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The evolution of roles☆

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A R T I C L E I N F O

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A B S T R A C T

We propose a novel formalization of roles in social networks that unifies the most commonly used definitions of role equivalence. As one consequence, we obtain a single, straightforward proof that role equivalences form lattices. Our formalization focuses on the evolution of roles from arbitrary initial conditions and thereby generalizes notions of relative and iterated roles that have been suggested previously. In addition to the unified structure result this provides a potential micro-foundation for the emergence of roles. Considering the genesis of roles may explain, and help overcome, the problem that social networks rarely exhibit interesting role equivalences of the traditional kind. Finally, we discuss extensions of the role concept to multivariate networks with actor and tie attributes by example.

1. Introduction

Role equivalences have been developed as formalizations of the concepts of social role and position. They are based on the idea that role-equivalent actors are related to others in similar or compatible ways (Lerner, 2005). This assumption, however, has been interpreted and formalized in various ways. These formalizations are of two fundamentally different types (Borgatti and Everett, 1992; Faust, 1988): One defines equivalent actors to have the same relations to identical others (*structural equivalence*, Lorrain and White, 1971), while the other expects equivalent actors to have the same relations to equivalent others (*general equivalences*, Faust, 1988). General equivalences in turn can be defined on, e.g., the global role structure, expecting equivalent individuals to have relationships to others with the same roles (*regular equivalence*, White and Reitz, 1983), or the involvement of individuals in the same types of direct or indirect relationships to others (*ego algebras*, Breiger and Pattison, 1986; or *Winship-Pattison equivalence*, Winship and Mandel, 1983).

While all these conceptions of role are rigorous and mathematically elegant, non-trivial instances are rare in empirical research, since vertices are more likely to be almost, but not quite equivalent in observed networks (Ferligoj et al., 2011). Therefore, many relaxations of equivalence have been developed that measure similarity instead of equivalence (*structural similarity*, Brandes and Lerner, 2010; Leicht et al., 2006; or *social distance*, Burt, 1976) or search partitions that approximate perfect solutions (e.g., *generalized blockmodeling*, Ferligoj et al., 2011).

Here, we take a different approach. One cause for the rarity of non-trivial role equivalences is that their definition is self-referential. Instead of relaxing equivalence to some kind of similarity or approximating perfect solutions, we break up the self-referentiality found within the definition of role equivalence. This leads to a generalized notion of roles relative to some arbitrary given equivalence, and via repeated applications of relative roles, a process of role formation. This formalization of roles not only unifies commonly used definitions of role equivalence, but also allows us to investigate common structural properties and enables us to derive a general proof that role equivalences form lattices.

This paper is organized as follows. Section 2 covers basic mathematical concepts and describes several commonly used notions of role equivalence. In Section 3, we derive a generalized definition of roles by dissolving self-referentiality, and introduce a process of role formation via repeated evaluations of relative roles. We inspect structural properties of this general conceptualization of roles and prove the lattice structure of role equivalences in Section 4. Section 5 establishes that our general formalization in fact unifies many commonly used definitions of role equivalences by demonstrating that they are particular instantiations of it. Utilizing the expressiveness of the formalization, we show how we can derive error-tolerant versions of commonly used definitions of role equivalence by means of an example, and use an error-tolerant equivalence to identify roles on an empirical network that are

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not discovered by established role equivalences in Section 6. Next, we illustrate how to incorporate actor and tie attributes of multivariate networks in the discovery of roles in Section 7. Finally, we give our conclusions and discuss further extensions as well as future directions in Section 8.

2. Preliminaries

We provide a short overview of basic definitions and concepts in this section. For further background reading, we suggest to the reader Diestel (2010) on graphs, Grätzer (2002) on orders and lattices and Lerner (2005) on role equivalences.

2.1. Graphs, equivalences, and lattices

We focus on simple undirected and unweighted graphs $G = (V, E)$, composed of a finite set of vertices V and a set of edges $E\subseteq\binom{V}{2} = \{\{i,j\} : i, j \in V, i \neq j\}$. We denote the number of vertices by $n = |V|$ and the number of edges by $m = |E|$.

A *neighbor* of a vertex i is a vertex j connected to i via an edge $\{i, j\} \in E$. In this case, we say that i and j are *adjacent*. The *neighborhood* $N(i)$ is the set of all neighbors of i, and the *degree* $\deg(i) = |N(i)|$ of i is the cardinality of its neighborhood.

An *equivalence* is a binary relation that is reflexive, symmetric and transitive. For two equivalences $\approx, \approx'$ with $\approx\subseteq\approx'$, we say that $\approx$ *refines* $\approx'$ and that $\approx'$ *coarsens* $\approx$. If $\approx\subseteq V\times V$ is an equivalence relation, then $[i]_\approx = \{j \in V : j \approx i\}$ is called the *equivalence class* of $i \in V$. We note that there is a one-to-one relationship between equivalence relations and partitions via equivalence classes, and may refer to them interchangeably. We use $\mathcal{P}$ to denote the set of all equivalences $\approx\subseteq V\times V$, and denote the transitive closure of a relation R by $[R]^*$.

A *lattice* is a partially ordered set closed under two operations, a *meet* $x \sqcap y$ and a *join* $x \sqcup y$, that yield the unique greatest lower bound (infimum) and least upper bound (supremum) for x and y. The set of equivalence relations on the set of vertices of a graph form a lattice because they are partially ordered by refinement with intersection and the transitive closure of union serving as the meet and join operation.

2.2. Role equivalences

A *role equivalence* is a partition of the vertices of a graph into equivalence classes such that two vertices are equivalent only if their neighborhoods are considered equivalent according to some specific definition. The following are three common examples with an illustration in Fig. 1. Additional definitions of neighborhood equivalence are considered in Section 5.

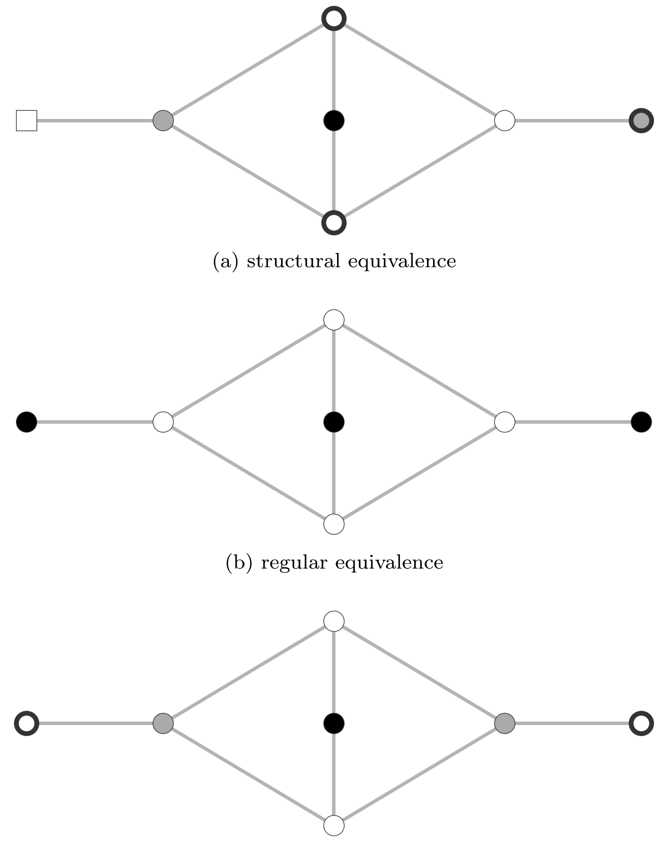
The strictest definition of neighborhood equivalence is equality. A role equivalence $\approx$ is called *structural* (Lorrain and White, 1971), if any two equivalent vertices have the exact same neighborhood, i.e., for all i, $j \in V$ we have that

$$i \approx j \Leftrightarrow N(i) = N(j).$$

An important relaxation of neighborhood equality is the presence of neighbors from the same classes rather than the same neighbors. A role equivalence $\approx$ is called *regular* (White and Reitz, 1983), if any two equivalent vertices are adjacent only to vertices from the same equivalence classes, i.e., for all $i, j \in V$ we have that

$$i \approx j \Rightarrow \begin{array}{l} \text{for each } k \in N(i) \text{ there exists} \\ \ell \in N(j) \text{ such that } \ell \approx k. \end{array}$$

Inbetween these two notions of role equivalence is *automorphic role equivalence* (Everett, 1985; Winship, 1988), where two vertices $i, j \in V$ may be equivalent if and only if there exists an automorphism $\varphi : V \to V$ such that $\varphi(i) = j$. Note that this implies that the neighbors can be matched into automorphically equivalent pairs themselves.



(a) structural equivalence

(b) regular equivalence

(c) automorphic equivalence

Fig. 1. Structural, regular, and automorphic equivalence illustrated. Equivalent vertices are drawn in the same style.

For many definitions of role equivalence, a graph admits multiple different role equivalences of the same kind, and if so these role equivalences form a lattice structure with respect to refinement (Everett and Borgatti, 1994). While an elegant concept mathematically, non-trivial role equivalences are rarely encountered in empirical network research.

3. Role formation

Underlying role equivalence is the fundamental idea that vertices in the same roles have neighborhoods equivalent in some manner. As Borgatti and Everett (1992) observed, there have been at least two different fundamental ways to operationalize this idea: On the one hand, equivalent vertices are required to have relationships with *identical* others in structural equivalence. On the other hand, equivalent vertices only need to have relationships with *the same kinds* of others in regular equivalence and similar role equivalence concepts. Put differently, neighbors of a vertex are not replaceable by any other vertices in structural equivalence, and in regular equivalence-like notions, neighbors are substitutable by any other vertices of the same kind.

In this section, we first refine this idea of substitutability and develop a general representation of substitution rules underlying notions of role. Then we break up the inherent self-referentiality in traditional role equivalences using a generalized notion of relative roles based on substitutions, and subsequently complement it with a variant of iterated roles to win back the original concept.

3.1. Notions of substitutability

Commonly used kinds of role equivalence particularly differ in the notions of substitutability that underlie their definitions of neighborhood equivalence. Fig. 2 illustrates this for structural and regular equivalences. Structural equivalence stipulates that two equivalent vertices must have the same neighbors. This means that neighbors of the former vertex must appear as neighbors of the latter and vice versa, so any of the neighbors is irreplaceable (Fig. 2a). Meanwhile, regular equivalence only requires that equivalent vertices must in turn have equivalent neighbors. Thus, we must be able to find for each neighbor of the former vertex an equivalent neighbor of the latter and vice versa (Fig. 2b). Similarly, many other notions of role equivalence impose some notion of substitutability among neighborhoods.

Fig. 2b also depicts one possible assignment of neighbors of one vertex to some choice of substitutes from the neighbors of the other. In the same spirit, we model a particular substituting vertex assignment for the whole neighborhood as a mapping $\sigma : V \to V$ and call it a *substitution*. We interpret such a substitution in the sense that we substitute a vertex $k \in V$ in one neighborhood by the vertex $\sigma(k)$ in the other. For structural equivalence, the only admissible substitution is the identity mapping, since all vertices are irreplaceable. For many other notions of substitutability, however, there is not only one admissible substitutive assignment between neighborhoods, but a plurality of them; e.g., for a regular equivalence $\approx$, any substitution σ that substitutes vertices enacting the same roles, i.e., satisfies $\sigma(k) \in [k]_\approx$ for all vertices $k \in V$, is admissible. When deciding neighborhood equivalence for a pair of vertices $i, j \in V$, we therefore generally deal with sets of substitutions $\Sigma_{ij}(\approx) \subseteq (V \to V)$ which contain all substitutions admissible under the selected notion of substitutability. The following definition formalizes this idea, imposing some additional constraints for well-behavedness.

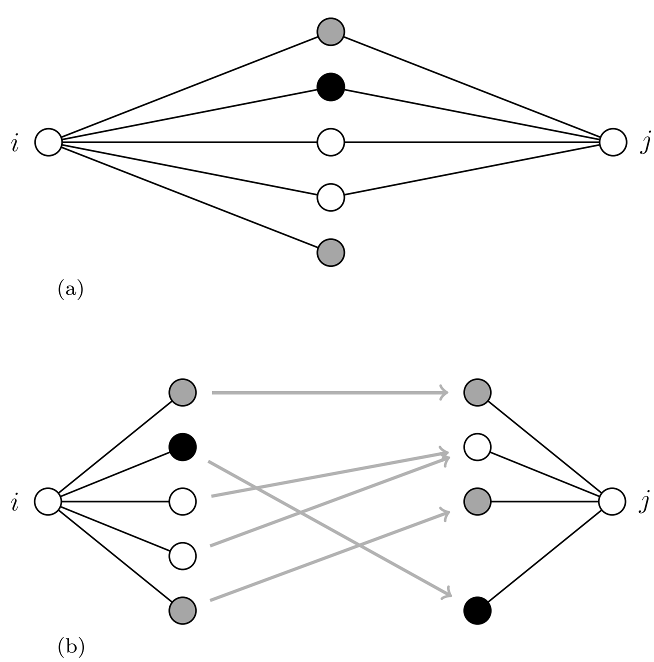


Fig. 2. Substitution rules underlying structural and regular equivalence illustrated from the neighborhood of vertex *i* to vertex *j*. The roles of neighbors are depicted by vertex color.
(a) Structural equivalence: Equivalent vertices must have the same neighbors; thus, no other vertex can stand in for any of *i*'s neighbors, or put differently, a neighbor is only substitutable by itself, independent of any enacted roles. In the depicted example, one neighbor of *i* is not a neighbor of *j* and substitution from *i* to *j* fails.
(b) Regular equivalence: Equivalent vertices must have neighbors enacting the same roles, which implies that neighbors of *i* are potentially substitutable by any vertex in the same roles. The arrows from neighbors of *i* to those of *j* produce one admissible manner of substitution under this rule.

Definition 1. Let $\approx \subseteq V \times V$ be an equivalence relation on a set V. A family $\Sigma_{ij}(\approx) \subseteq (V \to V)$, $i, j \in V$ of subsets of all *substitutions* is called a family of *admissible substitutions*, if

1. the family is closed under transitive composition, i.e., $\sigma_{ij} \in \Sigma_{ij}(\approx)$ and $\sigma_{jk} \in \Sigma_{jk}(\approx)$ implies that the *transitive composition* $\sigma_{jk} \circ \sigma_{ij} : x \mapsto \sigma_{jk}(\sigma_{ij}(x))$ is contained in $\Sigma_{ik}(\approx)$ for all $i, j, k \in V$, and
2. sets $\Sigma_{ii}(\approx)$ contain the identity mapping for all $i \in V$.

While we introduced this concept of substitutions on structural and regular equivalence, it also generalizes to other definitions of role equivalence by controlling admissible substitutions suitably. We may incorporate distinctions and commonalities between vertices or kinds of vertices (especially vertices enacting same or different roles) by way of making these substitutable or not. By restricting substitutions to permutations, we can factor in the number of vertices of some kind among neighbors. Similarly, we can suitably account for attributes on neighbors and incident ties, or tolerate some minor discrepancies in neighborhoods. In this fashion, we can not only describe substitutions underlying classic definitions of role equivalence, but also have a powerful tool at hand to recombine different aspects of them or express other mechanisms of substitutability, allowing us to describe new notions of roles.

3.2. Generalized roles

A major obstacle for the presence of non-trivial role equivalences in observed networks is that the definition of equivalence is self-referential: whether two vertices can be considered equivalent depends on neighbors being identical (structural equivalence) or equivalent in *the same relation* (regular, automorphic, and most other equivalences). We now dissolve this reflexivity and derive a unified notion of role equivalence that generalizes previous ones.

The following definition makes sure that role-equivalent vertices have neighbors that are allowed to substitute for each other.

Definition 2. Let $G = (V, E)$ be a simple undirected graph. Given any equivalence relation $\sim \subseteq V \times V$ and a family $\Sigma_{ij}(\sim)$ of admissible substitutions, the *role equivalence relative to* $\Sigma_{ij}(\sim)$ is defined by

$$i \approx j \Leftrightarrow \begin{cases} \text{there exist } \sigma_{ij} \in \Sigma_{ij}(\sim), \sigma_{ji} \in \Sigma_{ji}(\sim) \text{ s.t.} \\ \sigma_{ij}(k) \in N(j) \text{ for all } k \in N(i) \text{ and} \\ \sigma_{ji}(\ell) \in N(i) \text{ for all } \ell \in N(j). \end{cases}$$

The definition of $\approx$ is clearly symmetric, and the identity mapping $\mathrm{id} \in \Sigma_{ii}(\sim)$ ensures that it is reflexive. Since the transitive composition of admissible substitutions yields only admissible substitutions due to closure under this operation, $\approx$ is also transitive and thus indeed an equivalence.

The definitions in Section 2.2 are designed to verify that a given equivalence $\approx$ satisfies conditions on the neighborhoods of *i* and *j*. Our new definition not only verifies but simultaneously establishes the relation $\approx$. Since the sufficiency of the neighborhood condition depends on admissible substitutions, this is different from so-called *perfect equivalence* (Everett and Borgatti, 1994) where the condition still depends on the equivalence itself. Rather, it should be seen as a generalization of the *relative equivalences* of Boyd and Everett (1999), where among other restrictions, the role equivalence is a refinement of the given equivalence.

By restricting the admissible substitutions in $\Sigma_{ij}(\sim)$, we can control how neighborhoods are being compared. This unifies the above, and most other, notions of role equivalence, and will be discussed in more detail in Section 5.

More importantly, our definition generalizes previous ones by allowing for the comparison of neighbors based on equivalences that need not be the role equivalence itself. We may even define admissible substitutions in terms of attribute data rather than structurally. An example would be a social network of individuals where each individual is labeled by a profession and two individuals are considered role-equivalent, if they have the same relationships with members of the same professions.

Since the role equivalences obtained from Definition 2 are parameterized by an equivalence and a restricted set of neighbor substitutions, they can also be viewed as the outcome of an operation $\diamond$ that turns an arbitrary vertex equivalence $\approx$ into a new equivalence $\diamond(\approx)$ based on neighborhood substitutability. We introduce two more operations that consider the given equivalence not only in the comparison of neighborhoods but refine or coarsen it based on neighborhoods.

Definition 3. Let $G = (V, E)$ be a simple undirected graph. Given any equivalence relation $\approx \subseteq V \times V$, denote by $\diamond(\approx)$ the role equivalence relative to a family $\Sigma_{ij}(\approx)$ of admissible substitutions. Define the *role restriction* as the intersection

$$\triangledown(\approx) = \approx \cap \diamond(\approx)$$

and the *role extension* as the transitive closure

$$\vartriangle(\approx) = [\approx \cup \diamond(\approx)]^*$$

of the union of $\approx$ and $\diamond(\approx)$.

3.3. Role evolution

All three operations defined in the previous subsection turn any partition of the vertices of a graph into a new one by comparing the neighborhoods of vertices. By repeatedly applying such operations we obtain a sequence of role equivalences, each one relative to the previous one, and thus an evolution of roles.

How roles evolve depends not only on the operation applied but, crucially, on the choice of admissible substitutions. Note that this set may be completely different for each application of an operation. In fact, we may choose the admissible substitutions anew for each application and thus mimick *iterated roles* (Borgatti et al., 1989), where role equivalences are defined on the graph of the equivalence classes of another role equivalence.

While it will be interesting to study particular families of admissible substitutions constructed from a common principle, our focus in this paper is the evolution of roles when the same notions of neighborhood equivalence are applied. Therefore, it will be sufficient to require the following property.

Definition 4. Let any $\approx_1, \approx_2, \ldots \subseteq V \times V$ be a family of equivalence relations on the same set V, An associated family $\Sigma_{ij}(\approx_t)$, $i, j \in V$, $t = 1, 2, \ldots$ of admissible substitutions is called *isotone*, if

$$\approx_r \subseteq \approx_s \Rightarrow \Sigma_{ij}(\approx_r) \subseteq \Sigma_{ij}(\approx_s)$$

for all $r, s \geq 1$ and $i, j \in V$.

An evolution of roles is obtained from repeated application of an operation $\diamond$, $\triangledown$, or $\vartriangle$. For an initial equivalence $\approx = \approx^{(0)}$, we denote $\approx^{(t)} = \diamond(\approx^{(t-1)}) = \diamond^t(\approx)$ for $t \geq 1$ and likewise for the other two processes. Throughout this paper, we assume that the associated family of admissible substitutions is isotone. Fig. 4 shows the evolution of the process for some given equivalence on the network depicted in Fig. 3. As Fig. 4 demonstrates, the process does not have to follow a single direction in the lattice of equivalences, but may alternatingly increase and decrease the number of equivalence classes in each step.

4. Stable roles and the lattices of role equivalences

Our general definition of roles relative to a given partition operates on the lattice of all vertex partitions ordered by refinement. By application of one of the three operations – relative role equivalence $\diamond$, role restriction $\triangledown$, or role extension $\vartriangle$ – we are moving about this lattice, and we say that an equivalence is a *stable role equivalence* with respect to one of the three operations if application of this operation does not alter it. We observe that stable role equivalences with respect to relative role equivalence are those that are stable with respect to all three operations.

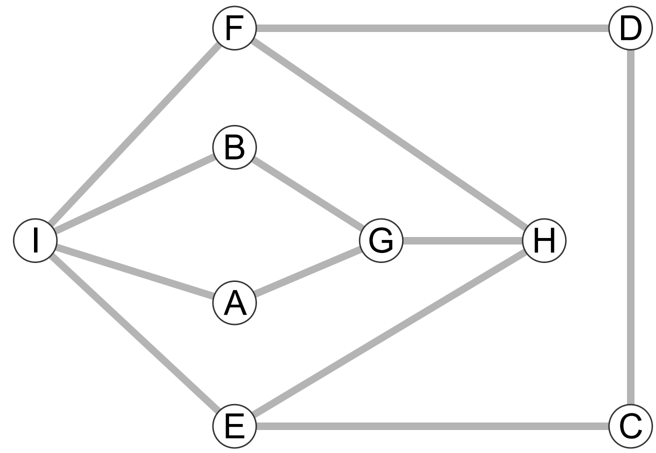


Fig. 3. Example graph used for illustrative purposes.

When following a process obtained from repeated applications of $\diamond$, $\triangledown$ or $\vartriangle$, we inevitably pass by an equivalence that was encountered before, since there is only a finite number of equivalences. This can imply that the process led to a stable role equivalence, but for the operation $\diamond$ it could also mean that a cycle was entered. Fig. 5 depicts two cycles in repeated applications of relative role equivalence $\diamond$. The cycle in Fig. 5b also demonstrates that equivalences traversed in the same cycle can differ in the number of equivalence classes they consist of.

Repeated applications of role restriction $\triangledown$ or extension $\vartriangle$, however, are guaranteed to converge to a stable role equivalence and can never enter a cycle. Since $\triangledown$ always refines and $\vartriangle$ always coarsens a given equivalence, repeated applications can only move through the lattice of equivalences in one direction. Processes induced by $\triangledown$ and $\vartriangle$ therefore must have reached a stable role equivalence after at most $n-1$ steps, because the size of any completely ordered subset of equivalences is bounded by n. We refer to these stable role equivalences that were reached from an initial equivalence $\approx$ as the *closure* $\vartriangle^*(\approx) = \lim_{t\to\infty} \vartriangle^t(\approx) = \vartriangle^{n-1}(\approx)$ and the *interior* $\triangledown^*(\approx) = \lim_{t\to\infty} \triangledown^t(\approx) = \triangledown^{n-1}(\approx)$ of $\approx$. The closure is the smallest stable role equivalence w.r.t. role extension that coarsens $\approx$, while the interior is the greatest stable role equivalence w.r.t. role restriction that refines $\approx$. Closure and interior bound the movement of the processes obtained from $\diamond$, $\triangledown$ and $\vartriangle$ within the lattice of equivalences. By straightforward induction, repeated application of the operations satisfy

$$\triangledown^*(\approx) \subseteq \triangledown^t(\approx) \subseteq \diamond^t(\approx) \subseteq \vartriangle^t(\approx) \subseteq \vartriangle^*(\approx) \quad \text{for} \quad t \geq 0.$$

We will show that the stable role equivalences with respect to the three operations form lattices themselves. This unifies and generalizes a number of previous results for role equivalences. In general, however, these are not sublattices because meet or join need to be adapted.

The crucial tool is a special case of Tarski's Fixed-Point Theorem. It presupposes an operation with the following property.

Definition 5. An operation $f : \mathcal{P} \to \mathcal{P}$ is called *isotone*, if

$$\approx_1 \subseteq \approx_2 \Rightarrow f(\approx_1) \subseteq f(\approx_2)$$

for any two equivalences $\approx_1, \approx_2 \in \mathcal{P}$.

The three operations $\diamond$, $\triangledown$ and $\vartriangle$ are isotone because we assume that the family of admissible substitutions is. Put another way, moving from

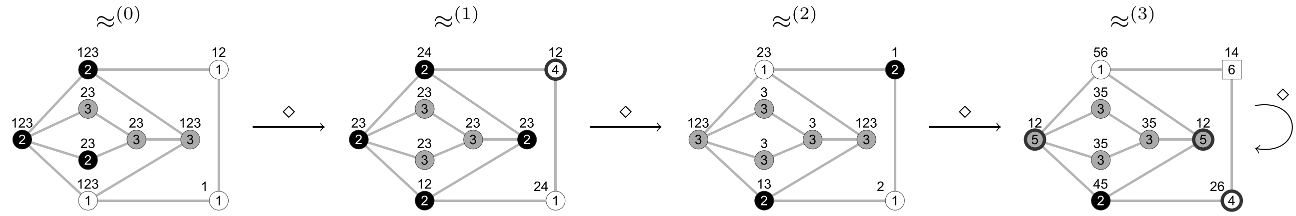


Fig. 4. Process progression on network Fig. 3 starting from equivalence $\approx^{(0)}$ when substitutions between vertices of the same class are admissible. Equivalent vertices have same vertex style and are labeled by the same digit. Labels next to the vertices are a short description of all the classes in their neighborhoods. Vertices with equal labels next to them will thus be assigned into the same roles in the next iteration, as substitution between all neighbors is possible.

a role equivalence $\approx_1$ to a coarser role equivalence $\approx_2$ can result in the introduction of new, but never in the removal of previously present admissible substitutions. Thus, vertices that are role-equivalent relative to $\approx_1$ are also role-equivalent relative to $\approx_2$, because the substitutions that established neighborhood equivalence with respect to $\approx_1$ are also admissible substitutions of $\approx_2$. Consequently, relative role equivalence $\diamond$, role restriction $\triangledown$ (only refining based on relative role equivalence) and role extension $\vartriangle$ (only coarsening based on relative role equivalence) are isotone operations.

A fixed point of an operation $f : \mathcal{P} \rightarrow \mathcal{P}$ is an element $x \in \mathcal{P}$ such that $f(x) = x$. Since stable role equivalences are unchanged by an application of the respective operation $\diamond$, $\triangledown$ or $\vartriangle$, they constitute the fixed points of these three operations.

Tarski's theorem below provides a characterization of the fixed point structure of an isotone operation.

Theorem 1. (Tarski, 1955) *Let $f : \mathcal{P} \rightarrow \mathcal{P}$ be an isotone operation on the lattice $\mathcal{P}$ of equivalences. Then the set F of fixed points of f is non-empty and forms itself a lattice ordered by refinement. Moreover,*

$$\left[\bigcup_{\approx \in F} \approx\right]^* = \left[\bigcup_{\substack{\approx \in \mathcal{P} \\ \approx \subseteq f(\approx)}} \approx\right]^* \in F$$

and

$$\bigcap_{\approx \in F} \approx \quad = \quad \bigcap_{\substack{\approx \in \mathcal{P} \\ \approx \supseteq f(\approx)}} \approx \quad \in F.$$

Applying Tarski's result to the three operators $\diamond$, $\triangledown$ and $\vartriangle$ lets us characterize the structure of stable role equivalences as follows.

Theorem 2. *The role equivalences that are stable w.r.t. $\diamond$, $\triangledown$ and $\vartriangle$ form non-empty lattices ordered by refinement. In these lattices of stable role equivalences, the join or meet are different from those in $\mathcal{P}$:*

1. *In the lattice obtained from $\diamond$, we have $\vartriangle^*([\approx_1 \cup \approx_2]^*)$ for join and $\triangledown^*(\approx_1 \cap \approx_2)$ for meet.*
2. *In the lattice obtained from $\triangledown$, we have $[\approx_1 \cup \approx_2]^*$ for join and $\triangledown^*(\approx_1 \cap \approx_2)$ for meet.*
3. *In the lattice obtained from $\vartriangle$, we have $\vartriangle^*([\approx_1 \cup \approx_2]^*)$ for join and $\approx_1 \cap \approx_2$ for meet.*

Moreover, the minimum (maximum) element in the lattice of role equivalences stable w.r.t. relative role equivalence is also the minimum (maximum) in the lattice w.r.t. role extension (restriction).

Proof. The lattice structures and the result for maximum and minimum stable role equivalences follow immediately from Theorem 1. However, we still have to identify the join and meet operations of the lattices. We only provide a proof for the meet operations here, since the join operations can be determined by a symmetric argument.

In general, observe that any isotone operation $f : \mathcal{P} \rightarrow \mathcal{P}$ has the property that

$$\approx_1 \cap \approx_2 = f(\approx_1) \cap f(\approx_2) \supseteq f(\approx_1 \cap \approx_2) \tag{1}$$

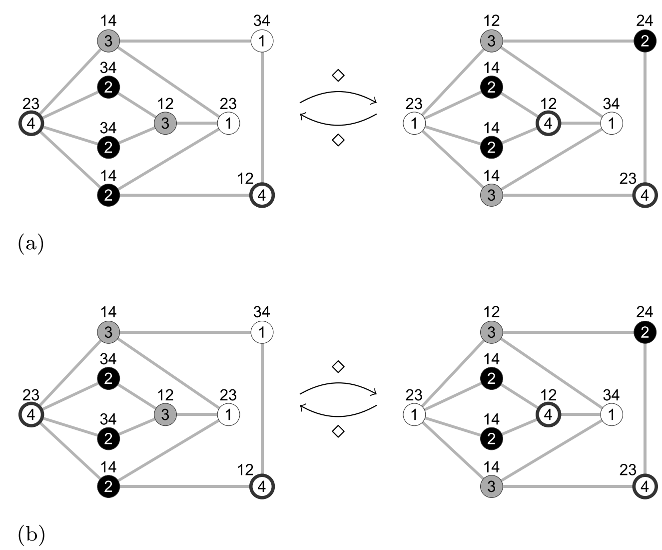


Fig. 5. Two cycles in the process obtained from relative role equivalence $\diamond$ when substitutions between vertices of the same class are admissible. Equivalent vertices have same vertex style and are labeled by the same digit. Labels next to the vertices are a short description of all the classes in their neighborhoods. Vertices with equal labels next to them will thus be assigned into the same roles in the next iteration, as substitution between all neighbors is possible.
(a) Cycle between equivalences with the same number of equivalence classes.
(b) Cycle between equivalences with different numbers of equivalence classes.

for any two fixed points $\approx_1, \approx_2 \in \mathcal{P}$ of f. Since f is isotone, it follows from $\approx_1 \cap \approx_2 \supseteq f(\approx_1 \cap \approx_2)$ that $f(\approx_1 \cap \approx_2) \supseteq f^2(\approx_1 \cap \approx_2)$, and thus we have

$$\approx_1 \cap \approx_2 \supseteq f(\approx_1 \cap \approx_2) \supseteq f^2(\approx_1 \cap \approx_2) \supseteq \cdots \tag{2}$$

Because the set of equivalences is finite, the sequence must reach a fixed point $f^t(\approx_1 \cap \approx_2)$ for some $t \geq 0$. Furthermore, due to the fact that f is isotone, any fixed point $\sim \subseteq \approx_1 \cap \approx_2$ of f also satisfies the inequality $\sim = f(\sim) \subseteq f(\approx_1 \cap \approx_2)$ and hence $\sim \subseteq f^t(\approx_1 \cap \approx_2)$ by induction, so $f^t(\approx_1 \cap \approx_2)$ is the coarsest fixed point of f that refines $\approx_1$ and $\approx_2$. Thus, $f^t(\approx_1 \cap \approx_2)$ is the meet of $\approx_1$ and $\approx_2$ in the fixed point lattice of f.

We now transfer these general findings to relative role equivalence, role restriction and role extension.

For the role restriction operation $\triangledown$, the meet corresponds to the interior $\triangledown^*(\approx_1 \cap \approx_2)$, since it denotes the stable role equivalence reached through repeated applications of $\triangledown$.

For relative role equivalence $\diamond$, we know from (2) that operator iteration only repeatedly refines. Thus, repeated applications of $\diamond$ and $\triangledown$ produce the same sequence of equivalences here, so $\triangledown^*(\approx_1 \cap \approx_2)$ is also the meet of two stable role equivalences $\approx_1$, $\approx_2$ w.r.t. relative role equivalence $\diamond$.

Finally, role extension $\vartriangle$ satisfies the inequality $\approx \subseteq \vartriangle(\approx)$ for any equivalence $\approx$. Combined with (1), it follows that $\approx_1 \cap \approx_2 = \vartriangle(\approx_1 \cap \approx_2)$

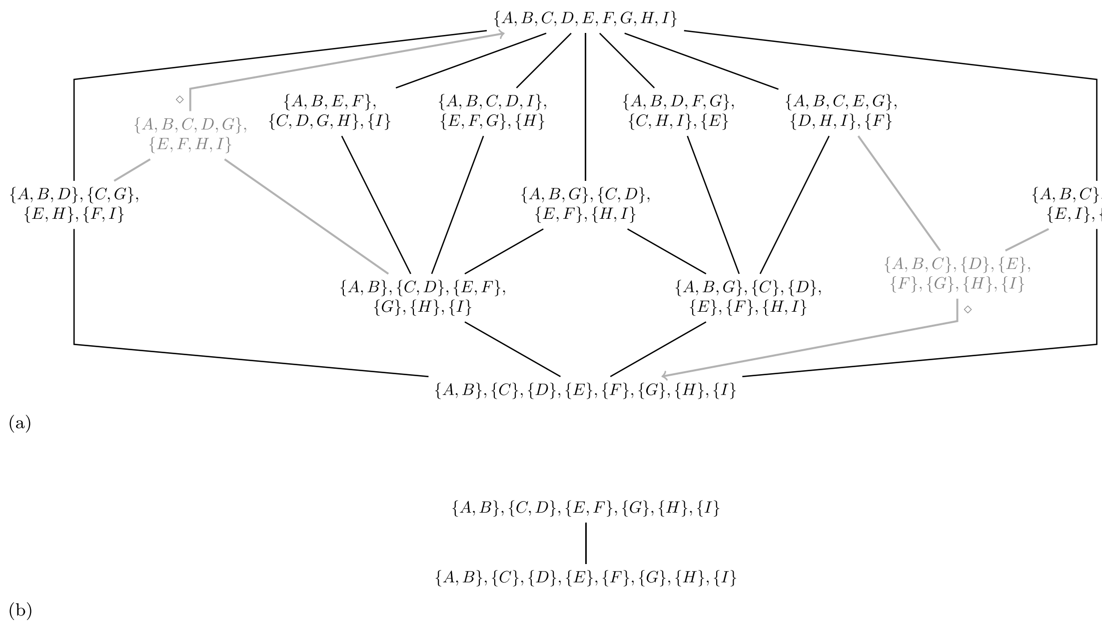


Fig. 6. Lattice structure of stable role equivalences in the network given in Fig. 3 for different choices of admissible substitutions. Each equivalence is represented by its equivalence classes.
(a) Lattice w.r.t. relative role equivalence $\diamond$ when substitutions between vertices of the same class are admissible. These equivalences are also known as perfect equivalences (see Section 5.2). Additionally, one join and one meet of two stable role equivalences in the lattice of all equivalences are depicted in gray. The join and meet are not stable role equivalences themselves, and arrows point to the actual join and meet in this lattice which are reached after an additional application of the operation $\diamond$.
(b) Lattice w.r.t. relative role equivalence $\diamond$ when substitutions are admissible between the same number of vertices from each class. The equivalences are also known as exact perfect equivalences (see Section 5.3).

for two stable role equivalences $\approx_1, \approx_2$ w.r.t. role extension, so $\approx_1 \cap \approx_2$ denotes the meet. □

Fig. 6 depicts lattice structures of stable role equivalences on the network given in Fig. 3. Evidently, the lattices are different when different sets of admissible substitutions are chosen. In addition, one join and one meet of two stable role equivalences in the lattice of all equivalences are depicted in gray in Fig. 6a, but these are actually not stable role equivalences. However, applying the operation $\diamond$ to these and thus following the gray arrows will lead to the actual join and meet in the lattice of stable role equivalences.

5. Established role equivalences

Many commonly used definitions of role equivalence reduce to stable role equivalences for particular selections of substitutions. We demonstrate that our general formalization of roles in fact unifies many of them. This also establishes structural properties such as lattice structure or existence of interior or closure operators for these notions of roles that have usually been shown for each notion individually. We provide select proofs for some crucial notions of role equivalence in Appendix A.

5.1. Structural equivalence

In the strictest version of neighborhood equivalence, neighborhood equality, substitutions between different neighbors are not allowed. Thus we have

$$\Sigma_{\mathrm{ij}}(\approx) = \{\mathrm{id}\}.$$

For this choice of admissible substitutions, the only stable role equivalence w.r.t. relative role equivalence is the structural equivalence. Furthermore, the role equivalences that are stable w.r.t. role restriction are the so-called structural colorations (Everett and Borgatti, 1994). Finally, the stable role equivalences w.r.t. role extension are the coarsenings of the structural equivalence.

5.2. Regular, perfect and ecological equivalence

Relaxing the assumptions of neighborhood equality, a regular equivalence allows for equivalent vertices to have neighbors in the same equivalence classes rather than having exactly the same neighbors. Put another way, a neighbor of one vertex can be replaced by any neighbor of the other in the same class. This suggests the following family of admissible substitutions:

$$\Sigma_{\mathrm{ij}}(\approx) = \{\sigma \in (V \to V) : \sigma(k) \in [k]_{\approx}, k \in V\}. \tag{3}$$

Regular equivalences are exactly the stable role equivalences w.r.t. role restriction under this family of admissible substitutions. Furthermore, two other established examples of role equivalences actually form the stable roles for the other two operations. The stable role equivalences w.r.t. role extension are called ecological equivalences (Everett and Borgatti, 1994). A role equivalence $\approx$ is *ecological* if we have for all $i, j \in V$ that

$$\left.\begin{array}{l}\text{for each } k \in N(i) \text{ there exists } \ell \in N(j) \text{ such that } \ell \approx k \\ \text{for each } k \in N(j) \text{ there exists } \ell \in N(i) \text{ such that } \ell \approx k\end{array}\right\} \Rightarrow i \approx j.$$

Furthermore, stable role equivalences w.r.t. relative role equivalence are those role equivalences which are both regular and ecological, and are also known as *perfect equivalences* (Everett and Borgatti, 1994).

5.3. Exact equivalences

While regular equivalence ensures the presence of neighbors from the same classes for equivalent vertices, the number of neighbors from the same class can differ greatly between regularly equivalent vertices. Exact equivalences (Everett and Borgatti, 1996) tighten the requirements of regular equivalence again and additionally suppose that equivalent neighborhoods have the same number of neighbors from each class. We represent this through permutations in substitutions, as permutations ensure that a neighbor of one vertex is allowed to substitute for at most one neighbor of the other. A family of substitutions admitting the exact equivalences is given by

$$\Sigma_{ij}(\approx) = \left\{\sigma \in (V \to V) : \begin{array}{l} \sigma \text{ is permutation and} \\ \sigma(k) \in [k]_{\approx}, \quad k \in V \end{array}\right\}.$$

Under this family of admissible substitutions, stable role equivalences of role restriction form the exact regular equivalences. A role equivalence $\approx$ is said to be *exact regular* if we have for any vertices $i, j \in V$ that

$$i \approx j \Rightarrow |N(i) \cap [k]_{\approx}| = |N(j) \cap [k]_{\approx}| \quad \text{for each} \quad k \in V.$$

Similarly, exact ecological equivalences are the stable role equivalences of the role extension operation $\vartriangle$. A role equivalence $\approx$ is called *exact ecological* if any vertices $i, j \in V$ satisfy

$$|N(i) \cap [k]_{\approx}| = |N(j) \cap [k]_{\approx}| \quad \text{for each} \quad k \in V \Rightarrow i \approx j.$$

Finally, the stable role equivalences of relative role equivalence $\diamond$ are both exact regular and exact ecological. Such equivalences are also called *exact perfect*.

5.4. Weak role equivalence

Weak role equivalence (Lerner, 2005) relaxes the assumptions underlying regular equivalence further. While regular equivalence requires that equivalent vertices are adjacent to role-equivalent neighbors, weak role equivalence assumes that it is sufficient for equivalence between two vertices that both are connected to some vertex or both are connected to no vertex all. Formally, an equivalence $\approx$ is called *weak role equivalence* if we have for all $i, j \in V$ that

$$i \approx j \Rightarrow \text{for each} \quad k \in N(i) \quad \text{there exists} \quad \ell \in N(j).$$

Conceptually, this means that any neighbor of one vertex can substitute for any neighbor of the other to establish neighborhood equivalence. The following admissible substitutions express this idea:

$$\Sigma_{ij}(\approx) = \{\sigma \in (V \to V)\}.$$

Under this family of admissible substitutions, the weak role equivalences constitute the stable role equivalences w.r.t. role restriction.

5.5. Degree equality

The analogue of exact equivalence for weak role equivalence is a role equivalence that considers vertices with the same number of neighbors as equivalent. In other words, they are equivalences that define neighborhood equivalence as degree equality. This means that a neighbor of one vertex can only substitute for one neighbor of the other vertex in neighborhood comparisons. Therefore, we express neighborhood equivalence as degree equality by allowing all permutations as admissible substitutions, i.e.,

$$\Sigma_{ij}(\approx) = \{\sigma \in (V \to V) : \sigma \text{ is } \text{ permutation}\}.$$

Then degree equality corresponds to the stable role equivalence w.r.t. relative role equivalence.

6. Error-tolerant equivalences

Meaningful role equivalences of the traditional kind are rarely found in empirical social networks: Minor deviations in edge patterns prevent pairs of vertices from being assigned the same role. We now demonstrate that the possibility to choose more general families of admissible substitutions substantially extends the range of applicability. By means of

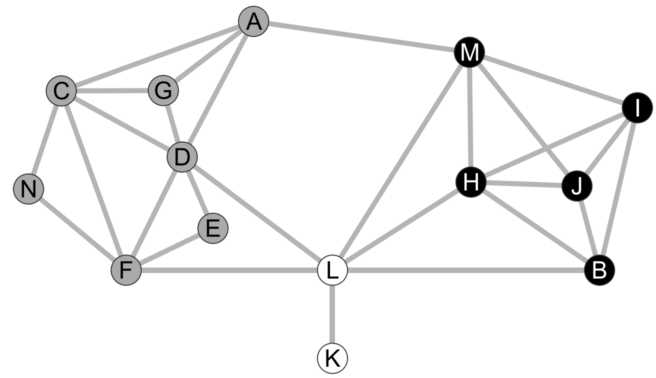


Fig. 7. Groups in a network of political actors as identified by Doreian and Albert (1989).

an example, we here specifically describe a technique to integrate some level of tolerance for minor discrepancies between neighborhoods into established notions of role equivalence.

Fig. 7 depicts a social network of Doreian and Albert (1989). It consists of 14 actors in a county in the U.S. Midwest, and the strong political ties between them. The political actors are labeled with letters A to N as assigned in the original study, from where we also take the partition into two camps and a pair of unaffiliated actors. This partition is neither a structural, exact, or regular equivalence, nor is any partition close to it. Note that pairs of vertices assigned to the same set are often almost, but not exactly equivalent under said role definitions. When initialized with the partition shown, the processes obtained from relative role equivalence or role restriction break up the groups almost entirely (except for the vertices I and J, that have ties to the same others apart from the tie connecting each other).

We now relax the notion of exact equivalence and derive an error-tolerant version of it by allowing substitutions with up to one error in pairwise comparison (i.e., up to one mis- or unmatched vertex in the neighborhood). However, substitutions defined in this way are not closed under transitive composition, and thus the relative roles operation $\diamond$ is not guaranteed to yield an equivalence. For this reason, we close the admissible substitution under transitive composition to ensure that the process yields equivalences. Formally, the family of admissible substitutions $\Sigma_{ij}(\approx)$ is defined as the smallest one closed under transitive composition that includes in $\Sigma_{ij}(\approx)$ all substitutions $\sigma : V \to V$ satisfying the following properties for all $i, j \in V$:

- Neighbors of i have to be substituted by neighbors of j, i.e.,

  $$\sigma(N(i)) \subseteq N(j).$$

- All but one of the neighbors of i have to be substituted equivalently and one-to-one by all neighbors of j but one, i.e.,

  $$|\{\sigma(k) : k \in N(i) \quad \text{and} \quad \sigma(k) \in [k]_{\approx}\}| \geq max(deg(i), deg(j)) - 1.$$

Since this family of admissible substitutions Σ_{ij} is isotone, it follows from the results in Section 4 that this error-tolerant version of exact equivalence exposes typical properties of role equivalences such as lattice structure.

Fig. 8 shows the evolution of roles when initialized with the partition reported by Doreian and Albert, yielding a stable role equivalence reproduced in Fig. 9. This equivalence is a refinement of the original partition adding two reasonable distinctions. First, while one alliance is retained, the other one is split into a connected core linked to L and a disconnected periphery attached to it. The other distinction is between

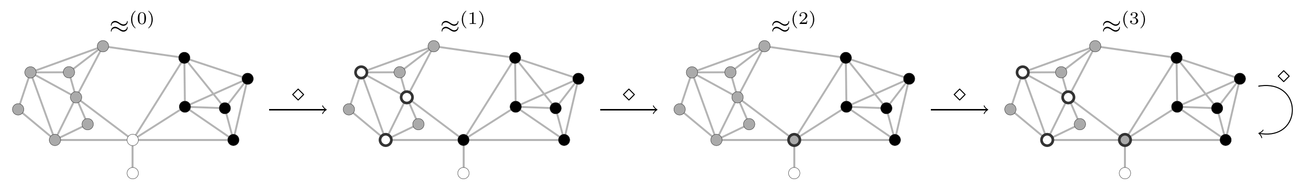


Fig. 8. Role evolution on the political network starting from some equivalence $\approx^{(0)}$. In each iteration, equivalent vertices have the same color and border thickness.

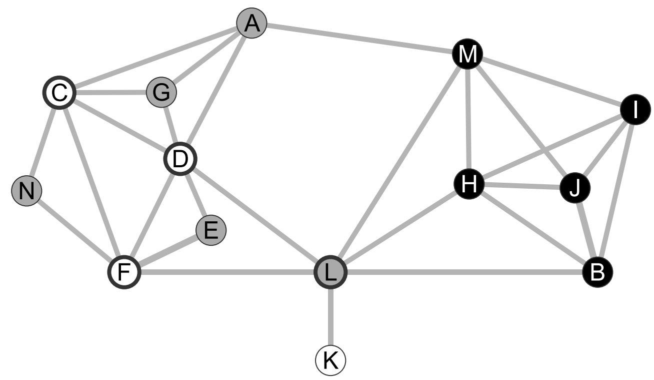


Fig. 9. The five roles identified by an error-tolerant variant of exact equivalence.

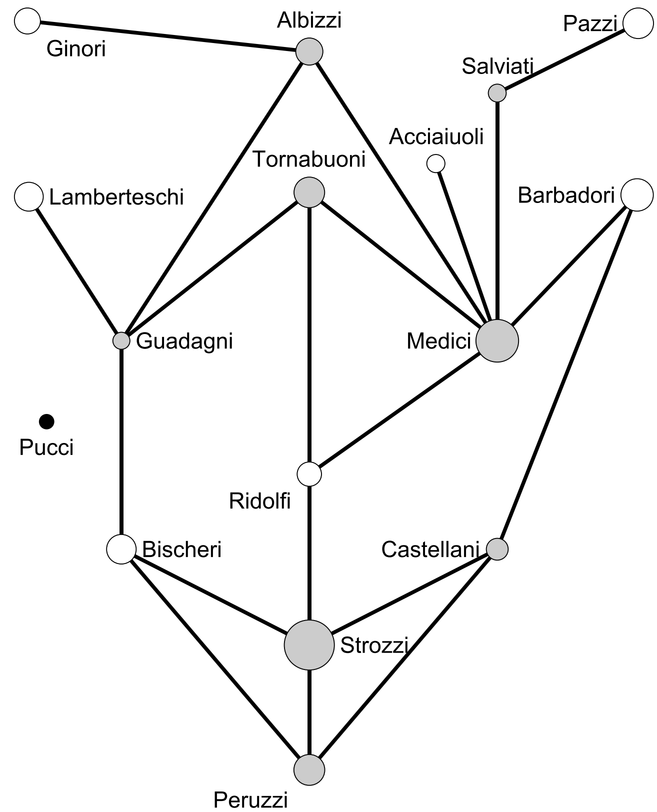


Fig. 10. Non-trivial regular role equivalence on the marriage network between Florentine families. The color of a vertex denotes the assigned role, while size represents the family wealth.

the well-connected L and its pendant K.

Observe that not all role-equivalent vertices have neighborhoods that differ by at most one error. However, because of closure under transitive composition, every two vertices with the same role are linked by a chain of vertices such that the neighborhoods of consecutive vertices satisfy the conditions of exact equivalence up to at most one error. Consider as an example political actors A and N in Fig. 9. At first glance, they appear non-equivalent, as their neighborhoods have two differences under the criteria of exact equivalence. However, G's neighborhood is at distance one from both, thus the chain from A via G to N establishes that A and N are role-equivalent.

The expressiveness of generalized roles thus enables us to cope with minor deviations between neighborhoods through adoption of suitable substitutions. The role equivalence found here is interpretable and meaningful, despite the fact that no such partition could be established using traditional concepts of role equivalence.

7. Handling multivariate networks

Above we have introduced the concept of generalized roles on undirected and unweighted networks, and discussed how it can be used, e.g., to handle minor discrepancies between the neighborhoods of vertices. One of the strengths of our generalized approach is that it is not limited to those simple kinds of networks, but naturally generalizes to directed, multivariate, and multirelational networks. In two examples on the same data set, we demonstrate how its expressiveness facilitates identifying roles that depend on (a) the relation between and ordinal attributes of vertices, or (b) ordinal actor and tie attributes simultaneously.

7.1. Actor attributes

The network we use as an example represents ties among Florentine families in the Fifteenth Century. This well-known data set is the result of extensive studies of Padgett (Padgett, 1987; Padgett and Ansell, 1993), from which Breiger and Pattison (1986) selected a subset of 16 families with some relational and attribute data. We focus here on one relation (marriage ties between these families) and one attribute (family wealth in the year 1427).

First, we note that applying traditional kinds of role equivalences such as regular or exact equivalence, including their error-tolerant variants along the lines of Section 6, yields almost exclusively trivial outcomes. An exception is a particular class of regular equivalences, each partitioning the families according to three roles. A typical example of a regular equivalence in this class is shown in Fig. 10. Note that any regular equivalence will assign the isolated Pucci family a role of its own, which leaves the other families to be partitioned into two roles. Since their induced subgraph is connected and has vertices of degree one, this necessarily leads to one role having ties only to the other role. Consistent with vertex colors in Fig. 10, call this the role the white role. Since the component is not bipartite, families in the other role have ties to those in the white role and their own, the gray role. Families with neighborhood degree one are necessarily in the white role and their neighbors in the gray role. While eight families are thus fixed to either the white or gray role, there is at least one regular equivalence assigning each of the remaining seven families to either the white or gray role. With such level of compositional arbitrariness, these regular equivalences do not seem to possess any interpretative value.

We therefore make use of one additional characteristic, the wealth of a family, to introduce distinctions in the value of relationships. Recall that role equivalence captures the idea that actors have the same (kind of) relationships with the same (kind of) others. While neighbors have to be exactly the same in structural equivalence, they only have to be from the same classes in regular equivalence. By distinguishing families based on their role and their wealth, we now introduce an intermediate level of equivalence.

If two vertices are regularly equivalent, all neighbors of one vertex have to be matched by a regularly equivalent neighbor of the other. We now tighten this condition by requiring that the substituting family is not only equivalent, but also at least as wealthy. As in Section 6, and to avoid difficulties arising from the fact that every class has a wealthiest family that cannot be matched, we tolerate a small number of mismatches.

Let integers w_i denote the wealth of the families represented by vertices $i \in V$. The idea is to allow substitution of neighbors only by wealthier families, i.e., a neighbor $k \in N(i)$ of i can only be matched by a neighbor $\ell \in N(j)$ of j for which $w_k \leq w_\ell$. We therefore define substitutions $\Sigma_{ij}(\approx)$ for pairs of vertices $i, j \in V$ w.r.t. a given equivalence $\approx \in \mathcal{P}$ by modifying the class-preserving substitutions (3) of regular equivalence as follows:

- Neighbors of i have to be substituted by neighbors of j, i.e.,

  $\sigma(N(i)) \subseteq N(j).$

- All but ϵ neighbors of i must be substitutable by equivalent wealthier neighbors of j, i.e., for all $\sigma \in \Sigma_{ij}(\approx)$,

  $deg(i) - |\{k \in N(i) : \sigma(k) \in [k]_\approx, w_k \leq w_{\sigma(k)}\}| \leq \epsilon.$

- For all $i, j \in V$,

  $\Sigma_{ij}(\approx) = \emptyset \Rightarrow \Sigma_{ji}(\approx) = \emptyset.$

Note that the first condition implies that transitive composition on $\Sigma_{ij}(\approx)$ is well-behaved, as it guarantees that the transitive composition of two substitutions $\sigma_{ik} \in \Sigma_{ik}(\approx)$ and $\sigma_{kj} \in \Sigma_{kj}(\approx)$ only attests the same role for i and j if the two substitutions σ_1 and σ_2 respectively already attest the same role for i and k as well as k and j. Additionally, the error tolerance in the second condition counts missing roles in the neighborhood and lack of wealth the same. The last condition ensures that the formed equivalence is indeed symmetric. Forming the closure of these substitutions under transitive composition, we finally obtain the desired admissible substitutions $\Sigma^*_{ij}(\approx)$. These satisfy the requirements of Definition 1, thus ensuring that an application of relative role equivalence $\diamond$ under $\Sigma^*_{ij}(\approx)$ yields an equivalence relation. Since $\Sigma^*_{ij}(\approx)$ is also isotone, this wealth-respecting and error-tolerant version of regular role equivalence exhibits the lattice structure and other common traits of role equivalences.

Does this specifically designed role criterion lead to any interesting role structures? If no ($\epsilon = 0$) or too many ($\epsilon > 1$) errors are allowed, only the all-singleton equivalence and the partition that splits off the isolated Pucci family from the others are stable. For $\epsilon = 1$, however, we find four nested role equivalences that are stable under relative role equivalence $\diamond$. The coarsest is again splitting only the Pucci from the rest, and the other three are depicted in Fig. 11.

The coarsest of these non-trivial stable equivalences partitions the families into four role classes. As usual, the isolated Pucci form a singleton class, having no marital ties to other families at all. The Strozzi, the richest family of Florence, and all families with direct marriage ties to them, compose a second group, which is refined into a core and boundary group in the nested stable equivalence. Except for the boundary-spanning Ridolfi, all families in this group have been identified as anti-Medici (Padgett and Ansell, 1993). The third and largest role

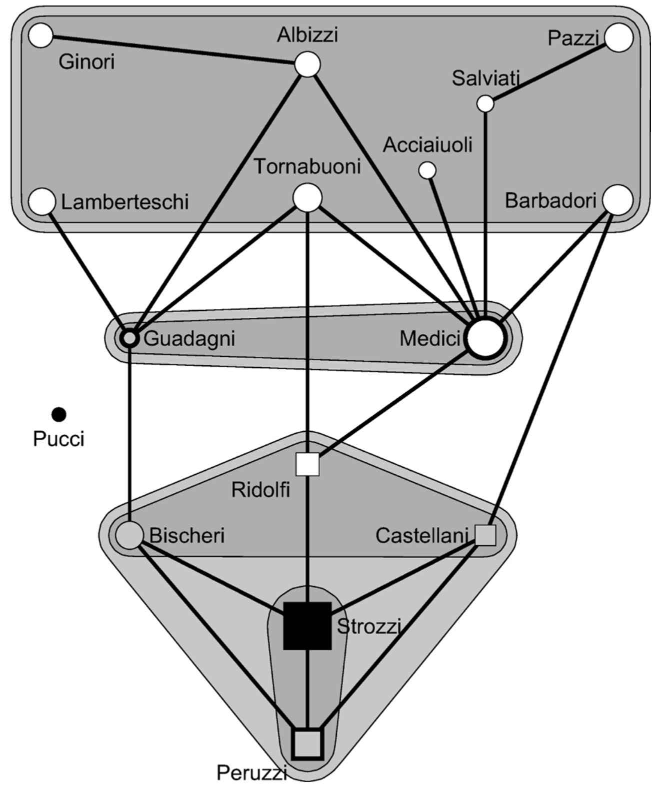


Fig. 11. Roles according to a wealth-respecting regular equivalence criterion allowing for small deviations. The finest equivalence is represented by vertex style, the two coarsenings by nested grouping of vertices. Vertex sizes represent family wealth.

comprises families that are not particularly rich and barely connected among each other. Except for the Lamberteschi family, this group is composed of families that belonged to the Medici faction or had split loyalties (Padgett and Ansell, 1993). The fourth and final group consists of the Medici and Guadagni, two rival families from the San Giovanni district of Florence (Padgett and Ansell, 1993). Notwithstanding the big difference in wealth between them, their roles in the network of marriages are similar: they share marital ties to two families, have high degree, and both serve as bridges between the third group of less prominent families and the peripheral subgroup of the Strozzi faction.

By integrating an ordinal attribute into the definition of role equivalence, we have thus accounted for potential differences in the benefits reaped from marrying into particular families, without insisting on their identities. With the exception of the Ridolfi and Lamberteschi families, this yielded stable role equivalences which are generally consistent with the account of Padgett and Ansell (1993).

7.2. Tie attributes

In their analysis of the Florentine families network, Breiger and Pattison (1986) also considered business ties besides marriage ties between the families. Previously, we neglected this essential relation in the lives of patricians and merchants and only determined roles based on marriages and family wealth. We now combine business and marriage ties with family wealth in the rules of substitutability to identify roles that respect both kinds of relationships as well as the wealth of families simultaneously.

We denote the graph of marriage ties by $M = (V, E_M)$ and the one of business ties by $B = (V, E_B)$ for the same families, and denote the composite graph on which we establish role equivalences by $G = (V, E_M \cup E_B)$.

Recall that we employed regular equivalence-like admissible substitutions that were defined in such a way that families were supposed to be substituted by wealthier families in the same roles. As an additional distinction now, we also consider the types of ties between these families: If a neighbor of one family is tied to it via marriages, then it should be substituted by a neighbor which is also connected to the other family by a marriage tie, and likewise for business ties. We additionally assume, however, that a marriage tie is a stronger and closer type of relationship than a business tie. For this reason, we again allow a neighbor tied to one family via some business to also be substituted by a neighbor with a marriage tie to the other family, as long as the marriage tie between the latter neighbor and family has not already been used to match a marriage tie between the former neighbor and family. As exact application of these criteria will only produce a lattice consisting of the trivial singleton equivalence, we also allow for up to ϵ mismatched ties (whether due to lack of matching tie of this type, mismatched roles or insufficient wealth).

Formally, we define the underlying set of substitutions $\Sigma_{ij}(\approx)$ for families $i, j \in V$ as follows:

- In the composite graph, neighbors of i have to be substituted by neighbors of j, i.e.,

  $$\sigma(N(i)) \subseteq N(j).$$

- All but ϵ marriage and business ties between families and their neighbors are matched according to the criteria described above. To this end, we first define an auxiliary function $d(S, T)$ for a given substitution σ and equivalence $\approx$ and sets of families $S \subseteq N(i)$, $T \subseteq N(j)$:

  $$d(S,T) := |S| - \left| \left\{ k \in S : \begin{array}{l} \sigma(k) \in [k]_{\approx} \cap T, \\ w_k \leq w_{\sigma(k)} \end{array} \right\} \right|$$

  This function $d(S, T)$ determines the number of errors in substituting neighbors S of i by neighbors T of j under the regular equivalence criterion and their ordering by family wealth.

  Observe that we can also account for the ordering between the two types of ties by selecting the neighbors S and T suitably, e.g., $d(N_M(i), N_M(j))$ denotes the number of errors in substituting marriages ties from i to j. Thus, we express the bound on the number of mismatched marriage and business ties by means of function $d(S, T)$ by

  $$d(N_M(i), N_M(j)) + d(N_B(i) \cap N_M(i), N_B(j)) + d(N_B(i) \backslash N_M(i), N_B(j) \cup N_M(j)) \leq \epsilon.$$

- For all $i, j \in V$,

  $$\Sigma_{ij}(\approx) = \varnothing \Rightarrow \Sigma_{ji}(\approx) = \varnothing.$$

In the second condition, we count ties missubstituted to the wrong type of tie, role in the neighborhood as well as lack of wealth the same and allow up to ϵ deviations. Otherwise, the construction is unchanged from Section 7.1. We close the substitutions under transitive composition and obtain the desired admissible substitutions $\Sigma^*_{ij}(\approx)$. These again satisfy the requirements of Definition 1. In the definition of relative role equivalence, we determine the substitutions with respect to the neighbors in the composite network G. Then the derived role equivalences exhibit the lattice structure and other common traits of role equivalences because $\Sigma^*_{ij}(\approx)$ is isotone.

For all choices of ϵ except $\epsilon = 2$, the lattice of stable equivalences under relative role equivalence consists of only one trivial or almost trivial role equivalence: the singleton partition ($\epsilon = 0$), an equivalence that places all families but Pazzi and Acciaiuoli into different classes

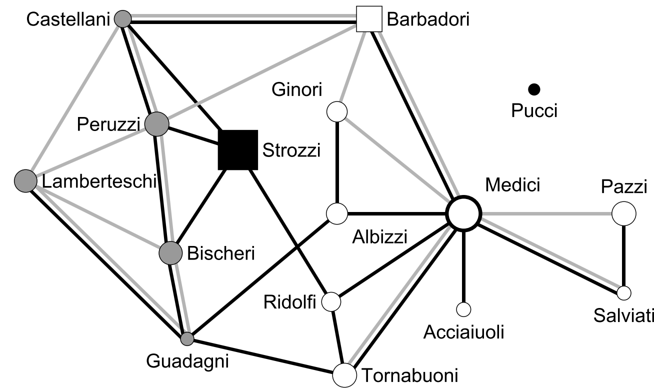


Fig. 12. Salient role equivalence on the network of marriage and business ties on between Florentine families. A marriage tie between families is represented by black, a business tie by a gray line. Equivalent families have the same vertex style. Vertex size represents family wealth.

($\epsilon = 1$), or a partition that puts all families but the Pucci family into the same class ($\epsilon > 2$). For $\epsilon = 2$, however, we observe a lattice consisting of eight stable equivalences (see Fig. 13). Based on the account in Padgett and Ansell (1993), Fig. 12 depicts one particularly salient equivalence in the lattice. It partitions the families into six classes: The Pucci family, not connected to anyone else, forms a singleton class. The wealthiest Medici and Strozzi families, powerful members of the opposed Medici and Oligarch faction, are in singleton classes. Except for the Albizzi and Salviati families with split loyalties between both factions, the families in white circles on the right joined the Medici faction. The families represented with gray circles on the left all belong to the Oligarch faction (Padgett and Ansell, 1993). Finally, the Barbadori family in a singleton class exposed split loyalties between both factions (Padgett and Ansell, 1993). We note that we would have also found the Medici faction, i.e., the class of families represented by white circles in Fig. 11, if we had omitted the ordering by family wealth from the equivalence definition, but we would have arrived at a different partitioning among the remaining families.

We have thus further refined the notion of role by incorporating two types of ties and respecting the ordering between them. In this manner, we did not only take family wealth and relationships between the families into consideration, but also factor in how marriages and business ties imply different degrees of strength in the bonds between families. Under this definition, we obtained a salient role equivalence very closely aligned with previous accounts.

However, the two examples in this section are mostly meant as a demonstration of the expressivity of our framework. Extensions to more rich kinds of networks and other notions of substitutability combining ties, actors and attributes in complex ways should be straightforward.

8. Conclusion

We introduced a new formalization of roles in networks that unifies established definitions of role equivalence. We first looked into rules of substitutability underlying notions of roles. Then we derived a generalized notion of relative roles based on substitutions by breaking up the typical self-referential definition of role equivalence, and from repeated applications of relative roles, we obtained a sequence of role equivalences. On the one hand, this gives us a procedure to derive roles in the network. On the other hand, we can interpret it as a process in which vertices with similar surroundings are influenced to enact the same roles, and thus as one mechanism at work in the formation of roles. In this sense, it can serve as a micro-foundation for the emergence of roles. By thus interpreting this sequence as an idealized process of role formation over time in which all actors adapt their roles due to influence exerted by their environment simultaneously, we obtain an evolution of roles.

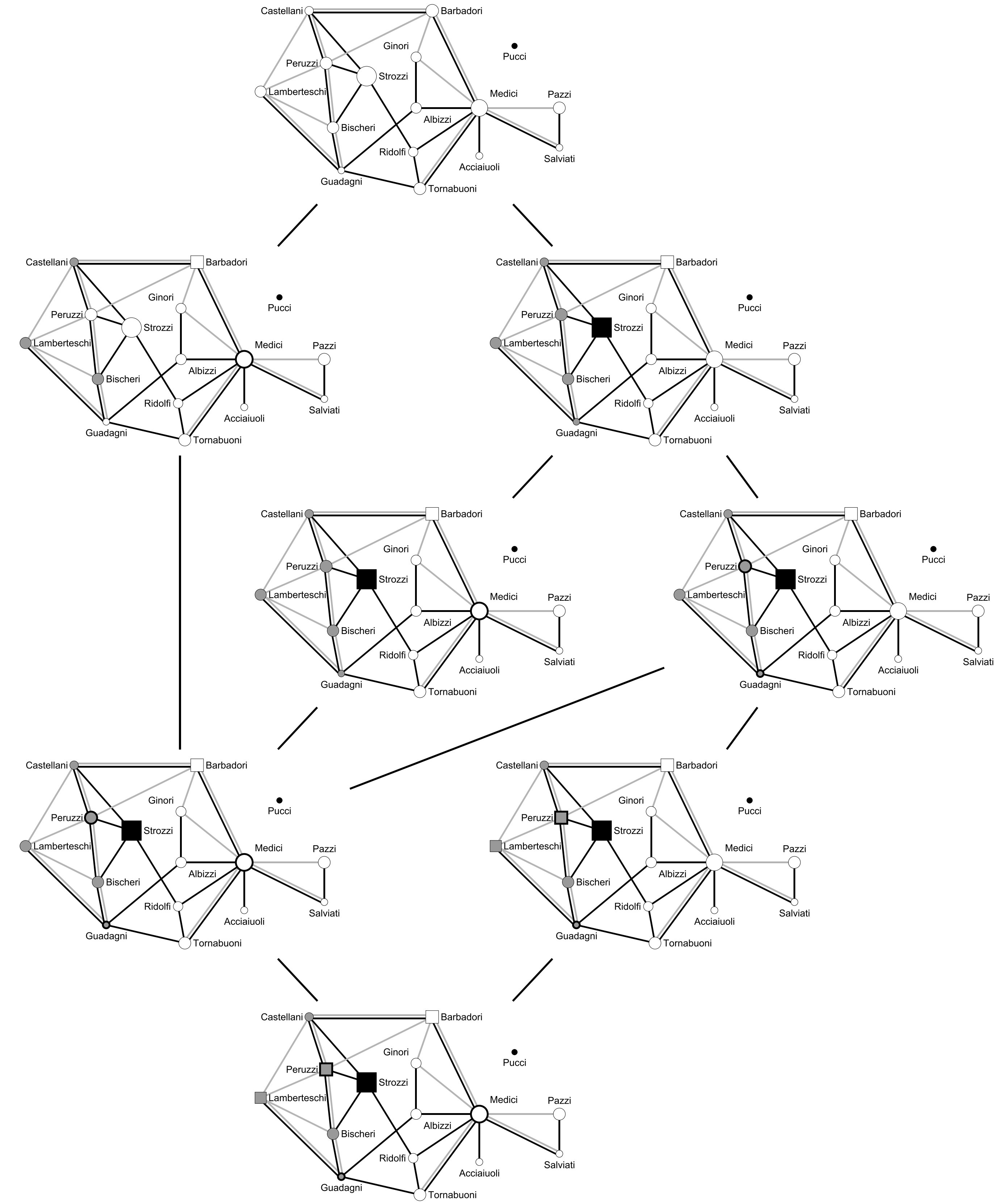


Fig. 13. Lattice of stable error-tolerant role equivalences as defined in Section 7.2 on the network of business and marriage ties between Florentine families for up to $\epsilon = 2$ errors. A marriage tie between families is represented by black, a business tie by a gray line. Family wealth is represented by vertex size, and equivalent families are depicted in the same vertex style.

This opens new opportunities for dealing with the rarity of established kinds of role equivalences through investigation of the evolution of roles and structures such as cycles in the role formation process. We also reclaimed the original concept of role equivalence as stable outcomes of such evolutions of roles, allowing us to investigate their structure. In particular, this enabled us to present a straightforward proof that role equivalences generally form lattices, a property that had been proven individually for different kinds of role equivalences before. We verified that our formalization is a unification of many established definitions of roles, such as structural, regular or exact equivalences, by demonstrating that they are particular instantiations of it.

Next, we utilized the expressiveness that the formalization offers via suitable adaption of admissible substitutions and presented an approach to derive error-tolerant versions of established equivalences. This allowed discovery of meaningful roles on an empirical network where traditional role equivalences did not bring about interesting instances.

Finally, we laid out a way to generalize to multivariate networks with actor and tie attributes. We first derived an error-tolerant notion of roles that combines an established role equivalence with an actor attribute by restricting admissible substitutions based on the actor attribute. Afterwards, we constructed an error-tolerant role equivalence that respects an ordinal actor attribute as well as two types of ties and the ordering between them by controlling admissible substitutions appropriately. In this way, we identified roles on an empirical data set consistent with previous accounts that would stay hidden when established equivalences or attributes are explored individually.

The error-tolerant role equivalences expose some properties similar to blockmodeling, in that both allow for some deviations from idealized notions of roles, but there are also some substantial differences between them. Notably, error-tolerant role equivalences and blockmodeling differ in the level they consider and count deviations: Blockmodeling takes a macroscopic view and aims to find a partition of vertices that minimizes a criterion function, which expresses the global deviation from some ideal blockmodel. The error-tolerant equivalences, on the other hand, bound errors for each pair of vertices individually on the micro level, and then derive the equivalence classes from the outcomes of these pairwise comparisons. At its core, error-tolerant role equivalence is thus established locally and not by minimizing the global number of deviations. In addition, there is a second, but more technical difference: In a blockmodeling application, the number of blocks is generally pre-specified by the user. In error-tolerant role equivalences specifically and in our proposed formalization of role equivalence in general, however, the number of equivalence classes is an implicit outcome of the selected notion of substitutability and the initial equivalence.

In the construction of error-tolerant equivalences, we closed the families of admissible substitutions under transitive composition. Since our focus in this article was on role equivalences, this step, while probably the most arguable, was necessary to ensure that the relative role equivalence, restriction and extension operations always yield equivalences. This is, however, just one among several possible approaches. Since role structures not restricted to partitions are conceivable, one other possibility, for instance, is to omit the closure step and explore the resulting non-transitive relation directly.

In general, the processes of role restriction and extension are guaranteed to converge after at most $(n-1)$ iterations, while we have not identified such a bound for the process of relative role equivalence (except for the trivial bound by the total number of partitions). In addition, the amount of computational effort for each iteration along any of the processes is bounded by the cost of performing all pairwise comparisons between vertices. Depending on the selected choice of admissible substitutions, it is therefore often computationally feasible to follow along these processes and identify the limits they reach. Indeed, the process of role restriction w.r.t. regular equivalence corresponds to an algorithmic procedure that has been suggested before by Boyd and Everett (1999) to compute the regular interior.

Of course, faster evaluation can be possible for instantiations of operations and processes under specific choices of substitutions. For instance, relative role equivalence, restriction and extension are computable in linear time for structural, regular and exact equivalence by using variations of the same algorithm (see Algorithm 21 in Lerner, 2005). Similarly, efficient algorithms are known to compute the interior, the limit of the role restriction process, for regular and exact equivalence in $\mathcal{O}((n+m)\log\ n)$ time (Cardon and Crochemore, 1982; Paige and Tarjan, 1987).

Our approach highlights the special status of automorphic equivalence among the established notions of role, as it is currently not unified by our general role formalization via simple and locally established substitutions. This is because automorphic equivalence is an inherently self-referential concept due to its definition via automorphisms, while an important component of our formalization is to dissolve this reflexivity. It appears that derived relations do not offer a solution, either, because at least those based on walks may fail to differentiate vertices that are not automorphically equivalent (Everett and Borgatti, 1990).

The expressiveness of our approach expands the possibilities and applicability of the role concept in complex settings and contexts. How to best make use of this increased flexibility – in particular, which formulation of role equivalence to select when – is subject of future research.

We presented this formalization of roles on undirected and unweighted graphs, and introduced some first extensions to undirected networks with ordinal actor and tie attributes. Generalizing these adaptions further enables extensions to more complex kinds of networks including directed and multirelational networks, interval-scaled attributes, as well as generalized network positions (Brandes, 2016). This also allows us to represent Winship-Pattison equivalence as an instantiation, but facilitates the definition of new notions of roles that combine different network aspects in meaningful ways. Detailed discussions of these extensions are, however, beyond the scope of this article.

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Appendix A. Proofs for admissible substitutions of established role equivalences

Here we exemplarily show for some classic notions of role equivalence that they are stable role equivalences under specific kinds of admissible substitutions as presented in Section 5. The same proof strategies can also be used to derive comparable results for the other role equivalences considered in that section.

We first generally characterize stable role equivalences under role restriction and extension. Afterwards, we prove for structural, regular and exact equivalences that they are stable under specific choices of admissible substitutions.

A.1 Stable role equivalences under role restriction and extension

By Definition 3, an equivalence $\approx$ is stable under role restriction if and only if

$$\approx = \nabla(\approx) = \approx \cap \diamond(\approx).$$

Expressed equivalently, $\approx$ is stable under role restriction if and only if

$$i \approx j \Rightarrow (i,j) \in \diamond(\approx) \quad \text{for all} \quad i,j \in V. \tag{A.1}$$

Similarly, an equivalence $\approx$ is stable under role extension if and only if

$$i \approx j \Leftarrow (i,j) \in \diamond(\approx) \quad \text{for all} \quad i,j \in V. \tag{A.2}$$

The stable role equivalences under role restriction and extension thus differ in whether they satisfy a necessary or a sufficient condition with respect to relative role equivalence.

A.2 Structural equivalence

Two vertices are structurally equivalent if and only if they have the same neighborhood. The following lemma identifies admissible substitutions under which this equivalence is the only stable one under relative role equivalence.

Lemma 1. *Under admissible substitutions $\Sigma_{ij}(\approx) = \{\mathrm{id}\}$, we have*

$$(i,j) \in \diamond(\approx) \Leftrightarrow N(i) = N(j) \quad \text{for all} \quad i,j \in V.$$

Proof. Let $i, j \in V$ and choose $\sigma_{ij} = \mathrm{id}$. We have $N(i) \subseteq N(j)$ if and only if $k \in N(i) \Rightarrow k \in N(j)$ for all $k \in V$, and this holds if and only if $k \in N(i) \Rightarrow \sigma_{ij}(k) \in N(j)$. If we likewise transform $N(i) \supseteq N(j)$ and combine both directions, we obtain that $N(i) = N(j)$ exactly corresponds to the definition of relative role equivalence $\diamond$ in Definition 2 under the certifying substitution id. □

Furthermore, an equivalence $\approx$ is a structural coloration if equivalent vertices must necessarily have the same neighborhood. It follows from Eq. (A.1) and Lemma 1 that the structural colorations are the stable role equivalences w.r.t. role restriction under the same admissible substitutions.

A.3 Regular equivalence

We now identify admissible substitutions under which relative role equivalence exactly describes the defining property of regular equivalence.

Lemma 2. *Under admissible substitutions*

$$\Sigma_{ij}(\approx) = \{\sigma \in (V \to V) : \sigma(k) \in [k]_{\approx}\},$$

we have $(i, j) \in \diamond(\approx)$ for all $i, j \in V$ if and only if

- *for each $k \in N(i)$ there exists $\ell \in N(j)$ such that $\ell \approx k$, and*
- *for each $k \in N(j)$ there exists $\ell \in N(i)$ such that $\ell \approx k$.*

Proof. Let $i, j \in V$. We show both directions individually.

On the one hand, suppose that $(i, j) \in \diamond(\approx)$. Then there exist substitutions $\sigma_{ij} \in \Sigma_{ij}(\approx)$, $\sigma_{ji} \in \Sigma_{ji}(\approx)$ such that for all $k \in V$ we have that $k \in N(i)$ implies $\sigma_{ij}(k) \in N(j)$ and $k \in N(j)$ implies $\sigma_{ji}(k) \in N(i)$. Since $\sigma_{ij}(k), \sigma_{ji}(k) \in [k]_{\approx}$, we obtain that for each $k \in N(i)$ there exists $\ell = \sigma_{ij}(k) \in N(j)$ such that $k \approx \ell$, and conversely for each $k \in N(j)$ there exists $\ell = \sigma_{ji}(k) \in N(i)$ such that $k \approx \ell$.

On the other hand, suppose that for each $k \in N(i)$ there exists $\ell \in N(j)$ such that $\ell \approx k$ and for each $k \in N(j)$ there exists $\ell \in N(i)$ such that $\ell \approx k$. Then, we construct substitution $\sigma_{ij} : V \to V$ as follows: For each $k \in N(i)$ we pick some $\ell \in N(j)$ with $k \approx \ell$ and set $\sigma_{ij}(k) = \ell$. For the remaining $k \in V \setminus N(i)$, we set $\sigma_{ij}(k) = k$, because $k \approx k$ holds trivially. Likewise, we construct substitution σ_{ji}. We observe that σ_{ij} is contained in $\Sigma_{ij}(\approx)$ and likewise σ_{ji} in $\Sigma_{ji}(\approx)$. Finally, plugging these mappings σ_{ij} and σ_{ji} into Definition 2 certifies that $(i, j) \in \diamond(\approx)$. □

It follows from Lemma 2 that the perfect equivalences are the stable role equivalences w.r.t. relative role equivalence. Combining Eq. (A.1) and Lemma 2, we obtain that regular equivalences are exactly the stable ones w.r.t. role restriction. Likewise, ecological equivalences are the stable role equivalences w.r.t. role extension by Eq. (A.2) and Lemma 2.

A.4 Exact equivalences

As above, we first identify admissible substitutions under which relative role equivalence corresponds to the defining property of the exact equivalences.

Lemma 3. *Under admissible substitutions*

$$\Sigma_{ij}(\approx) = \left\{ \sigma \in (V \to V) : \begin{matrix} \sigma \text{ is permutation and} \\ \sigma(k) \in [k]_\approx, k \in V \end{matrix} \right\},$$

we have $(i, j) \in \diamond(\approx)$ *for all* $i, j \in V$ *if and only if*

$$|N(i) \cap [k]_\approx| = |N(j) \cap [k]_\approx| \text{ for each } k \in V.$$

Proof. Let $i, j \in V$. Suppose $(i, j) \in \diamond(\approx)$, and let $\sigma_{ij} \in \Sigma_{ij}(\approx)$ and $\sigma_{ji} \in \Sigma_{ji}(\approx)$ denote admissible substitutions certifying that i and j are relatively role-equivalent. Let $k \in V$. By the definition of Σ_{ij}, we have that $\sigma_{ij}(N(i) \cap [k]_\approx) \subseteq [k]_\approx$, and since σ_{ij} certifies relative role equivalence, we have that $\sigma_{ij}(N(i) \cap [k]_\approx) \subseteq N(j)$. Therefore, the restriction $\sigma_{ij}|_{N(i)\cap[k]_\approx}$ has range $N(j) \cap [k]_\approx$, and it is injective because σ_{ij} is a permutation. Thus, $|N(i) \cap [k]_\approx| \le |N(j) \cap [k]_\approx|$. By an analogous argument for σ_{ji}, it holds that $|N(i) \cap [k]_\approx| \ge |N(j) \cap [k]_\approx|$. We hence conclude that $|N(i) \cap [k]_\approx| = |N(j) \cap [k]_\approx|$.

Conversely, let $P_1, \ldots, P_q$ denote the equivalence classes of $\approx$ and suppose that $|N(i) \cap P_r| = |N(j) \cap P_r|$ for $1 \le r \le q$. Then there exists a one-to-one mapping $\varphi_r : N(i) \cap P_r \to N(j) \cap P_r$ for any $1 \le r \le q$. We also observe that $|P_r \setminus N(i)| = |P_r \setminus N(j)|$, so there also exists a one-to-one mapping $\psi_r : P_r \setminus N(i) \to P_r \setminus N(j)$. We construct substitution σ_{ij} as follows:

$$\sigma_{ij} : V \to V : k \mapsto \begin{cases} \varphi_r(k) & \text{if } k \in P_r \cap N(i), 1 \le r \le q, \\ \psi_r(k) & \text{if } k \in P_r \setminus N(i), 1 \le r \le q. \end{cases}$$

Since the domains of the constituent bijective mappings of σ_{ij} form a partition of V and likewise their ranges, σ_{ij} must be a permutation. Furthermore, the construction ensures that σ_{ij} only maps elements from the same equivalence class to one another. Thus, σ_{ij} is contained among the admissible substitutions in $\Sigma_{ij}(\approx)$. Likewise, we can construct substitution $\sigma_{ji} \in \Sigma_{ji}(\approx)$.

The construction of σ_{ij} also ensures that neighbors of i are mapped to neighbors of j and vice versa for σ_{ji}. The substitutions σ_{ij} and σ_{ji} hence certify that $(i, j) \in \diamond(\approx)$ according to Definition 2. □

By Lemma 3, the stable role equivalences w.r.t. relative role equivalence are the exact perfect equivalences. Next, Eq. (A.1) and Lemma 3 yield that the exact regular equivalences are the stable role equivalences w.r.t. role restriction. Similarly, Eq. (A.1) and Lemma 3 imply that the exact ecological equivalences are the stable equivalences w.r.t. role extension.

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