by $V^{orb(G)} = V^G \oplus_i W_i$. By the results of [12], this is possible only if a certain 3-cocycle in the modular tensor category of the modules of V^G is trivial. To construct the actual VOA structure on $V^{orb(G)}$, an understanding of intertwining operators for twisted modules is needed. This question was addressed for the case when the automorphisms of different twisted modules commute [13]. More recently a lot of progress has been made in [14,15], which in particular gave the correct construction for the general, non-commuting case. See [16] for a nice overview on the status of orbifold VOAs. In this work, however, we will content ourselves with computing the character of $V^{orb(G)}$.

Following this overall approach, in [17] we started with VOAs based on extremal lattices of dimensions $d = 48$ and 72 [18–21], and picked the G to be cyclic groups. In this way, we constructed more than a hundred new examples of holomorphic VOAs with few light vectors. Although we did not find an extremal VOA, we found VOAs with much smaller dimensions than for unorbifolded lattice VOAs. Our construction of

$V^{orb(G)}$ in these cases was based on the theory of cyclic orbifolds, which was completely worked out in [22]. In particular, we could avoid constructing the twisted modules of V explicitly: their characters could be obtained from the known modular transformation of the characters of untwisted modules.

In this article, we return to the same lattice VOAs, but now orbifold by non-cyclic and even non-abelian automorphism groups G. In this way, we construct about fifty new VOAs, some of which improve significantly on the previously obtained ones in that their dimensions are much lower. For $c = 72$, as in [17], we can state one result of our constructions in the following form:

Theorem 1.1. *There exists a strongly rational holomorphic VOA with central charge $c = 72$ and*

$$\dim V_{(1)} = 0\ , \qquad \dim V_{(2)} = 12\ , \qquad \dim V_{(3)} = 200\ .$$

This VOA cannot be constructed as an abelian orbifold of a lattice VOA.

Note that the VOA in Theorem 1.1 improves over the VOA in the analogous theorem in [17], for which the dimensions were larger. We prove that this VOA cannot be obtained from an abelian orbifold of a lattice VOA by giving a lower bound on $\dim V_{(2)}$ for all such abelian orbifolds; it is thus a genuinely non-abelian construction. This indicates that the situation in $c = 72$ is different from $c = 24$, where all known 71 holomorphic VOAs can be constructed as cyclic lattice orbifolds [23–25].

For $c = 48$ we find several lattice orbifold VOAs with

$$\dim V_{(1)} = 0\ , \qquad \dim V_{(2)} = 48\ .$$

In [17] we found cyclic orbifolds which have the same character. We thus find no non-cyclic orbifolds that improve over cyclic orbifolds. One might of course try to improve by considering other lattices and their orbifolds. However, [26] suggests that other extremal lattices, if they exist, have smaller automorphism groups; [27] gives a similar result for Γ_{72}. Using non-extremal lattices on the other hand leads to larger dimensions coming from the short vectors of the lattice, which need to be eliminated by a powerful enough orbifold group. This gives at least some evidence that $\dim V_{(2)} = 48$ may be the optimal lattice orbifold result for $c = 48$ and that our $c = 72$ constructions may be fairly close to optimal for lattice orbifolds. Here of course we leave completely open the possibility of an altogether different construction that could lead to better results.

In the remainder of the introduction, let us describe our algorithm for constructing orbifolds of lattice VOAs in more detail and point out the key ingredients. To construct non-cyclic and non-abelian orbifolds of lattice VOAs, we proceed in several steps, relying on various results. The first step is to find a suitable subgroup of $\mathrm{Aut}(V_L)$ with which to orbifold. For this, we start with a group of lattice automorphisms $G < \mathrm{Aut}(L)$. The elements g of G then need to be lifted by a lift ι to $\iota(g) \in \mathrm{Aut}(V_L)$ [28,29]. In practice, we find it useful to focus on cases where the lifted group $\langle\iota(G)\rangle$ is isomorphic to the original group G. We give an explicit construction for ι.

The next step is to construct the g-twisted V_L-modules W_g. For this, we use the results of [30] and [31]. The central result here is that the full g-twisted module W_g can be constructed as an induced representation from a finite dimensional projective representation Ω_0 of a certain finite abelian group N.

Next we need to compute the action of the centralizer $C_G(g)$ on W_g: for general reasons we know that there is a projective representation $\phi_g(\cdot)$ of $C_G(g)$ on W_g [9,32,10]. To obtain the twining characters $T(g,h;\tau) = \mathrm{tr}_{W_g}\phi_g(h)q^{L_0-c/24}$ however, we need to construct ϕ_g explicitly. Because we know how $\phi_g(\cdot)$ acts on the twisted vertex operators, it is enough to compute the restriction $\phi_g(\cdot)|_{\Omega_0}$ to the finite dimensional space Ω_0. We achieve this by using Schur's lemma and the explicit representation of N on Ω_0. We then use $\phi_g(\cdot)$ to extract the irreducible V^G-modules from the twisted V-modules W_g using [9–11], and compute their characters explicitly, which allows us to read off their modular transformation properties.

In the final step, we extend V^G to a holomorphic VOA $V^{orb(G)}$ by adjoining a suitable set of the irreducible V^G-modules that we constructed in the previous step. This is not always possible: we need to use the results of [12] to check if there is an obstruction to this step, that is if in physics language the orbifold is 'anomalous'. The central idea is that the irreducible modules of V^G form a modular tensor category given by a Drinfeld double $D_\omega(G)$ twisted by some 3-cocycle $\omega \in H^3(G,U(1))$. The result of [12] then tells us that holomorphic extensions $V^{orb(G)}$ only exist if ω is trivial, and it also gives us the characters of all possible extensions. The condition that ω is trivial we can check from the modular transformation properties of the characters of the modules constructed above.

For many cases, it is not necessary to go through this whole procedure. If G is cyclic, then one can of course simply use the general theory of cyclic orbifolds worked out in [22], which simplifies things considerably: First of all, it is then unnecessary to construct the twisted modules and their twining characters, as it is enough to know the twining characters of the untwisted module, from which all other twining characters can be obtained by modular transformations [22,17]. Second, the condition $[\omega] = 1$ reduces to a simple condition on the conformal weight of the twisted module, either called the type-0 condition or the level-matching condition.

What we observe in this article is that in conjunction with [12], the cyclic methods of [22] can be applied not just to cyclic orbifold groups, but to a much larger class of non-abelian groups: We say G gives a *generalized cyclic* orbifold if for every pair $(a \in G, b \in C_G(a))$, we can find a cyclic subgroup $C < G$ such that $a, b \in C$, and moreover if the level-matching condition guarantees the vanishing of the obstruction to orbifolding. For such a G we can compute all twisted twining characters by considering orbifolds by various cyclic groups C. For generalized cyclic orbifolds, we, therefore, do not need to construct the twisted modules and their twining characters, and it is also much easier to check for holomorphic extensions. A large class of generalized cyclic orbifolds come from metacyclic groups of the form

$$G = \mathbb{Z}_q \rtimes \mathbb{Z}_p \ . \tag{1.3}$$

There are of course orbifolds that are not generalized cyclic, in which case we need our full algorithm: the simplest example is $\mathbb{Z}_2 \times \mathbb{Z}_2$, where any pair of distinct, non-identity elements (g_1, g_2) can not be embedded in a cyclic group. We illustrate this example and other examples that are not generalized cyclic for the smallest holomorphic VOA, namely the E_8 lattice VOA. Any orbifold of this VOA is of course highly constrained, since any holomorphic extension has to return the original E_8 VOA, which is the only holomorphic VOA of that central charge. It is therefore not at all surprising that only 14 subgroups of $\mathrm{Aut}(L_{E_8})$ have a so-called standard lift that gives trivial ω, all of which are abelian, and most of which are cyclic. We construct the orbifold for the smallest non-cyclic such group $\mathbb{Z}_3 \times \mathbb{Z}_3$. We note that $H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, U(1))$ is not trivial, which allows for the introduction of discrete torsion parametrized by $\psi \in H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, U(1))$ [12]. We find however that the contribution to the character controlled by the discrete torsion vanishes. This is necessary since every choice of ψ must return the original E_8 VOA, which is exactly what we find. To illustrate our methods, we also construct the twisted twining characters for some groups with obstructions, namely an S_3 and a $\mathbb{Z}_2 \times \mathbb{Z}_2$, and check explicitly that the modular transformations of their characters agree with the expected modular data.

This article is organized in the following way. In section 2, we briefly discuss the general theory of orbifolds and their extensions, and how it applies to cyclic orbifolds. In section 3 we introduce the notion of a generalized cyclic orbifold and discuss classes of examples of the form $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$. In section 4 we discuss the lifting of lattice automorphisms to VOA automorphisms. We discuss the lifting of a large class of non-abelian groups and give a construction of a splitting map. In section 5 we work out the construction of twisted modules of lattice VOAs, focusing on the construction of the defect representation. In section 6 we construct the projective representation ϕ_g of $C_G(g)$, which allows us to construct the twining characters of all twisted modules. In section 7 we apply these methods to some orbifolds of the E_8 lattice VOA. In section 8 we finally apply our methods to our main interest: constructing new holomorphic VOAs for $c = 48$ and 72. We construct a large number of generalized cyclic non-abelian orbifolds of extremal lattice VOAs. In particular, we construct the VOAs advertised in Theorem 1.1.

Acknowledgments: We thank Terry Gannon, Klaus Lux, Sven Möller and Nils Scheithauer for useful discussions. We thank Gerald Höhn and Yi-Zhi Huang for helpful discussions and comments on the draft. TG thanks the Department of Mathematics at University of Arizona for hospitality. We thank two anonymous referees for helpful comments on the draft. The work of TG is supported by the Swiss National Science Foundation Project Grant 175494.

2. Twisted modules, orbifolds and extensions

We want to orbifold a strongly rational holomorphic VOA V by some finite subgroup $G < \mathrm{Aut}(V)$ of its automorphisms. We call a VOA V *strongly rational* if it is rational,

C_2-cofinite, simple, self-contragredient and of CFT-type. The significance of this is that [33] established that the fusion rules for the modules of a strongly rational VOA satisfy the Verlinde formula [34] and hence that the modules form a modular tensor category [35]. By the combined results of [5–7], if G is a finite solvable group and V is strongly rational, then V^G is again strongly rational. This is believed to hold even if G is finite but not solvable; in the present work, however, G will always be solvable. If G is a non-trivial group, V^G will not be holomorphic. To construct new holomorphic VOAs $V^{orb(G)}$, we, therefore, need to find holomorphic extensions of V^G by adjoining a suitable set of its modules. In particular by [36] if V^G is strongly rational, its extensions are equivalent to haploid algebras with trivial twist over its module category. Hence we will need to study the module category $V^G - \text{mod}$ and understand, in particular, its simple objects and tensor category structure. To this end, it is necessary to introduce the notion of twisted modules.

2.1. Twisted modules, Schur-Weyl type duality and characters

Given a VOA V and an automorphism $g \in \text{Aut}(V)$ of order N, a g-twisted V-module W_g consists of a $\mathbb{C}$-graded vector space $W_g = \bigoplus_{\lambda\in\mathbb{C}} W_\lambda$ together with a linear map $Y_W : V \to (End(W))[[x^{1/N}, x^{-1/N}]]$ satisfying various axioms which can be found in [9]. In the following, the most important one is *twisting compatibility*, which states that

$$Y_W(v, x) = \sum_{k\in -r/N+\mathbb{Z}} v_k^W x^{-k-1} \text{ if } gv = \exp(2\pi i r/N)v \ , \qquad r = 0, \ldots N-1 \ . \tag{2.1}$$

There is then, of course, also the usual notion of an irreducible twisted module. We will denote by ρ_g the conformal weight of W_g.

Next we consider the action of a commuting automorphism h on g-twisted modules. Let us now take V to be a holomorphic C_2-cofinite VOA, g and h automorphisms of V such that $gh = hg$ and $(W_g, Y_W(\cdot, x))$ the unique irreducible g-twisted V-module. Then by the results of [9,32,10] there exists a linear map $\phi_g(h) : W_g \to W_g$ such that

$$\phi_g(h)Y_W(v, x)\phi_g(h)^{-1} = Y_W(hv, x) \ , \tag{2.2}$$

which is unique up to multiplication by a scalar. $\phi_g(\cdot)$ is a projective representation of the centralizer $C_G(g)$ of g with some cocycle $c_g(\cdot, \cdot)$.

The projective representation ϕ_g is useful to obtain the irreducible modules of the fixed point VOA V^G. From [9] we know that W_g decomposes into modules for $\mathbb{C}_{c_g}[C_G(g)]\otimes V^G$ as $C_G(g)$ through a Schur-Weyl duality

$$W_g = \bigoplus_{\chi\in \text{Irr}_{c_g}(C_G(g))} V_{[g,\chi]} \otimes W_{[g,\chi]}, \tag{2.3}$$

where Irr_{c_g} denotes the set of irreducible projective characters with 2-cocycle c_g, $V_{[g,\chi]}$ denotes the irreducible projective representation of the centralizer $C_G(g)$ with character

χ and $W_{[g,\chi]}$ is the corresponding irreducible module for the fixed-point subalgebra V^G. [10,11] then establishes that all irreducible modules of V^G are given by $V_{[g,\chi]}$ for some conjugacy class g and some projective irreducible representation χ of $C_G(g)$. In summary, the irreducible modules are labeled by $[g,\chi]$ and can be obtained from the twisted modules using the projector

$$\pi_\chi = \frac{1}{|C_G(g)|}\sum_{h\in C_G(g)} \overline{\chi}(h)\phi_g(h). \tag{2.4}$$

Note that this operator is independent of the choice of action on W_g: if we choose the action to be $\lambda(h)\phi_g(h)$ for some function $\lambda : C_G(g) \to U(1)$ instead, then the corresponding character becomes $\lambda(h)\chi(h)$, giving the same projection operator.

As an application of orbifold theory, we are interested in the construction of holomorphic, strongly rational VOAs with a small number of low-weight states. Hence, to test for this property, we will need to compute the characters of our new holomorphic extensions. The character of a (twisted) V-module W is given by

$$\mathrm{ch}_W(\tau) = \mathrm{tr}_{W_g} q^{L_0 - c/24}. \tag{2.5}$$

From Equations (2.3) and (2.4) it is clear that we will be able to express the character of any V^G-module and therefore of its holomorphic extensions in terms of the following twisted twining characters

$$T(g,h;\tau) = \mathrm{tr}_{W_g}\phi_g(h) q^{L_0 - c/24}. \tag{2.6}$$

In the most general case, determining these requires the explicit construction of the twisted modules and the action of automorphisms on them. However, we can greatly simplify computations by making use of the modular transformation properties of twining characters. In particular by [9], the graded trace functions transform under the action of a modular transformation $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ as

$$T(g,h;M\cdot\tau) = \sigma(g,h,M)T((g,h)\cdot M;\tau), \tag{2.7}$$

where $(g,h)\cdot M = (g^\alpha h^\gamma, g^\beta h^\delta)$ and the constants $\sigma(g,h,M)$ depend only on g, h and M and are determined by the choice of the action ϕ_g. Assuming we can compute these modular transformations, we now only need to construct one twining character for every orbit of characters under the modular group. In particular, if an orbit intersects the untwisted twining characters, we can determine its elements without having to construct twisted modules or the actions of automorphisms on them first. In the later parts of this paper, we will be particularly interested in groups for which every orbit intersects the untwisted twining characters.

2.2. MTCs and holomorphic extensions

The theory of holomorphic extensions of V^G was established in [12]. If V is strongly rational and holomorphic, then V^G is again strongly rational, but no longer holomorphic. Its irreducible modules form a modular tensor category governed by a 3-cocycle $\omega \in H^3(G;\mathbb{C}^*)$, isomorphic to a twisted Drinfeld double of the group G, $D^\omega(G)$-mod. This was originally proposed in [37], following up on work on the operator algebra of general orbifolds [38]. It has been established for ω trivial [39] and is believed to hold in general. For all orbifold VOAs $V^{orb(G)}$ we construct, ω will always be trivial, for reasons outlined below.

The cocycle ω then determines both the projective representations of the twisted modules, and the modular transformation properties of their characters: Denote $g^h = h^{-1}gh$, ${}^hg = hgh^{-1}$. From ω we obtain a family of two cocycles $c_g, g \in G$ via

$$c_g(h_1,h_2) = \omega(g,h_1,h_2)\omega(h_1,h_2,g^{h_1h_2})\bar{\omega}(h_1,g^{h_1},h_2). \tag{2.8}$$

In particular, we will assume in the following that the projective representation ϕ_g in the g-twisted module is chosen such that its 2-cocycle is given by (2.8). This, in turn, determines the modular transformation properties of the characters of $[a,\chi]$ through the S and T matrices

$$S_{[a_1,\chi_1],[a_2,\chi_2]} = \frac{1}{|G|}\sum_{g_i\in cl(a_i), g_1g_2=g_2g_1} \bar{\chi}_1(h_1)\bar{\chi}_2(h_2)\frac{c_{g_1}(k_1^{-1},h_1)c_{g_2}(k_2^{-1},h_2)}{c_{g_1}(g_2,k_1^{-1})c_{g_2}(g_1,k_2^{-1})}, \tag{2.9}$$

$$T_{[a_1,\chi_1],[a_2,\chi_2]} = e^{-2\pi ic/24}\delta_{[a_1,\chi_1],[a_2,\chi_2]}\chi_1(a_1)/\chi_1(1). \tag{2.10}$$

Here we define k_i through $g_i = a_i^{k_i}$, and $h_1 := {}^{k_i}g_2, h_2 := {}^{k_2}g_1$.

As mentioned above, V^G is strongly rational, but not holomorphic. Our goal is to extend V^G to a holomorphic VOA $V^{orb(G)}$ by adjoining a suitable set of irreducible V^G-modules $[a,\chi]$ to V^G. This is not always possible, as there can be obstructions. In physics, such cases are called ‘anomalous’. The theory of such extensions is described in [12]. Holomorphic extensions are given by Corollary 2 of [12]: they occur if ω is trivial. In such a situation we can also choose ‘discrete torsion’ $\psi \in Z^2(G,U(1))$. The extension is given by

$$V^{orb(G),\psi} = \bigoplus_{k\in Cl(G)} W_{[k,\beta_k^\psi]}, \tag{2.11}$$

where $\beta_g^\psi(h) = \psi(g,h^g)\bar{\psi}(h,g)$. (Note that [12] allowed for a choice of subgroups $K < G$. We always choose $K = G$, since we will consider all possible subgroups of the automorphism group anyway.) If $H^2(G,U(1))$ is non-trivial, we can thus recover several possibly non-isomorphic extensions. In all cases we consider however $H^2(G,U(1))$ is either trivial or does not lead to different VOAs, so that we will drop ψ in our future notation and simply write $V^{orb(G)}$.

To have a holomorphic extension of V^G, it is thus necessary that ω is in the trivial class. We can then choose the representative $\omega = 1$, so that $c_g(\cdot,\cdot) = 1$, which means that the projective characters χ are actually linear. The extension with discrete torsion $\psi = 1$ then simply consists of the $C_G(g)$-invariant part of the g-twisted modules W_g,

$$V^{orb(G),1} = \bigoplus_{k\in Cl(G)} W_k^{C_G(k)} \ . \tag{2.12}$$

Moreover, the transformation matrices are given by

$$S_{[a_1,\chi_1],[a_2,\chi_2]} = \frac{1}{|G|} \sum_{g_i\in cl(a_i), g_1g_2=g_2g_1} \bar{\chi}_1(h_1)\bar{\chi}_2(h_2) \ , \tag{2.13}$$

$$T_{[a_1,\chi_1],[a_2,\chi_2]} = e^{-2\pi ic/24}\delta_{[a_1,\chi_1],[a_2,\chi_2]}\chi_1(a_1)/\chi_1(1) \ . \tag{2.14}$$

We can, in principle, determine the transformation matrices S and T by constructing all twisted modules and extracting the 2-cocycles c_g. We can then check if these data are compatible with $[\omega] = 1$. Note, however, that in [40] Schauenburg and Mignard showed that fusion data alone does not fix the cohomological twist. The problem of how to determine $[\omega]$ in practice is, in fact, unsolved.

Therefore, we will restrict ourselves to groups, for which the level-matching condition is sufficient to guarantee the vanishing of the cohomological obstruction as in the cyclic case. More advanced methods of determining the cohomological obstruction are discussed in [41].

2.3. Cyclic orbifolds

In this section, we will briefly review cyclic orbifolds in our tensor categorical framework. The theory of cyclic orbifolds was fully developed in [22] using a purely vertex operator algebraic approach without making assumptions about the tensor category structure of the module categories. Let a be a generator of $\mathbb{Z}_N$. The cohomology of $\mathbb{Z}_N$ is given by

$$H^3(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N \tag{2.15}$$

and is generated by the 3-cocycle

$$\omega(a^j, a^k, a^l) = \exp\left(\frac{2\pi i}{N^2} j[k + l - \langle k + l\rangle_N]\right) \ , \tag{2.16}$$

where $\langle\cdot\rangle_N$ denotes reduction modulo N. A general 3-cocycle ω_p, $p = 0, \dots N - 1$,

$$\omega_p(a^j, a^k, a^l) = \exp\big(\frac{2\pi i p}{N^2} j[k + l - \langle k + l\rangle_N]\big) \ , \tag{2.17}$$

descends to the 2-cocycles

$$c_{a^j}(a^k, a^l) = \exp\big(\frac{2\pi i p}{N^2} j[k + l - \langle k + l\rangle_N]\big) \ . \tag{2.18}$$

For $[a^j, \chi_l]$ the projective characters with cocycle (2.18) are $\chi_l(a^k) = e^{2\pi i (pjk/N + lk)/N}$, so that the modular data is given by

$$T_{[a^j,\chi_l],[a^j,\chi_l]} = e^{-2\pi i c/24} \exp(2\pi i (pj^2 + Njl)/N^2) \tag{2.19}$$

and

$$S_{[g_1,\chi_1],[g_2,\chi_2]} = \frac{1}{N} \bar{\chi}_1(g_2)\bar{\chi}_2(g_1) \ . \tag{2.20}$$

From these equations, it follows that for a cyclic group of order N the cohomological twist vanishes if and only if the T-matrix is of order N. Then the vanishing of the cohomological obstruction is also equivalent to all irreducible V^G-modules being $\frac{1}{N}\mathbb{Z}$-graded. In the physics literature this condition is therefore referred to as the *level-matching* condition. Furthermore, the type $r \in \mathbb{Z}_N$ of the orbifold is defined as

$$r = N^2 \rho_a \mod N \ , \tag{2.21}$$

where ρ_a is the conformal weight of the a-twisted module W_a. [22] showed that ω is also trivial if and only if $r = 0$. This is referred to as the type-0 condition. It is clear that for general orbifolds, the level-matching (or equivalently type-0) condition is necessary, but not sufficient for the vanishing of the cohomological twist. For the remainder of this paper, we will be particularly interested in orbifolds for which level-matching does guarantee the vanishing of the cohomological twist. We will refer to such orbifolds as satisfying the *level-matching property*.

By our discussion at the end of subsection 2.2, in order to calculate characters of cyclic orbifolds, we need to construct the twisted twining characters $T(g, h; \tau)$ for any $g, h \in \mathbb{Z}_n$ and understand their modular transformation properties. [22] showed that any twisted twining character of the form $T(g, h; \tau)$ can be obtained from the untwisted characters $T(e, k; \tau)$ by using the modular transformations

$$T(g, h; T \cdot \tau) = \sigma(g, h, T)T(g, gh; \tau) \tag{2.22}$$

and

$$T(g, h; S \cdot \tau) = \sigma(g, h, S)T(h, g^{-1}; \tau) \ , \tag{2.23}$$

where the complex numbers $\sigma(g, h) = 1$ can then be fixed by the type r of the orbifold for a suitable choice of the action of automorphisms on twisted modules. We will refer to this property of cyclic orbifolds as the *character property*. For a detailed discussion also refer to [17].

Note in particular, that we can always infer the conformal weight ρ_g of a g-twisted module from the modular transformation properties of untwisted twining characters. Hence it is possible to prove existence and compute characters of cyclic orbifolds without having to construct twisted modules at all.

3. Generalized cyclic orbifolds

3.1. Generalized cyclic orbifolds

In subsection 2.3, we explained that for cyclic orbifolds, we can infer the vanishing of the cohomological twist from the level-matching condition alone and we can construct characters from untwisted twining characters using their modular transformation properties. We are interested in orbifolds by non-cyclic groups, which means that in general, we need to explicitly construct twisted modules. We will discuss how to do this for lattice orbifolds in section 5. However, in this section we point out that the character and level-matching properties mentioned above apply to a larger class of orbifold groups: we will call such orbifolds *generalized cyclic*. Note that in the case of non-cyclic groups, we say that G satisfies the level-matching condition if it holds for all cyclic subgroups of G as defined in the previous section. From a practical point of view, these orbifolds are particularly attractive since proving their existence and calculating their characters is comparatively simple.

Let us begin with the character property. We would like to classify all finite groups G such that for any two commuting elements $g, h \in G$, there exists an $k \in G$ and $M \in \mathrm{SL}(2, \mathbb{Z})$ such that

$$T(g, h; \tau) = \sigma(g, h, M) T(e, k; M \cdot \tau) \tag{3.1}$$

for some complex numbers $\sigma(g, h, M)$. From equation (2.7), it is easy to see that G has this property if and only if any two commuting elements $g, h \in G$ generate a cyclic subgroup which in turn is equivalent to G being a group with periodic cohomology [42,43]. Note that such groups have been fully classified and constructed [44]. In particular, a group has periodic cohomology if and only if for every odd prime p, the Sylow p-subgroup is cyclic and the Sylow 2-subgroup is either cyclic or a generalized quaternion group.

Note that groups with periodic cohomology have trivial $H^2(G, U(1))$. This agrees with the fact that generalized cyclic orbifolds cannot have discrete torsion, since their characters are completely fixed by the untwisted modules. Note that the converse however is not true: The group $(\mathbb{Z}_7 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_3$, where $\mathbb{Z}_3$ acts on both $\mathbb{Z}_7$-factors by squaring, has trivial $H^2(G, U(1))$, but does not give a generalized cyclic orbifold.

Next, we would like to find groups for which level-matching is sufficient to guarantee the vanishing of the cohomological twist. Unfortunately, in this case, obtaining a full classification does not seem feasible, so we will only construct a subset of these groups. To do so we will need the following classical result on detecting group cohomology from subgroups:

Lemma 3.1. *Let G be a finite group and let $S \subseteq G$ be a subgroup that contains a Sylow p-subgroup for some prime p. Let $H^k(G,\mathbb{C}^*)_{(p)}$ denote the p-primary part of the k-th cohomology group. Then the restriction map $\alpha \mapsto \alpha|_S : H^k(G,\mathbb{C}^*)_{(p)} \to H^k(S,\mathbb{C}^*)_{(p)}$ is an injection whose image is a direct summand.*

It follows immediately that a group G has the level-matching property if for every prime p dividing the order of G the corresponding Sylow p-subgroup is cyclic. Such groups are called Z-groups, and they have also been fully classified [45]. In fact, all Z-groups are metacyclic groups of the form $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, where p and q are coprime. Clearly, Z-groups also have periodic cohomology so that orbifolds by Z-groups are generalized cyclic orbifolds in our sense.

Note that generalized quaternion groups have cyclic third cohomology groups and it can readily be shown that they do not satisfy the level-matching property. Consequently, a group G whose Sylow 2-subgroup is a generalized quaternion group can only possess the level-matching property if $H^3(G,\mathbb{C}^*)_{(2)}$ is trivial. Hence by Lemma 3.1, a group G gives a generalized cyclic orbifold if and only if it is a Z-group or a group with periodic cohomology such that the 2-primary part of its cohomology $H^3(G,\mathbb{C}^*)_{(2)}$ is trivial.

3.2. Metacyclic orbifolds

The main class of such generalized cyclic orbifolds that we will discuss come from metacyclic groups of the form $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ defined by the relations

$$a^q = 1 \tag{3.2}$$

$$A^p = 1 \tag{3.3}$$

$$AaA^{-1} = a^\phi \ , \tag{3.4}$$

where $\phi \in \mathbb{Z}$ satisfies $\phi^p \equiv 1 \pmod q$ and $(\phi - 1)$ is coprime to q. If $\phi = 1$, then of course this simply gives the direct product. For p, q coprime these have periodic cohomology [44].

Now we want to describe the 3-cohomology classes for $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ where q and p are coprime. The third cohomology group for the group $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ such that q and p are coprime is given by

$$H^3(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p, U(1)) \cong \mathbb{Z}_p \times \mathbb{Z}_r \ , \tag{3.5}$$

where $r = (\phi^2 - 1, q)$ [46]. Then from the arguments of the previous section or using explicit expressions for the generators of H^3 it is straightforward to establish the following lemma:

Lemma 3.2. *For any coprime p and q, $H^3(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p, U(1))$ is cyclic, and $\omega \in H^3(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p, U(1))$ is trivial if and only if the type-0 condition is satisfied for the cyclic subgroups $\mathbb{Z}_p$ and $\mathbb{Z}_r$.*

For the groups listed in section 8, this lemma gives a simple criterion for determining whether the cohomological twist is trivial.

There is one other case which we will use later on: namely, if $p = q$, and $\phi = 1$. In that case, we recover the abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$, which does not lead to a generalized cyclic orbifold. Still, we can find an analogous lemma for the cohomological twist:

Lemma 3.3. *For abelian groups of the form $\mathbb{Z}_p \times \mathbb{Z}_p$, the H^3-twist ω vanishes if and only if the cyclic groups $\langle a\rangle, \langle A\rangle$ and $\langle aA\rangle$ are all of type 0.*

Proof. Direct computation using the explicit expressions for the generators of $H^3(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{C}^*) \cong \mathbb{Z}_p^3$ given *e.g.* in [47]. □

3.3. A character formula for metacyclic orbifolds

For future convenience, let us derive a character formula for metacyclic orbifolds in terms of cyclic orbifold characters. This is useful because it allows rapid computation using computer algebra. To this end, we classify the conjugacy classes of $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ and derive an explicit expression for the character of the holomorphic extension in terms of characters of orbifolds by cyclic subgroups. Note that for generalized cyclic orbifolds, we expect that the character can always be expressed in terms of characters of cyclic orbifolds. For simplicity, however, we will assume that q is prime. Analogous formulas can be easily derived for q non-prime.

Proposition 3.4. *Let q be prime. Assume that $\omega = 1$ for the orbifold $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$. The character of the orbifold $V^{orb(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}$ is given by*

$$\chi^{orb(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}(\tau) = \chi^{orb(\mathbb{Z}_p)}(\tau) - \frac{1}{r}(\chi^{orb(\mathbb{Z}_{\frac{p}{r}})}(\tau) - \chi^{orb(\mathbb{Z}_{\frac{pq}{r}})}(\tau)) , \tag{3.6}$$

where $r = (\phi^2 - 1, q)$.

Proof. Let $\mathrm{ch}_{W_g^G}(\tau)$ be the character of the G-invariant subspace of the irreducible g-twisted module. By [12] the orbifold character is given by $\chi^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}(\tau) = \sum_{g\in \mathrm{cl}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p))} \mathrm{ch}_{W_g^{C_G(g)}}(\tau)$. We write the orbifold group $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ in the presentation with generators a, A, B satisfying $a^q = 1, A^r = B, B^{\frac{p}{r}} = 1, AaA^{-1} = a^\phi, BaB^{-1} = a$. It has three types of conjugacy classes: 1) one class $[A^i]$ for every $i \in \mathbb{Z}_p$ with $r \nmid i$, with centralizer $C_G(A^i) = \mathbb{Z}_p$; 2) one class $[a^i B^j]$ for every pair (a^i, B^j) of a coset representative $a^i \in \frac{\mathbb{Z}_q^*}{\langle\phi\rangle_{\text{mult}}}$ (where $\langle\phi\rangle_{\text{mult}}$ denotes the multiplicative cyclic subgroup of $\mathbb{Z}_q^*$ generated by ϕ) and a (possibly identity) element $B^j \in \mathbb{Z}_{\frac{p}{r}}$, with centralizer $C_{\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p}(a^i B^j) = \mathbb{Z}_{\frac{pq}{r}}$; 3) one central class $[B^i]$ of length 1 for every element $B^i \in \mathbb{Z}_{\frac{p}{r}}$. To see this, note that the general conjugation of an element is given by

$$a^x A^y a^u A^v B^w A^{-y} a^{-x} = a^{u\phi^y - x(\phi^v - 1)} A^v B^w. \tag{3.7}$$

First take $v \neq 0$. Then $\phi^v - 1$ is invertible modulo q and the element $a^u A^v B^w$ is conjugate to $A^v B^w$ for any u. The centralizers of these representatives are exactly the group $\mathbb{Z}_p = \langle A\rangle$ so that the length of $[A^i]$ is q. Next take $v = 0$, $u \neq 0$. Then

$$a^x A^y a^u B^w A^{-y} a^{-x} = a^{u\phi^y} B^w \tag{3.8}$$

shows that there are $\frac{q-1}{r} \times \frac{p}{r}$ conjugacy classes representatives $a^u B^w$, $u \in \frac{\mathbb{Z}_q^*}{\langle\phi\rangle}_{\text{mult}}$. The centralizers of those conjugacy classes are generated by a and B and are therefore the cyclic group $\mathbb{Z}_{\frac{pq}{r}} = \langle aB\rangle$. Finally, $u = v = 0$ gives the $\frac{p}{r}$ central classes B^w whose centralizer is the entire group $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$.

It follows that we can write

$$\begin{aligned}\chi^{\text{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}(\tau) = \sum_{B^j \in \mathbb{Z}_{\frac{p}{r}}} \text{ch}_{W_{B^j}^{\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p}}(\tau) + \sum_{A^j, r\nmid j} \text{ch}_{W_{A^j}^{\mathbb{Z}_p}}(\tau) \\ + \frac{1}{r} \sum_{a^i B^j : a^i \in \mathbb{Z}_q^*, B^j \in \mathbb{Z}_{\frac{p}{r}}} \text{ch}_{W_{a^i B^j}^{\mathbb{Z}_{pq/r}}}(\tau)\ ,\end{aligned} \tag{3.9}$$

where in the last line we used the fact that $|\,\mathbb{Z}_q^* / \langle\phi\rangle_{\text{mult}}| = |\,\mathbb{Z}_q^*|/r$. To rewrite the character invariant under the entire group $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$, we use the fact that it has q maximal subgroups of order p that are all conjugate to $\mathbb{Z}_p = \langle A\rangle$, and one maximal subgroup $\mathbb{Z}_{\frac{pq}{r}} = \langle aB\rangle$ of order $\frac{pq}{r}$. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ is the union of all these maximal subgroups, and their pairwise intersections all equal to the center $\langle B\rangle$. Using the fact that the twining characters $T(g, h; \tau)$ only depend on the conjugacy class of h, we can thus decompose the first character as

$$\text{ch}_{W_{B^j}^{\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p}}(\tau) = \frac{1}{pq}\big(q \sum_{d=0}^{p-1} T(B^j, A^d; \tau) + \sum_{b=0}^{\frac{pq}{r}-1} T(B^j, (aB)^b; \tau)\big) - q \sum_{c=0}^{\frac{p}{r}-1} T(B^j, B^c; \tau) \tag{3.10}$$

$$= \text{ch}_{W_{B^j}^{\mathbb{Z}_p}}(\tau) - \frac{1}{r}\big(\text{ch}_{W_{B^j}^{\mathbb{Z}_{p/r}}}(\tau) - \text{ch}_{W_{B^j}^{\mathbb{Z}_{pq/r}}}(\tau)\big). \tag{3.11}$$

Hence

$$\chi^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}(\tau) = \sum_{B^j \in \mathbb{Z}_{\frac{p}{r}}} \mathrm{ch}_{W^{\mathbb{Z}_p}_{B^j}}(\tau) - \frac{1}{r}\big(\mathrm{ch}_{W^{\mathbb{Z}_{p/r}}_{B^j}}(\tau) - \mathrm{ch}_{W^{\mathbb{Z}_{pq/r}}_{B^j}}(\tau)\big) \tag{3.12}$$

$$+ \sum_{A^j, r\nmid j} \mathrm{ch}_{W^{\mathbb{Z}_p}_{A^j}}(\tau) + \frac{1}{r} \sum_{a^i B^j : a^i \in \mathbb{Z}_q^*, B^j \in \mathbb{Z}_{\frac{p}{r}}} \mathrm{ch}_{W^{\mathbb{Z}_{pq/r}}_{a^i B^j}}(\tau) \tag{3.13}$$

$$= \sum_{A^j \in \mathbb{Z}_p} \mathrm{ch}_{W^{\mathbb{Z}_p}_{A^j}}(\tau) + \frac{1}{r} \sum_{a^i B^j \in \mathbb{Z}_{\frac{pq}{r}}} \mathrm{ch}_{W^{\mathbb{Z}_{pq/r}}_{a^i B^j}}(\tau) - \frac{1}{r} \sum_{B^j \in \mathbb{Z}_{\frac{p}{r}}} \mathrm{ch}_{W^{\mathbb{Z}_{p/r}}_{B^j}}(\tau) \tag{3.14}$$

$$= \chi^{\mathrm{orb}(\mathbb{Z}_p)}(\tau) - \frac{1}{r}(\chi^{\mathrm{orb}(\mathbb{Z}_{\frac{p}{r}})}(\tau) - \chi^{\mathrm{orb}(\mathbb{Z}_{\frac{pq}{r}})}(\tau)). \quad \square \tag{3.15}$$

There are some cases in section 8 for which q is not prime. They turn out to be of the form $\mathbb{Z}_q \rtimes_{-1} \mathbb{Z}_4$, and it turns out that (3.6) is still correct for them.

4. Lifting theory for lattice VOAs

In this section, we will discuss the lifting of a finite subgroup G of $\mathrm{Aut}(L)$ to a finite subgroup $\hat{G}$ of $\mathrm{Aut}(V_L)$. In order to construct new orbifold VOAs, we need good control over the structure of the lifted group $\hat{G}$. It is, therefore, a natural goal to find a lift $\hat{G}$ that mostly retains the structure of G or, ideally, is isomorphic to it. We will consider this in general and prove that such well-behaved lifts exist for metacyclic groups.

4.1. Lifting map

Let us now discuss lattice VOAs and their automorphisms. Let L be an even lattice and $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ the ambient vector space. For an automorphism $g \in \mathrm{Aut}(L)$, define the fixed-point sublattice $L_g = L \cap \pi_g \mathfrak{h}$ and its orthogonal complement $L_g^\perp = L \cap (1 - \pi_g)\mathfrak{h}$, where π_g is the orthogonal projector onto the g-invariant subspace.

Then, as a vector space, the lattice VOA associated to L is given by

$$V_L \cong M_{\mathfrak{h}}(1,0) \otimes \hat{L}, \tag{4.1}$$

where $M_{\mathfrak{h}}(1,0)$ is the Heisenberg VOA associated to $\mathfrak{h}$ and $\hat{L}$ the twisted group algebra associated to L with product

$$\mathfrak{e}_\alpha \mathfrak{e}_\beta = \epsilon(\alpha, \beta)\mathfrak{e}_{\alpha+\beta}, \tag{4.2}$$

where $\epsilon : L \times L \to \{\pm 1\}$ is a 2-cocycle satisfying

$$\epsilon(\alpha, \alpha) = (-1)^{\langle \alpha, \alpha \rangle / 2} \tag{4.3}$$

and

$$\epsilon(\alpha,\beta)/\epsilon(\beta,\alpha) = (-1)^{\langle\alpha,\beta\rangle} \tag{4.4}$$

for $\alpha, \beta \in L$. We choose this 2-cocycle to be the bimultiplicative function $\epsilon : L \times L \to \{\pm 1\}$ defined by

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} (-1)^{\langle \alpha_i | \alpha_j \rangle} & \text{if } i > j \\ (-1)^{\langle \alpha_i | \alpha_i \rangle /2} & \text{if } i = j \\ 1 & \text{if } i < j, \end{cases} \tag{4.5}$$

where $\{\alpha_1, \ldots, \alpha_d\}$ is a basis for L.

The idea is to obtain automorphisms of the lattice VOA V_L by lifting automorphisms of the underlying lattice L. In what follows we will denote VOA automorphisms with hatted letters and the underlying lattice automorphism with unhatted letters. From [29] we know that for lattice VOAs, $\mathrm{Aut}(\hat{L})$ and $\mathrm{Aut}(L)$ are related by

$$1 \to \mathrm{Hom}(L, \mathbb{C}^*) \to \mathrm{Aut}(\hat{L}) \to \mathrm{Aut}(L) \to 1 \ . \tag{4.6}$$

$\mathrm{Aut}(V_L)$ is then obtained by extending $\mathrm{Aut}(\hat{L})$ by inner automorphisms. We will be considering extremal lattices where there are no inner automorphisms coming from lattice vectors, so that in this case $\mathrm{Aut}(V_L) = \mathrm{Aut}(\hat{L})$. To proceed, we pick a subgroup $G < \mathrm{Aut}(L)$ to get the short exact sequence

$$1 \to \mathrm{Hom}(L, \mathbb{C}^*) \to \hat{G} \to G \to 1 \ , \tag{4.7}$$

with $\hat{G} < \mathrm{Aut}(\hat{L})$. To lift G, we construct a lift

$$\iota : G \to \hat{G} \qquad g \mapsto \hat{g}, \tag{4.8}$$

where $\hat{g} \in \mathrm{Aut}(\hat{L})$ acts on $\hat{L}$ as

$$\hat{g} e_\alpha = \eta_g(\alpha)^{-1} e_{\alpha g} \ . \tag{4.9}$$

Here $\eta_g(\cdot)$ is a function from L to $\mathbb{C}^*$ satisfying

$$\frac{\eta_g(\alpha)\eta_g(\beta)}{\eta_g(\alpha+\beta)} = \frac{\epsilon(\alpha,\beta)}{\epsilon(g\alpha, g\beta)} \ . \tag{4.10}$$

This condition ensures that $\hat{g}$ is indeed an automorphism of the VOA. Note that $\frac{\epsilon(\alpha,\beta)}{\epsilon(\alpha g,\beta g)} \in B^2(L, \mathbb{C}^*)$ is a 2-coboundary so that a function $\eta_g : L \to \mathbb{C}^*$ satisfying equation (4.10) always exists. One solution can be constructed as follows: Let $\{\alpha_1, \ldots, \alpha_d\}$ be a basis for L and let $\epsilon_g : L \times L \to \{\pm 1\}$ be the bimultiplicative function defined by

$$\epsilon_g(\alpha_i, \alpha_j) = \begin{cases} \frac{\epsilon(\alpha_i,\alpha_j)}{\epsilon(\alpha_i g,\alpha_j g)} & \text{if } i > j \\ 1 & \text{otherwise} \ . \end{cases} \tag{4.11}$$

Then $\eta_g(\alpha) = \epsilon_g(\alpha, \alpha)$ satisfies Equation (4.10).

This η_g is not unique: for any $\xi_g \in \mathrm{Hom}(L, \mathbb{C}^*)$, $\xi_g \eta_g$ again satisfies (4.10). Furthermore, it is clear that any two solutions to (4.10) differ by an element of $\mathrm{Hom}(L, \mathbb{C}^*)$. A general element of $\mathrm{Aut}(\hat{L})$ is therefore of the form $\hat{g}\xi_g$ with action on $\hat{L}$ given by

$$\hat{g}\xi_g e_\alpha = \xi_g(\alpha)\eta_g(\alpha)^{-1} e_{\alpha g} \ . \tag{4.12}$$

The group multiplication of lifted lattice automorphisms is given by

$$\hat{g}\hat{h} = \widehat{gh} s(g, h), \tag{4.13}$$

where

$$s(g, h) = \frac{\eta_h(\cdot)^{-1}\eta_g(h\cdot)^{-1}}{\eta_{gh}(\cdot)^{-1}}. \tag{4.14}$$

It can be readily verified that $s(g, h)$ is indeed an element of $\mathrm{Hom}(L, \mathbb{C}^*)$. In practice, we can describe $\xi_g \in \mathrm{Hom}(L, \mathbb{C}^*)$ by the choice of a vector v_g such that $\xi_g(\alpha) = e^{2\pi i\langle \alpha, v_g\rangle}$.

Denoting by L_g the fixed point lattice of g, we call a lift such that

$$\eta_g(\alpha) = 1 \ , \qquad \alpha \in L_g \ , \tag{4.15}$$

a *standard lift*. In what follows, we will always work with such standard lifts. Given an arbitrary lift η_g, we can turn it into a standard lift $\tilde{\eta}_g$ as follows: By equation (4.10), the restriction $\eta_g|_{L_g}$ to the fixed-point lattice is a homomorphism. Hence for any $\xi_g \in \mathrm{Hom}(L, \mathbb{C}^*)$ such that $\xi_g|_{L_g} = \eta_g|_{L_g}^{-1}$ the lift $\xi_g(\cdot)\eta_g(\cdot)$ is a standard lift. In particular, note that by Lemma 5.1 any basis of L_g can be extended to a basis of L and hence such a ξ_g exists. Often the mere existence of a standard lift is enough for the construction of twisted module characters; in particular, it turns out that for the construction described in section 3, we only need the action on the fixed point lattice.

At the end of the day, we are not only interested in the lifted elements $\iota(g)$ by themselves, but in the group $\langle\iota(G)\rangle$ that they generate. We will be particularly interested in the case where ι splits the short exact sequence (4.7) so that the lifted group $\iota(G)$ is isomorphic to G. If ι is not splitting, we will have to work with the lifted group $\langle\iota(G)\rangle < \hat{G}$ instead, which is bigger than G. A typical case when this can occur is if G is cyclic, generated by an element g of even order. If we insist on satisfying (4.15), it is then sometimes necessary to define $\eta_g(\cdot)$ such that the order of $\hat{g}$ is doubled, $ord(\hat{g}) = 2ord(g)$ [22].

4.2. Lifting metacyclic groups

Finding a lift that splits is particularly important if G yields a generalized cyclic orbifold and we want $\iota(G)$ to retain this property. In view of section 3, let us therefore discuss the lifting theory of groups of the form

$$G = \mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p \ . \tag{4.16}$$

In Appendix A we discuss how to construct a splitting lift such that

$$\iota(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p) \cong \mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p \ . \tag{4.17}$$

The outcome is that in principle there can be an obstruction to such a splitting lift. We will work with lifts that leave the order of the generators invariant, such that $\hat{g}$ and $\hat{h}$ still satisfy (3.3) and (3.2). To ensure (3.4), we use that the relation $\hat{g}\hat{h}\hat{g}^{-1} = \hat{h}^\phi$ is equivalent to η_g, η_h satisfying

$$\eta_g(\alpha g^{-1}h)\overline{\eta}_g(\alpha g^{-1})\eta_h(\alpha g^{-1}) \prod_{i=0}^{\phi-1} \overline{\eta}_h(\alpha h^i) = 1. \tag{4.18}$$

To obtain such η_g, η_h, we construct homomorphisms ξ_g, ξ_h such that the new lifts $\tilde{\eta}_g = \eta_g \xi_g$ and $\tilde{\eta}_h = \eta_h \xi_h$ do satisfy Equation (A.6). In order for the new $\tilde{\eta}_g$ and $\tilde{\eta}_h$ to still be standard lifts we demand that $\xi_g(L_g) = \xi_h(L_h) = 1$. We find that even though there could be an obstruction to finding such ξ_g, ξ_h, for all examples we consider, this obstruction vanishes, so that we can find a splitting lift.

For a generalized cyclic orbifold as described in section 3, note that the mere existence of this lift is enough since we only need to know its action on the untwisted sector. For groups that do not fall in this category, such as for instance $\mathbb{Z}_3 \times \mathbb{Z}_3$, we will need the explicit expression for the lift, which is provided in Proposition Appendix A.3.

5. Constructing twisted modules

The advantage of generalized cyclic orbifolds as discussed in section 3 is that it is unnecessary to explicitly construct the twisted modules. However, some of the orbifold groups that we are interested in are not of that form. Let us, therefore, describe how to construct $\hat{g}$-twisted modules of lattice VOAs and the action of $C_{\iota(G)}(\hat{g})$.

Let L be an even, unimodular lattice, g a lattice automorphism of order n. Also, let V_L be the lattice VOA corresponding to L and $\hat{g}$ be a standard lift of g. We denote the unique irreducible $\hat{g}$-twisted V_L-module by $W_{\hat{g}}$. The twisted modules $W_{\hat{g}}$ for lattice VOAs were constructed in [30,31,8] as induced representations of a smaller representation Ω_0 of a finite group, the so-called defect representation. We will briefly review this construction, using the notation of [31], and then give an explicit construction of Ω_0 using technology from [48].

Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and let g act linearly on $\mathfrak{h}$. Let $\mathfrak{h}_j$ be the eigenspace of g to the eigenvalue ξ_n^{-j}.

Definition 5.1. The g-twisted Heisenberg current algebra $\hat{\mathfrak{h}}_g$ consists of states

$$\hat{\mathfrak{h}}_g = \mathrm{span}\{v_m : v \in \mathfrak{h}, m \in \frac{j}{n} + \mathbb{Z} \ \text{ if } gv = \xi_n^{-j} v\} \ , \tag{5.1}$$

together with the bracket

$$[v_m, v'_k] = \langle \pi_{nm} v | v' \rangle m \delta_{m,-k} \ , \tag{5.2}$$

where π_{nm} is the projector onto the g-eigenspace to the eigenvalue ξ_n^{-nm}. As an operator v_m has weight $-m$.

To construct the twisted lattice vertex operators, we want to define the analog of the twisted group algebra (4.2) used in the construction of the lattice VOA. However in order to accommodate the requirement of twist compatibility (2.1), the relevant structure has to be slightly larger than one might otherwise expect.

Definition 5.2. The twisted lattice group $G_{\hat{g}} = \mathbb{C}^\times \times \exp \mathfrak{h}_0 \times L$, where $\mathfrak{h}_0 = \pi_g \mathfrak{h}$ is the orthogonal projection on the g-invariant subspace, consists of elements of the form $ce^{v_0}U_\alpha$, together with the relations

$$U_\alpha U_\beta = \epsilon(\alpha, \beta) B(\alpha, \beta)^{-1} U_{\alpha+\beta} \ , \tag{5.3}$$

where $\epsilon(\alpha, \beta)$ can be chosen as in (4.5) and

$$B(\alpha, \beta) = n^{-\langle \alpha | \beta \rangle} \prod_{k=1}^{n-1} (1 - \xi_n^k)^{\langle \alpha g^k | \beta \rangle} \ , \tag{5.4}$$

$$e^{v_0} e^{w_0} = e^{(v+w)_0} \ , \tag{5.5}$$

$$e^{v_0} U_\alpha e^{-v_0} = \exp(\langle v | \alpha \rangle) U_\alpha \ , \tag{5.6}$$

and

$$U_{\alpha g} = \eta_g(\alpha) U_\alpha \exp\big(2\pi i (b_\alpha + \alpha_0)\big) \ , \tag{5.7}$$

where

$$b_\alpha = \frac{|\pi_g(\alpha)|^2}{2} \in \mathbb{C} \ . \tag{5.8}$$

Additionally, let the twisted lattice operators act on $\hat{\mathfrak{h}}_g$ by

$$[h_m, U_\alpha] = \delta_{m,0} \langle \pi_g(h) | \alpha \rangle U_\alpha, \quad \text{for all } \alpha \in L, h \in \mathfrak{h}. \tag{5.9}$$

Note that by using (5.3), the twist compatibility condition (5.7) can also be expressed as

$$U_{\alpha(1-g)} = \eta_g(\alpha)^{-1} \epsilon(\alpha(1-g), \alpha g) B(\alpha(1-g), \alpha g)^{-1} \exp(2\pi i b_\alpha) \exp(-2\pi i \alpha_0). \tag{5.10}$$

Define the subgroup of $G_{\hat{g}}$ corresponding to the coinvariant sublattice $L_g^{\perp}$

$$G_{\hat{g}}^{\perp} = \{cU_\mu : c \in \mathbb{C}, \mu \in L_g^{\perp}\}. \tag{5.11}$$

Then in [31] it is shown that as a $\hat{\mathfrak{h}}_g$-module, $W_{\hat{g}}$ is induced from its vacuum subspace

$$\Omega = \{w \in W_{\hat{g}} | v_m w = 0, v_m \in \hat{\mathfrak{h}}_g, m \geq 0\}. \tag{5.12}$$

Ω, in turn, is a $G_{\hat{g}}$-module that decomposes into $\exp \mathfrak{h}_0$ eigenspaces

$$\Omega = \sum_{\alpha \in \pi_g(L)} \Omega_\alpha \ , \tag{5.13}$$

where $\Omega_\alpha = \{w \in \Omega | e^{h_0} w = e^{\langle h|\alpha\rangle} w \text{ for all } h \in \mathfrak{h}_0\}$ such that

$$U_\beta \Omega_\alpha = \Omega_{\alpha + \pi_g \beta}. \tag{5.14}$$

It follows that all the homogeneous subspaces Ω_α are $G_{\hat{g}}^{\perp}$-modules and Ω can be induced from Ω_0 (or any Ω_α for that matter).

Note that from (5.7) it follows that $G_{\hat{g}}^{\perp}$ is a central extension of the finite abelian group

$$N := L_g^{\perp} / L(1-g) \tag{5.15}$$

whose isomorphism class is determined by the skew

$$C(\alpha, \beta) = U_\alpha U_\beta U_\alpha^{-1} U_\beta^{-1} = \prod_{k=0}^{n-1} (-\xi_n^k)^{-\langle \alpha g^k | \beta \rangle}. \tag{5.16}$$

In particular, Equation (4.44) in [31] shows that $C(\alpha, \beta)$ restricts to a non-degenerate, alternating, bimultiplicative form on N, endowing it with the structure of a symplectic module.

$W_{\hat{g}}$ is irreducible if and only if Ω_0 is an irreducible projective representation of N. We now want to construct Ω_0 (or any Ω_α for that matter) as a projective representation Ω_0 of N of dimension $d(g) := \sqrt{|N|}$. We will call Ω_0 the defect representation. We will see that Ω_0 is the unique projective representation of N with commutator $C(\cdot, \cdot)$.

For future convenience let us give a basis for the group N using the Smith normal form of the matrix $A = 1 - g$:

Lemma 5.1. *Let g be an automorphism of L of order n, $A = 1-g$. Then we can find two bases $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k, \gamma_1, \ldots, \gamma_{n-k}\}$ and $\{\alpha_1, \ldots, \alpha_k, \delta_1, \ldots, \delta_{n-k}\}$ of L such that A acts as*

$$\tilde{\alpha}_i A = s_i \alpha_i, \quad i = 1, \ldots, k \tag{5.17}$$

and

$$\gamma_j A = 0, \quad j = 1, \ldots, n-k. \tag{5.18}$$

The fixed-point lattice L_g is given by

$$L_g = \mathrm{span}_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_{n-k}) \tag{5.19}$$

and $L_g^\perp$, its orthogonal complement in L, by

$$L_g^\perp = \mathrm{span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_k). \tag{5.20}$$

Moreover $N = L_g^\perp / L(1-g)$ is generated by the elements $\{[\alpha_1], \ldots, [\alpha_k]\}$, where $[\alpha_i]$ has order s_i.

Proof. We write $S = P\ A\ Q$ to obtain the Smith normal form $S = \mathrm{diag}(s_1, s_2, ..., s_k, 0, \ldots, 0)$ of A. This gives two bases $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k, \gamma_1, \ldots, \gamma_{n-k}\}$ and $\{\alpha_1, \ldots, \alpha_k, \delta_1, \ldots, \delta_{n-k}\}$ of L (given by the row vectors of P and Q^{-1} respectively) with the action (5.17) and (5.18) of A.

Clearly, $\mathrm{span}_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_{n-k}) \subseteq L_g$. So to prove the first statement, we need to show that there is no vector $v \in \mathrm{span}_{\mathbb{Z}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ such that $v(1-g) = 0$. This follows directly from the linear independence of the $\{\alpha_i\}$. Clearly, $\mathrm{span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_k) \subseteq L_g^\perp$. So to prove the second statement, we need to show that there is no vector $w \in \mathrm{span}_{\mathbb{Z}}(\delta_1, \ldots, \delta_{n-k})$ such that $w \in L_g^\perp$. But because $\mathrm{Rank}(L_g^\perp) = \mathrm{Rank}(L(1-g))$ for any such vector there has to exist an integer n_w such that $n_w w \in L(1-g)$. This contradicts the linear independence of the basis elements of derived from Q^{-1}. Hence the second statement follows. The last statement immediately follows from (5.17) and (5.18). $\square$

5.1. Defect representation

We now want to construct the defect representation Ω_0 (or any Ω_α for that matter) as an irreducible projective representation of N. We will see that Ω_0 is the unique projective representation of N with commutator $C(\cdot,\cdot)$ and that it is of dimension $d(g) = \sqrt{|N|}$.

N is a finite group with the alternative, bimultiplicative, nondegenerate form $C : N \times N \to \mathbb{C}$. As a first step, we will present some results on the structure of N as for instance presented in [48].

Lemma 5.2. *N admits a basis $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ such that*

$$o(a_i) = o(b_i) = n_i, \tag{5.21}$$

where $n_i | n_{i+1}$ and such that

$$C(a_i, b_j) = \delta_{i,j}\xi_{n_i}^{l_i}, \tag{5.22}$$

where $\gcd(n_i, l_i) = 1$ *and*

$$C(a_i, a_j) = C(b_i, b_j) = 1. \tag{5.23}$$

Proof. By the fundamental theorem of finitely generated abelian groups there exists a basis $\{a_1, \ldots, a_r\}$ of N such $ord(a_i) = n_i$, $n_r = n_k$ and $n_i \mid n_{i+1}$. Then there exists an $a_j =: b_1$ such that

$$C(a_1, b_1) = \chi, \tag{5.24}$$

where χ is a primitive n_1-th root of unity. To see this, assume $C(a_1, a_i)$ is at most an m-th root of unity, for some proper divisor $m \mid n_1$ and for all i. Then $C(ma_1, a_i) = 1$ for all i and hence C is degenerate, giving a contradiction. This also implies that $ord(b_1) = ord(a_1)$.

Hence for any basis element a_i, $i \neq 1, j$ there are positive integers $k_{i,1}$ and $k_{i,j}$ such that

$$C(a_i, a_1) = \chi^{k_{i,1}} \tag{5.25}$$

and

$$C(a_i, a_j) = \chi^{k_{i,j}}. \tag{5.26}$$

Now set

$$\tilde{a}_i = a_i + k_{i,1}a_j - k_{i,j}a_1. \tag{5.27}$$

Note that $k_{i,1}a_j$ and $k_{i,j}a_1$ and hence also $\tilde{\alpha}_i$ are elements of order n_i. Then $\{a_1, b_1, \tilde{a}_2, \ldots, \tilde{a}_{r-1}\}$ is a basis of N. The theorem follows by induction. □

We can now apply this to the basis of Lemma 5.1 to obtain a basis $\{[\alpha_1], \ldots, [\alpha_{k/2}], [\beta_1], \ldots, [\beta_{k/2}]\}$ of N that satisfies the properties from Lemma 5.2.

To construct the irreducible projective representation (ρ, Ω_0) of N we want to decompose N in the following way. Using the fact that C is a nondegenerate, bimultiplicative, alternating form, by Lemma 5.2 we know that there exists a subgroup $A \leq N$ generated by $\{[\alpha_1], \ldots, [\alpha_{k/2}]\}$ and an isomorphism γ such that

$$\gamma : N \to A + \check{A} \qquad [\mu] \mapsto (x_{[\mu]}, \chi_{[\mu]}) \ , \tag{5.28}$$

where $\check{A}$ denotes the group of irreducible characters of A, generated by $\{[\beta_1], \ldots, [\beta_{k/2}]\}$, such that the bilinear form C is given by

$$C(\gamma^{-1}(x,\chi),\gamma^{-1}(y,\psi)) = \psi(x)\chi(y)^{-1}. \tag{5.29}$$

From this decomposition we can construct the sought after projective representation Ω_0:

Lemma 5.3. *Let $\Omega_0 = \mathbb{C}[\check{A}]$. Then the unique irreducible projective representation (ρ, Ω_0) of N with skew C is given by*

$$\rho([\mu])e_\psi = \psi_{[\mu]}(x_{[\mu]})e_{\chi\psi_{[\mu]}}, \tag{5.30}$$

where $(x_{[\mu]}, \chi_{[\mu]}) = \gamma([\mu])$.

Proof. We find that ρ satisfies the product relation

$$\rho([\mu])\rho([\nu])e_\psi = \phi_{[\nu]}(x_{[\mu]})\psi(x_{[\mu]}y_{[\nu]})e_{\chi_{[\mu]}\phi_{[\nu]}\psi} \tag{5.31}$$

$$= \phi_{[\nu]}(x_{[\mu]})\rho([\mu+\nu])e_\psi \tag{5.32}$$

so that ρ does indeed define a projective representation for the group N with commutator given by C. Then the characters are given by

$$\mathrm{Tr}(\rho([\mu])) = \begin{cases} \sum_{\psi\in\check{A}} \psi(x_{[\mu]}), & \text{if } \chi_{[\mu]} = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{5.33}$$

from which it follows $\|\rho\|^2 = 1$, so that ρ is indeed irreducible. Finally, the uniqueness of ρ follows from the fact that $\dim(\rho)^2 = |N|$. □

Finally, we want to extend this representation ρ of $N = L_g^\perp/L(1-g)$ to a representation $\tilde{\rho}$ of $L_g^\perp$ satisfying the twist compatibility condition (5.7) so that we can make the identification

$$\tilde{\rho} : L_g^\perp \to \mathrm{End}(\Omega_0), \qquad \mu \mapsto U_\mu \tag{5.34}$$

with the corresponding twisted lattice operator in the sense of Definition 5.2.

Due to the uniqueness of the irreducible projective representation of N the representations $\tilde{\rho}$ and ρ differ at most by a change of section, i.e. there is a function $\lambda : L_g^\perp \to \mathbb{C}^*$ such that

$$U_\mu = \lambda(\mu)\rho([\mu]). \tag{5.35}$$

We need to be careful in our choice of λ, since the U_μ need to satisfy both (5.3) and (5.10), which is equivalent to λ satisfying the two conditions

$$\lambda(\mu)\lambda(\nu)\chi_\nu(x_\mu) = \epsilon(\mu,\nu)B(\mu,\nu)^{-1}\lambda(\mu+\nu) \tag{5.36}$$

and

$$\lambda(\alpha(1-g)) = \eta_g(\alpha)^{-1}\epsilon(\alpha(1-g), \alpha g)B(\alpha(1-g), \alpha g)^{-1}\exp(2\pi i b_\alpha)\ . \tag{5.37}$$

In order to proceed to find a λ satisfying the conditions (5.36) and (5.37) we first note that a straightforward computation establishes that if the operators $U_{\alpha(1-g)}$ and $U_{\beta(1-g)}$ satisfy the twist compatibility condition (5.10), then so does $U_{(\alpha+\beta)(1-g)}$.

By Lemma 5.1 there exists a basis $\{\mu_i\}$ of $L_g^\perp$ such that $\{s_i\mu_i\}$ is a basis of $L(1-g)$, where $\{s_i\}$ are the elementary divisors of $1-g$. Hence if we determine $\lambda(\mu_i)$ for a basis μ_i of $L_g^\perp$ (and hence N) such that all $U_{s_i\mu_i}$ satisfy twist compatibility, we can repeatedly apply (5.3) to obtain a full set of twist compatible lattice operators. In particular, if $\{\alpha_i\}$ is a basis of L and we assume that the fixed set U_{α_i} satisfy (5.3) and (5.10), then a general lattice operator is given by

$$U_{(\sum_i n_i\alpha_i)} = \prod_{i<j}\big(\epsilon(\alpha_i,\alpha_j)B(\alpha_i,\alpha_j)\big)^{n_in_j}\prod_i U_{\alpha_i}^{n_i}\big(\epsilon(\alpha_i,\alpha_i)B(\alpha_i,\alpha_i)\big)^{\frac{n_i(n_i-1)}{2}}. \tag{5.38}$$

A straightforward computation shows that these operators indeed satisfy (5.3) and (5.10).

It is thus enough to ensure (5.36) and (5.37) on a basis of $L_g^\perp$. An explicit expression for the λ that achieves this is given by:

Lemma 5.4. *Let $\{\mu_i\}$ be a basis of $L_g^\perp$ such that $s_i\mu_i \in L(1-g)$ for all i, where $\tilde\mu_i(1-g) = s_i\mu_i$. Then the solution of Equations (5.36) and (5.37) $\lambda(\mu_i)$ is uniquely determined up to an s_i-th root of unity by*

$$\lambda(\mu_i) = \eta_g(\tilde\mu_i)^{-\frac{1}{s_i}}\epsilon(\mu_i,\tilde\mu_i)B(\mu_i,\tilde\mu_i)^{-1}\big(\epsilon(\mu_i,\mu_i)B(\mu_i,\mu_i)\big)^{\frac{s_i+1}{2}}\exp(\frac{2\pi i}{s_i}b_{\tilde\mu_i})\ . \tag{5.39}$$

Proof. By Equation (5.36) we have

$$\lambda(s_i\mu_i) = \lambda(\mu_i)^{s_i}\big(\epsilon(\mu_i,\mu_i)B(\mu_i,\mu_i)\big)^{\frac{s_i(s_i-1)}{2}}, \tag{5.40}$$

where we may choose a basis such that $\chi_{\mu_i}(x_{\mu_i})$ and by Equation (5.37)

$$\begin{aligned}
\lambda(s_i\mu_i) &= \lambda(\tilde\mu_i(1-g)) && (5.41)\\
&= \eta_g(\tilde\mu_i)^{-1}\epsilon(\tilde\mu_i(1-g),\tilde\mu_i g)B(\tilde\mu_i(1-g),\tilde\mu_i g)^{-1}\exp(2\pi i b_{\tilde\mu_i}) && (5.42)\\
&= \eta_g(\tilde\mu_i)^{-1}\epsilon(s_i\mu_i,\tilde\mu_i g)B(s_i\mu_i,\tilde\mu_i g)^{-1}\exp(2\pi i b_{\tilde\mu_i}) && (5.43)\\
&= \eta_g(\tilde\mu_i)^{-1}\epsilon(s_i\mu_i,\tilde\mu_i - s_i\mu_i)B(s_i\mu_i,\tilde\mu_i - s_i\mu_i)^{-1}\exp(2\pi i b_{\tilde\mu_i}) && (5.44)\\
&= \eta_g(\tilde\mu_i)^{-1}\big(\epsilon(\mu_i,\tilde\mu_i)B(\mu_i,\tilde\mu_i)^{-1}\big)^{s_i}\big(\epsilon(\mu_i,\mu_i)B(\mu_i,\mu_i)\big)^{s_i^2}\exp(2\pi i b_{\tilde\mu_i})\ . && (5.45)
\end{aligned}$$

Hence it follows that

$$\lambda(\mu_i)^{s_i} = \eta_g(\tilde{\mu}_i)^{-1}\big(\epsilon(\mu_i,\tilde{\mu}_i)B(\mu_i,\tilde{\mu}_i)^{-1}\big)^{s_i}\big(\epsilon(\mu_i,\mu_i)B(\mu_i,\mu_i)\big)^{\frac{s_i(s_i+1)}{2}}\exp(2\pi i b_{\tilde{\mu}_i}), \tag{5.46}$$

so that in particular $\lambda(\mu_i)$ is determined up to an s_i-th root of unity. □

From this we can construct U_{μ_i} for a basis μ_i, and thus by (5.4) all the operators U_μ.

5.2. A basis for $W_{\hat{g}}$

Finally let us give a basis for $W_{\hat{g}}$ starting from $(\tilde{\rho}, \Omega_0)$. First we find a sublattice $\Lambda \subset L$ such that

$$L = \Lambda + L_g^{\perp} \ . \tag{5.47}$$

Then we can decompose any $\alpha \in L$ as $\alpha = \gamma + \mu$ for some $\gamma \in \Lambda$ and $\mu \in L_g^{\perp}$. Note that in general we cannot choose Λ to be L_g. Let $\{e_\chi : \chi \in \check{A}\}$ be a basis for Ω_0. We can reconstruct $W_{\hat{g}}$ by

Proposition 5.5. *A basis for $W_{\hat{g}}$ is given by*

$$\{v^1_{-m_1}\dots v^k_{-m_k}U_\gamma e_\chi : v^i_{-m_i} \in \hat{\mathfrak{h}}_g, m_i > 0, \gamma \in \Lambda, \chi \in \check{A}\}. \tag{5.48}$$

The weight of such a state is given by

$$m_1 + \ldots + m_k + |(\pi_g\gamma)|^2/2 + \rho_g, \tag{5.49}$$

recovering the familiar result for the character.

A general twisted lattice operator U_α then acts on (5.48) in the obvious way.

6. Twining characters

6.1. The action of automorphisms on twisted modules

Let V be a C_2-cofinite VOA, $\hat{g}$ and $\hat{h}$ automorphisms of V such that $\hat{g}\hat{h} = \hat{h}\hat{g}$ and $(W_{\hat{g}}, Y_W(\cdot, x))$ an irreducible $\hat{g}$-twisted V-module. Then by the results of [9,32,10] there exists a linear map $\phi_{\hat{g}}(\hat{h}) : W_{\hat{g}} \to W_{\hat{g}}$ such that

$$\phi_{\hat{g}}(\hat{h})Y_W(v,x)\phi_{\hat{g}}(\hat{h})^{-1} = Y_W(\hat{h}v, x) \ , \tag{6.1}$$

which is unique up to multiplication by a scalar. $\phi_{\hat{g}}(\cdot)$ is a projective representation of $C_{\iota(G)}(\hat{g})$. We need to explicitly work out what this representation is for the construction at hand.

Let us now insert a twining element $\hat{h} \in C_{\iota(G)}(\hat{g})$. Acting with $\phi_{\hat{g}}(\hat{h})$ on a vector in (5.48) we can use (6.1) to find

$$\phi_{\hat{g}}(\hat{h})U_\alpha\phi_{\hat{g}}(\hat{h})^{-1} = \eta_h(\alpha)U_{\alpha h} \tag{6.2}$$

and

$$\phi_{\hat{g}}(\hat{h})v_{-n}\phi_{\hat{g}}(\hat{h})^{-1} = (v \cdot h)_{-n} \ . \tag{6.3}$$

We therefore have

Lemma 6.1. *Let $\hat{g}, \hat{h} \in \mathrm{Aut}(V_L)$ be two automorphisms such that $\hat{g}\hat{h} = \hat{h}\hat{g}$ lifted from commuting lattice automorphisms $g, h \in \mathrm{Aut}(L)$. Then $W_{\hat{g}}$ carries an action $\phi_{\hat{g}}(\hat{h})$ of $\hat{h}$ that satisfies (6.1) and is given by*

$$\phi_{\hat{g}}(\hat{h})v_{-n_1}\dots v_{-n_k}U_\alpha e_\chi = (vh)_{-n_1}\dots(vh)_{-n_k}\eta_h(\alpha)^{-1}U_{\alpha h}\phi_{\hat{g}}(\hat{h})|_{\Omega_0}e_\chi \ . \tag{6.4}$$

We thus need to find the action of $\phi_{\hat{g}}(\hat{h})$ on Ω_0. Note that $\phi_{\hat{g}}(\hat{h})|_{\Omega_0}$ is a $d(g) \times d(g)$-matrix $O_{\hat{h}} \in GL(\Omega_0)$. To compute $O_{\hat{h}}$, we note that $\tilde{\rho}(\cdot h)$ is again an irreducible representation of $L_g^\perp$ of dimension $d(g)$ with the same cocycle, and in fact also an irreducible representation of N. Since there is only one irreducible representation of dimension $d(g)$, Schur's lemma tells us that $\tilde{\rho}(\cdot h)$ and $\tilde{\rho}(\cdot)$ are related by a change of basis automorphism $O_{\hat{h}}$ on Ω, which is unique up to an overall factor in $\mathbb{C}^*$. From this it follows immediately that $\phi_{\hat{g}}|_{\Omega_0} : \hat{h} \mapsto O_{\hat{h}}$ is indeed a projective representation of $C_{\iota(G)}(\hat{g})$ on Ω_0.

To construct $O_{\hat{h}}$, we construct $U_{\mu_i}|_{\Omega_0}$ for a basis of $L_g^\perp$ and then use

$$O_{\hat{h}}U_{\mu_i}O_{\hat{h}}^{-1} = \eta_h(\mu_i)U_{\mu_i h} \tag{6.5}$$

to obtain constraints on the matrix $O_{\hat{h}}$. Schur's lemma guarantees that if we repeat this procedure for enough μ_i, it will fix $O_{\hat{h}}$ up to an overall phase $\mathbb{C}^*$. Repeating this for all $\hat{h} \in C_{\iota(G)}(\hat{g})$ gives us the projective representation $\phi_{\hat{g}}|_{\Omega_0}$.

From the matrices $O_{\hat{h}}$ we can read off the cocycle of the representation $c_{\hat{g}}(\hat{h}_1, \hat{h}_2)$. In principle, we could now try to reconstruct the obstruction cocycle ω. Instead, we will do something simpler. We are interested only in orbifolds that allow a holomorphic extension, that is for which ω is trivial. In such a case we can choose $c_{\hat{g}}(\hat{h}_1, \hat{h}_2) = 1$. To see if this is compatible with our orbifold, we try to find coboundaries such that $c_{\hat{g}}(\hat{h}_1, \hat{h}_2) = 1$. This means that ω can only be trivial if all projective representations are secretly linear representations. This provides a test if ω can be trivial.

6.2. Twining characters

We now have all the ingredients to give an expression for the twining character

$$T(\hat{g},\hat{h};\tau) = \mathrm{tr}_{W_{\hat{g}}} \phi_{\hat{g}}(\hat{h}) q^{L_0 - c/24} \tag{6.6}$$

that can be explicitly computed, which we can use for the specific examples discussed in sections 7 and 8.

Lemma 6.2. *Let $\hat{h}$ be a V_L-automorphism that commutes with $\hat{g}$, and $\phi_{\hat{g}}(\hat{h})$ the corresponding action on $W_{\hat{g}}$. Then states of the form*

$$\{v_{-n_1}\dots v_{-n_k} U_\alpha e_\chi : n_1,\dots,n_k \in ..., \alpha \in \Lambda, e_\chi \in \mathbb{C}[\hat{A}], v_{-n_i} \in \hat{\mathfrak{h}}_g^-\} \tag{6.7}$$

contribute to the trace of $\phi_{\hat{g}}(\hat{h})$ only if $\alpha \in \Lambda$ is such that

$$\pi_g(\alpha) = \pi_g(h\alpha), \tag{6.8}$$

that is $\pi_g(\alpha) \in \mathfrak{h}_g^0 \cap \mathfrak{h}_h^0$ or equivalently $(1-h)\alpha \in L_g^\perp$. We denote the sublattice of Λ such that (6.8) is satisfied by Λ_h.

Proof. In order for the state to contribute to the character we require that

$$[\alpha h] = [\alpha], \tag{6.9}$$

where $[\alpha]$ denotes the class of α in $\frac{L}{L_g^\perp}$. In other words we have that

$$\alpha h = \alpha + \beta^\perp, \tag{6.10}$$

for some $\beta^\perp \in L_g^\perp$. $\square$

Proposition 6.3. *The character of $\phi_{\hat{g}}(\hat{h})$ on the twisted module $W_{\hat{g}}$ is given by*

$$T(\hat{g},\hat{h};\tau) = T_H(\hat{g},\hat{h};\tau) T_L(\hat{g},\hat{h};\tau). \tag{6.11}$$

Here the lattice contribution to the twining character is given by

$$T_L(\hat{g},\hat{h};\tau) = \sum_{\alpha\in\Lambda_h} q^{\frac{|\alpha_0|^2}{2}} \eta_h(\alpha)^{-1} \epsilon_g(\alpha,\alpha(h-1)) B_g(\alpha,\alpha(h-1)) \mathrm{Tr}|_\Omega\big(U_{\alpha(h-1)}\phi_{\hat{g}}(\hat{h})\big). \tag{6.12}$$

The Heisenberg contribution to the twining character is given by

$$T_H(\hat{g},\hat{h};\tau) = q^{\rho_g} \prod_{(i,j)\in\Psi} \prod_{k=0}^{\infty} \frac{1}{1-\xi_m^j q^{k+\frac{i}{n}}}, \tag{6.13}$$

where the conformal weight ρ_g is given by

$$\rho_g = \sum_j \frac{j(n-j)}{4n^2}\dim(\mathfrak{h}_j), \tag{6.14}$$

where n and m are the orders of g and h, respectively, Ψ are the sets of pairs $(i,j) \in \{0,\dots,n-1\} \times \{0,\dots,m-1\}$ such that ξ_n^i and ξ_m^j are simultaneous eigenvalues of g and h, and $\mathfrak{h}_j$ is the eigenspace of g to the eigenvalue ξ_n^{-j}.

Proof. For T_L we use Lemma 6.2 and the fact that for $\alpha \in \Lambda_h$,

$$\begin{aligned} &\phi_{\hat{g}}(\hat{h})v_{-n_1}\dots v_{-n_k}U_\alpha e_\chi \\ &\quad = (vh)_{-n_1}\dots(vh)_{-n_k}\eta_h(\alpha)^{-1}U_{\alpha h}\phi_{\hat{g}}(\hat{h})e_\chi && (6.15)\\ &\quad = (vh)_{-n_1}\dots(vh)_{-n_k}\eta_h(\alpha)^{-1}\epsilon_g(\alpha,\alpha(h-1))B_g(\alpha,\alpha(h-1))U_\alpha U_{\alpha(h-1)}\phi_{\hat{g}}(\hat{h})e_\chi. && (6.16)\end{aligned}$$

For T_H, let $v \in \mathfrak{h}$ be a simultaneous eigenvector of g and h with eigenvalues ξ_n^i and ξ_m^j, respectively. Then by Definition 5.1 the weights of the operators corresponding to v in $\hat{\mathfrak{h}}_g$ are given by $\frac{i}{n} + \mathbb{Z}_{\geq 0}$. Then the contribution to the trace of h due to descendants of v is given by

$$\begin{aligned} T_v(\hat{g},\hat{h};\tau) &= q^{\rho_g}(1 + \xi_m^j q^{\frac{i}{n}} + \xi_m^{2j} q^{\frac{2i}{n}} + \dots)(1 + \xi_m^j q^{1+\frac{i}{n}} + \xi_m^{2j} q^{2+\frac{2i}{n}} + \dots)\dots && (6.17)\\ &= \prod_{k=0}^{\infty} \frac{1}{1-\xi_m^j q^{k+\frac{i}{n}}}. && (6.18)\end{aligned}$$

The desired result follows by multiplying over all simultaneous eigenvectors. □

7. Warm-up: orbifolds of E_8

To illustrate the procedure discussed above, let us discuss orbifolds of the E_8 lattice VOA as a warm-up. This is the smallest even unimodular lattice, which makes our computation easier. This case is strongly constrained since the only holomorphic VOA of central charge 8 is the E_8 lattice VOA. Any orbifold thus only has two possible outcomes: either ω is not trivial, so that there is no holomorphic extension, or we recover the original unorbifolded VOA. We will give an example for both possibilities.

Investigating all 62092 conjugacy classes of subgroups of $\mathrm{Aut}(L_{E_8})$, we find that under standard lifts only 14 are non-anomalous, of which 10 are cyclic. The non-cyclic cases are isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_6$ and $\mathbb{Z}_5 \times \mathbb{Z}_{10}$. We will investigate the smallest one of these as well as anomalous groups isomorphic to S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

7.1. An S_3-orbifold

First consider an orbifold of the E_8 lattice VOA by a group $S_3 = \mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$ generated by elements s and t of cycle types $C_s = 2^4$ and $C_t = 1^2 3^2$ (see equation (8.1) for a definition of the cycle type). S_3 has three conjugacy classes, $[e]$, $[s]$, $[t]$, containing elements of order 1, 2, and 3 respectively. Its third cohomology is given by $H^3(S_3, U(1)) = \mathbb{Z}_6$. The cyclic subgroups generated by s and t have types $r_s = 1$ and $r_t = 2$, from which it follows that ω is non-trivial. We can therefore not obtain a holomorphic orbifold. However, we can still obtain the twining characters for S_3:

$$T(\hat{e},\hat{e};\tau) = \frac{\theta_{\mathrm{E8}}(\tau)}{\eta(\tau)^8} \qquad T(\hat{e},\hat{s};\tau) = \frac{\theta_{D_4}(\tau)}{\eta(2\tau)^4}$$

$$T(\hat{e},\hat{t};\tau) = \frac{\theta_{A_2}(\tau)^2}{\eta(\tau)^2\eta(3\tau)^2} \tag{7.1}$$

$$T(\hat{s},\hat{e};\tau) = 2\frac{\theta_{D_4}(\frac{\tau}{2})}{\eta(\frac{\tau}{2})^4} \qquad T(\hat{s},\hat{s};\tau) = 2\xi_3\frac{\theta_{D_4}(\frac{\tau+1}{2})}{\eta(\frac{\tau+1}{2})^4} \tag{7.2}$$

$$T(\hat{t},\hat{e};\tau) = \frac{\theta_{A_2}(\frac{\tau}{3})^2}{\eta(\tau)^2\eta(\frac{\tau}{3})^2} \qquad T(\hat{t},\hat{t};\tau) = \xi_3\frac{\theta_{A_2}(\frac{\tau+1}{3})^2}{\eta(\tau+1)^2\eta(\frac{\tau+1}{3})^2}$$

$$T(\hat{t},\hat{t}^2;\tau) = \xi_3^2\frac{\theta_{A_2}(\frac{\tau+2}{3})^2}{\eta(\tau+2)^2\eta(\frac{\tau+2}{3})^2}\ . \tag{7.3}$$

Let us now write down the characters of the irreducible modules. There are 3 irreducible representation of S_3: the trivial, signum and 2-dimensional standard representation. The other centralizers have the usual cyclic characters. Using the irreducible characters with the 2-cocycles descended from ω, we find

$$\chi_{\hat{e},0}(\tau) = \frac{T(\hat{e},\hat{e};\tau) + 3T(\hat{e},\hat{s};\tau) + 2T(\hat{e},\hat{t};\tau)}{6} \tag{7.4}$$

$$\chi_{\hat{e},\mathrm{sgn}}(\tau) = \frac{T(\hat{e},\hat{e};\tau) - 3T(\hat{e},\hat{s};\tau) + 2T(\hat{e},\hat{t};\tau)}{6} \tag{7.5}$$

$$\chi_{\hat{e},2}(\tau) = \frac{T(\hat{e},\hat{e};\tau) - T(\hat{e},\hat{t};\tau)}{3} \tag{7.6}$$

$$\chi_{\hat{s},0}(\tau) = \frac{T(\hat{s},\hat{e};\tau) - \xi_4 T(\hat{s},\hat{s};\tau)}{2} \tag{7.7}$$

$$\chi_{\hat{s},1}(\tau) = \frac{T(\hat{s},\hat{e};\tau) + \xi_4 T(\hat{s},\hat{s};\tau)}{2} \tag{7.8}$$

$$\chi_{\hat{t},0}(\tau) = \frac{T(\hat{t},\hat{e};\tau) + \xi_9^{-2}T(\hat{t},\hat{t};\tau) + \xi_9^{-4}T(\hat{t},\hat{t}^2;\tau)}{3} \tag{7.9}$$

$$\chi_{\hat{t},1}(\tau) = \frac{T(\hat{t},\hat{e};\tau) + \xi_9 T(\hat{t},\hat{t};\tau) + \xi_9^2 T(\hat{t},\hat{t}^2;\tau)}{3} \tag{7.10}$$

$$\chi_{\hat{t},2}(\tau) = \frac{T(\hat{t},\hat{e};\tau) + \xi_9^4 T(\hat{t},\hat{t};\tau) + \xi_9^{-1}T(\hat{t},\hat{t}^2;\tau)}{3}, \tag{7.11}$$

where ξ_n denotes a primitive n-th root of unity. Using the ordering above, the transformation matrices under T and S indeed agree with the ones obtained in [38,49] with $p = 1$.

7.2. A $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbifold

Next, we consider an orbifold of the E_8 lattice VOA by a group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by elements g and h both of cycle type $C_g = C_h = 2^4$. The types of the cyclic subgroups generated by g, h and gh are given $r_g = r_h = 1$ and $r_{gh} = 0$, respectively. Thus the cohomological twist ω is non-trivial. There are 16 twisted twining characters, and they form the following orbits under $\mathrm{SL}(2,\mathbb{Z})$-transformations:

$$\begin{aligned}
&\mathcal{O}_{\hat{e},\hat{e}} = \{T(\hat{e},\hat{e};\tau)\}\ , \mathcal{O}_{\hat{e},\hat{g}} = \{T(\hat{g},\hat{e};\tau), T(\hat{e},\hat{g};\tau), T(\hat{g},\hat{g};\tau)\}\ ,\\
&\mathcal{O}_{\hat{e},\hat{h}} = \{T(\hat{h},\hat{e};\tau), T(\hat{e},\hat{h};\tau), T(\hat{h},\hat{h};\tau)\}\ , \mathcal{O}_{\hat{e},\widehat{gh}} = \{T(\widehat{gh},\hat{e};\tau), T(\hat{e},\widehat{gh};\tau), T(\widehat{gh},\widehat{gh};\tau)\}\ ,\\
&\mathcal{O}_{\hat{g},\hat{h}} = \{T(\hat{g},\hat{h};\tau), T(\hat{g},\widehat{gh};\tau), T(\hat{h},\hat{g};\tau), T(\hat{h},\widehat{gh};\tau), T(\widehat{gh},\hat{g}), T(\widehat{gh},\hat{h})\}.
\end{aligned} \tag{7.12}$$

Of those orbits, all but the last one intersect the untwisted sector and can be computed using cyclic orbifolds. We will therefore not give the full expressions.

For the last orbit, however, we need to use our full algorithm: that is, we construct the $\hat{g}$ and $\hat{h}$-twisted modules, compute the projective representations $\phi_{\hat{g},\hat{h}}(\cdot)$, and compute the twining characters as described in sections 5 and 6. In practice, we do this by computing the first few coefficients in the q expansion and matching them experimentally to modular expressions, giving

$$T(\hat{g},\hat{h};\tau) = T(\hat{h},\hat{g};\tau) = 2\left(\frac{\eta(\tau)^2}{\eta(\tau/2)\eta(2\tau)}\right)^4 \tag{7.13}$$

and

$$T(\hat{g},\widehat{gh};\tau) = T(\hat{h},\widehat{gh}) = e^{2\pi i/12}2\left(\frac{\eta(\tau)^2}{\eta(\tau/2+1/2)\eta(2\tau)}\right)^4\ . \tag{7.14}$$

The product $\widehat{gh}$ however is not a standard lift, so that we leave the construction of the $\widehat{gh}$-twisted module and its characters to future work. From the transformation properties of the above expressions, we conjecture that they will be given by

$$T(\widehat{gh},\hat{g};\tau) = T(\widehat{gh},\hat{h};\tau) = e^{-2\pi i/6}\left(\frac{\eta(\tau)^2}{\eta(\tau/2-1/2)\eta(\tau/2)}\right)^4\ . \tag{7.15}$$

7.3. A $\mathbb{Z}_3 \times \mathbb{Z}_3$-orbifold

Finally we consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold generated by two elements g and h of cycle type $1^{-1}3^3$. We find that all cyclic subgroups have type 0 and hence

$$\omega = 1 \ . \tag{7.16}$$

Again, this orbifold is not generalized cyclic, and we have $H^2(\mathbb{Z}_3 \times \mathbb{Z}_3) = \mathbb{Z}_3 \neq 0$, which allows for discrete torsion. This seems to pose a bit of a puzzle: since there is only one holomorphic VOA of central charge 8, all choices of discrete torsion need to give the same holomorphic character. The resolution to this is that the contribution of the characters that is controlled by discrete torsion vanishes.

As before, we can arrange all 81 twining characters into orbits of $\mathrm{SL}(2,\mathbb{Z})$, giving 6 orbits, one of which is simply $\{T(\hat{e},\hat{e};\tau)\}$. Of the remaining ones, 4 have length 8 and intersect the untwisted sector, and thus correspond to the orbifolds by the 4 cyclic subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The representatives for these four orbits $T(\hat{e},\hat{g};\tau)$, $T(\hat{e},\hat{h};\tau)$, $T(\hat{e},\widehat{gh};\tau)$ and $T(\hat{e},\widehat{g^2h};\tau)$ are all equal and given by

$$T(\hat{e},\hat{g};\tau) = T(\hat{e},\hat{h};\tau) = T(\hat{e},\widehat{gh};\tau) = T(\hat{e},\widehat{g^2h};\tau) = \frac{\theta_{A_2}(\tau)}{\eta(\tau)^{-1}\eta(3\tau)^3} \ . \tag{7.17}$$

The fifth orbit with representative $T(\hat{g},\hat{h};\tau)$ has length 48 and does not intersect the untwisted sector. We therefore again need to use our full algorithm to compute twisted modules and their twining characters explicitly. It turns out however that all twining characters are constant. The reason for this is that for all pairs $(\hat{g}_1,\hat{g}_2)$ that appear in this orbit, the lattice Λ complementary to $L_{g_1}^{\perp}$ does not have any g_2-invariant vectors so that the generalized theta function is a constant. Moreover, the eta functions turn out to give a constant, so that the overall character is indeed in turn constant. Considering the T-transformation as described in [22], we see that characters of the form $T(\hat{g},\hat{h};\tau)$ and $T(\hat{g},\widehat{gh};\tau)$ for any $\hat{g},\hat{h} \in \mathbb{Z}_3 \times \mathbb{Z}_3$ simply differ by a factor of a third root of unity. When we decompose the orbit into 16 groups of the form $\{T(\hat{g},\hat{h};\tau), T(\hat{g},\widehat{gh};\tau), T(\hat{g},\widehat{g^2h};\tau)\}$ for suitable pairs of elements $(\hat{g},\hat{h})$, each of them yields a vanishing sum of three third roots of unity. The upshot is thus that this orbit, whose contribution is in principle controlled by discrete torsion, never contributes to the character of $V^{orb(G)}$.

There is in fact a quicker way to see this: since this orbit does not intersect the untwisted module, and the conformal weights of the twisted modules are all strictly positive, no term $q^{-1/3}$ can appear, but only terms with higher powers of q. There is no modular function with the right multiplier system and only higher powers so that it follows that the sum over the orbit necessarily vanishes.

8. Non-abelian orbifolds at $c = 48$ and $c = 72$

Let us now turn to the main application of our methods: the systematic construction of holomorphic orbifolds of lattice VOAs of central charge $c = 48$ and $c = 72$. One of our goals is to come as close to an extremal VOA as possible. We therefore focus on extremal lattices, and orbifold by subgroups G of their automorphism groups. Extremal lattices $P_{48p}, P_{48q}, P_{48m}, P_{48n}$ in $d = 48$ and P_{72} in $c = 72$ and their automorphisms are constructed in [18–21], and their generators can be found in [50]. Using MAGMA [51], we first extract all subgroups of $\mathrm{Aut}(L)$ of the form $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ with q, p coprime. We then restrict to groups for which no lifted element $\hat{g}$ has its order doubled. By the procedure in section 4.2, we can lift them to automorphism groups of V_L which were isomorphic to $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$. We then only keep the ones which have $[\omega] = 1$. For these we use Theorem 3.4 to compute the characters of $V^{orb(G)}$.

In the following tables, we list the orbifold characters for all such groups. Different groups can give the same character; in these cases, we did not check if the orbifold VOAs just happen to have the same character, or if they are actually isomorphic. In the last column, we give the polar and constant part of the character, from which of course the entire character can be reconstructed. In columns two and four we give the cycle type of the cyclic subgroups $\mathbb{Z}_q$ and $\mathbb{Z}_p$ written as $\prod_{t|n} t^{b_t}$ where b_t can be read off from the characteristic polynomial of the generator g of $\mathbb{Z}_n$,

$$\mathcal{P}_g(x) = \prod_{t|n} (x^t - 1)^{b_t} \ . \tag{8.1}$$

8.1. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$-orbifolds for $d = 48$

8.1.1. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$-orbifolds for the lattice P_{48p}

q	$\mathbb{Z}_q$	p	$\mathbb{Z}_p$	ϕ	$\mathrm{ch}_{V^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}}(q)$
23	$1^2 23^2$	132	$2^1 4^{-1} 6^{-1} 12^1 22^1 44^1 66^{-1} 132^1$	5	$q^{-2} + 120 + \mathcal{O}(q)$
23	$1^2 23^2$	66	$1^2 2^{-2} 3^{-2} 6^2 11^2 22^{-2} 33^{-2} 66^2$	2	$q^{-2} + 168 + \mathcal{O}(q)$
23	$1^2 23^2$	44	$2^{-2} 4^2 22^{-2} 44^2$	5	$q^{-2} + 48 + \mathcal{O}(q)$
23	$1^2 23^2$	33	$1^{-2} 3^2 11^{-2} 33^2$	2	$q^{-2} + 48 + \mathcal{O}(q)$
23	$1^2 23^2$	22	$1^{-4} 2^4 11^{-4} 22^4$	2	$q^{-2} + 168 + \mathcal{O}(q)$
23	$1^2 23^2$	11	$1^4 11^4$	2	$q^{-2} + 4q^{-1} + 168 + \mathcal{O}(q)$
23	$1^2 23^2$	12	$2^{12} 4^{-12} 6^{-12} 12^{12}$	-1	$q^{-2} + 120 + \mathcal{O}(q)$
23	$1^2 23^2$	4	$2^{-24} 4^{24}$	-1	$q^{-2} + 48 + \mathcal{O}(q)$
3	$1^{-24} 3^{24}$	4	$2^{-24} 4^{24}$	-1	$q^{-2} + 576 + \mathcal{O}(q)$
11	$1^4 11^4$	4	$2^{-24} 4^{24}$	-1	$q^{-2} + 96 + \mathcal{O}(q)$
11	$1^4 11^4$	12	$2^{12} 4^{-12} 6^{-12} 12^{12}$	-1	$q^{-2} + 168 + \mathcal{O}(q)$
33	$1^{-2} 3^2 11^{-2} 33^2$	4	$2^{-24} 4^{24}$	-1	$q^{-2} + 96 + \mathcal{O}(q)$

8.1.2. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$-orbifolds for the lattice P_{48q}

q	$\mathbb{Z}_q$	p	$\mathbb{Z}_p$	ϕ	$\mathrm{ch}_{V^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}}(q)$
47	$1^1 47^1$	46	$1^{-2} 2^2 23^{-2} 46^2$	2	$q^{-2} + 72 + \mathcal{O}(q)$
47	$1^1 47^1$	23	$1^2 23^2$	2	$q^{-2} + 2q^{-1} + 120 + \mathcal{O}(q)$
23	$1^2 23^2$	4	$2^{-24} 4^{24}$	-1	$q^{-2} + 48 + \mathcal{O}(q)$

8.1.3. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$-orbifolds for the lattice P_{48m}

q	$\mathbb{Z}_q$	p	$\mathbb{Z}_p$	ϕ	$\mathrm{ch}_{V^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}}(q)$
3	$1^{-24}3^{24}$	20	$2^6 4^{-6} 10^{-6} 20^6$	-1	$q^{-2}+264+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+576+\mathcal{O}(q)$
15	$1^{-4}3^4 5^{-4}15^4$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+360+\mathcal{O}(q)$
5	$1^8 5^8$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+264+\mathcal{O}(q)$
5	$1^{-2}5^{10}$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+216+\mathcal{O}(q)$
5	$1^{-2}5^{10}$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+216+\mathcal{O}(q)$
5	$1^8 5^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
5	$1^8 5^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+288+\mathcal{O}(q)$
5	$1^{-2}5^{10}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+144+\mathcal{O}(q)$
5	$1^{-2}5^{10}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+144+\mathcal{O}(q)$

8.1.4. $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$-orbifolds for the lattice P_{48n}

q	$\mathbb{Z}_q$	p	$\mathbb{Z}_p$	ϕ	$\mathrm{ch}_{V^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}}(q)$
5	$1^{-12}5^{12}$	52	$2^2 4^{-2} 26^{-2} 52^2$	-1	$q^{-2}+120+\mathcal{O}(q)$
13	$1^{-4}13^4$	20	$2^6 4^{-6} 10^{-6} 20^6$	-1	$q^{-2}+120+\mathcal{O}(q)$
13	$1^{-4}13^4$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+168+\mathcal{O}(q)$
13	$1^{-4}13^4$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+168+\mathcal{O}(q)$
39	$1^2 3^{-2} 13^{-2} 39^2$	4	$2^{-24}4^{24}$	-1	$q^{-2}+96+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	52	$2^2 4^{-2} 26^{-2} 52^2$	-1	$q^{-2}+168+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	28	$2^4 4^{-4} 14^{-4} 28^4$	-1	$q^{-2}+168+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	28	$2^4 4^{-4} 14^{-4} 28^4$	-1	$q^{-2}+168+\mathcal{O}(q)$
35	$1^2 5^{-2} 7^{-2} 35^2$	4	$2^{-24}4^{24}$	-1	$q^{-2}+96+\mathcal{O}(q)$
7	$1^{-8}7^8$	20	$2^6 4^{-6} 10^{-6} 20^6$	-1	$q^{-2}+168+\mathcal{O}(q)$
7	$1^{-8}7^8$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+264+\mathcal{O}(q)$
7	$1^{-8}7^8$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+264+\mathcal{O}(q)$
7	$1^{-8}7^8$	12	$2^{12}4^{-12}6^{-12}12^{12}$	-1	$q^{-2}+264+\mathcal{O}(q)$
21	$1^4 3^{-4} 7^{-4} 21^4$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
21	$1^4 3^{-4} 7^{-4} 21^4$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
21	$1^4 3^{-4} 7^{-4} 21^4$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	28	$2^4 4^{-4} 14^{-4} 28^4$	-1	$q^{-2}+264+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	28	$2^4 4^{-4} 14^{-4} 28^4$	-1	$q^{-2}+264+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	28	$2^4 4^{-4} 14^{-4} 28^4$	-1	$q^{-2}+264+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	26	$2^{-2}26^2$	-1	$q^{-2}+84+\mathcal{O}(q)$
7	$1^{-8}7^8$	10	$2^{-6}10^6$	-1	$q^{-2}+84+\mathcal{O}(q)$
13	$1^{-4}13^4$	4	$2^{24}4^{-24}$	-1	$q^{-2}+96+\mathcal{O}(q)$
7	$1^{-8}7^8$	6	$2^{-12}6^{12}$	-1	$q^{-2}+132+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	14	$2^{-4}14^4$	-1	$q^{-2}+132+\mathcal{O}(q)$
7	$1^{-8}7^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
7	$1^{-8}7^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
7	$1^{-8}7^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
7	$1^{-8}7^8$	4	$2^{-24}4^{24}$	-1	$q^{-2}+192+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+288+\mathcal{O}(q)$
5	$1^{-12}5^{12}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+288+\mathcal{O}(q)$
3	$1^{-24}3^{24}$	4	$2^{-24}4^{24}$	-1	$q^{-2}+576+\mathcal{O}(q)$

8.2. $\mathbb{Z}_q \rtimes \mathbb{Z}_p$-orbifolds for $d = 72$

q	$\mathbb{Z}_q$	p	$\mathbb{Z}_p$	ϕ	$\mathrm{ch}_{V^{\mathrm{orb}(\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p)}}(q)$
7	$1^{-12}7^{12}$	78	$3^2 6^{-2} 39^{-2} 78^2$	2	$q^{-3} + 12q^{-1} + 200 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	30	$3^{-4} 6^4 15^{-4} 30^4$	2	$q^{-3} + 32q^{-1} + 304 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	30	$3^6 6^{-6} 15^{-6} 30^6$	2	$q^{-3} + 36q^{-1} + 568 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	39	$3^{-2} 39^2$	2	$q^{-3} + 12q^{-1} + 328 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	15	$3^4 15^4$	2	$q^{-3} + 32q^{-1} + 848 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	15	$3^{-6} 15^6$	2	$q^{-3} + 36q^{-1} + 824 + \mathcal{O}(q)$
3	3^{24}	26	$1^{-2} 2^{-2} 13^2 26^2$	-1	$q^{-3} + 48q^{-1} + 1344 + \mathcal{O}(q)$
3	3^{24}	26	$1^2 2^{-4} 13^{-2} 26^4$	-1	$q^{-3} + 48q^{-1} + 1272 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	6	$3^{-24} 6^{24}$	2	$q^{-3} + 144q^{-1} + 1040 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	6	$3^{-24} 6^{24}$	2	$q^{-3} + 144q^{-1} + 1040 + \mathcal{O}(q)$
3	3^{24}	10	$1^4 2^4 5^4 10^4$	-1	$q^{-3} + 8q^{-2} + 152q^{-1} + 4320 + \mathcal{O}(q)$
3	3^{24}	10	$1^6 2^{-12} 5^{-6} 10^{12}$	-1	$q^{-3} + 144q^{-1} + 3336 + \mathcal{O}(q)$
3	3^{24}	10	$1^{-6} 2^{-6} 5^6 10^6$	-1	$q^{-3} + 144q^{-1} + 3456 + \mathcal{O}(q)$
3	3^{24}	10	$1^{-4} 2^8 5^{-4} 10^8$	-1	$q^{-3} + 136q^{-1} + 3360 + \mathcal{O}(q)$
7	$1^{-12}7^{12}$	3	3^{24}	2	$q^{-3} + 144q^{-1} + 3376 + \mathcal{O}(q)$
3	3^{24}	2	$1^{-24} 2^{48}$	-1	$q^{-3} + 600q^{-1} + 14496 + \mathcal{O}(q)$
3	3^{24}	2	$1^{24} 2^{24}$	-1	$q^{-3} + 24q^{-2} + 648q^{-1} + 17376 + \mathcal{O}(q)$

8.3. Bounds for abelian orbifolds

At $d = 24$ it turns out that all 71 holomorphic VOAs can be constructed as cyclic orbifolds of lattice VOAs [23–25]. It is a reasonable question to ask whether that might also be the case at higher central charge. As we will show now, the answer to this question is no. To this end let us first establish the following lemma: Consider a lattice VOA at central charge d and let $M_{\mathfrak{h}}(1,0)$ be its Heisenberg sub-VOA.

Lemma 8.1. *For any finite, abelian group G acting orthogonally on the Heisenberg VOA $M_{\mathfrak{h}}(1,0)$ the dimensions of the G-invariant homogeneous subspaces are bounded from below by*

$$\dim(M_{\mathfrak{h}}(1,0)_{(2)}) \geq \frac{d}{2} \tag{8.2}$$

and

$$\dim(M_{\mathfrak{h}}(1,0)_{(3)}) \geq d. \tag{8.3}$$

In particular, the bounds are sharp.

Proof. Let $\{v_i\}$ denote the set of simultaneous eigenvectors of the elements of G.

$\mathfrak{h}_{(2)}$ is spanned by states of the form $(v_i)_{-2}\mathfrak{e}_0$ and $(v_i)_{-1}(v_j)_{-1}\mathfrak{e}_0$. Clearly, none of the former states will be invariant if G contains a fixed-point-free element. Among the latter, elements of the form $(v_i)_{-1}(\overline{v_i})_{-1}\mathfrak{e}_0$, where $\bar{\cdot}$ denotes componentwise complex conjugation, are clearly G-invariant yielding at least $\frac{d}{2}$ invariant elements. If G contains an element g with no real eigenvalues and whose characteristic polynomial is squarefree then there are exactly $\frac{d}{2}$ states of this form and they contain all g-invariant states at level 2. Hence the bound is sharp.

$\mathfrak{h}_3$ is spanned by states of the form $(v_i)_{-3}\mathfrak{e}_0$, $(v_i)_{-2}(v_j)_{-1}\mathfrak{e}_0$ and $(v_i)_{-1}(v_j)_{-1}(v_k)_{-1}\mathfrak{e}_0$. By a similar reasoning to above the first two types of states contribute at least a total of d states. A state of the last type is invariant under an element g if the respective eigenvalues satisfy $\lambda_i\lambda_j\lambda_k = 1$. Now let g have even order n and let all its eigenvalues be primitive n-th roots of unity. Then the product of any 3 eigenvalues is an odd power of a primitive n-th root of unity and there no invariant states. Hence the bound $\dim(\mathfrak{h}_{(3)}) \geq d$ is sharp. □

For cyclic groups, it is, in fact, possible to give exact expressions for the dimensions of the invariant low-weight homogeneous subspaces. The characteristic polynomial of a lattice automorphism g is a product of cyclotomic polynomials $\Psi_t(x)$. We say that an automorphism g of order n with characteristic polynomial

$$\mathcal{P}_g(x) = \prod_{t|n} \Psi_t(x)^{b_t}, \quad b_t \in \mathbb{N} \tag{8.4}$$

has cyclotomic type $\prod_{t|n} t^{b_t}$.

Lemma 8.2. *Let g be an automorphism of order n and cyclotomic type $\prod_{t|n} t^{b_t}$. Then the character of $\mathfrak{h}^g$ is given by*

$$\chi_{\mathfrak{h}^g}(q) = 1 + b_1 q + \left(\frac{3}{2}b_1 + \frac{1}{2}b_2 + \frac{1}{2}\sum_{t|n}\varphi(t)b_t^2\right)q^2 + O(q^3)\ , \tag{8.5}$$

where φ denotes Euler's totient function. □

Hence we find that for any finite abelian automorphism group of lifted lattice automorphisms the number of low-weight states is bounded from below. In particular, this establishes that the VOA at central charge $d = 72$ with character $\chi_V(\tau) = q^{-3} + 12q^{-1} + 200 + \mathcal{O}(q)$ constructed above cannot be constructed as a cyclic orbifold of a lattice VOA by a lifted lattice automorphism, thus proving Theorem 1.1.

Appendix A. Lifting semidirect products of cyclic groups $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$

In view of section 3.2, let us discuss the lifting theory of groups of the form

$$G = \mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p\ . \tag{A.1}$$

In this section we discuss how to construct a splitting lift such that

$$\widehat{\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p} \cong \mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p \ . \tag{A.2}$$

Finding such a lift that splits is particularly important if G possesses the level-matching and character properties and we want $\hat{G}$ to retain them. This amounts to constructing lifts $\hat{g}$ and $\hat{h}$ satisfying the group relations, namely

$$\hat{h}^q = 1 \tag{A.3}$$

$$\hat{g}^p = 1 \tag{A.4}$$

$$\hat{g}\hat{h}\hat{g}^{-1} = \hat{h}^\phi \ . \tag{A.5}$$

For simplicity, we would additionally like to impose that $\hat{g}$ and $\hat{h}$ are standard lifts without order doubling, such that the first two group relations are automatically satisfied. When $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_p$ has periodic cohomology, any element is conjugate to a power of one of the generators so that every element is a standard lift if the generators are. In general, this is however not the case. It is straightforward to generalize our construction for the case of non-standard lifts.

As for the last relation, $\hat{g}\hat{h}\hat{g}^{-1} = \hat{h}^\phi$, it can be readily verified that it is equivalent to η_g and η_h satisfying

$$\eta_g(\alpha g^{-1}h)\overline{\eta}_g(\alpha g^{-1})\eta_h(\alpha g^{-1}) \prod_{i=0}^{\phi-1} \overline{\eta}_h(\alpha h^i) = 1. \tag{A.6}$$

To obtain such η_g and η_h, we construct homomorphisms ξ_g and $\xi_h \in \mathrm{Hom}(L, \mathbb{C}^*)$ such that the new lifts $\tilde{\eta}_g = \eta_g\xi_g$ and $\tilde{\eta}_h = \eta_h\xi_h$ do satisfy Equation (A.6). In order for the new $\tilde{\eta}_g$ and $\tilde{\eta}_h$ to still be standard lifts we demand that $\xi_g(L_g) = \xi_h(L_h) = 1$. Note that this amounts to exploring the space of crossed homomorphisms $\xi : \mathbb{Z}_q \rtimes_q \mathbb{Z}_p \to \mathrm{Hom}(L, \mathbb{C}^*)$ that classify the sections of $\mathbb{Z}_q \rtimes_q \mathbb{Z}_p$ in the extension $\mathrm{Hom}(L, \mathbb{C}^*) \cdot \mathbb{Z}_q \rtimes_q \mathbb{Z}_p$.

By careful analysis of the action of the group $\mathbb{Z}_q \rtimes_q \mathbb{Z}_p$ on a lattice using the Smith normal form, we find that there is an obstruction to the existence of such ξ_g and ξ_h and we construct the homomorphisms explicitly when they exist. For all examples we consider in chapter 8, this obstruction vanishes so that we can construct a splitting lift.

Note that for a generalized cyclic orbifold, the character of the holomorphic extension does not depend on the precise form of the lift, but is determined by the conjugacy classes of $\hat{g}$ and $\hat{h}$. Hence we only need to prove the existence of such a lift in order to compute the character. For orbifolds that do not fall into this category, such as for instance $\mathbb{Z}_3 \times \mathbb{Z}_3$, we will need the explicit expression for the lift, which is provided in Proposition Appendix A.3, since here the computation of characters does require the construction of twisted modules and the actions of automorphisms on them.

Let now $\hat{g}$ and $\hat{h}$ be standard lifts without order doubling and η_g and η_h the corresponding functions that will not satisfy (A.6). We therefore want to construct homomorphisms ξ_g and ξ_h such that the new lifts $\tilde{\eta}_g = \eta_g\xi_g$ and $\tilde{\eta}_h = \eta_h\xi_h$ do satisfy

Equation (A.6). In order for the new $\tilde{\eta}_g$ and $\tilde{\eta}_h$ to still be standard lifts we demand that $\xi_g(L_g) = \xi_h(L_h) = 1$.

For convenience we will define the homomorphism f as the failure of $\hat{g}$ and $\hat{h}$ to satisfy Equation (A.5)

$$f(\alpha) := \eta_g(\alpha g^{-1} h)\overline{\eta}_g(\alpha g^{-1})\eta_h(\alpha g^{-1}) \prod_{i=0}^{\phi-1} \overline{\eta}_h(\alpha h^i). \tag{A.7}$$

To satisfy (A.6), ξ_g and ξ_h must then satisfy

$$\xi_g(\alpha g^{-1}(h-1))\xi_h(\alpha(g^{-1} - \sum_{i=0}^{\phi-1} h^i)) = f(\alpha). \tag{A.8}$$

We now want to construct such homomorphisms ξ_h, ξ_g. To do this, we will want to use the Smith normal form on the quotient L/L_h. Note that the action of g on this quotient is well defined since the projector $\pi_h = \sum_{i=0}^{q-1} h^i$ onto the h-invariant subspace commutes with g or, in other words, $gL_h \subset L_h$. Furthermore, we find that $f(\alpha) = 1$ for all $\alpha \in L_h$, so that f is also defined on the quotient.

Now in order to find ξ_g and ξ_h, let us define the following basis of L:

Lemma Appendix A.1. *There exist two bases* $\{\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_l, \epsilon_1, \dots, \epsilon_m\}$ *and* $\{\beta_1, \dots, \beta_k, \delta_1, \dots, \delta_l, \epsilon_1, \dots, \epsilon_m\}$ *of* L *together with positive integers* s_i *such that*

$$\alpha_i(g^{-1} - \sum_{i=0}^{\phi-1} h^i) - s_i\beta_i \in L_h \tag{A.9}$$

$$\gamma_j(g^{-1} - \sum_{i=0}^{\phi-1} h^i) \in L_h \tag{A.10}$$

$$\epsilon_j(g^{-1} - \sum_{i=0}^{\phi-1} h^i) \in L_h \ . \tag{A.11}$$

Proof. Let $L = \Lambda_h + L_h$ be a decomposition into the fixed-point sublattice L_h and a primitive sublattice $\Lambda_h \subset L$ as defined by Lemma 5.1, such that every element $\alpha \in L$ can be written as $\alpha = \gamma + \epsilon$ for some $\gamma \in L_h$ and $\epsilon \in \Lambda_h$. Pick $\{\epsilon_j\}$ to be a basis of L_h. (A.11) holds since g acts on L_h, and also establishes that $g^{-1} - \sum_{i=0}^{\phi-1} h^i$ acts on the quotient L/L_h. We can therefore use the Smith normal form for L/L_h, meaning that there exist bases $\{a_1, \dots, a_k, c_1, \dots, c_l\}$ and $\{b_1, \dots, b_k, d_1, \dots, d_l\}$ of L/L_h such that

$$a_i(g^{-1} - \sum_{i=0}^{\phi-1} h^i) = s_i b_i \tag{A.12}$$

and

$$c_j(g^{-1} - \sum_{i=0}^{\phi-1} h^i) = 0, \tag{A.13}$$

where $\{s_1, \ldots, s_k\}$ are the elementary divisors of $g^{-1} - \sum_{i=0}^{\phi-1} h^i$. Using the isomorphism $\Lambda_h \cong L/L_h$ as free $\mathbb{Z}$-modules, we can lift the above bases of L/L_h to bases $\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_l\}$ and $\{\beta_1, \ldots, \beta_k, \delta_1, \ldots, \delta_l\}$ of Λ_h. (A.9) and (A.10) are then automatically satisfied. □

Note that by Lemma 5.1 the basis $\{\gamma_i\}$ spans the complement of L_h in $\ker((g^{-1} - \sum_{i=0}^{\phi-1} h^i)(1-h))$ and the basis $\{\beta_i\}$ spans the complement of L_h in $\ker(g^{-1}(\sum_{i=0}^{p-1} M_h)(1-h))$, where $M_h = g\sum_{i=0}^{\phi-1}$ is an operator of order p on the quotient L/L_h.

Let us now discuss under what conditions it might be possible to construct such a lift.

We will prove the following lemma:

Lemma Appendix A.2. *There is no solution to (A.8) if $f(\alpha) \neq 1$ for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$.*

Proof. Note that by the group relations the equality

$$g^{-1}(h-1)(1-g) = -(g^{-1} - \sum_{i=0}^{\phi-1} h^i)(1-h) \tag{A.14}$$

holds. Hence for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$, it follows that $\alpha g^{-1}(h-1) \in L_g$ and $\alpha(g^{-1} - \sum_{i=0}^{\phi-1} h^i) \in L_h$. As ξ_g and ξ_h are both standard lifts, that is they vanish on L_g and L_h, respectively, it follows that

$$\xi_g(\alpha g^{-1}(h-1)) = 1 \tag{A.15}$$

and

$$\xi_h(\alpha(g^{-1} - \sum_{i=0}^{\phi-1} h^i)) = 1 \tag{A.16}$$

for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$. Hence the left side of Equation (A.8) vanishes for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$ so that there can only be a solution if the right side also vanishes. □

Using the basis of L constructed in Lemma Appendix A.1, we can now define possible solutions ξ_g, and ξ_h. Note that we find that a solution always exists if $f(\alpha) = 1$ for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$.

Proposition Appendix A.3. *There is a solution to (A.8) if and only if $f(\alpha) = 1$ for all $\alpha \in \ker(g^{-1}(h-1)(1-g))$. It is given by*

$$\xi_g(\alpha) = 1, \ \textit{for all } \alpha \in L \tag{A.17}$$

and

$$\xi_h(\beta_i) = f(\alpha_i)^{\frac{1}{s_i}} \ , \quad \xi_h(\delta_j) = 1 \ , \quad \xi_h(\epsilon_j) = 1 \ . \tag{A.18}$$

Proof. We prove this by evaluating (A.8) on the basis $\{\alpha_i, \gamma_j, \epsilon_j\}$. Note that the first factor in (A.8) vanishes because $\xi_g = 1$. We immediately have

$$f(\epsilon_j) = 1 = \xi_h(\epsilon_j(g^{-1} - \sum_{i=0}^{\phi-1} h^i)) \ . \tag{A.19}$$

By (A.9) we have

$$f(\alpha_i) = \xi_h(s_i\beta_i) = \xi_h(\alpha_i(g^{-1} - \sum_{i=0}^{\phi-1} h^i)) \tag{A.20}$$

and by (A.10)

$$f(\gamma_i) = 1 = \xi_h(\gamma_i(g^{-1} - \sum_{i=0}^{\phi-1} h^i)) \ . \quad \square \tag{A.21}$$

It turns out that for all the groups we consider the condition holds so that a splitting lift exists. Note also that for the abelian group $\mathbb{Z}_q \times \mathbb{Z}_p$ the condition is trivial and a splitting lift always exists.

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