A solution of the two-dimensional global asymptotic Jacobian stability conjecture

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1 Introduction

In this work we solve[1] the following problem which is known as the two dimensional Global Asymptotic Jacobian Stability Conjecture.

<u>*Problem 1:*</u> *Let* $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *be such that*

1) $\det Df(x) > 0$ *for all* $x \in \mathbb{R}^2$

2) $tr\, Df(x) < 0$ *for all* $x \in \mathbb{R}^2$

3) $f(0) = 0$

where $Df(x)$ *denotes the Jacobian matrix,* det *the determinant and tr the trace.*
Is it true that under the conditions 1) - 3) every solution of

$$\dot{x}(t) = f(x(t))$$

approaches 0 *as* $t \to \infty$ *?*

This problem and its n-dimensional reformulation goes back to Markus and Yamabe [MY] in 1960.
In the two dimensional case several authors achieved an affirmative answer to this problem under various additional assumptions. Krasovski [Kr] solved a related problem with condition 1) being replaced by a certain growth condition on f. Markus and Yamabe [MY] treated the case when one of the partial derivatives of f vanishes identically on $\mathbb{R}^2$. Hartman [Ha] used the stronger hypothesis that the symmetric part of $Df(x)$ is negative definite everywhere. His result is also valid in higher dimensions. Olech [Ol] solved the

[1] I acknowledge that Carlos Gutierrez has also solved this problem independently. Both he and the present author presented their proofs [Gu], [Fe] on the conference about RECENT RESULTS ON THE GLOBAL ASYMPTOTIC STABILITY JACOBIAN CONJECTURE, Università di Trento, I-38050 POVO (TN) ITALY, September 1993

problem affirmatively if f is bounded from below in some neighbourhood of infinity. A generalization to higher dimensions can be found in Hartman and Olech [HO]. In 1988 Meisters and Olech [MO] proved the conjecture for polynomial maps. The attention of the author was attracted to the problem by an article of Gasull, Llibre and Sotomayor [GLS] where the relation of this conjecture to several other problems was investigated. Barabanov [Ba] showed that this conjecture is false if $n \geq 4$.

In his work Olech proved in 1963 [Ol] that *problem 1* is equivalent to

Problem 2: *Let* $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *be such that*

1) $\det Df(x) > 0$ *for all* $x \in \mathbb{R}^2$

2) $tr\, Df(x) < 0$ *for all* $x \in \mathbb{R}^2$

is satisfied.
Is it true that f *is injective ?*

This gives the key to our solution. Theorem 1 which we will prove in this work is an affirmative answer to *problem 2*. Actually it is even more general: The hypotheses of *problem 2* are equivalent to assuming that the eigenvalues of $Df(x)$ have negative real parts for all $x \in \mathbb{R}^2$.
Therefore hypothesis 2) of our Theorem 1 is weaker than hypothesis 2) of *problem 2*. Furthermore, we only need it in a neighbourhood of infinity.

2 The Solution of the Problem

Theorem 1 *Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be such that*

1) $\det Df(x) > 0$ *for all* $x \in \mathbb{R}^2$.
(i.e. f is local diffeomorphism),

2) There is a compact set $K \subset \mathbb{R}^2$ such that
$Df(x)v \neq \lambda v$ *for all* $x \in \mathbb{R}^2 \backslash K$, $v \in \mathbb{R}^2 \backslash \{0\}$, $\lambda \in]0, \infty[$.
(i.e. $Df(x)$ has no real positive eigenvalues for all x in some neighbourhood of infinity.)

Then f is injective.

Proof of Theorem 1: We assume throughout the proof that f is not injective, but that our hypotheses 1) and 2) are true. (Actually, hypothesis 2) is not needed before Lemma 11.)
Finally, this will lead us to a contradiction.

Thus, if f is not injective we will find $x_0, x_1 \in \mathbb{R}^2$, $x_0 \neq x_1$, such that $f(x_0) = f(x_1)$. W.l.o.g. we may assume that $f(x_0) = f(x_1) = 0$.

The proof will be given using several definitions and lemmata as follows:

Definition 1 *We define the set C of curves as follows:*

$$C := \{\alpha \in C^1([0,1], \mathbb{R}^2) \mid \begin{array}{ll} \forall s \in [0,1] : \dot{\alpha}(s) \neq 0 & i) \\ \alpha(0) = x_0, \alpha(1) = x_1 & ii) \\ \alpha \textit{ is injective} & iii) \\ \alpha(]0,1[) \cap f^{-1}(0) = \emptyset\} & iv) \end{array}$$

Lemma 1 $C \neq \emptyset$

Proof: Let $\alpha_l(s) := (1-s)x_0 + sx_1$ be the straight line from x_0 to x_1. Then α obviously has all properties in order to be contained in C exept for iv).
Since f is a local homeomorphism, the set $f^{-1}(0)$ is discret. Therefore we can slightly modify α_l near the (finitely many) points of $\alpha_l(]0,1[) \cap f^{-1}(0)$ such that this set becomes empty (prop. iv).
Of course, this can be done in such a way that the other properties required for a curve to be in C remain valid.□

Definition and Lemma 1 *1) Every curve $\beta \in C^0(I, \mathbb{R}^2 \setminus \{0\})$ ($I =][a,b][$ being an arbitrary interval) induces an angle function*

$$\angle\beta \in C^0(I, \mathbb{R}), \quad \angle\beta(s) := \arg \beta_{\mathbb{C}}(s)$$

Here $\beta_{\mathbb{C}} \in C^0(I, \mathbb{C} \setminus \{0\}$ denotes the curve β composed with the canonical identification of $\mathbb{R}^2$ with the complexe plane $\mathbb{C}$. Furthermore, arg *denotes a continuous branch of the complex argument function. (Later we will use that* $\arg z = \mathrm{Im} \ln z$ *on every simple connected area of $\mathbb{C} \setminus \{0\}$ with an appropriatly chosen branch of the logarithm* ln.*)*
If $0 \in I$ we choose $\arg \beta_{\mathbb{C}}(0) \in [0, 2\pi[$ unless otherwise stated. Moreover, if $\beta \in C^1(I, \mathbb{R}^2 \setminus \{0\})$ then $\angle\beta \in C^1(I, \mathbb{R})$, too.

2) If $\beta \in C^1([a,b[,\mathbb{R}^2)$ (or $\beta \in C^1(]a,b],\mathbb{R}^2)$, resp.) is such that $0 \notin \beta(]a,b[)$ and

$$\beta(a) = 0, \quad \dot\beta(a) \neq 0 \ (or\ \beta(b) = 0,\ \dot\beta(b) \neq 0, resp.)$$

then

$$\lim_{s\searrow a} \angle\beta(s) = \angle\dot\beta(a) + 2\pi k \quad , k \in \mathbb{Z}.$$

$$(or\ \lim_{s\nearrow b} \angle\beta(s) = \angle\dot\beta(b) + \pi + 2\pi k,\ k \in \mathbb{Z},\ resp.)$$

Therefore we may extend the function $\angle\beta \in C^0(]a,b[,\mathbb{R})$ continuously to $[a,b[$ (or $]a,b]$, resp.) in this case.

Proof:

1) Since $0 \notin \beta(I) \arg \beta_{\mathbb{C}}$ is defined on I. The definition of $\angle\beta$ shows that $(\dot{\angle\beta}) = \operatorname{Im} \dot\beta/\beta$. Therefore, β being C^1 implies $\angle\beta$ being C^1, too.

2) Since β is differentiable at a and $\beta(a) = 0$ we know that
$\beta(s) = \dot\beta(a)(s-a) + \varphi(s-a)$, with a $\varphi \in o(\mathrm{id})$.
$\Rightarrow \angle\beta(s) \bmod 2\pi \beta_{\mathbb{C}} = \operatorname{Im}\ \ln \beta_{\mathbb{C}}(s) =$
$= \operatorname{Im}\ \ln(s-a)(\dot\beta_{\mathbb{C}}(a) + \varphi_{\mathbb{C}}(s-a)/(s-a) =$
$= \operatorname{Im}\ (\ln(s-a) + \ln(\dot\beta_{\mathbb{C}}(a) + \varphi_{\mathbb{C}}(s-a)/(s-a))) =$
$= \operatorname{Im}\ \ln(\dot\beta_{\mathbb{C}}(a) + \varphi_{\mathbb{C}}(s-a)/(s-a)) \rightarrow$
$\rightarrow \operatorname{Im}\ \ln \dot\beta_{\mathbb{C}}(a) =$
$= \angle\dot\beta(a) \bmod 2\pi \qquad$ as $s \searrow a$.
The proof for $s \nearrow b$ is analogous. However, since $(s-b) < 0$ in this case we obtain a summand π added.□

Definition and Lemma 2 *1) Every curve $\alpha \in C$ induces functions*
$\angle\dot\alpha,\ \angle f\circ\alpha,\ \angle\dot{(f\circ\alpha)} \in C^0([0,1],\mathbb{R})$
Moreover, $\angle f\circ\alpha \in C^1(]0,1[,\mathbb{R}]$.

2) We also define for every curve $\alpha \in C$ the function
$\Theta_\alpha \in C^0([0,1],\mathbb{R})$ by $\Theta_\alpha(s) := \angle\dot{(f\circ\alpha)}(s) - \angle f\circ\alpha(s)$
and observe that
$\Theta_\alpha(0) \bmod 2\pi = 0$ and $\Theta_\alpha(1) \bmod 2\pi = \pi$.

3) We will call the curve $\beta \in C^0([a,b],\mathbb{R}^2)$ piecewise regular (p.w. regular) if it is locally injective and if there exist s_i, $i = 1,\ldots,n$ such that $a = s_1 < \ldots < s_n = b$ and that $\beta_i := \beta \mid [s_i, s_{i+1}]$ is regular for all $i = 1,\ldots,n-1$. β_i being regular means that β_i is continuously differentiable (at s_i and s_{i+1} we consider one side differentials) and that $\dot\beta_i(s) \neq 0$ for all $s \in [s_i, s_{i+1}]$.
For such curves we may also define a unique tangent angle function $\angle\dot\beta$ as follows: If we assume that $s \in [s_i, s_{i+1}]$ then

$$\begin{aligned}\angle\dot\beta(s) \ &:= \ \angle\dot\beta_i(s) - \angle\dot\beta_i(s_i) + \\ &\quad + \sum_{k=1}^{i-1} \angle\dot\beta_k(s_{k+1}) - \angle\dot\beta_k(s_k) + \delta_{k+1}\end{aligned}$$

where δ_{k+1} denotes the "tangent angle jump" in the edge of β at s_{k+1}. We will define it in the following way: Let $\triangle_k\beta_{k-1}(h) := \beta_{k-1}(s_k) - \beta_{k-1}(s_k - h)$ and $\triangle_k\beta_k(h) := \beta_k(s_k + h) - \beta_k(s_k)$. Since $\dot\beta_{k-1}(s_k), \dot\beta_k(s_k) \neq 0$ there are unique functions $h_0, h_1 \in C^0([0,\epsilon[, [0,\infty[)$ with $h_0(0) = h_1(0) = 0$ such that

$$\| \triangle_k\beta_{k-1}(h_0(r)) \|_2 = r,$$

$$\| \triangle_k\beta_k(h_1(r)) \|_2 = r$$

for all $r \in [0,\epsilon[$ (implicit functiontheorem).
Let $\varphi_{|\pi|} \in]-\pi,\pi]$ denote the unique angle with $\varphi_{|\pi|} = \varphi \bmod 2\pi$. Since β is locally injective, the angle

$$\hat\delta_k(r) := (\angle \triangle_k\, \beta_k(h_1(r)) - \angle \triangle_k\, \beta_{k-1}(h_0(r)))_{|\pi|}$$

never equals π for all small $r > 0$. Therefore, $\hat\delta_k$ is continuous for such r and $\lim_{r\to 0}\hat\delta_k(r) \in [-\pi,\pi]$ exists.
Thus, we may finally define

$$\delta_k := \lim_{r\to 0}\hat\delta(r) \in [-\pi,\pi].$$

Proof:

1) $\alpha \in C \Rightarrow \dot\alpha, (f\circ\alpha)^{\cdot}$ are continuous and never 0 (see Definition 1) $\Rightarrow \angle\dot\alpha, \angle(f\circ\alpha)^{\cdot}$ are defined and continuous (see Definition and Lemma 1.1).
$\alpha \in C \Rightarrow f\circ\alpha \in C^1([0,1],\mathbb{R}^2)$, $0 \notin f\circ\alpha(]0,1[)$,
$f\circ\alpha(0) = f\circ\alpha(1) = 0 \Rightarrow \angle f\circ\alpha$ is defined on $[0,1]$ and is continuously differentiable (see Definition and Lemma 1.1) and 2)).

2) Is obvious from Definition and Lemma 1.2). □

Lemma 2 *If $\Theta_\alpha(s) \bmod 2\pi \in]0,\pi[$ ($\in]\pi,2\pi[$, resp.) for all $s \in]s_1,s_2[$ then $\angle f\circ\alpha$ is strictly increasing (strictly decreasing, resp.) on $[s_1,s_2]$.*

Proof: We conclude from our hypothesis that $0,1 \notin]s_1,s_2[$ since $\Theta_\alpha(s) \bmod 2\pi \in \{0,\pi\}$ if $s \in \{0,1\}$.
Therefore, using our definitions we may calculate:

$$\begin{aligned}
(\angle f\circ\alpha)^{\cdot} &= \operatorname{Im}\,(f\circ\alpha)^{\cdot}_{\mathbb{C}}/(f\circ\alpha)_{\mathbb{C}} = \\
&= \operatorname{Im}\,|\,(f\circ\alpha)^{\cdot}_{\mathbb{C}}\,|\,e^{i\angle(f\circ\alpha)^{\cdot}}/(f\circ\alpha)_{\mathbb{C}} = \\
&= \operatorname{Im}\,|\,(f\circ\alpha)^{\cdot}_{\mathbb{C}}/(f\circ\alpha)_{\mathbb{C}}e^{i\angle(f\circ\alpha)+i\Theta_\alpha} = \\
&= \operatorname{Im}\frac{|\,(f\circ\alpha)^{\cdot}_{\mathbb{C}}\,|\,\overline{(f\circ\alpha)_{\mathbb{C}}}}{|\,(f\circ\alpha)_{\mathbb{C}}\,|^2}\;\frac{(f\circ\alpha)_{\mathbb{C}}}{|\,(f\circ\alpha)_{\mathbb{C}}\,|}\quad e^{i\Theta_\alpha} = \\
&= \frac{|\,(f\circ\alpha)^{\cdot}_{\mathbb{C}}\,|}{|\,(f\circ\alpha)_{\mathbb{C}}\,|}\sin\Theta_\alpha
\end{aligned}$$

Applying Rolles theorem yields the assertion.□

(see Definition 2.4).
Assuming that there is no $s \in [s_1, s_2[$ with $s \in V_\alpha$ Lemma 2 shows that $\angle f \circ \alpha$ is strictly increasing (decreasing, resp.) in $[s_1, s_2]$.
Therefore

$$\angle f \circ \alpha(s_1) \neq \angle f \circ \alpha(s_2)$$

and we deduce from (1) that

$$\angle f \circ \alpha(s_2) - \angle f \circ \alpha(s_1) = 2\pi k$$

with a $k \in \mathbb{Z} \setminus \{0\}$.
Thus the continuity of $\angle f \circ \alpha$ implies that there must be an $s \in [s_1, s_2[$ such that

$$\angle f \circ \alpha(s) = \omega_\alpha \qquad \mod 2\pi$$

i.e. $s \in W_\alpha$.□

Definition and Lemma 4 *For all $s_1, s_2 \in [0,1]$, $s_1 < s_2$, we define $n_\alpha(s_1, s_2) \in \mathbb{N}$ by*

$$n_\alpha(s_1, s_2) := \begin{cases} \sharp([s_1, s_2[\cap (V_\alpha \cup W_\alpha)) & \text{if } s_2 < 1 \\ \sharp([s_1, s_2] \cap (V_\alpha \cup W_\alpha)) & \text{if } s_2 = 1 \end{cases}$$

Then, for every $s \in]0,1[$ and every $\alpha \in C^f$ it is true that

$$\sharp\, \alpha^{-1}(im\ \Gamma^f_\alpha(s, \cdot)) - 1 \leq n_\alpha(s_\alpha, s^\alpha)$$

where

$$s_\alpha := \min \alpha^{-1}(im\ \Gamma^f_\alpha(s, \cdot))$$
$$s^\alpha := \max \alpha^{-1}(im\ \Gamma^f_\alpha(s, \cdot)).$$

(i.e. the total number of intersections of α with the image of $\Gamma^f_\alpha(s, \cdot)$ minus 1 it at most the number of elements of $V_\alpha \cup W_\alpha$ which are between the first and the last intersection. Furthermore, we define $t_\alpha(s), t^\alpha(s) \in \Omega_\alpha(s)$ to be the unique elements such that

$$\Gamma^f_\alpha(s, t_\alpha(s)) = \alpha(s_\alpha) \text{ and } \Gamma^f_\alpha(s, t^\alpha(s)) = \alpha(s^\alpha), \text{resp.}$$

Proof: Since $\alpha \in C^f$ we know from Definition and Lemma 3.2) and 3) that $\sharp\, \alpha^{-1}(\text{im}\ \Gamma^f_\alpha(s, \cdot))$ is also finite (see Definition 2.4) in addition).
Moreover $s \in \alpha^{-1}(\text{im}\ \Gamma^f_\alpha(s, \cdot))$ by definition. Thus s_α, s^α are defined. Now, the assertion is a direct consequence of Lemma 3.□

Lemma 4 *1) $t_\alpha(s) = t^\alpha(s) \Rightarrow s_\alpha = s^\alpha$.*

2) $s_\alpha = 0$ and $s^\alpha = 1$ at the same time is impossible.

3) There is a neighbourhood $U_\alpha(s)$ of $\Gamma^f_\alpha(\{s\} \times [t_\alpha(s), t^\alpha(s)])$ in $\mathbb{R}^2$ such that

$$f \mid U_\alpha(s) : U_\alpha(s) \to f(U_\alpha(s))$$

is a diffeomorphism and
$f(U_\alpha(s))$ is a neighbourhood of $\Gamma_\alpha(\{s\} \times [t_\alpha(s), t^\alpha(s)])$.

Proof:

1) $t_\alpha(s) = t^\alpha(s) \Rightarrow \alpha(s_\alpha) = \alpha(s^\alpha)$ (see the def. of t_α, t^α in Definition and Lemma 4) $\Rightarrow s_\alpha = s^\alpha$ since α is injective.

3) $\Gamma_\alpha(s,\cdot)$ is injective by definition $\Rightarrow f \mid \text{im}\ \Gamma^f_\alpha(s,\cdot)$ is injective.
If the assertion would be false, we could construct two convergent (since $B := \Gamma^f_\alpha(\{s\} \times [t_\alpha(s), t^\alpha(s)]) \subset \text{im}\ \Gamma^f_\alpha(s,\cdot)$ is compact) sequences $x_n \to x \in B$, $y_n \to y \in B$ with $f(x_n) = f(y_n)$.
However, this implies $f(x) = f(y)$ which is a contradiction since we already know that $f \mid B$ is injective.

2) From the definitions we conclude that $\alpha(s_\alpha), \alpha(s^\alpha) \in U_\alpha(s)$.
If $s_\alpha = 0$ and $s^\alpha = 1$ we know that $\alpha(s_\alpha) \neq \alpha(s^\alpha)$ (α is injective), hence 3) implies that $f \circ \alpha(s_\alpha) \neq f \circ \alpha(s^\alpha)$, too.
However, $f \circ \alpha(0) = f \circ \alpha(1)$ by definition of α - a contradiction.□

Definition and Lemma 5 *(See figure 1) Let us assume that $\alpha \in C^f$ and $\bar{s} \in]0,1[$ are such that*

$$\sharp\, \alpha^{-1}(im\ \Gamma^f_\alpha(\bar{s},\cdot)) \geq 2 \tag{2}$$

Then we will construct a modified α (depending on $\bar{s}$), called $\alpha_{\text{mod}} \in C$ (associated with $\bar{s}$) such that

$$n_{\alpha_{\text{mod}}}(0,1) \leq n_\alpha(0,1) - (\sharp\alpha^{-1}(im\ \Gamma^f_\alpha(\bar{s},\cdot)) - 2).$$

This is as follows:
At first we define the curve $\beta_0 \in C^0([0,1], \mathbb{R}^2)$ by

$$\beta_0(s) := \begin{cases} f \circ \alpha(s) & \text{if } s \in [0, \bar{s}_\alpha] \cup [\bar{s}^\alpha, 1] \\ \Gamma_\alpha(\bar{s}, \tau(s)) & \text{if } s \in [\bar{s}_\alpha, \bar{s}^\alpha] \end{cases}$$

with $\tau(s) := (s - \bar{s}_\alpha)t^\alpha(\bar{s})/(\bar{s}^\alpha - \bar{s}_\alpha) + (s - \bar{s}^\alpha)t_\alpha(\bar{s})/(\bar{s}_\alpha - \bar{s}^\alpha)$.
Inequality (2) shows that $\bar{s}_\alpha \neq \bar{s}^\alpha$, hence $t_\alpha(\bar{s}) \neq t^\alpha(\bar{s})$. Therefore it is true that:

1) $\beta_0 \mid [\bar{s}_\alpha, \bar{s}^\alpha]$ is injective.

If we choose a neighbourhood $U_\alpha(\bar{s})$ of $\Gamma^f_\alpha(\{\bar{s}\} \times [t_\alpha(\bar{s}), t^\alpha(\bar{s})])$ according to Lemma 4.3) we can also find a neighbourhood $[\bar{s}_\alpha - \epsilon_\alpha, \bar{s}_\alpha + \epsilon^\alpha]$ relative to $[0,1]$ such that

2) $\beta_0([\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]) \subset f(U_\alpha(\bar{s}))$,
$\angle\beta_0([\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]) \subset [\angle\Gamma_\alpha(\bar{s},1) - \pi/4, \angle\Gamma_\alpha(\bar{s},1) + \pi/4] \bmod 2\pi$.

3) $\beta_0 \mid [\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$ is injective,

4) $[\bar{s}_\alpha - \epsilon_\alpha, \bar{s}_\alpha[\cap (V_\alpha \cup W_\alpha) =]\bar{s}^\alpha, \bar{s}^\alpha + \epsilon^\alpha] \cap (V_\alpha \cup W_\alpha) = \emptyset$.

At this stage we have to distinguish two cases:

<u>*case 1:*</u> *It is true that $\epsilon_\alpha, \epsilon^\alpha > 0$ and*
$\angle\beta_0(\bar{s}_\alpha - \epsilon_\alpha) < \angle\beta_0(\bar{s}_\alpha) = \angle\beta_0(\bar{s}^\alpha) < \angle\beta_0(\bar{s}^\alpha + \epsilon^\alpha)$
or vice versa, i.e. "$<$" replaced by "$>$".

case 2: *The condition for case 1 is not true. This means that either $\bar{s}_\alpha = 0$ or $\bar{s}^\alpha = 1$ or in one of the two inequalities above we have ">" and in the other we have "<".*

In both cases we can find a C^1-curve β arbitrarily close to β_0 such that the following is true:

5) $\beta(s) = \beta_0(s) = f \circ \alpha(s)$ *if* $s \in [0, \bar{s}_\alpha - \epsilon_\alpha]$ *or* $s \in [\bar{s}^\alpha + \epsilon^\alpha, 1]$

6) $\beta([\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]) \subset f(U_\alpha(\bar{s}))$
$\angle\beta([\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]) \subset [\angle\Gamma_\alpha(\bar{s}, 1) - \pi/4, \angle\Gamma_\alpha(\bar{s}, 1) + \pi/4] \bmod 2\pi$
(i.e. the modified part is contained in a half-cone.)

7) $\dot{\beta}(s) \neq 0$ *for all* $s \in [0, 1]$

8) $\beta \mid [\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$ *is injective*

9) *In case 1 (case 2, resp.) the function*

$$(\angle\dot{\beta}(\cdot) - \angle\beta(\cdot)) \bmod \pi \tag{3}$$

has no zero (exactly one zero, resp.) in $[\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$.
In case 2 this zero is transversal.
(Remember that (3) being zero is the condition for β being tangent at s to a straight ray emanating from 0.)

10) *There is at most one* $s \in [\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$ *such that*

$$\angle\beta(s) \bmod 2\pi = \omega_\alpha$$

in case 1 and there is no such s in case 2.

Finally, we define $\alpha_{\text{mod}} \in C^1([0,1], \mathbb{R}^2)$ by

$$\alpha_{\text{mod}}(s) := \begin{cases} (f \mid U_\alpha(\bar{s}))^{-1} \circ \beta(s) & \text{if } s \in [\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha] \\ \alpha(s) & \text{if } s \in [0, \bar{s}_\alpha - \epsilon_\alpha] \text{ or } s \in [\bar{s}^\alpha + \epsilon^\alpha, 1] \end{cases}$$

Then the following is true:

11) $\alpha_{\text{mod}} \in C$

12) $n_{\alpha_{\text{mod}}}(0, 1) \leq n_\alpha(0, 1) - (\sharp\alpha^{-1}(im\ \Gamma_\alpha^f(\bar{s}, \cdot) - 2)$

13) $V_{\alpha_{\text{mod}}}$ *has transversal elements only, i.e.* $\alpha_{\text{mod}} \in C^f$

14) $\angle f \circ \alpha_{\text{mod}}([\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]) \subset [\angle\Gamma_\alpha(\bar{s}, 1) - \pi/2, \angle\Gamma_\alpha(\bar{s}, 1) + \pi/2] \bmod 2\pi$ *(i.e. the f-image of the modified part of α_{mod} is contained in a half-cone.)*

Proof: (1) is already proven above.
If the neighbourhood $[\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$ is chosen to be small enough then (2) is true by continuity, (3) follows from (1) and from the fact that β_0 is locally injective and (4) is valid since V_α and W_α are finite sets.
Now using (2) and (3), it is easy to see that there are C^1-curves β arbitrarily close to β_0 which satisfy (5) - (10).
The precise construction is straightforward and is left to the reader since it would cause

unproportionally much notation here.
(11): Obviously, $\alpha_{\mathbf{mod}}$ is a C^1 map and the properties (i) and (ii) required of $\alpha_{\mathbf{mod}}$ to be contained in C are satisfied (see (7) and (5)).
Moreover, since β can be chosen to be arbitrarily close to β_0 property (iv) is also valid.
It remains to prove that $\alpha_{\mathbf{mod}}$ is injective (prop. (iii)): Using (8) the definition of $\alpha_{\mathbf{mod}}$ shows that we only have to prove that the unchanged parts $\alpha_{\mathbf{mod}} \mid [0, \bar{s}_\alpha - \epsilon]$ and $\alpha_{\mathbf{mod}} \mid [\bar{s}^\alpha + \epsilon^\alpha, 1]$, resp. cannot intersect the modified part $\alpha_{\mathbf{mod}} \mid [\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha]$.
However, if this would happen, then $\bar{s}_\alpha$ would not be the first intersection of α with $\operatorname{im} \Gamma^f_\alpha(\bar{s}, \cdot)$ or $\bar{s}^\alpha$ would not be the last, resp. (Use (5) and again the fact that β is arbitrarily close to β_0.)
(12): Using (9) and (10) we conclude from Definition and Lemma 3 and 4 that

$$n_{\alpha_{\mathbf{mod}}}(\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha) \leq 1.$$

At the same time we conclude from our hypothesis using Definition and Lemma 4 that

$$n_\alpha(\bar{s}_\alpha, \bar{s}^\alpha) \geq \sharp\alpha^{-1}(\operatorname{im} \Gamma^f_\alpha(\bar{s}, \cdot)) - 1.$$

Therefore, we may calculate

$$\begin{aligned} n_{\alpha_{\mathbf{mod}}}(0,1) &= n_{\alpha_{\mathbf{mod}}}(0, \bar{s}_\alpha - \epsilon_\alpha) + n_{\alpha_{\mathbf{mod}}}(\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha) + n_{\alpha_{\mathbf{mod}}}(\bar{s}^\alpha + \epsilon^\alpha, 1) \\ &\leq n_{\alpha_{\mathbf{mod}}}(0, \bar{s}_\alpha - \epsilon_\alpha) + 1 + n_{\alpha_{\mathbf{mod}}}(\bar{s}^\alpha + \epsilon_\alpha, 1) \end{aligned}$$

and

$$\begin{aligned} n_\alpha(0,1) &= n_\alpha(0, \bar{s}_\alpha) + n_\alpha(\bar{s}_\alpha, \bar{s}^\alpha) + n_\alpha(\bar{s}^\alpha, 1) \\ &\geq n_\alpha(0, \bar{s}_\alpha - \epsilon_\alpha) + \sharp\alpha^{-1}(\operatorname{im} \Gamma^f_\alpha(\bar{s}, \cdot)) - 1 + n_\alpha(\bar{s}^\alpha + \epsilon^\alpha, 1) \end{aligned}$$

Using the definition of $\alpha_{\mathbf{mod}}$ the assertion (12) follows.
(13) follows from (9) and (14) from (6). □

Definition and Lemma 6 *(See figure 1) For every $\bar{s} \in]0,1[$ and every curve $\alpha \in C^f$ we define*

$$\mu_\alpha(s) := \begin{cases} 2 & \textit{if } s \in]0,1[\cap V_\alpha \\ 1 & \textit{otherwise} \end{cases}$$

for all $s \in S_\alpha(\bar{s}) := \alpha^{-1}(im\ \Gamma^f_\alpha(\bar{s}, \cdot))$.
There exists a curve $\alpha_D \in C^f$ such that

$$\sum_{s \in S_{\alpha_D}(\bar{s})} \mu_{\alpha_D}(s) \leq 2$$

for all $\bar{s} \in]0,1[$.
(This means that the number of intersections of α_D with $im\ \Gamma^f_{\alpha_D}(\bar{s}, \cdot)$ is two at most. Interior intersections which are tangential at the same time are counted twice.)

Proof: We construct the desired α_D by a finite iteration of the modification process of Definition and Lemma 5:
We start with an arbitrarily chosen $\alpha \in C^f$. (Due to Definition and Lemma 3.2) this set is nonempty.)
If $\sum_{s \in S_\alpha(\bar{s})} \mu_\alpha(s) > 2$, $\bar{s} \in]0,1[$ and if $\sharp S_\alpha(\bar{s}) \leq 2$ there must be at least one $s \in S_\alpha(\bar{s})$ with $\mu(s) = 2$, i.e. $s \in]0,1[\cap V_\alpha$.

In other words, α intersects im $\Gamma^f_\alpha(\bar{s},\cdot)$ at s tangentially and the function $\Theta_\alpha(\cdot)$ mod π has a transversal zero at s (since $s \in]0,1[\cap V_\alpha$ and $\alpha \in C^f$).
This implies that the tangential intersection of α with im $\Gamma^f_\alpha(\bar{s},\cdot)$ at s is non-transversal. Therefore we can slightly modify α in a neighbourhood of s such that this intersection bifurcates into two intersections and such that V_α and W_α ($\alpha \in C^f \Rightarrow s \notin W_\alpha$, see Definition and Lemma 3.3)) remain unchanged in addition.
Thus we have seen that we may assume w.l.o.g. that $\sharp S_\alpha(\bar{s}) > 2$.
Now an application of Definition and Lemma 5.12) yields an $\alpha_{\mathbf{mod}} \in C^f$ such that $n_{\alpha_{\mathbf{mod}}}(0,1) \leq n_\alpha(0,1) - 1$.
Repeating this process if necessary we will finally arrive (since $n_\alpha(0,1)$ cannot become negative) at an $\alpha_D \in C^f$ such that the assertion is valid. □

Now we define a map (Definition 3) which relates the remaining two intersections of α_D with a lifted ray if they exist. Lemma 5 summarizes the most important properties of this map.

Definition 3 $A := \{s \in]0,1[\mid \sharp \alpha_D^{-1}(im\ \Gamma^f_{\alpha_D}(s,\cdot)) \geq 2\}$
Since always $s \in \alpha_D^{-1}(im\ \Gamma^f_{\alpha_D}(s,\cdot))$ and due to Definition and Lemma 6 we may define the following maps:
$a : A \to [0,1],\ a(s) := \alpha_D^{-1}(im\ \Gamma^f_{\alpha_D}(s,\cdot)) \backslash \{s\}$
(where we have identified one-element sets with their element) and
$b : A \to [0,\infty[$ *with $b(s)$ being the unique (since $\Gamma^f_{\alpha_D}(s,\cdot)$ is injective) element of $\Omega_{\alpha_D}(s)$ such that*

$$\alpha_D(a(s)) = \Gamma^f_{\alpha_D}(s, b(s)).$$

Lemma 5 *1)* $a(A) \cap V_{\alpha_D} \cap]0,1[= \emptyset,\ A \cap V_{\alpha_D} = \emptyset.$

2) A is open and a, b are continuous. Especially, $a^{-1}(0)$, $a^{-1}(1)$ are open, too.

3) If $a(s_1) = a(s_2) \notin \{0,1\}$ then $s_1 = s_2$.
Moreover a is monotone on every interval $I \subset A$ and is strictly monotone if $0, 1 \notin a(I)$.
In addition, $b(s) \neq 1$ everywhere.

4) $\angle f \circ \alpha_D$ and $\angle f \circ \alpha_D \circ a$ are monotone on every interval $I \subset A$.

5) For all $i = 1, \ldots, n-1$ there is an $\epsilon > 0$ such that $]v_i, v_i + \epsilon[\subset A$ ($]v_{i+1} - \epsilon, v_{i+1}[\subset A$, resp.) Moreover

$$a(s) \leq v_i \text{ and either } b(s) < 1 \text{ if } \Theta_{\alpha_D}(v_i) = 0 \mod 2\pi$$
$$\text{or} \quad b(s) > 1 \text{ if } \Theta_{\alpha_D}(v_i) = \pi \mod 2\pi$$

$$\text{(or } a(s) \geq v_{i+1} \text{ and either } b(s) < 1 \text{ if } \Theta_{\alpha_D}(v_{i+1}) = \pi \mod 2\pi$$
$$\text{or} \quad b(s) > 1 \text{ if } \Theta_{\alpha_D}(v_{i+1}) = 0 \mod 2\pi, \text{ resp.)}$$

for all $s \in]v_i, v_i + \epsilon[$ ($\in]v_{i+1} - \epsilon, v_{i+1}[$, resp.).
In addition

$$\lim_{s \to v_i} a(s) = v_i \qquad (\lim_{s \to v_{i+1}} a(s) = v_{i+1}, \text{ resp. }).$$

6) Let $I \subset A$ be an interval such that $v_i \in \bar{I}$ for some $i = 1, \ldots, n$. Then, for all $s \in I$

$$\angle f \circ \alpha_D \circ a(s) \leq \ (\geq, \textit{resp.}\ )\ \angle f \circ \alpha_D(s)$$

if $v_i = \inf I$ ($v_i = \sup I$, resp.) and $\angle f \circ \alpha_D$ is increasing on I or if $v_i = \sup I$ ($v_i = \inf I$, resp.) and $\angle f \circ \alpha_D$ is decreasing on I.

Proof:

1) If we assume that $a(s) \in V_{\alpha_D} \cap]0,1[$ then $\mu_{\alpha_D}(s) = 2$ and Definition and Lemma 6 shows that $a(s)$ must be the only element of $\alpha_D^{-1}(\text{im}\ \Gamma^f_{\alpha_D}(s,\cdot))$. However, $s \in \alpha_D^{-1}(\text{im}\ \Gamma^f_{\alpha_D}(s,\cdot))$ by definition - a contradiction.
If we assume that $s \in V_{\alpha_D}$ then $\mu_{\alpha_D}(s) = 2$ or $s \in \{0,1\}$. Therefore, Definition and Lemma 6 shows again that either $\alpha_D^{-1}(\text{im}\ \Gamma^f_{\alpha_D}(s,\cdot))\backslash\{s\} = \emptyset$ or $s \in \{0,1\}$. Both cases imply that $s \notin A$.

2) If $\alpha_D(a(s)) = \Gamma^f_{\alpha_D}(s, b(s))$ then assertion (1) shows that either $a(s) \in \{0,1\}$ or $a(s) \notin V_{\alpha_D}$. We recall that Ω_{α_D} is open in $]0,1[\times[0,\infty[$.
In the first case we take $a(s') := a(s) \in \{0,1\}$ for all $s \in [0,1]$. Then the implicit function theorem yields a continuous extension of b to some neighbourhood of s since $(\Gamma^f_{\alpha_D}(s,\cdot))^{\dot{}} \neq 0$.
In the second case we directly apply the implicit function theorem and obtain a continuous extension of a and b to some neighbourhood of s. This is due to the fact that $\dot{a}(s)$ and $(\Gamma^f_{\alpha_D}(s,\cdot))^{\dot{}}$ are not parallel if $a(s) \notin V_{\alpha_D}$.

3) Since $a(s_1) = a(s_2) \notin \{0,1\}$ we conclude that
$\Gamma^f_{\alpha_D}(s_1, b(s_1)) = \alpha_D(a(s_1)) = \alpha_D(a(s_2)) = \Gamma^f_{\alpha_D}(s_2, b(s_2))$
with $b(s_1), b(s_2) \neq 0$. Thus it is clear that $\text{im}\ \Gamma^f_{\alpha_D}(s_1,\cdot) = \text{im}\ \Gamma^f_{\alpha_D}(s_2,\cdot)$. Therefore, if $s_1 \neq s_2$ then $\text{im}\ \Gamma^f_{\alpha_D}(s_1,\cdot)$ would contain three different points, namely s_1, s_2 and $a(s_1) = a(s_2)$. This contradicts Definition and Lemma 6.
Now, monotonicity is obvious if we also use the connectedness of I and the continuity of a.
If $b(s) = 1$ then $\alpha_D(a(s)) = \Gamma^f_{\alpha_D}(s, b(s)) = \Gamma^f_{\alpha_D}(s, 1) = \alpha_D(s)$.
Since α_D is injective we conclude $a(s) = s$, a contradiction.

4) Assertion (1) implies that neither I nor $a(I)$ contain an element of V_{α_D} in their interior (rel $\mathbb{R}$). Therefore Lemma 2 shows that $\angle f \circ \alpha_D$ is monotone on I and on $a(I)$. Since a is also monotone (see assertion (3)) we are done.

5) Since the assertion is local and f is a local diffeomorphism we may consider the f-images of α_D and $\Gamma^f_{\alpha_D}$ as well. Thus if $v_{i+1} = 0$ it is easy to see that the assertion is true with $a(s) = 0$ and $b(s) = 0$ since $f \circ \alpha_D(0) = \Gamma_{\alpha_D}(s, 0)$.
If $v_{i+1} \neq 0$ then it is not difficult to see that the situation in a neighbourhood of $f \circ \alpha_D(v_i)$ is homeomorphic to the following one on $]0,1[^2$:
Let $f \circ \alpha_D(s) := ((s-1/2)^2, s-1/2)$, $\Gamma_{\alpha_D}(s,t) := ((s-1/2)^2, (s-1/2)+t-1)$ and $v_{i+1} = 1/2$. (case $\Theta_{\alpha_D}(v_{i+1}) = 0 \bmod 2\pi$). Then

$$f \circ \alpha_D(a(s)) = \Gamma_{\alpha_D}(s, b(s))$$

with $a(s) := 1-s$, $b(s) := 2-2s$. Now it is straightforward to verify the assertion. (We have only treated the assertion at v_{i+1} with $\Theta_{\alpha_D}(v_{i+1}) = 0 \bmod 2\pi$. However, the other cases are completely analogous.)

6) I may always be partitioned into $I = I_0 \cup I' \cup I_1$ with
$I_0 := a^{-1}(0) \cap I$, $I' := a^{-1}(]0,1[) \cap I$, $I_1 := a^{-1}(1) \cap I$.
However, since a is monotone on I we know that I_0, I', I_1 are intervals with

$$\sup I_0 = \inf I', \ \sup I' = \inf I_1 \tag{4}$$

if a is increasing and

$$\sup I_1 = \inf I', \ \sup I' = \inf I_0 \tag{5}$$

if a is decreasing on I.
In addition, at least one of the intervals I_0, I_1 is always empty. Otherwise, using the intermediate value theorem we could find an $s \in I$ with $s = a(s)$, a contradiction.
Our definitions show that

$$\angle f \circ \alpha_D \circ a(s) = \begin{cases} \angle f \circ \alpha_D(0) & \text{if } s \in I_0 \\ \angle f \circ \alpha_D(s) \bmod 2\pi & \text{if } s \in I' \\ \angle f \circ \alpha_D(1) & \text{if } s \in I_1 \end{cases} \tag{6}$$

We treat the case when $v_i = \inf I$ and $\angle f \circ \alpha_D$ is increasing on I:
Assertion (5) (i.e. $a(s) \leq v_i$ and $\lim_{s \to v_i} a(s) = v_i$) shows that a is decreasing. Therefore, we know that either $I = I_1 \cup I'$ or $I = I' \cup I_0$ with (5) being satisfied.
If $I = I_1 \cup I'$ (with I_1 assumed to be nonempty) we conclude

$$\inf I = v_i = \lim_{s \to v_i} a(s) = 1.$$

At the same time $I \subset]0,1[$ by definition. Therefore, I must be empty - a contradiction.
If $I = I' \cup I_0$ assertion (5) shows that

$$\lim_{s \to v_i} \angle f \circ \alpha_D \circ a(s) = \lim_{s \to v_i} \angle f \circ \alpha_D(s)$$

Therefore, (6) shows that

$$\angle f \circ \alpha_D \circ a(s) = \angle f \circ \alpha_D(s) \tag{7}$$

for all $s \in I'$ by continuation.
If $s \in I_0$ we calculate (using that $f \circ \alpha_D$ is increasing):

$$\angle f \circ \alpha_D(s) \geq \angle f \circ \alpha_D(\inf I_0) = \angle f \circ \alpha_D \circ a(\inf I_0)$$

$$= \angle f \circ \alpha_D(0) = \angle f \circ \alpha_D \circ a(s),$$

where the first equation follows from (7) by continuity. We also used (6).

The other three cases are completely analogous.□

The following lemma shows that w.l.o.g. we may assume that the remaining "oscillations" (i.e. the variation of $\angle f \circ \alpha_D$) are not too large in some sense.

Lemma 6 *W.l.o.g. we may assume that for all intervals $I \subset A$ such that $v_i \in \bar{I}$ (closure of I rel $\mathbb{R}$) (for some $i = 1, \ldots, n$) and for all $\bar{s} \in I$ it is true that*

$$|\ \angle f \circ \alpha_D(\bar{s}) - \angle f \circ \alpha_D(v_i)\ | \leq \pi.$$

Proof: If the assertion is not true for a v_i, I and an $\bar{s}$ we use Definition and Lemma 5 in order to replace α_D by its modification $(\alpha_D)_{\text{mod}}$ associated to $\bar{s}$. (This is possible since $\bar{s} \in I$ implies $\sharp\alpha_D^{-1}(\text{im } \Gamma^f_{\alpha_D}(\bar{s}, \cdot)) = 2$.) Clearly, all properties of α_D are conserved.
We show that this reduces

$$\text{Var }(\alpha_D) := \sum_{k=1}^{n-1} \mid \angle f \circ \alpha_D(v_{k+1}) - \angle f \circ \alpha_D(v_k) \mid$$

by $\pi/2$ at least. Since $v_i \in \bar{I}$ we conclude from Lemma 5 that $a(\bar{s}) \in [v_{i-1}, v_i]$ (or $a(\bar{s}) \in [v_i, v_{i+1}]$, resp.)
W.l.o.g. we will consider only the first case furtheron. Moreover,
$\sharp\alpha_D^{-1}(\text{im } \Gamma^f_{\alpha_D}(\bar{s}, \cdot)) = 2$ shows that $\bar{s}_\alpha = \bar{s}$ and $\bar{s}^\alpha = a(\bar{s})$ or vice versa. Since $(\alpha_D)_{\text{mod}}$ equals α_D exept for all $s \in]\bar{s}_\alpha - \epsilon_\alpha, \bar{s}^\alpha + \epsilon^\alpha[$ (with $\bar{s}_\alpha = \bar{s}$ and $\bar{s}^\alpha = a(\bar{s})$ or vice versa) only the (at most two) summands of Var (α_D), which contain v_i are affected by the modification.
Thus we calculate (using the abreviation $\check{s} := \angle f \circ \alpha_D(s)$ and recalling that we have defined $v_0 := v_1 = 0$, $v_{n+1} := v_n = 1$):

$$\begin{aligned}
&\mid \angle f \circ \alpha_D(v_i) - \angle f \circ \alpha_D(v_{i-1}) \mid + \mid \angle f \circ \alpha_D(v_{i+1}) - \angle f \circ \alpha_D(v_i) \mid \\
&\overset{(1)}{=} \mid \check{v}_i - \check{a}(\bar{s}) \mid + \mid \check{a}(\bar{s}) - \check{v}_{i-1} \mid + \mid \check{v}_{i+1} - \check{\bar{s}} \mid + \mid \check{\bar{s}} - \check{v}_i \mid \\
&\overset{(2)}{>} 0 + \mid \check{a}(\bar{s}) - \check{v}_{i-1} \mid + \mid \check{v}_{i+1} - \check{\bar{s}} \mid + \pi \\
&\overset{(3)}{\geq} \mid \check{a}(\bar{s}) - \check{v}_{i-1} \mid + \mid \check{v}_{i+1} - \check{\bar{s}} \mid + 2 \mid \check{\bar{s}} - \check{v}_{i,\text{mod}} \mid + \pi/2 \\
&\overset{(4)}{\geq} \mid \check{v}_{i,\text{mod}} - \check{a}(\bar{s}) \mid + \mid \check{a}(\bar{s}) - \check{v}_{i-1} \mid + \mid \check{v}_{i+1} - \check{\bar{s}} \mid + \mid \check{\bar{s}} - \check{v}_{i,\text{mod}} \mid + \pi/2 \\
&\overset{(5)}{=} \mid \check{v}_{i,\text{mod}} - \check{v}_{i-1} \mid + \mid \check{v}_{i+1} - \check{v}_{i,\text{mod}} \mid + \pi/2
\end{aligned}$$

Here, $v_{i,\text{mod}}$ denotes the single element of V_{α_D}, which is affected by the modification. Equation (1) is true since $\check{a}(\bar{s}) \in [v_{i-1}, v_i]$ and $\bar{s} \in [v_i, v_{i+1}]$ and since $\angle f \circ \alpha_D$ and $\angle f \circ \alpha_D \circ a$ are monotone (see Lemma 5).
Inequality (2) follows from our assumption that the assertion is not true.
Inequality (3) uses Definition and Lemma 5.14 and inequality (4) is a consequence of Lemma 5.6. Equation (5) is similar to equation (1).

Thus, our calculation shows that each time we apply this modification, we reduce Var (α_D) by $\pi/2$ at least. Since Var (α_D) cannot become negative, we finally will arrive at a curve, which satisfies the assertion. □

Now we aim at constructing an unbounded, injective curve γ such that the tangent angle of its image $f \circ \gamma$ rotates by an amount of $3\pi + \epsilon$ $(\epsilon > 0)$ at least if we follow the curve from a point close to the first "end" of γ to a point close to the other "end" (see Definition and Lemma 8). To this end we isolate an appropriate part of α_D, namely $\alpha_D \mid [a_1, a_2]$ (Definition 4, Lemma 8) and hang on the lifted rays, which pass through its ends (Definition and Lemma 7 - Lemma 9). Lemma 7 insures that the above mentioned rotation is 3π at least. Definition and Lemma 8 adds the ϵ-summand. This will be done by a slight rotation of one of the added rays.

Lemma 7 *(See figure 1) There are successive* $v_k, v_{k+1} \in V_{\alpha_D}$ *with* $k \in \{1, \ldots, n-1\}$ *such that* $\Theta_{\alpha_D}(v_k) \bmod 2\pi = 0$ *and either*

$$\Theta_{\alpha_D}(v_{k+1}) = \Theta_{\alpha_D}(v_k) + \pi,$$
$$\angle f \circ \alpha_D(v_k) < \angle f \circ \alpha_D(v_{k+1}) \quad or$$

$$\Theta_{\alpha_D}(v_{k+1}) = \Theta_{\alpha_D}(v_k) - \pi,$$
$$\angle f \circ \alpha_D(v_k) > \angle f \circ \alpha_D(v_{k+1}).$$

Throughout the rest of this work we assume w.l.o.g. that the first alternative holds. This can always be achieved by an orientation reversing reparametrization if necessary.

Proof: Due to Definition and Lemma 2.2 we know that $\Theta_{\alpha_D}(0) \bmod 2\pi = 0$. If $v_k, v_{k+1} \in V_{\alpha_D}$ are successive the continuity of Θ_{α_D} implies that $\mid \Theta_{\alpha_D}(v_{k+1}) - \Theta_{\alpha_D}(v_k) \mid$ must either equal π or equal 0.
If the latter would be true for every successive pair $v_i, v_{i+1} \in V_{\alpha_D}$ we could prove iteratively that $\Theta_{\alpha_D}(1) = \Theta_{\alpha_D}(0)$ (remember that $0, 1 \in V_{\alpha_D}$) which contradicts Definition and Lemma 2.2. Therefore, there is a first successive pair v_i, v_{i+1} such that $|\Theta_{\alpha_D}(v_{i+1}) - \Theta_{\alpha_D}(v_i)| = \pi$. Taking this pair, the first two parts of the assertion (i.e. the parts concerning Θ_{α_D} are obviously satisfied. The last part of the assertion is a direct consequence of Lemma 2 now. □

Definition 4

$$A_1 := a^{-1}([v_{k-1}, v_k]) \cap]v_k, v_{k+1}[$$

$$A_2 := a^{-1}([v_{k+1}, v_{k+2}]) \cap]v_k, v_{k+1}[$$

(Notice that $[v_{k-1}, v_k] = \{0\}$ if $k = 1$ as well as $[v_{k+1}, v_{k+2}] = \{1\}$ if $k + 1 = n$ since we have defined the notation $v_0 := v_1$, $v_{n+1} := v_n$)

Lemma 8 *1) $A_1 \cap A_2 = \emptyset$.*

2) $A_1, A_2 \neq \emptyset$. Especially, there are $\epsilon_1, \epsilon_2 > 0$ such that $]v_k, v_k + \epsilon_1[\subset A_1$ and $]v_{k+1} - \epsilon_2, v_{k+2}[\subset A_2$.

3) A_1, A_2 are open in $]v_k, v_{k+1}[$

4) There is an $\bar{r} \in]v_k, v_{k+1}[$ such that $\bar{r} \notin A_1 \cup A_2$.

Proof:

1) This is clear since $[v_{k-1}, v_k] \cap [v_{k+1}, v_{k+2}] = \emptyset$.

2) This is obvious from Lemma 5.5.

3) Lemma 5.1 shows that

$$a^{-1}([v_{k-1}, v_k]) = \begin{cases} a^{-1}(]v_{k-1}, v_k[) & \text{if } k > 2 \\ a^{-1}([0, v_2[) & \text{if } k = 2 \\ a^{-1}(0) & \text{if } k = 1 \end{cases}$$

Now Lemma 5.2 implies that A_1 is open. The proof for A_2 is analogous.

4) Since $]v_k, v_{k+1}[$ is connected 1),2) and 3) show that $]v_k, v_{k+1}[\setminus (A_1 \cup A_2)$ must be nonempty. □

Definition and Lemma 7 *(See figure 1) Let r_1 be the smallest element of $]v_k, v_{k+1}[\backslash A_1$ and r_2 be the greatest element of $]v_k, v_{k+1}[\backslash A_2$.*
They exist since these sets are left (right, resp.) closed (see Lemma 8.2) and 3)) and since $]v_k, v_{k+1}[\backslash(A_1 \cup A_2) \neq 0$ (see Lemma 8.4)).
Moreover, we know that

$$v_k < r_1 \leq \bar{r} \leq r_2 < v_{k+1}$$

Let $r_1^n \to r_1$, $r_2^n \to r_2$, resp. be sequences in A_1, A_2, resp.
We define

$$a_1 := \lim_{n\to\infty} a(r_1^n) \in [v_{k-1}, v_k], \quad b_1 := \lim_{n\to\infty} b(r_1^n)$$

$$a_2 := \lim_{n\to\infty} a(r_2^n) \in [v_{k+1}, v_{k+2}], \quad b_2 := \lim_{n\to\infty} b(r_2^n)$$

(These limits exist for a suitable subsequence at least.)
By continuity, we conclude that

$$\Gamma_{\alpha_D}(r_1, b_1) = f \circ \alpha_D(a_1),$$

$$\Gamma_{\alpha_D}(r_2, b_2) = f \circ \alpha_D(a_2).$$

Let $\Gamma_{\alpha_D}^{f,a_1}(r_1, \cdot)$ ($\Gamma_{\alpha_D}^{f,a_2}(r_2)$, resp.) denote the lift of $\Gamma_{\alpha_D}(r_1, \cdot)$ ($\Gamma_{\alpha_D}(r_2, \cdot)$, resp.) with respect to f however such that $\Gamma_{\alpha_D}(r_1, b_1)$ ($\Gamma_{\alpha_D}(r_2, b_2)$; resp.) is lifted to $\alpha_D(a_1)$ ($\alpha_D(a_2)$, resp.) Moreover, let $\Omega_{\alpha_D}^{a_1}(r_1) =]c_1, d_1[$ ($\Omega_{\alpha_D}^{a_2}(r_2) =]c_2, d_2[$, resp.) denote the maximal interval of existence. (Of course $b_1 \in \Omega_{\alpha_D}^{a_1}(r_1), b_2 \in \Omega_{\alpha_D}^{a_2}(r_2)$) Then the following is true:

1) $\alpha_D([a_1, a_2]) \cap \Gamma_{\alpha_D}^{f,a_1}(\{r_1\} \times [b_1, d_1[) = \alpha_D(a_1)$
   $\alpha_D([a_1, a_2]) \cap \Gamma_{\alpha_D}^{f,a_2}(\{r_2\} \times [b_2, d_2[) = \alpha_D(a_2)$

2) $\Gamma_{\alpha_D}^{f,a_1}(\{r_1\} \times [b_1, d_1[) \cap \Gamma_{\alpha_D}^{f,a_2}(\{r_2\} \times [b_2, d_2[) = \emptyset$

3) $b_1 \leq 1$, $b_2 \leq 1$

Proof:

1) We assume the contrary, i.e. there are an $s \in]a_1, a_2]$ and a $t \in [b_1, \infty[$ such that $\alpha_D(s) = \Gamma_{\alpha_D}^{f,a_1}(r_1, t)$. Then we know that this intersection is transversal:
   Indeed, since already $\alpha_D(a_1) = \Gamma_{\alpha_D}^{f,a_1}(r_1, b_1)$ we conclude from Definition and Lemma 6 that $s \notin]0, 1[\cap V_{\alpha_D}$. Our assumption says that $s \neq 0$. Assume $s = 1$. Then $f \circ \alpha_D(s) = 0$, hence $t = 0$ and therefore $b_1 = 0$. However the latter implies that $a_1 = 0$ and we arrive at a contradiction to Lemma 4.2) (with $s_\alpha = a_1$, $s^\alpha = s$).

   Thus we know that $s \in V_{\alpha_D}$ is impossible and therefore our intersection is transversal as asserted.
   Therefore, using that Ω_{α_D} (see Definition 2) is open, we conclude that this intersection continues to the lifts of nearby rays, i.e. to all lifted rays $\Gamma_{\alpha_D}^{f,a(r_1^n)}(r_1^n, \cdot)$ with n being large enough.
   Thus we deduce that α_D intersects im $\Gamma_{\alpha_D}^{f,a(r_1^n)}(r_1^n, \cdot)$ for all large n not only at r_1^n and at $a(r_1^n)$ but also at some s_n with $s_n \to s$.
   We know that $r_1^n \to r_1 \neq a_1$, $a(r_1^n) \to a_1$, $s_n \to s \neq a_1$. If we also assume that $s \neq r_1$ for the moment we deduce that $r_1^n, a(r_1^n), s_n$ are pairwise different for all large n. This yields a contradiction to Definition and Lemma 6 finally.
   Now we are left with proving that $s \neq r_1$. From our assumption $\alpha_D(s) \in$

im $\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot)$ we conclude that im $\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot) =$ im $\Gamma^{f}_{\alpha_D}(s,\cdot)$.
Therefore, if $s = r_1$ it would follow that
$\alpha_D(a_1) \in$ im $\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot) =$ im $\Gamma^{f}_{\alpha_D}(r_1,\cdot)$,
i.e. $r_1 \in A_1$ which contradicts the definition of r_1.
The proof of the second equation (with a_2 and r_2) is the same.

2) Since they are straight rays emanating from 0 the images of the curves $\Gamma_{\alpha_D}(r_1,\cdot)$ and $\Gamma_{\alpha_D}(r_2,\cdot)$ are either identical or have only the point 0 in common. Therefore, regarding their lifts we deduce that, if $\Gamma^{f,a_1}_{\alpha_D}(r_1,t_1) = \Gamma^{f,a_2}_{\alpha_D}(r_2,t_2)$ and not $t_1 = t_2 = 0$ then im $\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot) =$ im $\Gamma^{f,a_2}_{\alpha_D}(r_2,\cdot)$.
Thus, if 2) would not be valid we could conclude (following the curves backwards if necessary) that either $\alpha_D(a_1) \in \Gamma^{f,a_2}_{\alpha_D}(\{r_2\}\times[b_2,d_2[)$ or $\alpha_D(a_2) \in \Gamma^{f,a_1}_{\alpha_D}(\{r_1\}\times[b_1,d_1[)$.
However this contradicts assertion 1) which is already proven.

3) From Lemma 5.5 and 5.3 we conclude that $b(s) < 1$ for all $s \in]v_k, r_1[$ ($s \in]r_2, v_{k+1}[$, resp.) Now, the definition of b_1, b_2 implies the assertion. □

Definition 5 *(See figure 1) We define the curve* $\gamma_0 \in C^0(I_0 \cup [0,1] \cup I_1, \mathbb{R}^2)$ *by*

$$\gamma_0(s) := \begin{cases} \Gamma^{f,a_1}_{\alpha_D}(r_1, b_1 - s) & \text{if } s \in I_1 \\ \alpha_D(a_1(1-s) + a_2 s) & \text{if } s \in [0,1] \\ \Gamma^{f,a_2}_{\alpha_D}(r_2, b_2 + s - 1) & \text{if } s \in I_2 \end{cases}$$

There, I_1 and I_2 denote the following intervals:

$$I_1 := \{s \in]-\infty, 0] \mid b_1 - s \in \Omega^{a_1}_{\alpha_D}(r_1)\} =]b_1 - d_1, 0]$$

$$I_2 := \{s \in [1, \infty[\mid b_2 + s - 1 \in \Omega^{a_2}_{\alpha_D}(r_2)\} = [1, d_2 - b_2 + 1[$$

Of course, the maximum of I_1 is 0 and the minimum of I_2 is 1.

Lemma 9 *1) γ_0 is injective.*

2) $d_1 \leq 1$, i.e. $I_1 \subset]b_1 - 1, 0]$,
$d_2 \leq 1$, i.e. $I_2 \subset [1, 2 - b_2[$.
In particular, I_1 and I_2 are bounded.

Proof:

1) This is a direct consequence of Definition and Lemma 7.1).

2) Assume that $d_1 > 1$. Definition and Lemma 7.3 shows that $b_1 \leq 1$. Therefore, $\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot)$ is defined at 1 and Definition and Lemma 7.1) shows that
$\Gamma^{f,a_1}_{\alpha_D}(r_1,1) \neq \alpha_D(r_1)$.
Due to continuity we conclude that also $\Gamma^{f,a(r_1^n)}_{\alpha_D}(r_1^n,1) \neq \alpha_D(r_1^n)$ for all large enough n.
At the same time we know that $\Gamma^{f}_{\alpha_D}(r_1^n,\cdot) = \Gamma^{f,a(r_1^n)}_{\alpha_D}(r_1^n,\cdot)$
since $r_1^n \in A_1$. However, this yields a contradiction since $\Gamma^{f}_{\alpha_D}(r_1^n,1) = \alpha_D(r_1^n)$ by definition.
The proof for d_2 is the same. □

Definition and Lemma 8 *(See figure 2) If* $r_1 < r_2$ *we take* $\gamma := \gamma_0$.
If $r_1 = r_2$ *we improve* γ_0 *in the following way:*
We abreviate $\omega_0 := d_2 - b_2 + 1$ *(i.e.* $I_2 = [1, \omega_0[$*) and* $c := (1+\omega_0)/2$.
Now we take a C^1*-function* $\Psi : [1, \infty[\to [0,1]$ *with* $\Psi^{-1}(0) = [1, c/2]$ *and* $\Psi^{-1}(1) = [c, \infty[$.
For every $\eta \in]0, \infty[$ *we define* $\rho_\eta : [1, \infty[\to \mathbb{R}^2$ *by*

$$\rho_\eta(s) := \Gamma_{\alpha_D}(r_2, s + b_2 - 1) + \eta(s + b_2 - 2)\Psi(s)\Gamma_{\alpha_D}(r_2, 1)^\perp$$

where $(x,y)^\perp := (-y, x)$.
Let ρ_η^f *be defined to be the lift of* ρ_η *with respect to* f *such that* $\rho_\eta^f(1) = \gamma_0(1) \in f^{-1}(\rho_\eta(1))$ *and let* $[1, \omega_\eta[$ *denote the associated maximal interval of existence.*
Now we define

$$\gamma_\eta(s) := \begin{cases} \gamma_0(s) & \text{if } s \in I_1 \cup [0,1] \\ \rho_\eta^f(s) & \text{if } s \in [1, \omega_\eta[\end{cases}$$

Then there is a $\bar{\eta} > 0$ *such that the curve*

$$\gamma := \begin{cases} \gamma_0 \circ \beta & \text{if } r_1 < r_2 \\ \gamma_{\bar{\eta}} \circ \beta & \text{if } r_1 = r_2 \end{cases}$$

where $\beta : I_1 \cup [0,1] \cup [1, \omega_{\bar{\eta}}[\to I_1 \cup [0,1] \cup I_2$ *is an orientation preserving and regular reparametrization with* $\beta(0) = 0$, $\beta(1) = c$, *has the following properties:*

1) γ *is injective.*

2) γ *is proper and p.w. regular.*

3) *There is an* $\epsilon > 0$ *such that*
$\angle(f \circ \gamma)^{\cdot}(s_2) - \angle(f \circ \gamma)^{\cdot}(s_1) \geq 3\pi + \epsilon$ *for all* $s_2 \in I_2 \backslash \{1\}$, $s_1 \in I_1 \backslash \{0\}$.

Proof:

1) If $r_1 < r_2$ the assertion follows from Lemma 9.1). So let us consider the case $r_1 = r_2$. Since ρ_η is injective by definition we conclude that $\gamma_\eta \mid [1, \omega_\eta[= \rho_\eta^f$ is injective, too. From Lemma 9.1 we know that $\gamma_\eta \mid I_1 \cup [0, c/2] = \gamma_0 \mid I_1 \cup [0, c/2]$ is also injective. Thus, in order to prove the injectivity of γ_η for some $\eta > 0$ it remains to show that $\rho_\eta^f \mid]c/2, \omega_\eta[$ cannot intersect $\gamma_0 \mid I_1 \cup [0,1]$:
The definition of ρ_η shows that for all $\eta > 0$ the only intersection of $\rho_\eta \mid]c/2, \infty[$ and $\Gamma_{\alpha_D}(r_2, \cdot) = \Gamma_{\alpha_D}(r_1, \cdot)$ is

$$\rho_\eta(2 - b_2) = \Gamma_{\alpha_D}(r_2, 1).$$

However, the point $\Gamma_{\alpha_D}(r_2, 1)$ is neither lifted to $\Gamma_{\alpha_D}^{f,a_1}(r_1, \cdot)$ nor to $\Gamma_{\alpha_D}^{f,a_2}(r_2, \cdot)$ since $1 \notin \Omega_{\alpha_D}^{a_1}(r_1)$ and $1 \notin \Omega_{\alpha_D}^{a_2}(r_2)$ (see Lemma 9.2). Using Definition 5 this shows that $\rho_\eta^f \mid]c/2, \omega_\eta[$ cannot intersect $\gamma_0 \mid I_1$ nor $\gamma_0 \mid I_2$. Thus we are left with proving that $\rho_\eta^f \mid]c/2, \omega_\eta[$ cannot intersect $\gamma_0 \mid]0,1[$:

Let us assume the contrary and let $\rho_\eta^f(q_\eta) = \gamma_0(p_\eta)$ denote the first intersection. Then the injectivity of ρ_η^f and γ_0 implies that $\gamma_\eta \mid [p_\eta, q_\eta]$ is a closed Jordan curve dividing $\mathbb{R}^2$ into two components. We conclude that the bounded interior component G_c is on the right side of
$\gamma_\eta \mid [p_\eta, q_\eta]$: Indeed, $\gamma_0 \mid I_2$ branches off ρ_η^f to the left side (see the definition of ρ_η^f

together with Definition 5).
Since we already know that $\gamma_0 \,|\, I_2$ and $\rho^f_\eta \,|\,]c/2, \omega_\eta[$ cannot intersect and since γ_0 is injective we conclude that γ_0 cannot cross $\partial G_c = \gamma_0([p_\eta, c/2]) \cup \rho^f_\eta([c/2, q_\eta])$. Therefore, if G_c would be on the left side of $\gamma_\eta \mid [p_\eta, q_\eta]$ then $\gamma_0(I_2)$ would be contained in G_c which is impossible since γ_0 is proper and therefore $\gamma_0(I_2)$ unbounded.

Thus we conclude that $\gamma_\eta \mid [p_\eta, q_\eta]$ is a right oriented Jordan curve which implies that the total rotation of the tangent angle is -2π. Since f is orientation preserving the same is true for $f \circ \gamma_\eta \mid [p_\eta, q_\eta]$. (This can be shown by shrinking the curve (apply an isotopy $[0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$) into a neighbourhood of zero where f is a diffeomorphism.)
More precisely this means that

$$\angle(f \circ \gamma_0\dot{)}(1) - \angle(f \circ \gamma_0\dot{)}(p_\eta) + \triangle^1_\eta +$$

$$+\angle(f \circ \rho^f_\eta\dot{)}(q_\eta) - \angle(f \circ \rho^f_\eta\dot{)}(1) + \triangle^{p,q}_\eta = -2\pi$$

with $\triangle^1_\eta, \triangle^{p,q}_\eta$ denoting the tangent angle jumps of the two edges at $\gamma_\eta(1)$ and $\gamma_\eta(p_\eta) = \gamma_\eta(q_\eta)$. Therefore,

$$\triangle^1_\eta, \triangle^{p,q}_\eta \in [-\pi, \pi].$$

Thus we estimate

$$\angle(f \circ \gamma_0\dot{)}(1) - \angle(f \circ \gamma_0\dot{)}(p_\eta) \leq \angle(f \circ \rho^f_\eta\dot{)}(q_\eta) - \angle(f \circ \rho^f_\eta\dot{)}(1). \tag{8}$$

Now, we assume that assertion (1) is false. Then there is a sequence $\eta_k \to 0$ such that p_{η_k}, q_{η_k} exist for all $k \in \mathbb{N}$ (i.e. $\rho^f_{\eta_k} \,|\,]c/2, \omega_{\eta_k}[$ intersects $\gamma_0 \,|\,]0,1[$).
Since $p_{\eta_k} \in]0,1[$ we conclude that q_{η_k} also remains bounded. Therefore, passing to a subsequence if necessary, we may assume that $p_{\eta_k} \to p_0 \in [0,1]$ and $q_{\eta_k} \to q_0 \in [c/2, \infty[$. Clearly,the definition of ρ_η shows that

$$f \circ \gamma_0(p_0) = f \circ \rho^f_0(q_0). \tag{9}$$

Since $f \circ \rho^f_0 = \rho_0 \,|\, [1, \omega_0[$ is injective and $q_0 \neq 1$ it follows that $f \circ \rho^f_0(q_0) \neq f \circ \rho^f_0(1) = \rho_0(1) = f \circ \gamma_0(1)$ and thus we conclude that $p_0 \neq 1$, i.e. $p_0 \in [0,1[$.
Taking into account that $f \circ \rho^f_\eta = \rho_\eta$ it follows from the definition of ρ_η that the right-hand part of inequality (8) converges to 0 as $\eta \to 0$. Thus we conclude that

$$\angle(f \circ \gamma_0\dot{)}(1) - \angle(f \circ \gamma_0\dot{)}(p_0) \leq 0.$$

Using Definition 5 this may be reformulated as

$$\angle(f \circ \alpha_D\dot{)}(a_2) - \angle(f \circ \alpha_D\dot{)}(\tilde{p}_0) \leq 0 \tag{10}$$

with a $\tilde{p}_0 \in [a_1, a_2[$.
In the same way equation (9) translates to

$$f \circ \alpha_D(\tilde{p}_0) = \Gamma_{\alpha_D}(r_2, \tilde{q}_0)$$

with a $\tilde{q}_0 \in]b_2, \infty[$. Since $\tilde{q}_0 > 0$ this implies that $\Gamma_{\alpha_D}(r_2, \tilde{q}_0) \neq 0$ and therefore that (we use Definition 2.4)

$$\angle f \circ \alpha_D(\tilde{p}_0) = \angle \Gamma_{\alpha_D}(r_2, \tilde{q}_0) = \angle f \circ \alpha_D(r_2) \qquad (\text{mod} 2\pi). \tag{11}$$

Now we have to distinguish three cases:
case 1: $\tilde{p}_0 \in]v_k, v_{k+1}[$.
Since γ_0 is proper and injective it divides $\mathbb{R}^2$ into two components, one being to the right of γ_0 which we will call G_r and the other, G_l, being to the left. As already previously used in this proof, $\rho^f_{\eta_k}$ branches off γ_0 to the right side (obvious from the definition of ρ^f_η together with Definition 5). Therefore, $\rho^f_{\eta_k}(]c/2, q_{\eta_k}[) \subset G_r$ and the first intersection of $\rho^f_{\eta_k} \mid]c/2, \omega_{\eta_k}[$ with γ_0 at q_{η_k} must be from right to left. Since f is orientation preserving we conclude that $\triangle^{p,q}_{\eta_k} \in [-\pi, 0]$. Using that clearly $\triangle^{p,q}_{\eta_k} = \angle(f \circ \dot{\gamma_0})(p_{\eta_k}) - \angle(f \circ \dot{\rho}^f_{\eta_k})(q_{\eta_k})$ and also some redefinitions (Definition and Lemma 2.2, Definition 2.4, Definition 5 and the definition of ρ_η) it follows that $\Theta_{\alpha_D}(\tilde{p}_0) \bmod 2\pi = \lim_{k\to\infty} \triangle^{p,q}_{\eta_k} \in [-\pi, 0]$ by continuity. However Lemma 7 together with Lemma 2 shows that
$\Theta_{\alpha_D}(\tilde{p}_0) \bmod 2\pi \in]0, \pi[$, a contradiction.
Therefore, case 1 cannot occur.

In order to treat cases 2 and 3 we will frequently use that $\angle f \circ \alpha_D$ is strictly decreasing on $[a_1, v_k]$ and $[v_{k+1}, a_2]$ (DEC) and is strictly increasing on $[v_k, v_{k+1}]$ (INC) (Lemma 7 and Lemma 2). In addition, we need the following:
Lemma 5.6 shows that $\angle f \circ \alpha_D \circ a(r_1^n) \leq \angle f \circ \alpha_D(r_1^n)$ ($\angle f \circ \alpha_D \circ a(r_2^n) \geq \angle f \circ \alpha_D(r_2^n)$, resp.) for all $n \in \mathbb{N}$. Thus passing to the limit we deduce that

$$\angle f \circ \alpha_D(a_1) \leq \angle f \circ \alpha_D(r_1), \quad \angle f \circ \alpha_D(a_2) \geq \angle f \circ \alpha_D(r_2) \tag{12}$$

by continuity.

case 2: $\tilde{p}_0 \in [v_{k+1}, a_2[$.
Lemma 6 shows that $| \angle f \circ \alpha_D(r_2^n) - \angle f \circ \alpha_D(v_{k+1}) | \leq \pi$ for all $n \in \mathbb{N}$.
Thus, passing to the limit we conclude that

$$| \angle f \circ \alpha_D(r_2) - \angle f \circ \alpha_D(v_{k+1}) | \leq \pi \tag{13}$$

From (DEC) and inequality (12) it follows that
$\angle f \circ \alpha_D(\tilde{p}_0) > \angle f \circ \alpha_D(a_2) \geq \angle f \circ \alpha_D(r_2)$.
Using equation (11) this implies that

$$\angle f \circ \alpha_D(\tilde{p}_0) \geq \angle f \circ \alpha_D(r_2) + 2\pi.$$

Since $\angle f \circ \alpha_D(v_{k+1}) \geq \angle f \circ \alpha_D(\tilde{p}_0)$ (DEC) this leads to

$$\angle f \circ \alpha_D(v_{k+1}) \geq \angle f \circ \alpha_D(r_2) + 2\pi,$$

which contradicts inequality (13).
Therefore, case 2 cannot occur, too.

case 3: $\tilde{p}_0 \in [a_1, v_k]$.
From (DEC), (INC), inequality (12) and Definition and Lemma 7 it follows that

$$\begin{aligned} \angle f \circ \alpha_D(\tilde{p}_0) &\leq \angle f \circ \alpha_D(a_1) \\ &\leq \angle f \circ \alpha_D(r_1) \\ &\leq \angle f \circ \alpha_D(r_2) \leq \angle f \circ \alpha_D(a_2). \end{aligned} \tag{14}$$

Due to Lemma 7 and since v_k, v_{k+1} are transversal (see Definition and Lemma 3) we may also estimate

$$\Theta_{\alpha_D}(a_2) - \Theta_{\alpha_D}(\tilde{p}_0) \geq \Theta_{\alpha_D}(v_{k+1}) - \Theta_{\alpha_D}(v_k) = \pi. \tag{15}$$

Thus, using the definition of Θ_{α_D} (see Definition and Lemma 2) together with (14) and (15) we calculate

$$\angle(f \circ \alpha_D)^{\cdot}(a_2) - \angle(f \circ \alpha_D)^{\cdot}(\tilde{p}_0) =$$

$$= \angle f \circ \alpha_D(a_2) - \angle f \circ \alpha_D(\tilde{p}_0) + \Theta_{\alpha_D}(a_2) - \Theta_{\alpha_D}(\tilde{p}_0) \geq \pi,$$

which contradicts inequality (10).
This shows that case 3 is also impossible.
Therefore, there is an $\eta_0 > 0$ such that γ_η is injective for all η with $0 < \eta \leq \eta_0$.
However, in order to make it possible to prove assertion (3) we still have to improve this η:
Since $f \circ \gamma_0 \mid I_2$ is injective we conclude that the restriction $f \mid \gamma_0([1, c])$ is injective, too. Now, the same argumentation as in Lemma 4.3 shows that this injectivity of f even extends to a neighbourhood U of $\gamma_0([1, c])$, i.e. $f \mid U : U \to f(U)$ is a diffeomorphism. Clearly $f(U)$ is a neighbourhood of $f \circ \gamma_0([1, c])$.
Since $f \circ \gamma_\eta \mid [1, c] \to f \circ \gamma_0 \mid [1, c]$ uniformly as $\eta \to 0$, we can choose an $\bar{\eta}$ such that $f \circ \gamma_{\bar{\eta}}([1, c]) \subset f(U)$ and $0 < \bar{\eta} \leq \eta_0$. Since $\gamma_{\bar{\eta}}(1) = \gamma_0(1) \in U$ this insures that

$$\omega_{\bar{\eta}} > c. \tag{16}$$

2) Obvious.

3) A straight forward calculation using Lemma 7 shows that

$$\begin{aligned}\angle(f \circ \gamma_0)^{\cdot}(s_2) - \angle(f \circ \gamma_0)^{\cdot}(s_1) &= \\ &= 3\pi + f \circ \alpha_D(r_2) - f \circ \alpha_D(r_1)\end{aligned}$$

for all $s_2 \in I_2 \backslash \{1\}$, $s_1 \in I_1 \backslash \{0\}$.
Due to Lemma 2 and 7 we know that $\epsilon := f \circ \alpha_D(r_2) - f \circ \alpha_D(r_1) > 0$ if $r_2 > r_1$, which proves the assertion in this case.
If $r_1 = r_2$ we calculate for all $s \in [c, \omega_{\bar{\eta}}[\neq \emptyset$ (see (16)) that

$$\begin{aligned}\angle(f \circ \gamma_{\bar{\eta}})^{\cdot}(s) &= \angle \dot{\rho}_{\bar{\eta}}(s) \\ &= f \circ \alpha_D(r_2) + \bar{\eta} f \circ \alpha_D(r_2)^{\perp} \\ &= \angle(f \circ \gamma_0)^{\cdot}(s) + \arctan \bar{\eta}.\end{aligned}$$

This proves the assertion in the case $r_1 = r_2$ with $\epsilon := \arctan \bar{\eta}$.□

In order to make use of hypothesis 2) of Theorem (I) we now shift our curve γ completely into the region where hypothesis 2) holds, i.e. into $\mathbb{R}^2 \backslash K$.
More precisely we have the following

Lemma 10 *There is a curve $\bar{\gamma} \in C^0(I_1 \cup [0, 1] \cup I_2, \mathbb{R}^2)$ having the following properties:*

1) im $\bar{\gamma} \subset \mathbb{R}^2 \backslash K$

2) $\bar{\gamma}$ *is injective, proper and p.w. regular.*

3) There is an $\epsilon > 0$ *such that*

$$\angle(f \circ \bar{\gamma})^{\dot{}}(s_2) - \angle(f \circ \bar{\gamma})^{\dot{}}(s_1) \geq 3\pi + \epsilon$$

for all $s_2 \in \bar{I}_2\backslash\{1\}$, $s_1 \in \bar{I}_1\backslash\{0\}$ *with*
$\bar{I}_1 := I_1 \cap]-\infty, \bar{s}_1]$ *and* $\bar{I}_2 := I_2 \cap [\bar{s}_2, \infty[$ *for some*
$\bar{s}_1 \in I_1$ *and* $\bar{s}_2 \in I_2$.

Proof: In order to shift γ out of K we first construct a suitable diffeotopy:
Since K is compact and γ is proper and injective we find a $p \in \mathbb{R}^2\backslash \mathrm{im}\, \gamma$ and
$c := \min\{\| \gamma(s) - p \|_2 | \; s \in I_1 \cup [0,1] \cup I_2\}$ as well as
$d := 2\max\{\| x - p \|_2 | \; x \in K\}$ exists. Of course, $c > 0$.

Now we choose an auxiliary function $\rho \in C^1([0,\infty[, [0,\infty[)$ having the property that
$\rho'(s) > 0$ everywhere,
$\rho(c) = d$ and $\rho(s) = s$ if $s \geq 2d$ or $s = 0$ (It is easy to see that such functions exist.)
This enables us to define the diffeotopy $h \in C^1([0,1] \times \mathbb{R}^2, \mathbb{R}^2)$ by

$$h(t,x) := \begin{cases} ((1-t) \| x - p \|_2 + t\rho(\| x - p \|_2))(x-p)/ \| x - p \|_2 + p & \text{if } x \neq p \\ p & \text{if } x = p \end{cases}$$

Obviously, h inherits from ρ the following properties:

i) $D_x h$ is invertible everywhere and $h(t,\cdot)$ is a diffeomorphism for every $t \in [0,1]$.

ii) $\| h(1,x) - p \|_2 \geq d$ if $\| x - p \|_2 \geq c$,

iii) $h(t,x) = x$ if $t = 0$ or $\| x - p \|_2 \geq 2d$ or $x = p$.

Now we define $\bar{\gamma}(s) := h(1, \gamma(s))$.
Then assertion 1) is obvious from ii).
Assertion 2) follows from i), iii) and i),resp. together with Definition and Lemma 8.1) and 2).
Assertion 3): Since γ is proper

$$\bar{s}_1 := \max\{s \in I_1 \,|\, \| \gamma(s') - p \|_2 \geq 2d \text{ for all } s' \in I_1,\, s' \leq s\}$$

$$\bar{s}_2 := \min\{s \in I_2 \,|\, \| \gamma(s') - p \|_2 \geq 2d \text{ for all } s' \in I_2,\, s' \geq s\}$$

exist and we define
$\bar{I}_1 := I_1 \cap]-\infty, \bar{s}_1]$, $\bar{I}_2 := I_2 \cap [\bar{s}_2, \infty[$.
Then iii) shows that for all $s_1 \in \bar{I}_1\backslash\{0\}$, $s_2 \in \bar{I}_2\backslash\{1\}$, $t \in [0,1]$
$\angle(f \circ h(t,\gamma(\cdot)))^{\dot{}}(s_1) \bmod 2\pi = \angle(f \circ \gamma)^{\dot{}}(s_1) \bmod 2\pi$,
$\angle(f \circ h(t,\gamma(\cdot)))^{\dot{}}(s_2) \bmod 2\pi = \angle(f \circ \gamma)^{\dot{}}(s_2) \bmod 2\pi$
since $\bar{\gamma} \mid \bar{I}_1 = \gamma \mid \bar{I}_1$ and $\bar{\gamma} \mid \bar{I}_2 = \gamma \mid \bar{I}_2$.
Moreover, the angle $\angle(f \circ h(t,\gamma(\cdot)))^{\dot{}}(s)$ is defined everywhere (due to i)) and it is a continuous function of t and s. Therefore the two equations remain valid even if the mod 2π term is dropped and the assertion follows from Definition and Lemma 8.3) now.□

Now, hypothesis 2 of Theorem (1) allows to relate the tangent angle rotations along $f \circ \gamma$ and along γ. This leads to

Lemma 11

$$\angle\dot{\bar{\gamma}}(s_2) - \angle\dot{\bar{\gamma}}(s_1) \geq \pi + \epsilon \textit{ for all } s_2 \in \bar{I}_2\backslash\{1\},\ s_1 \in \bar{I}_1\backslash\{0\}$$

Proof:

$$\begin{aligned}\angle\dot{\bar{\gamma}}(s_2) - \angle\dot{\bar{\gamma}}(s_1) &= (\angle\dot{\bar{\gamma}}(s_2) - \angle(f\circ\bar{\gamma})^{\dot{}}(s_2)) - (\angle\dot{\bar{\gamma}}(s_1) - \angle(f\circ\bar{\gamma})^{\dot{}}(s_1)) + \\ &+ (\angle(f\circ\bar{\gamma})^{\dot{}}(s_2) - \angle(f\circ\bar{\gamma})^{\dot{}}(s_1))\end{aligned}$$

Due to Lemma 10.1) hypothesis 2) of our Theorem 1 shows that

$$(f\circ\bar{\gamma})^{\dot{}} = Df(\bar{\gamma}(s))\circ\dot{\bar{\gamma}}(s) \neq \lambda\dot{\bar{\gamma}}(s) \text{ for all } \lambda > 0.$$

This means that $(f\circ\bar{\gamma})^{\dot{}}(s)$ and $\dot{\bar{\gamma}}(s)$ never point into the same direction.
Therefore, there must be an open interval of lenght 2π, say $]\Psi, \Psi + 2\pi[$ such that $\angle\dot{\bar{\gamma}}(s) - \angle(f\circ\bar{\gamma})^{\dot{}}(s) \in]\Psi, \Psi + 2\pi[$ for all s in the domain of $\bar{\gamma}$.
Thus we know that the first two summands on the right side of our equation add up to at least -2π. Lemma 10.3) guarantees that the third summand is at least $3\pi + \epsilon.\square$.

Now, we are left with proving that Lemma 11 implies that the two end of γ will intersect, which contradicts the injectivity of γ. Hence Theorem (1) will be proven.

Proof of Theorem 1(continuation):
(See figure 3) Let $R := \max\bar{\gamma}([\bar{s}_1, \bar{s}_2]) + 1$
Then, for every $r \geq R$ we define the last intersection $\sigma_1(r)$ of $\bar{\gamma} \mid \bar{I}_1$ with the circle $\{x \in \mathbb{R}^2 \mid \| x \|_2 = r\}$ by

$$\sigma_1(r) := \max\{s \in \bar{I}_1 \mid \| \bar{\gamma}(s) \|_2 = r\}$$

and also the first intersection $\sigma_2(r)$ of $\bar{\gamma} \mid \bar{I}_2$ with the same circle by

$$\sigma_2(r) := \min\{s \in \bar{I}_2 \mid \| \bar{\gamma}(s) \|_2 = r\}$$

(Notice, that $\sigma_1(r), \sigma_2(r)$ always exist since $\bar{\gamma}$ is proper)

For every $r \geq R$ let ζ_r be a regular left-oriented (i.e. such that the interior is to the left) parametrization of the circle $\{x \in \mathbb{R}^2 \mid \| x \|_2 = r\}$ over $[\sigma_2(r), \sigma_2(r) + 2\pi]$ such that $\zeta_r(\sigma_2(r)) = \bar{\gamma}(\sigma_2(r))$ and $\zeta_r(\sigma_2(r) + 1) = \bar{\gamma}(\sigma_1(r))$.
Let $\iota_r : [\sigma_1(r), \sigma_2(r) + 1] \longrightarrow \mathbb{R}^2$ be the p.w. regular closed Jordan curve ($\bar{\gamma}$ is injective!)

$$\iota_r(s) := \begin{cases} \bar{\gamma}(s) & \text{if } s \in [\sigma_1(r), \sigma_2(r)] \\ \zeta_r(s) & \text{if } s \in [\sigma_2(r), \sigma_2(r) + 1] \end{cases}$$

and let Ω_r denote its open interior region.
We also need the rotation of the tangent angle at the two "edges" of ι_r:

$$\varphi_1(r) := (\angle\dot{\bar{\gamma}}(\sigma_1(r)) - \angle\dot{\zeta}_r(\sigma_2(r) + 1)) \bmod 2\pi \in [0, 2\pi[$$

$$\varphi_2(r) := (\angle\dot{\zeta}_r(\sigma_2(r)) - \angle\dot{\bar{\gamma}}(\sigma_2(r))) \bmod 2\pi \in [0, 2\pi[$$

Moreover, we use the abreviations

$$\varphi_\gamma(r) := \angle\dot{\bar{\gamma}}(\sigma_2(r)) - \angle\dot{\bar{\gamma}}(\sigma_1(r))$$

$$\varphi_\zeta(r) := \angle\dot{\zeta}_r(\sigma_2(r)+1) - \angle\dot{\zeta}_r(\sigma_2(r))$$

Lemma 11 shows that

$$\varphi_\gamma(r) \geq \pi + \epsilon \tag{17}$$

and since ζ_r is left-oriented it follows that

$$0 < \varphi_\zeta(r) < 2\pi. \tag{18}$$

Observing that $\bar{\gamma}$ enters Ω_r at $\sigma_1(r)$ and leaves Ω_r at $\sigma_2(r)$ it is clear that

$$0 \leq \varphi_i(r) \leq \pi \qquad (i = 1,2). \tag{19}$$

Since ι_r is a closed Jordan curve the total rotation of its tangent angle must be either 2π or -2π, i.e.

$$\varphi_\gamma(r) + \varphi_2(r) + \varphi_\zeta(r) + \varphi_1(r) = \pm 2\pi. \tag{20}$$

An easy calculation using the statements (17)-(20) yields that

$$\varphi_\gamma(r) + \varphi_2(r) + \varphi_\zeta(r) + \varphi_1(r) = 2\pi, \tag{21}$$

$$0 \leq \varphi_i(r) < \pi - \epsilon \qquad (i = 1,2), \tag{22}$$

$$0 < \varphi_\zeta(r) \leq \pi - \epsilon. \tag{23}$$

In addition equation (21) shows that Ω_r has to be to the left of ι_r.
Since Lemma 11 shows that
$\angle\dot{\bar{\gamma}}(\sigma_2(r)) \geq \angle\dot{\bar{\gamma}}(\sigma_1(R)) + \pi + \epsilon$ for all $r \geq R$
we conclude that $\bar{\varphi} := \inf_{r\geq R} \angle\dot{\bar{\gamma}}(\sigma_2(r))$ exists.
Moreover, we find an $r_0 \geq R$ such that

$$\angle\dot{\bar{\gamma}}(\sigma_2(r_0)) \leq \bar{\varphi} + \epsilon/8 \tag{24}$$

We also define

$$\Phi_1 := \bar{\varphi} - (\pi + \epsilon) + \epsilon/4 \tag{25}$$

$$\Phi_2 := \bar{\varphi} - \epsilon/4\,. \tag{26}$$

Now, we will prove the following three statements:

$$\Phi_2 - \Phi_1 = \pi + \epsilon/2 \tag{27}$$

$$\Phi_2 < \angle\dot{\bar{\gamma}}(\sigma_2(r)) < \Phi_2 + \pi \quad \text{if } r \geq r_0 \tag{28}$$

$$\Phi_1 - \pi < \angle\dot{\bar{\gamma}}(\sigma_1(r)) < \Phi_1 \quad \text{if } r \geq r_0 \tag{29}$$

Equation (27) is obvious from (25) and (26).
Using the definition of $\bar{\varphi}$ and definition (26) the left inequality in (28) follows. The right inequality in (29) follows from Lemma 11, from (24) and from definition (25).
In order to prove the right inequality in (28) we assume the contrary, namely

$$\angle\dot{\bar{\gamma}}(\sigma_2(r)) \geq \Phi_2 + \pi$$

From this we conclude that (using (21)-(23) in addition)

$$\begin{aligned} \angle\dot{\bar{\gamma}}(\sigma_1(r)) &= \angle\dot{\bar{\gamma}}(\sigma_2(r)) - \varphi_\gamma(r) = \\ &= \angle\dot{\bar{\gamma}}(\sigma_2(r)) - 2\pi + \varphi_\zeta(r) + \varphi_1(r) + \varphi_2(r) > \\ &> \Phi_2 - \pi \end{aligned}$$

However, using also (24) and (26) this would imply, that

$$\begin{aligned}\angle\dot{\bar{\gamma}}(\sigma_2(r_0)) - \angle\dot{\bar{\gamma}}(\sigma_1(r)) &< \bar{\varphi} + \epsilon/8 - (\Phi_2 - \pi) < \\ &< \pi + \epsilon\end{aligned}$$

which contradicts Lemma 11. Thus (28) is proven.
Similarly, we now assume that $\angle\dot{\bar{\gamma}}(\sigma_1(r)) \leq \Phi_1 - \pi$ in order to prove the left inequality in (29).
Using (21)-(23) and (27 again this would imply that

$$\angle\dot{\bar{\gamma}}(\sigma_2(r)) = \angle\dot{\bar{\gamma}}(\sigma_1(r)) + \varphi_\gamma(r) < \Phi_2$$

which is a contradiction to the left inequality in (28).

Now, we define an auxiliary curve γ_{aux} which equals $\bar{\gamma}$ in some middle part but has ends which are straight lines with the angles Φ_1 and Φ_2, resp.:

$$\gamma_{\mathrm{aux}}(s) := \begin{cases} \bar{\gamma}(\sigma_2(r_0)) + (s - \sigma_2(r_0))v_2 & \text{if } s \in [\sigma_2(r_0), \infty[\\ \bar{\gamma}(s) & \text{if } s \in [\sigma_1(r_0), \sigma_2(r_0)] \\ \bar{\gamma}(\sigma_1(r_0)) + (s - \sigma_1(r_0))v_1 & \text{if } s \in]-\infty, \sigma_1(r_0)] \end{cases}$$

There, $v_1, v_2 \in \mathbb{R}^2$ are taken such that

$$\angle v_i = \Phi_i \bmod 2\pi, \ i = 1, 2$$

Equation (27) shows that the two "ends" of γ_{aux} intersect, i.e.
$\gamma_{\mathrm{aux}}(c_1) = \gamma_{\mathrm{aux}}(c_2)$ with a $c_1 < \sigma_1(r_0)$ and a $c_2 > \sigma_2(r_0)$.
Then $\gamma_{\mathrm{aux}} \mid [c_1, c_2]$ clearly is a left-oriented closed Jordan curve.
Let Ω_{aux} denote its open interior region and let $r_c := \| \gamma_{\mathrm{aux}}(c_1) \|_2$.

Now we prove the following auxiliary assertion:

$$\text{if } r \in [r_0, r_c[\text{ and } \bar{\gamma}([\sigma_1(r), \sigma_2(r)]) \subset \overline{\Omega_{\mathrm{aux}}} \tag{30}$$

$$\text{then } \zeta_r(]\sigma_2(r), \sigma_2(r)+1[) \subset \Omega_{\mathrm{aux}} \tag{31}$$

$$\text{and } \Omega_r \subset \Omega_{\mathrm{aux}} \tag{32}$$

First, we need the following two facts:

i) It is clear from our definitions that ζ_r has exactly two transversal intersections with $\partial\Omega_{\mathrm{aux}}$ for every $r \in [r_0, r_c[$.
One intersection, say at t_1, is with $\gamma_{\mathrm{aux}} \mid [c_1, \sigma_1(r_0)] \subset \partial\Omega_{\mathrm{aux}}$ where ζ_r leaves Ω_{aux} (since this intersection is from left to right) and the other, say at t_2, is with $\gamma_{\mathrm{aux}} \mid [\sigma_2(r_0), c_2]$ where ζ_r enters Ω_{aux} again (from right to left). (Remember that Ω_{aux} is to the left of γ_{aux}.)

ii) If (31) would not be true, then (30) shows that $\zeta_r \mid [\sigma_2(r), \sigma_2(r)+1]$ would intersect $\partial\Omega_{\mathrm{aux}}$, since $\zeta_r(\sigma_2(r)) = \bar{\gamma}(\sigma_2(r))$ and $\zeta_r(\sigma_2(r)+1) = \bar{\gamma}(\sigma_1(r))$.

Altogether, i) and ii) show, that if (31) would not be true, then
$\zeta_r([\sigma_2(r), \sigma_2(r)+1])$ would even contain the whole arc $\zeta_r([t_1, t_2])$.
However, it is easy to see from our definitions that $\zeta_r([t_1, t_2])$ and therefore that $\zeta_r([\sigma_2(r), \sigma_2(r)+1])$ is more than a half circle which contradicts inequality (23). Therefore (31) must be true. Since (30) and (31) show that $\partial\Omega_r \subset \overline{\Omega_{\mathrm{aux}}}$ the Jordan curve theorem implies (32).

Let $[r_0, \bar{r}][$ be the maximal interval such that

$$\bar{\gamma}([\sigma_1(r), \sigma_2(r)]) \subset \overline{\Omega_{\mathrm{aux}}} \qquad \text{for all } r \in [r_0, \bar{r}][\tag{33}$$

iii) $[r_0, \bar{r}][$ contains r_0:
This is obvious from the definition of Ω_{aux}.

iv) $[r_0, \bar{r}][$ is closed:
It is obvious from the definition that σ_1 and σ_2 are monotonically decreasing and increasing, resp.
Now, a straightforward proof shows that σ_1 and σ_2 are left-sided continuous, i.e. $\sigma_i(r_n) \to \sigma_i(r)$ for every sequence $r_n \nearrow r$. This implies the assertion at once.

v) $[r_0, \bar{r}][$ is right-open in $[r_0, r_c]$:
Let $r_n \searrow r \in [r_0, \bar{r}][$ be any sequence in $[r_0, r_c]$. If $\bar{r} = r_c$ we are done. So let us assume $\bar{r} < r_c$.
The definition of σ_i $(i = 1, 2)$ shows that
$\sigma_1(r_n) \nearrow \bar{\sigma}_1 \leq \sigma_1(r)$ and $\sigma_2(r_n) \searrow \bar{\sigma}_2 \geq \sigma_2(r)$.

If $\bar{\sigma}_i = \sigma_i(r)$ for $i = 1$ or 2 then inequality (29) $(i = 1)$ or (28) $(i = 2)$, resp. shows that $\dot{\bar{\gamma}}(\bar{\sigma}_i)$ points outward of Ω_{aux} or inward in Ω_{aux}, resp.
Hence, $\gamma(\sigma_i(r_n)) \in \overline{\Omega_{\text{aux}}}$ $(i = 1$ or 2, resp.) for almost all n.

If $\bar{\sigma}_2 > \sigma_2(r)$ then the definition of σ_2 shows that $\dot{\bar{\gamma}}(\sigma_2(r))$ must be parallel to the tangent of the circle im ζ_r at $\bar{\gamma}(\sigma_2(r))$, i.e. we know that either $\varphi_2(r) = 0$ or $\varphi_2(r) = \pi$.
However inequality (22) shows that only $\varphi_2(r) = 0$ is possible.
This implies that $\bar{\gamma}$ enters $\Omega_r \cup \zeta_r(]\sigma_2(r), \sigma_2(r) + 1[)$ at $\bar{\gamma}(\sigma_2(r))$ and we may even conclude that

$$\bar{\gamma}(]\sigma_2(r), \bar{\sigma}_2]) \subset \Omega_r \cup \zeta_r(]\sigma_2(r), \sigma_2(r) + 1[).$$

Indeed, $\partial\Omega_r = \bar{\gamma}([\sigma_1(r), \sigma_2(r)]) \cup \zeta_r(]\sigma_2(r), \sigma_2(r) + 1[)$ by definition. Therefore, $\gamma \mid]\sigma_2(r), \bar{\sigma}_2]$ could leave Ω_r only at some point of $\zeta_r(]\sigma_2(r), \sigma_2(r) + 1[)$ since $\bar{\gamma}$ is injective (see Lemma 10).
However, this is impossible since the definition of $\bar{\sigma}_2$ implies that $\| \bar{\gamma}(s) \|_2 \leq r$ for all $s \in]\sigma(r), \bar{\sigma}_2]$.

A completely analogous proof shows that

$$\bar{\gamma}([\bar{\sigma}_1, \sigma_1(r)[) \subset \Omega_r \cup \zeta_r(]\sigma_2(r), \sigma_2(r) + 1[) \text{ if } \bar{\sigma}_1 < \sigma_1(r).$$

Now the auxiliary assertion shows that $\bar{\gamma}([\bar{\sigma}_1, \sigma_1(r)[)$ and $\bar{\gamma}(]\sigma_2(r), \bar{\sigma}_2])$ are contained in Ω_{aux}.
Therefore, $\bar{\gamma}(\sigma_i(r_n)) \in \Omega_{\text{aux}}$ $(i = 1, 2)$ for almost all n.

Finally, we conclude from iii) - v) that $[r_0, \bar{r}][= [r_0, r_c]$, i.e.

$$\bar{\gamma}([\sigma_1(r), \sigma_2(r)]) \subset \overline{\Omega_{\text{aux}}} \text{ for all } r \in [r_0, r_c].$$

Since $\bar{\gamma}(\sigma_i(r_c)) \in \text{im } \zeta_{r_c}$ $(i = 1, 2)$ in addition and since im $\zeta_{r_c} \cap \overline{\Omega_{\text{aux}}} = \{\gamma_{\text{aux}}(c_1)\}$ (i.e. it contains exactly one element) we conclude that

$$\bar{\gamma}(\sigma_1(r_c)) = \bar{\gamma}(\sigma_2(r_c)).$$

This means that $\bar{\gamma}$ is not injective - a contradiction to Lemma 10.2 which proves our Theorem 1.□

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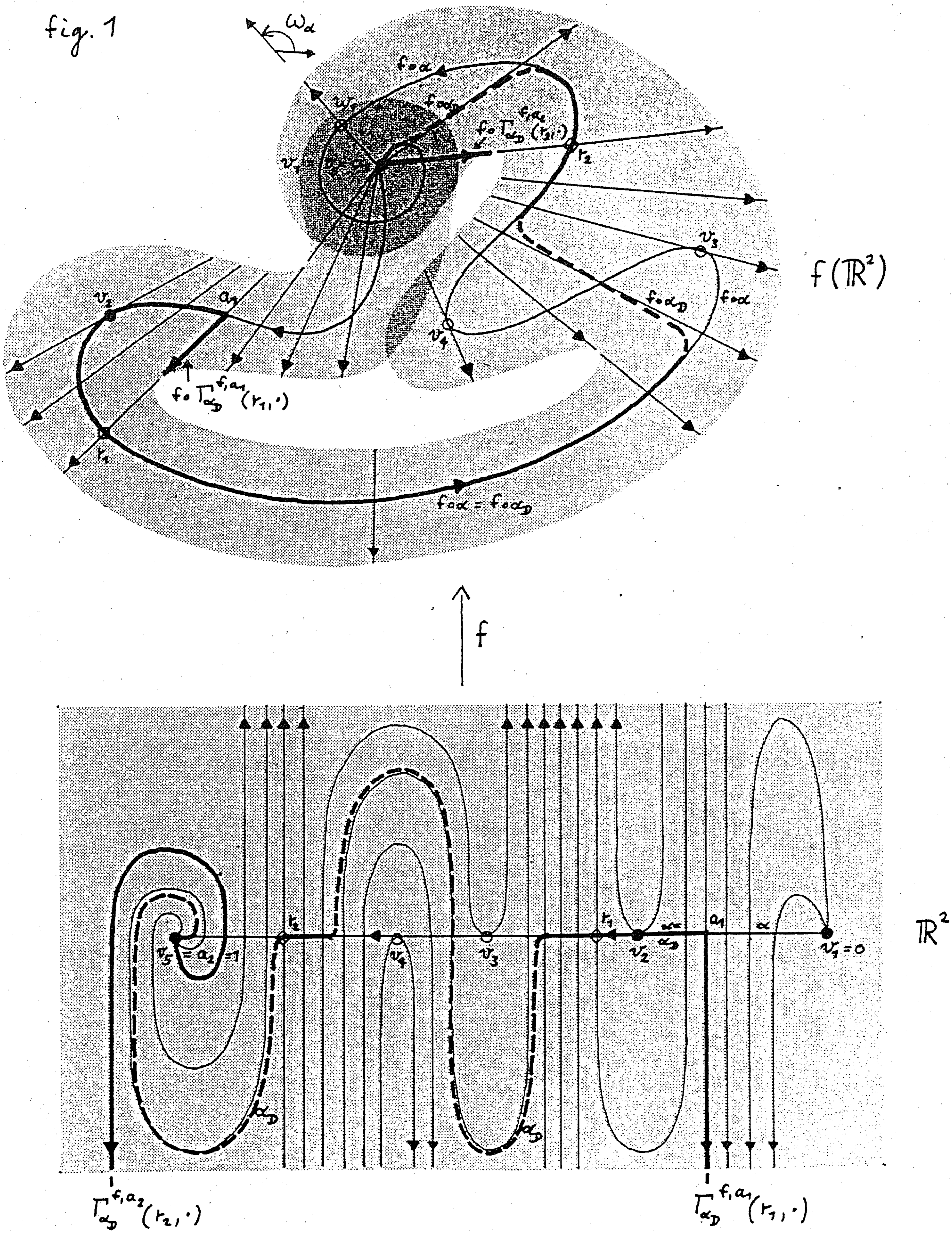
fig. 1



ω_α
$f\circ\alpha$
$f\circ\alpha_D$
$f\circ\Gamma^{f,a_2}_{\alpha_D}(r_2,\cdot)$
r_2
v_3
$f(\mathbb{R}^2)$
$f\circ\alpha_D$
$f\circ\alpha$
v_2
a_1
v_4
$f\circ\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot)$
r_1
$f\circ\alpha = f\circ\alpha_D$
f
$\mathbb{R}^2$
r_2
$v_5 = a_2 = 1$
v_4
v_3
r_1
v_2
$\alpha = \alpha_D$
a_1
α
$v_1 = 0$
α_D
α_D
$\Gamma^{f,a_2}_{\alpha_D}(r_2,\cdot)$
$\Gamma^{f,a_1}_{\alpha_D}(r_1,\cdot)$

fig. 2

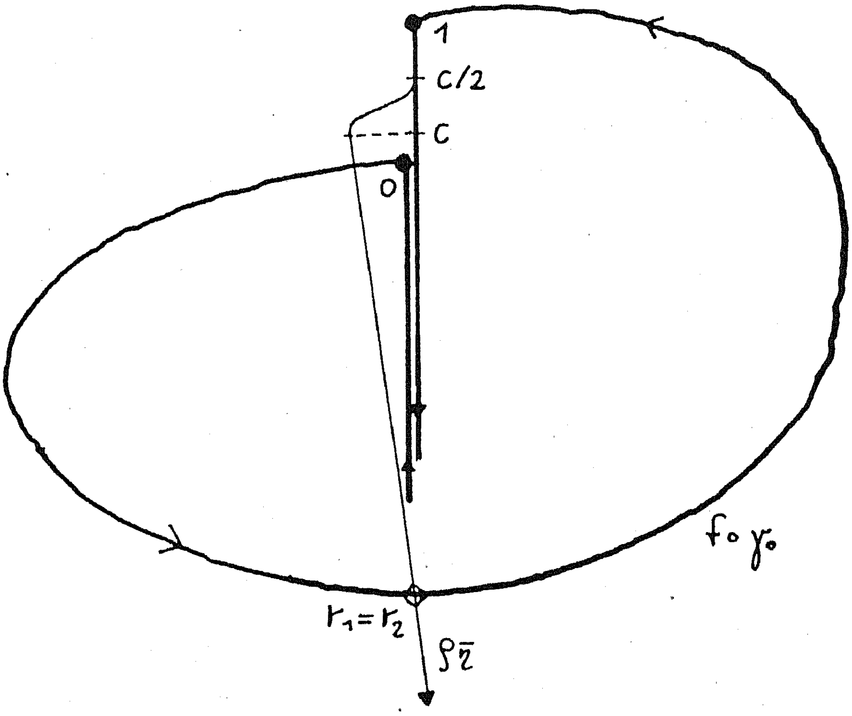


fig. 3