Adaptive Learning-based Model Predictive Control of Uncertain Interconnected Systems

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Adaptive Learning-based Model Predictive Control of Uncertain Interconnected Systems

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presented by

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To my family and friends

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Abstract

Safety-critical interconnected systems are becoming increasingly important in control theory, as they can represent many infrastructure systems that underpin our society and economy. Due to their complex interconnected nature, using centralized controllers with these systems may lead to various issues and complications. Hence, increasing effort has been devoted to developing distributed and decentralized controllers for safety critical, interconnected systems. In particular, distributed and decentralized model predictive control (MPC) schemes have been proposed to control these systems in the presence of constraints. In these schemes, interconnected systems are decomposed into several smaller coupled subsystems, each of which has its own local controller. Due to their complexity, interconnected systems are also subject to various uncertainties in the model and the topology as well as external disturbances due to additive noise and reference trajectories. These uncertainties and disturbances might lead to closed-loop performance degradation or even feasibility and stability problems. In this thesis, we discuss how to address these challenges and overcome these problems for uncertain interconnected systems within the MPC framework.

In the first part of this thesis, we present reconfigurable distributed MPC schemes which avoid feasibility problems in tracking applications. Various efforts have been devoted to developing distributed MPC schemes for reference tracking. In these schemes, terminal sets are usually computed offline and used in the MPC online phase to guarantee recursive feasibility and asymptotic stability. Maximal invariant terminal sets do not necessarily respect the distributed structure of the network, hindering the distributed implementation of the controller. On the other hand, structured terminal sets respect the distributed structure, but may lead to conservative schemes. We propose here novel distributed MPC schemes for reference tracking of interconnected systems, where the terminal ingredients are reconfigured online depending on the closed-loop states to alleviate the aforementioned issues. Although the resulting optimal control problems are nonconvex infinite-dimensional problems, we show that they can be approximated by quadratic programs which are amenable to distributed optimization. To show their efficacy, the proposed schemes are compared in simulation to standard tracking MPC schemes.

In the second part of this thesis, we introduce a passivity-based plug-and-play MPC scheme which preserves the stability of networks with varying topology. The proposed scheme allows subsystems to occasionally join and leave the network while preserving asymptotic stability and recursive feasibility and comprises two main phases. In the

redesign phase, passivity-based control is used to ensure that asymptotic stability of the network is preserved. In the transition phase, reconfigurable terminal ingredients are used to ensure that the distributed MPC problem is initially feasible after the PnP operation. The efficacy of the proposed scheme is evaluated in simulation by comparing it to a benchmark scheme. The developed plug-and-play MPC scheme makes use of a passivity-based decentralized linear state feedback control scheme for discrete-time interconnected systems, which is also introduced in the second part of the thesis. This passivity-based decentralized controller is synthesized by locally solving a semidefinite program offline for each subsystem in a decentralized fashion. This program comprises local conditions ensuring that the corresponding subsystem is locally passive. Passivity is ensured with respect to a local virtual output which is different from the local actual output. The program also comprises local conditions ensuring that the local passivity of all subsystems implies the asymptotic stability of the whole system. The performance of the proposed controller is evaluated on a case study in DC microgrids.

Building on all the aforementioned schemes, we finally conclude in the third part of this thesis by proposing an adaptive learning-based MPC framework which improves the performance of interconnected systems online in the presence of model uncertainties and external disturbances. In particular, we develop a novel adaptive learning-based MPC scheme for interconnected systems which can be decomposed into several smaller dynamically coupled subsystems with uncertain coupling. The proposed scheme is mainly divided into two online phases; a learning phase and an adaptation phase. Set membership identification is used in the learning phase to learn an uncertainty set that contains the coupling strength using online data. In the adaptation phase, rigid tube-based robust MPC is used to compute the optimal predicted states and inputs. Besides computing the optimal trajectories, the MPC ingredients are adapted in the adaptation phase taking the learnt uncertainty set into account. These MPC ingredients include the prestabilizing controller, the rigid tube, the terminal ingredients and the tightened state and input constraints. The recursive feasibility of the proposed scheme as well as the stability of the corresponding closed-loop system are discussed. The developed scheme is compared in simulations to existing schemes including robust, adaptive and learning-based MPC.

Sommario

I sistemi interconnessi critici per la sicurezza stanno diventando sempre più importanti nella teoria del controllo, in quanto possono rappresentare molti sistemi infrastrutturali che sono alla base della nostra società ed economia. A causa della loro complessa natura interconnessa, l'utilizzo di controllori centralizzati con questi sistemi può portare a vari problemi e complicazioni. Pertanto, sono stati dedicati sforzi crescenti allo sviluppo di controllori distribuiti e decentralizzati per sistemi interconnessi critici per la sicurezza. In particolare, sono stati proposti schemi di Model Predictive Control (MPC) distribuiti e decentralizzati per controllare questi sistemi in presenza di vincoli. In questi schemi, i sistemi interconnessi sono scomposti in diversi sottosistemi più piccoli, accoppiati fra loro, ognuno dei quali ha il proprio controllore locale. A causa della loro complessità, i sistemi interconnessi sono inoltre soggetti a varie incertezze nel modello e nella topologia, nonché a disturbi esterni dovuti a rumore additivo e bruschi cambiamenti nella traiettoria di riferimento. Queste incertezze e disturbi potrebbero portare a un degrado delle prestazioni ad anello chiuso o persino a problemi di fattibilità e stabilità. In questa tesi, discutiamo come affrontare queste sfide e superare questi problemi per sistemi interconnessi incerti all'interno del framework MPC.

Nella prima parte di questa tesi, presentiamo schemi MPC distribuiti riconfigurabili e con capacità di inseguimento che evitano problemi di fattibilità nel tracciare le applicazioni. Vari sforzi sono stati dedicati allo sviluppo di schemi MPC distribuiti per l'inseguimento di riferimenti. In questi schemi, i terminal set sono solitamente calcolati offline e utilizzati nella fase online dell'MPC per garantire fattibilità ricorsiva e stabilità asintotica. I terminal set massimi invarianti non rispettano necessariamente la struttura distribuita della rete, ostacolando l'implementazione distribuita del controllore. D'altra parte, terminal set strutturati rispettano la struttura distribuita, ma possono portare a schemi conservativi. Proponiamo qui nuovi schemi MPC distribuiti con capacità di inseguimento di riferimenti per sistemi interconnessi, in cui gli ingredienti terminali vengono riconfigurati online a seconda dello stato in anello chiuso per alleviare i problemi di cui sopra. Sebbene il problemi di controllo ottimo risultanti siano problemi a dimensione infinita non convessi, dimostriamo che può essere approssimato da programmi quadratici che sono pronta a schemi di ottimizzazione distribuita. Per dimostrare la loro efficacia, gli schemi proposti vengono confrontati in simulazione con i comuni schemi MPC di tracciamento.

Nella seconda parte di questa tesi, introduciamo uno schema MPC plug-and-play

basato sulla teoria della passività che preserva la stabilità delle reti con topologia variabile. Lo schema proposto consente ai sottosistemi di entrare e uscire occasionalmente dalla rete preservando la stabilità asintotica e la fattibilità ricorsiva e comprende due fasi principali. Nella fase di riprogettazione, viene utilizzato il controllo basato sulla passività per garantire il mantenimento della stabilità asintotica della rete. Nella fase di transizione, vengono utilizzati componenti terminali riconfigurabili per garantire che il problema MPC distribuito sia inizialmente fattibile dopo l'operazione PnP. L'efficacia dello schema proposto viene valutata in simulazione confrontandolo con uno schema di riferimento. Lo schema MPC plug-and-play sviluppato fa uso di uno schema di controllo tramite feedback di stato, lineare, decentralizzato, e basato sulla passività per sistemi interconnessi a tempo discreto, anch'esso introdotto nella seconda parte della tesi. Questo controllore decentralizzato basato sulla passività viene sintetizzato risolvendo localmente un programma semidefinito offline per ogni sottosistema in modo decentralizzato. Questo programma comprende le condizioni locali che assicurano che il sottosistema corrispondente sia localmente passivo. La passività è assicurata rispetto ad un'uscita virtuale locale diversa dall'uscita effettiva locale. Il programma garantisce anche che le condizioni locali che assicurano che la passività locale di tutti i sottosistemi implichino la stabilità asintotica dell'intero sistema. Le prestazioni del controllore proposto vengono valutate su un caso di studio in microreti DC.

Basandoci su tutti gli schemi di cui sopra, concludiamo infine nella terza parte di questa tesi proponendo un framework MPC basato sull'apprendimento adattamento che migliora le prestazioni dei sistemi interconnessi online in presenza di incertezze sul modello e disturbi esterni. In particolare, sviluppiamo un nuovo schema MPC basato sull'apprendimento adattamento per sistemi interconnessi che può essere scomposto in diversi sottosistemi più piccoli accoppiati dinamicamente con accoppiamento incerto. Lo schema proposto si articola principalmente in due fasi online; una fase di apprendimento e una fase di adattamento. Uno schema di set membership identification viene utilizzato nella fase di apprendimento per apprendere un set di incertezze che contiene il peso di accoppiamento utilizzando dati online. Nella fase di adattamento, viene utilizzato uno schema MPC robusto basato su tubi rigidi per calcolare gli stati e gli input ottimali previsti. Oltre a calcolare le traiettorie ottimali, i componenti dell'MPC vengono adattati nella fase di adattamento tenendo conto dell'incertezza appresa. Questi componenti MPC includono il controllore di prestabilizzazione, il tubo rigido, i componenti terminali e i vincoli ristretti di stato e ingresso. Vengono discusse la fattibilità ricorsiva dello schema proposto e la stabilità del corrispondente sistema ad anello chiuso. Lo schema sviluppato viene confrontato nelle simulazioni con schemi esistenti, tra cui MPC robusto, adattamento e basato sull'apprendimento.

Contents

Abbreviation

ADMM	Alternating Direction Method of Multipliers
LMI	Linear Matrix Inequality
LP	Linear Program
LQR	Linear Quadratic Regulator
LTI	Linear Time-invariant
MPC	Model Predictive Control
MSD	Mass-Spring-Damper
OCP	Optimal Control Problem
PnP	Plug-and-play
QP	Quadratic Program
RPI	Robust Positive Invariant
SDP	Semidefinite Program
SMI	Set Membership Identification
SOCP	Second-order Cone Program

Notation

I_n	Identity Matrix of size n
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_{>0}$	Set of positive real numbers
$\mathbb{R}_{\geq 0}$	Set of non-negative real numbers
$\mathbb{N}$	Set of Natural Numbers
$\mathbb{N}_+$	Set of Positive Natural Numbers
$\mathbb{Z}$	Set of Integers
$\mathbb{Z}_+$	Set of Positive Integers
$\text{int}(\cdot)$	Set Interior Operator
$\text{vec}(\cdot)$	Matrix-to-Vector Operator
$\text{diag}(\cdot)$	Operator for creating block-diagonal matrices
$\text{diag-vec}(\cdot)$	Operator for creating a vector with the diagonal elements of a matrix
$\text{trace}(\cdot)$	Trace Operator
$\mathbb{S}^m$	Set of Symmetric Matrices
$\mathbb{S}^m_+$	Set of Symmetric Positive Semidefinite Matrices
$\mathbb{S}^m_{++}$	Set of Symmetric Positive Definite Matrices
$\mathbb{D}^m$	Set of Diagonal Matrices
$\mathbb{D}^m_+$	Set of Diagonal Positive Semidefinite Matrices
$\mathbb{D}^m_{++}$	Set of Diagonal Positive Definite Matrices
$(\cdot)^\top$	Matrix Transpose Operator
$\|\cdot\|$	Vector Euclidean Norm
$\|\cdot\|_P$	Weighted Vector Euclidean Norm with a matrix P
$\|\cdot\|_F$	Frobenious Norm of a Matrix
$\sum(\cdot)$	Scalar Summation Operator
$(\cdot,\cdot)$	Vector Concatenation Operator
$\otimes$	Kronecker Product
$\cap$	Set Intersection
$\cup$	Set Union
$\times$	Cartesian Prodcuct Operator
$\oplus$	Minkowski Sum
$\ominus$	Pontryagin Difference
$\setminus$	Set Difference Operator

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CHAPTER 1

Overview

Many infrastructure systems have become nowadays indispensable for the progress of civilizations since these systems are extensively daily used by the society. Examples of such systems include, but are not limited to, power, water, and transportation networks. Hence, there are great motives and tremendous efforts to improve the performance of these networks. The main challenge behind enhancing the efficiency of such networks is their complex interconnected nature. However, the massive advancements in the computational and communication technologies have paved the way for further performance improvements. In particular, more advanced control schemes can be developed for such systems by making use of the latest technologies.

Model Predictive Control (MPC) is an advanced control strategy which solves a constrained optimization problem repeatedly to optimally control a given system [RMD17; BBM17; KC16]. In particular, a discrete-time open-loop optimal control problem (OCP) is solved over a finite-time horizon at each time instant taking the current state of the system into account. The output of this OCP is a control sequence whose first entry is applied to the system. At the next time instant, the whole procedure is repeated in a receding horizon fashion. MPC is well known for its ability to control a broad range of systems, handle the optimization requirements in a systematic manner, take state and input constraints into consideration and make use of the predicted behavior of the system. Due to its strong theoretical guarantees and successful industrial applications, MPC has recently gained huge attention. Furthermore, recent technological advances (e.g. computing hardware and software) and mathematical developments (e.g. computational algorithms for optimization) have made the practical implementation of MPC easier. Therefore, MPC is found to be a promising contender to control complex interconnected systems.

MPC is typically designed in a centralized fashion, that is, a central online optimal control problem is solved for the whole system at each time instant to globally compute the optimal control input. When dealing with complex interconnected systems, using centralized MPC schemes might be cumbersome and suffers several drawbacks in many cases. First, such schemes schemes are not scalable. In other words, their complexity

increase enormously as the size of the problem increases. In addition, they are not robust against failure since the whole network is influenced by any failure in the control unit. Moreover, they suffer privacy issues since the information of the whole network is shared and communicated with the same entity. These schemes can also face many problems due to data collection and time delays. Last but not least, they are not efficient when it comes to networks whose topology changes from time to time. This is because an entirely new control design is required if any slight change in the network occurs.

To circumvent the aforementioned drawbacks, other MPC variants are proposed for controlling complex interconnected systems. These variants include distributed, decentralized and hierarchical MPC [MN14; Sca09; NM14; Chr+13]. The idea behind these variants is to decompose the overall system into several smaller coupled subsystems and to design a local controller for each subsystem. In distributed MPC, each local controller can communicate with a set of other local controllers online. Distributed MPC is characterized by its ability to efficiently control strongly coupled networks. In hierarchical MPC, the lower level local controller of each subsystem communicates with a higher level central coordinator. On the contrary, the local controllers of all subsystems communicate neither with a central coordinator nor with each other in decentralized MPC. Hence, decentralized MPC is characterized by its reduced communication and computational demands. Different versions of the aforementioned variants have been developed over the years. For example, some versions are developed for dynamically coupled systems, whereas others are developed for systems coupled through the constraints. Some versions are cooperative in the sense that the local controllers behave for the benefit of the overall system, whereas others are noncooperative in the sense that the local controllers behave selfishly.

Despite the past massive efforts, control design of complex constrained interconnected systems is still a challenging task and a topic of ongoing research for various reasons. Due to their complexity, detailed accurate models of such systems can be difficult to obtain and/or complicated to use. Hence, simplified nominal models are usually used instead for the control design. In this case, the discrepancy between the accurate and nominal models should be taken into account to yield satisfactory performance. Hence, novel MPC schemes should be developed for complex constrained interconnected systems in the presence of parametric uncertainties and additive disturbances. Moreover, the topology of complex interconnected systems may vary over time. In other words, various agents may occasionally join and leave the system. These plug-in and plug-out operations might destabilize the overall system in some cases. Thus, novel MPC schemes should be proposed to preserve the asymptotic stability of interconnected systems in the presence of topology changes. In addition, interconnected systems might be required to track non-zero target points. Tracking such points might lead to feasibility problems. Even in the absence of feasibility problems, the system might still not be able to track the target signal successfully. Therefore, novel MPC schemes with increased flexibility should be

designed.

The goal of this thesis is to address the challenges outlined above as follows.

(a) We attempt to design reconfigurable distributed MPC schemes such that interconnected systems can successfully track reference trajectories while avoiding feasibility problems.

(b) We aim to develop passivity-based plug-and-play (PnP) MPC schemes which preserve the stability of interconnected systems with varying topology.

(c) We aspire to propose adaptive learning-based MPC schemes which can improve the performance of interconnected systems with parametric uncertainties and additive disturbances online as more data are collected.

1.1 Outline and Contributions

This PhD thesis is divided into three main parts, each of which addresses one of the aforementioned three points. The summary, outline and contributions of each part is listed below.

1.1.1 Part I: Reference Tracking Control

In the first part of this thesis, we address the problem of reference tracking for complex interconnected systems which can be decomposed into several smaller subsystems with state coupling through the dynamics and constraints. To address this problem for such systems, we develop novel reconfigurable distributed MPC schemes for reference tracking. The main feature of the developed schemes is that the terminal ingredients (i.e. set, controller and cost) can be reconfigured online at each time instant.

Outline. In Chapter 2 we start with a simpler version of the considered problem. In particular, we consider the regulation problem of interconnected systems to the origin. The developed schemes in this chapter serve as the building block for the tracking schemes in Chapter 3. Section 2.1 sheds the light on the advantages and disadvantages of the MPC terminal ingredients. Section 2.2 discusses the regulation problem of interconnected systems. In Section 2.3, we present two distributed MPC schemes where the terminal sets are reconfigured online at each time instant. In Section 2.4, we validate the efficiency of the proposed schemes. In Chapter 3, Section 3.1 reviews the existing tracking distributed MPC schemes in the literature as well as the existing methods to alleviate the conservatism imposed by terminal sets. Section 3.2 discusses the tracking problem of interconnected systems. In Section 3.3, we introduce two tracking distributed MPC

schemes with reconfigurable terminal ingredients. We discuss the recursive feasibility of the proposed schemes and the asymptotic stability of the closed-loop dynamics under the proposed schemes in Section 3.4. We then discuss the distributed implementation of the proposed schemes and comment on their feasible regions in Section 3.5 and Section 3.6, respectively. Finally, we investigate the efficacy of the proposed schemes in Section 3.7.

Contributions. The main contributions of Part I of the thesis are contained in Chapter 2 and Chapter 3, and are detailed as follows.

In Chapter 2, we develop two distributed MPC schemes for regulating interconnected systems to the origin. In the developed schemes, we reconfigure the local terminal set of each subsystem at each time instant depending on the current state of the system. In particular, we update the center and size of the terminal set online by considering them as decision variables in the OCP. We also show in simulations that the developed schemes outperform other existing schemes which also update the terminal set online.

In Chapter 3, we propose two tracking distributed MPC schemes for interconnected systems. Besides computing the optimal control inputs, we reconfigure the MPC terminal ingredients online while preserving the convexity of the OCP of the proposed schemes. In particular, we update online the center and volume of the terminal set as well as the terminal control gain and the affine term of the terminal controller by considering them as decision variables in the OCP. We add extra constraints to the OCP to ensure the positive invariance of the updated terminal set and the asymptotic stability of the updated terminal dynamics. We prove that the resulting MPC scheme is recursively feasible and the corresponding closed-loop dynamics is asymptotically stable. We show that the proposed schemes are amenable to distributed optimization. We compare the feasible regions of the proposed schemes to other existing schemes in the literature. We also show in simulations that the proposed schemes outperform standard tracking distributed MPC schemes with fixed terminal ingredients.

1.1.2 Part II: Plug-and-play Control

In the second part of this thesis, we address the problem of stabilizing interconnected systems in the presence of topology changes due to the occasional plug-in and plug-out operations of various agents. We consider here interconnected systems which can be decomposed into several smaller dynamically coupled subsystems whose coupling is described by the Laplacian interconnection. To address this problem for such systems, we develop novel passivity-based plug-and-play (PnP) MPC schemes for varying topology networks. The developed scheme can preserve the stability of the system and the feasibility of the controller independent of the varying network topology.

Outline. Before introducing the PnP distributed MPC scheme in Chapter 5, we present a passivity-based decentralized control scheme in Chapter 4 for the same class of interconnected systems. Section 4.1 sheds the light on the notion of passivity-

based control and the problem of passivity preservation under discretization. Section 4.2 introduces the class of interconnected systems with the Laplacian interconnection. In Section 4.3, we synthesize a decentralized controller for the considered systems using passivity theory. We investigate the efficacy of the proposed scheme using a DC Microgrid case study in Section 4.4. Section 5.1 reviews the existing PnP MPC schemes in the literature. Section 5.2 presents the considered PnP problem. In Section 5.3, we propose a PnP algorthim based on the passivity-based controller introduced in Chapter 4 and the reconfigurable distributed MPC scheme developed in Part I. We validate the efficiency of the proposed scheme in Section 5.4.

Contributions. The main contributions of Part II of the thesis are contained in Chapter 4 and Chapter 5, and are detailed as follows:

In Chapter 4, we develop a passivity-based decentralized controller for discrete-time interconnected systems. We synthesize this controller in a decentralized fashion where each subsystem solves a local convex optimization problem. We ensure the passivity of each subsystem by making use of a local virtual output. We ensure that the passivity of all subsystems guarantees the stability of the overall system by imposing constraints on the local virtual output and the dissipation rate of each subsystem. We show that the proposed controller can yield a wide range of optimal and robust behaviors.

In Chapter 5, we propose a two-phase PnP MPC algorithm which ensures the asymptotic stability of varying-topology networks and the recursive feasibility of the corresponding MPC scheme. We ensure stability by developing a redesign phase which makes use of the passivity-based control scheme in Chapter 4. We also ensure feasibility by developing a transition phase which makes use of the reconfigurable distributed MPC scheme in Part I. We compare the developed algorithm to a standard PnP MPC algorithm existing in the literature using a network of mass-spring-damper systems.

1.1.3 Part III: Adaptive Learning-based Control

In the third part of this thesis, we address the problem of improving the performance of uncertain interconnected systems online as more data is collected. We consider here interconnected systems which are subject to additive disturbances and parametric uncertainties in the coupling terms. To address this problem for such systems, we develop an adaptive learning-based MPC scheme in which the uncertainty sets are learnt online and the MPC ingredients are adapted accordingly. The developed schemes improves the performance (e.g. optimality and reference tracking) of interconnected systems while ensuring safety (e.g. stability, robustness and constraint satisfaction).

Outline. Chapter 6 proposes an adaptive learning-based MPC for the class of interconnected systems with uncertain coupling. Section 6.1 briefly reviews the existing literature on MPC for uncertain systems. Section 6.2 introduces the considered class of interconnected systems. In Section 6.3, we develop an adaptive learning-based MPC

scheme for this class of interconnected systems. In Section 6.4, we discuss the stability of the closed-loop system under the proposed controller. Finally, we investigate the efficacy of the proposed scheme in Section 6.5.

Contributions. The main contributions of Part III of the thesis are contained in Chapter 6, and are detailed as follows.

In Chapter 6, we develop a two-phase adaptive learning-based MPC scheme for interconnected systems with uncertain couplings. In this scheme, we learn the uncertainty set of each subsystem online and adapt the MPC ingredients accordingly. We learn the uncertainty set in the learning phase by making use of SMI. We then adapt the MPC ingredients in the adaptation phase which also computes the optimal control input. We show that the proposed controller is recursively feasible and the corresponding closed-loop system is input-to-state stable. Finally, we compare the developed scheme to other existing schemes including robust MPC, adaptive MPC and learning-based MPC.

In the last part of this thesis, we give some concluding remarks on each of the three main parts. We also discuss future directions which can be pursued based on the findings of this thesis.

1.2 Publications

This thesis contains a selected collection of results obtained during the studies of the author as a doctoral candidate. The corresponding articles on which this thesis is based are listed below.

1.2.1 Part I: Reference Tracking Control

The reconfigurable distributed MPC schemes for regulation in Chapter 2 and for tracking in Chapter 3 were developed in collaboration with A. Eichler, G. Banjac, F. Cordiano and J. Lygeros. We also acknowledge R. Smith and G. Darivianakis for the fruitful discussions on the topic.

[AEL20] A. Aboudonia, A. Eichler, and J. Lygeros. "Distributed Model Predictive Control with Asymmetric Adaptive Terminal Sets for the Regulation of Large-scale Systems". In: IFAC-PapersOnLine 53.2 (2020), pp. 6899–6904.

[Abo+22b] A. Aboudonia, A. Eichler, F. Cordiano, G. Banjac, and J. Lygeros. "Distributed Model Predictive Control with Reconfigurable Terminal Ingredients for Reference Tracking". In: IEEE Transactions on Automatic Control 67.11 (2022), pp. 6263–6270.

[Abo+22a] A. Aboudonia, G. Banjac, A. Eichler, and J. Lygeros. "Online Compu-

tation of Terminal Ingredients in Distributed Model Predictive Control for Reference Tracking". In: 2022 European Control Conference (ECC). IEEE. 2022, pp. 847–852.

1.2.2 Part II: Plug-and-play Control

The passivity-based decentralized controller for discrete-time interconnected systems in Chapter 4 and the PnP MPC scheme for varying topology networks in Chapter 5 were developed in collaboration with A. Martinelli, N. Hoischen and J. Lygeros.

[AML20] A. Aboudonia, A. Martinelli, and J. Lygeros. "Passivity-Based Decentralized Control for Discrete-Time Large-Scale Systems". In: IEEE Control Systems Letters 5.6 (2020), pp. 2072–2077. (Also, [AML21] A. Aboudonia, A. Martinelli, and J. Lygeros. "Passivity-based decentralized control for discrete-time large-scale systems". In: 2021 American Control Conference (ACC). IEEE. 2021, pp. 2037–2042.)

[Abo+22c] A. Aboudonia, A. Martinelli, N. Hoischen, and J. Lygeros. "Reconfigurable Plug-and-play Distributed Model Predictive Control for Reference Tracking". In: 2022 IEEE 61st Conference on Decision and Control (CDC). IEEE. 2022, pp. 1130–1135.

1.2.3 Part III: Adaptive Learning-based Control

The adaptive learning-based MPC scheme in Chapter 6 was developed in collaboration with J. Lygeros. We also acknowledge R. Smith, G. Banjac, A. Parsi and A. Iannelli for the fruitful discussions on the topic.

[AL23] A. Aboudonia and J. Lygeros. "Adaptive Learning-based Model Predictive Control for Uncertain Interconnected Systems: A Set Membership Identifi- cation Approach". (Submitted to Automatica).

1.2.4 Other publications

The following papers were published by the author during his doctoral studies, but are not included in this thesis.

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[Bün+20] F. Bünning, B. Huber, P. Heer, A. Aboudonia, and J. Lygeros. "Experimental demonstration of data predictive control for energy optimization and thermal comfort in buildings". In: Energy and Buildings 211 (2020), p. 109792.

[Bün+21a] F. Bünning, C. Pfister, A. Aboudonia, P. Heer, and J. Lygeros. "Comparing Machine Learning based Methods to standard Regression Methods for MPC on a virtual Testbed". In: Book of Abstracts. KU Leuven; Ghent University; Daidalos Peutz; Boydens engineering. 2021, p. 30346.

[Bün+21b] F. Bünning, A. Schalbetter, A. Aboudonia, M. H. de Badyn, P. Heer, and J. Lygeros. "Input convex neural networks for building MPC". In: Learning for Dynamics and Control. PMLR. 2021, pp. 251–262.

[Hub+21] B. Huber, F. Bünning, A. Decoussemaeker, P. Heer, A. Aboudonia, and J. Lygeros. "Benchmarking of data predictive control in a real-life apartment during heating season". In: Journal of Physics: Conference Series. Vol. 2042. 1. IOP Publishing. 2021, p. 012024.

[Par+21] A. Parsi*, A. Aboudonia*, A. Iannelli, J. Lygeros, and R. S. Smith. "A distributed framework for linear adaptive MPC". In: 2021 60th IEEE Conference on Decision and Control (CDC). IEEE. 2021, pp. 460–465.

[Bün+22] F. Bünning, B. Huber, A. Schalbetter, A. Aboudonia, M. H. de Badyn, P. Heer, R. S. Smith, and J. Lygeros. "Physics-informed linear regression is competitive with two Machine Learning methods in residential building MPC". In: Applied Energy 310 (2022), p. 118491.

* Equal Contribution

Part I

Reference Tracking Control

CHAPTER 2

Reconfigurable Distributed MPC for Regulation

2.1 Introduction

Thanks to its flexibility, versatility and strong theoretical properties, MPC [RMD17; BBM17; KC16] has been used over the past years in many practical applications such as robotics (e.g. see [PG00; Wie06; ANT12]), energy management (e.g. [PZ14; Sch+14; ZW15]) and biology (e.g. [Hov+04]). Besides, many MPC variants have been developed including, but not limited to, robust MPC [BM99], stochastic MPC [Mes16], and economic MPC [EDC14].

Control of interconnected systems is currently an active area of research due to its wide variety of applications (e.g. energy and mobility networks) [MN14]. Various control techniques have been developed to control such systems by regulating their states to the origin [Sca09]. Among these techniques is Model Predictive Control (MPC), that aims to optimize performance while ensuring safety [RMD17]. MPC is typically designed in a centralized fashion with one optimization problem solved for the overall system. For interconnected systems, centralized MPC may lead to various issues and complications as mentioned in Chapter 1. To overcome these difficulties, various efforts have been devoted to developing distributed MPC schemes for interconnected systems [Chr+13]. In these schemes, the system is decomposed into several smaller coupled subsystems, each of which has a local controller which can share information with a set of other local controllers [NM14].

Due to the increasing interest in MPC, various efforts have been devoted to ensuring the closed loop stability of plants controlled using MPC as well as the satisfaction of the state and input constraints of these plants at all times [May+00]. For this purpose, many MPC schemes use sufficiently long horizons [KC16]. However, this may result in high computational cost. Alternatively, others are equipped with terminal ingredients (i.e. a terminal cost and/or a terminal constraint) computed offline for ensuring asymptotic sta-

bility and recursive feasibility. The latter has been extensively used for centralized MPC (e.g. see [KG88; RM93; SD87]) and extended to distributed MPC [Con+12; Fer+13]. However, the computation of the terminal ingredients in the aforementioned distributed MPC schemes either require offline centralized computations [Fer+13] or lead to small feasible regions [Con+16] and hence, deteriorate the closed-loop performance.

To circumvent these issues, several studies have considered computing the terminal set for distributed MPC online (e.g. see [TM17; LKF15]). Recently, a distributed MPC scheme with adaptive terminal ingredients is proposed in [DEL19]. In addition to the terminal set, a linear terminal controller is determined at each time instant in this scheme and updated online based on the current state of the system, yielding a larger domain of attraction.

In this chapter, we present two reconfigurable distributed MPC schemes for regulating constrained interconnected systems where the terminal set of each subsystem is reconfigured online at each time instant. The main advantage of these approaches over the one introduced in [DEL19] is that the reconfigurable terminal set is not necessarily centered at the origin. Instead, the center of the terminal set, together with its size, are assumed to be decision variables to be determined online. This, however, hinders the possibility of considering the terminal control gain as a decision variable. Otherwise, the convexity of the optimal control problem is not preserved.

The terminal set invariance and constraint satisfaction are guaranteed through imposing additional constraints formulated as LMIs in the online optimal control problem. Similar to [DEL19], the first scheme uses a conservative description of the constraints while deriving the LMIs by converting the linear constraints to quadratic ones. On the contrary, the second scheme is more relaxed in the sense that the LMIs are derived using the actual linear constraints directly without converting them to quadratic constraints which are more conservative. The effectiveness of these approaches is evaluated by means of a simulation example. It is found that the online computation of the terminal set center results in enlarging the feasible region of the developed schemes when compared to that of [DEL19] although the terminal control gain is not a decision variable anymore.

2.2 Regulation Problem

We introduce here the regulation problem of constrained interconnected systems considered in this chapter. We consider an interconnected system which admits a separable structure and thus, can be decompsed into M subsystems. For each subsystem $i \in \{1, ..., M\}$, a set $\mathcal{N}_i$ of neighbours is defined comprising the i-th subsystem itself as well as all other subsystems coupled with the i-th subsystem through the dynamics and/or the constraints. The i-th subsystem is described as a discrete-time linear

time-invariant system given by,

$$x_i(t+1) = A_{N_i} x_{N_i}(t) + B_i u_i(t), \tag{2.1}$$

where $t \in \mathbb{N}_+$ is the time index, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $x_{N_i} \in \mathbb{R}^{n_{N_i}}$ are the state vector of the i-th subsystem, the input vector of the i-th subsystem and the state vector of the subsystems in the set $\mathcal{N}_i$, respectively. The system matrices $A_{N_i} \in \mathbb{R}^{n_i \times n_{N_i}}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ are assumed to be known. The state and input constraint sets of each subsystem are given by,

$$\begin{aligned} x_{N_i}(t) \in \mathcal{X}_{N_i} &= \{x_{N_i} \in \mathbb{R}^{n_{N_i}} : G_{N_i} x_{N_i} \leq g_{N_i}\}, \\ u_i(t) \in \mathcal{U}_i &= \{u_i \in \mathbb{R}^{m_i} : H_i u_i \leq h_i\}, \end{aligned} \tag{2.2}$$

where the matrices $G_{N_i} \in \mathbb{R}^{q_i \times n_{N_i}}$, $H_i \in \mathbb{R}^{r_i \times m_i}$ and vectors $g_{N_i} \in \mathbb{R}^{q_i}$, $h_i \in \mathbb{R}^{r_i}$ are assumed to be known. The origin is assumed to be contained in the interior of the constraint sets. We also assume that the inputs of the different subsystems are coupled neither through the dynamics, nor through the constraints; indeed this assumption can be imposed without loss of generality, because inputs can always be decoupled by introducing new auxiliary variables [DEL19].

Our main aim is to regulate the system to the origin. We therefore impose a quadratic cost function in the states and the inputs. To maintain the distributed structure of the optimal control problem, the local cost function of the i-th subsystem is assumed to be a function of the states of the subsystems in the set $\mathcal{N}_i$ and the inputs of the i-th subsystem. Let $N \in \mathbb{N}_+$ be the prediction horizon. The local cost function of the i-th subsystem is designed to be,

$$J_i(t) = \sum_{k=0}^{N-1} \left[\|x_{N_i}(k|t)\|^2_{Q_{N_i}} + \|u_i(k|t)\|^2_{R_i} \right] + \|x_i(N|t)\|^2_{P_i},$$

where $x_i(k|t)$ and $u_i(k|t)$ are the k-step ahead predicted state and input vectors of the i-th subsystem at time instant t and $x_{N_i}(k|t)$ comprises the k-step ahead predicted states of the subsystems in the set $\mathcal{N}_i$ at time instant t. The local cost function matrices $Q_{N_i} \in \mathbb{R}^{n_{N_i} \times n_{N_i}}$ and $R_i \in \mathbb{R}^{m_i \times m_i}$ are selected offline based on the optimization requirements. The local terminal cost matrix $P_i \in \mathbb{R}^{n_i \times n_i}$ is designed offline in such a way that the cost function $J(t) = \sum_{i=1}^{M} J_i(t)$ can be used as a Lyapunov function for the plant controlled by MPC.

Denoting the global state and input vectors of the whole system as $x = [x_1^\top, ..., x_M^\top]^\top \in \mathbb{R}^n$ and $u = [u_1^\top, ..., u_M^\top]^\top \in \mathbb{R}^m$ respectively, the mappings $U_i \in \{0,1\}^{n_i \times n}$, $W_i \in \{0,1\}^{n_{N_i} \times n}$ and $V_i \in \{0,1\}^{m_i \times m}$ can be defined to relate the local variables of the i-th subsystem to the global variables as follows,

$$x_i = U_i x, \quad x_{N_i} = W_i x, \quad u_i = V_i u. \tag{2.3}$$

Note that the matrices Q_{N_i}, R_i and P_i are selected such that $Q = \sum_{i=1}^{M} W_i^\top Q_{N_i} W_i \in \mathbb{S}_{++}^n$, $R = \sum_{i=1}^{M} V_i^\top R_i V_i \in \mathbb{S}_{++}^m$ and $P = \sum_{i=1}^{M} U_i^\top P_i U_i \in \mathbb{S}_{++}^n$. Note also that the matrices of the global dynamics can be constructed as follows, $A = [(A_1 W_1)^\top, \dots, (A_M W_M)^\top]^\top$ and $B = \text{diag}(B_1, \dots, B_M)$.

To ensure the asymptotic stability of the closed-loop system and the recursive feasibility of the proposed distributed MPC, the terminal state $x_i(N|t)$ of the i-th subsystem is constrained to lie in an ellipsoidal terminal set as follows,

$$x_i(N|t) \in \mathcal{X}_{f_i} = \{x_i \in \mathbb{R}^{n_i} : \|x_i - c_i\|_{P_i}^2 \leq \alpha_i^2\}, \tag{2.4}$$

where $\alpha_i \in \mathbb{R}$ and $c_i \in \mathbb{R}^{n_i}$ represent the size and center of the terminal set, respectively. This ellipsoidal terminal set is required to be invariant under the terminal controller $\kappa_i(x_{N_i}) = K_{N_i} x_{N_i}$. Assuming that $\mathscr{X}_{f_i}(P_i, K_{N_i})$ is the set of ellipsoidal terminal sets $\mathcal{X}_{f_i}$ which are invariant under the terminal controller $\kappa_i(x_{N_i})$, we impose the constraint,

$$\mathcal{X}_{f_i} \in \mathscr{X}_{f_i}(P_i, K_{N_i}). \tag{2.5}$$

We assume that the terminal control gain K_{N_i} and the matrix P_i have been designed offline as shown in the end of this section and we seek c_i and α_i online such that $\mathcal{X}_{f_i}$ satisfies (2.5).

In conclusion, the global online OCP for regulation is formulated as,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &s.t. \begin{cases} \left.\begin{aligned} &x_i(k+1|t) = A_{N_i} x_{N_i}(k|t) + B_i u_i(k|t), \\ &x_{N_i}(k|t) \in \mathcal{X}_{N_i}, \quad u_i(k|t) \in \mathcal{U}_i, \end{aligned}\right\} & \begin{aligned} &\text{for all } k \in \{0, ..., N-1\}, \\ &\text{for all } i \in \{1, ..., M\}, \end{aligned} \\ \left.\begin{aligned} &x_i(0|t) = x_{0_i}, \quad x_i(N|t) \in \mathcal{X}_{f_i}, \\ &\mathcal{X}_{f_i} \in \mathscr{X}_{f_i}(P_i, K_{N_i}), \end{aligned}\right\} \text{for all } i \in \{1, ..., M\}, \end{cases} \end{aligned} \tag{2.6}$$

where $x_{0_i} \in \mathbb{R}^{n_i}$ is the current state of the i-th subsystem. The decision variables of this optimal control problem at time instant t are the predicted state trajectory $x_i(k|t)$ for all $i \in \{1, ..., M\}$ and $k \in \{0, ..., N\}$, the predicted input trajectory $u_i(k|t)$ for all $i \in \{1, ..., M\}$ and $t \in \{0, ..., N-1\}$, the terminal set size α_i for all $i \in \{1, ..., M\}$ and the terminal set center c_i for all $i \in \{1, ..., M\}$. Note that the terminal set depends on the current state of the system because this set is computed by solving the online optimal control problem whose solution is a function of the current state. The last constraint in (2.6) is ensured by means of convex optimization tools in the next section.

In the above MPC formulation, the systems matrices A_{N_i}, B_i, the constraint matrices G_{N_i}, H_i, the constraint vectors g_{N_i}, h_i, the cost function matrices Q_{N_i}, R_i are all known for all $i \in \{1, ..., M\}$. The terminal cost matrix P_i and the stabilizing terminal control gain K_{N_i} are computed offline by solving a semidefinite program following [Con+12;

Con+16]. More specifically, these ingredients are computed such that $V(x) = \|x\|_P^2$ with $P = \text{diag}(P_1, ..., P_M) \in \mathbb{R}^{n \times n}$ is a Lyapunov function of the dynamics (2.1) under the controller $\kappa_i(x_{N_i}) = K_{N_i} x_{N_i}$ in the absence of constraints. With this choice, the terminal controller is stabilizing and the terminal cost upper bounds the infinite horizon cost [Con+12; Con+16]. The semidefinite program is given by,

$$\max_{E_i, Y_i, H_i, S_i} \sum_{i=1}^{M} \text{trace}(E_i) \text{ s.t. } \begin{cases} \forall i \in \{1, ..., M\} \\ E_i \geq \epsilon_i, \ (2.8), \ H_i \leq S_i, \\ \sum_{j \in \mathcal{N}_i} W_{ij} S_j W_{ij}^\top \leq 0, \end{cases} \tag{2.7}$$

where trace$(\cdot)$ is the trace operator of a matrix, ϵ_i are arbitrarily small positive constants, $E_i = P_i^{-1}$, $Y_i = K_{N_i} E_{N_i}$, $H_i \in \mathbb{R}^{n_{N_i} \times n_{N_i}}$ and $S_i \in \mathbb{R}^{n_{N_i} \times n_{N_i}}$ are decision variables, $E_{N_i} = \sum_{j \in \mathcal{N}_i} W_{ij}^\top E_j W_{ij}$, S_i is assumed to be block diagonal and,

$$\begin{bmatrix} W_i U_i^\top E_i U_i W_i^\top + H_i & E_{N_i} A_{N_i}^\top + Y_i^\top B_i^\top & E_i Q_i^{1/2} & Y_i^\top R_i^{1/2} \\ A_{N_i} E_{N_i} + B_i Y_i & E_i & 0 & 0 \\ Q_i^{1/2} E_i & 0 & I_{N_i} & 0 \\ R_i^{1/2} Y_i & 0 & 0 & I_{m_i} \end{bmatrix} \geq 0. \tag{2.8}$$

2.3 Control Design

In this section, we develop two reconfigurable distributed MPC schemes for regulating interconnected systems by modifying the online OCP (2.6) and replacing (2.5) with a set of LMIs to render the online OCP a semidefinite program. For this purpose, we recall that (2.5) ensures that the local terminal sets $\mathcal{X}_{f_i}$ for all $i \in \{1, ..., M\}$ are positively invariant. As mentioned in Section 2.2, this condition is required for the closed-loop system to be asymptotically stable. Proposition 2.1 shows the conditions to ensure the positive invariance of these terminal sets. Hence, (2.5) is satisfied if the conditions in Proposition 2.1 are satisfied.

Proposition 2.1 ([DEL19]). *Define the sets $\mathcal{X}_{f_{N_i}} = \times_{j \in \mathcal{N}_i} \mathcal{X}_{f_j}$. Each local terminal set $\mathcal{X}_{f_i}$ is positively invariant if for each $i \in \{1, ..., M\}$ and for all $x_{N_i} \in \mathcal{X}_{f_{N_i}}$,*

$$(A_{N_i} + B_i K_{N_i}) x_{N_i} \in \mathcal{X}_{f_i}, \tag{2.9a}$$

$$x_{N_i} \in \mathcal{X}_{N_i}, \tag{2.9b}$$

$$K_{N_i} x_{N_i} \in \mathcal{U}_i. \tag{2.9c}$$

Consequently, the global terminal set $\mathcal{X}_f = \times_{i \in \{1, ..., M\}} \mathcal{X}_{f,i}$ is positively invariant.

Condition (2.9a) ensures that the terminal set $\mathcal{X}_{f_i}$ is invariant, whereas conditions (2.9b) and (2.9c) show that all the state and input constraints are satisfied inside the

terminal set, respectively. In the sequel, we derive LMIs for each of the conditions in Proposition 2.1 to replace (2.5) in the OCP (2.6). Embedding these LMIs in the online OCP (2.6) guarantees the positive invariance of the terminal set. While the LMI corresponding to (2.9a) is the same for the two developed schemes, the LMIs corresponding to (2.9b) and (2.9c) differ. For both schemes, the derived LMIs depend on the following quantities: $\alpha = \text{diag}(\alpha_1 I_{n_1}, \ldots, \alpha_M I_{n_M})$, $c = [c_1^\top, \ldots, c_M^\top]^\top$, $\alpha_{N_i} = W_i \alpha W_i^\top$ and $c_{N_i} = W_i c$.

Condition (2.9a) can be represented using an LMI as shown in the following proposition.

Proposition 2.2. *For each subsystem $i \in \{1, ..., M\}$, the terminal set invariance condition (2.9a) holds if there exist $\lambda_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ such that,*

$$\begin{bmatrix} P_i^{-1}\alpha_i^{1/2} & (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2}) & [(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i] \\ (A_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2} + B_i K_{\mathcal{N}_i}\alpha_{\mathcal{N}_i}^{1/2})^\top & \sum_{j\in\mathcal{N}_i}\lambda_{ij}P_{ij} & 0 \\ (A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})c_{\mathcal{N}_i} - c_i]^\top & 0 & \alpha_i^{1/2} - \sum_{j\in\mathcal{N}_i}\lambda_{ij} \end{bmatrix} \geq 0. \tag{2.10}$$

Proof. By making use of the description of the ellipsoidal terminal set in (2.4), condition (2.9a) is given by,

$$\|(A_{N_i} + B_i K_{N_i})x_{N_i} - c_i\|_{P_i}^2 \leq \alpha_i^2 \text{ for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|_{P_j} \leq \alpha_j^2, \tag{2.11}$$

We now define an auxiliary vector $s_i \in \mathbb{R}^{n_i}$ for the state vector x_i of each subsystem as follows,

$$x_i = c_i + \alpha_i s_i. \tag{2.12}$$

By concatenation, the following relation also holds,

$$x_{N_i} = c_{N_i} + \alpha_{N_i} s_{N_i}. \tag{2.13}$$

By substituting these auxiliary vectors in (2.11), the invariance condition is written as,

$$\begin{aligned} \|(A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})s_{N_i}\|_{P_i}^2 + \|(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i\|_{P_i}^2 \\ + [(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top P_i (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})s_{N_i} \leq \alpha_i^2, \\ \text{for all } j \in \mathcal{N}_i,\ s_j : \|s_j\|_{P_j}^2 \leq 1. \end{aligned}$$

Using the mapping equations in (2.3) and multiplying the above equation by α_i^{-1} gives,

$$\begin{aligned} \|(A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})s_{N_i}\|_{P_i\alpha_i^{-1}}^2 + \|(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i\|_{P_i\alpha_i^{-1}}^2 \\ + (A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top P_i \alpha_i (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i}^{1/2})s_{N_i} \leq \alpha_i \\ \text{for all } j \in \mathcal{N}_i,\ s_{N_i} : \|s_{N_i}\|_{P_{ij}}^2 \leq 1. \end{aligned} \tag{2.14}$$

where $P_{ij} = W_i U_j^\top P_j U_j W_i^\top$. By applying the S-procedure [Boy+94] to (2.14), the invariance condition for each subsystem $i \in \{1, ..., M\}$ holds if there exist $\lambda_{ij} \geq 0, j \in \mathcal{N}_i$ such that,

$$\sum_{j \in \mathcal{N}_i} \lambda_{ij} \begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} M_{i_{11}} & M_{i_{12}} \\ M_{i_{12}}^\top & M_{i_{22}} \end{bmatrix} \geq 0, \tag{2.15}$$

where,

$$\begin{aligned} M_{i_{11}} &= (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top P_i \alpha_i^{-1} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i}), \\ M_{i_{12}} &= (A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top P_i \alpha_i^{-1} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i}), \\ M_{i_{22}} &= [(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top P_i \alpha_i^{-1} [(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i] - \alpha_i. \end{aligned}$$

Equation (2.15) can be rearranged as follows,

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \lambda_{ij} P_{ij} & 0 \\ 0 & \alpha_i - \sum_{j \in \mathcal{N}_i} \lambda_{ij} \end{bmatrix} - \begin{bmatrix} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top \\ [(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top \end{bmatrix} P_i \alpha_i^{-1} \begin{bmatrix} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top \\ [(A_{N_i} + B_i K_{N_i})c_{N_i} - c_i]^\top \end{bmatrix}^\top \geq 0. \tag{2.16}$$

Applying the Schur complement [Boy+94] to (2.16) leads to the LMI (2.10). □

Note that the terminal predicted state $x_i(N|t)$ has to satisfy,

$$\|x_i(N|t) - c_i\|_{P_i}^2 \leq \alpha_i^2. \tag{2.17}$$

By means of the Schur complement ([Boy+94]), an equivalent form to (2.17) can be reformulated as,

$$\begin{bmatrix} P_i^{-1}\alpha_i & x_i(N|t) - c_i \\ (x_i(N|t) - c_i)^\top & \alpha_i \end{bmatrix} \geq 0. \tag{2.18}$$

This LMI is derived to be compatible with the LMI (2.10), derived in Proposition 2.2, which is a function of α_i and c_i. In the sequel, we present the two novel distributed MPC schemes. Each scheme uses a different set of LMIs which ensures the conditions (2.9b) and (2.9c). Both schemes, however, use the LMIs (2.10) and (2.18) derived above.

2.3.1 MPC Variant with Asymmetric Terminal Set

We start with the first scheme which we call "Reconfigurable Distributed MPC with Asymmetric Terminal Sets". This scheme is given this name since the terminal set is not necessarily centered at the origin and hence, is not symmetric about the origin. This scheme ensures condition (2.9b) using the LMI derived in the following proposition.

Proposition 2.3. *Denote the p-th row of the matrix G_{N_i} by $G^p_{N_i}$ and the p-th element of the vector g_{N_i} by $g^p_{N_i}$. For each subsystem $i \in \{1, ..., M\}$, condition (2.9b) holds if there exist $\tau^p_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ and $p \in \{1, 2, ..., q_i\}$ such that,*

$$\begin{bmatrix} g^p_{N_i} & G^p_{N_i}\alpha_{N_i} & G^p_{N_i}c_{N_i} \\ \alpha_{N_i}G^{p^\top}_{N_i} & \sum\limits_{j\in\mathcal{N}_i}\tau^p_{ij}P_{ij} & 0 \\ c^\top_{N_i}G^{p^\top}_{N_i} & 0 & g^p_{N_i} - \sum\limits_{j\in\mathcal{N}_i}\tau^p_{ij} \end{bmatrix} \geq 0, \quad \text{for all } p \in \{1, \dots, q_i\}. \tag{2.19}$$

Proof. Consider the auxiliary vectors s_i and the concatenated auxiliary vectors s_{N_i} defined in (2.12) and (2.13) and note that (2.9b) holds if for all $p \in \{1, \dots, q_i\}$,

$$G^p_{N_i}x_{N_i} \leq g^p_{N_i}, \quad \text{for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|^2_{P_j} \leq \alpha^2_j. \tag{2.20}$$

By substituting for these auxiliary vectors, (2.20) becomes,

$$G^p_{N_i}(c_{N_i} + \alpha_{N_i}s_{N_i}) \leq g^p_{N_i}, \quad \text{for all } j \in \mathcal{N}_i,\ s_j : \|s_j\|^2_{P_j} \leq 1.$$

Following [DEL19], a sufficient condition for satisfying the above condition can be obtained by squaring the inequality. Using the mapping equations in (2.3), the resulting implication is given by,

$$(c_{N_i} + \alpha_{N_i}s_{N_i})^\top G^{p^\top}_{N_i}\left(g^p_{N_i}\right)^{-1}G^p_{N_i}(c_{N_i} + \alpha_{N_i}s_{N_i}) \leq g^p_{N_i} \text{ for all } \ s_{N_i} : \|s_{N_i}\|^2_{P_{ij}} \leq 1.$$

By applying the S-procedure [Boy+94] to the above implication, the state constraints of each subsystem $i \in \{1, ..., M\}$ are satisfied inside the terminal set if there exist $\tau^p_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ and $p \in \{1, \dots, q_i\}$ such that,

$$\sum_{j\in\mathcal{N}_i}\tau^p_{ij}\begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \alpha_{N_i}G^{p^\top}_{N_i}g^{p^{-1}}_{N_i}G^p_{N_i}\alpha_{N_i} & \alpha_{N_i}G^{p^\top}_{N_i}g^{p^{-1}}_{N_i}G^p_{N_i}c_{N_i} \\ c^\top_{N_i}G^{p^\top}_{N_i}g^{p^{-1}}_{N_i}G^p_{N_i}\alpha_{N_i} & c^\top_{N_i}G^{p^\top}_{N_i}g^{p^{-1}}_{N_i}G^p_{N_i}c_{N_i} - g^l_{N_i} \end{bmatrix} \geq 0,\ \forall p \in \{1, \dots, q_i\}.$$

This LMI can be rearranged and expressed as,

$$\begin{bmatrix} \sum\limits_{j\in\mathcal{N}_i}\tau^p_{ij}P_{ij} & 0 \\ 0 & g^p_{N_i} - \sum\limits_{j\in\mathcal{N}_i}\tau_{ij} \end{bmatrix} - \begin{bmatrix} \alpha_{N_i}G^{p^\top}_{N_i} \\ c^\top_{N_i}G^{p^\top}_{N_i} \end{bmatrix} g^{p^{-1}}_{N_i}\begin{bmatrix} G^p_{N_i}\alpha_{N_i} & G^p_{N_i}c_{N_i} \end{bmatrix} \geq 0,\ \forall p \in \{1, \dots, q_i\}.$$

By applying the Schur complement [Boy+94] to the above inequality, the LMI in (2.19) is reached. □

Similarly, condition (2.9c) can be represented using an LMI as shown in the following proposition.

Proposition 2.4. *Denote the l-th row of the matrix H_i by H_i^l and the l-th element of the vector h_i by h_i^l. For each subsystem $i \in \{1, \dots, M\}$, condition (2.9c) holds if there exist $\rho_{ij}^l \geq 0$ for all $j \in \mathcal{N}_i$ and $l \in \{1, \dots, r_i\}$ such that,*

$$\begin{bmatrix} h_i^l & H_i^l K_{N_i} \alpha_{N_i} & H_i^l K_{N_i} c_{N_i} \\ \alpha_{N_i} K_{N_i}^\top H_i^{l^\top} & \sum_{j \in \mathcal{N}_i} \rho_{ij}^l P_{ij} & 0 \\ c_{N_i}^\top K_{N_i}^\top H_i^{l^\top} & 0 & h_i^l - \sum_{j \in \mathcal{N}_i} \rho_{ij}^l \end{bmatrix} \geq 0, \text{ for all } l \in \{1, \dots, r_i\}. \tag{2.21}$$

Proof. Note that condition (2.9c) holds if for all $l \in \{1, 2, ..., r_i\}$,

$$H_i^l K_{N_i} x_{N_i} \leq h_i^l, \quad \text{for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|_{P_j}^2 \leq \alpha_j^2.$$

Thus, the proof follows that of Proposition 2.3 by replacing τ_{ij}^p, $g_{N_i}^p$ and $G_{N_i}^p$ with ρ_{ij}^l, h_i^l and $H_i^l K_{N_i}$, respectively, where ρ_{ij}^l for all $i \in \{1, \dots, M\}$, $j \in \mathcal{N}_i$ and $l \in \{1, \dots, r_i\}$ are the S-procedure variables. □

The derived matrix inequalities (2.10), (2.19) and (2.21) are linear with respect to the decision variables c_i and α_i. Hence, c_i and α_i are considered as decision variables in this setting without affecting the convexity of the problem. However, it is not possible to achieve convex conditions, and thus a convex optimization problem, when considering the terminal control gain K_{N_i} as a decision variable. This fact is due to the existence of the terms $K_{N_i}\alpha_{N_i}$ and $K_{N_i}c_{N_i}$ which would result in a nonconvex problem if the gain K_{N_i} is assumed to be a decision variable.

The LMIs derived in [DEL19] are the same as the LMIs (2.10), (2.19) and (2.21) evaluated at $c_i = 0$ for all $i \in \{1, \dots, M\}$. Therefore, Propositions 2.2, 2.3 and 2.4 are generalizations of those found in [DEL19]. Note, however, that the feasible region obtained using (2.10), (2.19) and (2.21) does not necessarily include that obtained using the LMIs in [DEL19]. This is because the the control gain K_{N_i} is a decision variable in [DEL19], but it is not here.

In conclusion, the online OCP of the reconfigurable distributed MPC scheme with asymmetric terminal sets is given by,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &s.t. \begin{cases} x_i(0|t) = x_{0_i},\ (2.1),\ (2.2),\ (2.10),\ (2.18),\ (2.19),\ (2.21), \\ \text{for all } i \in \{1, \dots, M\},\ t \in \{0, \dots, T\},\ p \in \{1, \dots, q_i\},\ l \in \{1, \dots, r_i\}. \end{cases} \end{aligned} \tag{2.22}$$

The following theorem shows that this MPC scheme is recursively feasible and the closed-loop system is asymptotically stable whenever the optimization problem is initially feasible.

Theorem 2.1. *The distributed MPC problem with asymmetric adaptive terminal sets is recursively feasible and the closed-loop system under this MPC controller is asymptotically stable.*

Proof. The proof follows from the standard MPC feasibility and stability arguments (e.g. see [RMD17]). □

In the standard MPC arguments, recursive feasibility is ensured by proving that the tail sequence at a specific time instant is a feasible solution at the next time instant. Asymptotic stability is ensured by considering the cost function $J(t) = \sum_{i=1}^{M} J_i(t)$ as a Lyapunov function. Denoting the optimal global cost, state and input at time t by $J^*(t)$, $x^*(t)$ and $u^*(t)$, one can verify that $J^*(t+1) - J^*(t) \leq J^s(t) - J^*(t) \leq -\|x^*(t)\|_Q^2 - \|u^*(t)\|_R^2$ where $J^s(t)$ is the cost corresponding to the tail sequence at time t.

2.3.2 Relaxed MPC Variant with Assymetric Terminal Set

We now move to the second scheme which we call "Relaxed Reconfigurable Distributed MPC with Asymmetric Terminal Sets". Recall that, in the previous scheme, the LMIs which correspond to (2.9b) and (2.9c) are derived after transforming the linear state and input constraints into quadratic ones. Note, however, that using the linear constraints directly is less conservative because transforming these constraints into quadratic ones imposes unnecessary additional constraints. In the considered scheme, the linear constraints are used directly and hence, more relaxed matrix inequalities are derived for the conditions (2.9b) and (2.9c). The resulting matrix inequalities are linear and hence, a convex online OCP can still be reached. In the sequel, we derive this set of relaxed LMIs.

We start with condition (2.9b) which is ensured using the LMI derived in the following proposition.

Proposition 2.5. *Denote the p-th row of the matrix G_{N_i} by $G_{N_i}^p$ and the p-th element of the vector g_{N_i} by $g_{N_i}^p$. For each subsystem $i \in \{1, ..., M\}$, condition (2.9b) holds if there exist $\sigma_{ij}^p \geq 0$ for all $j \in \mathcal{N}_i$ and $p \in \{1, \ldots, q_i\}$ such that,*

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \sigma_{ij}^p P_{ij} & \frac{1}{2}\alpha_{N_i} G_{N_i}^{p\top} \\ \frac{1}{2} G_{N_i}^p \alpha_{N_i} & g_{N_i}^p - G_{N_i}^p c_{N_i} - \sum_{j \in \mathcal{N}_i} \sigma_{ij}^p \end{bmatrix} \geq 0, \text{ for all } p \in \{1, \ldots, q_i\}. \tag{2.23}$$

Proof. Consider the auxiliary vectors s_i and the concatenated auxiliary vectors s_{N_i} defined in (2.12) and (2.13) and note that condition (2.9b) holds if for all $p \in \{1, 2, ..., q_i\}$,

$$G_{N_i}^p x_{N_i} \leq g_{N_i}^p, \quad \text{for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|_{P_j}^2 \leq \alpha_j^2. \tag{2.24}$$

Substituting for these auxiliary vectors, (2.24) becomes,

$$G_{N_i}^p (c_{N_i} + \alpha_{N_i} s_{N_i}) \leq g_{N_i}^p, \quad \text{for all } j \in \mathcal{N}_i,\ s_j : \|s_j\|_{P_j}^2 \leq 1.$$

Using the mapping equations in (2.3), the above implication can be expressed as,

$$G^p_{N_i}\alpha_{N_i}s_{N_i} + G^p_{N_i}c_{N_i} \leq g^p_{N_i}, \quad \text{for all } s_{N_i} : \|s_{N_i}\|^2_{P_{ij}} \leq 1.$$

By applying the S-procedure [Boy+94] to the above implication, the state constraints of each subsystem $i \in \{1, ..., M\}$ are satisfied inside the terminal set if there exist $\sigma^p_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ and $p \in \{1, \ldots, q_i\}$ such that,

$$\sum_{j\in\mathcal{N}_i} \sigma^p_{ij} \begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\alpha_{N_i}G^{p\top}_{N_i} \\ \frac{1}{2}G^p_{N_i}\alpha_{N_i} & G^p_{N_i}c_{N_i} - g^p_{N_i} \end{bmatrix} \geq 0, \text{ for all } p \in \{1, \ldots . q_i\}$$

Rearranging the above LMI results in (2.23). □

Similarly, condition (2.9c) can be ensured using an LMI, as shown in the following proposition.

Proposition 2.6. *Denote the l-th row of the matrix H_i by H^l_i and the l-th element of the vector h_i by h^l_i. For each subsystem $i \in \{1, ..., M\}$, condition (2.9c) holds if there exist $\beta^l_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ and $l \in \{1, \ldots, r_i\}$ such that,*

$$\begin{bmatrix} \sum\limits_{j\in\mathcal{N}_i} \beta^l_{ij}P_{ij} & \frac{1}{2}\alpha_{N_i}K^\top_{N_i}H^{l\top}_i \\ \frac{1}{2}H^l_iK_{N_i}\alpha_{N_i} & h^l_{N_i} - H^l_iK_{N_i}c_{N_i} - \sum\limits_{j\in\mathcal{N}_i}\beta^l_{ij} \end{bmatrix} \geq 0, \text{ for all } l \in \{1, \ldots, r_i\}. \tag{2.25}$$

Proof. Note that condition (2.9c) holds if for all $l \in \{1, 2, ..., r_i\}$,

$$H^l_iK_{N_i}x_{N_i} \leq h^l_{N_i}, \text{ for all } j \in \mathcal{N}_i, \; x_j : \|x_j - c_j\|^2_{P_j} \leq \alpha^2_j,$$

Hence, the proof follows that of Proposition 2.5 by replacing σ^k_{ij}, $g^k_{N_i}$ and $G^k_{N_i}$ with β^l_{ij}, h^l_i and $H^l_iK_{N_i}$, respectively, where β^l_{ij} for all $i \in \{1, \ldots, M\}$, $j \in \mathcal{N}_i$ and $l \in \{1, \ldots, r_i\}$ are the S-procedure variables. □

In summary, the online OCP of the relaxed reconfigurable distributed MPC with asymmetric terminal sets is given by,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &s.t. \begin{cases} x_i(0|t) = x_{0_i}, \; (2.1), \; (2.2), \; (2.10), \; (2.18), \; (2.23), \; (2.25), \\ \text{for all } i \in \{1, \ldots, M\}, \; t \in \{0, \ldots, N\}, \; k \in \{1, \ldots, q_i\}, \; l \in \{1, \ldots, r_i\}. \end{cases} \end{aligned} \tag{2.26}$$

The following theorem shows that this MPC scheme is recursively feasible and the closed loop system is asymptotically stable whenever the optimization problem is initially feasible.

Theorem 2.2. *The relaxed reconfigurbale distributed MPC scheme with asymmetric terminal sets is recursively feasible and the closed-loop system under this MPC controller is asymptotically stable.*

Proof. The proof follows that of Theorem 2.1. □

Feasible Regions

When LMIs (2.19) and (2.21) are derived based on the quadratic state and input constraints in the first scheme, the decision variable c_i is found to lie in the off-diagonal terms. When the relaxed LMIs (2.23) and (2.25) are derived directly using the actual linear constraints in the second scheme, the decision variable c_i is found to appear in the diagonal terms. In both cases, the variable c_i is constrained to move in some directions. In the former case, the center c_i is constrained to move along the direction perpendicular to the hyperplane defining the state/input constraint. In the latter case, the center c_i is constrained to move in the direction perpendicular and pointing towards the hyperplane defining the state/input constraint. Thus, by intuition, the relaxed optimal control problem may indeed have a larger feasible region. Notice that the LMI (2.10) remains the same with c_i appearing in the off-diagonal terms since the invariance condition (2.11) is quadratic by definition.

Distributed Implementation

Although the OCPs (2.22) and (2.26) are expressed centrally, it is still possible to solve them in a distributed fashion using a distributed optimization technique such as the alternating direction method of multipliers (ADMM) (see [Boy+11] for more details). When using ADMM, each subsystem has its own local controller which performs the following steps.

1. The local controller of subsystem $i \in \{1, \dots, M\}$ solves a local optimization problem to compute a local copy of $x_j(k|t)$, α_j and c_j for all $j \in \mathcal{N}_i$ and $k \in \{0, \dots, N\}$ as well as $u_i(k|t)$ for all $k \in \{0, \dots, N-1\}$.

2. The local controller of subsystem $i \in \{1, \dots, M\}$ sends the values of the local copies of $x_j(k|t)$, α_j and c_j for all $k \in \{0, \dots, N\}$ to the corresponding subsystem $j \in \mathcal{N}_i$.

3. The local controller of subsystem $i \in \{1, \dots, M\}$ computes the average values of $x_i(k|t)$, α_i and c_i for all $k \in \{0, \dots, N\}$ using the values of the local copies sent by its neighbours.

4. The local controller of subsystem $i \in \{1, \dots, M\}$ sends the computed average values of $x_i(k|t)$, α_i and c_i for all $k \in \{0, \dots, N\}$ to its neighbours.

5. The local controller of subsystem $i \in \{1, \ldots, M\}$ updates dual variables, each of which correspond to one of its local copies computed in the first step, using the average values and the values of its local copies.

The above procedure is repeated until consensus is reached on the local copies of the same variable. The values of the local copies differ in each iteration because they depend on the average values and the values of the dual variables computed in the previous iteration. In the first iteration, the average values and the values of the dual variables are initialized (e.g. set to zero). Note that consensus is guaranteed asymptotically given the above steps. Due to limited computational capabilities, the above procedure is terminated either after a predefined number of iterations or after a predefined time has passed. These predefined parameters can be determined, for example, based on the sampling time of the controller.

2.4 Simulation Results

In this section, the effectiveness of the reconfigurable distributed MPC with asymmetric terminal set (denoted by R-ASM) and the relaxed version (denoted by R-RLX) is illustrated by means of a simulation example. To emphasize their efficacy, these two schemes are compared to the one developed in [DEL19] (denoted by R-SYM).

We consider a discrete-time linear time-invariant system where the dynamics of the subsystems are given by,

$$\begin{aligned}
&x_1(t+1) = 2x_1(t) + 0.5x_2(t),\\
&x_i(t+1) = 0.5x_{i-1}(t) + 2x_i(t) + 0.5x_{i+1}(t), \text{ for all } i \in \{2, ..., M-1\},\\
&x_M(t+1) = 0.5x_{M-1}(t) + 2x_M(t).
\end{aligned}$$

The state and input constraints of the subsystems are as follows,

$$\begin{aligned}
&-5 \leq x_i(t) \leq 5, \text{ for all } i \in \{1, ..., M\},\\
&-0.25 \leq u_i(t) \leq 1, \text{ for all } i \in \{1, ..., M\}.
\end{aligned}$$

The system and constraint matrices in (2.1) and (2.2) can be derived in the obvious way. The cost function matrices are selected such that $Q = I_n$ and $R = I_m$ where I_n and I_m are identity matrices of size n and m, respectively. The prediction horizon is selected to be $N = 2$. The terminal cost and controller are computed by solving the SDP (2.7).

Two test scenarios are considered here. In the first one, we consider the case in which $M = 2$ to compare R-SYM, R-ASM and R-RLX, whereas we consider the case in which $M = 9$ in the second one to demonstrate the efficacy of the proposed approach for larger systems. While the global OCP is solved centrally in the first scenario to compare the

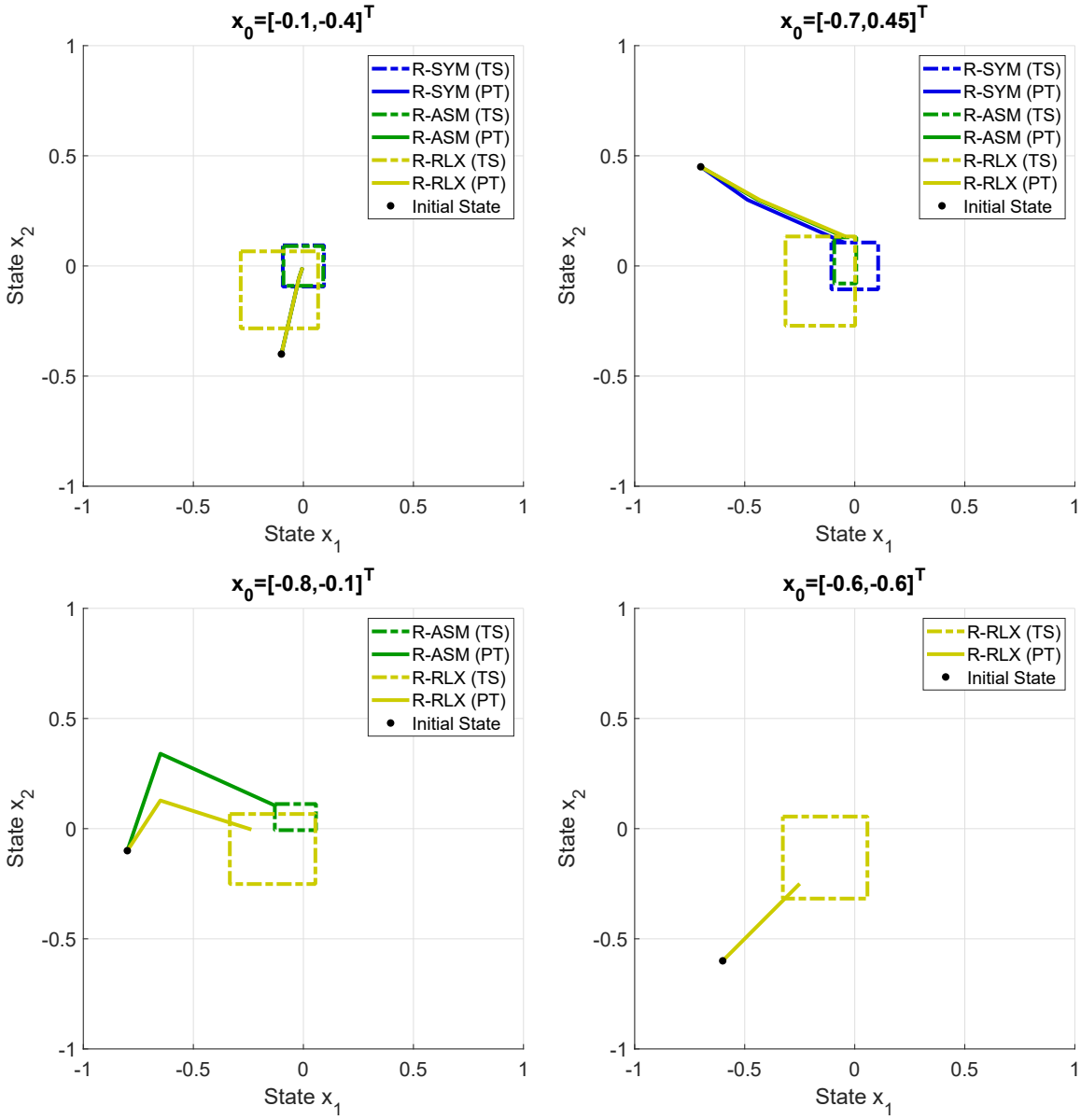

Figure 2.1: Predicted state trajectories (PT) and terminal sets (TS) of three MPC schemes; R-SYM (Blue), R-ASM (Green) and R-RLX (Yellow) for four different initial conditions and a prediction horizon of $T = 2$ when solving the online OCP centrally once.

Table 2.1: Closed-loop cost for different schemes and initial conditions

Initial Conditions	$x_0 = \begin{bmatrix} -0.1 \\ -0.4 \end{bmatrix}$	$x_0 = \begin{bmatrix} -0.7 \\ 0.45 \end{bmatrix}$	$x_0 = \begin{bmatrix} -0.8 \\ -0.1 \end{bmatrix}$	$x_0 = \begin{bmatrix} -0.6 \\ -0.6 \end{bmatrix}$
R-SYM	0.2531	1.2660	-	-
R-ASM	0.2531	1.2033	1.6930	-
R-RLX	0.2531	1.1939	1.4998	1.7140

three MPC schemes, ADMM is used in the second scenario to show that the proposed schemes are amenable to distributed optimization.

Figure 2.1 shows the optimal predicted state trajectory (refered to as PT) and the terminal set (refered to as TS) of the three schemes for four different initial conditions when the online OCP is solved once. The terminal sets are described by rectangles and not ellipsoids because each of them is the product of two ellipsoidal sets in one dimension. Note that the terminal set of one scheme is not the same for all the initial conditions. This emphasizes that the terminal set depends on the current state of the system. In each case, the terminal set will be updated online at the next time instant based on the new state of the system.

The online OCP is initially feasible for all the schemes and the predicted state trajectories are the same when the initial condition is $x_0 = [-0.1 \ -0.4]^\top$. Notice that the R-SYM terminal set is partially hidden behind that of R-ASM for this initial condition. The terminal set of R-RLX is clearly not centered at the origin and is found to be larger than the terminal sets of the other two schemes. This is because the terminal set is not constrained to be centered at the origin as in R-SYM and the LMIs derived for R-RLX are relaxed compared to those for R-ASM. Although the terminal set of R-ASM is almost centered at the origin, this is not necessarily the case as shown when the initial condition is $x_0 = [-0.7 \ 0.45]^\top$. For this initial condition, the online OCP is still feasible for all schemes. However, the predicted state trajectory of R-SYM is different from the other two schemes and hence, reflects the conservativeness of R-SYM. The terminal set of R-ASM is smaller than that of R-SYM in this case so that it can include some parts in the state space which can not be reached using the terminal set of R-SYM.

For the initial condition $x_0 = [-0.8 \ -0.1]^\top$, R-SYM is not initially feasible due to the constraint that the center of its terminal set should be the origin. On the other hand, R-ASM is initially feasible with the center of its terminal set not located at the origin. Similarly, R-RLX is also initially feasible and its terminal set is larger than that of R-ASM. When the initial condition is $x_0 = [-0.6 \ -0.6]^\top$, R-RLX is the only initially feasible scheme showing that its domain of attraction comprises some parts in the state space that are not included in the domain of attraction of the other two schemes.

Table 2.1 shows the closed-loop cost for the different schemes and initial conditions when the online OCP is solved recursively for 10 time steps. The symbol (-) is used

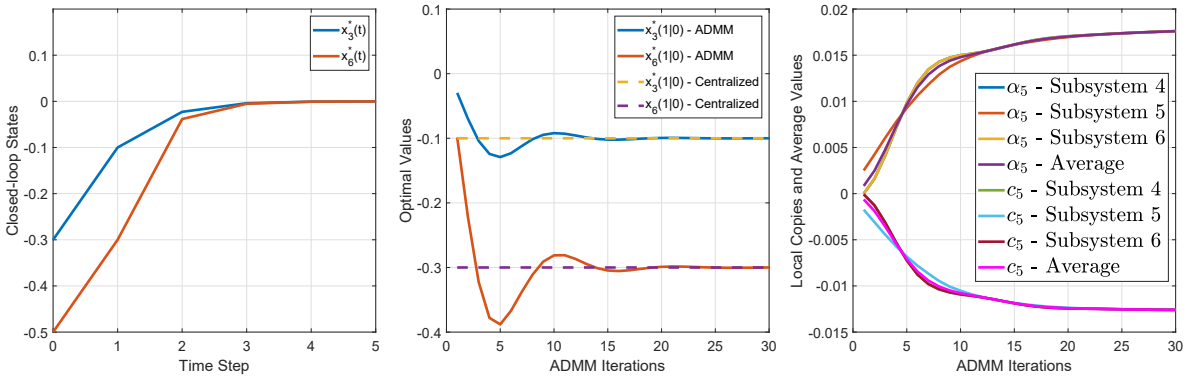

Figure 2.2: (left) Closed-loop state trajectories of the third and sixth subsystems starting from $x_{0_i} = [-0.3 \ -0.5]^\top$ for all $i \in \{1, \dots, M\}$ when solving the online OCP of R-RLX recursively using ADMM, (middle) Average values of the local copies of the second element of the predicted state trajectories of the third and sixth subsystems corresponding to the ADMM iterations at the first time step and the corresponding centralized solutions, (right) Values of the local copies of the terminal set size and center of the fifth subsystem as well as their average values corresponding to the ADMM iterations at the first time step.

when the corresponding scheme is infeasible for the considered initial condition. When the initial condition is $x_0 = [-0.1 \ -0.4]^\top$, the cost of all the schemes is the same because the optimal state and input trajectories are the same independent of the scheme applied. In the case of $x_0 = [-0.7 \, 0.45]^\top$, the cost of R-SYM is higher showing the relative suboptimality of R-SYM with respect to the other two schemes. For $x_0 = [-0.8 \ -0.1]^\top$, the cost of R-ASM is higher than that of R-RLX because R-ASM results in a smaller terminal set leading to a suboptimal solution. Finally, for $x_0 = [-0.6 \ -0.6]^\top$, R-RLX is the only feasible scheme as shown in Figure 2.1.

Figure 2.2 shows the result of the second test scenario when $M = 9$. We focus here on R-RLX since it outperforms the other schemes in the first test scenario. In the left figure, the closed-loop state trajectories of the third and sixth subsystems are shown when R-RLX is implemented recursively for ten time steps using ADMM. As shown, the system states converge to the origin illustrating the recursive feasibility of R-RLX and the asymptotic stability of the corresponding closed-loop system. In the middle figure, the average values of the local copies of the second element of the predicted state trajectories of the third and sixth subsystems corresponding to each ADMM iteration at the first time step are shown. Note that these average values converge to the optimal values obtained by solving the global OCP centrally. In other words, the optimal centralized solutions are reached using distributed optimization. In the right figure, the values of the local copies of the terminal set size and center of the fifth subsystem as well as their average values corresponding to each ADMM iteration at the first time step are shown.

Despite reaching consensus, more iterations are required so that the values of these local copies as well as their average values converge to the steady state value.

CHAPTER 3

Reconfigurable Distributed MPC for Tracking

3.1 Introduction

Various distributed MPC schemes have been proposed for constrained interconnected systems (e.g. see [NM14; MN14]). This is because distributed MPC has several advantages, such as increased privacy, robustness against failure and scalability when controlling such systems as discussed in Chapter 1. Although many of these schemes are developed for regulation problems, reference tracking is found to be crucial in many applications. Thus, several distributed MPC schemes have been developed for reference tracking (see, e.g. [RP16a; RP16b; FBS13; KF13]).

In [Fer+13], a tracking distributed MPC scheme is developed for linear time-invariant systems where the maximal invariant terminal set for tracking developed in [Lim+08] is used. This polytopic terminal set, however, does not necessarily respect the distributed structure of the system and might couple all subsystems. Various efforts have been devoted to synthesizing terminal sets which respect the distributed structure when using tracking distributed MPC. Polytopic terminal sets are used in [Cha+21] where the interconnected system is decomposed into several clusters, each comprising a set of subsystems. However, the coupling between the clusters is considered as an uncertainty yielding conservative polytopic terminal sets. A tracking distributed MPC scheme with ellipsoidal terminal sets is also developed in [Con+13]. Although this scheme respects the distributed structure of the system, it leads to relatively small feasible regions.

Various methods have been developed to alleviate the conservatism imposed by terminal sets and enlarge the resulting feasible regions. These methods include using a reference governor [NLK18; Di +18], dynamic terminal set transformation [SLG14], generalized terminal ingredients [MF16; FT13] and construction of terminal sets using feasible trajectories [BLA15]. Although the aforementioned methods are developed for centralized MPC schemes, some are extended to distributed MPC schemes as in [TM17; DEL19; WO16] where the terminal ingredients are computed online. However, these

schemes are mainly developed for regulation problems.

In this chapter, we introduce two reconfigurable distributed MPC schemes for reference tracking of interconnected systems with a distributed structure. Unlike [Fer+13], the terminal ingredients are designed to respect the distributed structure while alleviating the conservatism of [Cha+21; Con+13]. Inspired by [DEL19], conservatism is alleviated by reconfiguring the terminal ingredients online at each time instant based on the current state of the system.

In the first scheme, the terminal set, controller and cost are updated by considering them as decision variables in the online OCP. Additional constraints are imposed in the OCP to ensure the positive invariance of the reconfigured terminal set and the asymptotic stability of the reconfigured terminal dynamics. Although the resulting OCP is infinite-dimensional, it can be formulated as a semi-infinite program by restricting the terminal ingredients to ellipsoidal sets, affine controllers and quadratic costs. Ellipsoidal terminal sets are used here since they can be easily defined using the level sets of the Lyapunov function. Using robust optimization tools, the infinite number of constraints is then transformed into a finite number of matrix inequalities yielding a finite, albeit non-convex mathematical program. This is in turn shown to be equivalent to an SDP through a change of variables. To improve computational performance, the resulting SDP can be further approximated by a QP using an SDP scalability method which is based on diagonal dominance [AH17].

Unlike the first scheme, the terminal control gain of the affine controller in the second scheme is computed offline and no longer considered a decision variable in the online OCP. The resulting OCP has fewer decision variables in this case, but also fewer constraints; in particular, the constraints required to ensure the asymptotic stability of the terminal dynamics are no longer added in the OCP. Moreover, the terminal set is no longer required to be centered at an equilibrium point. In addition, constraint satisfaction inside the terminal set is ensured by a set of linear inequalities instead of LMIs without using diagonal dominance. The flexibility of the terminal set center maintains a large MPC feasible region. Moreover, the reduction in the number of decision variables and constraints (LMIs in particular) compared to the first scheme results in a remarkable reduction in the online computational cost. Hence, when solving the OCP using distributed optimization techniques such as ADMM, more iterations can be performed within the available sampling time, leading to better convergence to the optimal solution.

The recursive feasibility of the proposed schemes and the asymptotic stability of the corresponding closed-loop system are proven in the presence of reconfigurable terminal ingredients. The feasible regions of the developed schemes are compared to other existing schemes in the literature. The proposed schemes are evaluated via simulation and their efficacy is evaluated using a toy example, a benchmark example and a power network case study. The toy example is used to illustrate the idea behind reconfiguring the terminal ingredients online for tracking problems. The benchmark example is used to

compare the first scheme to other existing schemes in the literature. The power network case study is used to compare the two proposed schemes.

3.2 Tracking Problem

In this section, we introduce the tracking problem of complex constrained interconnected systems. We consider here the same class of systems considered in Chapter 2. In particular, we consider discrete-time linear time-invariant systems subject to polytopic state and input constraints. We assume that these systems can be decomposed into a set of M subsystems, each of which has a set of neighbours $\mathcal{N}_i$ for all $i \in \{1, \dots, M\}$. Two subsystems are considered as neighbours if the states of one appear in the dynamics and/or state constraints of the other. We assume that $i \in \mathcal{N}_i$ for all $i = \{1, \dots, M\}$. We denote the state and input vectors of the i-th subsystem by $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$, respectively. We also define $x_{N_i} \in \mathbb{R}^{n_{N_i}}$ to be a concatenated state vector comprising the states of the subsystems in the set $\mathcal{N}_i$. Hence, the dynamics of the i-th subsystem is given by,

$$x_i(t+1) = A_{N_i} x_{N_i}(t) + B_i u_i(t), \tag{3.1}$$

where $t \in \mathbb{N}_+$ is the time index and the matrices $A_{N_i} \in \mathbb{R}^{n_i \times n_{N_i}}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ are assumed to be known. Similarly, the constraints of the i-th subsystem are also given by,

$$\begin{aligned} x_{N_i}(t) \in \mathcal{X}_{N_i} &= \{x_{N_i} \in \mathbb{R}^{n_{N_i}} : G_{N_i} x_{N_i} \leq g_{N_i}\}, \\ u_i(t) \in \mathcal{U}_i &= \{u_i \in \mathbb{R}^{m_i} : H_i u_i \leq h_i\}, \end{aligned} \tag{3.2}$$

where $G_{N_i} \in \mathbb{R}^{q_i \times n_i}$, $H_i \in \mathbb{R}^{q_i \times m_i}$, $g_{N_i} \in \mathbb{R}^{q_i}$ and $h_i \in \mathbb{R}^{r_i}$ are assumed to be known. Without loss of generality, input coupling is not considered and the subsystems are assumed to be coupled only through the states; in the presence of coupled inputs, additional auxiliary variables can be introduced to ensure this assumption is met as in [DEL19].

Assuming that the i-th subsystem is required to track the target point x_{r_i}, the local cost function of this subsystem is chosen to be,

$$\begin{aligned} J_i(t) = \sum_{k=0}^{N-1} \Big(\|x_{N_i}(k|t) - x_{e_{N_i}}(t)\|_{Q_i}^2 + \|u_i(k|t) - u_{e_i}(t)\|_{R_i}^2 \Big) \\ + \|x_i(N|t) - x_{e_i}(t)\|_{P_i}^2 + \|x_{e_i}(t) - x_{r_i}\|_{S_i}^2, \end{aligned} \tag{3.3}$$

where N is the prediction horizon, $x_i(k|t)$ and $u_i(k|t)$ are the k-step ahead predicted state and input vectors of the i-th subsystem at time instant t and $x_{N_i}(k|t)$ comprises the k-step ahead predicted states of the subsystems in the set $\mathcal{N}_i$ at time instant t. The local cost function matrices $Q_i \in \mathbb{S}_{++}^{n_i}$, $R_i \in \mathbb{S}_{++}^{m_i}$ and $S_i \in \mathbb{S}_{++}^{n_i}$ are predefined to satisfy the optimization requirements and the terminal cost matrix $P_i \in \mathbb{S}_{++}^{n_i}$ is computed by solving the SDP (2.7). The pair $(x_{e_i}(t), u_{e_i}(t))$ is an artificial equilibrium point to which

the i-th subsystem aims to converge at time instant t. This artificial equilibrium is considered as a decision variable in the OCP. The last term in the cost function (3.3) aims to drive this equilibrium towards the target point. This form of the cost will become apparent later while proving stability and comparing the feasibility regions. We restrict attention to admissible equilibrium points that lie in the interior of the corresponding constraints. In particular, the point $(x_{e_i}(t), u_{e_i}(t))$ is an admissible equilibrium point under a stabilizing feedback policy $\kappa_i : X_{N_i} \to U_i$ if,

$$\begin{aligned} x_{e_i}(t) &= Ax_{e_{N_i}}(t) + Bu_{e_i}(t), \\ u_{e_i}(t) &= \kappa(x_{e_{N_i}}(t)) \\ x_{e_{N_i}}(t) &\in \xi\mathcal{X}_{N_i}, \\ u_{e_i}(t) &\in \xi\mathcal{U}_i, \end{aligned} \tag{3.4}$$

where $\xi \in (0, 1)$ is chosen arbitrarily close to one. We assume that there exists a feedback policy $\kappa_{r_i}(\cdot)$ such that the point $(x_{r_i}, \kappa_{r_i}(\cdot))$ is an admissible equilibrium point. Note that this implies that the sets X_{N_i} and U_i have non-empty interior. To guarantee that the proposed MPC is recursively feasible and the corresponding closed-loop dynamics is asymptotically stable, we require that,

$$x_i(N|t) \in \mathcal{X}_{f_i}, \tag{3.5}$$

where $\mathcal{X}_{f_i}$ is a positively invariant terminal set under the stabilizing terminal controller $\kappa_i(x_{N_i})$.

Defining the global state vector $x(t) = [x_1(t), \ldots, x_M(t)]^\top \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_M} = \mathbb{R}^n$ and the global input vector $u(t) = [u_1(t), \ldots, u_M(t)]^\top \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_M} = \mathbb{R}^m$ at time instant t, the dynamics of the overall system is given by,

$$x(t+1) = Ax(t) + Bu(t). \tag{3.6}$$

The global matrices A and B can be constructed from the local matrices A_{N_i} and B_i for all $i \in \{1, \ldots, M\}$ in the obvious way. We assume that the pair (A, B) is stabilizable. The local variables of the i-th subsystem can be extracted from the global variables using the mappings $U_i \in \{0, 1\}^{n_i \times n}$, $W_i \in \{0, 1\}^{n_{N_i} \times n}$ and $V_i \in \{0, 1\}^{m_i \times m}$ where,

$$\begin{aligned} x_i &= U_i x, \\ x_{N_i} &= W_i x, \\ u_i &= V_i u. \end{aligned} \tag{3.7}$$

Similarly, the states and input constraints of the overall system (3.6) are given by,

$$\begin{aligned} x(t) \in \mathcal{X} &= \{x \in \mathbb{R}^n : Gx \leq g\}, \\ u(t) \in \mathcal{U} &= \{u \in \mathbb{R}^m : Hu \leq h\}, \end{aligned}$$

The global matrices $G \in \mathbb{R}^{q\times n}$, $H \in \mathbb{R}^{r\times m}$ and the vectors $g \in \mathbb{R}^q$, $h \in \mathbb{R}^r$ can be obtained from the corresponding local matrices and vectors in the obvious way. By defining the set $\mathcal{X}_i = \mathcal{X}_{\mathcal{N}_i} \times \mathbb{R}^{\sum_{i\in\mathcal{M}\backslash\mathcal{N}_i} n_i}$ where $\mathcal{M}$ is the set of all subsystems and $\mathcal{M}\backslash\mathcal{N}_i$ is the set of all subsystems excluding the neighbours of the i-th subsystem, one can verify that,

$$\begin{aligned}\mathcal{X} &= \mathcal{X}_1 \cap ... \cap \mathcal{X}_i \cap ... \cap \mathcal{X}_M,\\ \mathcal{U} &= \mathcal{U}_1 \times ... \times \mathcal{U}_i \times ... \times \mathcal{U}_M.\end{aligned}$$

The global cost function is given by,

$$\begin{aligned}J(t) = \sum_{i=1}^{M} J_i(t) = \sum_{k=0}^{N-1} \left(\|x(k|t) - x_e(t)\|_Q^2 + \|u(k|t) - u_e(t)\|_R^2\right)\\ + \|x(N|t) - x_e(t)\|_P^2 + \|x_e(t) - x_r\|_S^2,\end{aligned}$$

where $Q \in \mathbb{S}_{++}^n$, $R \in \mathbb{S}_{++}^m$, $P \in \mathbb{S}_{++}^n$ and $S \in \mathbb{S}_{++}^n$ can be constructed using the corresponding local matrices in the obvious way. The global artificial equilibrium point (x_e, u_e) satisfy,

$$\begin{aligned}x_e(t) &= Ax_e(t) + Bu_e(t),\\ u_e(t) &= \kappa(x_e(t)),\\ x_e(t) &\in \xi\mathcal{X},\\ u_e(t) &\in \xi\mathcal{U}.\end{aligned}$$

where $x_e(t) = [x_{e_1}^\top(t), \ldots, x_{e_M}^\top(t)]^\top$, $u_e(t) = [u_{e_1}^\top(t), \ldots, u_{e_M}^\top(t)]^\top$ and $\kappa(x) = [\kappa_1^\top(x_{N_1}), \ldots, \kappa_M^\top(x_{N_M})]^\top$. By definition, the target point $x_r = [x_{r_1}^\top, \ldots, x_{r_M}^\top]^\top$ is such that the pair $(x_r, \kappa_r(x_r)$ is a global admissible equilibrium point for the overall system. Finally, note that the global terminal set is defined as,

$$\mathcal{X}_f = \mathcal{X}_{f_1} \times \ldots \times \mathcal{X}_{f_M}.$$

Recall that we aim to determine the terminal set $\mathcal{X}_{f_i}$ and the corresponding terminal controller $\kappa_i(\cdot)$ online. To ensure that they are positively invariant and asymptotically stabilizing, respectively, additional constraints have to be imposed. The latter requirement is ensured by the following theorem. In this theorem, a continuous function $f : [0, d] \to [0, \infty]$ belongs to the class $\mathcal{K}$ if $f(\cdot)$ is strictly increasing and $f(0) = 0$. Furthermore, a continuous function $f : [0, d] \to [0, \infty]$ belongs to the class $\mathcal{K}_\infty$ if $f(\cdot)$ belongs to the class $\mathcal{K}$, $d = \infty$ and $\lim_{a\to\infty} f(a) = \infty$.

Theorem 3.1 ([JL09])**.** *Consider a distributed controller of the form $\kappa_i(x_{N_i}) : \mathbb{R}^{n_{N_i}} \to \mathbb{R}^{m_i}$. Assume that, for each subsystem $i \in \{1, ..., M\}$, there exist functions $V_i(x_i) : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $\gamma_i(x_{N_i}) : \mathbb{R}^{n_{N_i}} \to \mathbb{R}$ and $l_i(x_{N_i}, \kappa_i(x_{N_i})) : \mathbb{R}^{n_{N_i}+m_i} \to \mathbb{R}_+$ and functions $\delta_{1_i}(\cdot)$, $\delta_{2_i}(\cdot)$,*

$\delta_{3_i}(\cdot) \in \mathcal{K}_\infty$ *such that for all* $x_i \in \mathcal{X}_{f_i}$,

$$\begin{aligned}
\delta_{1_i}(\cdot) \leq V_i(x_i) \leq \delta_{2_i}(\cdot),\\
\delta_{3_i}(\cdot) \leq l_i(x_{N_i}, \kappa_i(x_{N_i})),\\
V_i(A_{N_i}x_{N_i} + B_i\kappa_i(x_{N_i})) - V_i(x_i) \leq -l_i(x_{N_i}, \kappa_i(x_{N_i})) + \gamma_i(x_{N_i}),\\
\sum_{i=1}^{M} \gamma_i(x_{N_i}) \leq 0.
\end{aligned}$$

Then, the function $V(x) = \sum_{i=1}^{M} V_i(x_i)$ *is a Lyapunov function for the closed-loop system under the control law* $\kappa_i(x_{N_i})$ *on the set* $\mathcal{X}_f = \times_{i\in\{1,\dots,M\}}\mathcal{X}_{f_i}$.

The positive invariance of the terminal set under the action of the terminal controller is ensured by means of the conditions in Proposition 2.1. For the sake of completeness, this proposition is repeated below.

Proposition 3.1 ([DEL19])**.** *Define the sets* $\mathcal{X}_{f_{N_i}} = \times_{j\in\mathcal{N}_i}\mathcal{X}_{f_j}$, *then, each local terminal set* $\mathcal{X}_{f_i}$ *is positively invariant under the action of the distributed controller* $\kappa_i(x_{N_i})$ *if for each subsystem* $i \in \{1, ..., M\}$ *and for all* $x_{N_i} \in \mathcal{X}_{f_{N_i}}$,

$$A_{N_i}x_{N_i} + B_i\kappa_i(x_{N_i}) \in \mathcal{X}_{f_i}, \tag{3.9a}$$
$$x_{N_i} \in \mathcal{X}_{\mathcal{N}_i}, \tag{3.9b}$$
$$\kappa_i(x_{N_i}) \in \mathcal{U}_i. \tag{3.9c}$$

Consequently, the global terminal set $\mathcal{X}_f$ *is also positively invariant.*

Condition (3.9a) ensures that the terminal set $\mathcal{X}_{f_i}$ is invariant, whereas conditions (3.9b) and (3.9c) ensure that the state and input constraints are satisfied inside the terminal set.

To complete the formulation of the online OCP, we simply need to add a constraint to initialise the first element of the predicted state trajectory to the current state of the system as follows,

$$x_i(0|t) = x_{0_i}, \tag{3.10}$$

where $x_{0_i} \in \mathbb{R}^{n_i}$ is the current state of the i-th subsystem.

In summary, the global online OCP is given by,

$$\begin{aligned}
&\min \sum_{i=1}^{M} J_i(t)\\
&\text{s.t.} \begin{cases} (3.1),\ (3.2),\ (3.4),\ (3.5),\ (3.10),\ \text{for all } i \in \{1,\dots,M\} \text{ and } k \in \{0,\dots,N-1\},\\ (3.8),\ (3.9) \text{ for all } i \in \{1,\dots,M\} \text{ and } x_j \in \mathcal{X}_{f_j}. \end{cases}
\end{aligned} \tag{3.11}$$

The OCP (3.11) is solved at each time instant yielding for all $i \in \{1, \ldots, M\}$, the optimal predicted state trajecotory $x_i^*(k|t)$ for all $k \in \{0, \ldots, N\}$, the optimal predicted input trajectory $u_i^*(k|t)$ for all $k \in \{0, \ldots, N-1\}$, the optimal predicted equilibrium point $(x_{e_i}^*(t), u_{e_i}^*(t))$, the optimal local terminal set $\mathcal{X}_{f_i}^*$ and the optimal local terminal controller $\kappa_i^*(\cdot)$. Clearly, (3.11) is an intractable optimisation problem; we discuss the steps we propose to approximate it by a tractable problem in the subsequent sections.

3.3 Control Design

In this section, we present two reconfigurbale distributed MPC schemes for reference tracking of interconnected systems. These two schemes are developed by approximating the intractable OCP (3.11) by a tractable one. The developed schemes differ in the set of assumptions considered to reach a tractable OCP. The different assumptions result in a different set of decision variables and additional constraints as discussed in the rest of this section.

The online OCP (3.11) is infinite-dimensional, with an infinite number of decision variables (the terminal set $\mathcal{X}_{f_i}$ and the terminal controller $\kappa_i(\cdot)$ for all $i \in \{1, \ldots, M\}$) and an infinite number of constraints ((3.8) and (3.9) have to hold for each state inside the terminal set). To transform the infinite-dimensional problem into a finite-dimensional one, the terminal set is restricted to the class of ellipsoidal sets and the terminal controller is restricted to the class of affine controllers. In this case, the local terminal set of the i-th subsystem for all $i \in \{1, ..., M\}$ is defined as,

$$\mathcal{X}_{f_i} = \{x_i \in \mathbb{R}^{n_i} : \|x_i - c_i\|_{P_i}^2 \leq \alpha_i^2\}, \tag{3.12}$$

where $\alpha_i \in \mathbb{R}$ and $c_i \in \mathbb{R}^{n_i}$ define the size and center of the local terminal set of the corresponding subsystem, respectively, and are considered as decision variables. On the other side, the matrix P_i is not a decision variable. Indeed, P_i is computed offline by solving the SDP (2.7). Note that this ellipsoidal set is not necessarily centered at the target point x_{r_i}. Likewise, the terminal controller of the i-th subsystem for all $i \in \{1, \ldots, M\}$ is restricted to the class of affine controllers,

$$\kappa_i(x_{N_i}) = K_{N_i} x_{N_i} + d_i, \tag{3.13}$$

where $K_{N_i} \in \mathbb{R}^{m_i \times n_{N_i}}$ is the terminal control gain matrix and $d_i \in \mathbb{R}^{m_i}$ is the affine term of the terminal controller. While K_{N_i} and d_i are both considered decision variables in the first scheme, only d_i is considered as a decision variable in the second scheme. In this scheme, the matrix K_{N_i} is computed offline together with the matrix P_i by solving the SDP (2.7).

Given (3.13), the constraints (3.4) become,

$$\begin{aligned} x_{e_i} &= A_{N_i} x_{e_{N_i}} + B_i u_{e_i}, \\ u_{e_i} &= K_{N_i} x_{e_{N_i}} + d_i, \\ x_{e_{N_i}} &\in \xi \mathcal{X}_{N_i}, \\ u_{e_i} &\in \xi \mathcal{U}_i. \end{aligned} \tag{3.14}$$

Given (3.12), the terminal constraint (3.5) modifies to,

$$\|x_i(N|t) - c_i\|_{P_i}^2 \leq \alpha_i^2. \tag{3.15}$$

This constraint can be equivalently represented by means of the Schur complement as,

$$\begin{bmatrix} P_i^{-1}\alpha_i & x_i(N|t) - c_i \\ x_i(N|t) - c_i & \alpha_i \end{bmatrix} \geq 0. \tag{3.16}$$

Considering (3.12) and (3.13), the positive invariance constraints (3.9) in Proposition 3.1 modify to,

$$\text{for all } i \in \{1, \ldots, M\} \text{ and } x_i \in \mathcal{X}_{f_i},$$

$$\|(A_{N_i} + B_i K_{N_i}) x_{N_i} + B_i d_i - c_i\|_{P_i}^2 \leq \alpha_i^2, \tag{3.17a}$$

$$G_{N_i} x_{N_i} \leq g_{N_i}, \tag{3.17b}$$

$$H_i(K_{N_i} x_{N_i} + d_i) \leq h_i. \tag{3.17c}$$

Considering the variables $\Delta x_i = x_i - x_{e_i}$, $\Delta u_i = u_i - u_{e_i}$ and $\Delta x_{N_i} = x_{N_i} - x_{e_{N_i}}$, we define the shifted dynamics,

$$\Delta x_i(t+1) = A_{N_i} \Delta x_{N_i}(t) + B_i \Delta u_i(t).$$

We ensure that the terminal controller (3.13) is stabilizing by applying Theorem 3.1 to the shifted dynamics. Recall from (3.14) that $d_i = -K_{N_i} x_{N_i} + u_{e_i}$ and hence, $\kappa_i(x_{N_i}) = K_{N_i} x_{N_i} + d_i = K_{N_i} \Delta x_{N_i} + u_{e_i}$. Consequently, we consider,

$$\begin{aligned} \Delta\kappa_i(x_{N_i}) &= \kappa_i(x_{N_i}) - u_{e_i} = K_{N_i} \Delta x_{N_i}, \\ V_i(\Delta x_i) &= \|\Delta x_i\|_{P_i}^2, \\ l_i(\Delta x_{N_i}, \Delta\kappa_i(x_{N_i})) &= \|\Delta x_{N_i}\|_{Q_i}^2 + \|K_{N_i} \Delta x_{N_i}\|_{R_i}^2 = \|\Delta x_{N_i}\|_{(Q_i + K_{N_i}^\top R_i K_{N_i})}^2, \\ \gamma_i(\Delta x_{N_i}) &= \|\Delta x_{N_i}\|_{\Gamma_i}^2, \\ \delta_{1_i}(\Delta x_i) &= \lambda_{1_i} \|\Delta x_i\|^2, \\ \delta_{2_i}(\Delta x_i) &= \lambda_{2_i} \|\Delta x_i\|^2, \\ \delta_{3_i}(\Delta x_{N_i}) &= \lambda_{3_i} \|\Delta x_{N_i}\|^2, \end{aligned}$$

where $\Gamma_i \in \mathbb{S}_{++}^{n_{N_i}}$ and λ_{1_i}, λ_{2_i}, λ_{3_i} are positive scalars. Hence, conditions (3.8) in Theorem 3.1 become,

$$\text{for all } i \in \{1, \dots, M\} \text{ and } x_i \in \mathcal{X}_{f_i},$$

$$\lambda_{1_i}\|\Delta x_i\|^2 \leq \|\Delta x_i\|_{P_i}^2 \leq \lambda_{2_i}\|\Delta x_i\|^2, \tag{3.18a}$$

$$\lambda_{3_i}\|\Delta x_{N_i}\|^2 \leq \|\Delta x_{N_i}\|^2_{(Q_i+K_{N_i}^\top R_i K_{N_i})}, \tag{3.18b}$$

$$\|A_{N_i}\Delta x_{N_i} + B_i \Delta\kappa_i(x_{N_i})\|_{P_i}^2 - \|\Delta x_i\|_{P_i}^2 \leq -\|\Delta x_{N_i}\|^2_{(Q_i+K_{N_i}^\top R_i K_{N_i})} + \|\Delta x_{N_i}\|_{\Gamma_i}^2, \tag{3.18c}$$

$$\sum_{i=1}^{M} \|\Delta x_{N_i}\|_{\Gamma_i}^2 \leq 0. \tag{3.18d}$$

While Γ_i is considered as a decision variable in the online OCP, the scalars λ_{1_i}, λ_{2_i} and λ_{3_i} are computed offline. Indeed, conditions (3.18a) and (3.18b) are satisfied with $\lambda_{1_i} = \lambda_{min}(P_i)$, $\lambda_{2_i} = \lambda_{max}(P_i)$ and $\lambda_{3_i} = \lambda_{min}(Q_i + K_{N_i}^\top R_i K_{N_i})$ where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ are the minimum and maximum eigenvalues of the corresponding matrix, respectively. Note that $\lambda_{1_i}\|\Delta x_i\|^2$, $\lambda_{2_i}\|\Delta x_i\|^2$ and $\lambda_{3_i}\|\Delta x_{N_i}\|^2 \in \mathcal{K}_\infty$ in this case since $P_i \in S_{++}^{n_i}$, $Q_i \in S_{++}^{n_{N_i}}$ and $R_i \in S_{++}^{m_i}$.

In summary, the global online OCP is given by,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &\text{s.t.} \begin{cases} (3.1),\ (3.2),\ (3.10),\ (3.14),\ (3.16),\ \text{for all } i \in \{1, ..., M\} \text{ and } k \in \{0, ..., N-1\}, \\ (3.17),\ (3.18),\ \text{for all } i \in \{1, \dots, M\} \text{ and } x_i \in \mathcal{X}_{f_i}. \end{cases} \end{aligned} \tag{3.19}$$

One can see that (3.19) is a finite-dimensional conservative approximation of (3.11), as the terminal sets and the terminal controllers are no longer optimized over arbitrary sets and functions. Instead, for $i \in \{1, ..., M\}$, we optimize for the terminal set size α_i, the terminal set center c_i, the affine term d_i of the terminal controller and the auxiliary matrix Γ_i together with the predicted state trajectory $x_i(k|t)$, $k \in \{1, ..., N\}$, the predicted input trajectory $u_i(k|t)$, $k \in \{1, ..., N-1\}$ and the artificial equilibrium point $(x_{e_i}(t), u_{e_i}(t))$. Recall that we also optimize for the terminal control gain matrix K_{N_i} in one of the developed schemes. On the contrary, this matrix is fixed and computed offline in the other scheme. Since in the process the feasible set is reduced, (3.19) provides an upper bound of the cost in (3.11).

The onlince OCP (3.19) still has an infinite number of constraints represented by the robust constraints (3.17) and (3.18) since these two constraints have to hold for all states inside the terminal set. In the sequel, these robust constraints are transformed into a finite number of matrix inequalities. For this purpose, the following matrices and vectors are used: $\alpha = diag(\alpha_1 I_{n_1}, ..., \alpha_M I_{n_M})$, $c = [c_1^\top, ..., c_M^\top]^\top$, $\alpha_{N_i} = W_i \alpha W_i^\top$ and $c_{N_i} = W_i c$.

We start with the robust constraint (3.17a). This constraint can be transformed into a matrix inequality as shown in the following proposition. The resulting matrix inequality is used in both schemes. The only difference is that the matrix K_{N_i} existing in the resulting matrix inequality is a decision variable in one scheme and fixed in the other.

Proposition 3.2. *The terminal set invariance condition (3.17a) of the i-th subsystem given by,*

$$\|(A_{N_i}+B_iK_{N_i})x_{N_i}+B_id_i-c_i\|_{P_i}^2\leq\alpha_i^2,\ \text{for all}\ \ j\in\mathcal{N}_i\ \text{and}\ x_j:\|x_j-c_j\|_{P_j}^2\leq\alpha_j^2, \tag{3.20}$$

holds if there exist scalars $\rho_{ij}\geq 0$ for all $j\in\mathcal{N}_i$ such that,

$$\begin{bmatrix} P_i^{-1}\alpha_i & (A_{N_i}\alpha_{N_i}+B_iK_{N_i}\alpha_{N_i}) & (A_{N_i}c_{N_i}+B_i(K_{N_i}c_{N_i}+d_i)-c_i)\\ * & \sum\limits_{j\in\mathcal{N}_i}\rho_{ij}P_{ij} & 0\\ * & * & \alpha_i-\sum\limits_{j\in\mathcal{N}_i}\rho_{ij}\end{bmatrix}\geq 0. \tag{3.21}$$

Proof. For the i-th subsystem, define the auxiliary vector $s_i\in\mathbb{R}^{n_i}$ such that,

$$x_i=c_i+\alpha_i s_i, \tag{3.22}$$

and $s_{N_i}\in\mathbb{R}^{n_{N_i}}$ such that,

$$x_{N_i}=c_{N_i}+\alpha_{N_i}s_{N_i}. \tag{3.23}$$

Notice that $s_i=U_iW_i^\top s_{N_i}$. Substituting these auxiliary vectors in (3.20) yields,

$$\begin{aligned}&\|(A_{N_i}\alpha_{N_i}+B_iK_{N_i}\alpha_{N_i})s_{N_i}\|_{P_i}^2+\|(A_{N_i}+B_iK_{N_i})c_{N_i}+B_id_i-c_i\|_{P_i}^2\\&\quad+2\left((A_{N_i}+B_iK_{N_i})c_{N_i}+B_id_i-c_i\right)^\top P_i(A_{N_i}\alpha_{N_i}+B_iK_{N_i}\alpha_{N_i})s_{N_i}\leq\alpha_i^2,\\&\qquad\qquad\text{for all}\ j\in\mathcal{N}_i,\ \|s_j\|_{P_j}^2\leq 1.\end{aligned}$$

Using the mapping equations (3.7) and multiplying the resulting inequality by α_i^{-1} leads to,

$$\begin{aligned}&\|(A_{N_i}\alpha_{N_i}+B_iK_{N_i}\alpha_{N_i})s_{N_i}\|_{P_i\alpha_i^{-1}}^2+\|A_{N_i}c_{N_i}+B_i(K_{N_i}c_{N_i}+d_i)-c_i\|_{P_i\alpha_i^{-1}}^2\\&\quad+2\left(A_{N_i}c_{N_i}+B_i(K_{N_i}c_{N_i}+d_i)-c_i\right)^\top P_i\alpha_i^{-1}(A_{N_i}\alpha_{N_i}+B_iK_{N_i}\alpha_{N_i})s_{N_i}\leq\alpha_i,\\&\qquad\qquad\text{for all}\ j\in\mathcal{N}_i,\ \|s_{N_i}\|_{P_{ij}}^2\leq 1.\end{aligned}$$

Applying the S-Lemma [Boy+94] results in,

$$\sum_{j\in\mathcal{N}_i}\rho_{ij}\begin{bmatrix}P_{ij}&0\\0&-1\end{bmatrix}-\begin{bmatrix}M_{1_i}&M_{2_i}^\top\\M_{2_i}&M_{3_i}\end{bmatrix}\geq 0,$$

where,

$$M_{1_i} = (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top P_i \alpha_i^{-1}(A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})$$
$$M_{2_i} = [A_{N_i}c_{N_i} + B_i(K_{N_i}c_{N_i} + d_i) - c_i]^\top P_i \alpha_i^{-1}(A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})$$
$$M_{3_i} = [A_{N_i}c_{N_i} + B_i(K_{N_i}c_{N_i} + d_i) - c_i]^\top P_i \alpha_i^{-1}[A_{N_i}c_{N_i} + B_i(K_{N_i}c_{N_i} + d_i) - c_i] - \alpha_i.$$

Rearranging the above matrix inequality results in,

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_i} \rho_{ij}P_{ij} & 0 \\ 0 & \alpha_i - \sum_{j\in\mathcal{N}_i}\rho_{ij} \end{bmatrix}$$
$$- \begin{bmatrix} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top \\ [A_{N_i}c_{N_i} + B_i(K_{N_i}c_{N_i} + d_i) - c_i]^\top \end{bmatrix} P_i\alpha_i^{-1} \begin{bmatrix} (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i}) \\ [A_{N_i}c_{N_i} + B_i(K_{N_i}c_{N_i} + d_i) - c_i] \end{bmatrix}^\top \geq 0.$$

Applying the Schur complement [Boy+94] leads to (3.21). □

We now move to conditions (3.17b) and (3.17c) which ensure that all states inside the terminal set satisfy the state constraints and the terminal control input evaluated at these states satisfies the input constraints. We also consider the conditions in (3.18) which ensure that the terminal dynamics is asymptotically stable. We aim to replace these robust conditions with a finite number of constraints. Since K_{N_i} is a decision variable in one scheme and fixed in the other, the resulting constraints differ in the two schemes. It is shown below that these conditions are represented using a set of matrix inequalities in the first scheme and scalar inequalities in the second scheme.

3.3.1 MPC variant with reconfigurable terminal control gain

We introduce here the first scheme where the terminal control gain K_{N_i} is considered as a decision variable and reconfigured online at each time instant. First, we show how the robust conditions (3.17b) and (3.17c) can be replaced by a set of matrix inequalities. We start with the robust constraint (3.17b) in the following proposition.

Proposition 3.3. *Let $G^p_{N_i}$ be the p-th row of the matrix G_{N_i} and $g^p_{N_i}$ the p-th element of the vector g_{N_i}. The p-th state constraint of the i-th subsystem given by,*

$$G^p_{N_i}x_{N_i} \leq g^p_{N_i} \text{ for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|^2_{P_j} \leq \alpha_j^2,$$

holds if there exist $\sigma^p_{ij} \geq 0$ for all $j \in \mathcal{N}_i$ such that,

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_i}\sigma^p_{ij}P_{ij} & \frac{1}{2}\alpha_{N_i}G^{p\top}_{N_i} \\ \frac{1}{2}G^p_{N_i}\alpha_{N_i} & g^p_{N_i} - G^p_{N_i}c_{N_i} - \sum_{j\in\mathcal{N}_i}\sigma^p_{ij} \end{bmatrix} \geq 0. \tag{3.24}$$

Proof. The proof follows that of Proposition 2.5. □

We now move to the robust constraint (3.17c) in the following proposition.

Proposition 3.4. *Let H_i^l be the l-th row of the matrix H_i and h_i^l the l-th element of the vector h_i. The l-th input constraint of the i-th subsystem given by,*

$$H_i^l K_{N_i} x_{N_i} + H_i^l d_i \leq h_i^l \text{ for all } j \in \mathcal{N}_i \text{ and } x_j : \|x_j - c_j\|_{P_j}^2 \leq \alpha_j^2, \tag{3.25}$$

holds if there exist $\tau_{ij}^l \geq 0$ for all $j \in \mathcal{N}_i$ such that,

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_i} \tau_{ij}^l P_{ij} & \frac{1}{2}\alpha_{N_i} K_{N_i}^\top H_i^{l\top} \\ \frac{1}{2} H_i^l K_{N_i} \alpha_{N_i} & h_i^l - H_i^l (K_{N_i} c_{N_i} + d_i) - \sum_{j\in\mathcal{N}_i} \tau_{ij}^l \end{bmatrix} \geq 0. \tag{3.26}$$

Proof. Recall the definitions of the auxiliary vectors s_i and s_{N_i} in (3.22) and (3.23). Substituting for these vectors, the condition in (3.25) becomes,

$$H_i^l K_{N_i}(c_{N_i} + \alpha_{N_i} s_{N_i}) + H_i^l d_i \leq h_i^k \text{ for all } j \in \mathcal{N}_i \text{ and } s_j : \|s_j\|_{P_j}^2 \leq 1.$$

Using the maps in (3.7), the above condition is given by,

$$H_i^l K_{N_i} \alpha_{N_i} s_{N_i} + H_i^l K_{N_i} c_{N_i} + H_i^l d_i \leq h_i^l \text{ for all } \|s_{N_i}\|_{P_{ij}}^2 \leq 1.$$

Applying the S-procedure [Boy+94] leads to,

$$\sum_{j\in\mathcal{N}_i} \tau_{ij}^l \begin{bmatrix} P_{ij} & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\alpha_{N_i} K_{N_i}^\top H_i^{l\top} \\ \frac{1}{2} H_i^l K_{N_i} \alpha_{N_i} & H_i^l K_{N_i} c_{N_i} + H_i^l d_i - h_i^l \end{bmatrix} \geq 0.$$

Rearranging the above LMI results in (3.26). □

Finally, we discuss the conditions in (3.18) which ensure the asymptotic stability of the terminal dynamics.

Proposition 3.5. *The terminal dynamics is asymptotically stable (i.e. (3.18) hold) if for all $i \in \{1, \ldots, M\}$,*

$$\begin{bmatrix} W_i U_i^\top P_i^{-1} U_i W_i^\top + \alpha_{N_i} \Gamma_i \alpha_i^{-1} \alpha_{N_i} & \alpha_{N_i} A_{N_i}^\top + \alpha_{N_i} K_{N_i}^\top B_i^\top & \alpha_{N_i} Q_i^{1/2} & \alpha_{N_i} K_{N_i}^\top R_i^{1/2} \\ A_{N_i}\alpha_{N_i} + B_i K_{N_i} \alpha_{N_i} & P_i^{-1}\alpha_i & 0 & 0 \\ Q_i^{1/2}\alpha_{N_i} & 0 & \alpha_i I_{n_{N_i}} & 0 \\ R_i^{1/2} K_{N_i} \alpha_{N_i} & 0 & 0 & \alpha_i I_{m_i} \end{bmatrix} \geq 0, \tag{3.27}$$

and if there exist block-diagonal matrices $T_i \in \mathbb{R}^{n_{N_i} \times n_{N_i}}$ for all $i \in \{1, ..., M\}$ such that,

$$\alpha_{N_i} \Gamma_i \alpha_i^{-1} \alpha_{N_i} \leq T_i \text{ for all } i \in \{1, \ldots, M\}, \tag{3.28a}$$

$$\sum_{j\in\mathcal{N}_i} U_i W_j^\top T_j W_j U_i^\top \leq 0 \text{ for all } i \in \{1, \ldots, M\}. \tag{3.28b}$$

Proof. Recall that conditions (3.18a) and (3.18b) are satisfied for all $i \in \{1, \dots, M\}$ with $\lambda_{1_i} = \lambda_{min}(P_i)$, $\lambda_{2_i} = \lambda_{max}(P_i)$ and $\lambda_{3_i} = \lambda_{min}(Q_i + K_{N_i}^\top R_i K_{N_i})$ where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ are the minimum and maximum eigenvalues of the corresponding matrix, respectively. Hence, these conditions can be omitted from the OCP since they are satisfied by construction.

Recall also that $\Delta\kappa_i(x_{N_i}) = K_{N_i}\Delta x_{N_i}$. Using the maps in (3.7), conditions (3.18c) and (3.18d) are given by,

$$\|\Delta x_{N_i}\|^2_{W_i U_i^\top P_i U_i W_i^\top - (A_{N_i}+B_i K_{N_i})^\top P_i (A_{N_i}+B_i K_{N_i}) - (Q_i + K_{N_i}^\top R_i K_{N_i}) + \Gamma_i} \geq 0, \tag{3.29a}$$

$$\|\Delta x\|^2_{\left(\sum_{i=1}^M W_i^\top \Gamma_i W_i\right)} \leq 0. \tag{3.29b}$$

where $\Delta x = x - x_e$ is the global shifted state vector. By following the proof of Proposition 4 in [DEL19], conditions (3.29a) and (3.29b) can be replaced, respectively, by the matrix inequalities (3.27) and

$$\sum_{i=1}^{M} W_i^\top \alpha_{N_i} \Gamma_i \alpha_i^{-1} \alpha_{N_i} W_i \leq 0. \tag{3.30}$$

Following [Con+16], we introduce the block-diagonal matrices T_i as upper bounds of $\alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i}$ by requiring inequality (3.28a). Thus, condition (3.30) can be ensured by means of the inequalities (3.28b). □

Finally, to ensure that the S-lemma decision variables μ_{ij}, ρ_{ij}^k and τ_{ij}^l are non-negative, we require that for all $i \in \{1, \dots, M\}$, $j \in \mathcal{N}_i$, $p \in \{1, \dots, q_i\}$ and $l \in \{1, \dots, r_i\}$,

$$\mu_{ij} \geq 0, \ \rho_{ij}^p \geq 0, \ \tau_{ij}^l \geq 0. \tag{3.31}$$

In summary, the global online OCP becomes

$$\begin{aligned} &\min \ \sum_{i=1}^{M} J_i(t) \\ &\text{s.t.} \begin{cases} \text{for all } \ k \in \{0, \dots, N-1\} \text{ and } i \in \{1, \dots, M\}, \\ \text{for all } j \in \mathcal{N}_i, \ p \in \{1, \dots, q_i\} \text{ and } l \in \{1, \dots, r_i\}, \\ (3.1), \ (3.2), \ (3.10), (3.14), (3.16), \\ (3.21), \ (3.24), \ (3.26), \ (3.27), \ (3.28a), \ (3.28b), \ (3.31). \end{cases} \end{aligned} \tag{3.32}$$

Unlike (3.19), the online OCP (3.32) has a finite number of constraints since the robust constraints in (3.19) are replaced by a finite number of matrix inequalities in (3.32). Note that (3.32) provides an upper bound of the optimal cost in (3.19), as the feasible set has been restricted once more through the use of the S-procedure in Propositions 3.2, 3.3, 3.4 and 3.5. It is important to mention that the decision variables in (3.32) are, for all $i \in \{1, \dots, M\}$, the predicted state trajectory $x_i(k|t)$ for all $k \in \{0, ..., N\}$,

the predicted input trajectory $u_i(k|t)$ for all $k \in \{0, ..., N-1\}$, the artificial equilibrium point (x_{e_i}, u_{e_i}), the terminal set size α_i, the terminal set center c_i, the terminal control gain matrix K_{N_i}, the affine term d_i of the terminal controller, the auxiliary matrices Γ_i and T_i and the S-procedure variables μ_{ij}, ρ_{ij}^p and τ_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1, \dots, q_i\}$ and $l \in \{1, \dots, r_i\}$.

Even though (3.32) is finite-dimensional with a finite number of constraints, it is non-convex due to the nonlinear combinations of decision variables $K_{N_i}\alpha_{N_i}$, $K_{N_i}c_{N_i}$, $K_{N_i}x_{e_{N_i}}$ and $\alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i}$ in the constraint (3.14) and the matrix inequalities (3.21), (3.24), (3.26), (3.27) and (3.28). It can, however, be transformed into a convex problem through the change of variables,

$$\begin{aligned} v_{1_i} &= \alpha_{N_i}, \\ v_{2_i} &= c_{N_i}, \\ v_{3_i} &= K_{N_i}\alpha_{N_i}, \\ v_{4_i} &= K_{N_i}c_{N_i} + d_i, \\ v_{5_i} &= \alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i}, \\ v_{6_i} &= T_i. \end{aligned} \tag{3.33}$$

The above map is bijective as long as $\alpha_i > 0$ for all $i \in \{1, ..., M\}$; in this case, the original variables α_i, c_i, K_{N_i}, d_i, Γ_i and T_i can be retrieved from the newly-defined ones v_{j_i} for all $j \in \{1, \dots, 6\}$ as follows,

$$\begin{aligned} \alpha_{N_i} &= v_{1_i}, \\ c_{N_i} &= v_{2_i}, \\ K_{N_i} &= v_{3_i}v_{1_i}^{-1}, \\ d_i &= v_{4_i} - v_{3_i}v_{1_i}^{-1}v_{2_i}, \\ \Gamma_i &= v_{1_i}^{-1}v_{5_i}v_{1_i}^{-1}(U_iW_i^\top v_{1_i}W_iU_i^\top)_{11}, \\ T_i &= v_{6_i}, \end{aligned}$$

where $(\cdot)_{11}$ denotes the top-left element in a matrix. Notice that it is always possible to find $\alpha_i > 0$ because the equilibrium point is assumed to be in the interior of the state and input constraint sets as implied by (3.14). To express the constraint (3.14) as a linear combination of the variables (3.33), the artificial equilibrium point is constrained to be at the center of the terminal set (i.e. $x_{e_i} = c_i$ for all $i \in \{1, \dots, M\}$). In this case,

the constraint (3.14) together with the constraint that $\alpha_i > 0$ and $x_{e_i} = c_i$ become,

$$\begin{aligned} U_i W_i^\top v_{2_i} &= A_{N_i} v_{2_i} + B_i u_{e_i}, \\ u_{e_i} &= v_{4_i}, \\ v_{2_i} &\in \xi \mathcal{X}_{N_i}, \\ v_{4_i} &\in \xi \mathcal{U}_i, \\ \left(U_i W_i^\top v_{1_i} W_i U_i^\top\right)_{11} &\geq \epsilon, \\ x_{e_i} &= U_i W_i^\top v_{2_i}. \end{aligned} \tag{3.34}$$

where ϵ is an arbitrarily small positive scalar. Using the maps (3.7) and (3.33), the matrix inequalities (3.16), (3.21), (3.24), (3.26), (3.27) and (3.28) are given, respectively, by,

$$\begin{bmatrix} P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top & x_i(N|t) - U_i W_i^\top v_{2_i} \\ \left(x_i(N|t) - U_i W_i^\top v_{2_i}\right)^\top & (U_i W_i^\top v_{1_i} W_i U_i^\top)_{11} \end{bmatrix} \geq 0, \tag{3.35}$$

$$\begin{bmatrix} P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top & (A_{N_i} v_{1_i} + B_i v_{3_i}) & \left(A_{N_i} v_{2_i} + B_i v_{4_i} - U_i W_i^\top v_{2_i}\right) \\ (A_{N_i} v_{1_i} + B_i v_{3_i})^\top & \sum_{j \in \mathcal{N}_i} \mu_{ij} P_{ij} & 0 \\ \left(A_{N_i} v_{2_i} + B_i v_{4_i} - U_i W_i^\top v_{2_i}\right)^\top & 0 & (U_i W_i^\top v_{1_i} W_i U_i^\top)_{11} - \sum_{j \in \mathcal{N}_i} \mu_{ij} \end{bmatrix} \geq 0, \tag{3.36}$$

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \rho_{ij}^p P_{ij} & \frac{1}{2} v_{1_i} G_{N_i}^{p\top} \\ \frac{1}{2} G_{N_i}^p v_{1_i} & g_{N_i}^p - G_{N_i}^p v_{2_i} - \sum_{j \in \mathcal{N}_i} \rho_{ij}^p \end{bmatrix} \geq 0, \text{ for all } p \in \{1, \ldots, q_i\}, \tag{3.37}$$

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \tau_{ij}^l P_{ij} & \frac{1}{2} v_{3_i}^\top H_i^{l\top} \\ \frac{1}{2} H_i^l v_{3_i} & h_i^l - H_i^l v_{4_i} - \sum_{j \in \mathcal{N}_i} \tau_{ij}^l \end{bmatrix} \geq 0, \text{ for all } l \in \{1, \ldots, r_i\}, \tag{3.38}$$

$$\begin{bmatrix} W_i U_i^\top P_i^{-1} U_i W_i^\top + v_{5_i} & (A_{N_i} v_{1_i} + B_i v_{3_i})^\top & v_{1_i} Q_i^{\frac{1}{2}} & v_{3_i}^\top R_i^{\frac{1}{2}} \\ A_{N_i} v_{1_i} + B_i v_{3_i} & P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top & 0 & 0 \\ Q_i^{\frac{1}{2}} v_{1_i} & 0 & (U_i W_i^\top v_{1_i} W_i U_i^\top)_{11} I_{n_{N_i}} & 0 \\ R_i^{\frac{1}{2}} v_{3_i} & 0 & 0 & (U_i W_i^\top v_{1_i} W_i U_i^\top)_{11} I_{m_i} \end{bmatrix} \geq 0, \tag{3.39}$$

$$v_{5_i} \leq T_i, \tag{3.40}$$

$$\sum_{j \in \mathcal{N}_i} U_i W_j^\top v_{6_j} W_j U_i^\top \leq 0. \tag{3.41}$$

Note that the matrix inequalities (3.35), (3.36), (3.37), (3.38), (3.39), (3.40) and (3.41) are linear with respect to the newly-defined decision variables (3.33). Note also that, by substituting (3.34) in the LMI (3.36), this LMI becomes block diagonal.

Based on the above arguments, the global online OCP problem can be represented as a convex finite-dimensional optimization problem with a finite number of constraints as follows,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &s.t. \begin{cases} \text{for all } k \in \{0,\dots,N-1\} \text{ and } i \in \{1,\dots,M\}, \\ \text{for all } j \in \mathcal{N}_i,\ p\ \{1,\dots,q_i\} \text{ and } l \in \{1,\dots,r_i\}, \\ (3.1),\ (3.2),\ (3.10),\ (3.31),\ (3.34),\ (3.35) \\ (3.36), (3.37), (3.38),\ (3.39),\ (3.40),\ (3.41). \end{cases} \end{aligned} \tag{3.42}$$

This leads to the MPC scheme in Algorithm 1. Note that the cost of (3.42) is once more an upper bound on the cost of (3.32), due to the additional constraint that the artificial equilibrium point lies in the center of the terminal set, imposed by (3.34).

Algorithm 1 Reconfigurable Distributed MPC for Tracking (A1)

Input: For all $i \in \{1,\dots,M\}$, system matrices A_{N_i} and B_i, cost function matrices Q_i, R_i and S_i, constraint sets $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$, maps U_i, V_i and W_i, initial condition x_{0_i} and prediction horizon T

Output: For all $i \in \{1,\dots,M\}$, vectors $x_i(k|t)$ for all $k \in \{0,\dots,N\}$ and $u_i(k|t)$ for all $k \in \{0,\dots,N-1\}$, variables v_{j_i} for all $j \in \{1,...,6\}$ and scalars μ_{ij}, ρ_{ij}^p and τ_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1,...,q_i\}$ and $l \in \{1,...,r_i\}$.

1: **Initialization** For all $i \in \{1,\dots,M\}$, compute the matrix P_i using (2.7).
2: **while** true **do**
3: Solve the online OCP (3.42).
4: Apply the first control input $u(0|t)$ to the plant (3.1).
5: Measure the new state $x_0 \leftarrow Ax_0 + Bu(0|t)$.
6: **end while**

Unlike standard MPC schemes that require the solution of QPs, the online OCP (3.42) of the developed MPC scheme is an SDP due to the LMIs (3.35)-(3.41). In general, an SDP is more difficult to solve compared to a QP. To improve computational performance, SDPs can be approximated using various approaches (e.g. see [MHA20]). Here, we use an SDP scalability method based on diagonal dominance [AH17] to approximate the developed SDP by a QP [AH17; AM19]. In particular, we apply diagonal dominance to the LMIs (3.35)-(3.41) to replace them by a finite number of linear inequalities. In the sequel, we use $[\cdot]_{sk}$ to refer to the element in the s-th row and k-th column of a matrix and $|\cdot|$ to refer to a matrix with the absolute values of the elements in the original

matrix. By defining the bounds $b_{1_i} \in \mathbb{R}^{n_i}$, $b_{2_i} \in \mathbb{R}^{m_i \times n_{N_i}}$, $b_{3_i} \in \mathbb{S}^{n_{N_i}}$ and $b_{4_i} \in \mathbb{S}^{n_{N_i}}$ such that,

$$\begin{aligned} -b_{1_i} \leq x_i(N|t) - U_i W_i^\top v_{2_i} \leq b_{1_i}, \\ -[b_{2_i}]_{sk} \leq [v_{3_i}]_{sk} \leq [b_{2_i}]_{sk}, \\ -[b_{3_i}]_{sk} \leq [v_{5_i}]_{sk} \leq [b_{3_i}]_{sk}, \\ -[b_{4_i}]_{sk} \leq [v_{6_i}]_{sk} \leq [b_{4_i}]_{sk}, \end{aligned} \tag{3.43}$$

applying diagonal dominance to the LMIs (3.35)-(3.41) yields the linear inequalities,

$$\begin{aligned} 2[P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top]_{ss} \geq \sum_{k=1}^{n_i} [|P_i^{-1}| U_i W_i^\top v_{1_i} W_i U_i^\top]_{sk} + [b_{1_i}]_{s1} \text{ for all } s \in \{1, \dots, n_i\}, \\ [U_i W_i^\top v_{1_i} W_i U_i^\top]_{11} \geq \sum_{k=1}^{n_i} [b_{1_i}^\top]_{1k}, \end{aligned} \tag{3.44}$$

$$\begin{aligned} 2[P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top]_{ss} \geq \sum_{k=1}^{n_i} [|P_i^{-1}| U_i W_i^\top v_{1_i} W_i U_i^\top]_{sk} + \sum_{k=1}^{n_{N_i}} [|A_{N_i}| v_{1_i}]_{sk} + \sum_{k=1}^{m_i} [|B_i| b_{2_i}]_{sk} \\ \text{for all } s \in \{1, \dots, n_i\}, \\ 2\left[\sum_{j \in \mathcal{N}_i} \mu_{ij} P_{ij}\right]_{ss} \geq \sum_{j \in \mathcal{N}_i} \sum_{k=1}^{n_{N_i}} [\mu_{ij} |P_{ij}|]_{sk} + \sum_{k=1}^{n_{N_i}} [v_{1_i} |A_{N_i}^\top|]_{sk} + \sum_{k=1}^{n_{N_i}} [b_{2_i}^\top |B_i^\top|]_{sk} \\ \text{for all } s \in \{1, \dots, n_{N_i}\}, \\ [U_i W_i^\top v_{1_i} W_i U_i^\top]_{11} \geq \sum_{j \in \mathcal{N}_i} \mu_{ij}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} 2\left[\sum_{j \in \mathcal{N}_i} \rho_{ij}^p P_{ij}\right]_{ss} \geq \sum_{j \in \mathcal{N}_i} \sum_{k=1}^{n_{N_i}} [\rho_{ij}^p |P_{ij}|]_{sk} + \frac{1}{2} [v_{1_i} |G_{N_i}^{p^\top}|]_{s1} \text{ for all } \begin{cases} p \in \{1, \dots, q_i\} \\ s \in \{1, \dots, n_{N_i}\} \end{cases}, \\ g_{N_i}^p - G_{N_i}^p v_{2_i} - \sum_{j \in \mathcal{N}_i} \rho_{ij}^p \geq \sum_{k=1}^{n_{N_i}} \frac{1}{2} [|G_{N_i}^p| v_{1_i}]_{1k} \text{ for all } p \in \{1, \dots, q_i\}, \end{aligned} \tag{3.46}$$

$$\begin{aligned} 2\left[\sum_{j \in \mathcal{N}_i} \tau_{ij}^l P_{ij}\right] \geq \sum_{j \in \mathcal{N}_i} \sum_{k=1}^{n_{N_i}} [\tau_{ij}^l |P_{ij}|]_{sk} + \frac{1}{2} [b_{2_i}^\top |H_i^{l^\top}|]_{s1} \text{ for all } \begin{cases} l \in \{1, \dots, r_i\} \\ s \in \{1, \dots, n_{N_i}\} \end{cases}, \\ h_i^l - H_i^l v_{4_i} - \sum_{j \in \mathcal{N}_i} \tau_{ij}^l \geq \sum_{k=1}^{n_{N_i}} \frac{1}{2} [|H_i^l| b_{2_i}]_{1k} \text{ for all } l \in \{1, \dots, r_i\}, \end{aligned} \tag{3.47}$$

$$
\begin{aligned}
2[W_i U_i^\top P_i^{-1} U_i W_i^\top + v_{5_i}]_{ss} \geq \sum_{k=1}^{n_{N_i}} [W_i U_i^\top |P_i^{-1}| U_i W_i^\top + b_{3_i}]_{sk} \\
+ \sum_{k=1}^{n_{N_i}} [v_{1_i} |A_{N_i}^\top|]_{sk} + \sum_{k=1}^{n_{N_i}} [b_{2_i}^\top |B_i^\top|]_{sk} \\
+ \sum_{k=1}^{n_{N_i}} [v_{1_i} |Q_i^{\frac{1}{2}^\top}|]_{sk} + \sum_{k=1}^{m_i} [b_{2_i}^\top |R_i^{\frac{1}{2}^\top}|]_{sk} \\
\text{for all } s \in \{1, \dots, n_{N_i}\},
\end{aligned}
$$

$$
2[P_i^{-1} U_i W_i^\top v_{1_i} W_i U_i^\top]_{ss} \geq \sum_{k=1}^{n_i} [|P_i^{-1}| U_i W_i^\top v_{1_i} W_i U_i^\top]_{sk} + \sum_{k=1}^{n_{N_i}} [|A_{N_i}| v_{1_i}]_{sk} + \sum_{k=1}^{m_i} [|B_i| b_{2_i}]_{sk}
$$
$$
\text{for all } s \in \{1, \dots, n_i\},
$$

$$
\begin{aligned}
[U_i W_i^\top v_{1_i} W_i U_i^\top]_{11} \geq \sum_{k=1}^{n_{N_i}} [|Q_i^{\frac{1}{2}}| v_{1_i}]_{sk} \text{ for all } s \in \{1, \dots, n_{N_i}\}, \\
[U_i W_i^\top v_{1_i} W_i U_i^\top]_{11} \geq \sum_{k=1}^{n_{N_i}} [|R_i^{\frac{1}{2}}| b_{2_i}]_{sk} \text{ for all } s \in \{1, \dots, m_i\},
\end{aligned} \tag{3.48}
$$

$$
2[v_{6_i}]_{ss} \geq \sum_{k=1}^{n_{N_i}} [b_{4_i}]_{sk} + \sum_{k=1}^{n_{N_i}} [b_{3_i}]_{sk} \text{ for all } s \in \{1, \dots, n_{N_i}\}, \tag{3.49}
$$

$$
-2 \sum_{j \in \mathcal{N}_i} [U_i W_j^\top v_{6_j} W_j U_j^\top]_{ss} \geq \sum_{j \in \mathcal{N}_i} \sum_{k=1}^{n_{N_i}} [U_i W_j^\top b_{4_j} W_j U_j^\top]_{sk} \text{ for all } s \in \{1, \dots, n_{N_i}\}, \tag{3.50}
$$

In this case, the online OCP can be cast as a QP as follows,

$$
\begin{aligned}
&\min \sum_{i=1}^{M} J_i(t) \\
&s.t. \begin{cases}
\text{for all } k \in \{0, \dots, N-1\} \text{ and } i \in \{1, \dots, M\}, \\
\text{for all } j \in \mathcal{N}_i,\ p\ \{1, \dots, q_i\} \text{ and } l \in \{1, \dots, r_i\}, \\
(3.1),\ (3.2),\ (3.10),\ (3.31),\ (3.34), \\
(3.43),\ (3.44),\ (3.45),\ (3.46),\ (3.47),\ (3.48),\ (3.49),\ (3.50).
\end{cases}
\end{aligned} \tag{3.51}
$$

This leads to the MPC scheme in Algorithm 2. Notice that the cost of (3.42) is upper bounded by the cost of (3.51), due to the application of the SDP scalability method that is based on diagonal dominance. In summary, the costs of the various programs

considered satisfy,

$$\begin{aligned}(3.11) \leq &(3.19) \text{ \{ellipsoidal sets, affine controllers\}}\\ \leq &(3.32) \text{ \{S-procedure\}}\\ \leq &(3.42) \text{ \{centered equilibrium\}}\\ \leq &(3.51) \text{ \{diagonal dominance\}}.\end{aligned}$$

Algorithm 2 Reconfigurable Distributed MPC for Tracking (A2)

Input: For all $i \in \{1, \dots, M\}$, system matrices A_{N_i} and B_i, cost function matrices Q_i, R_i and S_i, constraint sets $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$, maps U_i, V_i and W_i, initial condition x_{0_i} and prediction horizon T

Output: For all $i \in \{1, \dots, M\}$, vectors $x_i(k|t)$ for all $k \in \{0, \dots, N\}$ and $u_i(k|t)$ for all $k \in \{0, \dots, N-1\}$, variables v_{j_i} for all $j \in \{1, ..., 6\}$, bounds b_{j_i} for all $j \in \{1, \dots, 4\}$ and scalars μ_{ij}, ρ_{ij}^p and τ_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1, ..., q_i\}$ and $l \in \{1, ..., r_i\}$.

1: **Initialization** For all $i \in \{1, \dots, M\}$, compute the matrix P_i using (2.7).
2: **while** true **do**
3: Solve the online OCP (3.51).
4: Apply the first control input $u(0|t)$ to the plant (3.1).
5: Measure the new state $x_0 \leftarrow Ax_0 + Bu(0|t)$.
6: **end while**

Although the MPC problems (3.42) and (3.51) are written centrally, they can be solved online using distributed optimization algorithms (e.g. see [NL18]) thanks to their distributed structure. Some of these algorithms can be used without requiring a central coordinator such as the distributed primal-dual algorithm [AH16] and some variants of ADMM [Boy+11]. ADMM is used here due to its better convergence properties [Ban+19]. In this case, each subsystem solves a local optimization problem iteratively at each time instant while communicating only with its neighbours until consensus among shared variables is reached. The shared variables between the two neighbours i and j are α_i, α_j, c_i, c_j, $U_j W_i^\top T_i W_i U_j^\top$, $U_i W_j^\top T_j W_j U_i^\top$, x_{e_i}, x_{e_j}, $x_i(k|t)$ and $x_j(k|t)$ for all $k \in \{0, ..., N-1\}$. For more details, see Section 3.5 where the distributed impelementation of the developed schemes using ADMM is discussed in detail.

3.3.2 MPC variant with off-centered terminal set

We now move to the second scheme where the stabilizing terminal control gain matrix K_{N_i} is synthesized offline and considered as a fixed parameter in the online OCP. To derive this scheme, we start with the semi-infinite program (3.19) which has a finite

number of decision variables, but an infinite number of constraints represented by the robust constraints (3.17) and (3.18).

Recall that the conditions in (3.18) aim to ensure that the terminal controller $\kappa_i(x_{N_i}) = K_{N_i}x_{N_i} + d_i$ for all $i \in \{1, \dots, M\}$ is stabilizing. Since the stabilizing control gain matrix K_{N_i} is already computed offline and d_i does not affect the stability of the terminal dynamics (i.e. d_i does not appear in (3.18)), the conditions in (3.18) are no longer required in the online OCP in this case. On the other side, the invariance conditions (3.17) are still required. Condition (3.17a) can still be replaced by the matrix inequality (3.21) as in the first scheme. The only difference now is that K_{N_i} is not a decision variable anymore in this matrix inequality. Instead of replacing (3.17b) and (3.17c) by the matrix inequalities (3.24) and (3.26), we show here that these robust constraints can be replaced by a finite number of scalar inequalities. We start with (3.17b) in the following proposition.

Proposition 3.6. *Let the p-th row of the matrix G_{N_i} be denoted by $G^p_{N_i}$ and the p-th element of the vector g_{N_i} by $g^p_{N_i}$. For the i-th subsystem, condition (3.17b) given by,*

$$G^p_{N_i}x_{N_i} \leq g^p_{N_i} \text{ for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|^2_{P_j} \leq \alpha_j^2 \text{ and } p \in \{1, \dots, q_i\}, \tag{3.52}$$

holds if and only if,

$$G^p_{N_i}c_{N_i} + \sum_{j\in\mathcal{N}_i} \|G^p_{N_i}W_iU_j^\top P_j^{-1/2}\|_2\alpha_j \leq g^p_{N_i} \text{ for all } p \in \{1, ..., q_i\}. \tag{3.53}$$

Proof. For all $j \in \mathcal{N}_i$, we define the auxiliary vector $s_j \in \mathbb{R}^{n_j}$ such that,

$$x_j = c_j + P_j^{-1/2}s_j,$$

and hence,

$$x_{N_i} = c_{N_i} + P_{N_i}^{-1/2}s_{N_i},$$

where $P_{N_i} = \sum_{j\in\mathcal{N}_i} W_iU_j^\top P_jU_jW_i^\top$ and $s_{N_i} = \sum_{j\in\mathcal{N}_i} W_iU_j^\top s_j$. Using the aforementioned auxiliary vectors, condition (3.52) can be written as,

$$G^p_{N_i}c_{N_i} + G^p_{N_i}P_{N_i}^{-1/2}s_{N_i} \leq g^p_{N_i} \text{ for all } j \in \mathcal{N}_i,\ s_j^\top s_j \leq \alpha_j^2 \text{ and } p \in \{1, ..., q_i\}.$$

By making use of the definitions of P_{N_i} and s_{N_i}, we reach that,

$$G^p_{N_i}c_{N_i} + \sum_{j\in\mathcal{N}_i} G^p_{N_i}W_iU_j^\top P_j^{-1/2}s_j \leq g^p_i \text{ for all } j \in \mathcal{N}_i,\ s_j^\top s_j \leq \alpha_j^2 \text{ and } p \in \{1, ..., q_i\}.$$

Following [Ban+20; BBV04], this robust constraint is satisfied if and only if (3.53) holds. □

We now move to condition (3.17c) in the following proposition.

Proposition 3.7. *Let the l-th row of the matrix H_i be denoted by H_i^l and the l-th element of the vector h_i by h_i^l. For the i-th subsystem, condition (3.17c) given by,*

$$H_i^l(K_{N_i}x_{N_i} + d_i) \leq h_i^l \text{ for all } j \in \mathcal{N}_i,\ x_j : \|x_j - c_j\|_{P_j}^2 \leq \alpha_j^2 \text{ and } l \in \{1, \dots, r_i\},$$

holds if and only if,

$$H_i^l K_{N_i} c_{N_i} + H_i^l d_i + \sum_{j \in \mathcal{N}_i} \|H_i^l K_{N_i} W_i U_j^\top P_j^{-1/2}\|_2 \alpha_j \leq h_i^l \text{ for all } l \in \{1, \dots, r_i\}. \tag{3.54}$$

Proof. The proof follows that of Proposition 3.6 by replacing G_{N_i} with $H_i K_{N_i}$ and g_{N_i} with $h_i - H_i d_i$. □

Note that the S-lemma is neither used with condition (3.17b) nor with condition (3.17c). Indeed, it is only used with condition (3.17a) and hence, an additional constraint is required to ensure that for all $i \in \{1, \dots, M\}$ and $j \in \mathcal{N}_i$,

$$\mu_{ij} \geq 0. \tag{3.55}$$

Based on the above arguments, the online OCP of this MPC scheme is given by,

$$\begin{aligned} &\min \sum_{i=1}^{M} J_i(t) \\ &s.t. \begin{cases} \text{for all } k \in \{0, \dots, N-1\} \text{ and } i \in \{1, \dots, M\}, \\ \text{for all } j \in \mathcal{N}_i,\ p\ \{1, \dots, q_i\} \text{ and } l \in \{1, \dots, r_i\}, \\ (3.1),\ (3.2),\ (3.10),\ (3.14)\ ,\ (3.15), \\ (3.21),\ (3.53),\ (3.54),\ (3.55), \end{cases} \end{aligned} \tag{3.56}$$

In this online OCP, the decision variables are, for all $i \in \{1, \dots, M\}$, the predicted state trajectory $x_i(k|t)$ for all $k \in \{0, ..., N\}$, the predicted input trajectory $u_i(k|t)$ for all $k \in \{0, ..., N-1\}$, the artificial equilibrium point (x_{e_i}, u_{e_i}), the terminal set size α_i, the terminal set center c_i, the affine term d_i of the terminal controller and the S-procedure variables μ_{ij} for all $j \in \mathcal{N}_i$.

It is important to mention that the constraints in (3.14) are all linear since K_{N_i} is no longer a decision variable and hence, no bilinear terms exist due to the multipilication of K_{N_i} and $x_{e_{N_i}}$. In this case, there is no need to constrain the artificial equilibrium point to be at the center of the terminal set (i.e. $c_i \neq x_{e_i}$ for all $i \in \{1, \dots, M\}$). Unlike (3.19) which uses the LMI (3.16) as the terminal constraint, we use here (3.15) which can be formulated as a second-order cone constraint. The scalar and matrix inequalities (3.21), (3.53) and(3.54) are all linear with respect to the decision variables and hence, the resulting online OCP (3.56) is an SDP and hence, convex. Note that since $c_i \neq x_{e_{N_i}}$ for all $i \in \mathcal{N}_i$, the matrix in (3.21) is not necessarily block-diagonal anymore. The online OCP (3.56) leads to the MPC scheme in Algorithm 3.

Algorithm 3 Reconfigurable Distributed MPC for Tracking (B1)

Input: For all $i \in \{1, \ldots, M\}$, system matrices A_{N_i} and B_i, cost function matrices Q_i, R_i and S_i, constraint sets $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$, maps U_i, V_i and W_i, initial condition x_{0_i} and prediction horizon T

Output: For all $i \in \{1, \ldots, M\}$, vectors $x_i(k|t)$ for all $k \in \{0, \ldots, N\}$, $u_i(k|t)$ for all $k \in \{0, \ldots, N-1\}$, x_{e_i}, u_{e_i}, c_i, d_i and scalars α_i, μ_{ij} for all $j \in \mathcal{N}_i$.

1: **Initialization** For all $i \in \{1, \ldots, M\}$, compute P_i and K_{N_i} using (2.7).
2: **while** true **do**
3: Solve the online OCP (3.56).
4: Apply the first control input $u(0|t)$ to the plant (3.1).
5: Measure the new state $x_0 \leftarrow Ax_0 + Bu(0|t)$.
6: **end while**

To reduce the computational effort of Algorithm 3, LMI (3.21) can be approximated using diagonal dominance [MHA20; AM19] by a set of linear inequalities given by,

$$
\begin{aligned}
2\left[P_i^{-1}\alpha_i\right]_{ss} &\geq \sum_{k=1}^{n_i}\left[|P_i^{-1}|\alpha_i\right]_{sk} + \sum_{k=1}^{n_{N_i}}\left[|A_{N_i}+B_iK_{N_i}|\alpha_{N_i}\right]_{sk} + [b_i]_{s1} \quad \text{for all } s \in \{1, ..., n_i\},\\
2\left[\sum_{j\in\mathcal{N}_i}\lambda_{ij}P_{ij}\right]_{ss} &\geq \sum_{k=1}^{n_{N_i}}\left[\sum_{j\in\mathcal{N}_i}\lambda_{ij}|P_{ij}|\right]_{sk} + \sum_{k=1}^{n_i}\left[\left(|A_{N_i}+ B_iK_{N_i}|\alpha_i\right)^\top\right]_{sk}\\
&\qquad\qquad \text{for all } s \in \{1, ..., n_{N_i}\},\\
\alpha_i - \sum_{j\in\mathcal{N}_i}\lambda_{ij} &\geq \sum_{k=1}^{n_i}\left[b_i^\top\right]_{1k},\\
-b_i \leq (A_{N_i}+B_iK_{N_i})c_{N_i} &+ B_id_i - c_i \leq b_i,
\end{aligned}
\tag{3.57}
$$

where $b_i \in \mathbb{R}^{n_i}$ is an additional decision variable, $[\cdot]_{sk}$ is the element in the s-th row and k-th column of a matrix and $|\cdot|$ is a matrix with the absolute values of the elements in the original matrix. In this case, the online OCP is given by the SOCP,

$$
\begin{aligned}
&\min \sum_{i=1}^{M} J_i(t)\\
&s.t. \begin{cases}
\text{for all } k \in \{0, \ldots, N-1\} \text{ and } i \in \{1, \ldots, M\},\\
\text{for all } j \in \mathcal{N}_i,\ p\ \{1, \ldots, q_i\} \text{ and } l \in \{1, \ldots, r_i\},\\
(3.1),\ (3.2),\ (3.10),\ (3.14),\ (3.15),\\
(3.53),\ (3.54),\ (3.55),\ (3.57).
\end{cases}
\end{aligned}
\tag{3.58}
$$

The resulting online OCP is an SOCP and not a QP due to constraint (3.15) which is formulated as a second-order cone constraint. The online OCP yields the MPC scheme

in Algorithm 4. In summary, the costs of the various programs considered satisfy,

$$\begin{aligned}(3.11) \leq &(3.19) \ \{\text{ellipsoidal sets, affine controllers}\}\\ \leq &(3.56) \ \{\text{S-procedure, fixed terminal control gain}\}\\ \leq &(3.58) \ \{\text{diagonal dominance}\}.\end{aligned}$$

Algorithm 4 Reconfigurable Distributed MPC for Tracking (B1)

Input: For all $i \in \{1, \ldots, M\}$, system matrices A_{N_i} and B_i, cost function matrices Q_i, R_i and S_i, constraint sets $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$, maps U_i, V_i and W_i, initial condition x_{0_i} and prediction horizon T

Output: For all $i \in \{1, \ldots, M\}$, vectors $x_i(k|t)$ for all $k \in \{0, \ldots, N\}$, $u_i(k|t)$ for all $k \in \{0, \ldots, N-1\}$, x_{e_i}, u_{e_i}, c_i, d_i, b_i and scalars α_i, μ_{ij} for all $j \in \mathcal{N}_i$.

1: **Initialization** For all $i \in \{1, \ldots, M\}$, compute P_i and K_{N_i} using (2.7).
2: **while** true **do**
3: Solve the online OCP (3.58).
4: Apply the first control input $u(0|t)$ to the plant (3.1).
5: Measure the new state $x_0 \leftarrow Ax_0 + Bu(0|t)$.
6: **end while**

Note that the feasible region of either the MPC scheme in Algorithm 1 or the one in Algorithm 3 does not coincide in that of the other. This is because the scheme in Algorithm 3 does not constrain the center of the terminal set to the equilibrium point defined by x_{e_i}, but uses a predefined terminal controller computed offline. On the contrary, the scheme in Algorithm 1 computes the terminal controller online, but constrains the center of the terminal set to the equilibrium point. Both schemes, however, have larger feasible sets than other schemes proposed in the literature (see Section 3.6 for more details). The same discussion above applies to the MPC schemes in Algorithm 2 and Algorithm 4.

It is worth mentioning that an attempt to use the simplified S-lemma [Ban+20] instead of the approximate S-lemma is made to derive Algorithm 3. In particular, the simplified S-lemma is intended to replace the robust constraint (3.17a) by a second-order cone constraint instead of the LMI (3.21) derived using the approximate S-lemma. This idea is inspired from [Ban+20] in which a reconfigurable centralized MPC is developed. In this case, the online OCP of the MPC scheme in Algorithm 3 would have been an SOCP instead of the SDP (3.56). Consequently, diagonal dominance would not have been used to approximate the SDP (3.56) by the SOCP (3.58) and derive the MPC scheme in Algorithm 4. The main advantage of this approach is that conservatism is alleviated when compared to the proposed approach. This is because that the simplified S-lemma provides necessary and sufficient conditions unlike the approximate S-lemma

and diagonal dominance which provide only sufficient conditions. Unlike the centralized case [Ban+20], the simplified S-lemma cannot be used in the distributed case for two reasons. First, the assumption on the simultaneous diagonalizability [Ban+20] is not necessarily satisfied. Even if this assumption is satisfied, the simplified S-lemma does not yield a second-order cone constraint in the distributed case. Instead, a non-convex constraint is reached hindering the possibility of deriving a convex online OCP.

3.4 Feasibility and Stability

In the previous section, we developed two reconfigurable distributed MPC schemes for reference tracking. The first one is the distributed MPC scheme with reconfigurable terminal control gain matrix and can be implemented in two different ways as in Algorithm 1 where the online OCP is an SDP or Algorithm 2 where the online OCP is a QP. The second one is the distributed MPC scheme with off-centered terminal sets and can be implemented in two different ways as in Algorithm 3 where the online OCP is an SDP or Algorithm 4 where the online OCP is an SOCP. In this section, we establish the recursive feasibility of all the developed MPC problems and the asymptotic stability of the corresponding closed-loop dynamics. First, we consider the first MPC scheme with the online OCP (3.42). We then show that the same arguments apply to the other schemes with the online OCPs (3.51), (3.56) and (3.58).

3.4.1 Recursive Feasibility

We start with showing that the MPC problem (3.42) is recursively feasible, that is, if it is initially feasible at time instant $t = 0$, then it is feasible at all future time instants $t \geq 0$. The following theorem elaborates the recursive feasibility of this scheme using standard MPC arguments (e.g. see [RMD17]).

Theorem 3.2. *The MPC scheme* (3.42) *is recursively feasible.*

Proof. Assume that the MPC problem (3.42) is initially feasible at time instant $t = 0$ and the values corresponding to the optimal solution for all $i \in \{1, \ldots, M\}$ are given by $x_i^*(k|t)$ for all $k \in \{0, \ldots, N\}$, $u_i^*(k|t)$ for all $k \in \{0, \ldots, N-1\}$, $x_{e_i}^*(t)$, $u_{e_i}^*(t)$, $\alpha_i^*(t)$, $c_i^*(t)$, $K_{N_i}^*(t)$, $d_i^*(t)$, $\Gamma_i^*(t)$, $T_i^*(t)$, μ_{ij}^*, $\rho_{ij}^{p^*}$ and $\tau_{ij}^{l^*}$ for all $j \in \mathcal{N}_i$, $p \in \{1, \ldots, q_i\}$ and $l \in \{1, \ldots, r_i\}$. Since the terminal set represented by $\alpha_i^*(t)$ and $c_i^*(t)$ for all $i \in \{1, \ldots, M\}$ is positively invariant, the state trajectory $\{x_i^*(1|t), \ldots, x_i^*(N|t), (A_{N_i} + B_i K_{N_i}^*(t))x_i^*(N|t) + B_i d_i^*(t)\}$ and the input trajectory $\{u_i^*(1|t), \ldots, u_i^*(N-1|t), K_{N_i}^*(t)x_i^*(N|t) + d_i^*(t)\}$ are feasible at the next time instant assuming that the other optimal values remain the same. In other words, the distributed MPC problem is feasible at the next time instant $t = 1$. By induction, the distributed MPC problem is feasible for all $t \geq 1$, or equivalently, recursively feasible. □

The recursive feasibility of the other MPC schemes (3.51), (3.56) and (3.58) can be established using the same arguments as shown below.

Theorem 3.3. *The MPC schemes* (3.51), (3.56) *and* (3.58) *are recursively feasible.*

Proof. The proof follows the lines of the proof of Theorem 3.2. In addition to the values corresponding to the optimal solution of the OCP (3.42) at time instant t, we also consider the optimal values $b^*_{j_i}(t)$ for all $j \in \{1,2,3,4\}$ to show the recursive feasibility of the OCP (3.51). Regarding the OCP (3.56), we consider for all $i \in \{1,\dots,M\}$ the optimal values $x^*_i(k|t)$ for all $k \in \{0,\dots,N\}$, $u^*_i(k|t)$ for all $k \in \{0,\dots,N-1\}$, $x^*_{e_i}(t)$, $u^*_{e_i}(t)$, $\alpha^*_i(t)$, $c^*_i(t)$, $d^*_i(t)$, μ^*_{ij} for all $j \in \mathcal{N}_i$. In addition to the optimal values of the OCP (3.56) at time instant t, we also consider the optimal value $b^*_i(t)$ to show the recursive feasibility of the OCP (3.58). □

3.4.2 Asymptotic Stability

We now discuss the stability of the closed-loop system under the MPC scheme (3.42). To prove that the target point is asymptotically stable, we first show that this point is stable using Lyapunov stability theory. We then show that this point is attractive by contradiction. The proof is inspired from [KC16; Lim+08].

Stability

To prove stability, we consider the augmented dynamics whose state vector is given by $[x^{*\top}(t), x^{*\top}_e(t), u^{*\top}_e(t)]^\top$ where $x^*(t) = x^*(0|t)$ and $u^*(t) = u^*(0|t)$. Note that $x^*(1|t) = x^*(0|t+1)$ in this case. In the following lemma, we show that the equilibrium point of this system given by the target point $[x_r^\top, x_r^\top, u_r^\top]^\top$ is stable where u_r is such that $x_r = Ax_r + Bu_r$. The proof follows from the standard MPC arguments (e.g. see [KC16]).

Lemma 3.1. *Consider the augmented closed-loop system with the state vector* $[x^{*\top}(t), x^{*\top}_e(t), u^{*\top}_e(t)]^\top$. *The equilibrium point* $[x_r^\top, x_r^\top, u_r^\top]^\top$ *of this system is stable.*

Proof. Let $x^*(k|t)$ for all $k \in \{0,\dots,N\}$, $u^*(k|t)$ for all $k \in \{0,\dots,N-1\}$, $x^*_e(t)$, $u^*_e(t)$ be the optimal values corresponding to the optimal solution of (3.42) at time instant t. Hence, the optimal cost at this time instant is given by,

$$J^*(t) = \sum_{k=0}^{N-1} \left(\|x^*(k|t) - x^*_e(t)\|^2_Q + \|u^*(k|t) - u^*_e(t)\|^2_R \right) + \|x^*(N|t) - x^*_e(t)\|^2|^2_P + \|x^*_e(t) - x_r\|^2_P.$$

Furthermore, define the tail sequence to be $x^*(k|t)$ for all $k \in \{1, \ldots, N\}$ and $u^*(k|t)$ for all $k \in \{1, \ldots, N-1\}$. Thus, the cost of the tail sequence is given by,

$$J^t(t) = \sum_{k=1}^{N-1} \left(\|x^*(k|t) - x_e^*(t)\|_Q^2 + \|u^*(k|t) - u_e^*(t)\|_R^2 \right) + \|x^*(N|t) - x_e^*(t)\|_P^2 + \|x_e^*(t) - x_r\|_P^2.$$

Subtracting $J^*(t)$ from $J^t(t)$ leads to,

$$J^t(t) - J^*(t) = -\|x^*(0|t) - x_e^*(t)\|_Q^2 - \|u^*(0|t) - u_e^*(t)\|_R^2.$$

Next, we define the function

$$\bar{J}(t) = \sum_{k=1}^{N} \left(\|x^*(k|t) - x_e^*(t)\|_Q^2 + \|u^*(k|t) - u_e^*(t)\|_R^2 \right) + \|x^*(N+1|t) - x_e^*(t)\|_P^2 + \|x_e^*(t) - x_r\|_P^2,$$

where

$$\begin{aligned} u^*(N|t) &= K^*(t)x^*(N|t) + d^*(t) \\ &= u_e^*(t) + K^*(t)(x^*(N|t) - x_e^*(t)), \\ x^*(N+1|t) &= x_e^*(t) + (A + BK^*(t))(x(N|t) - x_e^*(t)). \end{aligned}$$

Recall in the above equation that one of the constraints in (3.42) is $d = -Kx_e + u_e$. Note that the state and input sequences resulting in $\bar{J}(t)$ yield a feasible solution at time instant $t+1$. This is because $x^*(k|t)$ for all $k \in \{1, \ldots, N\}$ and $u^*(k|t)$ for all $k \in \{1, \ldots, N-1\}$ represent the tail sequence of the optimal solution at time instant t and $x^*(1|t) = x^*(0|t+1)$. In addition, the terminal state $x^*(N|t)$ belongs to the terminal set corresponding to the optimal solution at time instant t. Hence, the terminal control input $u^*(N|t) = K^*(t)x^*(N|t) + d^*(t)$ evaluated at the terminal state $x^*(N|t)$ is feasible. Moreover, the resulting state $x^*(N+1|t)$ lies inside the terminal set corresponding to the optimal solution at time instant t and hence, is also feasible. From (3.39), (3.40) and (3.41), one can verify that the matrix P satisfies,

$$P \geq (A + BK^*(t))^\top P(A + BK^*(t)) + Q + K^*(t)^\top RK^*(t),$$

or equivalently,

$$\begin{aligned} \|x^*(N|t) - x_e^*(t)\|_P^2 &\geq \|(A + BK^*(t))(x^*(N|t) - x_e^*(t))\|_P^2 \\ &\quad + \|x^*(N|t) - x_e^*(t)\|_Q^2 + \|K^*(t)(x^*(N|t) - x_e^*(t))\|_R^2 \\ &= \|(x^*(N+1|t) - x_e^*(t))\|_P^2 \\ &\quad + \|x^*(N|t) - x_e^*(t)\|_Q^2 + \|(u^*(N|t) - u_e^*(t))\|_R^2, \end{aligned}$$

and thus, $\bar{J}(t) \leq J^t(t)$.

Let $x^*(k|t+1)$ for all $k \in \{0,\dots,N\}$, $u^*(k|t+1)$ for all $k \in \{0,\dots,N-1\}$, $x_e^*(t)$, $u_e^*(t)$ be the values corresponding to the optimal solution of (3.42) at time instant $t+1$ and denote the corresponding optimal cost by $J^*(t+1)$. Clearly, $J^*(t+1) \leq \bar{J}(t)$ since $\bar{J}(t)$ is the cost of a feasible solution at time instant $t+1$ and $J^*(t)$ is the cost of the optimal solution at the same time instant. Since $x^*(t) = x^*(0|t)$ and $u^*(t) = u^*(0|t)$, then,

$$J^*(t+1) - J^*(t) \leq \bar{J}(t) - J^*(t) \leq J^t(t) - J^*(t) = -\|x^*(t) - x_e^*\|_Q^2 - \|u^*(t) - u_e^*\|_R^2. \quad (3.59)$$

Note that the optimal cost function evaluated at the equilibrium point $[x_r^\top, x_r^\top, u_r^\top]^\top$ is zero. Moroever, this function evaluated at any other point is positive since $Q \in \mathbb{S}_{++}^n$, $R \in \mathbb{S}_{++}^m$, $P \in \mathbb{S}_{++}^n$ and $S \in \mathbb{S}_{++}^n$. In addition, $J^*(t+1) - J^*(t)$ is zero if $x^*(t) = x_e^*$ and $u^*(t) = u_e^*$ and otherwise negative. Therefore, $J^*(t)$ is a Lyapunov function for the augmented system and the equilibrium point $[x_r^\top, x_r^\top, u_r^\top]^\top$ is stable. □

Attractivity

We now move to the second part where we prove that the target point defined by x_r is attractive. For this purpose, we show that the optimal state and input trajectories converge asymptotically to the optimal artificial equilibrium trajectory along the lines of [KC16]. Inspired by [Lim+08], we then show by contradiction that if the optimal state and input trajectories converge to the optimal artificial equilibrium trajectory, then the optimal artificial equilibrium trajectory converges to the target point. In other words, the optimal state trajectory converges to the target point.

We start by showing that the optimal state and input trajectories converge to the optimal artificial equilibrium in the following lemma.

Lemma 3.2. *The optimal closed-loop state trajectory $x^*(t)$ and the corresponding optimal closed-loop input trajectory $u^*(t)$ converge asymptotically to the optimal artificial equilibrium trajectory represented by $x_e^*(t)$ and $u_e^*(t)$, respectively.*

Proof. Compute $J^*(t+1) - J^*(t)$ for all the time instants $t \geq 0$ as shown in (3.59). Taking the telescopic sum leads to,

$$J^*(0) - \lim_{t\to\infty} J^*(t) \geq \sum_{t=0}^{\infty} \left(\|x^*(t) - x_e^*(t)\|_Q^2 + \|u^*(t) - u_e^*\|_R^2 \right).$$

If the MPC problem (3.42) is initially feasible, then $J^*(0)$ is finite and non-negative. Furthermore, since the MPC problem (3.42) is recursively feasible, then $J^*(t)$ is finite and non-negative for all $t \geq 0$. We therefore must have,

$$\lim_{t\to\infty} \left(\|x^*(t) - x_e^*(t)\|_Q^2 + \|u^*(t) - u_e^*(t)\|_R^2 \right) = 0.$$

Since $Q \in \mathbb{S}^n_{++}$ and $R \in \mathbb{S}^m_{++}$, then,

$$\lim_{t\to\infty}(x^*(t) - x^*_e(t)) = 0,$$
$$\lim_{t\to\infty}(u^*(t) - u^*_e(t)) = 0.$$

□

We now move to showing that the optimal artificial equilibrium trajectory converges to the target point in the following lemmas where $E(a,b) = \{x \in \mathbb{R}^n : \|x-a\|_P^2 \leq b\}$. The following proofs are inspired from [Lim+08], however (see lemma statements for the precise definitions):

- we prove in Lemma 3.3 that,

  $$E(x^*_e(t), \delta_a) \subseteq E(\bar{x}_e, \beta),$$

  and not just that,

  $$x^*_e(t) \in E(\bar{x}_e, \beta).$$

  Furthermore, additional steps are needed to establish the invariance of $E(\bar{x}_e, \beta)$.

- we prove in Lemma 3.4 that,

  $$\|x - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 < \|x^*_e(t) - x_r\|_S^2 \text{ for all } x \in E(x^*_e(t), \delta_a),$$

  instead of just proving that,

  $$\|x^*_e(t) - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 < \|x^k_e - x_r\|_S^2.$$

- we prove in Lemma 3.5 that,

  $$\lim_{t\to\infty}(x^*(t) - x^*_e(t)) = 0 \text{ implies } \lim_{t\to\infty}(x^*_e(t) - x_r) = 0,$$

  instead of just proving that,

  $$x^*(t) - x^*_e(t) = 0 \text{ implies } x^*_e(t) - x_r = 0.$$

Although the proof of Lemma 3.3 is similar to the one in [Lim+08], Lemma 3.4 and Lemma 3.5 require a new approach to the proof.

Lemma 3.3. *There exist $\underline{\lambda} < 1$, $\bar{\lambda} > 1$, $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, $\delta_a > 0$ and $\beta > 0$ such that the equilibrium point $(\bar{x}_e, \bar{u}_e) = (x_r + \lambda(x^*_e(t) - x_r), u_r + \lambda(u^*_e(t) - u_r))$ satisfies $E(x^*_e(t), \delta_a) \subset E(\bar{x}_e, \beta)$ where $E(\bar{x}_e, \beta)$ is a positively invariant set under the controller $K^*(t)x + \bar{d}$ and $\bar{d}$ satisfies the equation $\bar{d} = -K^*(t)\bar{x}_e + \bar{u}_e$.*

Proof. This proof is similar to that of Lemma 1 in [Lim+08]. Since $(x_e^*(t), u_e^*(t))$ is the artificial equilibrium corresponding to the optimal solution of (3.42) at time instant t, then,

$$(x_e^*(t), u_e^*(t)) \in \text{int}(\mathcal{X} \times \mathcal{U}).$$

Define $d_{min} \in [0, 1)$ as the smallest scalar such that,

$$(x_e^*(t), u_e^*(t)) \in d_{min}\mathcal{X} \times d_{min}\mathcal{U},$$

and let $\gamma \in (d_{min}, 1)$. Recall from the constraint (3.34) that $u_e^*(t) = K^*(t)x_e^*(t) + d^*(t)$. Hence, there exists $\delta_b > 0$ such that,

$$(x, K^*(t)x + d^*(t)) \in \gamma\mathcal{X} \times \gamma\mathcal{U} \text{ for all } x \in E(x_e^*(t), \delta_b).$$

Since $d^*(t) = -K^*(t)x_e^*(t) + u_e^*(t)$ from (3.34) and $\bar{d} = -K^*(t)\bar{x}_e + \bar{u}_e$ by assumption, then,

$$\bar{d} - d^*(t) = -K^*(t)(\bar{x}_e - x_e^*(t)) + (\bar{u}_e - u_e^*(t)).$$

Furthermore, since $\bar{x}_e = x_r + \lambda(x_e^*(t) - x_r)$ and $\bar{u}_e = u_r + \lambda(u_e^*(t) - u_r)$ by definition, then,

$$\bar{d} - d^*(t) = -(1 - \lambda)(d^*(t) - d_r).$$

where $d_r = -K^*(t)x_r + u_r$. Choose $\lambda \in \left[\underline{\lambda}, \bar{\lambda}\right]$ such that,

$$\bar{x}_e = x_r + \lambda(x_e^*(t) - x_r) \in \text{int}(E(x_e^*(t), 0.25\delta_b))$$

and

$$(0, -(1 - \lambda)(d^*(t) - d_r)) \in (1 - \gamma)\mathcal{X} \times (1 - \gamma)\mathcal{U}$$

where $\underline{\lambda}$ and $\bar{\lambda}$ are the minimum and maximum values satisfying the above two conditions. One can verify, through these conditions, that $\underline{\lambda} < 1$ and $\bar{\lambda} > 1$. Hence, there exists $\beta > 0$ such that,

$$x_e^*(t) \in \text{int}(E(\bar{x}_e, \beta)) \text{ and } E(\bar{x}_e, \beta) \subset E(x_e^*(t), \delta_b).$$

Consequently, there exists $\delta_a > 0$ such that,

$$E(x_e^*(t), \delta_a) \subset E(\bar{x}_e, \beta).$$

It remains to prove that $E(\bar{x}_e, \beta)$ is a positively invariant set under the controller $K^*(t)x + \bar{d}$. By definition, it is true that,

$$\|x - \bar{x}_e\|_P^2 \leq \beta \text{ for all } x \in E(\bar{x}_e, \beta).$$

Since $Q \in \mathbb{S}_{++}^n$ and $R \in \mathbb{S}_{++}^m$, it is also true that,

$$\|x - \bar{x}_e\|_P^2 - \|x - \bar{x}_e\|_{(Q+K^*(t)^\top RK^*(t))}^2 \leq \beta,$$

or equivalently,

$$\|x - \bar{x}_e\|^2_{(P-Q-K^*(t)^\top RK^*(t))} \leq \beta,$$

From (3.39), (3.40) and (3.41), one can verify that the matrix P satisfies,

$$P \geq (A + BK^*(t))^\top P(A + BK^*(t)) + Q + K^*(t)^\top RK^*(t),$$

and hence,

$$\|x - \bar{x}_e\|^2_{(A+BK^*(t))^\top P(A+BK^*(t))} \leq \beta,$$

or equivalently,

$$\|x^+ - \bar{x}_e\|^2_P \leq \beta,$$

where $x^+ = \bar{x}_e + (A + BK^*(t))(x - \bar{x}_e)$. Furthermore, for all $x \in E(\bar{x}_e, \beta)$,

$$\begin{aligned}(x, K^*(t)x + \bar{d}) &= (x, K^*(t)x + d^*(t)) + (0, \bar{d} - d^*(t)) \\ &= (x, K^*(t)x + d^*(t)) + (0, -(1-\lambda)(d^*(t) - d_r)) \\ &\in (\gamma\mathcal{X} \times \gamma\mathcal{U}) + ((1-\gamma)\mathcal{X} \times (1-\gamma)\mathcal{U}) \\ &= \mathcal{X} \times \mathcal{U}.\end{aligned}$$

□

Lemma 3.4. *Define $\sigma > 0$ such that $S > \sigma P$, $\delta_{max} = \min\left(\sigma, \frac{(1+3\sigma)-\sqrt{1+6\sigma+5\sigma^2}}{2}\right)$, $\delta_1 \in (0, \delta_{max})$ and assume that $\lambda \in \left(\frac{1+\delta_1-\sqrt{\delta_1^2-(1+3\sigma)\delta_1+\sigma^2}}{1+\sigma}, \frac{1+\delta_1+\sqrt{\delta_1^2-(1+3\sigma)\delta_1+\sigma^2}}{1+\sigma}\right)$. Then, there exist $\delta_a > 0$ such that $\|x - \bar{x}_e\|^2_P + \|\bar{x}_e\|^2_S < \|x^*_e - x_r\|^2_S$ for all $x \in E(x^*_e, \delta_a)$. Moreover, the range in which λ can be selected intersects the open set (0,1).*

Proof. Since $\bar{x}_e = x_r + \lambda(x^*_e(t) - x_r)$, then $x^*_e(t) - \bar{x}_e = (1-\lambda)(x^*_e(t) - x_r)$ and,

$$\|x^*_e(t) - \bar{x}_e\|^2_P = (1-\lambda)^2\|x^*_e(t) - x_r\|^2_P.$$

Furthermore,

$$\begin{aligned}x - \bar{x}_e &= (x - x^*_e(t)) + (x^*_e(t) - \bar{x}_e) \\ &= (x - x^*_e(t)) + (1-\lambda)(x^*_e(t) - x_r),\end{aligned}$$

and hence,

$$\|x - \bar{x}_e\|^2_P = \|x - x^*_e(t)\|^2_P + 2(1-\lambda)(x^*_e(t) - x_r)^\top P(x - x^*_e(t)) + (1-\lambda)^2\|x^*_e(t) - x_r\|^2_P.$$

Consider a constant $\delta_1 > 0$ and the set $\mathcal{X}_1$ defined as,

$$\mathcal{X}_1 = \left\{ x \in \mathbb{R}^n : \|x - x^*_e(t)\|^2_P \leq \delta_1\|x^*_e(t) - x_r\|^2_P, \right. \\ \left. |(x^*_e(t) - x_r)^\top P(x - x^*_e(t))| \leq \delta_1\|x^*_e(t) - x_r\|^2_P \right\}.$$

For every δ_1, we can select a small enough δ_a such that $E(x^*_e(t), \delta_a) \subset \mathcal{X}_1$ and hence,

$$\|x - \bar{x}_e\|^2_P \leq \|x^*_e(t) - x_r\|^2_{(\delta_1 P + 2\delta_1(1-\lambda)P + (1-\lambda)^2 P)} \text{ for all } x \in E(x^*_e(t), \delta_a).$$

Since $\bar{x}_e - x_r = \lambda(x_e^*(t) - x_r)$, then,

$$\|\bar{x}_e - x_r\|_S^2 = \lambda^2 \|x_e^*(t) - x_r\|_S^2,$$

and hence,

$$\|x - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 \leq \|x_e^*(t) - x_r\|^2_{(\delta_1 P + 2\delta_1(1-\lambda)P + (1-\lambda)^2 P + \lambda^2 S)} \text{ for all } x \in E(x_e^*(t), \delta_a).$$

To prove that $\|x - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 < \|x_e^*(t) - x_r\|_S^2$ for all $x \in E(x_e^*(t), \delta_a)$, it is required to find conditions on δ_1 and λ so that,

$$\|x_e^*(t) - x_r\|^2_{(\delta_1 P + 2\delta_1(1-\lambda)P + (1-\lambda)^2 P + \lambda^2 S)} < \|x_e^*(t) - x_r\|_S^2.$$

or equivalently,

$$(1-\lambda^2)S - (1-\lambda)^2 P - 2\delta_1(1-\lambda)P - \delta_1 P > 0.$$

Since $S > \sigma P$ by assumption, it suffices to ensure that,

$$(1-\lambda^2)\sigma - (1-\lambda)^2 - 2\delta_1(1-\lambda) - \delta_1 > 0,$$

or, equivalently,

$$-(1+\sigma)\lambda^2 + 2(1+\delta_1)\lambda + (\sigma - 1 - 3\delta_1) > 0.$$

Since the quadartic is concave in λ, its roots are required to be real and distinct so that there exists λ which satisfies the strict inequality. The roots are,

$$\frac{1+\delta_1}{1+\sigma} \pm \frac{\sqrt{(1+\delta_1)^2 + (1+\sigma)(\sigma - 1 - 3\delta_1)}}{1+\sigma}, \tag{3.60}$$

and are real and distinct as long as,

$$(1+\delta_1)^2 + (1+\sigma)(\sigma - 1 - 3\delta_1) > 0,$$

or, equivalently,

$$\delta_1^2 - (1+3\sigma)\delta_1 + \sigma^2 > 0.$$

This in turn is a convex quadratic in δ_1 whose roots,

$$\frac{(1+3\sigma) \pm \sqrt{(1+3\sigma)^2 - 4\sigma^2}}{2},$$

are real, distinct and positive since $\sigma > 0$ and hence,

$$(1+3\sigma)^2 - 4\sigma^2 = 1 + 6\sigma + 5\sigma^2 > 0.$$

If we then pick $\delta_1 \in \left(0, \frac{(1+3\sigma) - \sqrt{1+6\sigma+5\sigma^2}}{2}\right)$, the roots of (3.60) are real and distinct. Thus, for any $\sigma > 0$, there exists a small enough δ_1 such that there exists λ which satisfies the desired condition.

It remains to show that λ can be selected in the interval $(0, 1)$. For this, it suffices to prove that it is always possible to choose at least one of the roots in (3.60) to be between zero and one. Consider the larger root in (3.60) given by,

$$\left(1+\delta_1+\sqrt{(1+\delta_1)^2+(1+\sigma)(\sigma-1-3\delta_1)}\right)/(1+\sigma),$$

This root is always positive since,

$$(1+\delta_1)^2+(1+\sigma)(\sigma-1-3\delta_1)>0.$$

For this root to be smaller than 1, it is required that,

$$1+\sigma>1+\delta_1+\sqrt{(1+\delta_1)^2+(1+\sigma)(\sigma-1-3\delta_1)}.$$

Since $\delta_1<\sigma$, simplifying and squaring the above inequality reduces to $\delta_1(\sigma+1)>0$, which is always the case since σ and δ_1 are positive constants. In conclusion, for any positive σ and δ_1 such that $\sigma P\leq S$ and $\delta_1<\min\left(\sigma,\frac{(1+3\sigma)-\sqrt{1+6\sigma+5\sigma^2}}{2}\right)$, there exists $\lambda\in(0,1)$ such that,

$$(1-\lambda^2)\sigma-(1-\lambda)^2-2\delta_1(1-\lambda)-\delta_1>0,$$

and consequently,

$$\|x-\bar{x}_e\|_P^2+\|\bar{x}_e-x_r\|_S^2<\|x_e^*(t)-x_r\|_S^2 \text{ for all } x\in E(x_e^*(t),\delta_a)\subset\mathcal{X}_1.$$

□

Remark 3.1. *Both Lemma 3.3 and Lemma 3.4 impose constraints on λ. These constraints can, however, be made compatible with each other. In particular, Lemma 3.3 requires λ to be in a set containing one in its interior. On the other side, Lemma 3.4 requires λ to be in a range whose upper bound is between zero and one. Note that this upper bound tends to one as δ_1 tends to zero, that is,*

$$\lim_{\delta_1\to 0}\left(\frac{1+\delta_1}{1+\sigma}+\frac{\sqrt{(1+\delta_1)^2+(1+\sigma)(\sigma-1-3\delta_1)}}{1+\sigma}\right)=1.$$

Thus, if δ_1 is chosen sufficiently small, the upper bound in Lemma 3.4 (which tends to one) can be made higher than the lower bound in Lemma 3.3 (which is lower than one). The positive parameter δ_1 is only upper bounded in Lemma 3.4 and hence, can be chosen arbitrarily close to zero.

Remark 3.2. *Both Lemma 3.3 and Lemma 3.4 impose constraints on the value of the parameter δ_a. In particular, Lemma 3.3 requires that $E(x_e^*(t),\delta_a)\subseteq E(\bar{x}_e,\beta)$, whereas Lemma 3.4 requires that $E(x_e^*(t),\delta_a)\subseteq\mathcal{X}_1$. Note that these conditions can be made compatible with each other by choosing a small enough δ_a. One can also verify that, if δ_a*

satisfies both conditions, the set $E(x_e^*(t), \delta_a)$ is positively invariant under the controller $K^*(t)x + d^*(t)$. The invariance condition can be shown in the same way as the invariance condition of the set $E(\bar{x}_e, \beta)$ in Lemma 3.3. Moreover, the constraint satisfaction condition holds since $E(x_e^*(t), \delta_a) \subset E(\bar{x}_e, \beta) \subset E(x_e^*(t), \delta_b)$ and for all $x \in E(x_e^*(t), \delta_b)$, $(x, K^*(t)x + d^*(t)) \in \gamma\mathcal{X} \times \gamma\mathcal{U}$ according to Lemma 3.3.

Lemma 3.5. *If for a given initial state x_0, the sequence of optimal solutions to (3.42) is such that $\lim_{t\to\infty}(x^*(t) - x_e^*(t)) = 0$, then $\lim_{t\to\infty}(x_e^*(t) - x_r) = 0$.*

Proof. Assume, for the sake of contradiction, that $\lim_{t\to\infty}(x^*(t) - x_e^*(t)) = 0$ but the sequence of optimal artificial equilibrium points $\{x_e^*(t)\}_{t=0}^{\infty}$ either does not converge, or does but its limit is not the target point x_r. In both cases, there exists $\delta_c > 0$ such that for infinitely many t,

$$\|x_e^*(t) - x_r\|_P^2 \geq \delta_c.$$

Since $\lim_{t\to\infty}(x^*(t) - x_e^*(t)) = 0$, it is always possible to pick an arbitrarily large t such that,

$$x^*(t) \in E(x_e^*(t), \delta_a),$$

where $\|x_e^*(t) - x_r\|^2 \geq \delta_c$ and δ_a satisfies the condition $E(x_e^*(t), \delta_a) \subset E(\bar{x}_e, \beta)$ in Lemma 3.3 and the condition $E(x_e^*(t), \delta_a) \subseteq \mathcal{X}_1$ in Lemma 3.4. According to Remark 3.2, it is always the case that the selected $\delta_a > 0$ makes the set $E(x_e^*(t), \delta_a)$ positively invariant under the controller $K^*(t)x + d^*(t)$.

Since $x^*(t) \in E(x_e^*(t), \delta_a)$, the optimal cost $J^*(t)$ is given by,

$$J^*(t) = \sum_{k=0}^{N-1} \left(\|x^*(k|t) - x_e^*(t)\|_Q^2 + \|x^*(k|t) - x_e^*(t)\|_{K^*(t)^\top R K^*(t)}^2 \right) + \|x^*(N|t) - x_e^*(t)\|_P^2 + \|x_e^*(t) - x_r\|_S^2.$$

According to Lemma 3.3, $E(x_e^*(t), \delta_a) \subset E(\bar{x}_e, \beta)$ which is a positively invariant set under the controller $K^*(t)x + \bar{d}$. Thus, there exists a feasible solution starting from the initial condition $\hat{x}(t) = x^*(t) \in E(x_e^*(t), \delta_a)$ aiming to converge to the non-optimal equilibrium point $\bar{x}_e$. Denote the cost of this feasible solution as,

$$\hat{J}(t) = \sum_{k=0}^{N-1} \left(\|\hat{x}(k|t) - \bar{x}_e\|_Q^2 + \|\hat{x}(k|t) - \bar{x}_e\|_{K^*(t)^\top R K^*(t)}^2 \right) + \|\hat{x}(N|t) - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2.$$

Note that $J^*(t) < \hat{J}(t)$ since $J^*(t)$ is the optimal cost. One can also verify from (3.39), (3.40) and (3.41) that the matrix P satisfies the Lyapunov inequality,

$$P \geq (A + BK^*(t))^\top P (A + BK^*(t)) + Q + K^*(t)^\top R K^*(t)$$

and hence that,

$$\hat{J}(t) \leq \|x^*(t) - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2.$$

According to Lemma 3.4, for all $x \in E(x_e^*(t), \delta_a) \subset \mathcal{X}_1$,

$$\|x - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 < \|x_e^*(t) - x_r\|_S^2.$$

Since $x^*(t) \in E(x_e^*(t), \delta_a)$, then,

$$\|x^*(t) - \bar{x}_e\|_P^2 + \|\bar{x}_e - x_r\|_S^2 < \|x_e^*(t) - x_r\|_S^2.$$

Note that $\|x_e^*(t) - x_r\|_S^2 \leq J^*(t)$ which contradicts the optimality of $J^*(t)$. Indeed, $\hat{J}(t) < J^*(t)$ and there exists $\bar{x}_e = x_r + \lambda(x_e^*(t) - x_r)$ with $\lambda \in (0, 1)$ which corresponds to the optimal solution where the condition $\|\bar{x}_e - x_r\|_P^2 \geq \delta_c$ is no longer valid. □

Finally, we conclude the asymptotic stability of the closed-loop system under the MPC problem (3.42) in the following theorem where we use Lemma 3.1, Lemma 3.2 and Lemma 3.5.

Theorem 3.4. *The closed-loop system under the proposed MPC scheme (3.42) is asymptotically stable.*

Proof. For proving asymptotic stability, we prove that the closed-loop system is stable and that the closed-loop state trajectory asymptotically converges to the target point. Lemma 3.1 shows that the closed-loop system is stable using the optimal cost $J^*(t)$ as a Lyapunov function where,

$$J^*(t+1) - J^*(t) \leq 0.$$

Furthermore, Lemma 3.2 and Lemma 3.5 shows by contradiction that the closed-loop state trajectory asymptotically converges to the target point. In particular, Lemma 3.2 implies that,

$$\lim_{t \to \infty} (x^*(t) - x_e^*(t)) = 0$$

and Lemma 3.5 further implies that if $\lim_{t\to\infty}(x^*(t) - x_e^*(t)) = 0$, then,

$$\lim_{t \to \infty} (x_e^*(t) - x_r) = 0.$$

and hence,

$$\lim_{t \to \infty} (x^*(t) - x_r) = 0.$$

□

We also extend the previous result to the MPC schemes (3.51), (3.56) and (3.58) in the following theorem.

Theorem 3.5. *The closed-loop system under the proposed MPC schemes (3.51), (3.56) and (3.58) is asymptotically stable.*

Proof. Regarding the MPC problem (3.51), the proof follows that of Theorem 3.5. Concerning the MPC problems (3.56) and (3.58), the proof follows that of Theorem 3.5 expect that the terminal control gain matrix K computed offline replaces the reconfigurable terminal control gain matrix $K^*(t)$. □

3.5 Distributed Implementation

The global online OCPs (3.42), (3.51), (3.56) and (3.58) in Algorithm 1, Algorithm 2, Algorithm 3 and Algorithm 4, respectively, can be solved using one of the many distributed optimization techniques proposed in the literature [NL18]. Some of these techniques do not require a central coordinator such as the distributed primal-dual algorithm (DPDA) [AH19] and some variants of ADMM [Boy+11]; the numerical comparison in [Ban+19] suggests that ADMM outperforms DPDA for certain classes of problems. Motivated by this, we show how to implement the aforementioned OCPs using ADMM [Boy+11, Section 7], where each subsystem solves a local optimization problem iteratively at each time instant while sharing information with its neighbours until the optimal solution of the corresponding global online OCP is reached.

Although ADMM can be implemented in the same way for all the aforementioned schemes, these schemes differ in the information shared among the subsystems. In the sequel, we show the shared and non-shared variables for each scheme. For this purpose, recall the bijective map (3.33) and define the operator 'diag-vec' which arranges the diagonal elements of a matrix into a vector and the operator 'vec' which converts a matrix into a vector. Starting with (3.42), two neighbours i and j share the variables $x_i(k|t)$, $x_j(k|t)$, x_{e_i}, x_{e_j}, α_i, α_j, c_i, c_j, $U_jW_i^\top T_iW_iU_j^\top$ and $U_iW_j^\top T_jW_jU_i^\top$ for all $k \in \{1, ..., N\}$. Thus, the shared vector of the i-th subsystem is,

$$w_{N_i} = \begin{pmatrix} x_{N_i}(k|t)|_{k=\{0,\dots,N\}}, x_{e_{N_i}}, \text{diag-vec}(v_{1_i}), v_{2_i}, \\ \qquad \text{vec}\left(U_jW_i^\top v_{6_i} W_iU_j^\top\right)|_{j\in\mathcal{N}_i\setminus i}, \text{vec}\left(U_iW_j^\top v_{6_j} W_jU_i^\top\right)|_{j\in\mathcal{N}_i\setminus i} \end{pmatrix},$$

whereas its non-shared vector is,

$$v_i = \left(u_i(k|t)|_{t=\{0,\dots,N-1\}}, u_{e_i}, v_{3_i}, v_{4_i}, v_{5_i}, \mu_{ij}|_{j\in\mathcal{N}_i}, \rho_{ij}^p|_{j\in\mathcal{N}_i,\ p\in\{1,\dots,q_i\}}, \tau_{ij}^l|_{j\in\mathcal{N}_i,\ l\in\{1,\dots,r_i\}}\right);$$

to simplify notation we also define,

$$w_i = \left(x_i(k|t)|_{k=\{0,\dots,N\}}, x_{e_i}, \alpha_i, c_i, \text{vec}(v_{6_i})\right).$$

Since two neighbours share the same variables in (3.51), the shared vector w_{N_i} and the vector w_i remain the same. We add the variables b_{j_i} for all $j \in \{1, 2, 3, 4\}$ introduced due to using diagonal dominance to the non-shared vector v_i.

On the other side, two neighbours i and j share the variables $x_i(k|t)$, $x_j(k|t)$, x_{e_i}, x_{e_j}, α_i, α_j, c_i and c_j for all $k \in \{1, ..., N\}$ in (3.56). Hence, we define the vectors,

$$w_{N_i} = (x_{N_i}(k|t)|_{k=\{0,...,N\}}, x_{e_{N_i}}, \text{diag-vec}(\alpha_{N_i}), c_{N_i}),$$
$$v_i = (u_i(k|t)|_{t=\{0,...,N-1\}}, u_{e_i}, d_i, \lambda_{ij}|_{j\in\mathcal{N}_i}),$$
$$w_i = (x_i(k|t)|_{k=\{0,...,N\}}, x_{e_i}, \alpha_i, c_i).$$

Since two neighbours share the same variables in (3.58), the shared vector w_{N_i} and the vector w_i remain the same. We add the variables b_i introduced due to using diagonal dominance to the non-shared vector v_i. After defining the shared and non-shared vectors for the i-th subsystem in each scheme, we show briefly how to solve these schemes using ADMM; see [Boy+11, Section 7] for more details.

First, we define for each subsystem the local augmented cost function $\tilde{J}_i(w_{N_i}, v_i)$, which encodes both its local cost function $J_i(t)$ and its constraints through indicator functions. Hence, the online OCP is given by,

$$\min_{w_{N_i}, v_i} \sum_{i=1}^{M} \tilde{J}_i(w_{N_i}, v_i). \tag{3.61}$$

We then define a local copy for the shared vector of the i-th subsystem and denote it by $w_{N_i}^{(i)}$. We use this local copy in the augmented cost function of the i-th subsystem. To obtain a feasible solution, the local copies of the same decision variable existing in different local augmented cost functions should be equal. Hence, we define a global copy z comprising all shared decision variables. We refer to the subvector of z corresponding to w_{N_i} as z_{N_i} and to the subvector of z corresponding to w_i as z_i. For two neighbours i and j, the vectors z_{N_i} and z_{N_j} overlap. The vectors z_i (a subvector of z_{N_i}) and z_j (a subvector of z_{N_j}) do not overlap however. In this case, the online OCP (3.61) becomes,

$$\min_{w_{N_i}^{(i)}, v_i} \sum_{i=1}^{M} \tilde{J}_i(w_{N_i}^{(i)}, v_i) \ s.t. \ w_{N_i}^{(i)} = z_{N_i} \text{ for all } i \in \{1, ..., M\},$$

and its augmented Lagrangian is given by,

$$L = \sum_{i=1}^{M} L_i(w_{N_i}^{(i)}, v_i, z_{N_i}, y_{N_i}),$$

where y_{N_i} is the Lagrange multiplier vector computed by the i-th subsystem, ρ is an ADMM parameter and,

$$L_i(w_{N_i}^{(i)}, v_i, z_{N_i}, y_{N_i}) = \tilde{J}_i(w_{N_i}^{(i)}, v_i) + \tfrac{\rho}{2}\|w_{N_i}^{(i)} - z_{N_i} + \tfrac{1}{\rho}y_{N_i}\|_2^2.$$

Using the augmented Lagrangian, an iterative ADMM algorithm is developed where each subsystem performs three steps in each iteration. First, each subsystem solves a

Algorithm 5 ADMM Implementation of Reconfigurable Distributed MPC for tracking

Input: System matrices A_{N_i} and B_i, constraint sets $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$, cost function matrices Q_i, R_i and S_i, map U_i, V_i and W_i, initial condition x_{0_i}, prediction horizon T, ADMM step size ρ, ADMM maximum time T_{max}.

Output: Shared vector w_{N_i} and non-shared vector v_i.

Using (2.7), compute P_i for (3.42) & (3.51) or P_i & K_{N_i} for (3.56) & (3.58).
Set $z_{N_i}[0] = 0$ and $y_{N_i}[0] = 0$.
while true **do**
  Set $k = 1$.
  do
    Set $z_{N_i}^{\text{prev}} = z_{N_i}[k-1]$ and $y_{N_i}^{\text{prev}} = y_{N_i}[k-1]$.
    $(w_{N_i}^{(i)}[k], v_i[k]) = \underset{w_{N_i}^{(i)}, v_i}{\arg\min}\, L_i\left(w_{N_i}^{(i)}, v_i, z_{N_i}^{\text{prev}}, y_{N_i}^{\text{prev}}\right)$.
    Share $w_j^{(i)}$ and $w_i^{(j)}$ with $j \in \mathcal{N}_i$.
    $z_i[k] = \frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} w_i^{(j)}[k]$.
    Share $z_i[k]$ with all $j \in \mathcal{N}_i$.
    $y_{N_i}[k] = y_{N_i}[k-1] + \rho\left(w_{N_i}^{(i)}[k] - z_{N_i}^{(i)}[k]\right)$.
    Set $k = k + 1$.
  until time limit T_{max} **is reached.**
  Set $z_{N_i}[0] = z_{N_i}[k]$ and $y_{N_i}[0] = y_{N_i}[k]$.
  Apply the control input $u_i(0|t)$ to the plant (3.1).
  Measure the new state $x_{0_i} \leftarrow A_{N_i}x_{0_{N_i}} + B_i u_i(0|t)$.
end while

local optimization problem to update its local variables $w_{N_i}^{(i)}$ and v_i. Then, each subsystem updates the subvector z_i of the global copy z using local information only. Finally, each subsystem updates its Lagrange multiplier vector y_{N_i} locally. At each timestep, the ADMM algorithm terminates after a predefined time determined by the sampling time available for computations. Algorithm 5 shows how the i-th subsystem uses ADMM to implement the proposed scheme. At each time instant, we denote the value of the variable x at the k-th iteration by $x[k]$ in this algorithm.

3.6 Feasible Regions

In this section, we compare the feasible regions of the proposed schemes to those of existing ones in the literature. In particular, the feasible region of the MPC scheme (3.42) is compared to that of the one developed in [DEL19]. Furthermore, the feasible regions of the proposed MPC scheme (3.56) and the MPC scheme (2.26) in Chapter 2 are compared.

For the sake of completeness, we mention the online OCP of the MPC scheme developed in [DEL19] which is given by,

$$\min \sum_{i=1}^{M} J_i(t),$$

subject to,

$$\text{for all } k \in \{0,\dots,N-1\},\ i \in \{1,\dots,M\},$$

$$\text{for all } j \in \mathcal{N}_i,\ p \in \{1,\dots,q_i\},\ l \in \{1,\dots,r_i\},$$

$$x_i(0|t) = x_{0_i}, \quad x_i(k+1|t) = A_{N_i} x_{N_i}(k|t) + B_i u_i(k|t), \tag{3.62a}$$

$$G_{N_i} x_{N_i}(k|t) \le g_{N_i}, \quad h_i u_i(k|t) \le h_i, \tag{3.62b}$$

$$\begin{bmatrix} P_i^{-1}\alpha_i & x_i(N|t) \\ x_i(N|t)^\top & \alpha_i \end{bmatrix} \ge 0, \tag{3.62c}$$

$$\alpha_i - \sum_{j\in\mathcal{N}_i} \theta_{ij} \ge 0, \quad \begin{bmatrix} P_i^{-1}\alpha_i & (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i}) \\ (A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i})^\top & \sum\limits_{j\in\mathcal{N}_i} \theta_{ij} P_{ij} \end{bmatrix} \ge 0, \tag{3.62d}$$

$$g_i^p - \sum_{j\in\mathcal{N}_i} \phi_{ij}^p \ge 0, \quad \begin{bmatrix} g_i^p & G_{N_i}^p \alpha_{N_i} \\ \alpha_{N_i} G_{N_i}^{p^\top} & \sum\limits_{j\in\mathcal{N}_i} \phi_{ij}^p P_{ij} \end{bmatrix} \ge 0, \tag{3.62e}$$

$$h_i^l - \sum_{j\in\mathcal{N}_i} \psi_{ij}^l \ge 0, \quad \begin{bmatrix} h_i^l & H_i^l K_{N_i}\alpha_{N_i} \\ \alpha_{N_i} K_{N_i}^\top H_i^{l^\top} & \sum\limits_{j\in\mathcal{N}_i} \psi_{ij}^l P_{ij} \end{bmatrix} \ge 0, \tag{3.62f}$$

$$\begin{bmatrix} W_i U_i^\top P_i^{-1} U_i W_i^\top + \alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i} & \alpha_{N_i} A_{N_i}^\top \alpha_{N_i} K_{N_i}^\top B_i^\top & \alpha_{N_i} Q_i^{1/2} & \alpha_{N_i} K_{N_i}^\top R_i^{1/2} \\ A_{N_i}\alpha_{N_i} + B_i K_{N_i}\alpha_{N_i} & P_i^{-1}\alpha_i & 0 & 0 \\ Q_i^{1/2}\alpha_{N_i} & 0 & \alpha_i I_{n_{N_i}} & 0 \\ R_i^{1/2} K_{N_i}\alpha_{N_i} & 0 & 0 & \alpha_i I_{m_i} \end{bmatrix} \ge 0, \tag{3.62g}$$

$$\alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i} \le T_i, \quad \sum_{j\in\mathcal{N}_i} U_i W_j^\top T_j W_j U_i^\top \le 0, \tag{3.62h}$$

$$\alpha_i \ge \epsilon, \quad \theta_{ij} \ge 0, \quad \phi_{ij}^k \ge 0, \quad \psi_{ij}^l \ge 0. \tag{3.62i}$$

where

$$J_i(t) = \sum_{k=0}^{N-1} \left(\|x_{N_i}(k|t)\|_{Q_i}^2 + \|u_i(k|t)\|_{R_i}^2 \right) + \|x_i(N|t)\|_{P_i}^2.$$

The decision variables in this convex OCP are for all $i \in \{1,\dots,M\}$, $x_i(k|t)$ for all $k \in \{0,\dots,N\}$, $u_i(k|t)$ for all $k \in \{0,\dots,N-1\}$, α_i, K_{N_i}, Γ_i, T_i, θ_{ij}, ϕ_{ij}^p and ψ_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1,\dots,q_i\}$ and $l \in \{1,\dots,r_i\}$. This OCP is written according to the notation used in this thesis and will be used in Theorem 3.6. For the fairness of comparison, we replace the constraint $\sum_{i=1}^{M} W_i^\top \alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i} W_i \le 0$ in [DEL19] with (3.62h). This idea is introduced in [Con+16] to make the resulting MPC schemes amenable to distributed optimization and is used while deriving the MPC problem (3.42) in Proposition 3.5. We

also add the constraint $\alpha_i \geq \epsilon$ to (3.62), otherwise the decision variable $\alpha_{N_i}\Gamma_i\alpha_i^{-1}\alpha_{N_i}$ in (3.62g) is undefined.

We start by showing that the feasible region of the MPC scheme (3.42) contains that of the MPC scheme (3.62) developed in [DEL19] in the following theorem.

Theorem 3.6. *For a given initial condition, the proposed MPC problem (3.42) is feasible if the MPC problem (3.62) developed in [DEL19] is feasible.*

Proof. For a given initial condition $x_0 = [x_{0_1}^\top, \dots, x_{0_M}^\top]^\top$, assume that for all $i \in \{1, ..., M\}$, $x_i(k|t)$ for all $k \in \{0, ..., N\}$, $u_i(k|t)$ for all $k \in \{0, ..., N-1\}$, α_i, K_{N_i}, Γ_i, T_i, θ_{ij}, ϕ_{ij}^p and ψ_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1, ..., q_i\}$ and $l \in \{1, ..., r_i\}$ represent a feasible solution of the MPC problem (3.62) in [DEL19]. This solution satisfies the constraints (3.1), (3.2), (3.10), (3.39), (3.40) and (3.41) in the MPC problem (3.42) since the same set of constraints (i.e. (3.62a), (3.62b), (3.62g) and (3.62h)) occurs in the MPC problem in (3.62). Assuming that $x_{e_i} = 0$, $u_{e_i} = 0$, $c_i = 0$ and $d_i = 0$ and since $\alpha_i \geq \epsilon$ for all $i \in \{1, \dots, M\}$, constraint (3.34) in (3.42) is satisfied since $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$ contain the origin in their interior by assumption. Furthermore, assuming that $\mu_{ij} = \theta_{ij}$ for all $i \in \{1, \dots, M\}$ and $j \in \mathcal{N}_i$, the aforementioned feasible solution also satisfies (3.35) and (3.36) in (3.42) since these constraints become the same as (3.62c) and (3.62d) in (3.62).

Let $\rho_{ij}^p = 0.5\phi_{ij}^p$ for all $i \in \{1, \dots, M\}$, $j \in \mathcal{N}_i$ and $p \in \{1, \dots, q_i\}$. Hence, the LMI in (3.62e) becomes,

$$\begin{bmatrix} g_{N_i}^p & G_{N_i}^p \alpha_{N_i} \\ \alpha_{N_i} G_{N_i}^{p\top} & 2\sum_{j\in\mathcal{N}_i} \rho_{ij}^p P_{ij}, \end{bmatrix} \geq 0.$$

In addition, the scalar inequality in (3.62e) implies,

$$\begin{bmatrix} g_{N_i}^p - 2\sum_{j\in\mathcal{N}_i} \rho_{ij}^p & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

Adding the LMIs above yields,

$$\begin{bmatrix} 2g_{N_i}^p - 2\sum_{j\in\mathcal{N}_i} \rho_{ij}^p & G_{N_i}^p \alpha_{N_i} \\ \alpha_{N_i} G_{N_i}^{p\top} & 2\sum_{j\in\mathcal{N}_i} \rho_{ij}^p P_{ij}, \end{bmatrix} \geq 0.$$

Multiplying the resulting LMI by 0.5 confirms the satisfaction of LMI (3.37) in (3.42). The same procedures can be followed to prove that (3.62f) in (3.62) implies (3.38) in (3.42) when $\tau_{ij}^l = 0.5\psi_{ij}^l$ for all $i \in \{1, \dots, M\}$, $j \in \mathcal{N}_i$ and $l \in \{1, \dots, r_i\}$. Since $\mu_{ij} = \theta_{ij}$, $\rho_{ij}^p = 0.5\phi_{ij}^p$ and $\tau_{ij}^l = 0.5\psi_{ij}^l$ for all $i \in \{1, \dots, M\}$, $j \in \mathcal{N}_i$, $p \in \{1, \dots, q_i\}$ and $l \in \{1, \dots, r_i\}$, constraint (3.31) in (3.42) holds from (3.62i) in (3.62). □

Next, we show in the following theorem that the feasible region of the MPC scheme (3.56) contains that of the MPC scheme (2.26) proposed in Chapter 2.

Theorem 3.7. *For a given initial condition, the proposed MPC problem* (3.56) *is feasible if the MPC problem in* (2.26) *is feasible.*

Proof. For a given initial condition $x_0 = [x_{0_1}^\top, \dots, x_{0_M}^\top]^\top$, assume that for all $i \in \{1, ..., M\}$, $x_i(k|t)$ for all $k \in \{0, ..., N\}$, $u_i(k|t)$ for all $k \in \{0, ..., N-1\}$, α_i, c_i, λ_{ij}, σ_{ij}^p and ν_{ij}^l for all $j \in \mathcal{N}_i$, $p \in \{1, ..., q_i\}$ and $l \in \{1, ..., r_i\}$ represent a feasible solution of the MPC problem (2.26). This feasible solution satisfies constraints (3.1), (3.2) and (3.10) in (3.56) since the same constraints (i.e. (2.1), (2.2) and the constraint that $x_i(0|t) = x_{0_i}$ for all $i \in \{1, \dots, M\}$) occur in (2.26). Assuming that $x_{e_i} = 0$ and $u_{e_i} = 0$ for all $i \in \{1, \dots, M\}$, constraint (3.14) in (3.56) is also satisfied since $\mathcal{X}_{N_i}$ and $\mathcal{U}_i$ contain the origin in their interior by assumption. Using the Schur complement, one can verify that constraint (3.15) in (3.56) and (2.18) in (2.26) are equivalent and hence, the aforementioned feasible solution satisfies (3.15) is (3.56).

Assuming for all $i \in \{1, \dots, M\}$ that $d_i = 0$ and $\mu_{ij} = \lambda_{ij}$ for all $j \in \mathcal{N}_i$, the feasible solution also satisfies (3.21) in (3.56) since this constraint becomes the same as (2.10) in (2.26). Furthermore, note that constraints (3.53) and (3.54) in (3.56) are equivalent to conditions (3.9b) and (3.9c) assuming polytopic constraints, affine functions and ellipsoidal sets. On the other side, constraints (2.23) and (2.25) in (2.26) are sufficient conditions for (2.9b) and (2.9c) assuming polytopic constraints, affine functions and ellipsoidal sets. Since (3.9b) and (3.9b) are the same conditions as (2.9b) and (2.9c) (Proposition 2.1 in Chapter 2 and Proposition 3.1 in Chapter 3 are the same), then constraints (2.23) and (2.25) in (2.26) imply (3.53) and (3.53) in (3.56). Finally, since $\mu_{ij} = \lambda_{ij}$ and $\lambda_{ij} \geq 0$ for all $i \in \{1, \dots, M\}$ and $j \in \mathcal{N}_i$, then constraint (3.55) in (3.56) holds. □

3.7 Simulation Results

In this section, we investigate the efficacy of the novel schemes proposed in this chapter and compare them to other existing schemes in the literature. For this purpose, we make use of three different simulation examples; a toy example, a benchmark example and a power network case study. In the toy example, we visualize the behaviour of the closed-loop systems under the proposed controllers. Then, we explore the performance and computational complexity of the proposed approaches in the benchmark example. Finally, we compare the proposed approaches using the power network case study. For ease of notation, we refer in this section to the developed schemes in Algorithm 1, Algorithm 2, Algorithm 3 and Algorithm 4 by T-A1, T-A2, T-B1, T-B2, respectively. For comparison, we consider the centralized MPC scheme in [Lim+08] where the maximal invariant terminal set is computed offline (denoted here by T-CNT) and the distributed MPC scheme in [Con+13] where ellipsoidal terminal sets are computed offline (denoted here by T-DST). We perform the following simulations using MATLAB [MAT18] with

YALMIP [Löf04], MOSEK [ApS19] and GUROBI [Gur22] on a computer equipped with 16-GB RAM and a 1.9-GHz Intel core i7-8550U processor. Unless otherwise stated, all distributed MPC schemes are solved using ADMM [Boy+11].

3.7.1 Toy Example

The main aim of this example is to visualize the performance of T-A1 and T-A2 and compare them to T-CNT and T-DST. The dynamics of the toy example is given by,

$$x_1^+ = 2x_1 + 0.5x_2 - u_1,$$
$$x_2^+ = 0.5x_1 + 2x_2 - u_2,$$

with state and input constraints,

$$-5 \leq x_i \leq 5 \text{ for } i \in \{1, 2\},$$
$$-0.25 \leq u_i \leq 1 \text{ for } i \in \{1, 2\}.$$

The system is divided into two neighbouring subsystems with states x_1 and x_2 and inputs u_1 and u_2, respectively. The matrices of the cost function are chosen to be $Q_1 = Q_2 = 0.5I_2$, $R_1 = R_2 = 0.1$ and $S_1 = S_2 = 1$, the target point $x_r = [0\ 0]^\top$ and the prediction horizon $T = 2$. The matrix P is computed by solving (2.7).

Figure 3.1 shows the evolution of the predicted state trajectories and terminal sets of T-A1 when implemented recursively for 10 time steps starting from $[x_1\ x_2] = [1.1\ 0.1]$. This initial state is outside the maximal invariant terminal set (shown in dotted black) used with T-CNT. Note that the closed-loop state trajectories (shown in solid lines) converge to the target point (i.e. the origin) and the terminal sets converge to a set containing this target point. Although T-CNT and T-A2 yield similar closed-loop trajectories, T-DST is found to be initially infeasible starting from this initial condition. This indicates that the feasible region of T-DST is possibly smaller than those of the other three approaches. Although ellipsoidal terminal sets are utilized, the terminal sets appear as rectangles in Figure 3.1 since they are the Cartesian products of two one-dimensional ellipsoids.

Figure 3.2 compares the predicted state trajectories and terminal sets of T-A1, T-A2, T-CNT and T-DST when their optimal control problems are solved once starting from an initial condition $x_1 = 0.7$ and $x_2 = 0.3$, that is chosen such that all schemes are initially feasible. Although T-A1 and T-A2 lead to very similar predicted trajectories to that of T-CNT, T-DST results in a predicted trajectory with higher open-loop cost. This is mainly because the terminal set of T-DST is found to be relatively conservative, i.e. closer to the origin compared to those of T-A1 and T-A2. Notice that the terminal set of T-CNT is the maximal invariant terminal set shown in Figure 3.1 in dotted black. Note also that the terminal set of T-A1 and that of T-A2 (which is very small in Figure 3.2)

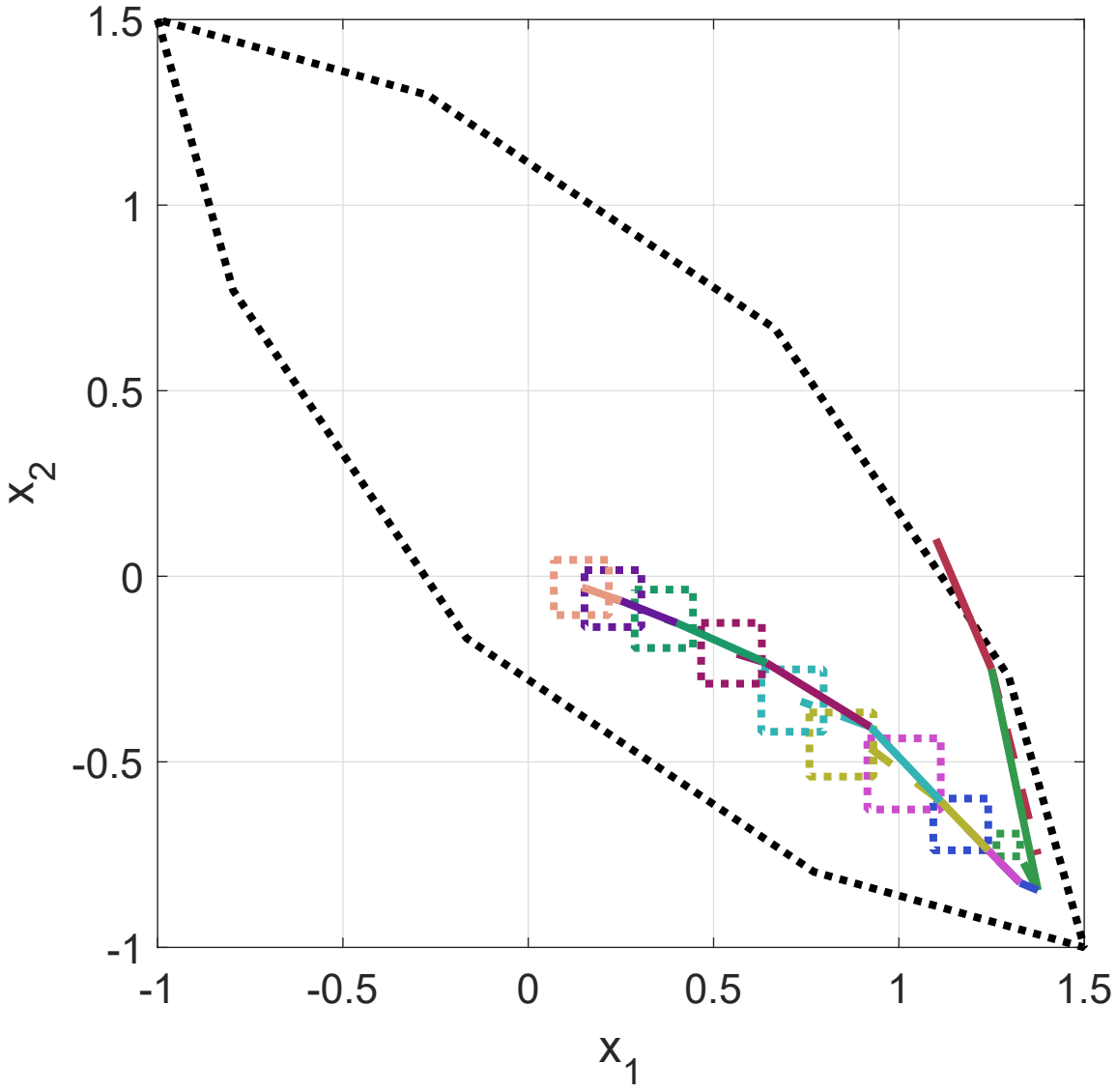

Figure 3.1: Evolution of predicted state trajectories and terminal sets for T-A1 when solved recursively for 10 timesteps (solid: first prediction step, dashed: second prediction step, dotted: terminal set).

are different since the cost functions of these MPC problems are not strongly convex with respect to the size and center of the terminal set.

3.7.2 Benchmark Example

In this example, we aim to compare the performance and computational complexity of T-A1, T-A2, T-CNT and T-DST using a benchmark example. We consider here a 7-subsystem interconnected system whose topology is shown in Figure 3.3. The dynamics of the i-th subsystem (partially adopted from Chapter 2 in [KC16]) is given by,

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t) + \sum_{j \in N_i \setminus i} A_{ij} x_j(t),$$

where,

$$A_i = \begin{bmatrix} 1.3 & 2 \\ 0 & 1.15 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$

Notice that these dynamics can be represented as in (3.1) in the obvious way. The i-th subsystem is subject to the constraints,

$$\begin{bmatrix} -8 \\ -8 \end{bmatrix} \leq x_i \leq \begin{bmatrix} 8 \\ 8 \end{bmatrix},$$

$$-1 \leq u_i \leq 1.$$

The cost function weights are given by $Q = I_{14}$, $R = 10I_7$ and $S = 10I_{14}$ where I_e is an identity matrix of size e. The matrix P is computed offline by solving (2.7). The target point x_r is chosen to be the origin.

First, we solve the OCP of each scheme $s \in \{\text{T-CNT, T-DST, T-A1, T-A2}\}$ centrally for one time step to compare the open-loop cost J_s^{olc} obtained by each scheme when solved to optimality. We use a prediction horizon $T = 5$ for all schemes. Figure 3.4 shows the suboptimality index between the centralized scheme T-CNT and the distributed schemes $s \in \{\text{T-DST, T-A1, T-A2}\}$, defined as,

$$\text{SI} = \frac{J_s^{olc} - J_{\text{T-CNT}}^{olc}}{J_{\text{T-CNT}}^{olc}},$$

for the initial conditions $x_{0_i} = [-0.2r \; 0.015r]^\top$ for all $r \in \{1, ..., 14\}$. The index of T-DST is shown only for $r \in \{1, ..., 5\}$ because this scheme is not feasible for the other initial conditions. As the initial condition moves further from the target point, the open-loop cost of T-DST becomes higher than those of T-A1 and T-A2. This demonstrates the conservatism imposed by T-DST compared to T-A1 and T-A2. Notice also that T-A2 is more conservative than T-A1 due to using diagonal dominance.

Second, we solve the MPC problems of T-DST, T-A1 and T-A2 using ADMM (see Section 3.5 for more details) to compare the performance and computational complexity

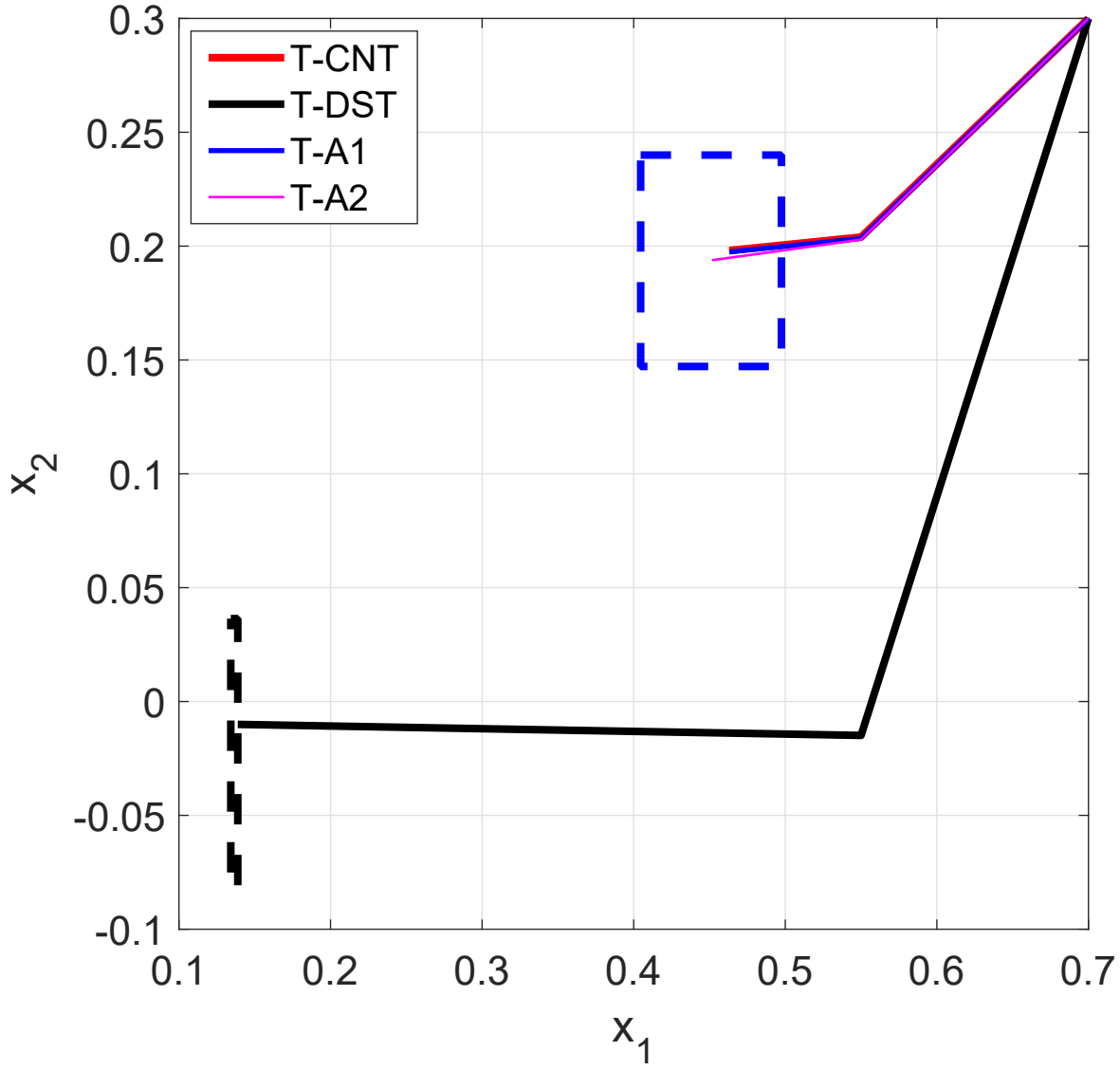

Figure 3.2: Comparison of the predicted state trajectories and terminal sets of T-A1, T-A2, T-CNT and T-DST.

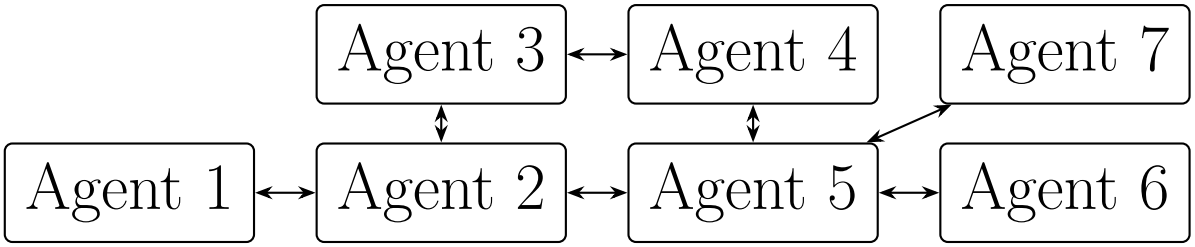

Figure 3.3: Topology of the considered interconnected system

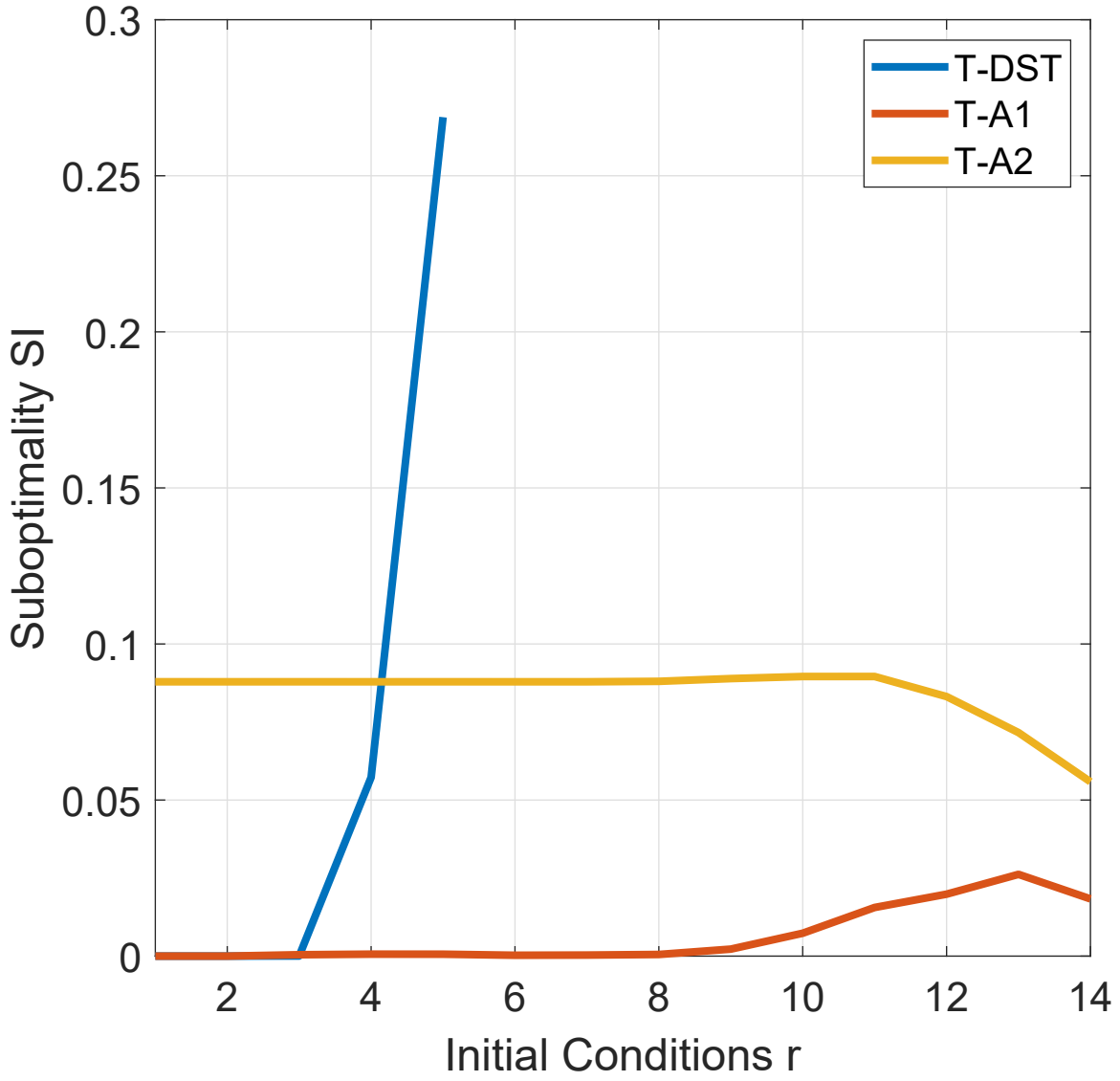

Figure 3.4: Suboptimality indexes of T-A1, T-A2 and T-DST with respect to T-CNT.

Table 3.1: Comparison of the four considered distributed MPC schemes in terms of the number of feasible initial conditions r_{fs}, the mean μ_c and standard deviation σ_c of J_s^{dif} and the mean μ_t and standard deviation σ_t of T_s^{admm}

	r_{fs}	μ_c	σ_c	μ_t	σ_t
DST_5-[Con+13]	5	0.0047	0.0031	0.2539	0.0063
DST_{20}-[Con+13]	12	0.0385	0.0087	0.5003	0.0258
RTI	14	0.0089	0.0058	2.8309	0.0962
RTI+DD	14	0.0050	0.0029	0.4455	0.0132

of the distributed schemes. We run the ADMM algorithm for $T_{sim} = 10$ simulation time steps with the ADMM parameter $\rho = 1000$ for 100 iterations and denote the closed-loop cost of the distributed scheme s obtained using ADMM by J_s^{admm}; note that J_s^{admm} converges only asymptotically to the closed-loop cost,

$$J_s^{clc} = \sum_{t=1}^{T_{sim}} \left[\|x^*(t) - x_r\|_Q^2 + \|u^*(t) - u_r\|_R^2 \right].$$

We denote the time required by the 5-th subsystem per time step to implement ADMM for scheme s by T_s^{admm}. We choose the 5-th subsystem as it has the largest number of neigbhours. Since using longer prediciton horizons is one way of reducing the conservatism imposed by T-DST, we consider two versions of T-DST; T-DST_5 with $N = 5$ and T-DST_{20} with $N = 20$.

Table 3.1 compares the distributed MPC schemes in terms of J_s^{admm} and T_s^{admm} by computing the mean and standard deviation of $J_s^{dif} = |J_s^{admm} - J_s^{clc}|/J_s^{clc}$ and T_s^{admm} over all initial conditions for which scheme s is feasible. Despite using longer prediction horizons, T-DST_{20} is still only feasible for $r \in \{1, ..., 12\}$. The scheme T-A2 has better convergence properties and smaller computational cost compared to T-A1, but the latter comes at a fraction of the open-loop cost (see Figure 3.4). Although the convergence properties of T-DST_5 are better than those of T-A1, they are similar to those of T-A2. All schemes, however, converge faster than T-DST_{20} possibly due to the larger number of shared variables in T-DST_{20}. The convergence properties of T-DST_{20} could potentially be improved by tuning the ADMM parameters, however T-DST_5 and T-DST_{20} still yield smaller feasible regions and possibly higher closed-loop costs. While the feasible region of T-DST_{20} can be enlarged by further increasing the prediction horizon, this would come at an additional computational cost, which is already higher than T-A2 (though not T-A1). We note that T-CNT requires less time ($\leq 0.1s$ per time step) than all distributed schemes (3.1) due to the ADMM iterations; the distributed schemes, however, generally have other advantages (e.g. robustness against failure).

We conclude this benchmark example by comparing the closed-loop performance when using the schemes; T-CNT, T-DST_5, T-DST_{20}, T-A1 and T-A2 in the case of non-

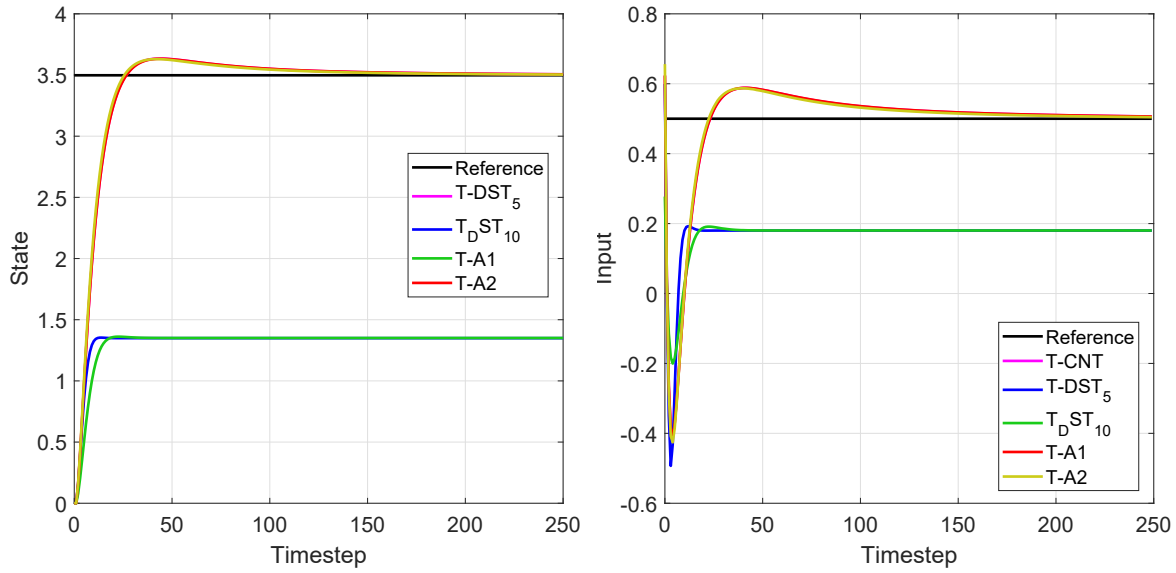

Figure 3.5: Closed-loop trajectories of the 5-th subsystem when using T-CNT, T-DST$_5$, T-DST$_{20}$, T-A1 and T-A2.

zero reference tracking. We have so far performed the comparison when the target point is the origin. In this last scenario, we require the system to track the target point $x_r = (I_n - A)^{-1}Bu_r$ where I_n is an identity matrix of size n and $u_r = [0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5\ 0.5]^\top$. Figure 3.5 shows the closed-loop state and input trajectories of the 5-th subsystem when using all the considered tracking MPC schemes. Note that the closed-loop trajectories of T-CNT is hidden behind those of T-A1. While the system is able to reach the desired target point when using T-CNT, T-A1 and T-A2, the system fails to reach the desired point when using T-DST$_5$ and T-DST$_{20}$. This is because the terminal ingredients of T-DST$_5$ and T-DST$_{20}$ are computed offline and hence, do not depend on the current state and the target point. This results in a terminal set which do not include the desired target point even when using a longer prediction horizon since the terminal set is not a function of the prediction horizon. On the other hand, T-A1 results in identical closed-loop trajectories to T-CNT and hence, no conservatism is imposed compared to T-CNT when using T-A1. This shows the effectiveness of computing the terminal ingredients online. While T-A2 is able to reach the desired target point, its trajectories are different from T-CNT and T-A1 and hence, it is more conservative due to using diagonal dominance.

3.7.3 Power Network Case Study

Finally, we evaluate the efficacy of the proposed schemes T-B1 and T-B2 by comparing them to T-A1 in terms of performance and computational cost; the above numerical results already suggest that these novel schemes have a larger feasible region and tends

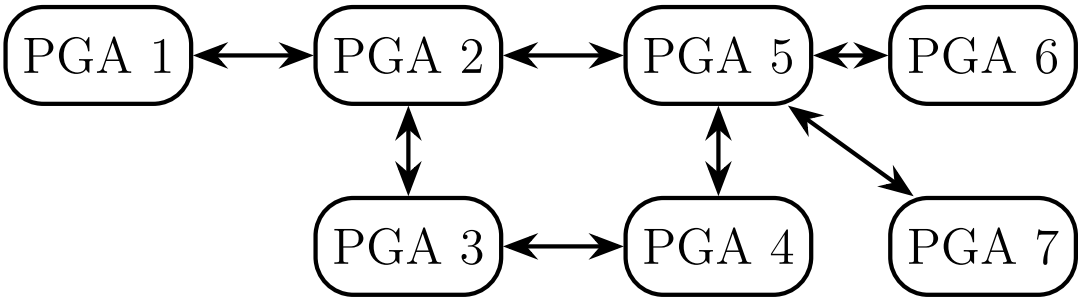

Figure 3.6: Power network topology

to show better performance compared to existing schemes in the literature.

We use a power network case study comprising a set of power generation areas (PGAs) [RFF13b]. Each PGA represents one subsystem in the network whose topology is shown in Figure 3.6. The state and input vectors of the i-th PGA are,

$$x_i = \begin{bmatrix} \Delta\theta_i, \\ \Delta\omega_i, \\ \Delta P_{M_i} - \Delta P_{L_i}, \\ \Delta P_{V_i} - \Delta P_{L_i} \end{bmatrix}, \quad u_i = \Delta P_{R_i} - \Delta P_{L_i},$$

where $\Delta\theta_i$ represents the angular displacement deviation, $\Delta\omega_i$ the angular velocity deviation, ΔP_{M_i} the mechanical power deviation, ΔP_{V_i} the steam valve position deviation, ΔP_{L_i} the load change deviation and ΔP_{R_i} the reference set power deviation. Following [RFF13a], the dynamics of the i-th PGA is given by,

$$\dot{x}_i = A_{ii}x_i + B_i u_i + \sum_{j \in N_i \setminus i} A_{ij}x_j,$$

where,

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\sum_{j\in\mathcal{N}_i} \frac{P_{ij}}{2H_i} & \frac{-D_i}{2H_i} & \frac{1}{2H_i} & 0 \\ 0 & 0 & \frac{-1}{T_{t_i}} & \frac{1}{T_{t_i}} \\ 0 & \frac{-1}{R_{t_i}T_{g_i}} & 0 & \frac{-1}{T_{g_i}} \end{bmatrix}, \; B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{T_{g_i}}, \end{bmatrix}, \; A_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{P_{ij}}{2H_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The PGA parameters H_i, D_i, R_{t_i}, T_{t_i} and T_{g_i} are listed in Table 3.1 for each PGA. The parameter P_{ij} describes the coupling between the two neighbours i and j where $P_{ij} = P_{ji}$, $P_{12} = 4$, $P_{23} = 2$, $P_{25} = 1$, $P_{34} = 2$ $P_{45} = 2$ $P_{56} = 3$ $P_{56} = 3$. For more information about the definitions of the aforementioned variables and parameters, we refer the reader to [RFF13a].

It is easy to verify that the continuous-time dynamics of each PGA has the same structure of the discrete-time dynamics (3.1). To preserve this structure after discretization, we use the Frobenius-norm-based discretization method [Sou+15] with a sampling time

Table 3.2: Power network parameters

PGA	H_i	D_i	R_{t_i}	T_{t_i}	T_{g_i}	p_i^{max}
1	12	0.05	0.7	0.65	0.1	0.5
2	10	0.0625	0.9	0.4	0.1	0.65
3	8	0.8	0.9	0.3	0.1	0.65
4	8	0.8	0.7	0.6	0.1	0.55
5	8	0.8	0.9	0.3	0.1	0.65
6	10	0.0625	0.9	0.4	0.1	0.65
7	12	0.05	0.7	0.65	0.1	0.5

of one second. Each PGA is subject to the constraints, $|\Delta\theta_i| \leq 0.1$ and $|\Delta P_{R_i}| \leq p_i^{max}$ where p_i^{max} of all PGAs are listed in Table 3.1. The weights of the cost function are given by $R_i = 0.1$, $S_i = \text{diag}(1000, 1000, 10, 10)$, $W_{ij}Q_{N_i}W_{ij}^\top = 0.99S_c$ if $i = j$ and $0.01S_c$ if $i \neq j$. The matrix P_i and the controller K_{N_i} are computed offline for T-B1 and T-B2 by solving (2.7). The prediction horizon is given by $T = 5$.

First, we compare the closed-loop cost J_s^{clc} for all $s \in \{\text{T-A1, T-B1, T-B2}\}$ for 25 randomly-generated target points x_{r_i} where,

$$J_s^{clc} = \sum_{i=1}^{M} \sum_{t=0}^{T_{sim}} \left(\|x_{N_i}^*(t) - x_{r_{N_i}}\|_{Q_i}^2 + \|u_i^*(t) - u_{r_i}\|_{R_i}^2 \right).$$

We solve the OCP of all considered schemes recursively for $T_{sim} = 10$ time steps and centrally to compare the costs when solved to optimality. It is found that all schemes yield almost the same closed-loop cost. Next, we solve the OCP of all schemes in a distributed fashion as described in Section 3.5. We denote the resulting closed-loop cost obtained by scheme s by J_s^{admm}. Note that J_s^{admm} converges to J_s^{clc} only asymptotically in the number of ADMM iterations. Hence, we choose a termination condition based on a pre-defined maximum time $T_{\max}$ to imitate the amount of computation time available for each sampling time.

Figure 3.7 shows a boxplot for the suboptimality $|J_s - J_s^{\text{admm}}|/J_s$ of each scheme s over the 25 randomly generated scenarios where $T_{\max} = 0.2r$ seconds and $r \in \{1, ..., 10\}$ (Equivalently 20% to 200% of the sampling time of 1 second). Note that T-B2 has better convergence properties than T-B1 which, in turn, outperforms T-A1. This is mainly because ADMM can perform the highest number of iterations with T-B2 and the lowest number of iterations with T-A1. This is obvious in Figure 3.8 which shows a boxplot for the number of ADMM iterations performed within $T_{\max}$ at each time step for each initial condition. The number of iterations of T-A1 is the lowest possibly due

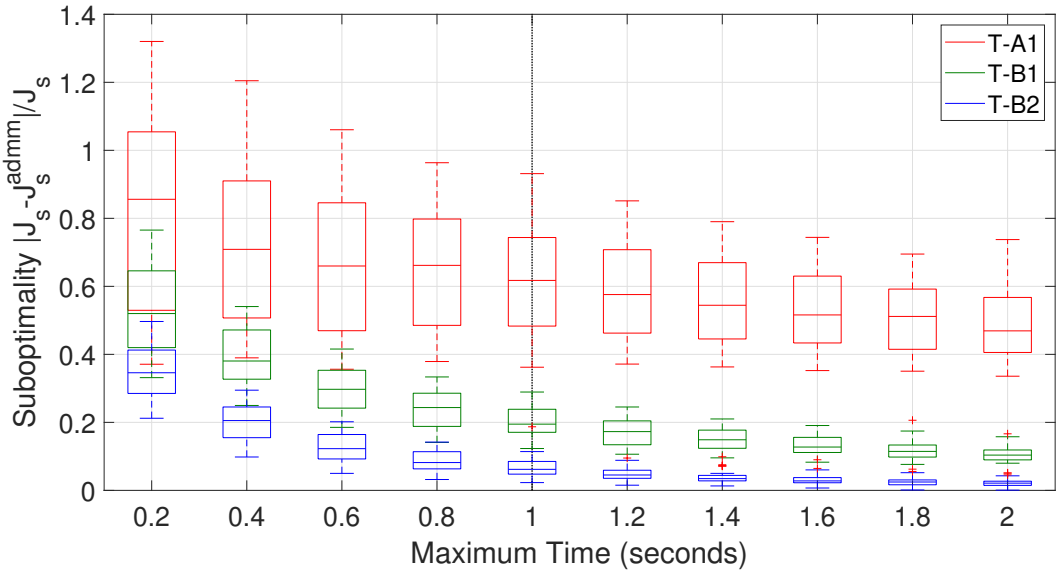

Figure 3.7: Median, interquartile range ($25\% - 75\%$), minimum, maximum and outliers of the suboptimality $|J_s^{admm} - J_s|/J_s$ vs the maximum computation time per timestep for the three schemes; the dotted black line refers to the used sampling time of 1 second.

to the larger number of constraints (in particular, LMIs) and decision variables. On the other hand, the number of iterations of T-B2 is the highest possibly because the resulting OCP is SOCP-representable. Notice that the communication and actuation time is not considered here since all computations are performed in simulations on a single processor. Note, however, that all schemes are using the same ADMM algorithm and communicating the same information. Hence, their communication demand is expected to be almost the same.

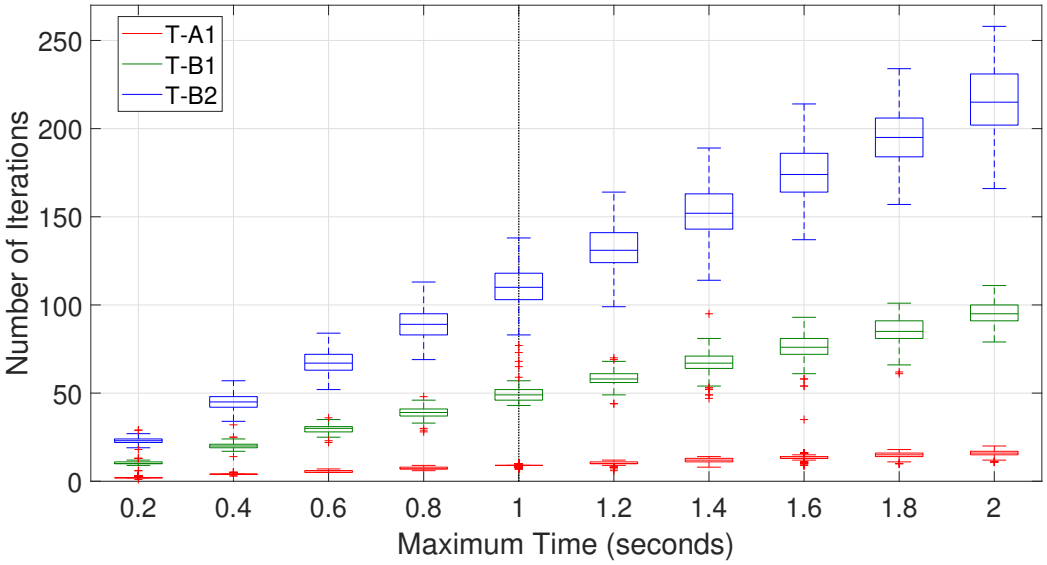

Figure 3.8: Median, interquartile range ($25\% - 75\%$), minimum, maximum and outliers of the number of ADMM iterations vs the maximum computation time per timestep for the three schemes; the dotted black line refers to the used sampling time of 1 second.

Part II

Plug-and-play Control

CHAPTER 4

Passivity-based Decentralized Control of Discrete-time Systems

4.1 Introduction

Passivity theory has proven to be useful for complex systems analysis and is found to be closely related to Lyapunov stability theory [Kha96; Kot+14; ERA09]. Hence, passivity-based control theory has been extensively used to design feedback controllers for linear and nonlinear systems [OJH97]. Such controllers have been used in many applications such as robotics [AOH07] and energy systems [MMW12]. Various efforts have been also devoted to developing robust [Bao+03] and adaptive [WL12] passivity-based controllers. Passivity theory has recently also contributed to developing distributed/decentralized control schemes for large-scale systems [Hat+15; Nah+20; DH20; KA17].

Many passivity-based control schemes are designed in continuous-time [Cho12; AMP16]. It is well-known, however, that the passivity properties of continuous-time systems are lost under discretization due to the resulting energy leakage of the zero-order-hold [Str+02]. Hence, various methods are developed in which passivity is preserved under discretization [De 02]. For example, passivity is preserved using relatively small sampling times in [Ois10]. Passivity is maintained in [KA07] by using the so-called inner-product equivalent sample and hold method. In [CF06], passivity is preserved by introducing a virtual output which is different from the actual output of the system. The aforementioned methods are developed for monolithic systems with a centralized controller.

In this chapter, we propose a passivity-based decentralized control scheme for a class of interconnected systems which can be decomposed into smaller dynamically-coupled subsystems. For each susbsystem, we synthesize a local state-feedback controller which depends on the states of the corresponding subsystem only, resulting in a decentralized architecture. We use passivity theory to synthesize these local controllers directly in discrete-time, unlike the aforementioned literature which considers passivating the continuous-time system and then discretizing it while maintaining passivity. The local controller of each subsystem is synthesized locally by solving a convex optimization

problem independently without the need to communicate with the other local controllers.

The optimization problem of each subsystem comprises conditions to ensure passivity of the corresponding subsystem. Passivity is ensured with respect to a virtual output which is different from the actual output of the subsystem. This virtual output is a combination of the actual outputs of the corresponding subsystem and its neighbours. Besides the passivizing control gains, the optimization problem is also solved for the storage function, the dissipation rate and the virtual output of the corresponding subsystem. Additional local constraints on the virtual output and the dissipation rate are imposed in the optimization problem of each subsystem to ensure that the local passivity of all subsystems guarantees the asymptotic stability of the overall system. The efficacy of the proposed controller is demonstrated by implementing it on a DC microgrid model.

One could also consider synthesising decentralised controllers in a centralised way. This would require the information about the global dynamics of the overall subsystem to be available centrally. Our approach obviates this need by also performing the synthesis of the decentralised controller in a decentralised manner. Furthermore, the proposed method does not suffer from the conservative performance associated with decentralized control approaches that treat the coupling terms as bounded disturbances (e.g. see [RFF13b]). Moreover, unlike methods that rely on communication and distributed optimisation (e.g. see [Con+16]), the proposed method requires minimal communication and safeguards the privacy of subsystems.

4.2 Networks with Laplacian Interconnection

We consider here the class of discrete-time interconnected systems which can be decomposed into a set of M subsystems described using the linear time-invariant (LTI) dynamics,

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i u_i(t) + F_i v_i(t), \\ y_i(t) &= C_i x_i(t), \\ v_i(t) &= \sum_{j \in \mathcal{N}_i^-} l_{ij} \left(y_j(t) - y_i(t) \right), \end{aligned} \tag{4.1}$$

where t is the time index, $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{m_i}$ are the state, input and output vectors of the i-th subsystem, respectively. For a given subsystem, the set $\mathcal{N}_i^-$ is the in-neighbour set, defined as the set of subsystems whose outputs affect the dynamics of this subsystem. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $F_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{m_i \times n_i}$ as well as the scalars l_{ij} are assumed to be known. We also assume that the pair (A_i, B_i) is controllable for all $i \in \{1, \ldots, M\}$. Note that we consider here the case in which the dimension of the output vectors of all subsystems is the same (i.e. $m_i = m_0$ for all $i \in \{1, \ldots, M\}$). Note also that this class of interconnected LTI systems is a special class of the general one considered in Part II. For the rest of this chapter, we ignore the

dependence of all variables on time t for simplicity and we refer to the dependence on time $t+1$ by $(\cdot)^+$. Defining the global state vector $x = [x_1^\top, \ldots, x_M^\top]^\top \in \mathbb{R}^n$, the global input vector $u = [u_1^\top, \ldots, u_M^\top]^\top \in \mathbb{R}^m$ and the global output vector $y = [y_1^\top, \ldots, y_M^\top]^\top \in \mathbb{R}^m$, the overall system dynamics is given by,

$$\begin{aligned} x^+ &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{4.2}$$

where the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ are obtained from the matrices and scalars in (4.1) in the obvious way.

The interconnection between subsystems can be represented by the weighted directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ where,

$$\begin{aligned} \mathcal{V} &= \{1, \ldots, M\}, \\ \mathcal{E} &\subseteq (\mathcal{V} \times \mathcal{V}), \\ \mathcal{W} &= \{l_{ij} \in \mathbb{R}, (i,j) \in \mathcal{E}\}, \end{aligned}$$

are the set of nodes, edges and weights of the graph $\mathcal{G}$, respectively. Each node in the graph represents a subsystem. An edge exists from the i-th node to the j-th node if the outputs of the i-th subsystem affect the dynamics of the j-th subsystem. The weight l_{ij} of this edge depends on the system parameters and indicates the strength of the coupling. For each node, the sets,

$$\begin{aligned} \mathcal{N}_i^+ &= \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}, \\ \mathcal{N}_i^- &= \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}, \\ \mathcal{N}_i &= \mathcal{N}_i^+ \cup \mathcal{N}_i^-, \end{aligned}$$

define the out-neighbour, in-neighbour and neighbour sets, respectively. The out-neighour set of a subsystem includes the subsystems whose dynamics are affected by the outputs of this subsystem. The Laplacian matrix $L \in \mathbb{R}^{M\times M}$ of the graph $\mathcal{G}$ describes the coupling structure between the subsystems and its entries are defined as,

$$L_{ij} = \begin{cases} \sum_{j\in\mathcal{N}_i} l_{ij}, & i = j, \\ -l_{ij}, & i \neq j,\ j \in \mathcal{N}_i^-, \\ 0, & i \neq j,\ j \notin \mathcal{N}_i^-. \end{cases}$$

The aim of this chapter is to synthesize for each $i \in \{1, \ldots, M\}$ a decentralized passivity-based control law,

$$u_i = K_i x_i, \tag{4.3}$$

where the control inputs of a given subsystem depend on the states of this subsystem only to ensure the asymptotic stability of the whole system. We also aim to synthesize these local controllers in a decentralized fashion. To this end, we recall the following definition.

Definition 4.1 ([Ali17]). *The discrete-time system (4.2) is strictly passive with respect to the input-output pair (u, y) if there exist a continuous storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ and a dissipation rate $\gamma : \mathbb{R}^n \to \mathbb{R}_{>0}$ with $\gamma(0) = 0$ such that,*

$$V(x^+) - V(x) \leq y^\top u - \gamma(x).$$

It is known that discrete time passivity generally requires feed-forward directly linking the input to the output of the system (a non-zero "D" matrix in linear systems [Ali17], or more generally zero relative degree [Nav05]). We note that such terms are not present in (4.1). We address this difficulty below through the introduction of virtual output variables.

4.3 Control Design

In this section, we synthesize the local control law (4.3) for all $i \in \{1, \ldots, M\}$ to asymptotically stabilize the overall system (4.2) in a decentralized fashion. For this purpose, we define for each subsystem the local virtual output,

$$z_i = y_i + D_i v_i = C_i x_i + D_i v_i,$$

where $D_i \in \mathbb{D}_{++}^{m_i}$ is a decision variable. The control synthesis is carried out by solving for each subsystem a local semidefinite program which guarantees that,

(I) The local controller (4.3) of a given subsystem strictly passivizes the dynamics (4.1) of this subsystem with respect to the local input-output pair (v_i, z_i).

(II) The strict passivity of all subsystems implies the asymptotic stability of the overall system, that is, asymptotic stability is achieved if the control input u_i of each subsystem passivizes this subsystem.

Note that the asymptotic stability of the overall system (4.2) (and not the asymptotic stability of the individual subsystems (4.1)) is considered in (II). This is because the coupling terms might destabilize the overall network even if each subsystem is asymptotically stable in the absence of coupling.

4.3.1 Passivity of Local Agents

First, we derive a matrix inequality for each subsystem which ensures (I) in the following lemma.

Lemma 4.1. *The i-th subsystem (4.1) is strictly passive with respect to the input-output pair (v_i, z_i) under the control law (4.3) if there exist $E_i \in \mathbb{S}^{n_i}_{++}$, $H_i \in \mathbb{D}^{n_i}_{++}$, $G_i \in \mathbb{R}^{m_i \times n_i}$ and $L_i \in \mathbb{D}^{m_i}_{++}$ such that,*

$$\begin{bmatrix} E_i & \frac{1}{2}E_iC_i^\top & (A_iE_i + B_iG_i)^\top & E_i \\ \frac{1}{2}C_iE_i & L_i & F_i^\top & 0 \\ (A_iE_i + B_iG_i) & F_i & E_i & 0 \\ E_i & 0 & 0 & H_i \end{bmatrix} \geq 0. \tag{4.4}$$

Proof. According to Definition 4.1, the closed-loop i-th subsystem under the controller $u_i = K_ix_i$ is strictly passive with respect to the input-output pair (v_i, z_i) if and only if there exist a positive semidefinite storage function $V_i(x_i)$ and a positive definite dissipation function $\gamma_i(x_i)$ such that,

$$V_i(x_i^+) - V_i(x_i) \leq v_i^\top z_i - \gamma_i(x_i). \tag{4.5}$$

The closed-loop dynamics of the i-th subsystem under the controller $u_i = K_ix_i$ is given by,

$$\begin{aligned} x_i^+ &= (A_i + B_iK_i)x_i + F_iv_i, \\ z_i &= C_ix_i + D_iv_i. \end{aligned} \tag{4.6}$$

Considering the positive definite quadratic functions $V_i(x_i) = \|x_i\|^2_{P_i}$ and $\gamma_i(x_i) = \|x_i\|^2_{\Gamma_i}$ with $P_i \in \mathbb{S}^{n_i}_{++}$ and $\Gamma_i \in \mathbb{D}^{n_i}_{++}$ and substituting (4.6) in (4.5) yield,

$$\begin{aligned} x_i^\top \left(P_i - (A_i + B_iK_i)^\top P_i(A_i + B_iK_i) - \Gamma_i\right) x_i \\ + 2v_i^\top \left(\frac{1}{2}C_i - F_i^\top P_i(A_i + B_iK_i)\right) x_i + v_i^\top \left(D_i - F_i^\top P_iF_i\right) v_i \geq 0. \end{aligned}$$

To satisfy the above inequality for all $x_i \in \mathbb{R}^{n_i}$ and $v_i \in \mathbb{R}^{m_i}$, we require that,

$$\begin{bmatrix} P_i - (A_i + B_iK_i)^\top P_i(A_i + B_iK_i) - \Gamma_i & \frac{1}{2}C_i^\top - (A_i + B_iK_i)^\top P_iF_i \\ \frac{1}{2}C_i - F_i^\top P_i(A_i + B_iK_i)^\top & D_i - F_i^\top P_iF_i \end{bmatrix} \geq 0. \tag{4.7}$$

Pre- and post-multiplying (4.7) by $\mathrm{diag}(P_i^{-1}, I_{m_i})$ where I_{m_i} is an identity matrix of size m_i and rearranging the resulting matrix inequality result in,

$$\begin{aligned} &\begin{bmatrix} P_i^{-1} - P_i^{-1}\Gamma_iP_i^{-1} & \frac{1}{2}P_i^{-1}C_i^\top \\ \frac{1}{2}C_iP_i^{-1} & D_i \end{bmatrix} \\ &\qquad - \begin{bmatrix} \left(A_iP_i^{-1} + B_iK_iP_i^{-1}\right)^\top \\ F_i^\top \end{bmatrix} P_i \begin{bmatrix} \left(A_iP_i^{-1} + B_iK_iP_i^{-1}\right) & F_i \end{bmatrix} \geq 0. \end{aligned}$$

Note that multiplying by $\mathrm{diag}(P_i^{-1}, I_{m_i})$ is valid since $P_i \in \mathbb{S}^{n_i}_{++}$. Applying the Schur complement and rearranging the resulting matrix inequality yield,

$$\begin{bmatrix} P_i^{-1} & \frac{1}{2}P_i^{-1}C_i^\top & (A_iP_i^{-1} + B_iK_iP_i^{-1})^\top \\ \frac{1}{2}C_iP_i^{-1} & D_i & F_i^\top \\ (A_iP_i^{-1} + B_iK_iP_i^{-1}) & F_i^\top & P_i^{-1} \end{bmatrix} - \begin{bmatrix} P_i^{-1} \\ 0 \\ 0 \end{bmatrix} \Gamma_i \begin{bmatrix} P_i^{-1} \\ 0 \\ 0 \end{bmatrix}^\top \geq 0. \tag{4.8}$$

Applying the Schur complement once more and defining the map,

$$\begin{aligned} E_i &= P_i^{-1}, \\ G_i &= K_i P_i^{-1}, \\ H_i &= \Gamma_i^{-1}, \\ L_i &= D_i, \end{aligned} \tag{4.9}$$

leads to (4.4). □

Although the matrix inequality (4.4) is not linear with respect to the variables P_i, K_i, Γ_i and D_i, it becomes linear with respect to the newly-defined variables E_i, G_i, H_i and L_i in the map (4.9). This map is bijective as long as P_i and Γ_i are nonsingular. These two conditions are satisfied by assumption in Lemma 4.1 since $P_i \in \mathbb{S}_{++}^{n_i}$ and $\Gamma_i \in \mathbb{D}_{++}^{m_i}$.

Although Definition 4.1 requires a positive semidefinite storage function $V_i(x_i) = x_i^\top P_i x_i$, a positive definite matrix P_i is used for three reasons; to be able to multiply (4.7) by $\operatorname{diag}(P_i^{-1}, I_{m_i})$, to define the bijective map (4.9) and because the matrices P_i are used later to define the Lyapunov function of the system. Furthermore, Definition 4.1 does not require a diagonal dissipation matrix Γ_i. However, it will become clear later that this additional constraint is required to reach a convex optimization problem.

Note that, under some assumptions, (4.7) is equivalent to the matrix inequality mentioned in [Kot+14] which ensures passivity of discrete-time systems. Note also that (4.7) demonstrates why strict passivity of the i-th subsystem with respect to the actual output y_i is not possible. If $D_i = 0$, the matrix inequality can only be satisfied if $F_i = 0$ and $C_i = 0$, that is only if the subsystems are decoupled and their outputs are always equal to zero. This motivates the introduction of the virtual output z_i above.

4.3.2 Stability of Overall Network

To ensure the asymptotic stability of the overall interconnected system under the strict passivity of all subsystems with respect to their virtual outputs, we introduce the following lemma. In the sequel, we use $P = \operatorname{diag}(P_1, \ldots, P_M)$, $\Gamma = \operatorname{diag}(\Gamma_1, \ldots, \Gamma_M)$, $C = \operatorname{diag}(C_1, \ldots, C_M)$ and $D = \operatorname{diag}(D_1, \ldots, D_M)$.

Lemma 4.2. *Assume that for all $i \in \{1, \ldots, M\}$, the i-th subsystem (4.1) under the controller $K_i x_i$ is strictly passive with respect to the input-output pair (v_i, z_i). The closed-loop dynamics of the overall system (4.2) is asymptotically stable if there exists $D \in \mathbb{D}_{++}^n$ such that,*

$$\begin{bmatrix} \Gamma - \epsilon_0 I_n + C^\top \tilde{L} C & C^\top \tilde{L}^\top \\ \tilde{L} C & D^{-1} \end{bmatrix} \geq 0, \tag{4.10}$$

where $\tilde{L} \in \mathbb{R}^{mM \times mM}$ consists of the submatrices $\tilde{L}_{ij} = l_{ij} I_{m_i} \in \mathbb{R}^{m_i \times m_i}$, I_n is an identity matrix of size n and ϵ_0 is a positive scalar.

Proof. The strict passivity of the i-th subsystem (4.1) under the controller $K_i x_i$ with respect to the input-output pair (v_i, z_i) implies that,

$$V_i(x_i^+) - V_i(x_i) \leq z_i^\top v_i - \gamma_i(x_i). \tag{4.11}$$

By defining the Lyapunov function,

$$V(x) = \sum_{i=1}^{M} V_i(x_i),$$

one can verify that $V(x) = \|x\|_P^2$. Note that $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$ since $P \in \mathbb{S}_{++}^n$. Summing up (4.11) for all $i \in \{1, \ldots, M\}$ leads to,

$$V(x^+) - V(x) = \sum_{i=1}^{M} V_i(x_i^+) - \sum_{i=1}^{M} V_i(x_i) \leq \sum_{i=1}^{M} z_i^\top v_i - \sum_{i=1}^{M} \gamma_i(x_i).$$

By defining the global function,

$$\gamma(x) = \sum_{i=1}^{M} \gamma_i(x_i),$$

one can verify that $\gamma(x) = \|x\|_\Gamma^2$, and hence,

$$V(x^+) - V(x) \leq z^\top v - x^\top \Gamma x,$$

where $z = [z_1^\top, ..., z_M^\top]^\top$ and $v = [v_1^\top, ..., v_M^\top]^\top$.

Recall that $z_i = C_i x_i + D_i v_i$ and $v_i = \sum_{j \in \mathcal{N}_i} l_{ij}(C_j x_j - C_i x_i)$ and hence,

$$\begin{aligned} z &= Cx + Dv, \\ v &= -\tilde{L} C x. \end{aligned}$$

In this case, we reach,

$$V(x^+) - V(x) \leq -x^\top (\Gamma + C^\top \tilde{L} C - C^\top \tilde{L}^\top D \tilde{L} C) x.$$

To guarantee the asymptotic stability of the closed loop dynamics of the overall system, it suffices to ensure that,

$$\Gamma + C^\top \tilde{L} C - C^\top \tilde{L}^\top D \tilde{L} C \geq \epsilon_0 I_n. \tag{4.12}$$

Since $D \in \mathbb{D}_{++}^m$ by assumption, the Schur Complement is applicable to (4.12) and yields (4.10). □

The matrix D_i appears in the diagonal terms in (4.4). Thus, the larger the eigenvalues of D_i are, the more likely the i-th subsystem is strictly passive. On the other side, $D^{-1} = \text{diag}(D_1^{-1}, \ldots, D_M^{-1})$ appears in the diagonal terms in (4.10). Thus, the larger the eigenvalues of D_i are, the less likely that the strict passivity of all subsystems implies the asymptotic stability of the overall system. In conclusion, the feed-forward decision variables D_i for all $i \in \{1, \ldots, M\}$ encode a trade-off between the strict passivity of the subsystems and the asymptotic stability of the overall system and can be chosen neither arbitrarily large nor arbitrarily small.

4.3.3 Computation of Stabilizing Controller

Next, we note that the matrix inequality (4.10) is nonlinear in the actual decision variables Γ and D as well as the newly-defined variables in (4.9), leading to a nonconvex optimization problem. Moreover, (4.10) couples all the subsystems because of the presence of the Laplacian matrix L in the off-diagonal terms. Thus, if this inequality is utilized, it has to be incorporated in the optimization problems of all subsystems implying that the synthesis is no longer decentralised.

To address these difficulties, we define the matrices $U = \tilde{L}C \in \mathbb{R}^{m \times n}$, $W = C^\top \tilde{L}^\top \in \mathbb{R}^{n \times m}$, $U_i \in \mathbb{R}^{m_i \times n}$ and $W_i \in \mathbb{R}^{n_i \times m}$ such that,

$$U = [U_1^\top, \ldots, U_M^\top]^\top,$$
$$W = [W_1^\top, \ldots, W_M^\top].$$

In the sequel, we denote the diagonal element in the j-th row of a matrix T_i by $[T_i]_j$ and the 1-norm of the j-th row by $|T_i|_j$.

Theorem 4.1. *The local control laws (4.3) for all $i \in \{1, \ldots, M\}$ asymptotically stabilize the global dynamics of the overall system (4.2) if for all $i \in \{1, \ldots, M\}$,*

$$E_i \in \mathbb{S}_{++}^n, \tag{4.13a}$$
$$H_i \in \mathbb{D}_{++}^n, \tag{4.13b}$$
$$L_i \in \mathbb{D}_{++}^m, \tag{4.13c}$$
$$\textit{LMI (4.4) holds}, \tag{4.13d}$$
$$[H_i]_j \leq \frac{1}{|W_i|_j + \epsilon_0}, \ \textit{for all } j \in \{1, ..., n_i\}, \tag{4.13e}$$
$$[L_i]_k \leq \frac{1}{|U_i|_k}, \ \textit{for all } k \in \{1, ..., m_i\} \ \textit{s.t.} \ |U_i|_k > 0. \tag{4.13f}$$

Proof. Considering the map (4.9), the conditions $P_i \in \mathbb{S}_{++}^{n_i}$, $\Gamma_i \in \mathbb{D}_{++}^{n_i}$ and $D_i \in \mathbb{D}_{++}^{m_i}$ are guaranteed because of the constraints (4.13a), (4.13b) and (4.13c). Thus, the strict

passivity of the i-th subsystem is ensured under the controller $K_i x_i$ using (4.13d) as indicated by Lemma 4.1.

Since $\Gamma_i \in \mathbb{D}_{++}^{n_i}$ and $D_i \in \mathbb{D}_{++}^{m_i}$, then $[\Gamma_i]_j$ for all $j \in \{1, ..., n_i\}$ and $[D_i]_k$ for all $k \in \{1, ..., m_i\}$ are invertible. Consequently, constraint (4.13f) implies that for all $k \in \{1, ..., m_i\}$,

$$[D_i^{-1}]_k = [L_i^{-1}]_k \geq |U_i|_k.$$

Similarly, constraint (4.13e) implies that for all $j \in \{1, ..., n_i\}$,

$$[\Gamma_i]_j - \epsilon_0 = [H_i^{-1}]_j - \epsilon_0 \geq |W_i|_j.$$

Considering the definitions of U_i and W_i in the above two inequalities and using diagonal dominance, one can verify that,

$$\begin{bmatrix} \Gamma - \epsilon_0 I_n & C^\top \tilde{L}^\top \\ \tilde{L} C & D^{-1} \end{bmatrix} \geq 0. \tag{4.14}$$

Since L is a Laplacian matrix, then $L \in \mathbb{S}_+^M$ by definition and hence, $\tilde{L} \in \mathbb{S}_+^{mM}$. Therefore, (4.14) implies (4.10). Hence, the strict passivity of all subsystems ensured by (4.13d) for all $i \in \{1, \ldots, M\}$ implies the asymptotic stability of the overall system using (4.13e) and (4.13f). □

Notice that the constraints in (4.13) are convex with respect to the decision variables E_i, G_i, H_i and L_i. Moreover, there are no common variables between the constraints of any two subsystems. In other words, each subsystem has its own variables which are not shared with other subsystems. Thus, adding any local convex function $f_i(\cdot)$ as a cost leads to a convex optimisation problem that can be solved independently by each subsystem. Indeed the cost function can be different for each subsystem, to reflect local preferences. It is also worth mentioning that other alternatives which ensure passivity of discrete-time systems, such as the KYB conditions in [Ali17], the matrix inequality in [Kot+14] and the matrix inequality (4.8) do not yield a convex program when replacing (4.4) in Theorem 4.1.

To solve the semidefinite program of one subsystem, the corresponding matrices U_i and W_i are required. These matrices only depend on the weights l_{ij} (which describe how this subsystem is affected by its in-neighbours) and l_{ji} (which describe how this subsystem affects its out-neighbours) as well as the matrices C_i of this subsystem and its neighbours. Thus, the semidefinite program of each subsystem requires limited information from its neighbouring subsystems. For many systems the physics of the underlying process imply that connections between subsystems are naturally symmetric ($l_{ij} = l_{ji}$); this is the case for DC microgirds considered below, but also for, e.g. thermal dynamics in buildings, action-reaction forces in mechanical systems, etc. In this case the Laplacian is symmetric and the information necessary for performing the decentralised synthesis is automatically almost available to each subsystem.

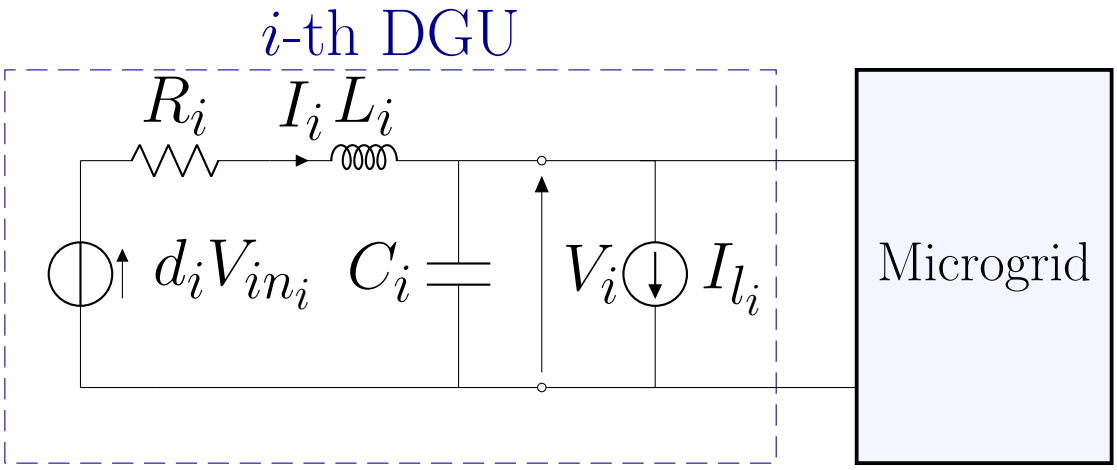

Figure 4.1: Electric circuit representing the averaged model of a DC/DC buck converter connected to the microgrid.

4.4 Simulation Results

We evaluate the proposed control scheme by applying it to a DC microgrid network comprising a number of dynamically-coupled distributed generation units (DGUs).

4.4.1 DC Microgrid Model

Each DGU consists of a DC voltage source and a buck converter as shown in Figure 4.1. The voltage source of the i-th DGU represents a renewable energy source which provides a constant voltage V_{in_i}. The buck converter of the i-th DGU is represented by an RLC circuit with a resistance R_i, an inductance L_i and a capacitance C_i. A switch is used to regulate the output voltage of the DGU by appropriately selecting the duty cycle d_i. The i-th DGU is assumed to support a constant current load which requires a current I_{l_i}. Two neighbouring DGUs i and j are connected through a resistive line with a resistance of R_{ij}.

For every DGU, let V_i and I_i be the output voltage and the converter current, respectively. To avoid any steady state error in the output voltages, each DGU is augmented with an integrator whose state is s_i. Considering the state vector $x_i = [V_i, I_i - I_{l_i}, s_i]^\top$ and the input $u_i = d_i - \frac{R_i I_{l_i}}{V_{in_i}}$, the averaged dynamics of the i-th DGU is given by,

$$\begin{aligned} \dot{x}_i &= A_{c_i} x_i + B_{c_i} u_i + F_{c_i} v_i, \\ y_i &= C_i x_i, \\ v_i &= \sum_{j \in \mathcal{N}_i} l_{ij}(y_j - y_i), \end{aligned} \tag{4.15}$$

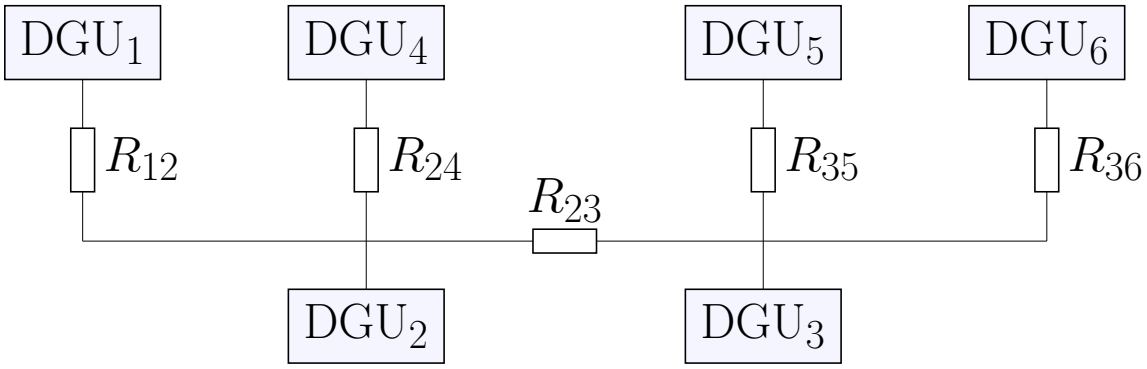

Figure 4.2: Microgrid structure.

where α_i is the integrator coefficient,

$$A_{c_i} = \begin{bmatrix} 0 & \frac{1}{C_i} & 0 \\ -\frac{1}{L_i} & -\frac{R_i}{L_i} & 0 \\ \alpha_i & 0 & 0 \end{bmatrix},$$

$$B_{c_i} = \begin{bmatrix} 0 \\ \frac{V_{in_i}}{L_i} \\ 0 \end{bmatrix}, \quad F_{c_i} = \begin{bmatrix} \frac{1}{C_i} \\ 0 \\ 0 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad l_{ij} = \frac{1}{R_{ij}}.$$

As mentioned above, DC Microgrids are represented using undirected graphs where $l_{ij} = l_{ji}$ and $\mathcal{N}_i^- = \mathcal{N}_i^+$. We consider here the six-DGU network given in [BN18] whose structure is shown in Figure 4.2.

4.4.2 Discretization Methods

The first difficulty to be addressed is time discretisation. Although the considered model (4.1) and the microgrid model (4.15) have the same structure, (4.1) is in discrete time and (4.15) is in continuous time. When applying exact discretization to (4.15), the matrices of the resulting discrete-time model are dense, compromising the distributed structure. When using the Euler discretization method, the model structure is maintained, however the model accuracy can be poor. Recently, considerable effort has been devoted to finding discrete-time models of good accuracy that preserve the continuous-time model structure [Sou+15; FCS13]. Here we compare four methods which preserve the model structure by computing the root mean squared error between the voltages and currents of all DGUs obtained by these methods and those obtained by exact discretization for impulsive, step and random inputs. We use a sampling time $T_s = 10^{-5}$ seconds and select the parameter $\alpha_i = \frac{1}{T_s}$ for all DGUs.

Table 4.1: The root mean squared error between the output voltages V_i and converter currents I_i of each model and those of the exact model in the case of an impulsive input, a step input and a random input.

	SN	FN	AM	LM
Impulse	8.45	3.83	5.06	0.06
Step	69.31	30.46	39.68	0.48
Random	34.81	19.2	21.46	0.33

We denote the first two methods by SN and FN. The methods SN and FN compute approximate discrete-time models by solving an optimization problem which minimizes, respectively, the spectral norm and the Frobenius norm of the error between the matrices of the exact discrete-time model and those of the approximate model [Sou+15]. The approximate model is constrained to respect the sparsity pattern and maintain the stability properties of the continuous-time model. We denote the other two methods by AM and LM. Besides sampling and holding the control inputs, the methods AM and LM sample and hold, respectively, the coupling terms $\sum_{j \in \mathcal{N}_i} \frac{1}{R_{ij}} y_j$ [FCS13] and the vector v_i in (4.1). Table 4.1 shows that this last method leads to the highest accuracy while maintaining the desired structure; this method is therefore selected for our controller design.

4.4.3 Controller Implementation

To compute the corresponding local controller, each DGU solves its local optimization problem. We solve these local problems using MATLAB [MAT18] with YALMIP [Löf04] and MOSEK [ApS19] on a computer equipped with 16-GB RAM and a 1.9-GHz Intel core i7-8550U processor. Although the LM model is used in the optimization problem, the resulting controller is applied to the exact discrete-time model to evaluate the closed-loop performance in simulation. We compare the proposed decentralized controller to the centralized discrete-time linear quadratic regulator (LQR). The LQR control gain matrix is given by,

$$K_c = -(B^\top P_c B + R)^{-1} B^\top P_c A,$$

where the matrix P_c is the unique positive-definite solution of the Riccati equation,

$$P_c = A^\top P_c A + Q - A^\top P B (B^\top P_c B + R)^{-1} B^\top P A.$$

The matrices Q and R are chosen to be the identity matrices I_n and I_m of size n and m, respectively.

While computing the local controller of each subsystem, we evaluate three different cost functions in the local optimization problems which include the constraints in (4.13).

The first one is,

$$f_i^a = 0 \text{ for all } i \in \{1, \dots, M\},$$

which is just used to find a feasible solution. The second function is,

$$f_i^b = \text{trace}(H_i) \text{ for all } i \in \{1, \dots, M\},$$

which aims to maximize the dissipation rate which is an indication of maximizing the passivity margin. Finally, the third cost function is,

$$f_i^c = \|E_i - E_{c_i}\|_F \text{ for all } i \in \{1, \dots, M\},$$

which tries to mimic the behaviour of the LQR by minimizing the Frobenius norm between the matrices E_i and $E_{c_i} = T_i P_c^{-1} T_i^\top$ where $T_i \in \{0, 1\}^{2\times 2}$ selects the diagonal submatrix corresponding to the i-th subsystem.

We perform 100 Monte Carlo simulations with the reference voltages changing initially from $50V$ to a random value between $49.95V$ and $50.05V$ and the load currents changing initially from $5A$ to a random value between $2.5A$ and $7.5A$. The goal is to regulate the output voltage of each DGU to the corresponding reference V_{r_i} in the presence of these loads.

To converge to the desired reference, the feedforward terms,

$$u_{f_i} = -\frac{V_{r_i}}{V_{in_i}} + K_i \begin{bmatrix} -V_{r_i} \\ 0 \\ 0 \end{bmatrix},$$

$$s_{f_i} = -V_{r_i},$$

are added to the control input u_i and the dynamics of the integrator state s_i, respectively. Although these terms lead to shifted coordinates, they change neither the system matrices nor the Laplacian matrix. Hence, neither passivity nor stability are affected since the constraints in (4.13) remain the same and hence, still hold. This matches the fact mentioned in [Jay+07] that an LTI system with shifted coordinates is passive if its associated system with non-shifted coordinates is passive. Note that the control input of one DGU is a function of its local variables and parameters only (i.e. gains, states and references).

For each simulation, the magnitude of the tracking error,

$$e = \sqrt{\sum_{k=0}^{T} \sum_{i=1}^{6} \left(\Delta V_i^{k^2} + \Delta I_i^{k^2} + \Delta s_i^{k^2} + \Delta u_i^{k^2} \right)},$$

is computed where,

$$\Delta V_i^k = V_i^k - V_{r_i},$$
$$\Delta I_i^k = I_i^k - I_{r_i},$$
$$\Delta s_i^k = s_i^k - s_{r_i},$$
$$\Delta u_i^k = u_i^k - u_{r_i},$$

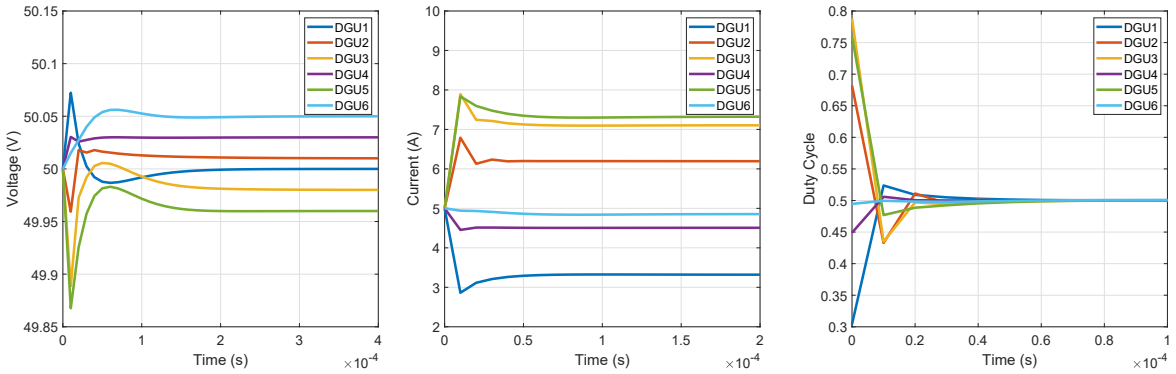

Figure 4.3: Output voltages (left), converter currents (middle) and duty cycles (right) of all DGUs when the cost $f_i^c(\cdot)$ is used for all $i \in \{1, \ldots, 6\}$.

T is the simulation time, I_{r_i}, s_{r_i} and u_{r_i} are the steady state values of the corresponding variables. We denote the magnitudes of the proposed controller with the cost functions f_i^a, f_i^b and f_i^c by e_{pbc}^a, e_{pbc}^b and e_{pbc}^c, respectively, and that of the LQR controller by e_{lqr}.

The closed-loop performance of one test scenario which uses the function $f_i^c(\cdot)$ is shown in Figure 4.3 that shows the output voltage V_i, converter current I_i and duty cycle d_i of all DGUs. In this scenario, the reference voltages are chosen to be $V_{r_i} = 50 + 0.01(i-1)(-1)^i$ where $i \in \{1, ..., 6\}$. Despite the uncertainties due to the discretization errors, the output voltages converge to the desired reference value. This shows the inherent robustness of our approach against discretization errors. Note that the other cost functions result in similar behaviours.

The mean μ_J^k and standard deviation σ_J^k of the suboptimality index,

$$J^k = \frac{e_{pbc}^k - e_{lqr}}{e_{lqr}} \text{ for all } k \in \{a, b, c\},$$

are given in Table 4.2. It is found that f_i^c results in a relatively good performance (i.e. small μ_J^c and σ_J^c). This could be because f_i^c tries to mimic the behavior of the LQR. We conjecture that suboptimality occurs because the control gains are not exactly the same since the proposed controller is decentralized whereas LQR is centralized. On the other hand, we also conjecture that f_i^b results in poor performance (i.e. large μ_J^b and σ_J^b) since it only maximizes the passivity margin.

Table 4.2 also shows the minimum eigenvalue $\underline{\lambda}^k$ of the dissipation rate matrix Γ which indicates how strict passivity is for each cost function. This eigenvalue can be considered as a measure of robustness, for example against uncertainties due to discretization errors that may lead to loss of passivity and stability. The function f_i^b results in a large eigenvalue, as opposed to f_i^c. Thus, we conjecture that f_i^b leads to a more robust controller compared to f_i^c.

When exploring the effect of the parameter ϵ_0, it is found that the system is under-

Table 4.2: Suboptimality mean J_m^k, suboptimality standard deviation J_s^k and minimum eigenvalue $\underline{\lambda}$ of the matrix Γ of the proposed controller for different cost functions.

	f_i^a	f_i^b	f_i^c
μ_J	0.05	0.13	0.02
σ_J	0.02	0.02	0.01
$\underline{\lambda}$	0.014	0.02	0.01

damped for small ϵ_0 and overdamped for large ϵ_0 when using $f_i^a(\cdot)$. In addition, larger ϵ_0 leads to slower convergence with larger overshoot. On the other hand, the performance is almost the same when using $f_i^b(\cdot)$ and $f_i^c(\cdot)$. For all cost functions, the optimization problems become infeasible for very large ϵ_0.

CHAPTER 5

Plug-and-play Distributed Model Predictive Control

5.1 Introduction

Plug-and-play (PnP) control schemes have received considerable attention for the control of varying-topology networks, where agents occasionally join and leave the network [Sto09; BTS11; Gol+18; Yua+19]. The need to integrate state and input constraints has also motivated research into combining MPC schemes and PnP algorithms for ensuring asymptotic stability and constraint satisfaction of such networks. Decentralized PnP MPC schemes were developed applicable mainly to weakly coupled networks, as they consider all the dynamic coupling terms as disturbances (e.g. see [RFF13b; RFF14; RF15]).

For the regulation of strongly coupled networks, distributed PnP MPC schemes were developed based on a two-phase PnP algorithm (e.g. see [Zei+13; Hu+18; Le +17]). Once a PnP request is received, the *redesign phase* computes offline the MPC terminal ingredients for a specific set of subsystems to ensure the asymptotic stability of the new network. Then, the *transition phase* computes online for the old network a steady state at which the PnP operation can take place and from which the MPC problem is initially feasible for the new network. If the optimization problems of both phases are feasible, the PnP request is approved and the system is steered to the computed steady state for the PnP operation to take place. Distributed MPC is then applied to the new network to compute the optimal control inputs of the new network in a distributed fashion.

We propose here a novel PnP algorithm for a class of interconnected systems with Laplacian interconnections. Our scheme is inspired by [Zei+13], but addresses reference tracking problems, instead of regulation problems. Unlike [Zei+13], the proposed PnP algorithm updates the local terminal sets online in the transition phase as discussed in Part I. This yields larger feasible regions in the transition phase where more PnP requests can be accommodated and also avoids the computation of a global terminal set which might require a central coordinator.

Furthermore, the redesign phase makes use of passivity theory to update the local terminal costs and controllers as discussed in Chapter 4. Hence, each subsystem whose terminal ingredients are updated solves only two optimization problems. This accelerates the PnP operation compared to [Zei+13], where an iterative procedure is used in the redesign phase. Unlike Chapter 4, we consider here a more general class of interconnected systems whose coupling can be described with more than one Laplacian.

5.2 Varying Topology Networks

We consider here interconnected systems with a distributed structure which can be decomposed into a set of M subsystems. Two subsystems are assumed to be neighbours if they are dynamically coupled through the states. The set of neighbours of the i-th subsystem is denoted by $\mathcal{N}_i$ and is always assumed to include the i-th subsystem. The notation $\mathcal{N}_i \backslash i$ refers to the set of neighbours of the i-th subsystem excluding the i-th subsystem itself. The i-th subsystem is described using the linear time-invariant dynamics,

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i u_i(t) + \sum_{s=1}^{p_i} F_{i,s} v_{i,s}(t), \\ v_{i,s}(t) &= \sum_{j \in \mathcal{N}_i \backslash i} l_{ij,s}(y_{j,s}(t) - y_{i,s}(t)), \\ y_{i,s}(t) &= C_{i,s} x_i(t), \end{aligned} \tag{5.1}$$

where $t \in \mathbb{Z}_+$ is the time index, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the state, input and output vectors of the i-th subsystem and $y_{i,s}$ refers to the s-th output of the i-th subsystem. The matrices A_i, B_i, vectors $F_{i,s}$, $C_{i,s}$ and scalars $l_{ij,s}$ are known and have appropriate dimensions. We assume that all subsystems have the same output dimension p_0 (i.e. $p_i = p_0$ for all $i \in \{1, \dots, M\}$). We also assume that the interconnection due to the s-th input-output pair $(v_{i,s}, y_{i,s})$ is described using the graph $\mathcal{G}_s$ with the Laplacian $L_s \in \mathbb{R}^{M \times M}$ whose entries are given by,

$$L_{ij,s} = \begin{cases} \sum\limits_{j \in \mathcal{N}_i \backslash i} l_{ij,s}, & i = j, \\ -l_{ij,s}, & i \neq j,\ j \in \mathcal{N}_i, \\ 0, & i \neq j,\ j \notin \mathcal{N}_i. \end{cases}$$

Note that if $j \in \mathcal{N}_i$, but the s-th output of the j-th subsystem does not affect the dynamics of the i-th subsystem, then $l_{ij,s}$ is set to zero.

The dynamics of the i-th subsystem in (5.1) can be also given by,

$$\begin{aligned} x_i(t+1) &= A_{N_i} x_{N_i}(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t), \end{aligned} \tag{5.2}$$

where $x_{N_i} \in \mathbb{R}^{n_{N_i}}$ is a state vector comprising the states of the subsystems in the set $\mathcal{N}_i$ and A_{N_i} can be constructed using A_i, $C_{i,s}$, $F_{i,s}$ and $l_{ij,s}$ in the obvious way. Hence, the considered class of interconnected systems is a special case of that considered in Part I. However, it is still more general than that considered in Chapter 4 where the graphs $\mathcal{G}_s$ for all $s \in \{1, \dots, p_0\}$ are the same and have the same Laplacian. The states and inputs of the i-th subsystem are constrained to polytopic sets as follows,

$$\begin{aligned} x_i(t) \in \mathcal{X}_i &= \{x_i \in \mathbb{R}^{n_i} : G_i x_i \leq g_i\}, \\ u_i(t) \in \mathcal{U}_i &= \{u_i \in \mathbb{R}^{m_i} : H_i u_i \leq h_i\}, \end{aligned} \tag{5.3}$$

where the matrices $G_i \in \mathbb{R}^{q_i \times n_i}$, $H_i \in \mathbb{R}^{r_i \times m_i}$ and the vectors $g_i \in \mathbb{R}^{q_i}$, $h_i \in \mathbb{R}^{r_i}$ are known. Note that there are no constraints coupling the states and inputs of different subsystems. Therefore, the considered constraints are again a special case of those considered in Part I which also considers coupling constraints through the states. However, this case is also more general than that in Chapter 4 where no constraints are considered.

Assuming that the i-th subsystem is required to track the target point $x_{r_i} \in \mathcal{X}_i$ whose corresponding input is $u_{r_i} \in \mathcal{U}_i$, we choose the cost function,

$$J_i(t) = \sum_{k=0}^{N-1} \left(\|x_i(k|t) - x_{e_i}\|_{Q_i}^2 + \|u_i(k|t) - u_{e_i}||_{R_i}^2 \right) + \|x_i(N|t) - x_{e_i}\|_{P_i}^2 + \|x_{e_i} - x_{r_i}\|_{S_i}^2,$$

where $x_i(k|t)$ and $u_i(k|t)$ are, respectively, the k-step ahead predicted state and input vectors of the i-th subsystem at time t, N is the prediction horizon and (x_{e_i}, u_{e_i}) is an artificial equilibrium of the i-th subsystem. Similar to Part I, the cost function matrices are chosen to satisfy $Q_i \in \mathbb{S}_{++}^{n_i}$, $R_i \in \mathbb{S}_{++}^{m_i}$, $P_i \in \mathbb{S}_{++}^{n_i}$ and $S_i \in \mathbb{S}_{++}^{n_i}$ and the artificial equilibrium of the i-th subsystem is required to satisfy,

$$\begin{aligned} x_{e_i} &= A_{N_i} x_{e_{N_i}} + B_i u_{e_i} \in \xi \mathcal{X}_i, \\ u_{e_i} &= K_i x_{e_i} + d_i \in \xi \mathcal{U}_i, \end{aligned} \tag{5.4}$$

where $\xi \in (0, 1)$, $K_i \in \mathbb{R}^{m_i \times n_i}$ and $d_i \in \mathbb{R}^{m_i}$ define the terminal controller $\kappa_i(x_i) = K_i x_i + d_i$ and $x_{e_{N_i}}$ is a vector comprising the equilibrium points of the subsystems in the set $\mathcal{N}_i$. Note that P_i and K_i are computed offline such that $V(x) = \|x\|_P^2$ with $x = [x_1^\top, \dots, x_M^\top]^\top$ and $P = \text{diag}(P_1, \dots, P_M)$ is a Lyapunov function for the overall system under the controller $\kappa_i(x_i) = K_i x_i$ in the absence of constraints. The local variables x_i, u_i and x_{N_i} can be extracted from the global variables $x = [x_1^\top, \dots, x_M^\top]^\top \in \mathbb{R}^n$ and $u = [u_1^\top, \dots, u_M^\top]^\top \in \mathbb{R}^m$ through the projections,

$$\begin{aligned} x_i &= U_i x, \\ x_{N_i} &= W_i x, \\ u_i &= V_i u, \end{aligned} \tag{5.5}$$

where $U_i \in \{0, 1\}^{n_i \times n}$, $W_i \in \{0, 1\}^{n_{N_i} \times n}$ and $V_i \in \{0, 1\}^{m_i \times m}$ are appropriately constructed.

To ensure the recursive feasibility of the MPC scheme and the asymptotic stability of the corresponding closed-loop system, the state $x_i(N|t)$ is constrained to lie in the terminal set $\mathcal{X}_{f_i}$ as follows,

$$x_i(N|t) \in \mathcal{X}_{f_i} = \{x_i \in \mathbb{R}^{n_i} : \|x_i - c_i\|_{P_i}^2 \leq \alpha_i^2\}$$

where α_i and c_i refer to the size and center of the local terminal set of the i-th subsystem, respectively, and are considered as decision variables in the online OCP. As shown in Part I, this constraint can be approximated using the Schur Complement and diagonal dominance as,

$$\begin{aligned} 2\alpha_i[P_i]_{ss} \geq |\bar{b}_i|_{s1} + \sum_{k=1}^{n_i}[|P_i|\alpha_i]_{sk} \text{ for all } s \in \{1, ..., n_i\}, \\ \alpha_i \geq \sum_{k=1}^{n_i}[\bar{b}_i]_{k1}, \\ -\bar{b}_i \leq x_i - c_i \leq \bar{b}_i, \end{aligned} \tag{5.6}$$

where $\bar{b}_i \in \mathbb{R}^{n_i}$ is a decision variable, $[\cdot]_{sk}$ refers to the element in the s-th row and k-th column of a matrix and $|\cdot|$ refers to a matrix with the absolute values of the elements in the original matrix. We ensure the positive invariance of the local terminal sets by satisfying for all $i \in \{1, \ldots, M\}$, $j \in \mathcal{N}_i$, $p \in \{1, ..., q_i\}$ and $l \in \{1, ..., r_i\}$,

$$\begin{aligned} 2\alpha_i \left[P_i^{-1}\right]_{ss} \geq \sum_{k=1}^{n_i}[|P_i^{-1}|\alpha_i]_{sk} + \sum_{k=1}^{n_{N_i}}[|A_{N_i} + B_iK_iU_iW_i^\top|\alpha_{N_i}]_{sk} + [b_i]_{s1} \\ \text{for all } s \in \{1, ..., n_i\}, \\ 2\sum_{j\in\mathcal{N}_i}[\lambda_{ij}P_{ij}]_{ss} \geq \sum_{j\in\mathcal{N}_i}\sum_{k=1}^{n_{N_i}}[\lambda_{ij}|P_{ij}|]_{sk} + \sum_{k=1}^{n_i}[\alpha_{N_i}|(A_{N_i} + B_iK_iU_iW_i^\top)^\top|]_{sk} \\ \text{for all } s \in \{1, ..., n_{N_i}\}, \\ -b_i \leq (A_{N_i} + B_iK_iU_iW_i^\top)c_{N_i} + B_id_i - c_i \leq b_i, \\ \alpha_i - \sum_{j\in\mathcal{N}_i}\lambda_{ij} \geq \sum_{k=1}^{n_i}[b_i]_k 1, \\ \lambda_{ij} \geq 0, \\ G_i^p c_i + \|G_i^p P_i^{-1/2}\|_2\alpha_i \leq g_i^p, \\ H_i^l K_i c_i + H_i^l d_i + \|H_i^l K_i P_i^{-1/2}\|_2\alpha_i \leq h_i^l, \end{aligned} \tag{5.7}$$

where $b_i \in \mathbb{R}^{n_i}$ and $\lambda_{ij} \in \mathbb{R}$ are additional decision variables, $P_{ij} = W_iU_j^\top P_jU_jW_i^\top$, G_i^p and H_i^l are the p-th and l-th rows of the matrices G_i and H_i respectively, g_i^p and h_i^l are the p-th and l-th entries of the vectors g_i and h_i. For the derivation of the constraints (5.7), we refer the reader to Part I.

In summary, the global online OCP is given by,

$$\min \sum_{i=1}^{M} J_i(t)$$
$$\text{s.t.} \begin{cases} \text{for all } \; k \in \{0,\dots,N-1\} \text{ and } i \in \{1,\dots,M\}, \\ \text{for all } j \in \mathcal{N}_i, \; p \in \{1,\dots,q_i\} \text{ and } l \in \{1,\dots,r_i\}, \\ x_i(0|t) = x_{0_i}, \; (5.2), \; (5.3), \; (5.4), \; (5.6), \; (5.7). \end{cases} \tag{5.8}$$

where x_{0_i} is the measured state of the i-th subsystem and the decision variables are for all $i \in \{1,\dots,M\}$, $x_i(k|t)$ for all $k \in \{0,\dots,N\}$, $u_i(k|t)$ for all $k \in \{0,\dots,N-1\}$, x_{e_i}, u_{e_i}, α_i, c_i, d_i, b_i, $\bar{b}_i$ and λ_{ij} for all $j \in \mathcal{N}_i$. Although the OCP (5.16) requires global information, it is still amenable to distributed optimization techniques such as consensus alternating direction method of multipliers (ADMM) and hence, can be solved in a distributed fashion [Boy+11].

If subsystems leave or join, the network topology changes and hence the cost and constraints in (5.16) also change. Thus, the asymptotic stability of the closed-loop system and the recursive feasibility of the MPC problem are no longer guaranteed. To address this difficulty, we develop below a PnP algorithm which allows changes in the network topology while ensuring asymptotic stability and recursive feasibility.

5.3 Control Design

We now discuss the PnP algorithm to be run whenever a set $\mathcal{J}$ of new subsystems and/or a set $\mathcal{L}$ of existing subsystems send PnP requests to the network. If the r-th PnP request is to be accepted, the subsystems of the r-th network after the PnP operation can be partitioned into three non-overlapping sets. These sets are,

- the plugged-in set $\mathcal{J}$ comprising the new subsystems joining the network,
- the neighbour set $\mathcal{Z}$ including the subsystems belonging to neither $\mathcal{J}$ nor $\mathcal{L}$ and whose set of neighbours contains at least one subsystem in $\mathcal{J}$ or $\mathcal{L}$,
- the non-neighbour set $\mathcal{O}$ comprising the subsystems belonging to neither $\mathcal{J}$ nor $\mathcal{Z}$.

Similar to [Zei+13], the proposed PnP algorithm comprises two phases: the redesign phase ensuring the asymptotic stability of the r-th network and the transition phase ensuring the recursive feasibility of the MPC problem. In the sequel, these two phases are discussed in detail.

5.3.1 Redesign Phase

In the redesign phase, the local terminal controllers of the plugged-in set $\mathcal{J}$ are designed. The controllers of the neighbour set $\mathcal{Z}$ are also modified to take into account the plugged-in set $\mathcal{J}$ and the plugged-out set $\mathcal{L}$. The newly-designed and modified terminal controllers are designed to ensure the asymptotic stability of the r-th network in the absence of constraints. The weights of the local stage costs of a specific set of subsystems are then updated to ensure the asymptotic stability of the r-th network in the presence of constraints. All parameters and variables mentioned in this section are those of the r-th network unless otherwise stated. For the rest of this section, we ignore the dependence of all variables on time t for simplicity and we refer to the dependence on time $t+1$ by $(\cdot)^+$.

First, we start with updating the terminal controllers of the subsystems in $\mathcal{J}$ and $\mathcal{Z}$ to asymptotically stabilize the r-th network in the absence of constraints. The terminal dynamics of the subsystems in $\mathcal{J}$ and $\mathcal{Z}$ under the control law $u_i = K_i x_i + d_i$ is given by,

$$x_i^+ = (A_i + B_i K_i)x_i + \sum_{s=1}^{p_i} F_{i,s} v_{i,s} + B_i d_i,$$

By appropriately shifting the equilibrium point, the dynamics becomes,

$$\Delta x_i^+ = (A_i + B_i K_i)\Delta x_i + \sum_{s=1}^{p_i} F_{i,s} \Delta v_{i,s}, \tag{5.9}$$

where,

$$\begin{aligned}\Delta v_{i,s} &= \sum_{j \in \mathcal{N}_i \backslash i} l_{ij,s}(\Delta y_{j,s} - \Delta y_{i,s}),\\ \Delta y_{i,s} &= C_{i,s}\Delta x_i.\end{aligned}$$

In addition to the subsystem outputs $\Delta y_{i,s}$, to exploit passivity we define the virtual outputs of each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ as,

$$\Delta z_{i,s} = C_{i,s}\Delta x_i + D_{i,s}\Delta v_{i,s} \text{ for all } s \in \{1, ..., p_i\}, \tag{5.10}$$

where $D_i = \text{diag}(D_{i,1}, ..., D_{i,p_i})$ is computed below. We also define,

$$\begin{aligned}\Delta z_i &= [\Delta z_{i,1}^\top, \ldots, \Delta z_{i,p_i}^\top]^\top,\\ \Delta v_i &= [\Delta v_{i,1}^\top, \ldots, \Delta v_{i,p_i}^\top]^\top,\\ F_i &= [F_{i,1}, \ldots, F_{i,p_i}],\\ C_i &= [C_{i,1}^\top, \ldots, C_{i,p_i}^\top]^\top.\end{aligned}$$

Definition 5.1 ([Ali17]). *The dynamics in (5.9) is strictly passive with respect to the pair $(\Delta v_i, \Delta z_i)$ if and only if there exist a storage function $V_i(\Delta x_i) : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ with*

$V(0) = 0$ *and a dissipation function* $\gamma_i(\Delta x_i) : \mathbb{R}^{n_i} \to \mathbb{R}_{>0}$ *with* $\gamma(0) = 0$ *such that,*

$$V_i(\Delta x_i^+) - V_i(\Delta x_i) \leq \sum_{s=1}^{p_i} \Delta v_{i,s} \Delta z_{i,s} - \gamma_i(\Delta x_i). \tag{5.11}$$

Note that we use virtual outputs because the dynamics in (5.1) can not be passivized using the actual outputs as discussed in Chapter 4.

Furthermore, we define,

$$L = \sum_{s=1}^{p_0} L_s \otimes e_s e_s^\top,$$

where $\otimes$ refers to the Kronecker product and $e_s \in \mathbb{R}^{p_0}$ is a unit vector whose s-th element is equal to one. One can show that the matrix L is positive semidefinite since both L_s and $e_s e_s^\top$ are positive semidefinite for all $s \in \{1, \ldots, p_0\}$. Note that L_s is positive semidefinite by definition since L_s is a Laplacian matrix and $e_s e_s^\top$ is positive semidefinite by construction. Considering $C = \text{diag}(C_1, ..., C_M)$, we define $L_C = LC$ and $C_L = C^\top L^\top$ which can be decomposed as follows,

$$L_C = [L_{C_1}^\top, ..., L_{C_M}^\top]^\top,$$
$$C_L = [C_{L_1}^\top, ..., C_{L_M}^\top]^\top,$$

where $L_{C_i} \in \mathbb{R}^{p_i \times n}$ and $C_{L_i} \in \mathbb{R}^{n_i \times M p_0}$. Finally, we define the scalars n_{ij} for all $i, j \in \{1, ..., M\}$ such that,

$$n_{ij} > 0 \text{ if } i \in \mathcal{N}_j \text{ and } n_{ij} = 0 \text{ otherwise},$$
$$\sum_{j=1}^{M} n_{ij} \leq 1 \text{ for all } i \in \{1, ..., M\}.$$

In the sequel, we denote the diagonal element in the s-th row of a matrix by $[\cdot]_s$ and the 1-norm of the s-th row by $|\cdot|_s$. We now make the following assumptions on the parameters and variables of the zeroth network before receiving any PnP request. Recall that all parameters and variables mentioned in this section are those of the r-th network unless otherwise stated as in the following assumptions.

Assumption 5.1. *The dynamics of the* i*-th subsystem under the control law* $u_i = K_i x_i + d_i$ *is strictly passive with respect to the input-output pair* $(\Delta v_i, \Delta z_i)$ *with quadratic storage function* $V_i(\Delta x_i) = \|\Delta x_i\|_{P_i}^2$ *and dissipation function* $\gamma_i(\Delta x_i) = \|\Delta x_i\|_{\Gamma_i}^2$*. The matrices* $P_i \in \mathbb{S}_{++}^{n_i}$, $\Gamma_i \in \mathbb{D}_{++}^{n_i}$ *and* $D_i \in \mathbb{D}_{++}^{n_i}$ *satisfy,*

$$[\Gamma_i^{-1}]_j \leq \frac{1}{|L_{C_i}|_j + \epsilon_i} \text{ for all } j \in \{1, ..., n_i\},$$
$$[D_i]_j \leq \frac{1}{|C_{L_i}|_j} \text{ for all } j \in \{1, ..., p_i\} \text{ such that } |C_{L_i}|_j > 0,$$

where $\epsilon_i > 0$ *can be chosen arbitrarily small for all* $i \in \{1, \ldots, M\}$.

Assumption 5.2. *The stage cost weights $Q_i \in \mathbb{S}^{n_i}_{++}$ and $R_i \in \mathbb{S}^{m_i}_{++}$ of the i-th subsystem satisfy,*

$$\sum_{j \in \mathcal{N}_i} n_{ij} W_i U_j^\top P_j U_j W_i^\top - W_i U_i^\top (Q_i + K_i^\top R_i K_i) U_i W_i^\top \\ - (A_{N_i} + B_i K_i U_i W_i^\top)^\top P_i (A_{N_i} + B_i K_i U_i W_i^\top) \geq 0.$$

Note that Assumption 5.1 and Assumption 5.2 can be ensured during the offline synthesis of the MPC scheme (5.16) for the zeroth network before any PnP operation. Assumption 5.1 ensures that the zeroth network under the controller $u_i = K_i x_i + d_i$ for all $i \in \{1, ..., M\}$ is asymptotically stable in the absence of constraints (see Chapter 4 for more details). Furthermore, Assumption 5.2 ensures that the zeroth network under the MPC scheme (5.16) is asymptotically stable in the presence of constraints. This is because adding the condition in Assumption 5.2 for all subsystems leads to the Lyapunov inequality for the overall network. This inequality is used to ensure the asymptotic stability of the overall network under the MPC scheme (5.16) (see Chapter 3 for more details).

The following theorem shows the conditions under which the r-th network under the controller $u_i = K_i x_i + d_i$ for all $i \in \{1, ..., M\}$ is asymptotically stable in the absence of constraints assuming that the $(r-1)$-th network is already asymptotically stable. Note that this assumption is valid for $r = 1$ as indicated in Assumption 5.1 and for $r > 1$ by induction as follows. If the conditions in the following theorem are not satisfied for one or more subsystems, the PnP request is rejected. The network remains the same and hence, asymptotic stability is preserved and the conditions in Assumption 5.1 hold until the next PnP request. If these conditions are satisfied for all subsystems, the PnP request can be accepted depending on the next steps in the PnP algorithm. If it is, the controllers are redesigned based on the conditions in Theorem 5.1 which imply the conditions in Assumption 5.1 for the r-th network after the PnP operation is completed, hence until the next PnP request.

Theorem 5.1. *Assume that the $(r-1)$-th network under the controller $u_i = K_i x_i + d_i$ for all $i \in \{1, ..., M\}$ is asymptotically stable in the absence constraints (i.e. Assumption 5.1 holds for the $(r-1)$-th network). Then, the r-th network under the controller $u_i = K_i x_i + d_i$ for all $i \in \{1, ..., M\}$ is asymptotically stable in the absence of constraints (i.e. Assumption 5.1 holds for the r-th network) if for each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ of this network, there exist $E_i \in \mathbb{S}^{n_i}_{++}$, $X_i \in \mathbb{D}^{n_i}_{++}$, $Y_i \in \mathbb{R}^{n_i \times m_i}$ and $D_i \in \mathbb{D}^{n_i}_{++}$ such that,*

$$\begin{bmatrix} E_i & \frac{1}{2} E_i C_i^\top & (A_i E_i + B_i Y_i)^\top & E_i \\ \frac{1}{2} C_i E_i & D_i & F_i^\top & 0 \\ (A_i E_i + B_i Y_i) & F_i & E_i & 0 \\ E_i & 0 & 0 & X_i \end{bmatrix} \geq 0, \tag{5.12a}$$

$$[X_i]_j \leq \frac{1}{|L_{C_i}|_j + \epsilon_i} \text{ for all } j \in \{1, ..., n_i\}, \tag{5.12b}$$

$$[D_i]_j \leq \frac{1}{|C_{L_i}|_j} \text{ for all } j \in \{1, ..., p_i\} \text{ such that } |C_{L_i}|_j > 0, \tag{5.12c}$$

where $\epsilon_i > 0$ *can be chosen arbitrarily small for all* $i \in \{1, \ldots, M\}$.

Proof. First, we derive conditions which ensure that all subsystems of the r-th network are strictly passive, or equivalently that (5.11) holds for all $i \in \{1, \ldots, M\}$. Note that this condition is satisfied for $\mathcal{O}$ since the dynamics of such subsystems in the r-th network remain the same as in the $(r-1)$-th network which satisfies Assumption 5.1. Hence, it suffices to ensure that (5.11) holds for $\mathcal{J}$ and $\mathcal{Z}$. Following Chapter 4, we take $V_i(\Delta x_i) = \|\Delta x_i\|_{P_i}^2$ and $\gamma_i(\Delta x_i) = \|\Delta x_i\|_{\Gamma_i}$, substitute (5.9) and (5.10) in (5.11), use the Schur Complement twice to reach (5.12a) where,

$$\begin{aligned} E_i &= P_i^{-1}, \\ Y_i &= K_i P_i^{-1}, \\ X_i &= \Gamma_i^{-1}. \end{aligned}$$

The conditions $E_i \in \mathbb{S}_{++}^{n_i}$ and $X_i \in \mathbb{D}_{++}^{n_i}$ ensure that $V_i(x_i) \geq 0$ and $\gamma_i(x_i) \geq 0$.

Second, we derive conditions which ensure that the strict passivity of all subsystems in the r-th network implies the asymptotic stability of this network. For this, we define,

$$\begin{aligned} \Delta x &= [\Delta x_1^\top, \ldots, \Delta x_M^\top]^\top, \\ \Delta z &= [\Delta z_1^\top, \ldots, \Delta z_M^\top]^\top, \\ \Delta v &= [\Delta v_1^\top, \ldots, \Delta v_M^\top]^\top, \\ C &= \text{diag}(C_1, \ldots, C_M), \\ D &= \text{diag}(D_1, \ldots, D_M), \\ P &= \text{diag}(P_1, \ldots, P_M), \\ \Gamma &= \text{diag}(\Gamma_1, \ldots, \Gamma_M). \end{aligned}$$

We also define the Lyapunov function,

$$V(\Delta x) = \sum_{i=1}^{M} V_i(\Delta x_i),$$

and the auxiliary function,

$$\gamma(\Delta x) = \sum_{i=1}^{M} \gamma_i(\Delta x_i).$$

One can verify that $V(\Delta x) = \|\Delta x\|_P^2$ and $\gamma(\Delta x) = \|\Delta x\|_\Gamma^2$. Under Assumption 5.1, (5.11) holds for all subsystems in $\mathcal{O}$ with quadratic storage and dissipation functions.

Moreover, (5.12a) implies that (5.11) holds for all subsystems in $\mathcal{J}$ and $\mathcal{Z}$ with quadratic storage and dissipation functions. By summing up (5.11) for all $i \in \{1, \ldots, M\}$, we reach,

$$V(\Delta x^+) - V(\Delta x) \leq \Delta v^\top \Delta z - \gamma(\Delta x),$$

where,

$$\Delta z = C\Delta x + D\Delta v,$$
$$\Delta v = -LC\Delta x.$$

To prove asymptotic stability of the r-th network under the controller $u_i = K_i x_i + d_i$ for all $i \in \{1, \ldots, M\}$ in the absence of constraints, it suffices to ensure that $\Delta v^\top \Delta z - \gamma(\Delta x) < 0$. or equivalently,

$$\Gamma + C^\top LC - C^\top L^\top DLC \geq \text{diag}(\epsilon_1 I_{n_1}, ...\epsilon_M I_{n_M}),$$

where $\epsilon_i > 0$ can be selected arbitrarily small for all $i \in \{1, \ldots, M\}$ and I_{n_i} is an identity matrix of size n_i. Following Chapter 4, we exploit the Schur Complement, diagonal dominance and the fact that $L \geq 0$, $X_i \in \mathbb{D}^{n_i}_{++}$ and $D_i \in \mathbb{D}^{n_i}_{++}$ to reach (5.12b) and (5.12c) for all subsystems in $\mathcal{J}$, $\mathcal{Z}$ and $\mathcal{O}$. Under Assumption 5.1, (5.12b) and (5.12c) are satisfied for all subsystems in $\mathcal{O}$ since the neighbours of these subsystems remain the same and hence, the corresponding matrices C_{L_i} and L_{C_i} do not change for all $i \in \mathcal{O}_i$. Hence, it suffices to ensure that (5.12b) and (5.12c) are satisfied for all subsystems in $\mathcal{J}$ and $\mathcal{Z}$. □

Based on Theorem 5.1, each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ computes its own P_i and K_i by solving a local optimization problem comprising an appropriately-chosen convex cost function and the local constraints in Theorem 5.1. The cost function can be selected, for example, to mimic the LQR behavior or to maximize the passivity margin (see Chapter 4 for more details). Furthermore, each subsystem in $\mathcal{L}$ computes its own P_i and K_i by solving the associated algebraic Riccati equation. If any of these problems is not feasible, the PnP request is rejected.

We now move to the second step of the redesign phase in which we update the cost functions of a specific set of subsystems to asymptotically stabilize the r^{th} network in the presence of constraints. For this purpose, we define the matrices,

$$P = \text{diag}(P_1, ..., P_M),$$
$$K = \text{diag}(K_1, ..., K_M),$$
$$Q = \text{diag}(Q_1, ..., Q_M),$$
$$R = \text{diag}(R_1, ..., R_M).$$

To ensure the asymptotic stability of the r^{th} network in the presence of constraints under the MPC controller (5.16), the Lyapunov condition,

$$V(x^+) - V(x) \leq -l(x),$$

should hold where,

$$V(x) = \|x\|_P^2,$$
$$l(x) = \|x\|_{(Q+K^\top RK)}^2$$

This, in turn, is satisfied if,

$$P - (A+BK)^\top P(A+BK) - Q - K^\top RK \geq 0.$$

This condition can be ensured by appropriately tuning the the cost function matrices Q and R. Although this inequality requires global information, it can be used to derive local conditions for each subsystem as shown in the following theorem.

The following theorem shows the conditions under which the r-th network under the MPC scheme (5.16) is asymptotically stable in the presence of constraints assuming that the $(r-1)$-th network is already asymptotically stable. Similar to Theorem 5.1, if the conditions in Theorem 5.2 are not satisfied for one or more subsystems, the PnP request is rejected. The network remains the same and, hence asymptotic stability is preserved and the conditions in Assumption 5.2 hold until the next PnP request. On the contrary, if the conditions in Theorem 5.2 are satisfied for all subsystems, the PnP request can be accepted depending on the next steps in the PnP algorithm. If it is, the stage cost weights are redesigned according to the conditions in Theorem 5.2 which imply that the conditions in Assumption 5.2 hold after the PnP operation is completed, hence until the next PnP request.

Theorem 5.2. *Assume that the $(r-1)$-th network under the MPC scheme (5.16) is asymptotically stable in the presence constraints (i.e. Assumption 5.2 holds for the $(r-1)$-th network). Then, the r-th network under the MPC scheme (5.16) is asymptotically stable in the presence of constraints (i.e. Assumption 5.2 holds for the r-th network) if for each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ as well as their neighbours, there exist $Q_i \in \mathbb{S}_{++}^{n_i}$ and $R_i \in \mathbb{S}_{++}^{m_i}$ such that,*

$$\sum_{j\in\mathcal{N}_i} n_{ij} W_i U_j^\top P_j U_j W_i^\top - W_i U_i^\top (Q_i + K_i^\top R_i K_i) U_i W_i^\top \\ - (A_{N_i} + B_i K_i U_i W_i^\top)^\top P_i (A_{N_i} + B_i K_i U_i W_i^\top) \geq 0. \quad (5.13)$$

Proof. According to Chapter 3, the r-th network under the MPC scheme (5.16) is asymptotically stable if,

$$P - (A+BK)^\top P(A+BK) - Q - K^\top RK \geq 0.$$

This inequality holds if and only if for all $x \in \mathbb{R}^n$,

$$x^\top \left(P - (A+BK)^\top P(A+BK) - Q - K^\top RK\right) x \geq 0$$

Due to the imposed structure on the matrices P_i, K_i, Q_i and R_i, this inequality is given by,

$$\sum_{i=1}^{M} x_i^\top P_i x_i - \sum_{i=1}^{M} x_i^\top (Q_i + K_i^\top R_i K_i) x_i \\ - \sum_{i=1}^{M} (A_{N_i} x_{N_i} + B_i K_i x_i)^\top P_i (A_{N_i} x_{N_i} + B_i K_i x_i) \geq 0,$$

or equivalently,

$$\sum_{i=1}^{M} \left(1 - \sum_{j=1}^{M} n_{ij}\right) x_i^\top P_i x_i + \sum_{i=1}^{M} \left(\sum_{j=1}^{M} n_{ij}\right) x_i^\top P_i x_i \\ - \sum_{i=1}^{M} \left(A_{N_i} x_{N_i} + B_i K_i x_i\right)^\top P_i (A_{N_i} x_{N_i} + B_i K_i x_i) \\ - \sum_{i=1}^{M} x_i^\top \left(Q_i + K_i^\top R_i K_i\right) x_i \geq 0.$$

Recall that by design,

$$\sum_{j=1}^{M} n_{ij} \leq 1.$$

Hence, it suffices to ensure that,

$$\sum_{i=1}^{M} \left(\sum_{j=1}^{M} n_{ij}\right) x_i^\top P_i x_i - \sum_{i=1}^{M} x_i^\top (Q_i + K_i^\top R_i K_i) x_i \\ - \sum_{i=1}^{M} (A_{N_i} x_{N_i} + B_i K_i x_i)^\top P_i (A_{N_i} x_{N_i} + B_i K_i x_i) \geq 0.$$

Using the map (5.5), the resulting inequality becomes,

$$\sum_{i=1}^{M} \left(\sum_{j=1}^{M} n_{ij}\right) x_{N_i}^\top W_i U_i^\top P_i U_i W_i^\top x_{N_i} \\ - \sum_{i=1}^{M} x_{N_i}^\top (A_{N_i} + B_i K_i U_i W_i^\top)^\top P_i (A_{N_i} + B_i K_i U_i W_i^\top) x_{N_i} \\ - \sum_{i=1}^{M} x_{N_i}^\top W_i U_i^\top (Q_i + K_i^\top R_i K_i) U_i W_i^\top x_{N_i} \geq 0.$$

One can show that this global inequality can be decomposed for all $i \in \{1, \dots, M\}$ into,

$$\sum_{j=1}^{M} n_{ij} x_{N_i}^\top W_i U_j^\top P_j U_j W_i^\top x_{N_i} \\ - x_{N_i}^\top (A_{N_i} + B_i K_i U_i W_i^\top)^\top P_i (A_{N_i} + B_i K_i U_i W_i^\top) x_{N_i} \\ - x_{N_i}^\top W_i U_i^\top (Q_i + K_i^\top R_i K_i) U_i W_i^\top x_{N_i} \geq 0.$$

Since $n_{ij} = 0$ if i and j are not neighbours, the above local inequalities hold if and only if the local LMIs (5.13) are satisfied for all $i \in \{1, \dots, M\}$. Since the $(r-1)$-th network satisfies Assumption 5.2, these LMIs are satisfied for all subsystems excluding those in $\mathcal{J}$ and $\mathcal{Z}$ as well as their neighbours. Hence, it suffices to ensure them for these subsystems. □

Following Chapter 3, one can ensure that the target point x_{r_i} for all $i \in \{1, \dots, M\}$ in the r-th network is asymptotically stable if (5.13) holds. We highlight here that the neighbours of $\mathcal{Z}$ which are in neither $\mathcal{J}$ nor $\mathcal{Z}$ should also recompute their matrices Q_i and R_i to satify the condition (5.13). This is because the first term in the condition (5.13) of each of these subsystems depends on the matrices P_i of its neighbours. By definition, at least one of these neighbours is in the set $\mathcal{Z}$ whose matrices P_i are modified as discussed in Theorem 5.1.

Since the subsystems in $\mathcal{L}$ leave the network and no longer have neighbours, their decoupled inequalities corresponding to (5.13) are given by,

$$P_i - (A_i + B_i K_i)^\top P_i (A_i + B_i K_i) - Q_i - K_i^\top R_i K_i \geq 0. \tag{5.14}$$

Note that (5.14) is the standard Lyapunov matrix inequaltiy for discrete-time systems. Notice also that each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ as well as their neighbours can satisfy (5.13) and each subsystem in $\mathcal{L}$ can satisfy (5.14) by solving a constrained optimization problem whose cost function is selected so that the desired performance specifications can be met. For example, one potential cost function is that which minimizes the Frobenius norm between the actual matrices Q_i and R_i (to be used in the online OCP) on one side and the desired matrices Q_{d_i} and R_{d_i} (determined based on the performance specifications) on the other side. In this case, each subsystem in $\mathcal{J}$ and $\mathcal{Z}$ as well as their neighbours solves the optimization problem,

$$\min_{Q_i, R_i} \|Q_i - Q_{d_i}\|_F^2 + \|R_i - R_{d_i}\|_F^2 \text{ s.t. } (5.13),\ Q_i \in \mathbb{S}_{++}^{n_i},\ R_i \in \mathbb{S}_{++}^{m_i}.$$

where $\|\cdot\|_F$ refers to the Frobenius norm of a matrix. Similarly, each subsystem in $\mathcal{L}$ solves the optimization problem,

$$\min_{Q_i, R_i} \|Q_i - Q_{d_i}\|_F^2 + \|R_i - R_{d_i}\|_F^2 \text{ s.t. } (5.14),\ Q_i \in \mathbb{S}_{++}^{n_i},\ R_i \in \mathbb{S}_{++}^{m_i}.$$

The PnP request is rejected if any of these optimization problems is infeasible.

Finally, we highlight that the scalars n_{ij} for all $i \in \mathcal{J}$, $j \in \mathcal{N}_i$ and for all $i \in \mathcal{Z}$, $j \in \mathcal{N}_i \cap \mathcal{J}$ can be considered as decision variables in these optimization problems. While the local optimization problems of $\mathcal{L}$ and the neighbours of $\mathcal{Z}$ that are in neither $\mathcal{J}$ nor $\mathcal{Z}$ remain the same in this case, the local optimization problems of $\mathcal{J}$ and $\mathcal{Z}$ become,

$$\begin{aligned} &\min_{Q_i, R_i, n_{ij}|_{j\in\mathcal{N}_i}} \|Q_i - Q_{d_i}\|_F^2 + \|R_i - R_{d_i}\|_F^2 + \sum_{j\in\mathcal{N}_i} n_{ij}^2, \\ &\text{s.t. } (5.13),\ Q_i \in \mathbb{S}_{++}^{n_i},\ R_i \in \mathbb{S}_{++}^{m_i},\ n_{ij} \geq 0 \text{ for all } j \in \mathcal{N}_i, \end{aligned} \tag{5.15}$$

and,

$$\begin{aligned} &\min_{Q_i, R_i, n_{ij}|_{j\in\mathcal{N}_i\cap\mathcal{J}}} \|Q_i - Q_{d_i}\|_F^2 + \|R_i - R_{d_i}\|_F^2 + \sum_{j\in\mathcal{N}_i\cap\mathcal{J}} n_{ij}^2, \\ &\text{s.t. } (5.13),\ Q_i \in \mathbb{S}_{++}^{n_i},\ R_i \in \mathbb{S}_{++}^{m_i},\ n_{ij} \geq 0 \text{ for all } j \in \mathcal{N}_i \cap \mathcal{J}, \end{aligned} \tag{5.16}$$

respectively. Consequently, the PnP request is rejected if any of the optimization problems of $\mathcal{L}$ or the neighbours of $\mathcal{Z}$ that are in neither $\mathcal{J}$ nor $\mathcal{Z}$ is infeasible or if for any subsystem in $\mathcal{J}$ or $\mathcal{Z}$,

$$\sum_{j\in\mathcal{N}_i} n_{ij} > 1.$$

5.3.2 Transition Phase

In the transition phase, we compute for the old network and the new subsystems before joining the network a steady state x_{s_i} at which the PnP operation can take place. This steady state leads to safe PnP operations during which $x_i(t+1) = x_i(t)$ for all subsystems. To compute this steady state, an optimization problem is solved in a distributed manner.

In this problem, we ensure that the old network and the new subsystems before joining the network can reach their steady states starting from the current state. We also ensure that the terminal sets of the new network and the plugged-out subsystems after leaving the network are reachable from the steady state. Unlike [Zei+13], the terminal sets are considered as decision variables in this optimization problem.

It is important to mention that the steady state is an equilibrium point for the old network and the new subsystems before joining the network, but is no longer an equilibrium point for the new network and the plugged-out subsystems after leaving the network. In other words, the steady state is no longer an equilibrium point once the PnP operation takes place. In the sequel, the superscript $(\cdot)^o$ refers to the parameters of the old network before the PnP operations, whereas the superscript $(\cdot)^n$ refers to those of the new network after the PnP operations.

The prediction horizon in the transition phase is divided into two periods at $N_{pp} \in \{0, \ldots, N\}$. In the first period, we drive the current state of the old network and the new

subsystems to their steady state while satisfying their dynamics and constraints before the PnP operation. For this purpose, we consider the dynamics and constraints of the old network and the new subsystems before joining the network as follows,

$$\begin{aligned} &\text{for all } i \in \mathcal{M}^o \cup \mathcal{J} \\ &\text{for all } k \in \{0, \dots, N_{pp}-1\} \end{aligned} \begin{cases} x_i(0|t) = x_{0_i}, \\ x_i(k+1|t) = A^o_{N_i} x_{N_i}(k|t) + B^o_i u_i(k|t), \\ x_i(k|t) \in \mathcal{X}^o_i, \\ u_i(k|t) \in \mathcal{U}^o_i \\ x_i(N_{pp}|t) = x_{s_i}. \end{cases} \tag{5.17}$$

In the second period, we drive the states of the new network and the plugged-out subsystems to their terminal sets while satisfying their dynamics and constraints after the PnP operation. For this purpose, we consider the dynamics and constraints of the new network and the plugged-out subsystems after leaving the network as follows,

$$\begin{aligned} &\text{for all } i \in \mathcal{M}^n \cup \mathcal{L} \\ &\text{for all } k \in \{N_{pp}, ..., N-1\} \end{aligned} \begin{cases} x_i(N_{pp}|t) = x_{s_i}, \\ x_i(k+1|t) = A^n_{N_i} x_{N_i}(k|t) + B^n_i u_i(k|t), \\ x_i(k|t) \in \mathcal{X}^n_i, \\ u_i(k|t) \in \mathcal{U}^n_i, \\ x_i(N|t) \in \mathcal{X}^n_{f_i}. \end{cases} \tag{5.18}$$

To guarantee that the steady state x_{s_i} is a feasible equilibrium point for the old network and the new subsystems before joining the network, we consider for all $i \in \mathcal{M}^o \cup \mathcal{J}$ the constraints,

$$\begin{aligned} x_{si} &= A^o_i x_{s_{N_i}} + B^o_i u_{si}, \\ x_{s_i} &\in \mathcal{X}^o_i, \\ u_{s_i} &\in \mathcal{U}^o_i, \end{aligned} \tag{5.19}$$

where u_{s_i} is the input corresponding to the steady state x_{s_i} and $x_{s_{N_i}}$ is a concatenated vector which includes the steady states of the subsytems in the set $\mathcal{N}^o_i$. Finally, we also consider the artificial equilibrium constraints (5.4) and the terminal set constraints (5.7) for the new network and the plugged-out subsystems after leaving the network.

In summary, the transition phase is performed by solving the optimization problem,

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{M}^o \cup \mathcal{J}} \|x_{0_i} - x_{s_i}\|^2, \\ \text{s.t.} \quad & \begin{cases} (5.17),\ (5.18), \\ (5.19),\ \text{for all } i \in \mathcal{M}^o \cup \mathcal{J}, \\ (5.4),\ (5.7) \text{ for all } i \in \mathcal{M}^n \cup \mathcal{L},\ j \in \mathcal{N}_i,\ p \in \{1, \dots, q_i\},\ l \in \{1, \dots, r_i\}. \end{cases} \end{aligned} \tag{5.20}$$

Note that (5.6) is used to represent the last constraint in (5.18) for all $i \in \mathcal{M}^n \cup \mathcal{J}$. In (5.20), the decision variables of each subsystem are x_{s_i}, u_{s_i}, $x_i(k|t)$ for all $k \in \{0, \dots, N\}$, $u_i(k|t)$ for all $k \in \{0, \dots, N-1\}$, x_{e_i}, u_{e_i}, α_i, c_i, d_i, b_i, $\bar{b}_i$ and λ_{ij} for all $j \in \mathcal{N}_i$.

If the optimization problem (5.20) is feasible, the PnP request is accepted and the control sequence $u_i(k|t)$ for all $k \in \{0, \dots, N_{pp}\}$ is applied to all subsystems before the PnP operation to drive the whole network to steady state. Once the steady state is reached, the distributed MPC problem (5.16) is solved recursively for the new network. Similar to the distributed MPC problem (5.16), the transition phase can be solved in a distributed manner using ADMM [Boy+11].

Theorem 5.3. *The MPC problem (5.16) is recursively feasible with the new network starting from the steady state x_{s_i} if the optimization problem (5.20) is feasible.*

Proof. Let $x_i(k|t)$ for all $k \in \{0, \dots, N\}$ and $u_i(k|t)$ for all $k \in \{0, \dots, N-1\}$ be feasible trajectories of the i-th subsystem in the optimization problem (5.20). Therefore, the i-th subsystem can be steered to its steady state x_{s_i} in finite time since $x_i(N_{pp}|t) = x_{s_i}$. Moreover, let α_i and c_i refer to feasible size and center of the terminal set of the i-th subsystem in the optimization problem (5.20). In addition, let d_i be a feasible affine term of the terminal controller of the i-th subsystem in the optimization problem (5.20). Define the sequences $x_i^{tail} = (x_i(N+1|t), \dots, x_i(N+N_{pp}|t))$ and $u_i^{tail} = (u_i(N|t), \dots, u_i(N+N_{pp}-1|t))$ such that for all $k \in \{N+1, \dots, N+N_{pp}\}$,

$$x_i(k|t) = (A_{N_i}^n + B_i^n K_i^n U_i^n W_i^{n^\top})^{k-N} x_{N_i}(N|t) + \sum_{k=1}^{k-N} (A_{N_i}^n + B_i^n K_i^n U_i^n W_i^{n^\top})^{k-1} B_i^n d_i,$$

$$u_i(k-1|t) = K_i x_i(k-1|t) + d_i,$$

where $x_i(N|t)$ is the terminal state which is assumed to be feasible. Note that the sequence x_i^{tail} represents the evolution of the terminal dynamics and hence lies inside the terminal set described by α_i and c_i. Furthermore, the sequence u_i^{tail} refers to the corresponding control inputs. Thus, the state sequence $x_i(k|t)$ for all $k \in \{N_{pp}, \dots, N+N_{pp}\}$ and the input sequence $u_i(k|t)$ for all $k \in \{N_{pp}, \dots, N+N_{pp}-1\}$ together with α_i, c_i and d_i represent a feasible solution for the MPC problem (5.16) applied to the new network starting from the steady state. In other words, the MPC scheme (5.16) is initially feasible for the new network starting from the steady state. Recursive feasibility can thus be ensured using standard MPC arguments [KC16] (also refer to Chapter 3 for more details). □

5.4 Mass-Spring-Damper Network Case Study

In this section, the efficacy of the proposed PnP algorithm is illustrated using a network of mass-spring-damper (MSD) systems. The dynamics of each mass is given in continuous time by,

$$\dot{x}_i = A_{ii} x_i + B_i u_i + F_{i,1} \sum_{j \in \mathcal{N}_i \setminus i} l_{ij,1}(C_{j,1} x_j - C_{i,1} x_i) + F_{i,2} \sum_{j \in \mathcal{N}_i \setminus i} l_{ij,2}(C_{j,2} x_j - C_{i,2} x_i),$$

where for all $i \in \{1, \dots, M\}$ and $j \in \mathcal{N}_i \backslash i$, x_i comprises the position and velocity, u_i is the applied force,

$$
\begin{aligned}
A_{ii} &= \begin{bmatrix} 0 & 1 \\ \frac{k_i}{m_i} & \frac{c_i}{m_i} \end{bmatrix}, \\
B_i = F_{i,1} = F_{i,2} &= \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix}, \\
C_{i,1} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
C_{i,2} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
l_{ij,1} &= k_{ij}, \\
l_{ij,2} &= c_{ij}.
\end{aligned}
$$

The parameters m_i, k_i and c_i represent the mass, stiffness coefficient and damping coefficient of the i-th mass where $m_i = 1kg$, $k_i = 1kg/s^2$ and $c_i = 2kg/s$ for all $i \in \{1, \dots, M\}$. Similarly, the parameters k_{ij} and c_{ij} represent the stiffness and damping coefficients connecting the i-th and j-th mass where $k_{ij} = 0.4kg/s^2$ and $c_{ij} = 0.8kg/s$ for all $i \in \{1, \dots, M\}$ and $j \in \mathcal{N}_i \backslash i$. The position, velocity and input force of all masses are constrained between $-1m$ and $1m$, $-1m/s$ and $1m/s$, $-1N$ and $1N$, respectively. The system is discretized using the method in [RFF13b] to have an accurate discrete-time model while preserving the distributed structure of the system. The sampling time and prediction horizon are given by 0.1 seconds and 8 time steps, respectively. All optimization problems are solved using MATLAB [MAT18] with Yalmip [Löf04] and Mosek [ApS19] on a computer equipped with 16-GB RAM and a 1.9-GHz Intel core i7-8550U processor.

Our simulation considers a network of eight MSD systems with the topology shown in Figure 5.1. The simulation starts without the eighth mass. The seventh mass sends a plug-out request at $t = 0.6$ seconds followed by a plug-in request by the eighth mass at $t = 1.6$ seconds. Figure 5.2 shows the state trajectories of all masses starting from the initial condition $x_{0_i} = [0.125\ 0]^\top$ for all $i \in \{1, \dots, 8\}$ towards the target $x_{r_i} = [0\ 0]^\top$ for all $i \in \{1, \dots, 8\}$. Note that the network is converging to the target despite the PnP operations. From $t = 0.6$ till $t = 1$ and from $t = 1.6$ till $t = 2$, all subsystems run the transition phase and move to their steady states at which their velocities are zero so that subsystem 7 is plugged out and subsystem 8 is plugged in, respectively. Apart from these two periods, all subsystems run the MPC scheme to converge to the target point. The redesign phase is run offline in parallel to the MPC scheme before running the transition phase.

Next, we compare our algorithm to the one in [Zei+13]. First, we compare the redesign phase by running 100 simulations with different coupling strength $k_{18} = 2.5r$ and $c_{18} = 5r$ for all $r \in \{0.01, ..., 1\}$. We report the number of times n_R the redesign phase is feasible as well as the mean μ_R and standard deviation σ_R of the time taken

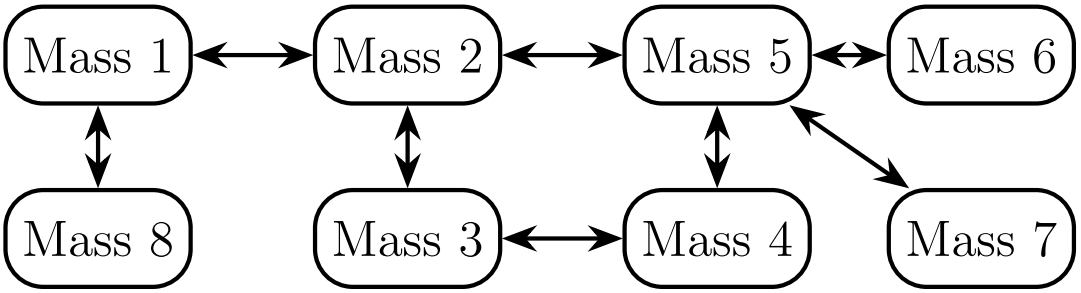

Figure 5.1: Network Topology

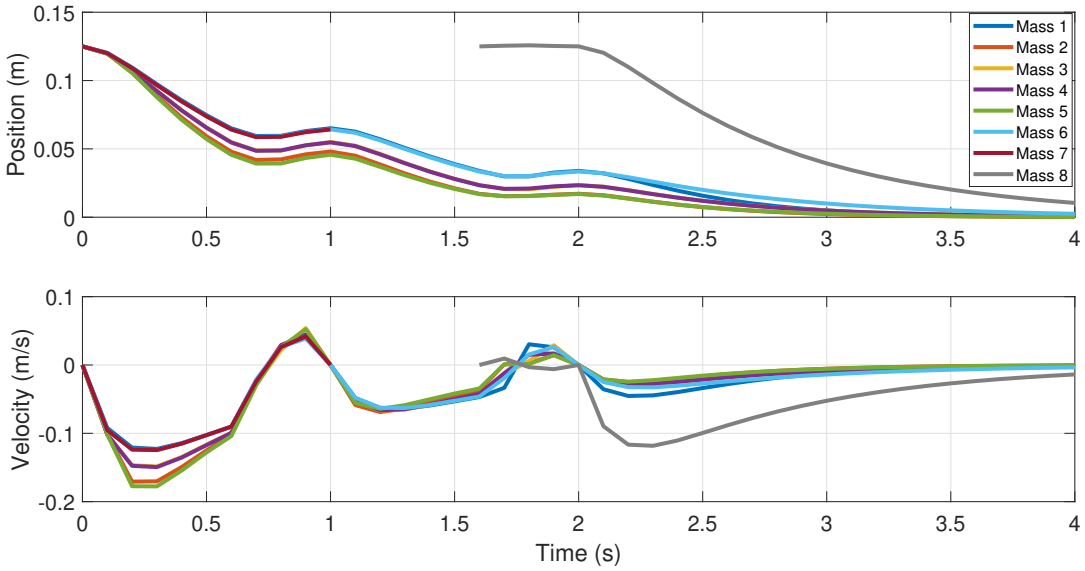

Figure 5.2: State trajectories of all masses

Table 5.1: The novel algorithm vs benchmark algorithm [Zei+13]

Redesign	Existing	New
n_R	100	82
μ_R (seconds)	29.73	0.008
σ_R (seconds)	3.07	0.0006

Transition	Existing	New
n_T	15	100
μ_T (seconds)	0.0024	0.003
σ_T (seconds)	0.0004	0.0004

in seconds by the redesign phase in Table 5.1. Unlike the algorithm of [Zei+13], our algorithm fails for strong couplings, but is much faster. This is because the optimization problem of our algorithm is conservative, but completely decentralized, unlike that of [Zei+13] that is distributed and solved using ADMM, requiring hundreds of iterations and hence, is computationally demanding. We also compare the transition phase by running 100 simulations with different initial conditions $x_{0_8} = 0.5r$ for all $r \in \{0.01, ..., 1\}$. Both algorithms use ADMM to solve the optimization problem of the transition phase. We report the number of times n_T the transition phase is feasible as well as the mean μ_T and standard deviation σ_T of the time required in seconds per ADMM step. As shown in Table 5.1, our algorithm requires slightly more time as it has more constraints to ensure the terminal set invariance. But, it is feasible for all initial conditions due to the additional flexibility of computing the terminal sets online.

Part III

Adaptive Learning-based Control

CHAPTER 6

Adaptive Learning-based Model Predictive Control

6.1 Introduction

Model Predictive Control (MPC) is an advanced control strategy which can optimally control a given system by solving an online optimal control problem (OCP) repeatedly at each time instant [RMD17; BBM17; KC16]. MPC heavily relies on the availability of an accurate model of the system, however in many cases, accurate models are either difficult to find (e.g. time varying systems) or complex to use (e.g. highly nonlinear systems). To mitigate this issue, various efforts have been devoted to finding MPC variants which do not require an accurate model. These variants include robust, stochastic and learning-based MPC schemes.

Robust MPC schemes make use of a simplified nominal model. The discrepancy between the accurate and nominal models is then considered as a source of uncertainty typically assumed to lie within a compact set. Based on this assumption, robust MPC schemes with constraint tightening were developed in [CRZ01], tube-based robust MPC schemes with rigid and homothetic tubes were proposed in [MSR05; Rak+12], whereas schemes for parametric uncertainties were developed in [KBM96]. Various robust MPC schemes were also developed with affine disturbance feedback policies instead of state feedback ones to alleviate conservatism [GKM06]. Robust MPC schemes were also extended to nonlinear systems [May+11], networked dynamical systems [RH07] and tracking applications [Lim+10].

Unlike robust MPC, stochastic MPC relies on a statistical description of the uncertainty. Inspired by the robust MPC ideas, stochastic MPC schemes with affine disturbance feedback policy [OJM08] and tube-based stochastic MPC schemes [Can+10] were developed. Stochastic programming-based approaches were also extensively utilized to develop sample-based stochastic MPC schemes where the stochastic dynamics are characterized by a finite set of random realizations of the uncertainty [BB09]. Stochastic MPC was also extended to systems with parametric uncertainty [Dai+18] and large-

scale systems [Dai+16].

Learning-based MPC schemes were proposed such that machine learning techniques could replace first-principles modelling to obtain an approximate system model using the available data. Such models were then used within the OCP. These machine learning algorithms include neural networks [PW11] and random forests [Sma+18]. Data predictive control schemes were also proposed in [CLD21; Ber+20] where the system model was derived using a finite number of data samples collected offline. The construction of this non-parametric model relies on a behavioural systems theory approach.

One drawback of these schemes is that the model is usually not updated online using collected data leading to potentially conservative performance. Recently, various adaptive and safe learning-based MPC schemes have been proposed to ensure safety (e.g. stability, robustness, constraint satisfaction) while improving closed-loop performance using data collected online. This goal is achieved by combining physics-based knowledge with machine learning algorithms and system identification techniques. In [Asw+13; Asw+12; BAT12], a provably safe and robust learning-based MPC scheme was proposed considering additive uncertainty. In this scheme, safety and performance were decoupled by considering two system models; a nominal model for ensuring safety and a learnt one for improving performance. The additive uncertainty in the latter was learnt by means of an oracle which can be either parametric (e.g. Kalman filters) or non-parametric (e.g. neural networks). In [LCA19; Köh+19], an adaptive MPC scheme was developed considering parametric uncertainty. This scheme combined robust MPC ideas with set membership identification to update the parametric uncertainty sets using online data. While these schemes were formulated in robust settings, others were formulated in stochastic settings. In such schemes, MPC was integrated with Gaussian process regression techniques to develop cautious MPC schemes for systems with additive disturbances [HKZ19; Kol+18].

In this chapter, we propose a novel adaptive learning-based MPC scheme for interconnected systems which can be decomposed into several smaller subsystems. Each subsystem is dynamically coupled with a set of other subsystems called the set of neighbours. The source of uncertainty in each subsystem is the strength of the coupling with its neighbours. This source of uncertainty exists in various classes of interconnected systems, such as varying-topology networks. The proposed scheme learns the uncertainty set online by applying distributed set membership identification on collected data and adapts the MPC ingredients accordingly. In particular, the developed control scheme adapts the rigid tube, the prestabilizing controller, the terminal set and the tightened state and input constraint sets by considering them as decision variables in the OCP. The developed scheme is compared in simulation to existing schemes including robust MPC, adaptive MPC and learning-based MPC using a network of dynamically coupled double integrators and found to lead to a better trade-off between closed-loop performance and computation cost.

6.2 Problem Formulation

6.2.1 Robust MPC

We review briefly the rigid tube-based robust MPC scheme [MSR05] that forms the basis for our scheme. The subscript $(\cdot)_c$ is used in this subsection to differentiate between the variables of the monolithic system used here and those of the interconnected system in the rest of the paper. Rigid tube-based robust MPC considers discrete-time linear time-invariant system whose dynamics is given by,

$$z_c(t+1) = A_c z_c(t) + B_c v_c(t) + E_c w_c(t), \tag{6.1}$$

where $z_c \in \mathbb{R}^n$, $v_c \in \mathbb{R}^m$ and $w_c \in \mathbb{R}^p$ are the state, input and disturbance vectors, respectively, $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $E_c \in \mathbb{R}^{n\times p}$ are the system matrices. The system is subject to the state and input constraints $z_c(t) \in \mathcal{Z}_c$ and $v_c(t) \in \mathcal{V}_c$ where $\mathcal{Z}_c$ and $\mathcal{V}_c$ are polytopic sets with the origin in their interior. The disturbance $w_c(t)$ lies in the polytopic set $\mathcal{W}_c$ which also includes the origin in its interior.

Following [MSR05], we define the nominal dynamics,

$$x_c(t+1) = A_c x_c(t) + B_c u_c(t), \tag{6.2}$$

where $x_c \in \mathbb{R}^n$ and $u_c \in \mathbb{R}^m$ are the nominal state and input vectors, respectively. Note that unlike the previous chapters, we use z and v to refer to the state and input vectors of the actual system, whereas x and u denote the state and input vectors of the nominal system. By introducing the controller,

$$v_c(t) = u_c(t) + K_c(z_c(t) - x_c(t)),$$

with the control gain $K_c \in \mathbb{R}^{m\times n}$ which stabilizes the nominal system (6.2), we define the error dynamics between the actual system (6.1) and the nominal system (6.2) which is given by,

$$e_c(t+1) = (A_c + B_c K_c) e_c(t) + E_c w_c(t), \tag{6.3}$$

where $e_c(t) = z_c(t) - x_c(t)$. We compute a robust positive invariant (RPI) set $\mathcal{Z}_{K_c}$ for the error dynamics (6.3) under the disturbance $w_c(t) \in \mathcal{W}_c$. We also define the tightened state and input constraint sets $\mathcal{X}_c \subseteq \mathcal{Z}_c \ominus \mathcal{Z}_{K_c}$ and $\mathcal{U}_c \subseteq \mathcal{V}_c \ominus K_c\mathcal{Z}_{K_c}$ and consider the tightened constraints $x_c \in \mathcal{X}_c$ and $u_c \in \mathcal{U}_c$. Note that the aforementioned set inclusion properties ensure that if $x_c(t) \in \mathcal{X}_c$ and $u_c(t) \in \mathcal{U}_c$ for all t, then $z_c(t) \in \mathcal{Z}_c$ and $v_c(t) = u_c(t) + K_c(z_c(t) - x_c(t)) \in \mathcal{V}_c$ for any $w_c(t) \in \mathcal{W}_c$. Finally, we compute a terminal set $\mathcal{X}_{f_c}$ for the nominal dynamics with the tightened constraint sets.

A robust MPC scheme can now be developed by solving at each time instant t the

optimal control problem,

$$\begin{aligned}
&\min \sum_{k=0}^{N-1} l_c\left(x_c(k|t), u_c(k|t)\right) + V_{f_c}\left(x_c(N|t)\right) \\
&x_c(k|t)|_{k\in\{0,\dots,N\}}, \\
&u_c(k|t)|_{k\in\{0,\dots,N-1\}} \\
&s.t. \begin{cases} \text{for all } k \in \{0,\dots,N-1\}, \\ z_c(t) - x_c(0|t) \in \mathcal{Z}_{K_c}, \\ x_c(k+1|t) = A_c x_c(k|t) + B_c u_c(k|t), \\ x_c(k|t) \in \mathcal{X}_c, \\ u_c(k|t) \in \mathcal{U}_c, \\ x_c(N|t) \in \mathcal{X}_{f_c}, \end{cases}
\end{aligned} \tag{6.4}$$

where N is the prediction horizon, $z_c(t)$ is the measured state, $x_c(k|t)$ and $u_c(k|t)$ are the k-step ahead predicted state and input vectors at time instant t, $l_c(\cdot,\cdot)$ is the stage cost and $V_{f_c}(\cdot)$ is the terminal cost. To guarantee robust constraint satisfaction, the control input $v_c(t) = u_c(0|t) + K_c(z_c(t) - x_c(0|t))$ is applied to (6.1) at each time instant.

6.2.2 Control of Interconnected Systems

We consider here the class of interconnected systems with discrete-time linear time-invariant dynamics. We assume that this system can be decomposed into M dynamically-coupled subsystems and that the i-th subsystem has a set of neighbours $\mathcal{N}_i$ including itself and all other subsystems with which it is dynamically coupled. We assume that the dynamics of the i-th subsystem is given by,

$$z_i(t+1) = A_i z_i(t) + B_i v_i(t) + E_i \sum_{j\in\mathcal{N}_i\setminus i} a_{ij} C_j z_j(t) + d_i(t), \tag{6.5}$$

where $z_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}^{m_i}$ and $d_i \in \mathbb{R}^{n_i}$ are the state, input and noise vectors of the i-th subsystem, $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{m_i\times n_i}$ and $E_i \in \mathbb{R}^{n_i\times m_i}$ are known subsystem matrices and $a_{ij} \in \mathbb{R}$ are unknown constant parameters which represent the coupling strengths. We assume that the scalars a_{ij} satisfy $0 \le a_{ij}^{min} \le a_{ij} \le a_{ij}^{max}$ where the bounds a_{ij}^{min} and a_{ij}^{max} are known. We also assume that the i-th subsystem is subject to the state and input constraints,

$$\begin{aligned}
z_i(t) &\in \mathcal{Z}_i = \{z_i : G_i z_i \le 1\}, \\
v_i(t) &\in \mathcal{V}_i = \{v_i : H_i v_i \le 1\},
\end{aligned}$$

where $G_i \in \mathbb{R}^{q_i\times n_i}$, $H_i \in \mathbb{R}^{r_i\times m_i}$, $\mathcal{Z}_i$ and $\mathcal{V}_i$ are compact and include the origin in their interior. Finally, the noise vector $d_i(t)$ is assumed to lie in a known compact polytopic set $\mathcal{D}_i = \{d_i : \Xi_i d_i \le 1\}$ which includes the origin in its interior.

The idea behind the rigid tube-based robust MPC scheme (6.4) can be used here to develop an MPC scheme for controlling the considered class of interconnected systems [RFF13b]. This can be done by considering the coupling terms in the dynamics (6.5) of each subsystem as disturbances. In this case, the dynamics of the i-th subsystem can be written as,

$$\begin{aligned} z_i(t+1) &= A_i z_i(t) + B_i v_i(t) + E_i w_i(t) + d_i(t), \\ w_i(t) &= \sum_{j \in \mathcal{N}_i \backslash i} a_{ij} C_j z_j(t) \in \mathcal{W}_i = \bigoplus_{j \in \mathcal{N}_i \backslash i} a_{ij}^{max} C_j Z_j, \end{aligned} \tag{6.6}$$

with the nominal dynamics given by,

$$x_i(t+1) = A_i x_i(t) + B_i u_i(t),$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the nominal state and input vectors of the i-th subsystem, respectively.

We consider here a tracking problem where the state is required to follow the target point x_{r_i} which is an equilibrium point of the nominal system. The target point x_{r_i} and the corresponding input u_{r_i} are assumed to satisfy the state and input constraints. For this purpose, we define the artificial equilibrium point (x_{e_i}, u_{e_i}) which satisfies the constraint,

$$x_{e_i} = A_i x_{e_i} + B_i u_{e_i}.$$

We also select the cost function,

$$\begin{aligned} &J_i \left(x_i(k|t)|_{k\in\{0,\dots,N\}}, u_i(k|t)|_{k\in\{0,\dots,N-1\}}, \; x_{e_i}, u_{e_i} \right) \\ &\qquad = \sum_{k=0}^{N-1} \left(\|x_i(k|t) - x_{e_i}\|_{Q_i}^2 + \|u_i(k|t) - u_{e_i}\|_{R_i}^2 \right) + \|x_{e_i} - x_{r_i}\|_{S_i}^2, \end{aligned}$$

where $x_i(k|t)$ and $u_i(k|t)$ are the k-step ahead predicted nominal state and input vectors of the i-th subsystem at time instant t.

Following the same concept of the rigid tube-based robust MPC scheme (6.4) and considering,

$$v_i(t) = u_i(t) + K_i(z_i(t) - x_i(t)),$$

we define for the i-th subsystem the error dynamics,

$$e_i(t+1) = (A_i + B_i K_i) e_i(t) + E_i w_i(t) + d_i(t), \tag{6.7}$$

where $e_i(t) = z_i(t) - x_i(t)$ and K_i is a stabilizing control gain. We consider a local

optimal control problem for the i-th subsystem,

$$\begin{aligned}
&\min_{\substack{x_i(k|t)|_{k\in\{0,\dots,N\}}, x_{e_i},\\ u_i(k|t)|_{k\in\{0,\dots,N-1\}}, u_{e_i}}} J_i\begin{pmatrix} x_i(k|t)|_{k\in\{0,\dots,N\}}, x_{e_i},\\ u_i(k|t)|_{k\in\{0,\dots,N-1\}}, u_{e_i}\end{pmatrix}\\
&s.t.\begin{cases}
\text{for all } k\in\{0,\dots,N-1\},\\
z_i(t)-x_i(0|t)\in\mathcal{Z}_{K_i},\\
x_i(k+1|t)=A_ix_i(k|t)+B_iu_i(k|t),\\
x_i(k|t)\in\mathcal{X}_i,\quad u_i(k|t)\in\mathcal{U}_i,\\
x_{e_i}=A_ix_{e_i}+B_iu_{e_i},\\
x_{e_i}\in\xi\mathcal{X}_i,\quad u_{e_i}\in\xi\mathcal{U}_i,\\
x_i(N|t)=x_{e_i},
\end{cases}
\end{aligned} \tag{6.8}$$

where $z_i(t)$ is the measured state of the i-th subsystem, $\mathcal{Z}_{K_i}$ is an RPI set for its error dynamics (6.7) under the disturbances $w_i(t)$ and $d_i(t)$, $\xi\in(0,1)$, $\mathcal{X}_i=\{x_i : G_ix_i\le g_i\}$ and $\mathcal{U}_i=\{u_i : H_ix_i\le h_i\}$ are its tightened constraint sets where g_i and h_i are positive scalars. Notice that the last four constraints in (6.8) indicate that $x_i(N|t)$ reaches an invariant terminal set which satisfies the state and input constraints. Note also that the global prestabilzing control gain is denoted by $K=\text{diag}(K_1,\dots,K_M)$. Similar to the MPC scheme (6.4), the control input $v_i(t)=u_i(0|t)+K_i(z_i(t)-x_i(0|t))$ is applied to (6.5) to guarantee robust constraint satisfaction.

The interconnected system can be described using a graph $\mathcal{G}(\mathcal{N},\mathcal{E},\mathcal{A})$ where $\mathcal{N}=\{1,\dots,M\}$ represents the set of nodes, $\mathcal{E}\subseteq(\mathcal{N}\times\mathcal{N})$ the set of edges and $\mathcal{A}=\{a_{ij}\in\mathbb{R},(i,j)\in\mathcal{E}\}$ the set of weights of the graph $\mathcal{G}$. Each subsystem is represented by a node. An edge exists between the i-th and j-th nodes if the i-th and j-th subsystems are neighbours. We define the Adjacency matrix $A_g\in\mathbb{R}^{M\times M}$ of the graph $\mathcal{G}$ such that the element in its i-th row and j-th column is given by,

$$A_{g_{ij}}=\begin{cases}0, & i=j \text{ or } (i\notin\mathcal{N}_j \text{ and } j\notin\mathcal{N}_i),\\ a_{ij}, & i\neq j \text{ and } (i\in\mathcal{N}_j \text{ or } j\in\mathcal{N}_i).\end{cases}$$

The degree matrix $D_g\in\mathbb{D}_+^M$ of the graph $\mathcal{G}$ is defined such that the diagonal element in the i-th row is given by $\sum_{j\in\mathcal{N}_i\setminus i}a_{ij}$. The Laplacian matrix $L_g\in\mathbb{R}^{M\times M}$ of the graph $\mathcal{G}$ is given by D_g-A_g and is positive semidefinite by definition. The considered system can be viewed as a graph with an uncertain adjacency matrix whose non zero entries are unknown but lie within known compact sets. The global dynamics is then given by,

$$z(t+1)=Az(t)+Bv(t)+d(t), \tag{6.9}$$

where $z(t)=[z_1^\top(t),\dots,z_M^\top(t)]^\top$, $v(t)=[v_1^\top(t),\dots,v_M^\top(t)]^\top$, $d(t)=[d_1^\top(t),\dots,d_M^\top(t)]^\top$, $A=A_{\text{dg}}+EA_dC$, $B=\text{diag}(B_1,\dots,B_M)$, $C=\text{diag}(C_1,\dots,C_M)$, $E=\text{diag}(E_1,\dots,E_M)$ and $A_{\text{dg}}=\text{diag}\,(A_{11},\dots,A_{MM})$.

6.2.3 Problem Formulation

At each time instant, the proposed scheme performs two phases: a learning phase to learn the model uncertainty and an adaptation phase to compute the optimal control action. We assume that every subsystem can communicate with its neighbours at each time instant. Although distributed MPC schemes can be developed in this case, such schemes require extensive computation and communication. Hence, decentralized MPC schemes are developed in this work where the different subsystems do not communicate in the adaptation phase. Communication is only limited to the learning phase to learn the model uncertainty which relies on the states of the neighbours.

In the learning phase, we aim to reduce the volume of the uncertainty set $\mathcal{W}_i$ in (6.6) by learning the bounds a_{ij}^{min} and a_{ij}^{max} of the uncertain parameter a_{ij} as more data is collected online. For this purpose, we develop a distributed set membership identification approach which learns these bounds by sharing information among the neighbours only once at each time instant. In the adaptation phase, we aim to adapt all the MPC ingredients which depend on the learnt uncertainty set and compute the local optimal control inputs taking the updated uncertainty set into consideration. The adapted MPC ingredients include the rigid tube represented by $\mathcal{Z}_{K_i}$, the prestabilizing controller K_i, the tightened state constraint set $\mathcal{X}_i$, the tightened input constraint set $\mathcal{U}_i$ and the terminal set x_{e_i}.

In the sequel, we use the ellipsoidal rigid tube,

$$\mathcal{Z}_{K_i} = \{e_i : e_i^\top Z_i e_i \leq \alpha_i^2\},$$

and the linear prestabilizing controller,

$$\kappa_i(x_i) = K_i x_i.$$

We assume that $K_i = K_{o_i} T_i$ where K_{o_i} is a full-rank stabilizing control gain and T_i is a positive definite square matrix. Recall also that the tightened polytopic state and control constraint sets are given, respectively, by $\mathcal{X}_i = \{x_i : G_i x_i \leq g_i\}$ and $\mathcal{U}_i = \{u_i : H_i x_i \leq h_i\}$. In addition to the decision variables in (6.8), α_i, T_i, g_i and h_i are also considered as decision variables in the developed scheme in order to update the MPC ingredients online; K_{o_i}, on the other hand, is assumed to be known in advance, computed, for example, with methods such as the one introduced in Chapter 4. When updating these ingredients, one should ensure satisfying the following conditions,

I. The set $\mathcal{Z}_{K_i}$ is a robust positive invariant set of the error dynamics (6.7).

II. The local matrices $A_i + B_i K_i$ for all $i \in \{1, \ldots, M\}$ is Schur.

III. The global matrix $A + BK$ is Schur.

IV. The Minkowski sum of the tightened state constraint set and the RPI set $\mathcal{Z}_{K_i}$ is contained in the actual state constraint set (i.e. $\mathcal{X}_i \bigoplus \mathcal{Z}_{K_i} \subseteq \mathcal{Z}_i$).

V. The Minkowski sum of the tightened input constraint set and the set $K_i\mathcal{Z}_{k_i}$ is contained in the actual input constraint set (i.e. $\mathcal{U}_i \bigoplus K_i\mathcal{Z}_{K_i} \subseteq \mathcal{V}_i$).

6.3 Control Design

First, we discuss the distributed set membership identification approach used to learn the uncertainty set online. We then derive the additional constraints added to the local online optimal control problem of each subsystem.

6.3.1 Learning Phase

We aim to learn the uncertainty set $\mathcal{W}_i$ in real time, as more data is collected online. According to (6.6), this set is a function of the bounds of the uncertain parameters. Hence, a better estimation of these bounds leads to a better description of the uncertainty set. Thus, we develop here a distributed identification technique based on set membership identification in order to improve the estimates of these bounds.

Assume that the bounds of the uncertain parameter a_{ij} at the previous time instant $t-1$ are $a_{ij}^{min}(t-1)$ and $a_{ij}^{max}(t-1)$. Our goal is to improve the estimates of these bounds at the current time instant t. For this purpose, we construct for every a_{ij} the feasible parameter set defined by the inequality,

$$F_{ij}a_{ij} \leq f_{ij}(t-1),$$

where,

$$F_{ij} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad f_{ij}(t-1) = \begin{bmatrix} a_{ij}^{max}(t-1) \\ a_{ij}^{min}(t-1) \end{bmatrix}.$$

We then define the vector θ_i which comprises all the uncertain parameters to be identified by the i-th subsystem (i.e. a_{ij}, $j \in \mathcal{N}_i \setminus i$). In this case, the feasible parameter set for the i-th subsystem is given by,

$$F_i\theta_i \leq f_i(t-1),$$

where F_i is a block diagonal matrix with the matrices F_{ij} for all $j \in \mathcal{N}_i \setminus i$ on the diagonal and f_i is a vector vertically concatenating the subvectors f_{ij} for all $j \in \mathcal{N}_i \setminus i$.

Recall that each subsystem can communicate with its neighbours in the learning phase. Hence, the states of the neighbours of the i-th subsystem at the previous time instant $t-1$ are available for the i-th subsystem at the current time instant t. Furthermore, the states and inputs of the i-th subsystem at the previous time instant $t-1$ are available. Finally, the states of the i-th subsystem at the current time instant t can be

measured. Recall also that $\mathcal{D}_i = \{d_i : \Xi_i d_i \leq 1\}$. Defining $\chi_i(t-1)$ to be a matrix horizontally concatenating the vectors $C_j z_j(t-1)$ for all $j \in \mathcal{N}_i \setminus i$ and,

$$\Omega_i(t-1) = -\Xi_i E_i \chi_i(t-1),$$
$$\omega_i(t-1) = 1 + \Xi_i A_i z_i(t-1) + \Xi_i B_i v_i(t-1) - \Xi_i z_i(t),$$

we can construct for the i-th subsystem the non-falsified parameter set,

$$\begin{aligned}\Delta_i &= \{a_{ij}\ \forall j \in \mathcal{N}_i \setminus i : z_i(t) - A_i z_i(t-1) - B_i v_i(t-1) - E_i \sum_{j \in \mathcal{N}_i \setminus i} a_{ij} C_j z_j(t-1) \in \mathcal{D}_i\} \\ &= \{\theta_i : -\Xi_i E_i \chi_i(t-1)\theta_i \leq 1 + \Xi_i A_i z_i(t-1) + \Xi_i B_i v_i(t-1) - \Xi_i z_i(t)\} \\ &= \{\theta_i : \Omega_i(t-1)\theta_i \leq \omega_i(t-1)\}.\end{aligned}$$

In order to update the bounds of its uncertain parameters, the i-th subsystem should solve $2(|\mathcal{N}_i|-1)$ optimization problems at time instant t where $|\cdot|$ denotes set cardinality. Each optimization problem updates one of its bounds. For each $j \in \mathcal{N}_i \setminus i$, the lower bound of the parameter a_{ij} is given by,

$$a_{ij}^{min}(t) = \min_{a_{ij}} \ e_{ij}^\top \theta_i \text{ s.t. } \begin{bmatrix} F_i \\ \Omega_i(t-1) \end{bmatrix} \theta_i \leq \begin{bmatrix} f_i(t-1) \\ \omega_i(t-1) \end{bmatrix},$$

whereas the upper bound of the parameter a_{ij} is given by,

$$a_{ij}^{max}(t) = \max_{a_{ij}} \ e_{ij}^\top \theta_i \text{ s.t. } \begin{bmatrix} F_i \\ \Omega_i(t-1) \end{bmatrix} \theta_i \leq \begin{bmatrix} f_i(t-1) \\ \omega_i(t-1) \end{bmatrix},$$

where e_{ij} is a unit vector which selects the parameter a_{ij} in the vector θ_i. Using asymmetric duality of linear programs, the lower bound becomes,

$$a_{ij}^{min}(t) = -\min_{\phi_{ij}^{min}} \begin{bmatrix} f_i(t-1) \\ \omega_i(t-1) \end{bmatrix}^\top \phi_{ij}^{min} \text{ s.t. } \begin{bmatrix} F_i \\ \Omega_i(t-1) \end{bmatrix}^\top \phi_{ij}^{min} = -e_{ij} \ \& \ \phi_{ij}^{min} \geq 0, \quad (6.10)$$

whereas the upper bound becomes,

$$a_{ij}^{max}(t) = \min_{\phi_{ij}^{max}} \begin{bmatrix} f_i(t-1) \\ \omega_i(t-1) \end{bmatrix}^\top \phi_{ij}^{max} \text{ s.t. } \begin{bmatrix} F_i \\ \Omega_i(t-1) \end{bmatrix}^\top \phi_{ij}^{max} = e_{ij} \ \& \ \phi_{ij}^{max} \geq 0, \quad (6.11)$$

where ϕ_{ij}^{min} and ϕ_{ij}^{max} are dual variables of appropriate dimensions.

Define the matrix $\mathbf{e}_i$ which vertically concatenates the matrices $[-e_{ij}\ e_{ij}]^\top$ for all $j \in \mathcal{N}_i \setminus i$ and the matrix ϕ_i which horizontally concatenates the matrices $[\phi_{ij}^{min}\ \phi_{ij}^{max}]$ for all $j \in \mathcal{N}_i \setminus i$. Define also the optimization variables ψ_{ij}^{min} and ψ_{ij}^{max} as well as the vector ψ_i which vertically concatenates the vectors $[\psi_{ij}^{min}\ \psi_{ij}^{max}]^\top$. The optimization problems (6.10)-(6.11) can then be compactly written as,

$$\begin{aligned} &\min_{\phi_i} \mathbf{1}^\top \psi_i \\ &\text{s.t. } \begin{cases} \phi_{ij}^{min} \geq 0,\ \phi_{ij}^{max} \geq 0,\ \forall\ j \in \mathcal{N}_i \setminus i, \\ \phi_i^\top \begin{bmatrix} f_i(t-1) \\ \omega_i(t-1) \end{bmatrix} \leq \psi_i, \quad \phi_i^\top \begin{bmatrix} F_i \\ \Omega_i(t-1) \end{bmatrix} = \mathbf{e}_i. \end{cases} \end{aligned} \quad (6.12)$$

where $\mathbf{1}$ is a vector of ones of appropriate dimensions. Note that $a_{ij}^{min} = \psi_{ij}^{min}$ and $a_{ij}^{max} = \psi_{ij}^{max}$. In conclusion, the i-th subsystem learns the uncertainty set $\mathcal{W}_i$ in (6.6) by solving at each time instant t the optimization problem (6.12) which identifies the uncertain parameter bounds. Notice that $a_{ij}^{min}(t) \geq a_{ij}^{min}(t-1)$ and $a_{ij}^{max}(t) \leq a_{ij}^{max}(t-1)$ and hence, $\mathcal{W}_i(t) \subseteq \mathcal{W}_i(t-1)$ for all t.

6.3.2 Adaptation Phase

As mentioned in Section 6.2.3, the online OCP is solved in the adaptation phase to adapt the MPC ingredients based on the learnt uncertainty set and compute the optimal control inputs. To adapt the MPC ingredients online while ensuring robust constraint satisfaction, the five conditions mentioned in Section 6.2.3 should be satisfied. In this section, we derive additional constraints which ensure the satisfaction of these conditions while maintaining the convexity of the online OCP (6.8).

We start with the condition which ensures the robust positive invariance of the set $\mathcal{Z}_{K_i}$. First, we define for the i-th subsystem the output vector $y_i = C_i z_i$ and the constraint set $\mathcal{Y}_i$, obtained from $\mathcal{Z}_i$ in the obvious way, such that $y_i \in \mathcal{Y}_i$. We then compute a minimum volume ellipsoidal set $\hat{\mathcal{Y}}_i = \{y_i : \|y_i - c_{y_i}\|_{Y_i}^2 \leq 1\}$ that contains the set $\mathcal{Y}_i$. For this purpose, the vertex description of $\mathcal{Y}_i$ is needed. Note that this is in general combinatorial in the linear inequalities needed to define $\mathcal{Y}_i$. However, the sets $\mathcal{Y}_i$ of all subsystems do not change over time and hence, their minimum volume ellipsoidal sets $\hat{\mathcal{Y}}_i$ can be computed once offline by solving a convex optimization problem for each. Finally, we construct the outer approximation $\hat{\mathcal{W}}_i(t) = \bigoplus a_{ij}^{max}(t)\hat{\mathcal{Y}}_j$ of the uncertainty set $\mathcal{W}_i(t) = \bigoplus a_{ij}^{max}(t)\mathcal{Y}_j$. Note that $\hat{\mathcal{W}}_i(t)$ is not necessarily an ellipsoidal set, but it is the Minkowski sum of the ellipsoidal sets,

$$\mathcal{W}_{ij} = a_{ij}^{max}(t)\hat{\mathcal{Y}}_j = \{w_{ij} : \|w_{ij} - a_{ij}^{max}(t)c_{y_j}\|_{Y_j}^2 \leq {a_{ij}^{max}}^2(t)\}.$$

Similarly, we also compute a minimum volume ellipsoidal set $\hat{\mathcal{D}}_i = \{d_i : \|d_i\|_{\hat{\Xi}_i}^2 \leq 1\}$ offline for the set $\mathcal{D}_i$ of the i-th subsystem.

In Proposition 6.1, we consider Condition I in Section 6.2.3 and derive constraints which ensure the robust positive invariance of the set $\mathcal{Z}_{K_i}$ for the error dynamics (6.7) under the disturbances $w_i \in \hat{\mathcal{W}}_i(t)$ and $d_i \in \hat{\mathcal{D}}_i$. Although this might be more conservative than considering the disturbances $w_i \in \mathcal{W}_i(t)$ and $d_i \in \mathcal{D}_i$, the convexity of the resulting OCP is preserved in this case. Another source of conservatism in Proposition 6.1 is using the approximate S-lemma to reach local convex constraints. Recall here that $K_i = K_{o_i} T_i$.

Proposition 6.1. *The set Z_{K_i} is robustly positively invariant under the error dynamics (6.7) and the uncertainty sets $\hat{\mathcal{W}}_i$ and $\hat{\mathcal{D}}_i$ if there exist non-negative scalars λ_i, λ_{d_i} and*

λ_{ij} *for all* $j \in \mathcal{N}_i \setminus i$ *such that,*

$$\begin{bmatrix} \lambda_i Z_i & * & * & * & * \\ 0 & Y_{\lambda_i} & * & * & * \\ 0 & 0 & \lambda_{d_i}\hat{\Xi}_i & * & * \\ 0 & -\tau_i^\top Y_{\lambda_i} & 0 & \alpha_i - \lambda_i - \lambda_{d_i} - \sum_{j\in\mathcal{N}_i\setminus i} \lambda_{ij}(1 - \|c_{y_j}\|^2_{Y_j}) a_{ij}^{max^2}(t) & * \\ A_i\alpha_i + B_i K_{o_i} T_i \alpha_i & \mathcal{E}_i & I_{n_i} & 0 & \alpha_i Z_i^{-1} \end{bmatrix} \geq 0. \tag{6.13}$$

Proof. The set Z_{K_i} is robustly positively invariant under the considered dynamics and uncertainty sets if and only if,

$$\|e_i^+\|^2_{Z_i} \leq \alpha_i^2 \text{ for all } \begin{cases} e_i : \|e_i\|^2_{Z_i} \leq \alpha_i^2, \\ w_{ij} : \|w_{ij} - a_{ij}^{max}(t) c_{y_j}\|^2_{Y_j} \leq a_{ij}^{max^2}(t), \ j \in \mathcal{N}_i \setminus i, \\ d_i : \|d_i\|^2_{\hat{\Xi}_i} \leq 1. \end{cases}$$

By defining $s_i = \alpha_i e_i$, the invariance condition becomes,

$$\|(A_i\alpha_i + B_i K_{o_i} T_i \alpha_i) s_i + E_i w_i + d_i\|^2_{Z_i} \leq \alpha_i^2,$$
$$\text{for all } \begin{cases} s_i : \|s_i\|^2_{Z_i} \leq 1, \\ w_{ij} : \|w_{ij} - a_{ij}^{max}(t) c_{y_j}\|^2_{Y_j} \leq a_{ij}^{max^2}(t), \ j \in \mathcal{N}_i \setminus i, \\ d_i : \|d_i\|^2_{\hat{\Xi}_i} \leq 1. \end{cases}$$

We now define the block diagonal matrix Y_{λ_i} with the submatrices $\lambda_{ij} Y_j$ for all $j \in \mathcal{N}_i \setminus i$ on the diagonal and the vector τ_i which vertically concatenates the subvectors $a_{ij}^{max}(t) c_{y_j}$. Using the S-lemma, the invariance condition is given by,

$$\bar{T}_i - \bar{S}_i \alpha_i^{-1} Z_i \bar{S}_i^\top \geq 0,$$

where,

$$\bar{T}_i = \begin{bmatrix} \lambda_i Z_i & 0 & 0 & 0 \\ 0 & Y_{\lambda_i} & 0 & -Y_{\lambda_i}\tau_i \\ 0 & 0 & \lambda_{d_i}\hat{\Xi}_i & 0 \\ 0 & -\tau_i^\top Y_{\lambda_i}^\top & 0 & \alpha_i - \lambda_i - \lambda_{d_i} - \sum_{j\in\mathcal{N}_i\setminus i} \lambda_{ij} a_{ij}^{max^2}(t)(1 - \|c_{y_j}\|^2_{Y_j}) \end{bmatrix},$$

and,

$$\bar{S}_i = \begin{bmatrix} (A_i\alpha_i + B_i K_{o_i} T_i \alpha_i)^\top \\ \mathcal{E}_i^\top \\ I_{n_i} \\ 0 \end{bmatrix},$$

I_{n_i} is an identity matrix of size n_i and $\mathcal{E}_i$ is a matrix horizontally concatenating the matrix E_i for $|\mathcal{N}_i| - 1$ times. Applying the Schur Complement yields (6.13). □

Next, we consider Condition II in Section 6.2.3 by deriving constraints which guarantee that $A_i + B_i K_i$ for all $i \in \{1, \ldots, M\}$ are Schur and condition III by deriving constraints ensuring that $A + BK$ is Schur. For this purpose, we rely on the passivity arguments of discrete-time interconnected systems. First, we derive a constraint which ensures that the noise-free dynamics of the i-th subsystem (i.e. (6.5) with $d_i(t) = 0$ for all t) under the controller $u_i(t) = K_{o_i} T_i x_i(t)$ is strictly passive. We show that this implies that $A_i + B_i K_i$ is Schur. We then derive other constraints which ensure that the strict passivity of all subsystems implies the asymptotic stability of the whole system ensuring that $A + BK$ is Schur. Before discussing the strict passivity of the i-th subsystem in Proposition 6.2, we define the virtual output,

$$\tilde{y}_i = C_i z_i + D_i w_i,$$

where $D_i \in \mathbb{D}_{++}^{m_i}$ is a decision variable.

Proposition 6.2. *The noise-free dynamics of the i-th subsystem,*

$$z_i(t+1) = (A_i + B_i K_{o_i} T_i) z_i(t) + E_i w_i(t),$$

is strictly passive with respect to the input w_i and output $\tilde{y}_i$ if,

$$\begin{bmatrix} (P_i - \Gamma_i)\alpha_i & 0.5 C_i^\top \alpha_i & (A_i \alpha_i + B_i K_{o_i} T_i \alpha_i)^\top \\ 0.5 C_i \alpha_i & D_i \alpha_i & E_i^\top \alpha_i \\ A_i \alpha_i + B_i K_{o_i} T_i \alpha_i & E_i \alpha_i & P_i^{-1} \alpha_i \end{bmatrix} \geq 0, \tag{6.14}$$

where $P_i \in \mathbb{S}_{++}^{n_i}$ and $\Gamma_i \in \mathbb{D}_{++}^{n_i}$ are the storage and dissipation matrices of the i-th subsystem, respectively.

Proof. First, we define a minimum volume ellipsoidal set $\hat{\mathcal{Z}}_i = \{z_i : \|z_i - c_{z_i}\|_{\hat{Z}_i} \leq 1\}$ for the state constraint set $\mathcal{Z}_i$. The noise-free dynamics of the i-th subsystem under the controller $u_i(t) = K_{o_i} T_i x_i(t)$ is strictly passive if,

$$\|z_i^+\|_{P_i}^2 - \|z_i\|_{P_i}^2 \leq \tilde{y}_i^\top w_i - \|z_i\|_{\Gamma_i}^2,$$
$$\text{for all } \begin{cases} z_i : \|z_i - c_{z_i}\|_{\hat{Z}_i}^2 \leq 1, \\ w_{ij} : \|w_{ij} - a_{ij}^{max}(t) c_{y_j}\|_{Y_j}^2 \leq a_{ij}^{max^2}(t), \ j \in \mathcal{N}_i \setminus i. \end{cases}$$

By substituting for the dynamics, the strict passivity condition is given by,

$$\|z_i\|_{L_{1_i}}^2 + 2 w_i^\top L_{2_i} z_i + \|w_i\|_{L_{3_i}}^2 \geq 0,$$
$$\text{for all } \begin{cases} z_i : \|z_i - c_{z_i}\|_{\hat{Z}_i}^2 \leq 1, \\ w_{ij} : \|w_{ij} - a_{ij}^{max}(t) c_{y_j}\|_{Y_j}^2 \leq a_{ij}^{max^2}(t), \ j \in \mathcal{N}_i \setminus i, \end{cases}$$

where,

$$
\begin{aligned}
L_{1_i} &= P_i - \Gamma_i - (A_i + B_i K_{o_i} T_i)^\top P_i (A_i + B_i K_{o_i} T_i), \\
L_{2_i} &= 0.5 C_i - E_i^\top P_i (A_i + B_i K_{o_i} T_i), \\
L_{3_i} &= D_i - E_i^\top P_i E_i.
\end{aligned}
$$

Now, we define the non-negative scalars ρ_i and σ_{ij} for all $j \in \mathcal{N}_i \setminus i$. We also define the block diagonal matrix Y_{σ_i} with the submatrices $\sigma_{ij} Y_j$ for all $j \in \mathcal{N}_i \setminus i$ on the diagonal and the vector τ_i which vertically concatenates the subvectors $a_{ij}^{max} c_{y_j}$ for all $j \in \mathcal{N}_i \setminus i$. Using the S-lemma, the passivity condition becomes,

$$
\begin{bmatrix}
\rho_i \hat{Z}_i + L_{1_i} & L_{2_i}^\top & -\rho_i \hat{Z}_i c_{z_i} \\
L_{2_i} & Y_{\sigma_i} + L_{3_i} & -Y_{\sigma_i} \tau_i \\
-\rho_i c_{z_i}^\top \hat{Z}_i & -\tau_i^\top Y_{\sigma_i}^\top & -\rho_i - \sum\limits_{j \in \mathcal{N}_i \setminus i} \sigma_{ij} (1 - \|c_{y_j}\|_{Y_j}^2) a_{ij}^{max^2}(t)
\end{bmatrix} \geq 0. \tag{6.15}
$$

Recall that the ellipsoidal set $\hat{\mathcal{Y}}_i$ contains the origin in its interior and hence, $\|c_{y_i}\|_{Y_i}^2 \leq 1$. Thus, the bottom right term in (6.15) is always negative hindering the satisfaction of this condition unless $\rho_i = 0$ and $\sigma_{ij} = 0$ for all $j \in \mathcal{N}_i \setminus i$. Setting these variables to zero yields,

$$
\begin{bmatrix}
L_{1_i} & L_{2_i}^\top \\
L_{2_i} & L_{3_i}
\end{bmatrix} \geq 0. \tag{6.16}
$$

Using the Schur complement and multiplying the resulting matrix by α_i leads to (6.14). □

Notice that (6.16) implies that for all $z_i \in \mathbb{R}^{n_i}$, $L_{1_i} \geq 0$ or equivalently,

$$
z_i^\top (P_i - (A_i + B_i K_{o_i} T_i)^\top P_i (A_i + B_i K_{o_i} T_i)) z_i \geq z_i^\top \Gamma_i z_i,
$$

where $\Gamma_i \in \mathbb{S}_{++}^{n_i}$ and hence, $A_i + B_i K_i$ is Schur. While P_i is computed offline together with K_{o_i}, for example as discussed in Chapter 4, Γ_i is considered as an additional decision variable in the online OCP. Note also that setting the S-lemma variables to zero means that the passivity condition is satisfied globally, which is not surprising for passivity of linear systems. One source of conservatism in Proposition 6.2 to satisfy Condition II is that the local controller is designed to not only stabilize the local dynamics, but also to passivize it. The benefit of passivizing the local dynamics will become obvious in the next proposition.

We now consider Condition III in Section 6.2.3 by deriving constraints which ensure that the strict passivity of all noise-free subsystems implies the asymptotic stability of the overall noise-free system. For this purpose, we consider the noise-free global dynamics,

$$
z(t+1) = A_{cl} z(t) + E w(t),
$$

where,

$$\begin{aligned} A_{cl} &= \text{diag}(A_{11} + B_1 K_{o_1} T_1, \ldots, A_{MM} + B_M K_{o_M} T_M), \\ E &= \text{diag}(E_1, \ldots, E_M), \\ w &= [w_1^\top, \ldots, w_M^\top]^\top. \end{aligned}$$

We also define the Lyapunov function,

$$V(z) = \|z\|_P^2,$$

where $P = \text{diag}(P_1, \ldots, P_M)$. Furthermore, we define the matrices,

$$\begin{aligned} \Gamma &= \text{diag}(\Gamma_1, \ldots, \Gamma_M), \\ C &= \text{diag}(C_1, \ldots, C_M), \\ D &= \text{diag}(D_1, \ldots, D_M). \end{aligned}$$

Recall that A_g and D_g are the adjacency and degree matrices, respectively. The global virtual output vector is given by,

$$\tilde{y} = Cz + Dw,$$

and the global coupling vector is given by,

$$w = A_g C z.$$

Finally, we construct A_g^{max} and D_g^{max} by replacing all uncertain parameters in A_g and D_g with their upper bounds. Consequently, we define,

$$\begin{aligned} U &= \text{diag}(U_1, \ldots, U_M) = C^\top D_g C, \\ V &= [V_1^\top, \ldots, V_M^\top]^\top = C^\top A_g^\top, \\ W &= [W_1^\top, \ldots, W_M^\top]^\top = A_g C, \end{aligned}$$

where $U_i \in \mathbb{R}^{m_i \times m_i}$, $V_i \in \mathbb{R}^{n_i \times m_i}$ and $W_i \in \mathbb{R}^{m_i \times n_i}$. We also construct U^{max}, V^{max} and W^{max} by replacing all uncertain parameters in U, V and W with their upper bounds. In the sequel, we denote the diagonal element in the j^{th} row of a matrix by $[\cdot]_j$ and the summation of the absolute values of the entries in the j^{th} row of a matrix by $|\cdot|_j$.

Proposition 6.3. *The strict passivity of the noise-free dynamics of all subsystems under the corresponding controller $u_i(t) = K_{o_i} T_i x_i(t)$,*

$$z_i(t+1) = (A_i + B_i K_{o_i} T_i) z_i(t) + E_i w_i(t),$$

implies the asymptotic stability of the noise-free dynamics of the overall system,

$$z(t+1) = A_{cl} z(t) + E w(t),$$

if,

$$\begin{aligned} &([\Gamma_i]_j - \epsilon)\alpha_i \geq |U_i^{max}|_j \alpha_i + |V_i^{max}|_j \alpha_i, \quad \textit{for all } j \in \{1, \ldots, n_i\} \\ &[D_i]_j \alpha_i \leq \alpha_i / |W_i^{max}|_j, \ \textit{for all } j \in \{1, \ldots, m_i\} : |W_i|_j \neq 0. \end{aligned} \tag{6.17}$$

Proof. The passivity of the noise-free dynamics of each subsystem implies that,

$$\|z_i^+\|_{P_i}^2 - \|z_i\|_{P_i}^2 \leq \tilde{y}_i^\top w_i - \|z_i\|_{\Gamma_i}^2.$$

Summing up these inequalities for all subsystems yields,

$$V(z^+) - V(z) \leq \tilde{y}^\top w - \|z\|_\Gamma.$$

To ensure asymptotic stability, it suffices to guarantee that,

$$\tilde{y}^\top w - \|z\|_\Gamma < 0.$$

Using the expressions of $\tilde{y}$ and w, this condition can be written as,

$$z^\top(\Gamma - C^\top A_g C - C^\top A_g^\top D A_g C)z \geq \epsilon z^\top z,$$

where $\epsilon > 0$. Recall that $L_g = D_g - A_g$ and hence,

$$z^\top(\Gamma + C^\top L_g C - C^\top D_g C - C^\top A_g^\top D A_g C)z \geq \epsilon z^\top z.$$

Note also that L_g is positive semidefinite by definition and hence, it is sufficient to ensure that,

$$z^\top(\Gamma - C^\top D_g C - C^\top A_g^\top D A_g C)z \geq \epsilon z^\top z.$$

This condition can be reformulated using the Schur Complement as,

$$\begin{bmatrix} \Gamma - C^\top D_g C - \epsilon I_n & C^\top A_g^\top \\ A_g C & D^{-1} \end{bmatrix} \geq 0. \tag{6.18}$$

Using diagonal dominance, the matrix inequality (6.18) can be approximated for each subsystem using the inequalities,

$$[\Gamma_i]_j - \epsilon \geq |U_i|_j + |V_i|_j \text{ for all } j \in \{1, \ldots, n_i\},$$
$$[D_i^{-1}]_j \geq |W_i|_j \text{ for all } j \in \{1, \ldots, m_i\}.$$

Multiplying the former by α_i and the latter by α_i^{-1} and inverting, these inequalities become,

$$([\Gamma_i]_j - \epsilon)\alpha_i \geq |U_i|_j \alpha_i + |V_i|_j \alpha_i \text{ for all } j \in \{1, \ldots, n_i\},$$
$$[D_i]_j \alpha_i \leq \alpha_i / |W_i|_j, \text{ for all } j \in \{1, \ldots, m_i\} : |W_i|_j \neq 0.$$

Note that the resulting inequalities are functions of the uncertain parameters. More conservative inequalities can be obtained by replacing the uncertain parameters with their upper bounds, leading to (6.17). □

In Proposition 6.3, (6.17) ensures that the Lyapunov condition is satisfied for the noise-free global dynamics $z(t+1) = A_{cl}z(t) + Ew(t)$ or equivalently, $z(t+1) = (A + BK)z(t)$ and hence, $A + BK$ is Schur. Sources of conservatism in this proposition to satisfy Condition III include implying $\tilde{y}^\top w - \|z\|_\Gamma < 0$ instead of $V(z^+) - V(z) < 0$, ignoring the term $C^\top L_g C$, using diagonal dominance and replacing the true values of the parameters with their upper bounds. These sources are utilized to eventually reach local convex constraints.

We now move to Conditions IV and V in Section 6.2.3. In the sequel, we denote the j^{th} row of the matrices G_i and H_i by $G_i^{(j)}$ and $H_i^{(j)}$ respectively.

Proposition 6.4. *The Minkowski sum of the tightened state constraint set $\mathcal{X}_i$ and the RPI set $\mathcal{Z}_{K_i}$ is contained in the actual state constraint set $\mathcal{Z}_i$ (i.e. $\mathcal{X}_i \bigoplus \mathcal{Z}_{K_i} \subseteq \mathcal{Z}_i$) if there exist non-negative scalars μ_{ij} for all $j \in \{1, \dots, q_i\}$ such that,*

$$\begin{bmatrix} \mu_{ij} Z_i & -0.5 G_i^{(j)^\top} \alpha_i \\ -0.5 G_i^{(j)} \alpha_i & 1 - \mu_{i_j} - g_i \end{bmatrix} \geq 0. \tag{6.19}$$

Proof. The Minkowski sum of $\mathcal{X}_i$ and $\mathcal{Z}_{K_i}$ is contained in $\mathcal{Z}_i$ if and only if for all $j \in \{1, \dots, q_i\}$,

$$G_i^{(j)}(x_{e_i} + x_{s_i}) \leq 1 \text{ for all } \begin{cases} x_{e_i} : \|x_{e_i}\|_{Z_i}^2 \leq \alpha_i^2, \\ x_{s_i} : G_i x_{s_i} \leq g_i. \end{cases}$$

By defining s_i and t_i such that $x_{e_i} = \alpha_i s_i$ and $x_{s_i} = g_i t_i$, this condition becomes for all $j \in \{1, \dots, q_i\}$,

$$G_i^{(j)} \alpha_i s_i + G_i^{(j)} g_i t_i \leq 1, \text{ for all } \begin{cases} s_i : s_i^\top Z_i s_i \leq 1, \\ t_i : G_i t_i \leq 1. \end{cases}$$

Using the S-lemma, this condition is given for all $j \in \{1, \dots, q_i\}$ by,

$$\begin{bmatrix} \mu_{ij} Z_i & 0 & -0.5 \alpha_i G_i^{(j)^\top} \\ 0 & 0 & \sum_{l=1}^{q_i} 0.5 \mu_{ijl} G_i^{(l)^\top} - 0.5 G_i^{(j)^\top} g_i \\ -0.5 \alpha_i G_i^{(j)} & \sum_{l=1}^{q_i} 0.5 \mu_{ijl} G_i^{(l)} - 0.5 G_i^{(j)} g_i & 1 - \mu_{ij} - \sum_{l=1}^{q_i} \mu_{ijl} \end{bmatrix} \geq 0. \tag{6.20}$$

Note that (6.20) holds for all $j \in \{1, \dots, q_i\}$ if for all $j \in \{1, \dots, q_i\}$,

$$\sum_{l=1}^{q_i} 0.5 \mu_{ijl} G_i^{(l)} = 0.5 G_i^{(j)} g_i.$$

By stacking this equality for all $j \in \{1, \dots, q_i\}$, we reach $g_i G_i = \mu_i G_i$ where μ_{ijl} is the entry of μ_i in the j-th row and l-th column. To satisfy this equality, we choose the diagonal elements of μ_i to be equal to g_i and the off-diagonal elements to be equal to zero. In this case, (6.20) reduces for all $j \in \{1, \dots, q_i\}$ to

$$\begin{bmatrix} \mu_{ij} Z_i & -0.5 \alpha_i G_{i_j}^\top \\ -0.5 \alpha_i G_{i_j} & 1 - \mu_{ij} - \mu_{ijj} \end{bmatrix} \geq 0. \tag{6.21}$$

Substituting μ_{ijj} by g_i in (6.21) yields (6.19). □

Proposition 6.5. *The Minkowski sum of the tightened input constraint set $\mathcal{U}_i$ and the set $K_{o_i}T_i\mathcal{Z}_{k_i}$ is contained in the actual input constraint set $\mathcal{V}_i$ (i.e. $\mathcal{U}_i \bigoplus K_{o_i}T_i\mathcal{Z}_{K_i} \subseteq \mathcal{V}_i$) if there exist positive scalars ν_{ij} for all $j \in \{1,\dots,r_i\}$ and a matrix $M_i \in \mathbb{S}^{m_i}_{++}$ such that,*

$$\begin{bmatrix} M_i^{-1} & K_i^\top \alpha_i \\ K_i\alpha_i & \nu_{ij} Z_i \end{bmatrix} \geq 0, \tag{6.22}$$

and,

$$\begin{bmatrix} M_i^{-1} & 0.5M_i^{-1}H_i^{(j)^\top} \\ 0.5H_i^{(j)}M_i^{-1} & 1-\nu_{ij}-h_i \end{bmatrix} \geq 0. \tag{6.23}$$

Proof. First, we define for all $j \in \{1,\dots,r_i\}$, $M_i \in \mathbb{S}^{m_i}_{++}$ and $\nu_{ij} > 0$ such that,

$$M_i \leq \nu_{ij}(K_{o_i}T_i\alpha_i Z_i^{-1}\alpha_i T_i^\top K_{o_i}^\top)^{-1}.$$

Since $T_i \in \mathbb{S}^{n_i}_{++}$, $\alpha_i > 0$ and K_{o_i} is full-rank, this inequality is invertible and hence for all $j \in \{1,\dots,r_i\}$,

$$M_i^{-1} - \nu_{ij}^{-1}K_{o_i}T_i\alpha_i Z_i^{-1}\alpha_i T_i^\top K_{o_i}^\top \alpha_i \geq 0$$

Applying the Schur Complement yields (6.22).

The Minkowski sum of $\mathcal{U}_i$ and $K_{o_i}T_i\mathcal{Z}_{K_i}$ is contained in $\mathcal{V}_i$ if and only if for all $j \in \{1,\dots,r_i\}$,

$$H_i^{(j)}(u_{e_i}+u_{s_i}) \leq 1, \text{ for all } \begin{cases} u_{e_i} : \|u_{e_i}\|^2_{(K_{o_i}T_iZ_i^{-1}T_i^\top K_{o_i}^\top)^{-1}} \leq \alpha_i^2, \\ u_{s_i} : H_iu_{s_i} \leq h_i. \end{cases}$$

By defining s_i and t_i such that $u_{e_i} = s_i$ and $u_{s_i} = h_it_i$, we reach for all $j \in \{1,\dots,r_i\}$ the condition,

$$H_i^{(j)}s_i + H_i^{(j)}h_it_i \leq 1 \text{ for all } \begin{cases} s_i : s_i^\top(K_{o_i}T_i\alpha_iZ_i^{-1}\alpha_iT_i^\top K_{o_i}^\top)^{-1}s_i \leq 1, \\ t_i : H_it_i \leq 1. \end{cases}$$

Using the S-lemma, we derive for all $j \in \{1,\dots,r_i\}$ the matrix inequality,

$$\begin{bmatrix} \nu_{ij}(K_{o_i}T_i\alpha_iZ_i^{-1}T_i^\top K_{o_i}^\top\alpha_i)^{-1} & 0 & -0.5H_i^{(j)^\top} \\ * & 0 & \sum_{l=1}^{r_i} 0.5\nu_{ijl}H_i^{(l)^\top} - 0.5H_i^{(j)^\top}h_i \\ * & * & 1-\nu_{ij}-\sum_{l=1}^{r_i}\nu_{ijl} \end{bmatrix} \geq 0. \tag{6.24}$$

Notice that (6.24) holds for all $j \in \{1,\dots,r_i\}$ if for all $j \in \{1,\dots,r_i\}$,

$$\sum_{l=1}^{r_i} 0.5\nu_{ijl}H_i^{(l)} = 0.5H_i^{(j)^\top}h_i.$$

By stacking this equality for all $j \in \{1, \dots, r_i\}$, we reach $h_i H_i = \nu_i H_i$ where ν_{ijl} is the entry of ν_i in the j-th row and l-th column. To satisfy this equality, we choose the diagonal elements of ν_i to be equal to h_i and the off-diagonal elements to be equal to zero. Recalling that $M_i \leq \nu_{ij}(K_i \alpha_i Z_i^{-1} K_i^\top \alpha_i)^{-1}$, (6.24) is satisfied if for all $j \in \{1, \dots, r_i\}$,

$$\begin{bmatrix} M_i & -0.5 H_i^{(j)^\top} \\ -0.5 H_i^{(j)} & 1 - \nu_{ij} - \nu_{ijj} \end{bmatrix} \geq 0. \tag{6.25}$$

Substituting ν_{ijj} by h_i in (6.25) and pre- and post-multiplying by $\text{diag}(M_i^{-1}, 1)$ yield (6.23). □

One source of conservatism in Proposition 6.4 is choosing the diagonal elements of μ_i to equal g_i and the off-digonal elements to equal zero. The same applies to ν_i in Proposition 6.5. Additionally, introducing M_i adds one more degree of conservatism in Proposition 6.5.

In conclusion, the optimal control problem of the i-th subsystem becomes,

$$\begin{aligned} &\min J_i \left(x_i(k|t)|_{k\in\{0,\dots,N\}}, u_i(k|t)|_{k\in\{0,\dots,N-1\}}, x_{e_i}, u_{e_i} \right) \\ &s.t. \begin{cases} \text{for all } k \in \{0, \dots, N-1\}, \\ z_i(t) - x_i(0|t) \in \mathcal{Z}_{K_i}, \\ x_i(k+1|t) = A_i x_i(k|t) + B_i u_i(k|t), \\ x_i(k|t) \in \mathcal{X}_i, \quad u_i(k|t) \in \mathcal{U}_i, \\ x_{e_i} = A_i x_{e_i} + B_i u_{e_i}, \\ x_{e_i} \in \xi \mathcal{X}_i, \quad u_{e_i} \in \xi \mathcal{U}_i, \\ x_i(N|t) = x_{e_i} \\ (6.13),\ \lambda_i \geq 0,\ \lambda_{d_i} \geq 0,\ \lambda_{ij} \geq 0 \ \text{ for all } j \in \mathcal{N}_i \setminus i \\ (6.14), (6.17),\ T_i \alpha_i \in \mathbb{S}^{n_i}_{++},\ \Gamma_i \alpha_i \in \mathbb{D}^{n_i}_{++},\ D_i \alpha_i \in \mathbb{D}^{n_i}_{++} \\ (6.19),\ \mu_{ij} \geq 0 \text{ for all } j \in \{1, \dots, q_i\} \\ (6.22), (6.23)\ M_i \in \mathbb{S}^{n_i}_{++},\ \nu_{ij} > 0 \text{ for all } j \in \{1, \dots, r_i\} \end{cases} \end{aligned} \tag{6.26}$$

The decision variables are $x_i(k|t)$ and $u_i(k|t)$ for all $k \in \{0, \dots, N\}$, x_{e_i}, u_{e_i}, g_i, h_i, α_i, $T_i \alpha_i$, $\Gamma_i \alpha_i$, $D_i \alpha_i$, M_i^{-1}, λ_i, λ_{d_i}, λ_{ij} for all $j \in \mathcal{N}_i \setminus i$, μ_{ij} for all $j \in \{1, \dots, q_i\}$ and ν_{ij} for all $j \in \{1, \dots, r_i\}$. Although the OCP might seem nonconvex using these decision variables, one can turn it into an SDP by redefining the decision variables and, if desired, reversing the transformation to recover the original variables from the solution. Similar to the OCP (6.8), the online OCP (6.26) is solved at each time instant and the optimal control input $v_i(t) = u_i(0|t) + K_{o_i} T_i (z_i(t) - x_i(0|t))$ is applied to the i-th subsystem.

6.4 Stability Analysis

In this section, we prove that the online OCP (6.26) is recursively feasible and the resulting closed-loop system is input-to-state stable. In the sequel, we use the superscript $(\cdot)^*$ to refer to the optimal solution.

Theorem 6.1. *The proposed MPC scheme with the online OCP (6.26) is recursively feasible.*

Proof. Assume that the proposed MPC scheme is initially feasible for all subsystems at time t_0 with the optimal state and input sequences $(x_i^*(0|t_0), \ldots, x_i^*(N|t_0))$ and $(u_i^*(0|t_0), \ldots, u_i^*(N-1|t_0))$. Recall that $x_i^*(N|t_0) = x_{e_i}^*$ satisfies $x_{e_i}^* = A_i x_{e_i}^* + B_i u_i^*$ and consider the tail sequence $(x_i^*(1|t_0), \ldots, x_i^*(N|t_0), x_{e_i}^*)$ and $(u_i^*(1|t_0), \ldots, u_i^*(N-1|t_0), u_{e_i}^*)$. Note that the tail sequence correspond to a feasible solution at the next time instant t_0+1. Furthermore, feasible values for all other decision variables can be obtained from the optimal solution at time t_0. The recursive feasibility of the proposed approach follows. □

The stability analysis is divided intro three main steps. First, we discuss the convergence of the nominal state trajectory $x^*(t) = [x_1^{*\top}(t), \ldots, x_M^{*\top}(t)]^\top$ to the artificial equilibrium trajectory $x_e^*(t) = [x_{e_1}^{*\top}(t), \ldots, x_{e_M}^{*\top}(t)]^\top$. Then, we discuss the convergence of the artificial equilibrium trajectory $x_e^*(t)$ to the target point $x_r = [x_{r_1}^{*\top}, \ldots, x_{r_M}^{*\top}]^\top$. Finally, we discuss the stability of the error dynamics between the actual state $z^*(t) = [z_1^{*\top}(t), \ldots, z_M^{*\top}(t)]^\top$ and the nominal state $x^*(t)$.

6.4.1 Stability of the Nominal System

We start the first step by defining the augmented nominal system whose state vector is $[(x^*(t) - x_e^*(t))^\top, x_e^{*\top}(t), u_e^{*\top}(t)]^\top$ where $u_e^*(t) = [u_{e_1}^{*\top}(t), \ldots, u_{e_M}^{*\top}(t)]^\top$. In Lemma 6.1, we discuss the stability of this system and the convergence of $x^*(t)$ to $x_e^*(t)$. Notice that $x^*(t) = x^*(0|t) = [x_1^{*\top}(0|t), \ldots, x_M^{*\top}(0|t)]^\top$ where $x_i^*(0|t)$ is the optimal 0-step ahead predicted state of the i-th subsystem at time instant t.

Lemma 6.1. *The augmented nominal system with the state vector $[(x^*(t) - x_e^*(t))^\top$, $x_e^{*\top}(t)$, $u_e^{*\top}(t)]^\top$ is stable. Furthermore, $\lim_{t\to\infty}(x^*(t) - x_e^*(t)) = 0$.*

Proof. Let the predicted states and inputs as well as the artificial equilibrium corresponding to the optimal solution of the OCP (6.26) of the i-th subsystem at time instant t be $x_i^*(k|t)$, $u_i^*(k|t)$, $x_{e_i}^*(t)$ and $u_{e_i}^*(t)$, respectively. The optimal cost is then,

$$J_i^*(t) = \sum_{k=0}^{N-1} \left(\|x_i^*(k|t) - x_{e_i}^*(t)\|_{Q_i}^2 + \|u_i^*(k|t) - u_{e_i}^*(t)\|_{R_i}^2 \right) + \|x_{e_i}^*(t) - x_{r_i}\|_{S_i}^2.$$

Also, the tail sequence cost is given by,

$$J_i^s(t) = \sum_{k=1}^{N-1} \left(\|x_i^*(k|t) - x_{e_i}^*(t)\|_{Q_i}^2 + \|u_i^*(k|t) - u_{e_i}^*(t)\|_{R_i}^2 \right) + \|x_{e_i}^*(t) - x_{r_i}\|_{S_i}^2,$$

or equivalently,

$$J_i^s(t) = \sum_{k=1}^{N} \left(\|x_i^*(k|t) - x_{e_i}^*(t)\|_{Q_i}^2 + \|u_i^*(k|t) - u_{e_i}^*(t)\|_{R_i}^2 \right) + \|x_{e_i}^* - x_{r_i}\|_{S_i}^2,$$

since $x_i^*(N|t) = x_{e_i}^*(t)$ and $u_i^*(N|t) = u_{e_i}^*(t)$.

Note that the predicted trajectories and artificial equilibrium used in $J_i^s(t)$ correspond to a feasible solution at time instant $t+1$ and let the optimal cost at time instant $t+1$ be $J_i^*(t+1)$. Hence,

$$J_i^*(t+1) - J_i^*(t) \leq J_i^s(t) - J_i^*(t) = -\|x_i^*(0|t) - x_{e_i}^*(t)\|_{Q_i}^2 - \|u_i^*(0|t) - u_{e_i}^*(t)\|_{R_i}^2 \leq 0.$$

Since $x^*(t) = x^*(0|t)$, summing up the above inequality for all $i \in \{1, \ldots, M\}$ yields,

$$J^*(t+1) - J^*(t) \leq -\|x^*(t) - x_e^*(t)\|_Q^2 - \|u^*(t) - u_e^*(t)\|_R^2 \leq 0,$$

where $J(t) = \sum_{i=1}^M J_i(t)$, $Q = \text{diag}(Q_1, \ldots, Q_M)$ and $R = \text{diag}(R_1, \ldots, R_M)$. Considering $J^*(t)$ as a Lyapunov function shows that the augmented global nominal system is stable.

Taking the telescopic sum of $J^*(t+1) - J^*(t)$ leads to,

$$J^*(t) - \lim_{t\to\infty} J^*(t) \geq \sum_{t=0}^{\infty} \|x^*(t) - x_e^*(t)\|_Q^2 + \|u^*(t) - u_e^*(t)\|_R^2.$$

Note that $J^*(t)$ is initially finite and non-negative if the OCP (6.26) is initially feasible. Since the OCP (6.26) is recursively feasible, $J^*(t)$ is finite and non-negative for all t. Since $Q \in \mathbb{S}_{++}^n$ and $R \in \mathbb{S}_{++}^m$, this means that we must have,

$$\lim_{t\to\infty} (x^*(t) - x_e^*(t)) = 0,$$
$$\lim_{t\to\infty} (u^*(t) - u_e^*(t)) = 0.$$

□

6.4.2 Attractivity of the Target Point

Second, we discuss the convergence of $x_e^*(t)$ to the target point x_r. For this purpose, we define $\tilde{x}_r$ to be the projection of x_r onto the set of equilibrium points contained in the set $\xi\mathcal{X}_i$ and $\tilde{u}_r$ to be the corresponding input. We also define the matrix $\tilde{P}$ such

that $\tilde{P} - (A+BK)^\top \tilde{P}(A+BK) - Q - K^\top RK \geq 0$. Note that such $\tilde{P}$ exists since K asymptotically stabilizes the actual global dynamics as shown in Proposition 6.3. Finally, we define $E(c,s) = \{x : \|x-c\|^2_{\tilde{P}} \leq s\}$. For brevity, we omit the dependence on t in the proof of Lemma 6.2.

Lemma 6.2. *If the optimal nominal state trajectory $x^*(t)$ and optimal artificial equilibrium trajectory $x^*_e(t)$ are such that $\lim_{t\to\infty}(x^*(t)-x^*_e(t)) = 0$, then $\lim_{t\to\infty}(x^*_e(t)-\tilde{x}_r) = 0$.*

Proof. According to the OCP (6.26), $(x^*_e, u^*_e) \in \text{int}(\mathcal{X}\times\mathcal{U})$. Hence, there always exists $E(x^*_e,\beta_1)$ such that for all $x \in E(x^*_e,\beta_1)$,

$$(x, K(x-x^*_e)+u^*_e) \in \gamma\mathcal{X}\times\gamma\mathcal{U},$$

where $\gamma \in (0,1)$. Note that $E(x^*_e,\beta_1)$ is invariant since it is defined using the level sets of a Lyapunov function. We now define an equilibrium point $(\bar{x}_e,\bar{u}_e) \in \text{int}(\mathcal{X}\times\mathcal{U})$ such that,

$$\|\bar{x}_e - \tilde{x}_r\|^2_S = \beta_2\|x^*_e - \tilde{x}_r\|^2_S,$$
$$\|\bar{x}_e - x^*_e\|^2_{\tilde{P}} = (1-\lambda)\|\tilde{x}_r - x^*_e\|^2_{\tilde{P}},$$

where $\beta_2 \in (0,1)$ and $\lambda \in (0,1)$ is chosen sufficiently large such that,

$$(0, K(\bar{x}_e - x^*_e) + \bar{u}_e - u^*_e) \in (1-\gamma)\mathcal{X}\times(1-\gamma)\mathcal{U}.$$

Next, we define $E(\bar{x}_e,\beta_3) \subset E(x^*_e,\beta_1)$ such that $x^*_e \in \text{int}(E(\bar{x}_e,\beta_3))$. This set is invariant since it is defined using the level sets of a Lyapunov function. Moreover, for all $x \in E(\bar{x}_e,\beta_3)$,

$$\begin{aligned}(x, K(x-\bar{x}_e)+\bar{u}_e) &= (x, K(x-x^*_e)+u^*_e) + (0, K(\bar{x}_e-x^*_e)+\bar{u}_e-u^*_e)\\ &\in (\gamma\mathcal{X}\times\gamma\mathcal{U}) \oplus ((1-\gamma)\mathcal{X}\times(1-\gamma)\mathcal{U}) = \mathcal{X}\times\mathcal{U}.\end{aligned}$$

We now define $\beta_4 > 0$ such that $S > \beta_4\tilde{P}$ and the set,

$$\hat{\mathcal{X}} = \{x : \|x-x^*_e\|^2_{\tilde{P}} \leq \beta_5\|x^*_e-\tilde{x}_r\|^2_{\tilde{P}} \;\&\; (x^*_e-\bar{x}_e)^\top\tilde{P}|x-x^*_e| \leq \beta_5\|x^*_e-\tilde{x}_r\|^2_{\tilde{P}}\},$$

where $0 < \beta_5 < \beta_4(1-\beta_2)/3$. Hence for all $x \in E(x^*_e,\beta_6) \subset \hat{\mathcal{X}}$,

$$\|x-\bar{x}_e\|^2_{\tilde{P}} = \|x-x^*_e\|^2_{\tilde{P}} + 2(x^*_e-\bar{x}_e)^\top\tilde{P}(x-x^*_e) + (1-\lambda)\|x^*_e-\tilde{x}_r\|^2_{\tilde{P}} \leq \|x^*_e-\tilde{x}_r\|^2_{3\beta_5\tilde{P}+(1-\lambda)\tilde{P}}.$$

Note that $E(x^*_e,\beta_6)$ can be also chosen to lie inside $E(\bar{x}_e,\beta_3) \subset E(x^*_e,\beta_1)$ and hence is invariant. Note also that for all $x \in E(x^*_e,\beta_6)$,

$$\|x-\bar{x}_e\|^2_{\tilde{P}} + \|\bar{x}_e-\tilde{x}_r\|^2_S \leq \|x^*_e-\tilde{x}_r\|^2_S,$$

if,

$$(1-\beta_2)S - (1-\lambda+3\beta_5)\tilde{P} \geq 0,$$

or equivalently,

$$\beta_4(1-\beta_2)-(1-\lambda+3\beta_5)\geq 0,$$

This can be achieved if,

$$\lambda \geq 1-(\beta_4(1-\beta_2)-3\beta_5),$$

Since $\beta_5 < \beta_4(1-\beta_2)/3$, this condition on λ does not contradict the aforementioned conditions.

We conclude the proof using contradiction. Assume that,

$$\lim_{t\to\infty}(x^*-x_e^*)=0,$$
$$\lim_{t\to\infty}(x_e^*-\tilde{x}_r)\neq 0.$$

Hence, it is always possible to pick an arbitrarily large t such that,

$$x^*\in E(x_e^*,\beta_6)\subset(\bar{x}_e,\beta_3).$$

In addition, there exists β_7 such that for infinitely many t,

$$\|x_e^*-\tilde{x}_r\|\geq\beta_7.$$

Since $E(x_e^*,\beta_6)$ and $E(\bar{x}_e,\beta_3)$ are both invariant and satisfy the state and input constraints, let J^* and $\bar{J}$ be the cost corresponding to the optimal solution and the feasible solution aiming to converge to $\bar{x}_e$ and note that $J^* < \bar{J}$. Note also that due to the Lyapunov inequality that $\tilde{P}$ satisfies,

$$\bar{J}\leq\|x^*-\bar{x}_e\|_{\tilde{P}}^2+\|\bar{x}_e-\tilde{x}_r\|_S^2.$$

Since $x^*\in E(x_e^*,\beta_6)$, then,

$$\|x^*-\bar{x}_e\|_{\tilde{P}}^2+\|\bar{x}_e-\tilde{x}_r\|_S^2\leq\|x_e^*-\tilde{x}_r\|_S^2.$$

But, $\|x_e^*-\tilde{x}_r\|_S^2\leq\|x_e^*-x_r\|_S^2\leq J^*$ which contradicts the optimality of J^*. □

Lemma 6.1 and Lemma 6.2 show that the target point of the augmented nominal system is asymptotically stable by showing that it is stable (Lemma 6.1) and attractive (Lemma 6.1 and Lemma 6.2).

6.4.3 Input-to-state Stability of the Actual System

Finally, we discuss the input-to-state stability of the global closed-loop system under the MPC scheme (6.26) for all subsystems. Recall from (6.9) that the global system is given by,

$$z(t+1)=(A_{\mathrm{dg}}+EA_dC)z(t)+Bv(t)+d(t).$$

Recall also that the optimal input is given by,

$$v^*(t) = u^*(t) + K^*(t)(z^*(t) - x^*(t)),$$

where,

$$K^*(t) = \text{diag}(K_{o_1} T_1^*(t), \dots, K_{o_M} T_M^*(t)),$$
$$u^*(t) = u^*(0|t) = [u_1^{*\top}(0|t), \dots, u_M^{*\top}(0|t)]^\top.$$

Hence, the global closed-loop system is given by,

$$z^*(t+1) = A_{\text{dg}} z^*(t) + Bu^*(t) + BK^*(t)(z^*(t) - x^*(t)) + EA_d C z^*(t) + d(t).$$

By defining $e^*(t) = z^*(t) - x^*(t)$, the closed-loop system becomes,

$$\begin{aligned} e^*(t+1) =& - x^*(t+1) + A_{\text{dg}}(e^*(t) + x^*(t)) + Bu^*(t) \\ & + BK^*(t)e^*(t) + EA_d C(e^*(t) + x^*(t)) + d(t) \\ =& (A_{\text{dg}} + EA_d C + BK^*(t))e^*(t) + d(t) \\ & - x^*(t+1) + (A_{\text{dg}} + EA_d C)x^*(t) + Bu^*(t) \\ =& (A + BK^*(t))e^*(t) + d(t) \\ & - x^*(t+1) + Ax^*(t) + Bu^*(t). \end{aligned}$$

Notice that $d(t)$, $x^*(t+1)$, $x^*(t)$ and $u^*(t)$ are all bounded as they are constrained to lie within bounded sets. Moreover, the considered system is switched since $K^*(t)$ is adapted at each time instant. Hence, the closed-loop system under the proposed MPC scheme can be considered as an uncertain switched system with bounded disturbances.

Lemma 6.3. *The global closed-loop system under the proposed MPC scheme,*

$$e^*(t+1) = (A + BK^*(t))e^*(t) + d(t) - x^*(t+1) + Ax^*(t) + Bu^*(t), \tag{6.27}$$

is input-to-state statble

Proof. First, we consider the disturbance-free system,

$$e^*(t+1) = (A + BK^*(t))e^*(t).$$

The matrix $A + BK^*(t)$ is always Schur for any $K^*(t)$ since the constraints derived in Propositions 6.2 and 6.3 ensure that at each time instant t,

$$P - (A + BK^*(t))^\top P(A + BK^*(t)) \geq \Gamma - C^\top A_g C - C^\top LDLC \geq \epsilon I_n.$$

Since this condition is always satisfied using the same matrix P, then $V(e^*) = \|e^*\|_P^2$ is a Lyapunov function for the disturbance-free system independent of $K^*(t)$.

Now, we consider the closed-loop system (6.27). Using the Lyapunov function $V(e^*)$ and defining $\eta(t) = d(t) - x^*(t+1) + Ax^*(t) + Bu^*(t)$, we can deduce that,

$$\begin{aligned} &V((A+BK^*(t))e^*(t)+\eta(t)) - V(e^*) \\ &\qquad \leq -(\epsilon-\delta)\|e^*(t)\|^2 - \delta\|e^*(t)\|^2 + 2\|\eta(t)\|_P\|(A+BK^*(t))e^*(t)\|_P + \|\eta(t)\|_P^2, \end{aligned}$$

where $0 < \delta < \epsilon$. To prove input-to-state stability, it suffices to ensure that,

$$\delta\|e^*(t)\|_2^2 \geq 2\|\eta(t)\|_P\|(A+BK^*(t))e^*(t)\|_P + \|\eta(t)\|_P^2,$$

for some $\zeta > 0$ such that $\|e^*(t)\| \geq \zeta$.

We denote the maximum eigenvalue of P by $\lambda_{max}(P)$ and the maximum norm of $\eta(t)$ by η_{max}. Since $\|(A+BK^*(t))e^*(t)\|_P < \|e^*(t)\|_P$ and $\|(\cdot)\|_P \leq \sqrt{\lambda_{max}(P)}\|(\cdot)\|$, it suffices instead to ensure that,

$$\delta\|e^*(t)\|^2 - 2\lambda_{max}(P)\eta_{max}\|e^*(t)\| - \lambda_{max}(P)\eta_{max}^2 \geq 0.$$

By computing the roots of the convex quadratic function on the left-hand side of the inequality, it is easy to verify that this inequality holds for,

$$\|e^*(t)\| \geq \zeta = \left(\lambda_{max}(P) + \sqrt{\lambda_{max}^2(P) + \delta\lambda_{max}(P)}\right)\eta_{max}/\delta$$

Hence, input-to-state stability is guaranteed. □

6.5 Simulation Results

In this section, we investigate the efficacy of the proposed scheme by comparing it to existing schemes in the literature. In particular, we compare the proposed scheme (ADP+LRN) to Robust MPC (ROB) [MSR05], Adaptive MPC (ADP) [Par+21] and Learning-based MPC (LRN) [Asw+13]. We solve all optimization problems using MATLAB with YALMIP [Löf04], MOSEK [ApS19] and GUROBI [Gur22] on a computer equipped with a 1.9-GHz Intel core i7-8550U processor.

In our simulation example, we use a network of five dynamically-coupled discrete-time double integrators as shown in Figure 6.1. We consider each double integrator as a subsystem whose dynamics is given by (6.5) where for all $i \in \{1,2,3,4,5\}$,

$$A_{ii} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; B_i = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \; C_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; E_i = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}.$$

All states and inputs are constrained between -2 and 2. The cost function matrices are given by $Q_i = I_2$, $R_i = 0.1$ and $S_i = 10I_2$ for all subsystems and the prediction horizon is 5. The noise $d(t) \in \mathbb{R}^2$ is extracted uniformly where the first entry is constrained between

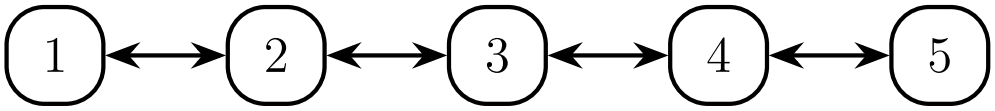

Figure 6.1: Double integrator network topology

-0.025 and 0.025 and the second entry is constrained between -0.05 and 0.05. The unknown true value of the parameters a_{12} and a_{54} is 1, whereas that of the parameters a_{21}, a_{23}, a_{32}, a_{34}, a_{43} and a_{45} is 0.5. The parameters a_{12} and a_{54} are assumed to lie initially between 0 and 4, whereas all other parameters are assumed to lie initially between 0 and 2. Each subsystem is required to follow a randomly generated piecewise constant reference whose value changes every 100 seconds and is constrained between -1.5 and 1.5. The total simulation time is 1000 seconds and the initial condition is $[0\ 0]^\top$ for all subsystems.

Figure 6.2 shows for each scheme the closed-loop performance of the third subsystem. While ADP+LRN and ADP follow the reference trajectory successfully, LRN and ROB only track small references. This is mainly because ADP+LRN and ADP update the tube and the tightened constraint sets online using the collected data. On the contrary, ROB updates nothing and LRN updates the model only while keeping the initial tube and tightened constraint sets unchanged. This has a direct effect on the closed-loop cost as shown in Figure 6.3 where ROB has the highest cost, followed by LRN whose cost is slightly lower due to updating the model. The cost of ADP+LRN and ADP is considerably lower due to the successful reference tracking. ADP has a slightly lower cost due to the absence of terminal constraints and the distributed nature of this scheme. This, however, requires using a distributed optimization technique to solve the distributed MPC problem of ADP. For this purpose, we use ADMM here and run it for sufficiently long time to ensure reaching the optimal solution.

Figure 6.4 shows the mean and standard deviation of the computation time required by each scheme and the corresponding closed-loop cost. We use the ADP scheme whose computation time is around 1 second in Figure 6.2 and Figure 6.3. It is found, however, that if we run ADMM for around 0.1 second only, we still get a relatively satisfactory performance. Running ADMM for less time result in closed-loop performance deterioration. On the other hand, all other schemes have lower computation time due to their decentralized nature which do not require using distributed optimization techniques such as ADMM. Since the OCP of ADP+LRN is a semidefinite program, ADP+LRN requires slightly higher computation compared to LRN and ROB whose OCPs are quadratic programs.

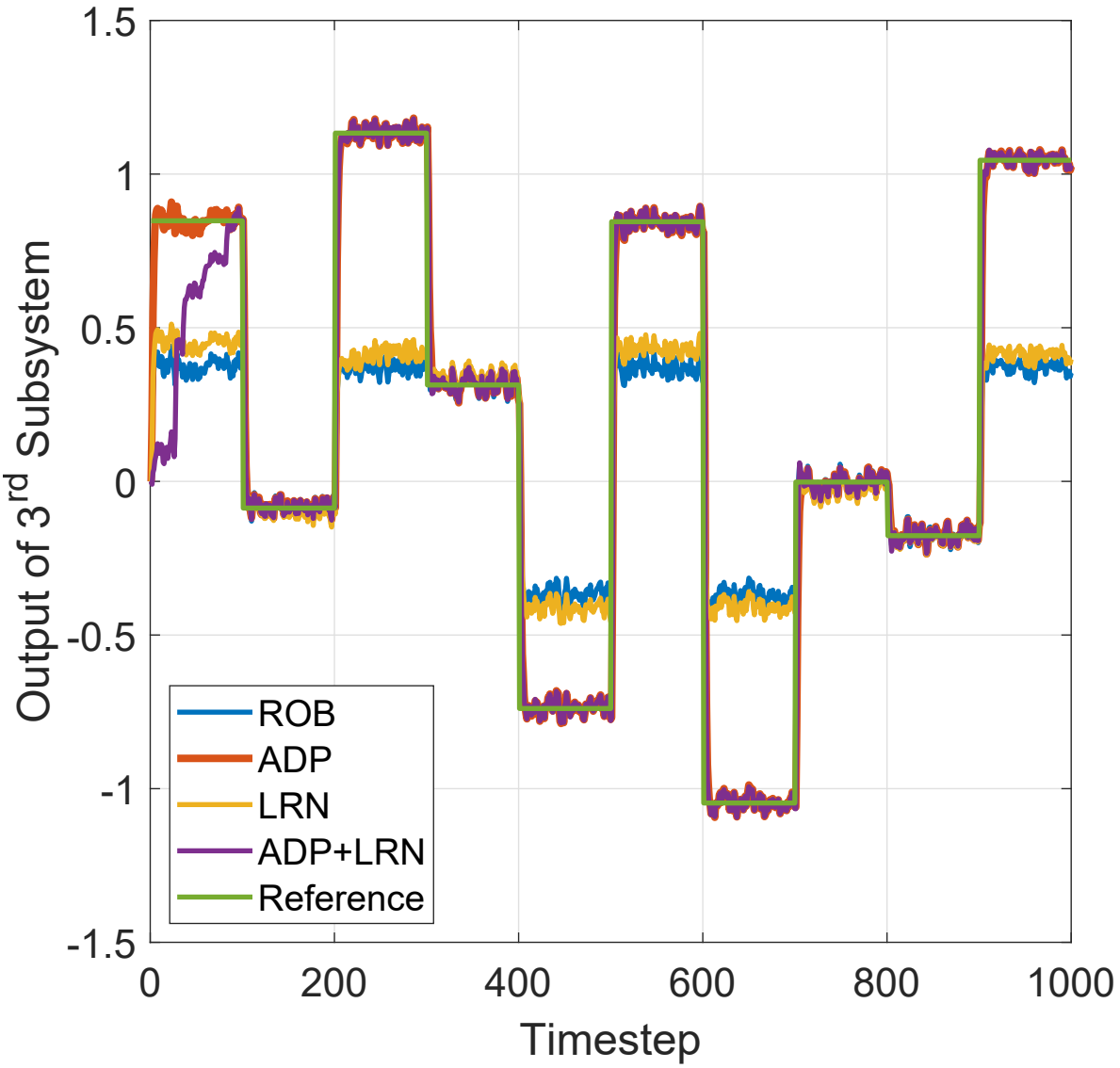

Figure 6.2: Output trajectory of the third subsystem using ROB, LRN, ADP and ADP+LRN.

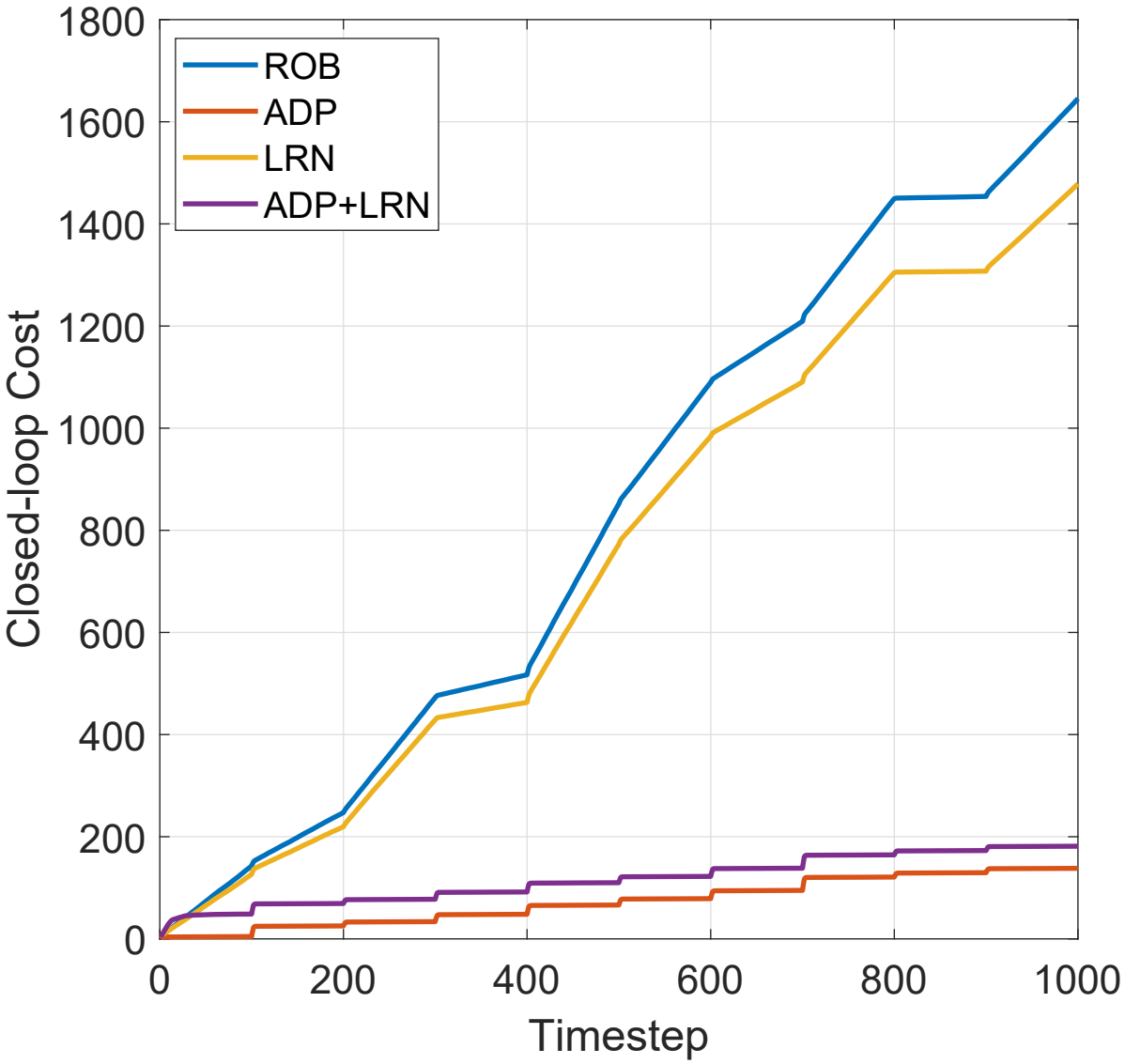

Figure 6.3: Global closed-loop cost using ROB, LRN, ADP and ADP+LRN.

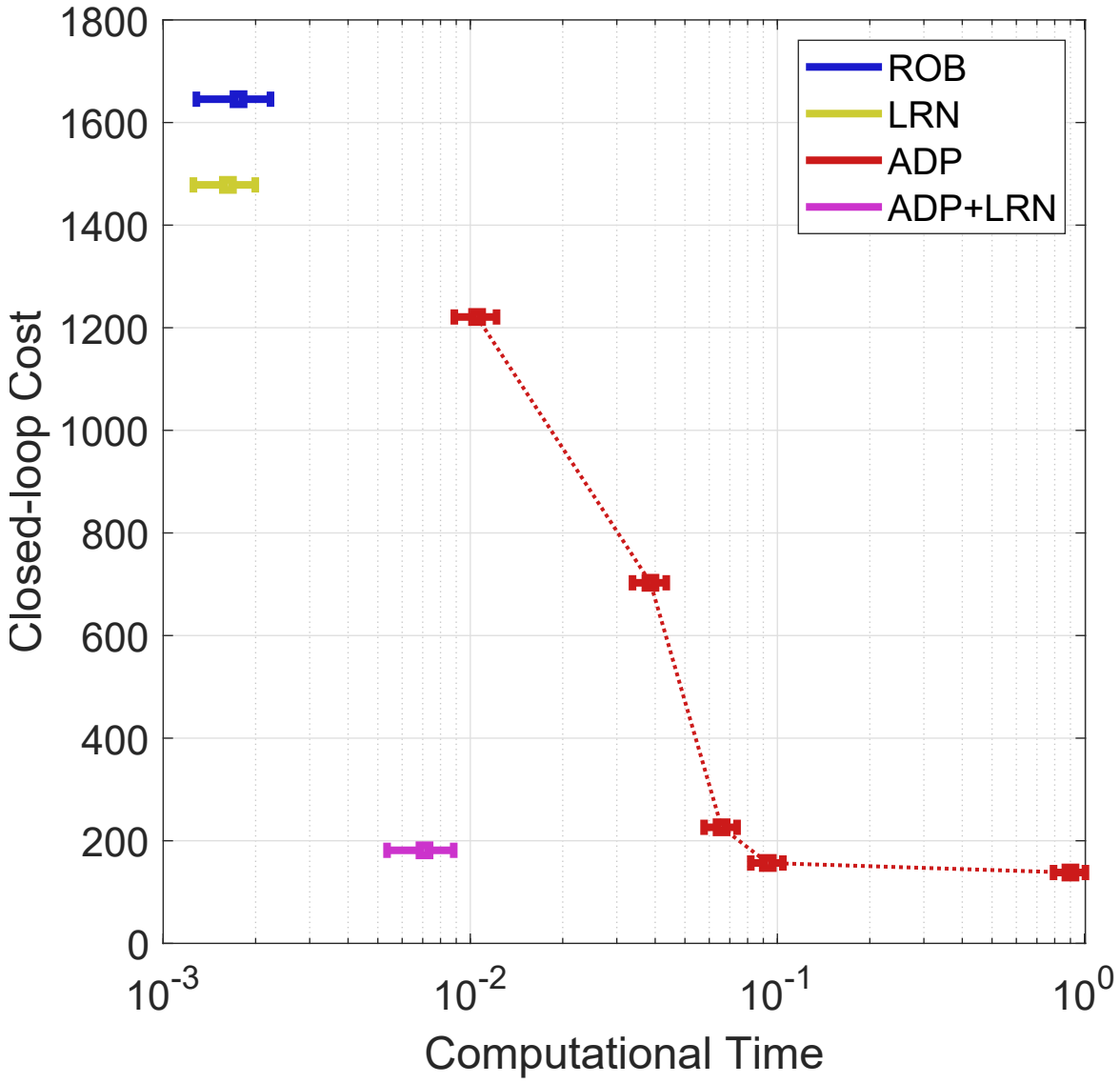

Figure 6.4: Computation complexity of ROB, LRN, ADP and ADP+LRN.

Part IV

Conclusions

CHAPTER 7

Conclusions and outlook

Control of uncertain interconnected systems is a challenging task and a subject of ongoing research. This is attributed to the existing model uncertainties and additive disturbances in these systems as well as their possibly varying topology and reference trajectories. All these challenges might lead to closed-loop performance degradation, feasibility issues and stability problems when dealing with these systems.

In this PhD thesis, we have addressed these challenges by developing novel distributed and decentralized MPC schemes for such systems. More specifically, we have developed reconfigruable distributed MPC schemes for interconnected systems in the first part of this thesis to avoid feasibility problems in tracking applications. We have then introduced a passivity-based plug-and-play MPC scheme for varying topology networks in the second part. This scheme avoids stability problems when the topology of the network is occasionally varying. Finally, we have presented an adaptive learning-based MPC scheme for uncertain interconnected systems with model uncertainties and additive disturbances in the third part. This scheme improves the closed-loop performance of the considered systems online by learning the uncertainty using collected data and adapting the MPC ingredients accordingly. In the sequel, we discuss the main concluding remarks and the future research directions in each of the three main parts of this thesis.

7.1 Part I: Reference Tracking Control

7.1.1 Conclusions

In Part I, we have introduced reconfigurable distributed MPC schemes for the regulation of interconnected systems where the terminal set is computed online by considering it as a decision variable in the online optimal control problem. The positive invariance of the terminal set is ensured by imposing additional constraints in the MPC problem.

Building upon the aforementioned schemes, we have proposed reconfigurable distributed MPC schemes for tracking applications of interconnected systems. In these schemes, the terminal ingredients (i.e. set, controller and cost) are updated online at

each time instant taking into account the current state of the system. Although the resulting online optimal control problem is found to be nonconvex and infinite dimensional in this case, we have shown that it can be approximated by a quadratic program. We have also shown that the proposed schemes are amenable to distributed optimization. Moreover, we have shown that the proposed schemes are recurseively feasible and the corresponding closed-loop dynamics are asymptotically stable. In addition, we have compared the feasible regions of the proposed schemes to those of existing schemes. It is found that the proposed schemes can have larger feasible regions compared to the standard MPC schemes while preserving similar computational complexity.

7.1.2 Outlook

System level synthesis (SLS) is an advanced control method that is well known for synthesizing distributed optimal controllers systematically for interconnected systems [Doy+17; And+19]. The main idea behind SLS is to design the whole closed-loop system response from the initial condition and disturbances to the states and inputs instead of only designing the feedback loop from the state to the input. SLS has been recently utilized for developing distributed and localized closed-loop MPC schemes for interconnected systems where the online optimal control problem is formulated as an optimization problem over the closed loop response of the system [AM20; AMA20]. It would be interesting to combine SLS with the framework proposed in this part and assess the benefits of using SLS with the developed schemes.

In all the proposed schemes, we have assumed that an accurate model of the considered interconnected system is known in advance. In practice, this model can be obtained using first principles modelling approaches or offline system identification methods. In some cases, these techniques might be relatively costly. Recently, data-enabled predictive control (DeePC) techniques have been developed to optimally control a given system without resorting to the aforementioned possibly costly techniques [CLD21; Ber+20]. Instead, a non-parametric model is found using a finite number of collected data points. More specifically, this model is derived using the fundamental lemma in behavioral systems theory. Various efforts have been devoted to developing DeePC schemes for interconnected systems (e.g. see [Hua+21]). Hence, it would be interesting to extend the developed schemes in this part using the notion of behavioral systems theory.

7.2 Part II: Plug-and-play Control

7.2.1 Conclusions

In Part II, we have presented a passivity-based controller for discrete-time interconnected systems where the control synthesis and operation are decentralised. We have used passivity theory for discrete-time systems directly to avoid the problem of passivity preservation under discretization. Each subsystem synthesizes its local controller by solving a semidefinite program comprising local conditions ensuring its passivity. Passivity is ensured with respect to a local virtual output which is different from the local actual output. Virtual outputs are used since strictly proper discrete-time LTI systems are not passive and can not be passivized. It is found, however, that passivity of all subsystems does not ensure stability of the overall network due to the introduced virtual outputs. Hence, the semidefinite program of each susbsystem also comprises local conditions ensuring that the strict passivity of all subsystems implies the asymptotic stability of the whole system. It is found that the proposed controller can lead to a wide range of behaviors ranging from optimal behaviors which mimic that of LQR to robust behaviors with high passivity margin.

Building upon the aforementioned scheme, we have developed a PnP algorithm for tracking applications of varying-topology networks. The algorithm comprises a redesign phase based on passivity-based arguments of discrete-time systems and a transition phase that makes use of reconfigurable terminal ingredients. Using this PnP algorithm, the recursive feasibility of the distributed MPC scheme and the asymptotic stability of the corresponding closed-loop system are ensured under the varying topology of the network. It is found that the novel redesign phase results in faster PnP operations and the novel transition phase adds more flexibility by accepting more PnP requests compared to a benchmark scheme.

7.2.2 Outlook

In this part of the thesis, we have restricted our attention to a specific class of interconnected systems whose coupling is described using the Laplacian interconnection. One exciting research direction is to extend these results to the general class of interconnected systems or other special classes such as those whose coupling is described using the skew-symmetric interconnection [Nah+20; Sol+18]. Another exciting research direction is to explore the possibility and assess the benefits of using the more general notion of QSR dissipativity to design decentralized controllers for interconnected systems [Kot+14]. Recently, there has been a growing attention to utilize data for determining dissipativity properties [KBA20]. It would be also interesting to explore the possibility of designing passivity-based controllers for interconnected systems using data.

Regarding the developed PnP MPC scheme, we have considered deterministic interconnected systems with neither model uncertainties nor external disturbances. It would be interesting to develop a PnP framework for uncertain interconnected systems. In this case, novel redesign and transition phases should be developed. In the redesign phase,

one could think of making use of robust passivity arguments to take the uncertainty into account [LS99]. In the transition phase, reaching a steady state in finite time might be no longer possible. Instead, one could think of driving the system to a robust positive invariant set for the PnP operation to take place instead of driving the system to a single steady state [Rak+05].

7.3 Part III: Adaptive Learning-based Control

7.3.1 Conclusions

In Part III, we have proposed a novel adaptive learning-based MPC scheme for uncertain interconnected systems. This two-phase scheme learns the uncertainty set online using collected data and adapts the MPC ingredients accordingly while computing the optimal predicted trajectories. In the learning phase, set membership identification is used to shrink the volume of the uncertainty set online. In the adaptation phase, rigid tube-based MPC is used to adapt the MPC ingredients and compute the optimal control actions taking the shrunk uncertainty set into account. The developed scheme is compared in simulations to existing schemes such as robust MPC, adaptive MPC and learning-based MPC and found to lead to a better trade-off between computational burden and closed-loop performance.

7.3.2 Outlook

We have used rigid tubes and affine state feedback policies in the adaptation phase of the developed scheme. In general, rigid tubes and affine state feedback policies might be more conservative than homothetic tubes and affine disturbance feedback policies [Rak+12; GKM06]. Hence, it would be interesting to develop adaptive learning-based MPC schemes which make use of these less conservative features. The main challenge would be to derive a convex online optimal control problem when these features are considered.

Furthermore, model uncertainties are updated in the learning phase by making use of set membership identification. Nevertheless, this technique can be used with special classes of uncertainties. One future research direction is to explore other system identification techniques and machine learning methods such as kinky inference [Bla+19], Gaussian process regression [HKZ19] and reinforcement learning [Bru+22] to consider more generic classes of uncertainties. Another future research direction is to exploit these developed ideas within the predictive safety filter framework [WZ21].

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