Cycle Integrals of Maass Cusp Forms and of Holomorphic Functions with Rational Periods

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Cycle Integrals of Maass Cusp Forms and of Holomorphic Functions with Rational Periods

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Abstract

Akin to the Fourier expansion of a modular form, the values of its cycle integrals determine the modular form completely. Given the importance of the Fourier coefficients of modular forms in arithmetic applications, it is thus not surprising that their cousins, the cycle integrals, are of crucial arithmetic importance.

We will extend the Katok-Sarnak formula (1991), which relates the Fourier coefficients of the Shintani lift of a Maass cusp from of weight zero for the full modular group to the cycle integrals of the Maass cusp form, to Maass cusp forms of non-zero weights and higher level.

In the second part, we investigate a formula of S. Katok (1985), which expresses the imaginary part of the cycle integral of the hyperbolic Poincarére series geometrically. We add a complementary counterpart to this interpretation by considering a suitable cycle integral of the Parson Poincarér series, a non-modular analog of the hyperbolic Poincaré series.

Finally, we will study an integer-valued analog of the classical Dedekind sums, the Hardy sums. Dedekind sums appear in the study of the cycle integrals of the non-modular Eisenstein series of weight 2. We prove distributional results of these Hardy sums, and give a geometrical interpretation of them.

Abstract (deutsch)

Wie die Fourierkoeffizienten einer Modulform beschreiben die Werte ihrer Zykelintegrale diese komplett. Da die Fourierkoeffizienten von Modulformen zahlreiche arithmetische Anwendungen haben, überrascht es nicht, dass ihre Verwandten, die Zykelintegrale, ebenso von arithmetischer Bedeutung sind.

Wir erweitern in dieser Dissertation die Katok-Sarnak-Formel (1991), die die Fourierkoeffizienten des Shintani-Lifts einer Maass-Spitzenform von Gewicht Null für die Gruppe $\mathrm{SL}_2(\mathbf{Z})$ als die Zykelintegrale der Maass-Spitzenformen ausdrückt, auf Gewicht ungleich Null und höhere Stufen.

Im zweiten Teil untersuchen wir eine Formel von S. Katok (1985), die eine den Imaginärteils des Zykelintegrals der hyperbolischen Poincaréreihe geometrisch ausdrückt. Dieser Interpretation fügen wir ein komplementäres Gegenstück hinzu, indem wir ein geeignetes Zykelintegral der Parson Poincaréreihe, einem nicht-automorphen Analogon der hyperbolischen Poincaréreihe, betrachten.

Schliesslich betrachten wir ganzzahlige Pendants zu den klassischen Dedekindsummen, die Hardysummen. Die Dedekindsummen treten bei Untersuchungen der Zykelintegrale der holomorphen Eisensteinreihen von Gewicht 2 auf. Wir beweisen Resultate über die Verteilung der Hardysummen und geben eine geometrische Deutung dieser.

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So it has been four years. It takes me a long time to write these last paragraphs of my thesis. I daydream and keep getting nostalgic. I think of Leonie and Gini, their laughter, smiles, and how close we were. I think of my friends, the lunches with Paddy, the cinema nights with Judith, picking mushrooms with Joel, angrily playing cards with Zoe, dinner with Niklas. The coming and going of the friends in the office: Riccardo, Giuliano, Yannick, Cynthia and Joao, Markus, Tim, Sebastian, Tal, Beat, Ana. Thank you for chatting, venting, complaining and explaining, and our hopeless attempts to keep the office tidy. I think of the Environmental Policy Lab and the good times we had when Rachael declared Gini and me to honorary members.

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Für meine Eltern.

Contents

CHAPTER 1

Introduction

Cycle integrals are simple but rich objects, which have many applications in the theory of automorphic forms. Let $f : \mathbf{H} \to \mathbf{C}$ be a holomorphic function on the upper half-plane, which is modular of weight 2 for some Fuchsian group Γ, i.e. $f(g.z) = (cz+d)^2 f(z)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and where $g.z = \frac{az+b}{cz+d}$ is the usual action by linear fractional transformations. Then the cycle integral $\displaystyle\int_{z_0}^{g.z_0} f(z)\, dz$ for some $z_0 \in \mathbf{H}$ is

(1) invariant under the choice of the path between z_0 and $g.z_0$, since $f(z)$ is holomorphic,
(2) invariant under the choice of the base point z_0 due to Cauchy's theorem and the modular property of f.

From (1) and (2) it follows that the cycle integral is invariant under conjugacy classes of $g \in \Gamma$.

If $f(z)$ is holomorphic and modular of weight $2k$ for $k \in \mathbf{Z}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the cycle integral is defined by $\displaystyle\int_{z_0}^{g.z_0} f(z)(cz^2+(d-a)z-b)^{k-1}\, dz$. An easy calculation shows that

$$(c(g.z)^2 + (d-a)(g.z) - b)^{k-1} = (cz+d)^{2-2k}(cz^2 + (d-a)z - b)^{k-1}$$

and hence the cycle integral $\displaystyle\int_{z_0}^{g.z_0} f(z)(cz^2+(d-a)z-b)^{k-1}\, dz$ is still invariant under the choice of the path, the choice of the base point $z_0 \in \mathbf{H}$, and invariant under conjugacy classes of $g \in \Gamma$.

In this thesis, we will mostly be concerned with phenomena appearing when we weaken the two conditions above, i.e. when $f(z)$ is not holomorphic or not modular. Both of these weakened conditions can be treated by *homogenizing* the cycle integral and thus making it a conjugacy class invariant again. The idea of homogenization also appears in the theory of quasimorphisms in bounded cohomology.

Although most of the theorems in this thesis hold for more general groups, we will limit ourselves to the case of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ for the rest of the introduction.

1. The Katok-Sarnak Formula for Non-Zero Weights

In the first part, we will weaken condition (1) above and consider the cycle integrals of Maass cusp forms $\varphi(z)$ of weight $2k$, which are not necessarily holomorphic but eigenfunctions of the weight $2k$ Laplace-Beltrami operator Δ_{2k} on $\mathbf{H}$.

For holomorphic cusp forms, cycle integrals appear in the Shintani lift in the theory of the Shimura correspondence. More precisely, let $k > 1$ and $h(z)$ be a holomorphic cusp form of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$, which transforms with the automorphy factor $J(g, z) = \frac{\Theta(g.z)}{\Theta(z)}$ for $g \in \Gamma_0(4)$ of the theta-function $\Theta(z) = \sum_{n \in \mathbf{Z}} e(n^2 z)$ (here, $e(z) = e^{2\pi i z}$ as usual). We write $S_{k+\frac{1}{2},\,4}$ for the space of holomorphic cusp forms of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$. Denote the Fourier expansion of $h \in S_{k+\frac{1}{2},\,4}$ around the cusp $i\infty$ by $h(z) = \sum_{n>0} a_h(n) e(nz)$. Through an ingenious application of a converse theorem, G. Shimura [**Shim73**] showed that for a fundamental discriminant d with $(-1)^k d > 0$ the function

$$\mathrm{Shim}_d(h)(z) = \sum_{n>0} \left(\sum_{m|n} \left(\frac{d}{m} \right) m^{k-1} a_h(n^2 d/m^2) \right) e(nz)$$

defines a cusp form of weight $2k$ for $\Gamma_0(2)$. For some $h(z)$ the image $\mathrm{Shim}_d(h)$ is actually a cusp form of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbf{Z})$.

A natural question is to find the inverse function of Shim_d. This function was first constructed by T. Shintani [**Shin75**] for cusp forms of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$ which are mapped to cusp forms of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$. Shintani's inverse map is given by a theta-lift, for which cycle integrals appear naturally as the Fourier coefficients of the half-integral weight forms through unfolding. However, given the disparity on the level, the question remains for which subspace of $S_{k+\frac{1}{2},\,4}$ the map Shim_d is actually an isomorphism. The answer was given by W. Kohnen in his dissertation [**Koh80**] (and later for higher level [**Koh82**]), namely that Shim_d will be an isomorphism when restricted to what we now refer to as the Kohnen plusspace

$$S^+_{k+\frac{1}{2},\,4} = \left\{ h(z) \in S_{k+\frac{1}{2},\,4} : a_h(n) \neq 0 \text{ only if } (-1)^k n \equiv 0, 1 \pmod 4 \right\}.$$

It is somewhat easier and more general to think of cycle integrals as associated to binary quadratic forms in this context. This setting will also cover the case of negative discriminants below. Let $Q(x, y) = [A, B, C](x, y) = Ax^2 + Bxy + Cy^2$ be an integral binary quadratic form of discriminant $D = B^2 - 4AC$. There is a canonical right-action of $\mathrm{SL}_2(\mathbf{Z})$ on the set $\mathcal{Q}_D$ of binary quadratic forms of discriminant D. The stabilizer Γ_Q of a form Q is finite if $D < 0$ and a cyclic group generated by an element $\gamma_Q \in \mathrm{SL}_2(\mathbf{Z})$ if $D > 0$. In fact, conjugacy classes of primitive hyperbolic elements of $\mathrm{SL}_2(\mathbf{Z})$ and classes of indefinite binary quadratic forms are in one-to-one correspondence, which will be explained in Chapter 2.

For a cusp form $f(z)$ of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$, d a fundamental discriminant and $d' \equiv 0, 1 \pmod 4$, we define the twisted trace of $f(z)$ by

$$\mathrm{Tr}_{d,d'}(f) = \sqrt{dd'} \sum_{Q \in \mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{Q}_{dd'}} \chi_d(Q) \int_{z_0}^{\gamma_Q . z_0} f(z) Q(z,1)^{k-1} dz, \quad \text{for } dd' > 0, \tag{1.1}$$

where χ_d is the genus character given by

$$\chi_d([A,B,C]) = \begin{cases} 0, & \text{if } (A,B,C,d) > 1, \\ \left(\frac{d}{r}\right), & \text{if } (A,B,C,d) = 1 \text{ and } [A,B,C] \text{ represents } r. \end{cases}$$

Theorem 1.1 (Kohnen). *Let $k > 1$ and let $d, d' > 0$ be integers such that $(-1)^k d' \equiv 0, 1 \pmod 4$ and so that $(-1)^k d$ is a fundamental discriminant. Let $h(z) \in S^+_{k+\frac{1}{2},4}$ and $a_h(d) f(z) = \mathrm{Shim}_d(h)(z)$. Then*

$$6(-1)^{\lfloor k/2 \rfloor} \sqrt{dd'}\, \overline{a_h(d)} a_h(d') = 2^k \, \mathrm{Tr}_{(-1)^k d,\, (-1)^k d'}(f).$$

W. Kohnen [**Koh85**] proved Theorem 1.1 for level N being odd and squarefree (where some of the definitions need to be adjusted). Theorem 1.1 is particularly interesting in light of an earlier result of J.-L. Waldspurger [**Wal81**], who proved through adelic methods that the Fourier coefficients of $g(z)$ are given by the traces of $\mathrm{Shim}_d(g)(z)$ up to a constant factor.

An obvious generalization of Theorem 1.1 is to consider the bigger class of Maass cusp forms instead, for which some interesting phenomena appear. Let now $k \in \mathbf{Z}$. A Maass cusp form $\psi(z)$ of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$ has the Fourier expansion

$$\psi(z) = \sum_{n \neq 0} b_\psi(n) (4\pi |n| y)^{-\left(\frac{k}{2} + \frac{1}{4}\right)} W_{\mathrm{sign}(n)\left(\frac{k}{2}+\frac{1}{4}\right),\, s-\frac{1}{2}}(4\pi |n| y) e(nx),$$

where $W_{\kappa,\mu}(y)$ is the W-Whittaker function and the spectral parameter $s \in \mathbf{C}$ is given by the eigenvalue of ψ under $\Delta_{k+\frac{1}{2}}$, i.e. $\Delta_{k+\frac{1}{2}}\psi = (s - k - 1/2)(1/2 - k - s)\psi$.[1] Under the map Shim_d defined above, the function $\mathrm{Shim}_d(\psi)(z)$ is a Maass cusp form of weight $2k$ and spectral point $2s - \frac{1}{2}$; see Theorem 5.4.

Let $\varphi(z)$ be a Mass cusp form of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$. Note that the integrand in $\int_{z_0}^{\gamma_Q . z_0} \varphi(z) Q(z,1)^{k-1} dz$ is no longer holomorphic and that thus the integral depends on the choice of z_0 and the choice of the path from z_0 to $\gamma_Q . z_0$. To make it a conjugacy class

[1] As usual, we will write $z = x + iy$ with $x \in \mathbf{R}$ and $y > 0$ for a $z \in \mathbf{H}$.

invariant, we can choose z_0 to lie on the geodesic S_Q connecting the two real roots of $Q(z,1)=0$ and pick the geodesic path from z_0 to $\gamma_Q.z_0$, i.e. integrating over the geodesic segment $C_Q = \Gamma_Q \setminus S_Q$. In fact, this is equivalent to homogenizing the integral:

$$\lim_{n\to+\infty} \int_{\gamma_Q^n.z_0}^{\gamma_Q^{n+1}.z_0} \varphi(z)Q(z,1)^{k-1}dz = \int_{C_Q} \varphi(z)Q(z,1)^{k-1}dz.$$

The orientation of C_Q is positive if $Q(1,0)>0$ and negative if $Q(1,0)<0$. If $Q(1,0)=0$, the discriminant $\mathrm{disc}(Q)$ is a square, so that the segment will be the entire geodesic line oriented from the root of $Q(z,1)=0$ to $i\infty$.

For $dd'<0$ there is no longer a geodesic segment associated to a binary quadratic form Q of discriminant dd', but instead a CM-point z_Q (the root of $Q(z,1)=0$ that lies in $\mathbf{H}$). Since the action by linear fractional transformations on a CM-point is compatible with the action on binary quadratic forms, i.e. $z_{Q\circ g} = g.z_Q$ for $g \in \mathrm{SL}_2(\mathbf{Z})$, the twisted trace

$$\mathrm{Tr}_{d,d'}(\varphi) = \sum_{Q\in \mathrm{SL}_2(\mathbf{Z})\setminus \mathcal{Q}_{dd'}} \chi_d(Q)|\Gamma_Q|^{-1}\,\varphi(z_Q), \quad dd'<0, \tag{1.2}$$

is well-defined for a Maass cusp form $\varphi(z)$ of weight zero.

THEOREM 1.2 (Katok-Sarnak). *Let $d \equiv 0,1 \pmod 4$ and φ be a Maass cusp form of weight zero for* $\mathrm{SL}_2(\mathbf{Z})$. *Then*

$$6\sqrt{d}\sum_{\psi} \frac{\overline{b_\psi(1)}b_\psi(d)}{\langle \psi,\psi\rangle} = \langle \varphi,\varphi\rangle^{-1}\cdot \begin{cases} 2\sqrt{\pi}\,\mathrm{Tr}_{1,d}(\varphi), & \text{if } d<0,\\ \mathrm{Tr}_{1,d}(\varphi), & \text{if } d>0,\end{cases}$$

where the sum runs over an orthogonal basis ψ of Maass cusp forms in $U^+_{\frac{1}{2},4}(s)$ such that $\mathrm{Shim}_d(\psi) = b_\psi(1)\varphi$.

Originally, S. Katok and P. Sarnak [**KS93**] proved Theorem 1.2 by using a theta-lift. The Katok-Sarnak formula has been generalized to higher level Maass cusp forms by Á. Biró [**Bir00**] (by the use of trace formulas) and to more general splitting of the discriminant $D=dd'$ with $d\neq d'$ by W. Duke, Ö. Imamoğlu, and Á. Tóth [**DIT16**] (by the theory of the resolvent kernel).

If we set $d=1$ in Theorem 1.2, we have $\mathrm{Tr}_{1,1}(\varphi) = \int_0^{i\infty}\varphi(z)\frac{dz}{z}$, since the class of $[0,1,0]$ is the only class in $\mathrm{SL}_2(\mathbf{Z})\setminus\mathcal{Q}_1$. Hence, the trace $\mathrm{Tr}_{1,1}(\varphi)$ is given in terms of the central value of the L-function $L(\varphi,w) = \sum_{n=1}^\infty \frac{a_\varphi(n)}{n^w}$. We get the following immediate corollary from the Katok-Sarnak formula.

COROLLARY 1.3 (Katok-Sarnak). *For each Maass cusp form φ of weight zero for* $\mathrm{SL}_2(\mathbf{Z})$*, the central L-value $L(\varphi, 1)$ is non-negative.*

A similar argument applied to Theorem 1.1 with $d = d'$ gives that the central value (at $w = k$) of the twisted L-function $L(f, d, w) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{a(n)}{n^w}$ of a holomorphic cusp form $f(z) = \sum_{n>1} a(n)e(nz)$ of weight $2k$ is non-negative. Hence, the case $d = d'$ includes information about the behavior of the L-functions attached to the cusp forms. The case $d = d' > 1$ for Maass cusp forms of weight zero was established by E.M. Baruch and Z. Mao [**BM10**] through a representation-theoretic argument.

The right hand side in Theorem 1.1 would trivially be zero if we did not assume that $d, d' > 0$, due to the cancellation arising from $\chi_d(-Q) = \mathrm{sign}(d)\chi_d(Q)$. The left hand side in Theorem 1.1 is also trivially zero for holomorphic cusp forms, as the negative Fourier coefficients vanish. For Maass cusp forms, however, the negative Fourier coefficients do not necessarily vanish. So an interesting question is to ask what the analog of Theorem 1.1 for Maass cusp forms of non-zero weight would be, as it would include the case of holomorphic cusp forms.

The results we mentioned are all for Maass cusp forms of weight zero, one obvious difficulty being that the twisted trace in (1.2) would not be well-defined for non-zero weights. In fact, this problem can be solved by first applying lowering operators (if $k > 0$) or raising operators (if $k < 0$) to $\varphi(z)$ making it an $\mathrm{SL}_2(\mathbf{Z})$-invariant function; see Definition 4.6. We will establish a Katok-Sarnak-type formula for non-zero weights and for all d, d' (also for $d = d'$), thus obtaining a generalization of all the above mentioned results and a finer picture of Kohnen's formula for $k > 0$. The argument is similar to the one used by W. Duke, Ö. Imamoḡlu, and Á. Tóth [**DIT16**], in which they explicitly calculated the cycle integrals of the Niebur Poincaré series. The Niebur Poincaré series appear as the coefficients in the Fourier expansion of the resolvent kernel of the Laplace operator and are eigenfunctions of the Laplace operator, but not square-integrable, which leads to convergence issues in the case $d = d'$ (since the geodesics of the cycle integrals are no longer compact). Instead, we will use the Selberg Poincaré series built out of the natural test-function $y^s e(mz)$ for $m > 0$, which are square-integrable, but no longer eigenfunctions of the Laplace operator. The arising technical difficulties can be handled by an idea going back to A. Selberg [**Sel62**] of approximating certain terms arising from the Selberg Poincaré series and realizing that the error goes to zero at the points of interest. This leads to the first theorem of this thesis.

THEOREM 1.4. *Let $k \in \mathbf{Z}$ and d, d' be integers such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$. Let φ be a Maass cusp form of weight $2k$ for* $\mathrm{SL}_2(\mathbf{Z})$ *with spectral parameter s such that* $\mathrm{Re}(s) > 0$. *Then*

$$6(-1)^{\lfloor k/2 \rfloor}\sqrt{dd'} \sum_{\mathrm{Shim}_d(\psi) = b_\psi(d)\varphi} \frac{\overline{b_\psi(d)} b_\psi(d')}{\langle \psi, \psi \rangle}$$

$$= \langle \varphi, \varphi \rangle^{-1} \cdot \begin{cases} 2\sqrt{\pi}\, \mathrm{Tr}_{(-1)^k d,\, (-1)^k d'}(\varphi), & \text{if } dd' < 0, \\ 2^k\, \mathrm{Tr}_{(-1)^k d,\, (-1)^k d'}(\varphi), & \text{if } d, d' > 0, \\ 2^{1-k}\, \mathrm{Tr}_{(-1)^k d,\, (-1)^k d'}\left(2iy^k \overline{\partial_{\overline{z}} \varphi}\right), & \text{if } d, d' < 0, \end{cases}$$

where the sum runs over an orthogonal basis ψ of the preimage of φ under Shim_d.

The theorem we will prove holds for higher level N as well; see Theorem 5.5. If φ is holomorphic, which is equivalent to $\partial_{\overline{z}}\varphi = 0$, we see that both the left and right hand side in the formula of Theorem 1.4 vanish trivially for $d, d' < 0$. Applying the lowering operator k-times has the same effect, i.e. that $\mathrm{Tr}_{(-1)^k d,\, (-1)^k d'}(\varphi) = 0$ for $dd' < 0$ for $\varphi(z)$ holomorphic, so again both sides of the formula vanish in this case.

2. Cycle Integrals of the Parson Poincaré Series

Instead of considering non-holomorphic objects as the integrands of the cycle integrals, we may also weaken the condition of modularity, but keep the integrand holomorphic. This makes the cycle integral path-invariant but dependent on the base point z_0.

Given any holomorphic function $f : \mathbf{H} \to \mathbf{C}$, a useful way to describe its transformation behavior under the weight $2k$ action of $\mathrm{SL}_2(\mathbf{Z})$ is by defining the weight $2k$ cocycle

$$r_{f,k}(g, z) = f(z)|_{2k} g - f(z) \quad \text{for } g \in \mathrm{SL}_2(\mathbf{Z}).^2 \tag{1.3}$$

The family of functions $r_{f,k}(g, z)$ satisfies the property

$$r_{f,k}(gg', z) = r_{f,k}(g, z)|_{2k} g' + r_{f,k}(g', z) \quad \text{for } g, g' \in \mathrm{SL}_2(\mathbf{Z}). \tag{1.4}$$

Given a family of functions $\{r(g, z) : g \in \mathrm{SL}_2(\mathbf{Z})\}$ satisfying (1.4), we call the function $f : \mathbf{H} \to \mathbf{C}$ in (1.3) its modular integral. There has been a good amount of research undertaken in constructing the modular integrals for families of cocycles. The earliest example of constructing the modular integrals was by M. Knopp through "generalized

[2] We use the common notation $f(z)|_{2k} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cz + d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$.

Poincaré series" [**Kno74**], thus proving that every family of cocycles (with certain additional growth conditions on the cocycles) arises from a function as in (1.3). Possibly the easiest construction to date is due to W. Duke, Ö. Imamoğlu, and Á. Tóth [**DIT10**] for rational cocycles, who gave the corresponding modular integral as an explicit Fourier series for every family of cocycles; see also the construction in W. Duke's paper [**Duk21**, Prop. 1] for the more general class of holomorphic cocycles.

We will limit ourselves to families of rational cocycles. A classical example of a modular integral with a rational cocycle is the weight 2 Eisenstein series

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)e(nz), \quad \text{where } \sigma_1(n) = \sum_{d|n} d, \tag{1.5}$$

for the cocycle $r\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}, z\right) = \frac{6}{\pi i}\frac{c}{cz+d}$. Another example of a weight 2 modular integral, involving cycle integrals of weakly holomorphic modular forms, was constructed by W. Duke, Ö. Imamoğlu, and Á. Tóth [**DIT11, DIT10**] as

$$F_Q(z) = \frac{2\sqrt{\mathrm{disc}(Q)}}{\pi}\sum_{n=1}^{\infty}\left(\int_{z_0}^{\gamma_Q.z_0} j_n(z)\frac{dz}{Q(z,1)}\right)e(nz), \tag{1.6}$$

where $j_n(z)$ denotes the unique weakly holomorphic modular function for $\mathrm{SL}_2(\mathbf{Z})$ whose Fourier expansion has the shape $j_n(z) = e(-nz) + O(e(z))$. Its cocycle is given by

$$r_Q(g,z) = F_Q(z)|_2 g - F_Q(z) = \frac{2}{\pi}\sum_{\substack{Q'\in[Q],\\ w'_{Q'}<g^{-1}.i\infty<w_{Q'}}}\left(\frac{1}{z-w_Q} - \frac{1}{z-w'_Q}\right),$$

where $w'_{Q'}$, $w_{Q'}$ are the roots of $Q'(z,1) = 0$. Interestingly, we can define the primitive cocycle of $r_Q(g,z)$ by the cycle integral

$$R_Q(g,z_0) = \int_{z_0}^{g.z_0} F_Q(z)dz, \tag{1.7}$$

i.e. $R_Q(g,z)$ is the unique weight zero cocycle satisfying (1.4) such that $\partial_z R_Q(g,z) = r_Q(g,z)$. Through an explicit calculation of the primitive cocycle $R_Q(g,z)$, T. Matsusaka [**Mat20**] proved the remarkable formula

$$\mathrm{Im}\left(\lim_{n\to+\infty}\int_{\gamma^n_{Q'}.z_0}^{\gamma^{n+1}_{Q'}.z_0} F_Q(z)dz\right) = \sum_{p\in[S_Q]\cap[S_{Q'}]} 1 \tag{1.8}$$

for any pair of indefinite binary quadratic forms Q, Q' with $[Q] \neq [Q']$ and where the sum runs over the intersection points p of $[S_Q]$ with $[S_{Q'}]$, the geodesics S_Q, $S_{Q'}$ projected onto $\mathrm{SL}_2(\mathbf{Z}) \setminus \mathbf{H}$, counted with multiplicity.

For higher weight $k > 1$ there is an example by L.A. Parson [**Par93**], where we average over the class of an indefinite binary quadratic form Q under the action of $\mathrm{SL}_2(\mathbf{Z})$. First, we define the sign of a quadratic form by $\mathrm{sign}([A, B, C]) = \mathrm{sign}(A)$ for $A \neq 0$ and $\mathrm{sign}([A, B, C]) = \mathrm{sign}(C)$ for $A = 0$. The Parson Poincaré series is then defined to be

$$F_Q^{2k}(z) = -\frac{\sqrt{\mathrm{disc}(Q)}}{\pi} \sum_{Q' \in [Q]} \frac{\mathrm{sign}(Q')}{Q'(z, 1)^k}. \tag{1.9}$$

For $k > 1$ the sum in (1.9) converges absolutely and locally uniformly in z, hence defines a holomorphic function.

The weight $2k$ cocycle for the Parson Poincaré series is given by

$$r_{k,Q}(g, z) = (F_Q^{2k}(z)|_{2k} g) - F_Q^{2k}(z) = \frac{2\sqrt{\mathrm{disc}(Q)}}{\pi} \sum_{\substack{Q' \in [Q], \\ w'_{Q'} < g^{-1}.i\infty < w_{Q'}}} \frac{\mathrm{sign}(Q')}{Q'(z, 1)^k} \tag{1.10}$$

for $g \in \mathrm{SL}_2(\mathbf{Z})$. The sum on the right hand side of (1.10) is finite and defines a holomorphic rational function on $\mathbf{H}$.

The Parson Poincaré series in (1.9) resemble the hyperbolic Poincaré series

$$P_Q^{2k}(z) = \sum_{Q' \in [Q]} \frac{1}{Q'(z, 1)^k},$$

which are holomorphic cusp forms of weight $2k$ and appear in W. Kohnen's construction of the kernel function of the Shimura correspondence [**Koh85**]. In fact, the family $\{P_Q^{2k}(z) : \mathrm{disc}(Q) > 0\}$ generates the space of cusp forms of weight $2k$, which has a clear advantage over the family of parabolic Poincaré series in that these hyperbolic Poincaré series exist for Fuchsian groups that do not contain parabolic elements. Furthermore, S. Katok [**Kat85**] found an interesting geometric expression for the cycle integrals of the $P_Q^{2k}(z)$, namely

$$\mathrm{Im}\left(\int_{z_0}^{\gamma_{Q'}.z_0} P_Q^{2k}(z) Q'(z, 1)^{k-1} dz \right) = (\mathrm{disc}(Q)\mathrm{disc}(Q'))^{\frac{k-1}{2}} \sum_{p \in [S_Q] \cap [S_{Q'}]} \mu_p^k P_{k-1}(\cos\theta_p), \tag{1.11}$$

where P_r denotes the r-th Legendre polynomial, $\theta_p(Q,Q')$ the angle of intersection at p measured counterclockwise from the tangent at $[S_Q]$ to the tangent of $[S_{Q'}]$ at p, and $\mu_p(Q,Q')$ is the signed topological intersection.[3]

Notice the similarity between T. Matsusaka's formula (1.8) and S. Katok's formula (1.11). Morally speaking, if we put $k=1$ in (1.11), we would get the sum over the signed topological intersections μ_p, so a "twisted" intersection number. Motivated by this observation, our second theorem generalizes T. Matsusaka's formula (1.8) to the higher weight case.

THEOREM 1.5. *Let Q, Q' be indefinite binary quadratic forms of discriminant D, $D'>0$ respectively and $k>1$. The homogenized cycle integral of the Parson Poincaré series evaluates to*

$$\operatorname{Im}\left(\lim_{n\to+\infty}\int_{\gamma_{Q'}^{n}.z_0}^{\gamma_{Q'}^{n+1}.z_0} F_Q^{2k}(z)Q_\sigma(z,1)^{k-1}dz\right)=(DD')^{\frac{k-1}{2}}\sum_{p\in[S_Q]\cap[S_{Q'}]}\mu_p^{k-1}P_{k-1}(\cos\theta_p).$$

T. Matsusaka's line of argument for (1.8) of explicitly calculating the primitive cocycle $R_Q(g,z)$ becomes hopelessly complicated in the case $k>1$, so the proof does not carry over. However, we can use an adaptation of S. Katok's proof, which is based on classical tools such as an unfolding argument and tools from complex analysis.

Previous to Matsusaka's formula (1.8), W. Duke, Ö. Imamoḡlu, and Á. Tóth [**DIT17**] expressed the number of intersections of the net of the geodesics associated to the class of $[Q]$ with the non-compact geodesic $S_{-d/c}$ from $-\frac{d}{c}$ to $i\infty$ as the central value of the twisted L-function

$$L_Q(w,-d/c)=\sum_{n=1}^{\infty}\frac{a_Q(n)e\left(-n\frac{d}{c}\right)}{n^w}$$

of the modular integral $F_Q(z)=\sum_{n=1}^{\infty}a_Q(n)e(nz)$ in (1.6). The explicit formula reads

$$\frac{1}{2\pi}\operatorname{Re}\left(L_Q(1,-d/c)\right)=\sum_{p\in[S_Q]\cap S_{-d/c}}1, \tag{1.12}$$

where the equivalence class is over the action of $\mathrm{SL}_2(\mathbf{Z})$ on the geodesic S_Q. We generalize (1.12) to the higher weight case as well.

The Parson Poincaré series $F_Q^{2k}(z)$ is 1-periodic and thus has a Fourier expansion of the shape $F_Q^{2k}(z)=\sum_{n=1}^{\infty}a_{2k,Q}(n)e^{2\pi inz}$. The Fourier coefficients $a_{2k,Q}(n)$ can be explicitly

[3]To be precise, pick Q'' in the class $[Q']$ such that the intersection point p corresponds to the intersection point of S_Q and $S_{Q''}$ and suppose that S_Q and $S_{Q''}$ are oriented clockwise (which means that $Q(1,0)$, $Q''(1,0)<0$). Then $\mu_p(Q,Q')=+1$ if the left endpoint of $S_{Q''}$ lies between the two endpoints of S_Q, and $\mu_p(Q,Q')=-1$ otherwise. The sign of $\mu_p(Q,Q')$ changes if the orientation is reversed.

computed in terms of Kloosterman sums and Bessel functions [**Par93**, Theorem 3.1] or in terms of cycle integrals of weakly holomorphic modular forms [**DIT10**, Theorem 3]. We define the twisted L-function of $F_Q^{2k}(z)$ by

$$L_{2k,Q}\left(w, -\frac{d}{c}\right) = \sum_{n=1}^{\infty} \frac{a_{2k,\,Q}(n)e\left(-n\frac{d}{c}\right)}{n^w}, \quad \mathrm{Re}(w) \gg 1.$$

We are interested in its central value at $w = k$, where the twisted L-function satisfies the following analog of (1.12).

THEOREM 1.6. *Let $k > 1$ be an odd integer, Q be an indefinite binary quadratic form of discriminant $D > 0$ and d, c be coprime integers with $c > 0$. We have*

$$\frac{(k-1)!i^{k-1}}{(2\pi)^k} \mathrm{Re}\left(L_{2k,Q}\left(k, -d/c\right)\right) = D^{\frac{k-1}{2}} \sum_{p \in [S_Q] \,\cap\, S_{-d/c}} P_{k-1}\left(\cos\theta_p\right),$$

where θ_p denotes the angle of intersection at the point p between the geodesic in the equivalence class of S_Q and the geodesic $S_{-d/c}$.

3. Hardy Sums

The primitive cocycle $R_F(g,z) = \displaystyle\int_z^{g.z} F(w)\, dw$ of a family of holomorphic rational cocycles $\{r(g,z) : g \in \mathrm{SL}_2(\mathbf{Z})\}$ of weight 2 with modular integral $F(z)$ is also arithmetically interesting without homogenization. In the simplest case, where $F(z) = E_2(z)$, the primitive cocycle is given by

$$\int_z^{g.z} E_2(w)\, dw = \frac{6}{\pi i} \log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) + \Phi(g), \;\; g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}). \tag{1.13}$$

Throughout this thesis, $\log(z)$ will denote the canonical branch of the logarithm, i.e. with argument in $(-\pi, \pi]$. It is easily checked that differentiating the right hand side in (1.13) yields the cocycle $\frac{6}{\pi i}\frac{c}{cz+d}$ of the Eisenstein series. The constant $\Phi(g)$ ensures that

$$R_{E_2}(gg', z) = R_{E_2}(g, g'.z) + R_{E_2}(g', z). \tag{1.14}$$

As is well known, the Eisenstein series is the logarithmic derivative of the η-function $\eta(z) = e^{\frac{\pi i}{12}z} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$, i.e. $\frac{\eta'(z)}{\eta(z)} = \frac{\pi i}{12} E_2(z)$. So what we observe in (1.13) is really the transformation behavior of $\log \eta(z)$. Since $\eta^{24}(z)$ transforms with a trivial multiplier system, we must have $\Phi(g) \in \mathbf{Z}$.

Explicitly, the constant $\Phi(g)$ is given by

$$\Phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \begin{cases} b, & \text{for } c = 0, \\ \frac{a+d}{c} - 12\mathrm{sign}(c)\ s(d, |c|), & \text{for } c \neq 0, \end{cases} \tag{1.15}$$

where

$$s(d, c) = \sum_{k=1}^{c-1} \left(\left(\frac{dk}{c}\right)\right)\left(\left(\frac{kd}{c}\right)\right),\ (d, c) = 1,\ c > 0, \tag{1.16}$$

with $((x)) = x - \lfloor x \rfloor - 1/2$ is a so-called *Dedekind sum*. We call the constant $\Phi(g)$ the *Dedekind symbol.*

The Dedekind sums $s(d, c)$ satisfy many symmetries, e.g., the reciprocity law

$$s(d, c) + s(c, d) = \frac{1}{12}\frac{c^2 + d^2 + 1}{cd} - \frac{1}{4}, \tag{1.17}$$

which follows from setting $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ and $g' = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in (1.14). The reciprocity law (1.17) can be proved using elementary arguments as well; see H. Rademacher and E. Grosswald's monograph [**RG72**]. However, the nature of Dedekind sums lies much deeper than being elementary finite sums that happen to satisfy pretty symmetries. T. Asai [**Asa70**] defined them, without any reference to automorphic forms, as arising from the cohomology of $\mathrm{SL}_2(\mathbf{R})$ and C. Burrin [**Burr17, Burr18**] defined generalizations of the Dedekind sums for non-arithmetic subgroups of $\mathrm{SL}_2(\mathbf{R})$, which also satisfy reciprocity laws. Moreover, Dedekind sums appear in a multitude of applications, e.g., in physics, geometry, topology, and computer science.

We will consider a natural analog of Dedekind sums resp. the Dedekind symbol (depending on the point of view). Namely, we will consider the boundary of $\log\theta(z)$ and $\log\theta_4(z)$, where

$$\theta(z) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z} \quad\text{and}\quad \theta_4(z) = \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i n^2 z}.$$

The transformation law for, say, $\log\theta_4(z)$ is given by

$$\begin{aligned}\log\theta_4(g.z) - \log\theta_4(z) &= \frac{\pi i}{12}\int_z^{g.z} \left(E_2\left(\frac{w}{2}\right) - E_2(w)\right) dw \\ &= \frac{1}{2}\log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) + \frac{\pi i}{4} S_4(d, c), \quad g = \left(\begin{smallmatrix} a & 2b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}),\end{aligned}$$

where

$$S_4(d,c) = \sum_{k=1}^{|c|-1} (-1)^{\lfloor \frac{kd}{c} \rfloor}, \ (d,c)=1, \ d \text{ odd} \tag{1.18}$$

is a so-called *Hardy sum*; see B. Berndt's paper [**Ber78**].[4] Surprisingly, the sum in (1.18) already appeared in an early paper of G.H. Hardy [**Har05**], which did not refer to automorphic forms at all.

Obviously, the Hardy sums $S_4(d,c)$ in (1.18) are integer-valued, but are only meaningful on the smaller set of coprime integers (c,d) for which d is odd. Those are the pairs that correspond to a matrix in $\Gamma^0(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) : b \equiv 0 \pmod 2 \right\}$, for which $\theta_4(z)$ exhibits automorphic behavior. As a matter of fact, $S_4(d,c) = 0$ if d is even.

On the other hand, since the inversion S does not lie in $\Gamma^0(2)$, there is no reason to expect a reciprocity law as in (1.17) for $S_4(d,c)$. Most of the properties of Dedekind sums are proven in one way or the other through the reciprocity law. The reciprocity law, together with the properties $s(d+c,c) = s(d,c)$ and $s(0,1) = 0$, defines the Dedekind sums uniquely, which allows to calculate them quickly through the Euclidean algorithm.

D. Hickerson [**Hic77**] used the reciprocity law to prove a representation of $s(d,c)$ in terms of the continued fraction expansion of $\frac{d}{c}$ (which again encodes Euclidean division), namely

$$s(d,c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a+d}{c} - \sum_{k=1}^{2n+1} (-1)^k a_k\right), \quad 0 < a < c, \ ad \equiv 1 \pmod{c}, \tag{1.19}$$

where the $a_i \geq 0$ are the partial quotients of the continued fraction expansion of

$$\frac{d}{c} = [0; a_1, ..., a_{2n+1}] = \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_{2n+1}}}}.$$

D. Hickerson then used the formula (1.19) to answer a question raised by H. Rademacher about the distribution of $(d/c, \ s(d,c))$ in $\mathbf{R} \times \mathbf{R}$.

THEOREM 1.7 (Hickerson). *The set* $\{(d/c, \ s(d,c)) : (d,c) = 1\}$ *is dense in* $\mathbf{R} \times \mathbf{R}$.

The same question can be asked about the pair $(d/c, \ S_4(d,c))$ in $\mathbf{R} \times \mathbf{Z}$, for which the answer is also affirmative.

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[4] Most of the theorems in the introduction have an analog for the Hardy sums $S(d,c) = \sum_{k=1}^{c-1} (-1)^{k+1+\lfloor \frac{kd}{c} \rfloor}$ appearing in the transformation law of $\log \theta(z)$, but for conciseness we focus on $S_4(d,c)$ exclusively in the introduction. Later on, we will always state the theorems for both sums, whenever possible.

THEOREM 1.8 (Meyer). *The set* $\{(d/c,\ S_4(d,c)) : (d,c) = 1,\ d \text{ odd}\}$ *is dense in* $\mathbf{R} \times \mathbf{Z}$.

J. Meyer's proof [**Mey97**] uses non-trivial results of L. Goldberg [**Gol82**] and W. Duke, J. Friedlander, and H. Iwaniec [**DFI95**] and, due to the absence of the reciprocity law, does not use any kind of representation of $S_4(d,c)$ in terms of continued fractions. We will identify a special kind of continued fraction expansion (what we call an expansion of $\Gamma^0(2)$-type), for which a formula for $S_4(d,c)$ in the style of (1.19) exists. We prove the following theorem.

THEOREM 1.9. *Let* d, c *be coprime integers and* d *be odd. Let* $a_0 \in \mathbf{Z}$ *and* $a_1, ..., a_n$ *be non-zero integers with* $|a_k| > 1$ *for* $k = 1, 3, ..., n-2$ *such that the continued fraction expansion of* $\frac{d}{c}$ *is equal to* $[0; a_1, 2a_2, a_3, ..., 2a_{n-1}, a_n]$. *The Hardy sums* $S_4(d,c)$ *take the form*

$$S_4(d,c) = (a_1 + a_3 + ... + a_n) + \sum_{k=1}^{n} (-1)^k \operatorname{sign}(a_k). \tag{1.20}$$

As an application of Theorem 1.9 we will give an alternative proof of Theorem 1.8.

More is known about the distributional properties of Dedekind sums. For instance, I. Vardi [**Var87**] showed that the fractional parts of $\{s(d,c) : 1 \le d < c \le N,\ (d,c) = 1\}$ are equidistributed on $\mathbf{R}/\mathbf{Z}$ as $N \to +\infty$. His argument reduces the problem to a question about sums of Kloosterman sums, which can be handled through the work D. Goldfeld and P. Sarnak [**GS83**]. We will show the analog of Vardi's result for Hardy sums by reducing the problem to the spectral theory of certain Eisenstein series. The location of the poles in the analytic continuation of these Eisenstein series encodes information about the growth rate of the Weyl sums.

THEOREM 1.10. *Let* $m > 1$. *The set* $\{S_4(d,c) \pmod m : 1 \le d < c \le N,\ (d,c) = 1,\ d \text{ odd}\}$ *is equidistributed on* $\mathbf{Z}/m\mathbf{Z}$ *with respect to the counting measure as* $N \to +\infty$.

Unfortunately, the literature on Hardy sums is not as vast as the literature on Dedekind sums, but the ubiquity of the θ-functions makes it probable that there is much to be discovered in this regard. One application that we found is that the Hardy sums $S_4(d,c)$ appear as an "error term" in counting the number of lattice points of $(2\mathbf{Z})^2$ that lie in the triangle

$$T(d,c) = \left\{ (x,y) \in \mathbf{R}^2_{\ge 0} : \frac{x}{d} + \frac{y}{c} < 1 \right\}$$

for $(d,c) = 1$, i.e. in the triangle with vertices $(0,0)$, $(d,0)$ and $(0,c)$. More precisely, we prove the following theorem.

THEOREM 1.11. *Let d, c be positive coprime integers.*

(1) *If d, c are both odd, then*

$$\#(T(d,c) \cap (2\mathbf{Z})^2) = \frac{1}{8}(cd + 2d + 2c + S_4(d,c) + S_4(c,d)).$$

(2) *If d is even, then*

$$\#(T(d,c) \cap (2\mathbf{Z})^2) = \frac{1}{4}\left(c + d + \frac{cd}{2} + S_4(c, d/2)\right).$$

Finally, we will give a completely geometric realization of the Hardy sums as signed intersection numbers of a non-compact geodesic in $\mathbf{H}$ with a fixed net of oriented geodesics (see Theorem 11.6). This was done independently, but much more abstract and for a bigger class of Dedekind-type sums, by C. Burrin and F. Von Essen [**BvE22**], who realized C. Burrin's generalized Dedekind sums as winding numbers of geodesics around a cusp of the modular surface. Our hands-on approach, however, allows us to prove certain symmetries of the Hardy sums by elementary geometry.

CHAPTER 2

Binary Quadratic Forms, Hyperbolic Matrices, and Continued Fractions

There is a three-way correspondence between binary quadratic forms of square-free positive discriminant, primitive hyperbolic matrices in the modular group, and the continued fraction expansion of a rational number. In the upcoming chapters we will often use the language which is most convenient for the purpose at hand. In this introductory section, we explain the basic properties of these three concepts and illustrate their interchangeability.

1. Binary Quadratic Forms

The study of binary quadratic forms has a long history in mathematics, dating back at least to Diophantus, who considered the problem of representing an integer by the sum of two squares. One of C.F. Gauss' seminal contributions to number theory was to develop a general theory of binary quadratic forms. Later on, the theory of binary quadratic forms proved to have connections to algebraic number theory, notably to real and imaginary quadratic fields.

A binary quadratic form $Q(x, y) = [A, B, C](x, y) = Ax^2 + Bxy + Cy^2$ is a homogeneous polynomial in two variables with integer coefficients A, B, C. The discriminant of Q is defined to be $\operatorname{disc}(Q) = B^2 - 4AC$.

There is a right-action of the modular group $\mathrm{SL}_2(\mathbf{Z})$ on the set of binary quadratic forms given by

$$(Q \circ g)(x, y) = Q(ax + by, cx + dy), \text{ where } g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}). \tag{2.1}$$

If two binary quadratic forms are equivalent under the action in (2.1), then they represent the same integers.

The discriminant of Q is invariant under the action in (2.1). Naturally, the congruence subgroup

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) : c \equiv 0 \pmod{N} \right\} \tag{2.2}$$

for $N \geq 1$ acts on the set of binary quadratic forms as well. Moreover, if $N|Q(1,0)$, then $N|(Q \circ g)(1,0)$ for $g \in \Gamma_0(N)$.

The set

$$\mathcal{Q}_{N,D} = \{[A,B,C] : B^2 - 4AC = D,\ N|A\} \tag{2.3}$$

is thus invariant under the action of $\Gamma_0(N)$. We write $\mathcal{Q}^+_{N,D}$ for the subset of forms $[A,B,C]$ in $\mathcal{Q}_{N,D}$ with $A > 0$. The subset $\mathcal{Q}^+_{N,D}$ is invariant under $\Gamma_0(N)$ for negative discriminant D, so we use the convention that $\mathcal{Q}_{N,D} = \mathcal{Q}^+_{N,D}$ for $D < 0$.

For any D the set $\mathcal{Q}^+_{N,D}$ is invariant under the action of $\Gamma_\infty = \left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} : n \in \mathbf{Z}\right\}$, the stabilizer subgroup of the cusp $i\infty$ under the action of $\Gamma_0(N)$ on the upper half plane $\mathbf{H}$ by linear fractional transformations. On several occasions we will need the following representation of the cosets of $\mathcal{Q}^+_{N,D}/\Gamma_\infty$, which we formulate in a lemma.

Lemma 2.1. *The map*

$$\{(A,B) : N|A > 0,\ B\ (2A),\ B^2 \equiv D\ (4A)\} \to \mathcal{Q}^+_{N,D}/\Gamma_\infty$$

sending (A,B) to $[A,B,(B^2-D)/4A]$ is a bijection.

Proof. This follows immediately from $[A,B,C]\circ\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} = [A, B+2An, C+Bn+An^2]$ for $n \in \mathbf{Z}$. □

The action of $\Gamma_0(N)$ on $\mathbf{H}$ is compatible with the action of $\Gamma_0(N)$ on $\mathcal{Q}_{N,D}$. Namely, consider the degree 2 polynomial $Q(z,1) = Az^2 + Bz + C$ for $Q \in \mathcal{Q}_{N,D}$ with $z \in \mathbf{H}$. If $D < 0$, the polynomial has two roots $-\frac{B}{2A} \pm \frac{\sqrt{D}}{2A}$, one of which lies in the upper half plane. We denote it by $z_Q = -\frac{B}{2A} + \frac{\sqrt{D}}{2|A|} \in \mathbf{H}$ and refer to it as a CM-point. We have

$$g.z_Q = z_{Q\circ g} \text{ for all } g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N), \tag{2.4}$$

which follows easily from $(Q \circ g)(z,1) = j(g,z)^2 Q(g.z,1)$, where $j(g,z) = cz + d$ is the automorphy factor. In particular, it follows from (2.4) that the group of automorphs $\Gamma_Q = \{g \in \Gamma_0(N) : Q \circ g = Q\}$ is trivial for any $Q \in \mathcal{Q}_{N,D}$ for $D < 0$, unless z_Q is an elliptic point in $\mathbf{H}$. This only happens for the forms $A(x^2+y^2)$ and $A(x^2+xy+y^2)$ with $A > 0$, in which case Γ_Q has order 2 resp. 3.

For $D > 0$ the polynomial $Q(z,1) = Az^2 + Bz + C$ for $Q \in \mathcal{Q}_{N,D}$ with $A \neq 0$ has two real roots $w'_Q < w_Q$. The geodesic

$$S_Q = \{z \in \mathbf{H} : A|z|^2 + B\,\mathrm{Re}(z) + C = 0\} \tag{2.5}$$

connects the two roots w'_Q and w_Q and satisfies

$$g.S_Q = S_{Q\circ g} \text{ for all } g \in \Gamma_0(N). \tag{2.6}$$

The roots $w'_Q < w_Q$ (and thus the geodesic S_Q) are fixed for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ subject to the conditions

$$cz^2 + (d-a)z - b = 0 \text{ for } z = w_Q,\ w'_Q,\ ad - bc = 1, \text{ and } N|c. \tag{2.7}$$

So $c = At$, $d - a = Bt$ and $b = -Ct$ for a $t \in \mathbf{Z}$ by the first condition in (2.7). The condition $ad - bc = 1$ is satisfied if and only if t and $u = a + d$ satisfy the Pell equation $u^2 - Dt^2 = 4$. Since all solutions to the Pell equation arise from a fundamental solution (u_0, t_0) (i.e. such that $u_0 \geq 1$ is minimal), the group of automorphs Γ_Q is a cyclic group $\{\pm\gamma_Q^n : n \in \mathbf{Z}\}$, where

$$\gamma_Q = \begin{pmatrix} \frac{u_0 + Bt_0}{2} & -Ct_0 \\ At_0 & \frac{u_0 - Bt_0}{2} \end{pmatrix}. \tag{2.8}$$

The requirement that $N|A$ ensures that γ_Q indeed lies in $\Gamma_0(N)$. In particular, if $D > 0$ is not a square, the group Γ_Q is infinite cyclic. For $D > 0$ being a square, the only solutions to the Pell equation are $(\pm 2, 0)$, hence Γ_Q is trivial.

There are only finitely many equivalence classes $\mathcal{Q}_{N,D}/\Gamma_0(N)$ for any D. For $N = 1$ and D a fundamental discriminant the number of classes in $\mathcal{Q}_{1,D}/\mathrm{SL}_2(\mathbf{Z})$ is equal to the (narrow) class number of the quadratic field $\mathbf{Q}(\sqrt{D})$. For $D = d^2$ being a positive square, the number of classes in $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$ is given in the next lemma. The proof relies on an argument of A. Biró [**Bir00**, p. 131].[1]

Lemma 2.2. *Let $N \geq 1$ and $d > 0$ with $(d, N) = 1$. The number of equivalence classes in $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$ is equal to $2^{\nu(N)}d$, where $\nu(N)$ is the number of distinct prime divisors of N.*

Proof. For the case $N = 1$, it is easy to see that $\{[0, d, \mu] : 0 \leq \mu < d\}$ is a set of representatives of $\mathcal{Q}_{1,\,d^2}/\Gamma_0(1)$. Moreover, since $(d, N) = 1$, we may as well pick the representatives $[0, d, \mu N]$ for $0 \leq \mu < d$.

For $N \geq 1$, the set of representatives of $\Gamma_0(1)/\Gamma_0(N)$ is given by

$$\left\{ g_{u,v} = \begin{pmatrix} u & * \\ v & * \end{pmatrix} \in \Gamma_0(1) : u|N,\ 1 \leq v \leq N/u \right\};$$

see, e.g., G. Shimura's monograph [**Shim71**, p. 25].

[1] A. Biró's argument for Lemma 2.2 and Lemma 2.3 below is quite condensed and we could not find it explained elsewhere in the literature, so we decided to give a more extensive proof.

Hence, we may pick

$$\{[0, d, \mu N] \circ g_{u,v} : u|N,\ 1 \leq v \leq N/u,\ 0 \leq \mu < d\}. \tag{2.9}$$

as a set of representatives for $\mathcal{Q}_{1,\,d^2}/\Gamma_0(N)$. To count the number of equivalence classes in $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$, it suffices to count which representatives in (2.9) lie in $\mathcal{Q}_{N,\,d^2}$, since the subset $\mathcal{Q}_{N,\,d^2} \subset \mathcal{Q}_{1,\,d^2}$ is closed under the action of $\Gamma_0(N)$.

We see that $([0, d, \mu N] \circ g_{u,v})(1, 0) = [0, d, \mu N](u, v) = duv + \mu N v^2 \equiv 0 \pmod N$ is possible if and only if $v \equiv 0 \pmod{N/u}$ (again, because $(d, N) = 1$). But since $(u, v) = 1$, we must also have $(u, N/u) = 1$, so the representatives of $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$ are given by

$$\{[0, d, \mu N] \circ g_{u,N/u} : u|N,\ (u, N/u) = 1,\ 0 \leq \mu < d\}. \tag{2.10}$$

We deduce that there are exactly $2^{\nu(N)}d$ classes in $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$. □

There is a second way of labeling the representatives of $\mathcal{Q}_{N,\,d^2}/\Gamma_0(N)$ than in (2.10) using Atkin-Lehner involutions, which will prove useful later. For $m > 1$ with $m|N$ and $(m, N/m) = 1$, we write $m\|N$. For any such $m\|N$, pick $\beta, \delta \in \mathbf{Z}$ such that $\delta m^2 - N\beta = m$ and let

$$W_m = \frac{1}{\sqrt{m}} \begin{pmatrix} m & \beta \\ N & m\delta \end{pmatrix} \tag{2.11}$$

be the Atkin-Lehner involution. It is readily checked using $N\beta \equiv 0 \pmod m$ that

$$W_m \Gamma_0(N) W_m^{-1} \subset \Gamma_0(N). \tag{2.12}$$

Moreover, the Atkin-Lehner involution satisfies

$$W_m^2 = \frac{1}{m} \begin{pmatrix} m^2 + N\beta & m\beta + m\beta\delta \\ Nm + Nm\delta & N\beta + m^2\delta^2 \end{pmatrix} \in \Gamma_0(N), \tag{2.13}$$

which makes it an involution when acting on a $\Gamma_0(N)$-invariant function. For $m, m'\|N$, it follows from (2.12) and (2.13) that

$$W_m W_{m'} \in W_{m'} W_m \Gamma_0(N).$$

Furthermore, if $m, m'\|N$ are coprime then $W_m W_{m'} \in W_{mm'}\Gamma_0(N)$, as can be verified by direct computation. Hence, for $m = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ with distinct prime factors, we have $W_m = W_{p_1^{r_1}} W_{p_2^{r_2}} \cdots W_{p_n^{r_n}} g$ for a $g \in \Gamma_0(N)$. In particular, we may also deduce that $W_m W_{m'} \in W_{\frac{mm'}{(m,m')}} \Gamma_0(N)$ for all $m, m'\|N$.

LEMMA 2.3. *Let $N \geq 1$ and $d \geq 1$ such that $(d, N) = 1$. The classes $\mathcal{Q}_{N,d^2}/\Gamma_0(N)$ are represented by*

$$\{[0, d, \mu] \circ W_m : 0 \leq \mu < d,\ m \| N\}. \tag{2.14}$$

PROOF. First, we need to check that $[0, d, \mu] \circ W_m$ with $0 \leq \mu < d$ and $m \| N$ indeed defines an element of $\mathcal{Q}_{N,\, d^2}$. We calculate

$$[0, d, \mu] \circ W_m = \left[N\left(d + \mu N/m\right), d\beta N/m + 2\mu\delta N + d\delta m, \beta d\delta + \mu\delta^2 m\right],$$

which has discriminant $d^2 \frac{(\beta N - \delta m^2)^2}{m^2} = d^2$.

The claimed number of representatives in (2.14) is exactly $2^{\nu(N)} d$, so it suffices to show that they are pairwise inequivalent under the action of $\Gamma_0(N)$ by Lemma 2.2. Moreover, the claim is trivial if $N = 1$, so we may assume that $N > 1$.

Let us check that $[0, d, \mu]$ and $[0, d, \mu'] \circ W_N$ are inequivalent for $1 \leq \mu, \mu' < d$. We have $[0, d, \mu'] \circ W_N = [N(d + \mu'), -d, 0]$. The binary quadratic form $[N(d + \mu'), -d, 0]$ is equivalent to $[N\mu', -d, 0]$. Suppose there is a $g = \left(\begin{smallmatrix} a' & b' \\ c'N & d' \end{smallmatrix}\right) \in \Gamma_0(N)$ such that $[N\mu, -d, 0] \circ g = [0, d, \mu]$, i.e.

$$[N\mu(a')^2 - da'c'N,\ 2N\mu a'b' - d(a'd' + b'c'N),\ N\mu(b')^2 - db'd'] = [0, d, \mu].$$

Comparing the second coefficient, we would have $-d \equiv d \pmod{2N}$, which is a contradiction to $(d, N) = 1$.

Suppose now, more generally, that there are $0 \leq \mu, \mu' < d$ and $m, m' \| N$ such that $[0, d, \mu] \circ W_m = [0, d, \mu'] \circ W_{m'} \circ g'$ for some $g' \in \Gamma_0(N)$. Equivalently, we may say that $[0, d, \mu] = [0, d, \mu'] \circ g \circ W_{\frac{mm'}{(m,m')}}$ for some $g = \left(\begin{smallmatrix} a' & b' \\ c'N & d' \end{smallmatrix}\right) \in \Gamma_0(N)$. To ease the notation, we write m instead of $\frac{mm'}{(m,m')}$.

In this case, we must have

$$\begin{aligned}
0 &= [0, d, \mu](1, 0) \\
&= ([0, d, \mu'] \circ g \circ W_m)(1, 0) \\
&= \frac{1}{m}[0, d, \mu'](a'm + b'N, c'Nm + d'N) \\
&= (d(a' + b'N/m) + \mu'(c'N + d'N/m)) \cdot (c'Nm + d'N).
\end{aligned}$$

If the term $d(a' + b'N/m) + \mu'(c'N + d'N/m)$ is zero, we have

$$da' \equiv 0 \pmod{N/m},$$

which implies that $m = N$, since $(a', N) = 1$ and $(d, N) = 1$, which we already excluded.

Thus, assume that $c'N + d'N/m = 0$. This implies that $d' \equiv 0 \pmod{m}$, which is absurd unless $m = 1$. Since $W_1 \in \Gamma_0(N) \subset \Gamma_0(1)$, we must have $\mu \equiv \mu' \pmod{d}$. $\square$

For $[A, B, C] \in \mathcal{Q}_{N,d^2}$, the geodesic $S_{[A,B,C]}$ defined in (2.5) is still a semi-circle if $A \neq 0$, but for $Q_\mu = [0, d, \mu]$ it is the geodesic

$$S_{Q_\mu} = \left\{ -\frac{\mu}{d} + it : t > 0 \right\}.$$

For any Q with $\mathrm{disc}(Q) > 0$, we orient S_Q counterclockwise if $A = Q(1, 0) > 0$ and clockwise if $A < 0$. If $A = 0$, like in the case of the Q_μ, the geodesic shall be oriented towards the cusp $i\infty$, i.e. from $-\frac{\mu}{d}$ to $i\infty$.

One of C.F. Gauss' key insights was to show that there is a composition operation of the classes of binary quadratic forms in $\mathrm{Cl}_D^+ = \mathcal{Q}_{1,D}/\Gamma_0(1)$, which makes Cl_D^+ a finite abelian group. Nowadays this composition law is more often formulated in terms of ideal classes of the ring of integers of a number field, which allows for a more general framework (as we already mentioned, for a fundamental discriminant D the group Cl_D^+ is the narrow class group of the quadratic number field $\mathbf{Q}(\sqrt{D})$, hence the notation).

Instead of considering equivalence classes for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{Q}_{1,D}$ as in (2.1), we may consider the coarser notion of $\mathrm{SL}_2(\mathbf{Q})$-equivalence on $\mathcal{Q}_{1,D}$. Obviously, these $\mathrm{SL}_2(\mathbf{Q})$-equivalence classes consist of equivalence classes in $\mathcal{Q}_{1,D}/\Gamma_0(1)$. We will refer to such a coarser class as a genus class.

For D a fundamental discriminant, the group of genus classes is isomorphic to the quotient $\mathrm{Cl}_D^+/(\mathrm{Cl}_D^+)^2$. The characters on this group are hence real. For our purposes, it is most convenient to phrase the genus characters of a general discriminant D in terms of representations of integers of the binary quadratic form. The genus characters for a discriminant D are in one-to-one correspondence with the splittings $D = dd'$, where d is a fundamental discriminant. For a binary quadratic form $[A, B, C]$ of discriminant D the genus character associated to the splitting $D = dd'$ is defined to be

$$\chi_d([A, B, C]) = \begin{cases} 0, & \text{if } (A, B, C, d) > 1, \\ \left(\frac{d}{r}\right), & \text{if } (A, B, C, d) = 1 \text{ and } [A, B, C] \text{ represents } r, \end{cases} \tag{2.15}$$

where $\left(\frac{d}{\cdot}\right)$ is the Kronecker symbol.

Let us check that χ_d is well-defined. Suppose the form $[A, B, C]$ represents two distinct integers r, s. We may assume w.l.o.g. that both r, s are properly represented, i.e. that there are coprime $a, c \in \mathbf{Z}$ such that $[A, B, C](a, c) = s$. Picking $g = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, consider $Q' = [A, B, C] \circ g = [s, B', C']$. Since $Q'(x, y)$ properly represents the same integers as $[A,B,C]$, there are $x_0, y_0 \in \mathbf{Z}$ such that $Q'(x_0,y_0)=r$. We calculate

$$4rs = 4sQ'(x_0,y_0) = (2sx_0 + B'y_0)^2 - dd'y_0^2,$$

so $\left(\frac{d}{r}\right) = \left(\frac{d}{s}\right)$. It is also clear that χ_d is invariant for classes $\mathcal{Q}_{N,D}/\Gamma_0(N)$.

We finish this section by an example calculation of values of χ_d for the case where $D = d^2$ for d a fundamental discriminant. Consider the classes of $\mathcal{Q}_{N,d^2}/\Gamma_0(N)$ as in Lemma 2.3. Obviously, the binary quadratic form $[0,d,\mu]\circ W_m$ represents the integer $([0,d,\mu]\circ W_m)(0,1) = \beta d\delta + \mu\delta^2 m$, so

$$\chi_d([0,d,\mu]\circ W_m) = \left(\frac{d}{\beta d\delta + \mu\delta^2 m}\right) = \left(\frac{d}{\mu}\right)\left(\frac{d}{m}\right). \tag{2.16}$$

2. Hyperbolic Matrices

In equation (2.7), we already saw that for a binary quadratic form $Q(x,y)$ – with $\mathrm{disc}(Q) > 0$ not a square – the fixed points of its primitive automorph γ_Q correspond to the roots of $Q(z,1) = 0$. In fact, there is a one-to-one correspondence between binary quadratic forms of positive square-free discriminant and primitive hyperbolic elements of $\mathrm{SL}_2(\mathbf{Z})$.

The non-trivial primitive automorph $\gamma_Q = \begin{pmatrix} \frac{u_0+Bt_0}{2} & -Ct_0 \\ At_0 & \frac{u_0-Bt_0}{2} \end{pmatrix}$ in (2.8) satisfies $\mathrm{Tr}(\gamma_Q) = u_0 > 2$, since $u_0^2 = 4 + Dt_0^2 > 4$ if $t_0 \neq 0$. It is hence a hyperbolic matrix in $\mathrm{SL}_2(\mathbf{Z})$.

Conversely, given a hyperbolic matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we may associate a binary quadratic form $Q_\gamma(x,y) = cx^2 + (d-a)xy - by^2$ to it. An easy calculation gives that

$$\mathrm{disc}(Q_\gamma) = \mathrm{Tr}(\gamma)^2 - 4 > 0.$$

Proposition 2.4. *There is a bijection φ between*

$$\bigcup_{\substack{D>0,\\ D \text{ square-free}}} \mathcal{Q}_{1,D} \quad \text{and} \quad \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) : \mathrm{Tr}(\gamma) > 2,\ \gamma \text{ primitive}\}$$

given by $\varphi(Q) = \gamma_Q$. Its inverse φ^{-1} is given by the map $\gamma \mapsto \frac{1}{(b,d-a,c)}Q_\gamma$.

Proof. Let $Q = [A,B,C]$ and $D = B^2 - 4AC$ be square-free (this implies that $(A,B,C) = 1$). Its primitive automorph $\gamma_Q = \begin{pmatrix} \frac{u_0-Bt_0}{2} & -Ct_0 \\ At_0 & \frac{u_0+Bt_0}{2} \end{pmatrix}$ is given in (2.8), where u_0, t_0 is the fundamental solution of the Pell equation $u_0^2 - Dt_0^2 = 4$. Thus, we have $\frac{1}{(At_0,Bt_0,Ct_0)}Q_{\gamma_Q}(x,y) = Ax^2 + Bxy + Cy^2$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ be primitive and hyperbolic and $m = (c, d-a, b)$. The Pell equation $u^2 - \frac{(a+d)^2-4}{m^2} t^2 = 4$ has the fundamental solution $u = a+d$ and $t = m$. Hence, $\gamma_{\frac{1}{m}Q_\gamma} = \gamma$. □

One advantage of considering primitive hyperbolic elements instead of binary quadratic forms, is that this viewpoint generalizes easily to Fuchsian groups of the first kind, i.e. to cofinite discrete subgroups $\Gamma < \mathrm{SL}_2(\mathbf{R})$. For any primitive hyperbolic $\gamma \in \Gamma$, we can associate a binary quadratic form $Q_\gamma(x,y)$ with real coefficients to it as before. The group Γ still acts on the set of binary quadratic forms of discriminant $\mathrm{Tr}(\gamma)^2 - 4$ with real coefficients as in (2.1); the action also preserves the discriminant of a binary quadratic form with real coefficients. The equivalence classes of binary quadratic forms $Q_\gamma(x,y)$ correspond to conjugacy classes in Γ. More precisely, for $g \in \Gamma$, we have

$$(Q_\gamma \circ g)(x,y) = Q_{g^{-1}\gamma g}(x,y), \tag{2.17}$$

which can be proved easily by recalling that the roots of $Q_\gamma(z,1) = 0$ are the fixed points of γ. In particular, this means that $(Q_\gamma \circ \gamma)(x,y) = Q_\gamma(x,y)$.

Denote the real fixed points of γ by $w'_\gamma < w_\gamma$. Let $R_\gamma \in \mathrm{SL}_2(\mathbf{R})$ be the matrix, which sends $R_\gamma.0 = w'_\gamma$ and $R_\gamma.i\infty = w_\gamma$. The matrix R_γ is unique up to right multiplication by a homothety $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ with $x \in \mathbf{R}$, i.e. transformations which fix the imaginary axis $i\mathbf{R}_{>0}$. As $R_\gamma^{-1}\gamma R_\gamma$ fixes the imaginary axis as well, we must have $R_\gamma^{-1}\gamma R_\gamma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ for a $\xi \in \mathbf{R}$, so R_γ diagonalizes the matrix γ.

Taking $R_\gamma = \frac{1}{\sqrt{w_\gamma - w'_\gamma}} \begin{pmatrix} w_\gamma & w'_\gamma \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$, we calculate

$$\begin{aligned} R_\gamma^{-1}\gamma R_\gamma &= \frac{1}{w_\gamma - w'_\gamma} \begin{pmatrix} aw_\gamma + b - w'_\gamma(cw_\gamma + d) & aw'_\gamma + b - w'_\gamma(cw'_\gamma + d) \\ aw_\gamma + b + w_\gamma(cw_\gamma + d) & aw'_\gamma + b + w_\gamma(cw'_\gamma + d) \end{pmatrix} \\ &= \begin{pmatrix} cw_\gamma + d & 0 \\ 0 & cw'_\gamma + d \end{pmatrix}, \end{aligned}$$

using that $Q_\gamma(w_\gamma, 1) = Q_\gamma(w'_\gamma, 1) = 0$. Hence, $\xi = \frac{a+d}{2} + \mathrm{sign}(c)\frac{\sqrt{(a+d)^2-4}}{2}$ and $\xi^{-1} = \frac{a+d}{2} - \mathrm{sign}(c)\frac{\sqrt{(a+d)^2-4}}{2}$. The binary quadratic form $Q_{R_\gamma^{-1}\gamma R_\gamma}$ is thus equal to $[0, \xi - \xi^{-1}, 0] = [0, \mathrm{sign}(c)\sqrt{D}, 0]$. It should be noted that $|\xi| > 1$ if and only if $c(a+d) > 0$.

The orientation of $S_\gamma = S_{Q_\gamma}$ is as before in terms of $\mathrm{sign}(c)$. However, for certain γ we can view the orientation in a more geometric/dynamic fashion in terms of attractive and repulsive fixed points of γ. Under repeated action of γ on the upper half plane with its boundary, every point that is not a fixed point will converge to the attractive fixed point.

LEMMA 2.5. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ *be a primitive hyperbolic element and* $w'_\gamma < w_\gamma$ *its real fixed points. For any* $z \in \mathbf{H} \cup \{i\infty\} \cup \mathbf{R} - \{w_1, w_2\}$*, we have*

$$\lim_{n\to+\infty} \gamma^n.z = \begin{cases} w_\gamma, & \text{if } c(a+d) > 0, \\ w'_\gamma, & \text{if } c(a+d) < 0. \end{cases}$$

PROOF. This follows from

$$\gamma^n = R_\gamma \begin{pmatrix} \xi^n & 0 \\ 0 & \xi^{-n} \end{pmatrix} R_\gamma^{-1} = \frac{1}{w_\gamma - w'_\gamma} \begin{pmatrix} w_\gamma\xi^n - w'_\gamma\xi^{-n} & -w'_\gamma w_\gamma \xi^n + w'_\gamma w_\gamma \xi^{-n} \\ \xi^n - \xi^{-n} & -w'_\gamma \xi^n + w_\gamma \xi^{-n} \end{pmatrix}$$

and $|\xi| > 1$ if and only if $c(a+d) > 0$. □

Lemma 2.5 shows that for γ a hyperbolic element with $\mathrm{Tr}(\gamma) < 0$, the geodesic S_γ is oriented from the repulsive fixed point to the attractive one.

2.1. The Signed Topological Intersection Number. Let S_1, S_2 be two positively oriented semi-circular geodesics in $\mathbf{H}$. If the two geodesics intersect, we define the signed topological intersection number of S_1 and S_2 to be $\mu(S_1, S_2) = +1$ if the left endpoint of S_2 lies between the endpoints of S_1 and $\mu(S_1, S_2) = -1$ otherwise. Clearly, we have $\mu(S_1, S_2) = -\mu(S_2, S_1)$. Whenever we change the orientation of one of the two geodesics S_1, S_2, the sign of $\mu(S_1, S_2)$ is reversed.

Let $\gamma, \sigma \in \Gamma$ be primitive hyperbolic elements. Both geodesics S_γ and S_σ have an orientation depending on the lower left entry of the matrix γ resp. σ. Let $[S_\gamma]$, $[S_\sigma]$ be the projections of the geodesics S_γ, S_σ respectively onto $\Gamma \setminus \mathbf{H}$. The intersection points p of $[S_\gamma]$ and $[S_\sigma]$ are in one-to-one correspondence with the intersection points of S_γ with the net $\{g.S_\sigma : g \in \Gamma_\gamma \setminus \Gamma / \Gamma_\sigma\}$. Let $g_p \in \Gamma_\gamma \setminus \Gamma/\Gamma_\sigma$ be the corresponding element to an intersection point p. The topological intersection number of $[S_\gamma]$ with $[S_\sigma]$ at p is defined as

$$\mu_p(\gamma, \sigma) = \mu(S_\gamma, g_p.S_\sigma).$$

The topological intersection $\mu_p(\gamma, \sigma)$ is a conjugacy class invariant in γ and σ. We can also express the topological intersection number in terms of binary quadratic forms.

LEMMA 2.6. *Let* $\gamma, \sigma \in \Gamma$ *be primitive hyperbolic elements, whose geodesics* S_γ *and* S_σ *intersect. Let* $w'_\sigma < w_\sigma$ *be the roots of* $Q_\sigma(z, 1) = 0$*. We have*

$$\mu(S_\gamma, S_\sigma) = -\mathrm{sign}(Q_\sigma)\mathrm{sign}(Q_\gamma(w'_\sigma, 1)) = \mathrm{sign}(Q_\sigma)\mathrm{sign}(Q_\gamma(w_\sigma, 1)).$$

PROOF. Suppose first that $\mathrm{sign}(Q_\gamma), \mathrm{sign}(Q_\sigma) > 0$. The root w'_σ lies between the endpoints of S_γ if and only if $\mathrm{sign}(Q_\gamma(w'_\sigma, 1)) < 0$ or, equivalently, $\mathrm{sign}(Q_\gamma(w_\sigma, 1)) > 0$. So in this case we have

$$\mu(S_\gamma, S_\sigma) = -\mathrm{sign}(Q_\gamma(w'_\sigma, 1)) = \mathrm{sign}(Q_\gamma(w_\sigma, 1)).$$

If we change the orientation of S_γ, then the sign of $\mathrm{sign}(Q_\gamma(w'_\sigma, 1))$ and $\mathrm{sign}(Q_\gamma(w_\sigma, 1))$ changes as well. If $\mathrm{sign}(Q_\sigma) < 0$, then

$$\mu(S_\gamma, S_\sigma) = -\mu(S_\gamma, S_{\sigma^{-1}}) = \mathrm{sign}(Q_\gamma(w'_{\sigma^{-1}}, 1)) = -\mathrm{sign}(Q_\gamma(w_{\sigma^{-1}}, 1)),$$

which proves the claim since $w'_{\sigma^{-1}} = w'_\sigma$ and $w_{\sigma^{-1}} = w_\sigma$. $\square$

3. Continued Fractions

In this section, we will show that the continued fraction expansions of a rational number is equivalent to writing a matrix as a word in terms of certain generators of the modular group $\mathrm{SL}_2(\mathbf{Z})$.

A continued fraction is defined to be

$$[a_0; a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}} \tag{2.18}$$

with $n \geq 0$, $a_0 \in \mathbf{Z}$ and $a_1, ..., a_n$ a sequence of non-zero integers. The integers $a_0, a_1 ..., a_n$ are called the partial quotients of the continued fraction.[2] The integer n is called its length. Naturally, $[a_0; a_1, ..., a_n]$ is a rational number.

PROPOSITION 2.7. *Let $r \in \mathbf{Q}$ be a rational number. There exist $a_0 \in \mathbf{Z}$ and $a_1, ..., a_n \geq 1$ with $n \geq 0$ such that*

$$r = [a_0; a_1, a_2, ..., a_n].$$

PROOF. Let $a_0 \in \mathbf{Z}$ be such that $r_1 = r - a_0 \in [0, 1)$. If $r_1 = 0$, we have $r = [a_0]$, so there is nothing more to show. Thus, we may suppose that $0 < r_1 < 1$.

Pick $a_1 \geq 1$ such that $r_2 = \frac{1}{r_1} - a_1 \in [0, 1)$. Again, if $r_2 = 0$, then $r = a_0 + \frac{1}{a_1} = [a_0; a_1]$, so there is nothing more to show. Thus, we may suppose that $0 < r_2 < 1$.

Continue this process to find $a_2, a_3, ... \geq 1$ and generate r_2, r_3, ..., until $r_n = 0$ for an $n \geq 0$. Then $r = [a_0; a_1, ..., a_n]$.

To see that this process comes to an end, write $r_i = \frac{h_i}{k_i} \in (0, 1)$ with $(h_i, k_i) = 1$ for each iteration r_i. In the next iteration, we have $r_{i+1} = \frac{k_i - a_{i+1} h_i}{h_i} = \frac{h_{i+1}}{k_{i+1}}$. But since $1 \leq k_{i+1} < h_i < k_i$, the size of the denominator of r_i decreases with each step. $\square$

[2]Often all but the zeroth partial quotient are required to be positive, but we differ from this convention.

The continued fraction expansion (c.f.e.) in Proposition 2.7 is not unique. For instance, if $a_n \neq 1$, then we may write

$$[a_0; a_1, a_2, ..., a_n] = [a_0; a_1, a_2, ..., a_n - 1, 1].$$

In particular, we can require each c.f.e. of a rational number to have odd length n.

The proof of Proposition 2.7 gives an explicit algorithm to find the c.f.e. of a rational number, which is essentially the extended Euclidean algorithm. A second way of proving the existence of a c.f.e. for a rational number is to look at the action of a matrix in $\mathrm{SL}_2(\mathbf{Z})$ in terms of its generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on the cusp $i\infty$. Recall that the proof that T and V generate $\mathrm{SL}_2(\mathbf{Z})$ again boils down to the Euclidean algorithm.

To make this explicit, write $r = \frac{a}{c}$ with $(a, c) = 1$ and $c > 0$. Then there are $b, d \in \mathbf{Z}$ such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. Write A as a word in T, V, i.e. $A = \pm T^{a_0} V^{a_1} T^{a_2} \cdots V^{a_n} T^{a_{n+1}}$ with $a_0, a_{n+1} \in \mathbf{Z}$ and $a_1, ..., a_n$ being non-zero integers.

The matrix A acting on the cusp $i\infty$ gives

$$\begin{aligned} \frac{a}{c} &= A.i\infty \\ &= \pm T^{a_0} V^{a_1} T^{a_2} \cdots V^{a_{n-2}} T^{a_{n-1}} V^{a_n}.i\infty \\ &= \pm T^{a_0} V^{a_1} T^{a_2} \cdots V^{a_{n-2}} T^{a_{n-1}}.\frac{1}{a_n} \\ &= \pm T^{a_0} V^{a_1} T^{a_2} \cdots V^{a_{n-2}}.\left(a_{n-1} + \frac{1}{a_n}\right) \\ &= \pm T^{a_0} V^{a_1} T^{a_2} \cdots T^{a_{n-3}}.\frac{1}{a_{n-2} + \frac{1}{a_{n-1} + \frac{1}{a_n}}} \\ &= [a_0; a_1, ..., a_n]. \end{aligned}$$

So each way of writing the matrix A as a word in T and V gives rise to a c.f.e. of $\frac{a}{c}$ of odd length n. Conversely, each c.f.e. of odd length gives rise to a word in T and V for a matrix $\begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

Similarly, considering the generators T and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbf{Z})$ instead of T and V, we get a different kind of c.f.e.. Again, writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^{b_0} S T^{b_1} S \cdots S T^{b_n} S T^{b_{n+1}}$ with $b_0, b_{n+1} \in \mathbf{Z}$ and $b_1, ..., b_n$ non-zero integers, we get a "negative" continued fraction by acting on the cusp $i\infty$:

$$\begin{aligned}\frac{a}{c} &= A.i\infty \\ &= \pm T^{b_0} S T^{b_1} S \cdots S T^{b_{n-1}} S T^{b_n} S.i\infty \\ &= \pm T^{b_0} S T^{b_1} S \cdots S T^{b_{n-1}} S T^{b_n}.0 \\ &= \pm T^{b_0} S T^{b_1} S \cdots S T^{b_{n-1}}. -\frac{1}{b_n} \\ &= \pm T^{b_0} S T^{b_1} S \cdots S. \left(b_{n-1} - \frac{1}{b_n} \right) \\ &= b_0 - \cfrac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_n}}}}.\end{aligned}$$

We will denote such a negative c.f.e. by $\frac{a}{c} = [\![b_0; b_1, ..., b_n]\!]$.

Part 1

The Shintani Lift of Maass Cusp Forms

CHAPTER 3

Maass Cusp Forms

1. Basic Properties

A natural generalization of the holomorphic cusp forms are the Maass cusp forms. These are eigenfunctions of the Laplace-Beltrami operator $\Delta_\kappa = -y^2(\partial_x^2 + \partial_y^2) + i\kappa y(\partial_x + i\partial_y)$, where $\kappa \in \frac{1}{2}\mathbf{Z}$ denotes their weight. They play a central role in the spectral theory of the hyperbolic surface $\Gamma_0(N) \setminus \mathbf{H}$ for $N \geq 1$.

We allow the weight κ of a Maass form to be half-integral with respect to the theta-cocycle

$$J(g, z) = \frac{\Theta(g.z)}{\Theta(z)} \text{ for } g \in \Gamma_0(4),$$

where $\Theta(z) = \sum_{n \in \mathbf{Z}} e(n^2 z)$. Explicitly, the θ-cocycle is given by

$$J(g, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where $\varepsilon_d = 1, i$ according to $d \equiv 1, 3 \pmod 4$. We define $\varphi|_\kappa g = J(g, z)^{-2\kappa}\varphi(g.z)$ for $g \in \Gamma_0(4)$, so that if κ is an even integer the $|_\kappa$-operator agrees with the $|_{2k}$-operator defined in the introduction.

Definition 3.1. A real-analytic function $\varphi : \mathbf{H} \to \mathbf{C}$ is called a *Maass form* of weight $\kappa \in \frac{1}{2}\mathbf{Z}$ and level N (with $4|N$ for $\kappa \notin \mathbf{Z}$) if it satisfies the following properties:

(1) φ is an eigenfunction of Δ_κ, i.e. $\Delta_\kappa \varphi = \lambda\varphi$ for a $\lambda \in \mathbf{C}$,
(2) φ is modular of weight κ, i.e. $(\varphi|_\kappa g)(z) = \varphi(z)$ for all $g \in \Gamma_0(N)$ and all $z \in \mathbf{H}$, and
(3) φ is of polynomial growth at the cusps, i.e. there exists a $C > 0$ such that $\varphi|_\kappa \sigma_\mathfrak{a} = O(y^C)$ as $y \to +\infty$ for every cusp $\mathfrak{a}$ of $\Gamma_0(N)$ and a $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbf{Z})$ such that $\sigma_\mathfrak{a}.i\infty = \mathfrak{a}$.

If, in addition, φ satisfies $\displaystyle\int_0^1 (\varphi|_\kappa \sigma_\mathfrak{a})(x + iy)dx = 0$ for each cusp $\mathfrak{a}$, we call φ a *Maass cusp form*.

It will be useful to consider the ξ_κ-operator in our study of Maass forms, which is defined as

$$\xi_\kappa = 2iy^\kappa \overline{\partial_{\overline{z}}}. \tag{3.1}$$

The Laplace operator Δ_κ can be realized as a composition of ξ_κ-operators.

Lemma 3.2. *For any weight $\kappa \in \frac{1}{2}\mathbf{Z}$, we have*

$$\Delta_\kappa = -y^2(\partial_x^2 + \partial_y^2) + i\kappa y(\partial_x + i\partial_y) = -\xi_{2-\kappa} \circ \xi_\kappa.$$

Proof. This is a calculation. Let $f : \mathbf{H} \to \mathbf{C}$ be a real-analytic function and denote $\partial_x f = f_x$ and $\partial_y f = f_y$. We then have

$$\begin{aligned} -\xi_{2-\kappa}(\xi_\kappa f) &= -iy^{2-\kappa}\overline{\left(\partial_x(iy^\kappa(\overline{f_x} - i\overline{f_y}) + i\partial_y(iy^\kappa(\overline{f_x} - i\overline{f_y}))\right)} \\ &= -iy^{2-\kappa}\left(-iy^\kappa(f_{xx} + if_{xy}) - \kappa y^{\kappa-1}(f_x + if_y) - y^\kappa(f_{xy} + if_{yy})\right) \\ &= -y^2(f_{xx} + f_{yy}) + i\kappa y(f_x + if_y). \end{aligned}$$

□

A key property of the Laplacian is that it commutes with the $|_\kappa$-operator. This is particularly useful when constructing Poincaré series out of eigenfunctions of Δ_κ (e.g., non-holomorphic Eisenstein series). The ξ_κ-operator allows us to split the calculation verifying this.

Proposition 3.3. *Let $\kappa \in \frac{1}{2}\mathbf{Z}$. For all $g \in \Gamma_0(1)$ and every real-analytic function $f : \mathbf{H} \to \mathbf{C}$ we have $(\xi_\kappa f)|_{2-\kappa} g = \xi_\kappa(f|_\kappa g)$. In particular, if f is invariant under the $|_\kappa$-operator for a subgroup $\Gamma_0(N)$, then $\xi_\kappa f$ is invariant for a subgroup $\Gamma_0(N)$ under the $|_{2-\kappa}$-operator.*

Proof. Let $g \in \Gamma_0(1)$. We calculate

$$\begin{aligned} (\xi_\kappa f)|_{2-\kappa} g &= 2i\mathrm{Im}(g.z)^\kappa J(g,z)^{2\kappa-2}\overline{(\partial_{\overline{z}} f)(g.z)} \\ &= 2i\frac{y^\kappa}{|J(g,z)|^{4\kappa}} J(g,z)^{2\kappa}\overline{J(g,\overline{z})^{-2}(\partial_{\overline{z}} f)(g.z)} \\ &= 2iy^\kappa\overline{J(g,z)^{-2\kappa}\partial_{\overline{z}} f(g.z)} \\ &= \xi_\kappa(f|_\kappa g). \end{aligned}$$

□

Corollary 3.4. *The operator Δ_κ commutes with the $|_\kappa$-operator for all $g \in \Gamma_0(1)$.*

PROOF. This follows from Propositions 3.2 and 3.3. $\square$

Denote the space of modular forms of weight $k \in \mathbf{Z}$ and level N by $M_{k,N}$ and the subspace of cusp forms by $S_{k,N}$. A function $f : \mathbf{H} \to \mathbf{C}$ is holomorphic if and only if $\xi_\kappa f = 0$. Hence, any $f \in M_{k,N}$ satisfies $\Delta_k f = 0$. Conditions (2) and (3) in Definition 3.1 hold by definition of a modular form. This implies that every holomorphic modular form is a Maass form and that every holomorphic cusp form is a Maass cusp form.

PROPOSITION 3.5. *Let φ be a Maass form of weight κ and level N with eigenvalue $\lambda = (s-\kappa/2)(1-\kappa/2-s)$ for an $s \in \mathbf{C}$. The function φ has a Fourier-expansion around a cusp $\mathfrak{a}$ of $\Gamma_0(N) \setminus \mathbf{H}$ of the form*

$$\varphi|_\kappa\sigma_\mathfrak{a} = a_{\varphi,\mathfrak{a}}(0)y^{s-\kappa/2}+b_{\varphi,\mathfrak{a}}(0)y^{1-s-\kappa/2}+\sum_{n\neq 0} a_{\varphi,\mathfrak{a}}(n)(4\pi|n|y)^{-\kappa/2}W_{\mathrm{sign}(n)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(nx),$$

where $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbf{Z})$ is such that $\sigma_\mathfrak{a}.i\infty = \mathfrak{a}$ and $W_{\kappa,\mu}(y)$ is the W-Whittaker function.

PROOF. Consider the stabilizer subgroup $\Gamma_\mathfrak{a} \subset \Gamma_0(N)$. Since $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}$ fixes the cusp $i\infty$, we must have $T \in \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}$. Since the function $\varphi|_\kappa\sigma_\mathfrak{a}$ is thus 1-periodic and real-analytic, it admits a Fourier expansion

$$\varphi|_\kappa\sigma_\mathfrak{a} = \sum_{n\in\mathbf{Z}} a_{\varphi,\mathfrak{a}}(n,y)e(nx).$$

Since $\Delta_\kappa\varphi = \lambda\varphi$ for a $\lambda \in \mathbf{C}$, Corollary 3.4 implies that

$$\partial_y^2 a_{\varphi,\mathfrak{a}}(n,y) + \frac{\kappa}{y}\partial_y a_{\varphi,\mathfrak{a}}(n,y) + \left(\frac{\lambda}{y^2} + \frac{2\pi n\kappa}{y} - 4\pi^2 n^2\right) a_{\varphi,\mathfrak{a}}(n,y) = 0 \text{ for } n \neq 0$$

and

$$-y^2\partial_y^2 a_{\varphi,\mathfrak{a}}(0,y) - \kappa y\partial_y a_{\varphi,\mathfrak{a}}(0,y) = \lambda a_{\varphi,\mathfrak{a}}(0,y).$$

The latter has the solution $a_{\varphi,\mathfrak{a}}(0)y^{\frac{1}{2}-\frac{\kappa}{2}+\sqrt{\frac{(\kappa-1)^2}{4}-\lambda}} + b_{\varphi,\mathfrak{a}}(0)y^{\frac{1}{2}-\frac{\kappa}{2}-\sqrt{\frac{(k-1)^2}{4}-\lambda}}$.

For $n \neq 0$, we may substitute $a_{\varphi,\mathfrak{a}}(n,y) = (4\pi|n|y)^{-\kappa/2}f_n(y)$ for which the differential equation becomes

$$f_n''(y) + \left(\frac{\lambda - \frac{\kappa^2}{4} + \frac{\kappa}{2}}{y^2} + \frac{2\pi n\kappa}{y} - 4\pi^2 n^2\right) f_n(y) = 0.$$

This is now a version of the Whittaker differential equation, which has the solution

$$f_n(y) = a_{\varphi,\mathfrak{a}}(n)W_{\mathrm{sign}(n)\frac{\kappa}{2},\sqrt{\frac{(\kappa-1)^2}{4}-\lambda}}(4\pi|n|y) + b_{\varphi,\mathfrak{a}}(n)M_{\mathrm{sign}(n)\frac{\kappa}{2},\sqrt{\frac{(\kappa-1)^2}{4}-\lambda}}(4\pi|n|y).$$

Since $\varphi|_k\sigma_\mathfrak{a}$ is polynomially bounded as $y \to +\infty$, we have $b_{\varphi,\mathfrak{a}}(n) = 0$ for $n \neq 0$. $\square$

We call the point $s \in \mathbf{C}$, such that $\lambda = (s - \kappa/2)(1 - \kappa/2 - s)$ is the eigenvalue of a Maass form φ, the *spectral parameter* of φ.

PROPOSITION 3.6. *The eigenvalue of any non-zero Maass cusp form φ is real and non-negative. Moreover, the eigenvalue of φ is positive if φ is not holomorphic.*

PROOF. By Stokes' Theorem and Lemma 3.2, we get

$$\lambda\langle\varphi, \varphi\rangle = \langle\Delta_k\varphi, \varphi\rangle = \langle\xi_k\varphi, \xi_k\varphi\rangle = \langle\varphi, \Delta_k\varphi\rangle = \overline{\lambda}\langle\varphi, \varphi\rangle,$$

by which it follows that $\lambda \geq 0$.

Since $\langle\xi_k\varphi, \xi_k\varphi\rangle > 0$ if and only if $\xi_k\varphi$ is not identically zero, the eigenvalue λ is zero if and only if φ is holomorphic. $\square$

Hence, for the spectral parameter $s = \sigma + it$ of a Maass cusp form, the imaginary part $t - 2\sigma t$ of $(s - \kappa/2)(1 - \kappa/2 - s) = s(1 - s) - \frac{\kappa}{2}\left(1 - \frac{\kappa}{2}\right)$ vanishes, which implies that $\sigma = \frac{1}{2}$ or $t = 0$. It was conjectured by A. Selberg that $\sigma = \frac{1}{2}$ is always the case. We call those eigenvalues for which $\sigma \neq \frac{1}{2}$ the *exceptional spectrum* of Laplace operator Δ_κ.

Let $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$ be the space of square-integrable functions on $\Gamma_0(N) \setminus \mathbf{H}$, which are invariant under the $|_\kappa$-operator. This is a Hilbert space with the Petersson inner product, i.e. with

$$\langle f, g\rangle = \int_{\Gamma_0(N)\setminus\mathbf{H}} f(z)\overline{g(z)}y^\kappa d\mu(z), \tag{3.2}$$

where $d\mu(z) = \frac{dxdy}{y^2}$ is the hyperbolic measure. As usual, two functions f, g are seen as identical in $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$ if their difference has norm zero. The operator Δ_κ admits to a self-adjoint extension to all of $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$ by Friedrich's Theorem [**Iwa21**, Thm. A.3].

Due to the exponential decay of Maass cusp forms (see Proposition 3.5) at the cusps of $\Gamma_0(N)$, every Maass cusp form of weight κ and level N is contained in $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$. We denote the subspace of $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$ of Maass cusp forms with spectral parameter $s \in \mathbf{C}$ by $U_{\kappa,N}(s)$.

PROPOSITION 3.7. *We have $U_{\kappa,N}(\kappa/2) = U_{\kappa,N}(1 - \kappa/2) = S_{\kappa,N}$.*

PROOF. By Stokes' Theorem it follows that $\langle\xi_\kappa\varphi, \xi_\kappa\varphi\rangle = \langle\Delta_\kappa\varphi, \varphi\rangle = 0$. This implies that $\xi_\kappa\varphi = 0$ and thus φ is holomorphic. $\square$

For $s \neq \frac{\kappa}{2}, 1 - \frac{\kappa}{2}$ the ξ_κ-operator encodes a symmetry between Maass cusp forms of weight κ and dual weight $2 - \kappa$.

PROPOSITION 3.8. *The operator ξ_κ is an isomorphism $U_{\kappa,N}(s) \to U_{2-\kappa,N}(s)$ for $s \neq \frac{\kappa}{2}, 1 - \frac{\kappa}{2}$.*

PROOF. Let $\varphi \in U_{\kappa,N}(s)$. By Proposition 3.2 we have

$$\Delta_{2-\kappa}\xi_\kappa\varphi = \xi_\kappa\Delta_\kappa\varphi = (s - \kappa/2)(1 - \kappa/2 - s)\xi_\kappa\varphi.$$

The function $\xi_\kappa\varphi$ is invariant under the $|_{2-\kappa}$-operator by Proposition 3.3 and it is easily verified that $\xi_\kappa\varphi$ is polynomially bounded at all cusps. The inverse of ξ_κ is given by $-\frac{1}{(s-\kappa/2)(1-\kappa/2-s)}\xi_{2-\kappa}$. □

The Fourier coefficients of the two "dual" Maass cusp forms φ and $\xi_\kappa\varphi$ are very similar, as we will show in the next proposition.

PROPOSITION 3.9. *Let $\varphi \in U_{\kappa,N}(s)$ and denote its n-th Fourier coefficient of the expansion around a cusp $\mathfrak{a}$ of $\Gamma_0(N)\setminus\mathbf{H}$ by $a_{\varphi,\mathfrak{a}}(n)$. The n-th Fourier coefficient of $\xi_\kappa\varphi$ is given by $-(4\pi|n|)^{1-\kappa}\overline{a_{\varphi,\mathfrak{a}}(-n)}$ for $n > 0$ and by $(s - \kappa/2)(1 - \kappa/2 - s)(4\pi|n|)^{1-\kappa}\overline{a_{\varphi,\mathfrak{a}}(-n)}$ for $n < 0$.*

PROOF. Let $\varphi|_\kappa\sigma_\mathfrak{a} = \sum_{n\neq 0} a_{\varphi,\mathfrak{a}}(n)(4\pi|n|y)^{-\frac{\kappa}{2}}W_{\mathrm{sign}(n)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(nx)$. Write

$$f_n(x, y) = a_{\varphi,\mathfrak{a}}(n)(4\pi|n|y)^{-\frac{\kappa}{2}}W_{\mathrm{sign}(n)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(nx).$$

Applying the ξ_κ-operator to $f_n(x, y)$, we obtain

$$\begin{aligned}
&iy^\kappa \cdot \overline{\partial_x f_n(x, y) + i\partial_y f_n(x, y)} \\
&= iy^\kappa \cdot \overline{2\pi i n f_n(x, y) - i\frac{\kappa}{2y}f_n(x, y) + ia_{\varphi,\mathfrak{a}}(n)(4\pi|n|y)^{-\frac{\kappa}{2}}e(nx)\partial_y W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)}.
\end{aligned}$$

We use that

$$\partial_y W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y) = \frac{1}{2y}\left(4\pi|n|y - \nu\right)W_{\nu,\mu}(4\pi|n|y) - \frac{1}{y}W_{\nu+1,\mu}(4\pi|n|y).$$

For $n < 0$, we hence obtain

$$\begin{aligned}
\xi_k f_n(x, y) &= iy^\kappa \cdot \overline{-ia_{\varphi,\mathfrak{a}}(n)(4\pi|n|)^{-\frac{\kappa}{2}}y^{-\frac{\kappa}{2}-1}W_{-\frac{\kappa}{2}+1,s-\frac{1}{2}}(4\pi|n|y)e(nx)} \\
&= -\overline{a_{\varphi,\mathfrak{a}}(n)}(4\pi|n|)^{1-\kappa}(4\pi|n|y)^{\frac{\kappa}{2}-1}W_{\frac{2-\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(-nx).
\end{aligned}$$

Suppose now that $n > 0$. Since

$$\begin{aligned}
W_{\frac{\kappa}{2}+1,s-\frac{1}{2}}(4\pi ny) &= (4\pi ny - \kappa)\,W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny) \\
&\quad + \frac{1}{4}\left(4(s - 1/2)^2 - (1 - \kappa)^2\right)W_{\frac{\kappa}{2}-1,s-\frac{1}{2}}(4\pi ny),
\end{aligned}$$

we obtain

$$\begin{aligned}\partial_y W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny) &= \left(2\pi n - \frac{\kappa}{2y}\right) W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny) - \left(4\pi n - \frac{\kappa}{y}\right) W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny)\\ &- \frac{1}{4y}\left(4(s-1/2)^2 - (1-\kappa)^2\right) W_{\frac{\kappa}{2}-1,s-\frac{1}{2}}(4\pi ny)\\ &= \left(\frac{\kappa}{2y} - 2\pi n\right) W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny)\\ &+ \frac{1}{y}(s-\kappa/2)(1-\kappa/2-s) W_{\frac{\kappa}{2}-1,s-\frac{1}{2}}(4\pi ny).\end{aligned}$$

Hence,

$$\xi_\kappa(f_n(x,y)) = (s-\kappa/2)(1-\kappa/2-s)\overline{a_{\varphi,\mathfrak{a}}(n)}(4\pi|n|)^{1-\kappa}(4\pi|n|y)^{\frac{\kappa}{2}-1} W_{-\frac{2-\kappa}{2},s-\frac{1}{2}}(4\pi ny)e(-nx).$$

□

It is a crucial point in applications that the spaces $S_{\kappa,N}$ are finite-dimensional. We will prove that the same is true for the spaces $U_{\kappa,N}(s)$. The next lemma gives a (crude) bound on the Fourier coefficients of a Maass cusp form, which will suffice to show that each Maass cusp form in $U_{\kappa,N}(s)$ is determined by finitely many of its Fourier coefficients (and hence the space is finite-dimensional).

LEMMA 3.10. *Let $\varphi \in U_{\kappa,N}(s)$ and let $a_{\varphi,\mathfrak{a}}(n)$ be the n-th Fourier coefficient around a cusp $\mathfrak{a}$ of $\Gamma_0(N) \setminus \mathbf{H}$. The Fourier coefficients are bounded by*

$$a_{\varphi,\mathfrak{a}}(n) \ll_s \langle \varphi, \varphi \rangle \cdot n^{\frac{\kappa}{2}} \text{ as } n \to \pm\infty.$$

PROOF. Let $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbf{Z})$ be such that $\sigma_{\mathfrak{a}}.i\infty = \mathfrak{a}$. Consider the integral

$$\int_Y^\infty \int_{-1}^1 |(\varphi|_\kappa \sigma_{\mathfrak{a}})(z)|^2 y^\kappa d\mu(z)$$

for any $Y > 0$ over the rectangle $R_Y = \{x + iy : -1 < x < 1,\ y > Y\}$. Let $\mathcal{F}$ be the standard fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$. A standard argument shows that

$$\{g \in \Gamma_0(1) : g.\mathcal{F} \cap R_Y \neq \emptyset\} \ll \frac{1}{Y},$$

hence also

$$\int_Y^\infty \int_{-1}^1 |(\varphi|_\kappa \sigma_{\mathfrak{a}})(z)|^2 y^\kappa d\mu(z) \ll \frac{1}{Y}\int_{\mathcal{F}} |(\varphi|_\kappa \sigma_{\mathfrak{a}})(z)|^2 y^k d\mu(z) \le \frac{1}{Y}\langle \varphi, \varphi \rangle. \tag{3.3}$$

Inserting the Fourier expansion of $\varphi|_\kappa\sigma_{\mathfrak{a}}$ on the left hand side of (3.3), we obtain

$$\begin{aligned}\int_Y^\infty\int_0^1 |(\varphi|_\kappa\sigma_{\mathfrak{a}})(z)|^2 y^\kappa \frac{dxdy}{y^2} &= \sum_{m\neq 0}(4\pi|m|)^{-\kappa}|a_{\varphi,\mathfrak{a}}(m)|^2\int_Y^\infty |W_{\operatorname{sign}(m)\frac{k}{2},s-\frac{1}{2}}(4\pi|m|y)|^2\frac{dy}{y^2}\\ &\geq (4\pi|n|)^{-\kappa+1}\,|a_{\varphi,\mathfrak{a}}(n)|^2\int_{4\pi|n|Y}^\infty |W_{\operatorname{sign}(n)\frac{k}{2},s-\frac{1}{2}}(y)|^2\frac{dy}{y^2}\end{aligned}$$

for any $n\neq 0$. Picking $Y=\frac{1}{|n|}$ proves the claim. □

Proposition 3.11. *The eigenspaces $U_{\kappa,N}(s)$ of Maass cusp forms with spectral parameter $s\in\mathbf{C}$ are finite dimensional.*

Proof. We argue by contradiction: Suppose that there is an $n_0\geq 1$ such that for all $\varphi\in U_{\kappa,N}(s)$ the Fourier coefficients $a_{\varphi,\mathfrak{a}}(n)=0$ for all $|n|\leq n_0$. If this leads to a contradiction, then each Maass cusp form is determined by a finite number of Fourier coefficients and hence the space $U_{\kappa,N}(s)$ is finite-dimensional.

A fundamental domain $\mathcal{F}_N$ for $\Gamma_0(N)$ is given by $\mathcal{F}_N=\bigcup_{\mathfrak{a}}\sigma_{\mathfrak{a}}.\mathcal{F}$, where $\mathcal{F}$ is the standard fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$ and the union is disjoint, running over all inequivalent cusps $\mathfrak{a}$ of $\Gamma_0(N)\setminus\mathbf{H}$.

Hence, we have

$$\begin{aligned}\langle\varphi,\varphi\rangle^2 &= \sum_{\mathfrak{a}}\int_{\mathcal{F}}|\varphi|_\kappa\sigma_{\mathfrak{a}}|^2 y^k d\mu(z)\\ &\leq \sum_{\mathfrak{a}}\sum_{|n|>n_0}(4\pi|n|)^{-\kappa}|a_{\varphi,\mathfrak{a}}(n)|^2\int_{\frac{\sqrt{3}}{2}}^\infty W_{\operatorname{sign}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi|n|y)\frac{dy}{y^2},\end{aligned}$$

and using Lemma 3.10 and that the Whittaker W-function is of exponential decay, we get

$$\langle\varphi,\varphi\rangle \ll_{N,k,s} \langle\varphi,\varphi\rangle e^{-n_0},$$

which leads to a contradiction for n_0 big enough. □

In Lemma 3.10 there is an implicit dependence on the spectral parameter s, hence the proof of Proposition 3.11 only shows that $U_{\kappa,N}(s)$ is finite-dimensional for a fixed spectral parameter $s\in\mathbf{C}$. It is easy to see from the self-adjointness of Δ_κ that the eigenspaces are orthogonal to each other. The space of Maass cusp forms $U_{\kappa,N}=\bigoplus_{s\in\mathbf{C}}U_{\kappa,N}(s)$ is, in fact, infinite-dimensional, but can be attacked one spectral parameter at a time.

2. Hecke-Theory

As with holomorphic cusp forms, there is a Hecke-theory on $U_{\kappa,N}(s)$ for all $s \in \mathbf{C}$. The realization of such a theory is a straightforward generalization from the theory of holomorphic cusp forms, as holomorphy does (almost) not play any role in Hecke-theory.

2.1. Hecke-Theory for Maass Cusp Forms of Even Integral Weight. Let $k \in \mathbf{Z}$ and $N \geq 1$. For a matrix $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$, we define the $|_{2k}$-operator on a function $f : \mathbf{H} \to \mathbf{C}$ as $f|_{2k}\alpha = \det(\alpha)^k j(\alpha, z)^{-2k} f(\alpha.z)$. For $n \geq 1$, we define the n-th Hecke operator $T(n)$ for level N by

$$f|_{2k}T(n) = \frac{1}{n} \sum_{\substack{ad=n,\ a\geq 1,\\ b(d),\ (a,N)=1}} f|_{2k}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \tag{3.4}$$

where "$b(d)$" in the sum is shorthand for summing over all $b \pmod d$.

The Hecke operators are designed so that the space $U_{2k,N}(s)$ admits a simultaneous eigenbasis of $\{T(n) : (n, N) = 1\}$ and so that the eigenvalues of $T(n)$ are given by the Fourier coefficients of the eigenform. They thus realize the Fourier coefficients of cusp forms as the spectrum of a family of operators.

The matrices $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n,\ a \geq 1,\ (a, N) = 1,\ b \pmod d\right\}$ in (3.4) are a set of representatives α_i for the double-coset decomposition $\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_0(N) = \bigcup_i \Gamma_0(N)\alpha_i$, where the right hand side is a disjoint union. If $\varphi \in U_{2k,N}(s)$, we must thus have $\varphi|_{2k}T(n)|_{2k}g = \varphi|_{2k}T(n)$ for all $g \in \Gamma_0(N)$.

It is easy to see as in Corollary 3.4 that the $T(n)$ and the Laplacian Δ_{2k} commute. Some elementary number-theoretic considerations show that the Fourier expansion under the action of $T(n)$ of $\varphi \in U_{2k,N}(s)$ is equal to

$$\varphi|_{2k}T(n) = \sum_{m\neq 0} \sum_{\substack{d|(m,n),\ d>0\\ (d,N)=1}} d^{-1} a_\varphi\left(\frac{mn}{d^2}\right)(4\pi|m|y)^{-k} W_{\mathrm{sign}(m)k, s-\frac{1}{2}}(4\pi|m|y)e(mx) \tag{3.5}$$

at the cusp $i\infty$ and similarly at the other cusps. In particular, $T(n)$ is a well-defined operator on $U_{2k,N}(s)$.

Proposition 3.12. *If $\varphi \in U_{k,N}(s)$ is an eigenfunction of $T(n)$ for $n \geq 1$, i.e. if there is a $\lambda_n \in \mathbf{C}$ such that $\varphi|_k T(n) = \lambda_n \varphi$, then $a_\varphi(n) = \lambda_n a_\varphi(1)$ and $a_\varphi(-n) = \lambda_n a_\varphi(-1)$.*

PROOF. The first positive and negative Fourier coefficients of $\varphi|_k T(n)$ are equal to

$$\lambda_n a_\varphi(1) = \sum_{\substack{d|(1,n), d>0,\\ (d,N)=1}} d^{-1} a\left(\frac{n}{d^2}\right) = a_\varphi(n)$$

and

$$\lambda_n a_\varphi(-1) = \sum_{\substack{d|(-1,n), d>0,\\ (d,N)=1}} d^{-1} a\left(-\frac{n}{d^2}\right) = a_\varphi(-n)$$

by (3.5). $\square$

We see from (3.5) that the Hecke operators satisfy

$$T(m)T(n) = \sum_{\substack{d|(m,n),\\ (d,N)=1}} T\left(\frac{mn}{d^2}\right), \; m,n \geq 1, \tag{3.6}$$

and hence commute. As in the holomorphic case [**CS17**], the operators $\{T(n) : (n,N) = 1\}$ are self-adjoint on $U_{2k,N}(s)$ with respect to the Petersson inner product. Thus, by the Spectral Theorem, the space $U_{2k,N}(s)$ admits an orthogonal basis of eigenforms of $\{T(n) : (n,N) = 1\}$.

2.1.1. *Even and Odd Maass Cusp Forms.* There is a second splitting of the Laplacian Δ_{2k} into raising and lowering operators instead of the ξ_{2k}-operators as in Corollary 3.4. These operators are defined by

$$L_{2k} = 2iy^2\partial_{\overline{z}} \text{ and } R_{2k} = \frac{2k}{y} + 2i\partial_z, \tag{3.7}$$

where L_{2k} is the lowering operator and R_{2k} is the raising operator.

PROPOSITION 3.13. *Let $f : \mathbf{H} \to \mathbf{C}$ be any real-analytic function. For any $g \in \mathrm{SL}_2(\mathbf{Z})$, we have $(L_{2k}f)|_{2k-2}g = L_{2k}(f|_{2k}g)$ and $(R_{2k}f)|_{2k+2}g = R_{2k}(f|_{2k}g)$.*

PROOF. This is a calculation. The first claim is an application of the chain-rule:

$$\begin{aligned}(L_{2k}f)|_{2k-2}g &= 2ij(g,z)^{2-2k}\mathrm{Im}(g.z)^2(\partial_{\overline{z}}f)(g.z)\\ &= 2ij(g,z)^{-2k}\frac{y^2}{j(g,\overline{z})^2}(\partial_{\overline{z}}f)(g.z)\\ &= 2iy^2\partial_{\overline{z}}(f|_{2k}g),\end{aligned}$$

The second claim follows from

$$\begin{aligned}
2i\partial_z f|_{2k} g &= -4kij(g,z)^{-2k-1} f(g.z)\partial_z j(g,z) + 2ij(g,z)^{-2k}\partial_z f(g.z)\\
&= -2kj(g,z)^{-2k-1} f(g.z)\frac{j(g,z)-j(g,\overline{z})}{y} + 2ij(g,z)^{-2k}\partial_z f(g.z)\\
&= -\frac{2k}{y} j(g,z)^{-2k} f(g.z) + \frac{2k}{\mathrm{Im}(g.z)} j(g,z)^{-2k-2} f(g.z) + 2ij(g,z)^{-2-2k}(\partial_z f)(g.z).
\end{aligned}$$

□

PROPOSITION 3.14. $\Delta_{2k} = L_{2k+2}R_{2k} + 2k = R_{2k-2}L_{2k}$.

PROOF. We use that $\partial_z\partial_{\overline{z}} = \partial_{\overline{z}}\partial_z = \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right)$. Then

$$\begin{aligned}
L_{2k+2}R_{2k} f &= 2iy^2\partial_{\overline{z}}\left(\frac{2k}{y} f(z) + 2i\partial_z f(z)\right)\\
&= -2kf(z) + 4kiy\partial_{\overline{z}} f(z) - 4y^2\partial_{\overline{z}}\partial_z f(z)\\
&= -2kf(z) + \Delta_{2k} f(z)
\end{aligned}$$

and

$$\begin{aligned}
R_{2k-2}L_{2k} f(z) &= \frac{2k-2}{y} 2iy^2\partial_{\overline{z}} f(z) - 4\partial_z y^2\partial_{\overline{z}} f(z)\\
&= 4kiy\partial_{\overline{z}} f(z) - 4iy\partial_{\overline{z}} f(z) + 4i\partial_{\overline{z}} f(z) - 4y^2\partial_z\partial_{\overline{z}} f(z)\\
&= \Delta_{2k} f(z).
\end{aligned}$$

□

Propositions 3.13 and 3.14 give an alternative proof of Corollary 3.4. Moreover, we see from Proposition 3.14 that for $\varphi \in U_{2k,N}(s)$, it follows that $\Delta_{2k+2}R_{2k}\varphi = (s-(k+1))(1-(k+1)-s)R_{2k}\varphi$ and $\Delta_{2k-2}L_{2k}\varphi = (s-(k-1))(1-(k-1)-s)L_{2k}\varphi$.

PROPOSITION 3.15. *Let $\varphi \in U_{2k,N}(s)$ and let $\mathfrak{a}$ be a cusp of $\Gamma_0(N)\setminus\mathbf{H}$. Then*

(1) $a_{L_{2k}\varphi,\mathfrak{a}}(n) = \begin{cases} (4\pi|n|)^{-1} a_{\varphi,\mathfrak{a}}(n), & \text{for } n < 0,\\ (s-k)(1-s-k)(4\pi|n|)^{-1} a_{\varphi,\mathfrak{a}}(n), & \text{for } n > 0; \end{cases}$

(2) *and* $a_{R_{2k}\varphi,\mathfrak{a}}(n) = \begin{cases} (s-k)(1-s-k)(4\pi|n|)\cdot a_{\varphi,\mathfrak{a}}(n), & \text{for } n < 0,\\ (4\pi|n|)\cdot a_{\varphi,\mathfrak{a}}(n), & \text{for } n > 0. \end{cases}$

PROOF. The proof of the first claim is very similar to the proof of Proposition 3.9. By Proposition 3.14, we have

$$(s-k)(1-s-k)a_{\varphi,\mathfrak{a}}(n) = a_{L_{2k+2}R_{2k}\varphi,\mathfrak{a}}(n)$$

and hence the second claim follows from the first claim. □

Propositions 3.13, 3.14, and 3.15 prove that $L_{2k}\varphi \in U_{2k-2,N}(s)$ and $R_{2k}\varphi \in U_{2k+2,N}(s)$ for any $\varphi \in U_{2k,N}(s)$.

Given a Maass cusp form $\varphi \in U_{2k,N}(s)$, it is easy to see that $y^{2k}\varphi\left(-\overline{z}\right)$ is a Maass cusp form in $U_{-2k,N}(s)$. Combinning this with a repeated application of raising resp. lowering operators gives a self-adjoint operator on $U_{2k,N}(s)$, whose eigenspaces split the spaces of Maass cusp forms of even integral weight into two orthognal subspaces of even and odd Maass cusp forms.

Proposition 3.16. *Let $\varphi \in U_{2k,N}(s)$ with $s \neq k, 1-k$. The map*

$$(X_{s,2k}\varphi)(z) = \frac{\Gamma(s-k)}{\Gamma(s+k)} \cdot \begin{cases} y^{-2k}(L_{2-2k} \circ L_{4-2k} \circ \ldots \circ L_{2k}\varphi)\left(-\overline{z}\right) & \text{for } k > 0, \\ y^{-2k}(R_{-2k-2} \circ R_{-2k-4} \circ \cdots \circ R_{2k}\varphi)\left(-\overline{z}\right) & \text{for } k \leq 0, \end{cases}$$

is a self-adjoint involution on the space $U_{2k,N}(s)$ and commutes with $\{T(n) : (n,N) = 1\}$ and Δ_{2k}.

Proof. As before, we can check that $y^{2k-2}(L_{2k}\varphi)(-\overline{z}) = -R_{-2k}y^{2k}\varphi(-\overline{z})$. Suppose that $k > 0$, so $(X_{s,2k}\varphi)(z) = (-1)^k \frac{\Gamma(s-k)}{\Gamma(s+k)} R_{2k-2} \circ R_{2k-4} \circ \cdots \circ R_{-2k}y^{2k}\varphi\left(-\overline{z}\right)$. Then

$$X_{s,2k}^2\varphi(z) = \left(\frac{\Gamma(s-k)}{\Gamma(s+k)}\right)^2 R_{2k-2} \circ R_{2k-4} \circ \cdots \circ R_{-2k} \circ L_{2-2k} \circ L_{4-2k} \circ \ldots \circ L_{2k}\varphi(z)$$

and successively applying Proposition 3.14 shows that $X_{s,2k}^2\varphi = \varphi$. The proof for $k < 0$ is the same.

The self-adjointness of $X_{s,2k}$ follows from Stokes' theorem. To see that $X_{2s,k}$ commutes with the Hecke operators, we may argue as in Proposition 3.13. □

Since $X_{s,2k}$ in Proposition 3.16 is an involution, all eigenforms of $X_{s,2k}$ have eigenvalue ± 1. We call all those eigenforms with eigenvalue $+1$ even Maass cusp forms and those with eigenvalue -1 odd Maass cusp forms. Since the Hecke operators $\{T(n) : (n,N) = 1\}$ commute with $X_{s,2k}$, the spectral theorem asserts that there is an orthogonal basis of $U_{2k,N}(s)$ of eigenfunctions of $\{T(n) : (n,N) = 1\}$ and $X_{s,2k}$.

One difficulty that arises when working with Maass cusp forms lies in the fact that their negative Fourier coefficients might not vanish, so a priori we need to treat them separately. However, for an odd or even cusp form, its negative Fourier coefficients are determined by their positive counterparts.

PROPOSITION 3.17. *Let $\varphi \in U_{2k,N}(s)$ be such that $X_{s,2k}\varphi = \pm\varphi$ and let $\mathfrak{a}$ be any cusp of $\Gamma_0(N) \setminus \mathbf{H}$. We have $a_{\varphi,\mathfrak{a}}(-n) = \pm\frac{\Gamma(s+k)}{\Gamma(s-k)} a_{\varphi,\mathfrak{a}}(n)$ for $n > 0$.*

PROOF. Let $k \geq 0$ and $n > 0$. By Proposition 3.15 and the assumption $X_{s,2k}\varphi = \pm\varphi$, comparing the Fourier expansions of $X_{s,2k}\varphi$ and $\pm\varphi$ yields

$$\begin{aligned}
&\frac{\Gamma(s-k)}{\Gamma(s+k)} y^{-2k} a_{L_{2-2k}\circ L_{4-2k}\circ\cdots\circ L_{2k}\varphi,\mathfrak{a}}(-n)(4\pi|n|y)^k W_{\mathrm{sign}(-n)(-k),s-\frac{1}{2}}(4\pi|n|y)e(nx)\\
&= a_{\varphi,\mathfrak{a}}(-n)(4\pi|n|y)^{-k} W_{k,s-\frac{1}{2}}(4\pi|n|y)e(nx)\\
&= \pm a_{\varphi,\mathfrak{a}}(n)(4\pi|n|y)^{-k} W_{k,s-\frac{1}{2}}(4\pi|n|y)e(nx).
\end{aligned}$$

The proof for $k < 0$ is similar. □

2.1.2. *Atkin-Lehner-theory.* For $M|N$ and a cusp form $\varphi \in U_{2k,N/M}(s)$, we have $\varphi(Mz) \in U_{2k,N}(s)$, which follows from $M\begin{pmatrix} a & b \\ cN & d\end{pmatrix}.z = \begin{pmatrix} a & bM \\ cN/M & d\end{pmatrix}.(Mz)$. Linear combinations of such Maass cusp forms arising from lower ("older") levels are said to be laying in the "old" space, which is defined to be

$$U^{\mathrm{old}}_{2k,N}(s) = \mathrm{span}\{\varphi(Mz) : M|N,\ M \neq 1\ \varphi \in U_{2k,N/M}(s)\}.$$

Its orthogonal complement in $U_{2k,N}(s)$ will be called the "new" space and denoted by $U^{\mathrm{new}}_{\kappa,N}(s)$.

The spaces $U^{\mathrm{old}}_{\kappa,N}(s)$ and $U^{\mathrm{new}}_{\kappa,N}(s)$ are stable under the action of the Hecke operators and the operator $X_{s,2k}$, so there exists an orthogonal basis of eigenforms on both spaces. The elements in these bases are called old- and newforms.

For any $\varphi \in U^{\mathrm{old}}_{\kappa,N}(s)$, we will always have $a_\varphi(1) = 0$ by construction. In fact, this is an equivalence for oldforms.

THEOREM 3.18. *Let $\varphi \in U_{2k,N}(s)$ be an odd or even Hecke eigenform for all $T(n)$ with $(n, N) = 1$. If $a_\varphi(1) = 0$, then $\varphi(z)$ is an oldform.*

PROOF. Since φ is odd or even, we also have $a_\varphi(-1) = 0$ by Proposition 3.17. By Proposition 3.5, we must thus have $a_\varphi(n) = 0$ for $(n, N) = 1$. The argument is now the same as in the holomorphic case [**Miy06**, Thm. 4.6.8]. □

The dependence on the first Fourier coefficients in Theorem 3.18 is crucial: Assume, e.g., that φ_1, φ_2 are both newforms of weight $2k$ and level N and furthermore have the same eigenvalues for the operators $\{T(n) : (n, N) = 1\}$ and $X_{s,2k}$. Pick $\alpha \in \mathbf{C}$ such that the first Fourier coefficient of $\alpha\varphi_1 - \varphi_2$ vanishes. By Theorem 3.18, the cusp form

$\alpha\varphi_1 - \varphi_2$ is simultaneously an oldform and a newform, so it must vanish. This property is usually called the "multiplicity one" property of newforms: If two newforms have the same eigenvalues for the operators $\{T(n) : (n,N) = 1\}$ and $X_{s,2k}$, they are equal up to multiplication by a scalar. The multiplicity one property also holds if the eigenvalues of $\{T(p) : p \nmid N\}$ of two newforms are equal, since the eigenvalues of $\{T(n) : (n,N) = 1\}$ are completely determined by the eigenvalues of $\{T(p) : p \nmid N\}$.

Let us give an application of the multiplicity one property of newforms. Recall the Atkin-Lehner involutions W_m for $m \| N$ defined in (2.11). From (2.12) it follows that the W_m are operators the space $U_{2k,N}(s)$ and from (2.13) we see that W_m is an involution.

PROPOSITION 3.19. *The Atkin-Lehner involutions commute with the Hecke operators $T(n)$ for $(n,N) = 1$ and the operator $X_{s,2k}$.*

PROOF. The proof that the Hecke operators $\{T(n) : (n,N) = 1\}$ commute with W_m for $m \| N$ is the same as in the holomorphic case [**CS17**, Prop. 13.2.6.b)]. To show that W_m commutes with $X_{s,2k}$, we argue as in Proposition 3.13. □

COROLLARY 3.20. *Each newform $\varphi \in U_{2k,N}(s)$ is an eigenfunction for the operators W_m with $m \| N$.*

PROOF. By Proposition 3.19 the newforms φ and $\varphi|_{2k} W_m$ have the same eigenvalues for the operators $\{T(n) : (n,N) = 1\}$ and $X_{s,2k}$. Hence, by the multiplicity one property of newforms, there is a $\mu_m \in \{\pm 1\}$ such that $\mu_m \varphi = \varphi|_{2k} W_m$, i.e. φ is an eigenfunction of the involution W_m. □

2.2. Hecke-Theory for Maass Cusp Forms of Half-Integral Weight. Let $k \in \mathbf{Z}$ and $N \geq 1$. For half-integral weight Maass cusp forms ψ of weight $k + \frac{1}{2}$ and level $4N$, there is a similar theory of Hecke operators as for $U_{2k,N}(s)$. However, instead of constructing them via double-cosets of $\Gamma_0(4N)$, we need to pass to the double-cover of the group $\Gamma_0(4N)$; see [**Shim73**, Sec. 1]. Explicitly, the Hecke operator $T(p^2)$ for a prime number p is defined by

$$\psi|_{k+\frac{1}{2}} T(p^2) = p^{k+\frac{1}{2}} \sum_{b(p^2)} \psi\left(\frac{z+b}{p^2}\right) + \sum_{r(p)} \varepsilon_p^{-2k-1} \left(\frac{-r}{p}\right) \psi\left(z + \frac{r}{p}\right) + p^{-k-\frac{1}{2}} \psi(p^2 z). \tag{3.8}$$

The operator $T(p)$ would be the zero operator [**Shim73**, Prop. 1.0, (1.14)]. The Hecke operators $\{T(p^2) : (p,4N) = 1\}$ again form a set of commutative self-adjoint operators on $U_{k+\frac{1}{2},4N}(s)$, so the space $U_{k+\frac{1}{2},4N}(s)$ admits an basis of eigenforms of $\{T(p^2) : (p,4N) = 1\}$.

2.2.1. *The Kohnen Plusspace.* Let $N \geq 1$ be odd. Consider the operator

$$\psi|_{k+\frac{1}{2}} L_{k+\frac{1}{2},4N} = \frac{2^{k-1}(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(1+i)^{2k+1}} \sum_{\nu(4)} \psi\left(z+\frac{1}{4}\right)|_{k+\frac{1}{4}} \begin{pmatrix} 1 & 0 \\ 4Nr & 1 \end{pmatrix}. \tag{3.9}$$

The operator $L_{k+\frac{1}{2},4N}$ is a self-adjoint operator on $U_{k+\frac{1}{2},4N}(s)$ with eigenvalues 1 and $-\frac{1}{2}$ [**Koh82**, Prop. 1], which commutes with the Hecke operators $\{T(p^2) : (p,N) = 1\}$ [**Koh82**, Sec. 3].

The eigenspace of $L_{k+\frac{1}{2},4N}$ of eigenvalue 1 is the so-called Kohnen plusspace

$$U^+_{k+\frac{1}{2},4N}(s) = \left\{\psi \in U_{k+\frac{1}{2},4N}(s) : b_\psi(n) \neq 0 \text{ only if } (-1)^k n \equiv 0, 1 \pmod 4\right\}. \tag{3.10}$$

Since the Hecke operators commute with the operator $L_{k+\frac{1}{2},4N}$, the space $U^+_{k+\frac{1}{2},4N}(s)$ admits a basis of Hecke eigenforms.

The action of projection operator $\mathrm{pr}^+_{k+\frac{1}{2},4N}$ onto the space $U^+_{k+\frac{1}{2},4N}(s)$ can be given explicitly in terms of the Fourier expansion of a Maass cusp form φ.

PROPOSITION 3.21. *Let $\alpha = \left(\frac{-4}{N}\right)$. For a cusp $\mathfrak{a}$ of $\Gamma_0(N) \setminus \mathbf{H}$, let $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbf{Z})$ be such that $\sigma_\mathfrak{a}.i\infty = \mathfrak{a}$. For $\psi \in U_{k+\frac{1}{2},4N}(s)$ write*

$$\begin{aligned}
\psi(z) &= \sum_{n\in\mathbf{Z}} b_\varphi(n)\ (4\pi|n|y)^{-\frac{2k+1}{4}} W_{\mathrm{sign}(n)\frac{2k+1}{4},s-\frac{1}{2}}(4\pi|n|y)e(nx),\\
\psi|_{k+\frac{1}{2}}\sigma_{-\frac{\alpha}{N}} &= \sum_{n\neq 0} b_{\psi,-\alpha/N}(n)\ (\pi|n|y)^{-\frac{2k+1}{4}} W_{\mathrm{sign}(n)\frac{2k+1}{4},s-\frac{1}{2}}(\pi|n|y)e(nx/4),\\
\varphi|_{k+\frac{1}{2}}\sigma_{\frac{1}{2N}} &= \sum_{(-1)^{\frac{k}{2}}n\equiv 1(4)} a_{\psi,1/2N}(n)\ (\pi|n|y)^{-\frac{2k+1}{4}} W_{\mathrm{sign}(n)\frac{2k+1}{4},s-\frac{1}{2}}(\pi|n|y)e(nx/4)
\end{aligned}$$

for its Fourier expansions around the cusps $i\infty$, $-\frac{\alpha}{N}$, and $\frac{1}{2N}$. Then the Fourier coefficients of $\psi|\mathrm{pr}^+_{k+\frac{1}{2},4N}$ around the cusp $i\infty$ are given by

$$\frac{2}{3}\left(b_\psi(n) + 2^{2k-1}(1-(-1)^k i)i^{\frac{n}{4}} b_{\psi,-\alpha/N}(n/4)\right), \text{ if } n \equiv 0 \pmod 4$$

and

$$\frac{2}{3}\left(b_\psi(n) + 2^{k-1}\left(\frac{(-1)^{\frac{k}{2}}n}{2}\right) b_{\psi,1/2N}(n)\right), \text{ if } (-1)^k n \equiv 1 \pmod 4.$$

PROOF. This is a straightforward adaptation of [**Koh85**, Prop. 3]. $\square$

CHAPTER 4

Parabolic Poincaré Series

It is a difficult problem to construct non-holomorphic Maass cusp forms, but we know from the Weyl law that the Laplace operator has infinitely many eigenvalues. The space of holomorphic cusp forms $S_{2k,N}$ with $k \in \mathbf{Z}$ and $N \geq 1$ is spanned by the parabolic Poincaré series, which are realized by averaging a holomorphic Γ_∞-invariant test-function over the cosets $\Gamma_\infty \setminus \Gamma$. To find a Maass cusp form in $U_{2k,N}(s)$, we might thus try to average a Γ_∞-invariant real-analytic test-function, which has eigenvalue $(s-k)(1-s-k)$ under Δ_{2k}, over the cosets $\Gamma_\infty \setminus \Gamma$. There are two natural choices for these test-functions, coming from the Fourier expansion of a Maass cusp form, namely

$$y^{s-k} \text{ and } y^{-k} W_{\operatorname{sign}(m)k, s-\frac{1}{2}}(4\pi|m|y)e(mx)$$

for $m \neq 0$. The first leads to non-holomorphic Eisenstein series, which are Maass forms, but not cuspidal. The second choice of test-function runs into convergence issues, since

$$y^{-k} W_{\operatorname{sign}(m)k, s-\frac{1}{2}}(4\pi|m|y)e(mx) = \frac{\Gamma(2s-1)}{\Gamma(s-k)} y^{1-k-s} + O(y^{\frac{3}{2}-k-s}), \text{ as } y \to 0. \tag{4.1}$$

So the only region of convergence of a Poincaré series built out of this test-function would be $\operatorname{Re}(s) < 0$, which cannot be a spectral parameter of a non-holomorphic Maass cusp form by Proposition 3.6.

Using the eigenfunction $\psi_m(z) = y^{-k} M_{\operatorname{sign}(m)k, s-\frac{1}{2}}(4\pi|m|y)e(mx)$ of Δ_{2k} to build a Poincaré series $F_m^\kappa(z,s) = \sum_{g \in \Gamma_\infty \setminus \Gamma_0(N)} \psi_m(z)|_{2k} g$ instead, we get absolute convergence for $\operatorname{Re}(s) > 1$, since the limiting form of the M-Whittaker function is

$$M_{\operatorname{sign}(m)k, s-\frac{1}{2}}(y) = O(y^s) \text{ as } y \to 0. \tag{4.2}$$

Poincaré series of this kind are called *Niebur Poincaré series.* However, they do not lie in $L^2(\Gamma_0(N) \setminus \mathbf{H}, \kappa)$ and are hence not Maass cusp forms. We might nevertheless expect that at spectral parameter $s \in \mathbf{C}$, they give rise to Maass cusp forms.

The test-function of the non-holomorphic Eisenstein series can be tweaked by considering $y^{s-k}e(mz)$ for $m > 0$ instead. The *Selberg Poincaré series* formed out of the test-function $y^{s-k}e(mz)$ is then square-integrable, but no longer an eigenfunction of Δ_{2k}.

Nevertheless, due to the spectral expansion of functions in $L^2(\Gamma_0(N)\setminus\mathbf{H},2k)$, we can recover the Maass cusp forms in $U_{2k,N}(s)$ if $s\in\mathbf{C}$ is a spectral parameter.

Both the Niebur and the Selberg Poincaré series are difficult to handle in some settings and easier to handle in others. The Niebur Poincaré series has a simpler Fourier expansion compared to the Selberg Poincaré series. However, the Selberg Poincaré series does not run into convergence issues when integrating it over a non-compact geodesic. It is hence more suited to study the cycle integrals of Maass cusp forms than the Niebur Poincaré series for square discriminants.

1. Niebur Poincaré Series

Let $\kappa\in\frac{1}{2}\mathbf{Z}$ and $N\geq 1$, divisible by 4 if $\kappa\notin\mathbf{Z}$. The Poincaré series of Niebur type for $m\neq 0$ is defined as

$$F_m^\kappa(z,s)=-\frac{\Gamma\left(s-\operatorname{sign}(m)\frac{\kappa}{2}\right)}{4\pi|m|\Gamma(2s)}\sum_{g\in\Gamma_\infty\setminus\Gamma_0(N)}y^{-\kappa/2}M_{\operatorname{sign}(m)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|m|y)e(mx)|_\kappa g \tag{4.3}$$

for $\operatorname{Re}(s)>1$. The Poincaré series $F_m^\kappa(z,s)$ is absolutely convergent for $\operatorname{Re}(s)>1$ by (4.2) and hence real-analytic. By Corollary 3.4 it is an eigenfunction of Δ_κ with eigenvalue $(s-\kappa/2)(1-\kappa/2-s)$.

The $F_m^\kappa(z,s)$ can be analytically continued in s to the whole complex plane with poles at the spectral parameters of Maass cusp forms of weight κ and level N. Their residues at the poles are equal to Maass cusp forms.

Proposition 4.1. *Let $m\neq 0$, $\kappa\in\frac{1}{2}\mathbf{Z}$, $N\geq 1$ (divisible by 4 if $\kappa\notin\mathbf{Z}$) and $s_0\neq\frac{\kappa}{2},1-\frac{\kappa}{2}$ be the spectral parameter of a Maass cusp form of weight κ and level N. The Niebur Poincaré series $F_m^\kappa(z,s)$ has a pole at s_0 with residue given by*

$$\operatorname{Res}_{s=s_0}(2s-1)F_m^\kappa(z,s)=-(4\pi|m|)^{-\kappa/2}\sum_\varphi\frac{\overline{a_\varphi(m)}}{\langle\varphi,\varphi\rangle}\varphi(z),$$

where the sum in φ runs over an orthogonal basis of Hecke eigenforms of $U_{\kappa,N}(s_0)$.

Proof. This result can be found in J. Fay's paper [**Fay77**, Sec. 3]. Our series $F_m^\kappa(z,s)$ in (4.3) corresponds to $y^{\kappa/2}\overline{F_m(z,\overline{s})}$ (with $k=\kappa/2$) in J. Fay's notation. $\square$

By comparing Fourier coefficients in Proposition 4.1 we can get to the Fourier coefficients of Maass cusp forms. In the next proposition, we will calculate the Fourier expansion of $F_m^\kappa(z,s)$.

PROPOSITION 4.2. *Let m be a nonzero integer and $\mathrm{Re}(s) > 1$. We have*

$$\begin{aligned}F_m^\kappa(z,s) = &-\frac{\Gamma\left(s-\operatorname{sign}(m)\frac{\kappa}{2}\right)}{4\pi|m|\Gamma(2s)}y^{-\frac{\kappa}{2}}M_{\operatorname{sign}(m)\frac{\kappa}{2},s-\frac12}(4\pi|m|y)e(mx)\\ &- e^{-\pi i\kappa/2}\frac{m^s\Gamma(2s-1)}{|m|\Gamma(s+\operatorname{sign}(m)\kappa/2)}L^\kappa_{m,0}(s)y^{1-s-\kappa/2}\\ &-\sum_{n\in\mathbf{Z}}\frac{e^{-\pi i\frac{\kappa}{2}}}{2\sqrt{|mn|}}\frac{\Gamma\left(s-\operatorname{sign}(m)\frac{\kappa}{2}\right)}{\Gamma\left(s+\operatorname{sign}(n)\frac{\kappa}{2}\right)}L^\kappa_{m,n}(s)y^{-\frac{\kappa}{2}}W_{\operatorname{sign}(m)\frac{\kappa}{2},s-\frac12}(4\pi|n|y)e(nx).\end{aligned}$$

The Fourier coefficient $L^\kappa_{m,n}(s)$ is given by the zeta-function

$$L^\kappa_{m,n}(s) = \begin{cases}\sum_{0<c\equiv 0(N)}\frac{K_\kappa(m,n;c)}{c}J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \operatorname{sign}(mn)>0,\\ \sum_{0<c\equiv 0(N)}\frac{K_\kappa(m,n;c)}{c^{2s}} & \text{if } n=0,\\ \sum_{0<c\equiv 0(N)}\frac{K_\kappa(m,n;c)}{c}I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \operatorname{sign}(mn)<0,\end{cases}$$

where

$$K_\kappa(m,n;c) = \begin{cases}\sum_{a(c)^*} e\left(\frac{ma+n\bar a}{c}\right) & \text{if } \kappa\in\mathbf{Z},\\ \sum_{a(c)^*}\varepsilon_a^{2\kappa}\left(\frac{c}{a}\right)^{2\kappa} e\left(\frac{ma+n\bar a}{c}\right) & \text{if } \kappa\in\frac12\mathbf{Z},\end{cases}$$

is the Kloosterman sum, which runs over all a modulo c for which an integer $\overline{a}$ exists such that $a\overline{a}\equiv 1 \pmod c$, and the functions $J_\nu(x)$, $I_\nu(x)$ are the Bessel-functions of the first kind.

PROOF. By Poisson summation we have

$$\begin{aligned}-\frac{4\pi\Gamma(2s)}{\Gamma(s-\operatorname{sign}(m)\kappa/2)}F_m^\kappa(z,s) &= y^{-\kappa/2}M_{\operatorname{sign}(m)\frac{\kappa}{2},s-\frac12}(4\pi|m|y)e(mx)\\ &+\sum_{n\in\mathbf{Z}}\sum_{0<c\equiv 0(N)}y^{1-\kappa/2}K_\kappa(m,n;c)I_\kappa(m,n,c,y)e(nx)\end{aligned}$$

with

$$\begin{aligned}&I_\kappa(m,n,c,y)\\ &=\int_{-\infty}^{\infty}\left(\frac{i-\xi}{i+\xi}\right)^{\kappa/2}M_{\operatorname{sign}(m)\frac{\kappa}{2},s-\frac12}\left(\frac{4\pi|m|}{c^2y(1+\xi^2)}\right)e\left(-\frac{m\xi}{c^2y(1+\xi^2)}-ny\xi\right)d\xi.\end{aligned}\tag{4.4}$$

The integral can be solved by the evaluation

$$\int_{-\infty}^{\infty}\left(\frac{1+i\xi}{1-i\xi}\right)^{\frac{2k+1}{4}} M_{\operatorname{sign}(m)\frac{2k+1}{4},s-\frac{1}{2}}\left(\frac{|\beta|}{(1+\xi^2)}\right) e\left(-\frac{\beta\xi}{\xi^2+1}+\alpha\xi\right)d\xi$$

$$= e^{-\pi i\frac{2k+1}{4}}\Gamma(2s)\cdot\begin{cases}\frac{2\pi\sqrt{\left|\frac{\beta}{\alpha}\right|}}{\Gamma(s-(2k+1)/4)}W_{-\operatorname{sign}(\alpha)\frac{2k+1}{4},s-\frac{1}{2}}(4\pi|\alpha|)J_{2s-1}\left(4\pi\sqrt{|\alpha\beta|}\right), & \text{if } \alpha\beta<0,\\ \pi 2^{2-2s}\frac{\Gamma(2s-1)}{\Gamma(s-(2k+1)/4))\Gamma(s+(2k+1)/4)}\beta^s, & \text{if } \alpha=0,\\ \frac{2\pi\sqrt{\left|\frac{\beta}{\alpha}\right|}}{\Gamma(s+(2k+1)/4)}W_{-\operatorname{sign}(\alpha)\frac{2k+1}{4},s-\frac{1}{2}}(4\pi|\alpha|)I_{2s-1}\left(4\pi\sqrt{\alpha\beta}\right), & \text{if } \alpha\beta>0,\end{cases}$$

which can be found in [**Bru02**, p. 33], setting $\alpha=-ny$ and $\beta=\frac{m}{c^2y}$. □

Even though the Fourier coefficients in Proposition 4.2 are fairly complicated, they are simpler than the ones of the Selberg Poincaré series below.[1] This (relative) simplicity is especially useful in calculations involving the Fourier coefficients of a Poincaré series.

For instance, let us apply the projection operator $\mathrm{pr}^+_{\kappa,N}$ in Proposition 3.21 to $F^\kappa_m(z,s)$. The projected Niebur Poincaré series will again have analytic continuation and its residue at the poles corresponding to spectral parameters of Maass cusp forms will be in terms of the Kohnen plusspace. Let $\kappa=k+\frac{1}{2}$ for $k\in\mathbf{Z}$ and the level be $4N$ with $N\geq 1$ odd for the rest of this section.

PROPOSITION 4.3. *Let $N\geq 1$ be odd, $m\neq 0$ such that $(-1)^k m\equiv 0,1 \pmod 4$, and* $\operatorname{Re}(s)>1$. *We have*

$$\begin{aligned}&F_m^{\frac{k+1}{2}}(z,s)|\mathrm{pr}^+\\ &=-\frac{\Gamma\left(s-\operatorname{sign}(m)\frac{k+1}{4}\right)}{6\pi|m|\Gamma(2s)}y^{-\frac{k+1}{4}}M_{\operatorname{sign}(m)\frac{k+1}{4},s-\frac{1}{2}}(4\pi|m|y)e(mx)+c^+(m,s)y^{1-s-\frac{2k+1}{4}}\\ &-\sum_{\substack{(-1)^{\frac{k}{2}}n\equiv 0,1(4),\\ n\neq 0}}\frac{\sqrt{2}(-1)^{\left\lfloor\frac{k+1}{4}\right\rfloor}\Gamma\left(s-\operatorname{sign}(m)\frac{k+1}{4}\right)}{3\sqrt{|mn|}\Gamma\left(s+\operatorname{sign}(n)\frac{k+1}{4}\right)y^{\frac{k+1}{4}}}L_{m,n}^{\frac{k+1}{2},+}(s)W_{\operatorname{sign}(m)\frac{k+1}{4},s-\frac{1}{2}}(4\pi|n|y)e(nx),\end{aligned}$$

where $c^+(m,s)$ is a function depending on m,s, and the Fourier coefficients

$$L_{m,n}^{\frac{k+1}{2},+}(s)=\begin{cases}\sum_{0<c\equiv 0(4N)}\frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c}J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \textit{if } \operatorname{sign}(mn)>0;\\ \sum_{0<c\equiv 0(4N)}\frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c}I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \textit{if } \operatorname{sign}(mn)<0,\end{cases}$$

[1]We will not calculate the Fourier expansion of the Selberg Poincaré series, as we will not use it, but the Fourier coefficients are essentially infinite series of zeta-functions; compare, e.g., with Selberg's original article [**Sel62**, Eq. 3.21].

where $K^{+}_{\frac{k+1}{2}}(m,n;c)$ *is defined by*

$$K^{+}_{\frac{k+1}{2}}(m,n;c) = (1-(-1)^{\frac{k}{2}}i)\left(1+\left(\frac{4}{c/4}\right)\right)\sum_{d(c)^*}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\frac{k+1}{2}} e\left(\frac{md+n\overline{d}}{c}\right),\ 4|c.$$

Proof. By Proposition 3.21 we need to calculate the Fourier expansions of $F_m^{k+\frac{1}{2}}(z,s)$ at the cusps $-\frac{\alpha}{N}$ and $\frac{1}{2N}$, where $\alpha = \left(\frac{-4}{N}\right)$. It should be observed that $\alpha = 1,-1$ according to $N \equiv 1,3 \pmod N$. Let $\sigma^{-\alpha/N} = \begin{pmatrix} 1 & 0 \\ -\alpha N & 1\end{pmatrix}$ and $\sigma^{1/2N} = \begin{pmatrix} 1 & 0 \\ 2N & 1\end{pmatrix}$.

To make the notation simpler, we write $\nu(c,d) = \left(\frac{c}{d}\right)\varepsilon_d^{-2k-1}$ and

$$\psi_m^{k+\frac{1}{2}}(y) = y^{-k-\frac{1}{2}} M_{\mathrm{sign}(m)\frac{2k+1}{4}, s-\frac{1}{2}}(4\pi|m|y).$$

The function $\nu(c,d)$ is d-periodic in the first variable. We have

$$\begin{aligned}
&-\frac{4\pi\Gamma(2s)}{\Gamma(s-\mathrm{sign}(m)(2k+1)/4)} F_m^{k+\frac{1}{2}}(z,s)|_{k+\frac{1}{2}}\sigma^{-\alpha/N}\\
&= (-\alpha N+1)^{-k-\frac{1}{2}}\psi_m^{k+\frac{1}{2}}(\mathrm{Im}(\sigma^{-\alpha/N}.z))e(m\mathrm{Re}(\sigma^{-\alpha/N}.z))\\
&+\sum_{\substack{c>0,\\ ad-4Nbc=1}} \frac{\nu(4Nc,d)\psi_m^{k+\frac{1}{2}}\left(\frac{y}{|N(4c-\alpha d)z+d|^2}\right)}{(N(4c-\alpha d)z+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{(a-b\alpha N)z+b}{N(4c-\alpha d)z+d}\right)\right),
\end{aligned}\tag{4.5}$$

since the expression $\log\left(4Nc\frac{z}{-\alpha Nz+1}+d\right)+\log(-\alpha Nz+1)-\log(N(4c-\alpha d)z+d)$ is constant in z and evaluates to zero at $z=0$.

Substituting $4c-\alpha d \to c$ and $a-b\alpha N \to c$, we get the following expression for the second sum in (4.5):

$$\begin{aligned}
&\sum_{\substack{c>-\alpha d,\\ c\equiv -\alpha d(4),\\ ad-Nbc=1}} \frac{\nu(N(c+\alpha d),d)\psi_m^{k+\frac{1}{2}}\left(\frac{y}{|4Ncz+d|^2}\right)}{(Ncz+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{az+b}{Ncz+d}\right)\right)\\
&= \sum_{\substack{c>0,\\ c\neq -\alpha d,\\ c\equiv -\alpha d(4),\\ ad-Nbc=1}} \frac{\nu(Nc,d)\psi_m^{k+\frac{1}{2}}\left(\frac{y}{|Ncz+d|^2}\right)}{(Ncz+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{az+b}{Ncz+d}\right)\right),
\end{aligned}$$

since the set $\{(a,b,c,d)\in \mathbf{Z}^4 : c>-\alpha d,\ c\equiv -\alpha d \pmod 4,\ ad-Nbc=1\}$ splits into $\{(a,b,c,d)\in \mathbf{Z}^4 : c>0,\ c>-\alpha d,\ c\equiv -\alpha d \pmod 4,\ ad-Nbc=1\}$ and $\{(a,b,c,d)\in \mathbf{Z}^4 : -c>0,\ c>-\alpha d,\ c\equiv -\alpha d \pmod 4,\ ad-Nbc=1\}$ disjointly and the latter is equal to $\{(a,b,c,d) \in \mathbf{Z}^4 : c > 0,\ c < -\alpha d,\ c \equiv -\alpha d \pmod 4),\ ad - Nbc = 1\}$ after substituting a, b, c, d with $-a, -b, -c, -d$ respectively. In the case that $c = -\alpha d$, we must have $d(a - 4N\alpha) = 1$, so $d = \pm 1$. We hence see that

$$\begin{aligned}
&-\frac{4\pi\Gamma(2s)}{\Gamma(s - \mathrm{sign}(m)(2k+1)/4)} F_m^{k+\frac{1}{2}}(z,s)|_{k+\frac{1}{2}}\sigma^{-\alpha/N} \\
&= \sum_{\substack{c>0,\\ c\equiv -\alpha d(4),\\ ad-Nbc=1}} \frac{\nu(Nc,d)\psi_m^{k+\frac{1}{2}}\left(\frac{y}{|Ncz+d|^2}\right)}{(Ncz+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{az+b}{Ncz+d}\right)\right) \\
&= \sum_{\substack{c>0,\\ c\equiv 1(2),\\ d\equiv -Nc(4),\\ ad-Nbc=1}} \frac{\nu(Nc,d)\psi_m^{k+\frac{1}{2}}\left(\frac{y}{|Ncz+d|^2}\right)}{(Ncz+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{az+b}{Ncz+d}\right)\right),
\end{aligned}$$

where the last step is justified since $\alpha N \equiv 1 \pmod 4$ from the definition of α, so $-d \equiv -N\alpha d \equiv Nc \pmod 4$ and so both d, c must be odd (N is odd).

Thus, by Poisson summation:

$$\begin{aligned}
&-\frac{4\pi\Gamma(2s)}{\Gamma(s - \mathrm{sign}(m)(2k+1)/4)} F_m^{k+\frac{1}{2}}(z,s)|_{k+\frac{1}{2}}\sigma^{-\alpha/N} \\
&= \sum_{\substack{c>0,\\ c\equiv 1(2),\\ d\equiv -Nc(4),\\ ad-Nbc=1}} \frac{\nu(Nc,d)4^{-\frac{2k+1}{4}}\psi_{4m}^{k+\frac{1}{2}}\left(\frac{y/4}{|4Nc\frac{z}{4}+d|^2}\right)}{(4Nc\frac{z}{4}+d)^{k+\frac{1}{2}}} e\left(4m\mathrm{Re}\left(\frac{a\frac{z}{4}+b/4}{4Nc\frac{z}{4}+d}\right)\right) \\
&= \sum_{n\in\mathbf{Z}} \sum_{\substack{c>0,\\ c\equiv 1(2)}} 4^{-\frac{2k+1}{4}} K'_{k+\frac{1}{2}}(4m,n;4Nc)(y/4)^{1-\frac{2k+1}{4}} I_{k+\frac{1}{2}}(4m,n,4Nc,y/4)e(nx/4),
\end{aligned}$$

where the integral $I_{k+\frac{1}{2}}(m,n,c,y)$ is given by (4.4) and

$$K'_{k+\frac{1}{2}}(m,n;4c) = \sum_{\substack{d(4c)^*,\\ d\equiv -c(4)}} \nu(c,d)e\left(\frac{md+n\overline{d}}{4c}\right).$$

The same argument for the Fourier expansion around the cusp $\frac{1}{2N}$ gives

$$
\begin{aligned}
&-\frac{4\pi\Gamma(2s)}{\Gamma(s-\operatorname{sign}(m)(2k+1)/4)} F_m^{k+\frac{1}{2}}(z,s)|_{k+\frac{1}{2}}\sigma^{1/(2N)} \\
&= \sum_{\substack{c>0,\\ c\equiv 1(2),\\ ad-2Nbc=1}} \frac{\nu(2Nc,d)\psi_{4m}^{k+\frac{1}{2}}\left(\frac{y}{|2Ncz+d|^2}\right)}{(2Ncz+d)^{k+\frac{1}{2}}} e\left(m\mathrm{Re}\left(\frac{az+b}{2Ncz+d}\right)\right) \\
&= \sum_{n\in\mathbf{Z}} \sum_{\substack{c>0,\\ c\equiv 1(2)}} 4^{-\frac{2k+1}{4}} K_{k+\frac{1}{2}}(4m,n;8Nc)(y/4)^{1-\frac{2k+1}{4}} I_{k+\frac{1}{2}}(4m,n,8Nc,y/4)e(nx/4).
\end{aligned}
$$

Kohnen [**Koh85**, p. 257] proved that for c being odd, we have

$$
K_{k+\frac{1}{2}}(m,n;4Nc) = \begin{cases} (1-(-1)^k i)i^{n/4}K'_{k+\frac{1}{2}}(4m,n/4;4Nc), & \text{if } n\equiv 0 \pmod 4,\\ \frac{1}{\sqrt{2}}\left(\frac{(-1)^k n}{2}\right)K_{k+\frac{1}{2}}(4m,n;8Nc), & \text{if } (-1)^k n\equiv 1 \pmod 4,\end{cases}
$$

which together with $e^{-\pi i\frac{2k+1}{4}} = (-1)^{\left\lfloor\frac{2k+1}{4}\right\rfloor}\sqrt{2}(1-(-1)^{\frac{k}{2}}i)$ proves the claim. $\square$

1.1. The Modified Kloosterman Zeta-function. Consider the modified Klooserman zeta-function

$$
Z^{+}_{k+\frac{1}{2}}(m,n;s) = \sum_{0<c\equiv 0(4N)} \frac{K^{+}_{k+\frac{1}{2}}(m,n;c)}{c^{2s}} \tag{4.6}
$$

for $m,n\neq 0$ and $k\in\mathbf{Z}$. The series in (4.6) is absolutely convergent for $\mathrm{Re}(s)>1$, which follows from the Weil bound for Kloosterman sums, i.e.

$$
K(m,n;c)\leq \tau(c)\sqrt{(m,n,c)}\sqrt{c},
$$

where $\tau(c)$ denotes the number of divisors of c.

Considering the Fourier coefficients of $L^{+}_{m,n}(s)$ in Proposition 4.3 we see from the asymptotic expansions

$$
J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right),\ I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) = \frac{(2\pi)^{2s-1}|mn|^{s-\frac{1}{2}}}{\Gamma(2s)c^{2s-1}} + O(c^{-2s-1}),\ \text{as } c\to+\infty,
$$

that

$$
L^{+}_{m,n}(s) = \frac{(2\pi)^{2s-1}|mn|^{s-\frac{1}{2}}}{\Gamma(2s)} Z^{+}_{k+\frac{1}{2}}(m,n;s) + H_{m,n,k}(s) \tag{4.7}
$$

for a function $H_{m,n,k}(s)$, which is holomorphic for $\mathrm{Re}(s) > 0$. Since $F_m^{k+\frac{1}{2}}(z,s)|\mathrm{pr}^+_{k+\frac{1}{2}}$ (and hence $L^+_{m,n}(s)$) has an analytic continuation, we get the analytic continuation of the modified Kloosterman zeta-function to the half-plane $\mathrm{Re}(s) > 0$. Its poles are located at the poles of $L^+_{m,n}(s)$. In particular, $Z^+_{k+\frac{1}{2}}(m,n;s)$ has poles at the spectral parameters s of Maass cusp forms of weight $k+\frac{1}{2}$ and level $4N$. The residue at $s \in \mathbf{C}$ being a spectral parameter are given in terms of Fourier coefficients of Maass cusp forms in $U^+_{k+\frac{1}{2},4N}(s)$.

PROPOSITION 4.4. *Let $n \neq 0$ be an integer such that $(-1)^k n \equiv 0, 1 \pmod 4$ and let $s_0 \in \mathbf{C}$ be a spectral parameter of a Maass cusp form of weight $k+\frac{1}{2}$ and level $4N$. The residue of the modified Kloosterman sum $Z^+_{k+\frac{1}{2}}(m,n;s)$ at $s = \frac{s_0}{2} + \frac{1}{4}$ is given by*

$$\frac{3(-1)^{\left\lfloor \frac{2k+1}{4} \right\rfloor} 2^{\frac{1}{2}-s_0-2k} \pi^{\frac{3}{4}-\frac{s_0}{2}-k} |mn|^{\frac{1}{2}-\frac{s_0}{2}-\frac{k}{2}}}{2\sqrt{2}} \frac{\Gamma\left(s_0-\frac{1}{2}\right)\Gamma\left(s_0+\mathrm{sign}(n)\frac{2k+1}{4}\right)}{\Gamma\left(s_0-\mathrm{sign}(m)\frac{2k+1}{4}\right)} \sum_{\psi} \frac{\overline{b_\psi(m)} b_\psi(n)}{\langle \psi, \psi \rangle},$$

where the sum in ψ runs over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s_0}{2}+\frac{1}{2}\right)$.

PROOF. This follows from Proposition 4.1, Proposition 4.3, and (4.7). □

We wrote $\frac{s_0}{2}+\frac{1}{4}$ for the spectral parameter in Proposition 4.4, as the spectral parameter changes under the Shimura lift. We will see in the next chapter that the cusp forms in $U^+_{k+\frac{1}{2},4N}\left(\frac{s_0}{2}+\frac{1}{2}\right)$ get mapped to $U_{2k,N}(s_0)$, so the way we wrote the spectral parameter in Proposition 4.4 is convenient.

2. Selberg Poincaré Series

Let $k \in \mathbf{Z}$ and $N \geq 1$ odd. From Proposition 4.1, we see that we can recover the cycle integrals of Maass cusp forms by calculating the cycle integral of the Niebur Poincaré series, which is easier since the latter is given as an explicit Poincaré series. However, the Niebur Poincaré series runs into the issue that its cycle integral is not convergent when integrated over a non-compact geodesic; so we cannot cover the square discriminant case.

For $m > 0$ we can instead define the Selberg Poincaré series

$$P_m^{2k}(z,s) = \sum_{g \in \Gamma_\infty \backslash \Gamma_0(N)} \left(y^{s-k} e(mz)\right)|_{2k} g, \tag{4.8}$$

which is absolutely convergent and converges uniformly in z on compact subsets of the upper half plane for $\mathrm{Re}(s) > 1$. The Selberg Poincaré series $P_m^{2k}(z,s)$ is no longer an eigenfunction of the Laplace operator but satisfies

$$\Delta_{2k} P_m^{2k}(z,s) = (s-k)(1-s-k) P_m^{2k}(z,s) + 4\pi m s P_m^{2k}(z,s), \tag{4.9}$$

which can be easily checked through Corollary 3.4.

The Niebur Poincaré series $F_m^{2k}(z,s)$ and the Selberg Poincaré series $P_m^{2k}(z,s)$ are interchangeable for our purposes, since

$$\phi_m(y,s) = (4\pi my)^s e^{-2\pi my} - M_{k,s-\frac{1}{2}}(4\pi my) = O(y^{s+1}), \text{ as } y \to 0,$$

and so the series $\sum_{g\in\Gamma_\infty\backslash\Gamma_0(N)}(y^{-k}\phi_m(y,s)e(mx))|_{2k}g$ converges absolutely for $\mathrm{Re}(s) > 0$. Hence, we immediately see that

$$P_m^{2k}(z,s) + \frac{(4\pi m)^{1-s}\Gamma(2s)}{\Gamma(s-k)}F_m^{2k}(z,s) \tag{4.10}$$

has an analytic continuation to $\mathrm{Re}(s) > 0$. Hence, Proposition 4.1 together with (4.10) gives the following proposition.

Proposition 4.5. *Let $k \in \mathbf{Z}$ and $m > 0$. The Selberg Poincaré series $P_m^{2k}(z,s)$ can be analytically continued to* $\mathrm{Re}(s) > 0$ *with poles at the spectral parameters s_0 of Maass cusp forms of weight $2k$ and level N. Its residue at s_0 is given by*

$$\mathrm{Res}_{s=s_0} P_m^{2k}(z,s) = (4\pi m)^{1-s_0-k}\frac{\Gamma(2s_0-1)}{\Gamma(s_0-k)}\sum_{\varphi}\overline{a_\varphi(m)}\frac{\varphi(z)}{\langle\varphi,\varphi\rangle},$$

where the sum in φ runs over an orthogonal basis of Hecke eigenforms in $U_{2k,N}(s_0)$.

One can extend the approach to prove Proposition 4.5 by using the power series expansion of the test function $y^{-k}M_{k,s-\frac{1}{2}}(4\pi my)$ to get the Niebur Poincaré series as a formal series of Selberg Poincaré series. E. Yoshida [**Yos91**, Prop. 2] showed that this formal series converges.

Another way of proving the analytic continuation of the Selberg Poincaré series is to use its spectral expansion in the space $L^2(\Gamma_0(N)\setminus\mathbf{H}, 2k)$, namely

$$P_m^{2k}(z,s) = \sum_{\varphi}\langle P_m^{2k}(\cdot,s),\varphi\rangle\varphi(z) + \sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle P_m^{2k}(\cdot,s), E_{\mathfrak{a}}^{2k}(\cdot,1/2+it)\rangle E_{\mathfrak{a}}^{2k}(z,1/2+it)dt, \tag{4.11}$$

where φ runs over an orthogonal basis of the space $U_{2k,N}$, $\mathfrak{a}$ runs over a set of pairwise inequivalent cusps of $\Gamma_0(N)\setminus\mathbf{H}$, and $E_{\mathfrak{a}}^{2k}(z,s) = \sum_{g\in\Gamma_{\mathfrak{a}}\backslash\Gamma_0(N)} y^{s-k}|_{2k}g$ is the non-holomorphic Eisenstein series at the cusp $\mathfrak{a}$. The non-holomorphic Eisenstein series $E_{\mathfrak{a}}^{2k}(z,s)$ admits an analytic continuation to the whole complex plane and has no poles for $\frac{1}{2} \leq \mathrm{Re}(s) < 1$. For $\varphi \in U_{2k,N}(s_0)$, unfolding the inner product gives

$$\begin{aligned}
\langle P_m^{2k}(\cdot, s), \varphi \rangle &= \int_0^\infty \int_0^1 y^s e^{2\pi m(x+iy)} \overline{\varphi(z)} y^{k-2} dx dy \\
&= \frac{\overline{a_\varphi(m)}}{(4\pi m)^{k+s-1}} \int_0^\infty y^{s-2} e^{-\frac{y}{2}} W_{k,s_0-\frac{1}{2}}(y) dy \\
&= \frac{\overline{a_\varphi(m)}}{(4\pi m)^{k+s-1}} \frac{\Gamma(s+s_0-1)\Gamma(s-s_0)\Gamma(1-s-k)}{\Gamma(s_0-k)\Gamma(1-s_0-k)},
\end{aligned}$$

from which we get the analytic continuation of the first term on the right hand side of (4.11). We also see that its poles for $\mathrm{Re}(s) > 0$ lie exactly at s_0 and $1 - s_0$, where s_0 is a spectral parameter. For the continuous part of the spectrum, write

$$E_{\mathfrak{a}}^{2k}(z, s) = y^{s-k} + \varphi_{\mathfrak{a}}(0, s) y^{1-s-k} + \sum_{n \neq 0} y^{-k} \varphi_{\mathfrak{a}}(n, s) W_{\mathrm{sign}(n)k, s-\frac{1}{2}}(4\pi |n| y) e(nx)$$

for the Fourier expansion of the non-holomorphic Eisenstein series. A similar argument as for the cuspidal part shows

$$\begin{aligned}
&\int_{-\infty}^\infty \langle P_m^{2k}(\cdot, s), E_{\mathfrak{a}}^{2k}(\cdot, 1/2+it) \rangle E_{\mathfrak{a}}^{2k}(z, 1/2+it) dt \\
&= -i(4\pi m)^{1-s-k} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s-\xi)\Gamma(s+\xi-1)\Gamma(1-s-k)}{\Gamma(\xi-k)\Gamma(1-\xi-k)} \overline{\varphi_{\mathfrak{a}}(m, \xi)} E_{\mathfrak{a}}(z, \xi) d\xi.
\end{aligned}$$

By the analytic continuation of the Γ-function, the integral on the right hand side can be analytically continued to $\mathrm{Re}(s) > \frac{1}{2}$. To continue it beyond the line $\mathrm{Re}(s) = \frac{1}{2}$, pick $\alpha > 0$ small enough. By Cauchy's theorem, we have for all $\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2} + \alpha$:

$$\begin{aligned}
&\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s-\xi)\Gamma(s+\xi-1)\Gamma(1-s-k)}{\Gamma(\xi-k)\Gamma(1-\xi-k)} \overline{\varphi_{\mathfrak{a}}(m, \xi)} E_{\mathfrak{a}}(z, \xi) d\xi \\
&= \int_{\frac{1}{2}+\alpha-i\infty}^{\frac{1}{2}+\alpha+i\infty} \frac{\Gamma(s-\xi)\Gamma(s+\xi-1)\Gamma(1-s-k)}{\Gamma(\xi-k)\Gamma(1-\xi-k)} \overline{\varphi_{\mathfrak{a}}(m, \xi)} E_{\mathfrak{a}}(z, \xi) d\xi \\
&- \frac{\Gamma(2s-1)}{\Gamma(s-k)} \varphi_{\mathfrak{a}}(m, s) E_{\mathfrak{a}}^{2k}(z, s).
\end{aligned}$$

The right hand side can now be analytically continued to the half-plane $\mathrm{Re}(s) < \frac{1}{2} + \alpha$. We see that $P_m^{2k}(z, s)$ can hence be analytically continued in s to the whole complex plane and that its poles for $\mathrm{Re}(s) < \frac{1}{2}$ only come from the discrete spectrum.

2.1. Traces of the Selberg Poincaré Series. In this section we prove the central technical result of the first part of this thesis, namely that the traces of the Selberg

Poincaré series are essentially given in terms of the modified Kloosterman zeta function in (4.6). For the notations on binary quadratic forms, the reader should consult section 1 in chapter 2.

Definition 4.6. Let $k \in \mathbf{Z}$ and $N \geq 1$ be odd. Let $f : \mathbf{H} \to \mathbf{C}$ be a real-analytic function of weight $2k$ and level N, d be a fundamental discriminant and d' such that $(-1)^k d' \equiv 0, 1 \pmod 4$.

(1) For $dd' < 0$ and $k \geq 0$, the twisted trace of f is defined by

$$\mathrm{Tr}_{d,d'} f = \sum_{Q \in \Gamma_0(N) \backslash \mathcal{Q}_{dd',N}} \chi_d(Q) |\Gamma_Q|^{-1} (L_2 \circ \cdots \circ L_{2k-2} \circ L_{2k} f)(z_Q).$$

For $k < 0$, the twisted trace $\mathrm{Tr}_{d,d'}(f)$ is similarly defined with raising operators instead of lowering operators.

(2) For $dd' > 0$, the twisted trace of f is defined by

$$\mathrm{Tr}_{d,d'}(f) = \sum_{Q \in \Gamma_0(N) \backslash \mathcal{Q}_{dd',N}} \chi_d(Q) \int_{C_Q} f(z) Q(z,1)^k d_Q z,$$

where the measure $d_Q z$ is given by $\frac{\sqrt{\mathrm{disc}(Q)}\, dz}{Q(z,1)}$.

It is necessary in Definition 4.6 to use lowering and raising operators in the case $dd' < 0$ for the trace to be well-defined. For f being modular of weight zero, we namely have $f(z_{Q \circ g}) = f(g.z_Q) = f(z_Q)$ for all $Q \in \mathcal{Q}_{dd',N}$ and $g \in \Gamma_0(N)$, so that it makes sense to sum over classes of binary quadratic forms.

With Definition 4.6 we are set to state the main result of this chapter.

Theorem 4.7. *Let $k \in \mathbf{Z}$, $N \geq 1$ be odd, $m > 0$, $(-1)^k d$ be a fundamental discriminant, and d' be an integer such that $(-1)^k d' \equiv 0, 1 \pmod 4$. Suppose $\mathrm{Re}(s) > 0$ and set $D = dd'$.*

(1) *If $D < 0$, the trace $\mathrm{Tr}_{(-1)^k d, (-1)^k d'} P_m^{2k}(z,s)$ equals*

$$\frac{(-1)^k \Gamma(s)}{\Gamma(s-k)} 2^{s-1} |D|^{k/2+s/2} \sum_{\substack{n \mid m, \\ (n,N)=1}} \left(\frac{(-1)^k d}{n} \right) \frac{1}{n^s} Z^+_{k+\frac{1}{2}} \left(\frac{m^2 d}{n^2}, d'; \frac{s}{2} + \frac{1}{4} \right) + H_1(s),$$

where $H_1(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

(2) *For $D = dd' > 0$ with $d > 0$, the trace $\mathrm{Tr}_{(-1)^k d, (-1)^k d'} P_m^{2k}(z,s)$ equals*

$$\frac{(-1)^k 2\pi D^{k/2+s/2} \Gamma(s)}{\Gamma\left(\frac{s+1}{2} + \frac{k}{2}\right) \Gamma\left(\frac{s+1}{2} - \frac{k}{2}\right)} \sum_{\substack{n \mid m, \\ (n,N)=1}} \left(\frac{(-1)^k d}{n} \right) n^{-s} Z^+_{k+\frac{1}{2}} \left(\frac{m^2 d}{n^2}, d'; \frac{s}{2} + \frac{1}{4} \right) + H_2(s),$$

where $H_2(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

(3) *For $D = dd' > 0$ with $d < 0$, the trace* $\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2-2k}(z,s)$ *equals*

$$-\frac{(-1)^k 2\pi D^{k/2+s/2}\Gamma(s)}{\Gamma\left(\frac{s}{2}+1-\frac{k}{2}\right)\Gamma\left(\frac{s}{2}+\frac{k}{2}\right)} \sum_{\substack{n|m,\\ (n,N)=1}} \left(\frac{(-1)^k d}{n}\right) n^{-s} Z^+_{k+\frac{1}{2}}\left(\frac{m^2 d}{n^2}, d'; \frac{s}{2}+\frac{1}{4}\right) + H_3(s),$$

where $H_3(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

With Proposition 4.5, Theorem 4.7, and Proposition 4.4 we immediately see that the twisted traces of Maass cusp forms of weight $2k$ and level N are given in terms of Fourier coefficients of Maass cusp form of weight $k+\frac{1}{2}$ and level $4N$. This fact will be worked out more precisely in the next chapter.

We will divide the proof into the case of negative and positive discriminant D. In the evaluation of the traces we will encounter Salie sums, which are defined to be

$$S_m(d,d';c) = \sum_{b\ (c),\ b^2\equiv dd'(c)} \chi_d\left(\left[\frac{c}{4}, b, \frac{b^2-dd'}{c}\right]\right) e\left(-\frac{2mb}{c}\right) \tag{4.12}$$

for c divisible by 4.

Note that for any $\varepsilon > 0$ the Salie sums are bounded in c by

$$S_m(d,d';c) \ll_\varepsilon c^\varepsilon \text{ as } c \to +\infty. \tag{4.13}$$

A crucial step in the proof of Theorem 4.7 will be a proposition on exponential sums relating the Salie sums to the Kloosterman sums $K^+_{k+\frac{1}{2}}(m,n;c)$ defined in Proposition 4.3.

Proposition 4.8 (Kohnen). *Let m, d, d' be non-zero integers and k an even integer. Suppose further that $(-1)^{\frac{k}{2}} d$ is a fundamental discriminant, $c \equiv 0 \pmod 4$, and $(-1)^{\frac{k}{2}} d' \equiv 0,1 \pmod 4$. Then we have*

$$S_m((-1)^{\frac{k}{2}} d, (-1)^{\frac{k}{2}} d'; c) = \sum_{n|(m,c/4)} \left(\frac{(-1)^{\frac{k}{2}} d}{n}\right) \sqrt{\frac{n}{c}} K^+_{\frac{k+1}{2}}\left(\frac{m^2 d}{n^2}, d'; \frac{c}{n}\right).$$

Proof. See [**Koh85**, Prop. 5]. □

2.1.1. *The Proof of Theorem 4.7 for $D < 0$.*

Proposition 4.9. *Assume that $D = dd' < 0$, where $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0,1 \pmod 4$. Then*

$$\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^0(z,s) = 2^{s-1}|D|^{s/2} \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^k d, (-1)^k d'; c)}{c^s} e^{-4\pi m \frac{\sqrt{|D|}}{c}}.$$

Proof. This is a calculation:

$$\begin{aligned}
&\sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}} \chi_{(-1)^k d}(Q)|\Gamma_Q|^{-1}P_m^0(z_Q,s)\\
&= \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}}\sum_{g\in\Gamma_\infty\backslash\Gamma} \chi_{(-1)^k d}(Q)|\Gamma_Q|^{-1}\mathrm{Im}(g.z_Q)^s e(mg.z_Q)\\
&= \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}}\sum_{g\in\Gamma_\infty\backslash\Gamma/\Gamma_Q} \chi_{(-1)^k d}(Q)\mathrm{Im}(z_{Q\circ g^{-1}})^s e(mz_{Q\circ g^{-1}})\\
&= \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{N,D}} \chi_{(-1)^k d}(Q)(\mathrm{Im} z_Q)^s e\left(mz_Q\right),
\end{aligned}$$

which by Lemma 2.1 yields the result. □

We will also need the following expressions for lowering resp. raising operators applied to $P_m^{2k}(z,s)$.

Lemma 4.10.

$$L_2\circ\cdots L_{2k-2}\circ L_{2k}P_m^{2k}(z,s) = \frac{(-1)^k\Gamma(s)}{\Gamma(s-k)}P_m^0(z,s), \text{ if } k>0, \text{ and}$$

$$R_{-2}\circ\cdots R_{2k+2}\circ R_{2k}P_m^{2k}(z,s) = (-1)^k\sum_{l=0}^{|k|}(4\pi m)^l\frac{\Gamma(s+l)}{\Gamma(s+l-k)}P_m^0(z,s+l), \text{ if } k<0.$$

Proof. We use the identities

$$L_{2k'}(y^{s-k}e(mz)) = -(s-k')y^{s+1-k}e(mz) \text{ for all } k'\in\mathbf{Z}$$

and

$$R_{2k'}(y^{s-k'}e(mz)) = -(s-k')y^{s+1-k'}e(mz) - 4\pi m y^{s+1-k'}e(mz) \text{ for all } k'\in\mathbf{Z}.$$

By applying the identities to (4.8) with Proposition 3.13, we get the claim for $\mathrm{Re}(s)>1$. Uniqueness of the analytic continuation yields the result for all complex s. □

The proof of Theorem 4.7 for $D<0$. Assume that $k>0$. After applying the first part of Lemma 4.10, we see

$$\begin{aligned}
\mathrm{Tr}_{(-1)^k d,(-1)^k d'}P_m^{2k}(z,s) &= \frac{(-1)^k|D|^{\frac{k}{2}}\Gamma(s)}{\Gamma(s-k)}\mathrm{Tr}_{(-1)^k d,(-1)^k d'}P_m^0(z,s)\\
&= \frac{(-1)^k\Gamma(s)}{\Gamma(s-k)}|D|^{k/2+s/2}\sum_{0<c\equiv 0(4N)}\frac{S_m((-1)^k d,(-1)^k d';c)}{c^s}e^{-4\pi m\frac{\sqrt{|D|}}{c}}.
\end{aligned}$$

Using the first order approximation $e^{-4\pi m\frac{\sqrt{|D|}}{c}} = 1 + O(1/c)$, we get

(4.14)
$$\sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^k d, (-1)^k d'; c)}{c^s} e^{-4\pi m\frac{\sqrt{|D|}}{c}} = \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^k d, (-1)^k d'; c)}{c^s} + H_1(s)$$

with $H_1(s)$ holomorphic for $\mathrm{Re}(s) > 0$ by (4.13). Applying Proposition 4.8 leads to the claim (1) of Theorem 4.7 in the case $k > 0$.

For $k < 0$ we arrive at (4.14) with the same argument as above, since for $l \geq 1$ the terms $P_m^0(z, s+l)$ in Lemma 4.10 are holomorphic in $\mathrm{Re}(s) > 0$. □

2.1.2. *The Proof of Theorem 4.7 for $D > 0$.* To calculate the twisted traces of the Selberg Poincaré series, we will first treat a slightly more general family of Poincaré series to show that the method applies generally to Poincaré series. Let $\phi : \mathbf{R}_{>0} \to \mathbf{R}$ be a smooth function such that $\phi(y) = O(y^{1-k+\varepsilon})$ as $y \to 0$ for any $\varepsilon > 0$. Then the Poincaré series

$$P_m^{2k}(z, \phi) = \sum_{g\in\Gamma_\infty\backslash\Gamma_0(N)} (\phi(y)e(mx))\,|_{2k} g \tag{4.15}$$

converges for any integer m.

Lemma 4.11. *Let $k \in \mathbf{Z}$, $(-1)^k d$ a fundamental discriminant, and $(-1)^k d'$ an integer such that $(-1)^k d' \equiv 0, 1 \pmod 4$.*

(1) *If $d < 0$ and $D = dd'$ is not a square, then $\mathrm{Tr}_{(-1)^k d, (-1)^k d'} P_m^{2k}(z, \phi) = 0$.*
(2) *If $d > 0$, then*

$$\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^k(z,\phi) = 2 \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{dd',N}} \chi_{(-1)^k d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^k d_Q z$$
$$+ \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{dd',N},\\ Q(1,0)=0}} \chi_{(-1)^k d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^k d_Q z.$$

Proof. The argument is a standard application of the unfolding method. The group of automorphs Γ_Q acts freely on $\Gamma_\infty \setminus \Gamma_0(N)$ on the right, and a direct calculation shows that

$$
\begin{aligned}
&\int_{C_Q} P_m^{2k}(z,\phi)Q(z,1)^k d_Q z \\
&= \sum_{g\in\Gamma_\infty\backslash\Gamma_0(N)} \int_{C_Q} j(g,z)^{-2k} e(m\mathrm{Re}(g.z))\phi(\mathrm{Im}(g.z))Q(z,1)^k d_Q z \\
&= \sum_{g\in\Gamma_\infty\backslash\Gamma_0(N)/\Gamma_Q} \sum_{n\in\mathbf{Z}} \int_{\gamma_Q^n.C_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^k d_Q z.
\end{aligned}
$$

Therefore

$$
\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,\phi) = \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{dd',N}} \chi_{(-1)^k d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^k d_Q z.
$$

The measure $d_Q z$ satisfies $d_{-Q}z = -d_Q z$ and for $A \neq 0$ the geodesics $S_{[A,B,C]}$ and $S_{[-A,-B,-C]}$ are the same semicircle but with opposite orientations. By

$$
\int_{S_{-Q}} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))(-Q(z,1))^k d_{-Q}z = (-1)^k \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^k d_Q z
$$

and $\chi_{(-1)^k d}(-Q) = (-1)^k \mathrm{sign}(d)\chi_{(-1)^k d}(Q)$, we either get cancellation of the integrals twisted by $\chi_{(-1)^k d}$ for Q and $-Q$ in the first case or the integrals twisted by $\chi_{(-1)^k d}$ for Q and $-Q$ are equal. $\square$

Because of the trivial cancellation we explained in the last paragraph of the proof of Lemma 4.11, we will be interested in factorizing the discriminant $D = (-1)^k d \cdot (-1)^k d'$ into factors with $d, d' > 0$. If we want to consider such a factorization for $d, d' < 0$, we will hence consider Poincaré series of dual weight $2-2k$ instead, so that

$$
D = (-1)^k d \cdot (-1)^k d' = (-1)^{1-k}(-d)\cdot(-1)^{1-k}(-d')
$$

and the twisted trace $\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2-2k}(z,\phi)$ does not vanish trivially in Lemma 4.11.

PROPOSITION 4.12. *Let $k \in \mathbf{Z}$, $N \geq 1$ be odd, $d > 0$ be such that $(-1)^k d$ is a fundamental discriminant, and $d' > 0$ be an integer such that $(-1)^k d' \equiv 0, 1 \pmod 4$. Set $D = dd'$.*

(1) *If $D = dd'$ is not a square then*

$$
\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,\phi) = D^{k/2} \sum_{0<c\equiv 0(4N)} S_m((-1)^k d,(-1)^k d';c)\Phi_m^{2k}\left(\frac{2\sqrt{D}}{c}\right)
$$

with

$$\Phi_m^{2k}(t) = (it)^k \int_0^\pi e\left(mt\cos\theta\right)\phi(t\sin\theta)(\sin\theta)^{k-1}e^{i\theta k}d\theta. \tag{4.16}$$

(2) *If* $d = d'$ *and* $(d, N) = 1$, *then*

$$\begin{aligned}\mathrm{Tr}_{(-1)^k d,(-1)^k d} P_m^{2k}(z,\phi) = d^k &\sum_{0<c\equiv 0(4N)} S_m((-1)^k d, (-1)^k d; c)\Phi_m^{2k}\left(\frac{2d}{c}\right)\\ &+2(id)^k G_m(d)\mathcal{M}(\phi)(k)\end{aligned}$$

where $\mathcal{M}(\phi)(s)$ *is the Mellin transform of* ϕ *and*

$$G_m(d) = \sum_{\mu(d)^*}\left(\frac{d}{\mu}\right)e\left(-m\frac{\mu}{d}\right)$$

is a Gauss sum.

Proof. The first claim follows from Lemma 2.1 and Lemma 4.11 after parametrising the semicircle $S_{[A,B,C]}$ by $\theta \in (0,\pi)$ such that

$$z = -\frac{B}{2A} + \frac{\sqrt{D}}{2|A|}e^{i\theta}$$

so that we get

$$\begin{aligned}&\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,\psi)\\ &= 2\sum_{\substack{A>0,\ B(2A),\\ B^2\equiv D(4A)}} \chi_{(-1)^k d}\left(\left[A, B, \frac{B^2-D}{4A}\right]\right)\times\\ &\int_0^\pi e\left(m\left(-\frac{B}{2A}+\frac{\sqrt{D}}{2A}\cos\theta\right)\right)\phi\left(\frac{\sqrt{D}}{2A}\sin\theta\right)\frac{D^k}{(4A)^k}(e^{i\theta}-1)^k(e^{i\theta}+1)^k\frac{d\theta}{\sin\theta}\\ &= 2\sum_{0<c\equiv 0(4N)} S_m((-1)^k, (-1)^k d'; c)\frac{2^k i^k D^k}{c^k}\times\\ &\int_0^\pi e\left(m\frac{2\sqrt{D}}{c}\cos\theta\right)\phi\left(\frac{2\sqrt{D}}{c}\sin\theta\right)(\sin\theta)^{k-1}e^{i\theta k}d\theta.\end{aligned}$$

To prove the second claim, i.e. $D = d^2$, we again use Lemma 4.11. The sum

$$\sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{d^2,N},\\ Q(1,0)=0}} \chi_{(-1)^k d}(Q)\int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^k d_Q z \tag{4.17}$$

runs over $\{Q_\mu = [0, d, \mu] : 0 \leq \mu < d\}$. We have $\chi_{(-1)^k d}(Q_\mu) = \left(\frac{d}{\mu}\right)$ by (2.16). Thus, the sum (4.17) is equal to

$$\begin{aligned}
&\sum_{\mu(d)} \left(\frac{d}{\mu}\right) \int_{\{-\frac{\mu}{d}+it:t>0\}} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q_\mu(z,1)^k d_{Q_\mu} z \\
&= \sum_{\mu(d)^*} \left(\frac{d}{\mu}\right) e\left(-m\frac{\mu}{d}\right) \int_0^\infty \phi(t)(id)^k t^k \frac{dt}{t} \\
&= G_m(d)(id)^k \mathcal{M}(\phi)(k).
\end{aligned}$$

□

We now consider $\phi(y) = y^{s-k}e^{-2\pi my}$ in (4.15), so that $P_m^{2k}(z,\phi) = P_m^{2k}(z,s)$. Except for the holomorphic case $s = k$ (Propositon 4.14), the evaluation of the transform

$$\int_0^\pi e\left(mt\cos\theta\right)\phi(t\sin\theta)(\sin\theta)^{k/2-1}e^{i\theta k/2}d\theta$$

is bypassed by the next lemma which allows us to express the main term contributing to the residue of each pole of $P_m^{2k}(z,s)$ on the strip $0 < \mathrm{Re}(s) < 1$.

Lemma 4.13. *Let $\Phi_m^{2k}(t)$ be defined as in (4.16). For* $\mathrm{Re}(s) > 0$, *we have*

$$\Phi_m^{2k}(t) = c_{2k}(s)t^s + O(t^{s+1}), \text{ as } t \to 0^+,$$

where

$$c_{2k}(s) = (-1)^k \frac{\pi\Gamma(s)}{2^{s-1}\Gamma\left(\frac{s+1}{2}+\frac{k}{2}\right)\Gamma\left(\frac{s+1}{2}-\frac{k}{2}\right)}.$$

Proof. Using the asymptotics $e(mt\cos\theta) = 1 + O(mt\cos\theta)$ and $e^{-2\pi mt\sin\theta} = 1 + O(mt\sin\theta)$ as $t \to 0^+$, we get

$$\Phi_m^{2k}(t) = i^{\frac{k}{2}}t^s\left(\int_0^\pi (\sin\theta)^{s-1}e^{ik\theta/2}d\theta + O(t)\right).$$

The proof is finished with the integral formula (see [**GR07**, p. 485, 3.892(1)])

$$\int_0^\pi e^{i\beta x}(\sin x)^{\nu-1}dx = \frac{\pi e^{i\pi\beta/2}\Gamma(\nu)}{2^{\nu-1}\Gamma\left(\frac{\nu+\beta+1}{2}\right)\Gamma\left(\frac{\nu-\beta+1}{2}\right)}, \ \mathrm{Re}(\nu) > 0,$$

with $\nu = s$ and $\beta = k$. □

Proof of Theorem 4.7 for $D > 0$. Suppose $d, d' > 0$. We use Proposition 4.12 and Lemma 4.13 to write

(4.18)

$$\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,s) = c_{2k}(s) D^{k/2} \sum_{0<c\equiv 0(4N)} S_m((-1)^k d, (-1)^k d'; c) \left(\frac{2\sqrt{dd'}}{c}\right)^s + H_2(s),$$

where $H_2(s)$ is a holomorphic function on $\mathrm{Re}(s) > 0$. In the case of $d = d'$, the Mellin transform term $(id)^k G_m(d)\mathcal{M}(\phi)(k)$ is holomorphic and is part of $H_2(s)$.

With Proposition 4.8, we may rewrite equation (4.18) as

$$\begin{aligned}&\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,s)\\ &= c_{2k}(s) D^{k/2} \sum_{n|m} \left(\frac{(-1)^k d}{n}\right) \left(\frac{2\sqrt{dd'}}{n}\right)^s \sum_{0<c\equiv 0(4N)} \frac{K^+_{k+\frac{1}{2}}\left(\frac{m^2 d}{n^2}, d'; c\right)}{c^{s+1/2}} + H_2(s)\end{aligned}$$

by substituting $c \to nc$.

Suppose now that $d, d' < 0$. By Proposition 4.12 for weight $2 - 2k$ and $-d, -d' > 0$ and Lemma 4.13, we get

(4.19)

$$\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2-2k}(z,s) = c_{2-2k}(s) \sum_{0<c\equiv 0(4N)} S_m((-1)^k d, (-1)^k d'; c) \left(\frac{2\sqrt{dd'}}{c}\right)^s + H_3(s),$$

where $H_3(s)$ is a holomorphic function on $\mathrm{Re}(s) > 0$. Once again, Proposition 4.8 gives the claim. □

Evaluating $\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^{2k}(z,s)$ at the spectral parameter $s = k > 1$ reduces to considering $\phi(y) = e^{-2\pi m y}$ in (4.15) so that $P_m^{2k}(z,\phi) = P_m^{2k}(z) = \sum_{g\in\Gamma_\infty\backslash\Gamma_0(N)} e(mz)|_{2k} g$, the holomorphic Poincaré series of weight $2k$. For completeness we also give the result in this case.

Proposition 4.14. *Suppose that $k > 1$, $m > 0$, $d, d' > 0$ with $d \neq d'$ such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$. We have*

$$\begin{aligned}&\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^k(z)\\ &= \frac{(-1)^k \Gamma(k) D^{\frac{k}{2}+\frac{3}{4}}}{2^{k-\frac{3}{2}} m^{k-\frac{1}{2}} \pi^{k-1}} \sum_{n|m} \left(\frac{(-1)^k d}{n}\right) \sum_{0<c\equiv 0(4N)} \frac{1}{c} K^+_{k+\frac{1}{2}}\left(\frac{m^2 d}{n^2}, d'; c\right) J_{k-\frac{1}{2}}\left(\frac{4\pi m n\sqrt{D}}{c}\right).\end{aligned}$$

If $d = d'$, we get the additional term $(id)^k (4\pi m)^{-k}\Gamma(k)$ on the right hand side.

PROOF. Suppose that $D = dd'$ is not a square. By Lemma 4.11, we get

$$\mathrm{Tr}_{(-1)^k d,(-1)^k d'} P_m^k(z) = 2 \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{D,N}^+} \chi_{(-1)^k d}(Q) \int_{S_Q} e(mz) Q(z,1)^k d_Q z.$$

After the change of variable $z \to \frac{\sqrt{D}z - B}{2A}$ for $Q = [A, B, C]$, one needs to evaluate

$$-\int_{-1}^{1} e\left(m\frac{\sqrt{D}}{2A}z\right)(z^2-1)^{k-1}dz = \frac{(-1)^k 2^{k-\frac{1}{2}}\Gamma(k)}{\pi^{k-1}} \left(\frac{\sqrt{D}m}{A}\right)^{-k+\frac{1}{2}} J_{k-\frac{1}{2}}\left(\frac{\pi m\sqrt{D}}{A}\right),$$

which yields the claim for D not being a square. The case when D is a square can be dealt with in the same fashion as in the proof of Proposition 4.12. □

CHAPTER 5

The Shimura Correspondence of Maass Cusp Forms

In this chapter, we will define the Shimura lift for Maass cusp forms and prove Theorem 1.4 and its corollaries.

1. The Shimura Lift

Let $k \in \mathbf{Z}$ and $N \geq 1$ be odd. G. Shimura [**Shim73**] discovered a connection between half-integral weight cusp forms and even weight cusp forms. As we saw, the essential identity of the Shimura correspondence is the proposition on exponential sums relating Salie sums to Kloosterman sums; see Proposition 4.8.

Extended to Maass cusp forms (G. Shimura's original work was concerned with holomorphic cusp forms), the Shimura correspondence allows for striking applications, most notably the equidistribution theorem of W. Duke [**Duk88**]. That breakthrough was only possible due to the result of H. Iwaniec [**Iwa87**], who showed that the Fourier coefficients of half-integral weight cusp forms exhibit non-trivial bounds (smaller than the growth rate of class numbers).

We will now define the Shimura lift rigorously and prove that it indeed defines a map from half-integral cusp forms to even weight cusp forms.

Definition 5.1. Let d be such that $(-1)^k d$ is a fundamental discriminant and $\psi \in U^+_{k+\frac{1}{2},4N}(s)$. We define the d-th Shimura lift of ψ by

$$\mathrm{Shim}_d(\psi) = \sum_{m\neq 0} (4\pi|m|y)^{-k} a(m) W_{\mathrm{sign}(m)k,2s-\frac{3}{2}}(4\pi|m|y)e(mx),$$

where the Fourier coefficients are given by

$$a(m) = \sum_{\substack{n|m,\\(n,N)=1}} n^{k-1}\left(\frac{(-1)^k d}{n}\right) b_\psi\left(\frac{dm^2}{n^2}\right) \quad \text{and} \quad a(-m) = \frac{\Gamma(s+k)}{\Gamma(s-k)} a(m)$$

for $m > 0$.

It is a priori not clear that the function $\mathrm{Shim}_d(\psi)$ in Definition 5.1 defines a Maass cusp form. We see from its Fourier expansion that it is an eigenfunction of Δ_{2k} with spectral parameter $2s-1$.

To prove that $\mathrm{Shim}_d(\psi)$ is indeed a Maass cusp form (of weight $2k$ and level N), we use Theorem 4.7 and the spectral theory of Poincaré series developed in Chapter 4. In the crucial Lemma 5.3 below, we relate the functions $\mathrm{Shim}_d(\psi)$ to the traces of Maass cusp forms of even weight.

We also see from Definition 5.1 that if $\mathrm{Shim}_d(\psi)$ were a Maass cusp form, it would have to be an even cusp form (see Proposition 3.17). In fact, the traces of odd Maass cusp forms evaluate to zero, as we show in the next lemma.

LEMMA 5.2. *Let d, d' be integers such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$. If $\varphi \in U_{2k,N}(s)$ is an odd form, then* $\mathrm{Tr}_{(-1)^k d,(-1)^k d'}(\varphi) = 0$.

PROOF. It suffices to see that $\mathrm{Tr}_{(-1)^k d,(-1)^k d'}(X_{s,2k}\varphi) = \mathrm{Tr}_{(-1)^k d,(-1)^k d'}(\varphi)$. For, say, non-negative weight k we use $y^{2k-2}(L_{2k}\varphi)(-\overline{z}) = -R_{-2k} y^{2k}\varphi(-\overline{z})$ to get

$$(X_{s,2k}\varphi)(z) = \frac{\Gamma(s-k)}{\Gamma(s+k)} (R_{2k-2} \circ \cdots \circ R_0)(L_2 \circ \cdots L_{2k-2}\varphi)(-\overline{z}).$$

Substituting the class of the binary quadratic form $[A, B, C]$ with the class of $[A, -B, C]$ and using Proposition 3.14 proves the claim. □

LEMMA 5.3. *Let d, d' be integers such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$. Let $s_0 \in \mathbf{C}$.*

(1) *If $dd' < 0$, then*

$$6(-1)^{\lfloor k/2 \rfloor}\sqrt{|dd'|}\sum_{\psi} \frac{\overline{b_\psi(d')}}{\langle \psi, \psi\rangle} \mathrm{Shim}_d(\psi)(z) = 2\sqrt{\pi}\sum_{\varphi} \mathrm{Tr}_{(-1)^k d,(-1)^k d'}(\varphi)\frac{\varphi(z)}{\langle \varphi, \varphi\rangle}.$$

(2) *If $d, d' > 0$, then*

$$6(-1)^{\lfloor k/2 \rfloor}\sqrt{|dd'|}\sum_{\psi} \frac{\overline{b_\psi(d')}}{\langle \psi, \psi\rangle} \mathrm{Shim}_d(\psi)(z) = 2^k\sum_{\varphi} \mathrm{Tr}_{(-1)^k d,(-1)^k d'}(\varphi)\frac{\varphi(z)}{\langle \varphi, \varphi\rangle}.$$

(3) *If $d, d' < 0$, then*

$$6(-1)^{\lfloor k/2 \rfloor}\sqrt{|dd'|}\sum_{\psi} \frac{\overline{b_\psi(d')}}{\langle \psi, \psi\rangle} \mathrm{Shim}_d(\psi)(z) = 2^{1-k}\sum_{\varphi} \mathrm{Tr}_{(-1)^k d,(-1)^k d'}(\xi_{2k}\varphi)\frac{\varphi(z)}{\langle \varphi, \varphi\rangle}.$$

The sums over ψ run over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s_0}{2}+\frac{1}{4}\right)$ and the sums over φ run over a orthogonal basis of Hecke eigenforms of the subspace of $U_{2k,N}(s_0)$ of even forms.

Proof. Suppose first that $s_0 \neq k, 1-k$ and suppose that $dd' < 0$. Let $m > 0$. From Theorem 4.7, we obtain

$$\begin{aligned}
&\operatorname{Res}_{s=s_0}\operatorname{Tr}_{(-1)^k d,(-1)^k d'}P_m^{2k}(z,s)\\
&= \frac{(-1)^k\Gamma(s_0)}{\Gamma(s_0-k)}2^{s_0-1}|dd'|^{k/2+s_0/2}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^k d}{n}\right)n^{-s_0}\operatorname{Res}_{s=s_0}Z^+_{k+\frac{1}{2}}\left(\frac{m^2d}{n^2},d';\frac{s}{2}+\frac{1}{4}\right)\\
&= \frac{(-1)^k\Gamma(s_0)}{\Gamma(s_0-k)}2^{s_0-1}|dd'|^{k/2+s_0/2}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^k d}{n}\right)n^{-s_0}4\operatorname{Res}_{s=\frac{s_0}{2}+\frac{1}{4}}Z^+_{k+\frac{1}{2}}\left(\frac{m^2d}{n^2},d';s\right).
\end{aligned}$$

Applying Proposition 4.5 and Proposition 4.4, we then get

$$\begin{aligned}
&(4\pi m)^{1-s_0-k}\frac{\Gamma(2s_0-1)}{\Gamma(s_0-k)}\sum_{\varphi}\overline{a_\varphi(m)}\frac{\operatorname{Tr}_{(-1)^k d,(-1)^k d'}\varphi(z)}{\langle\varphi,\varphi\rangle}\\
&= \frac{3(-1)^{\lfloor k/2\rfloor}2^{-2k}\pi^{-s_0-k}m^{1-s_0-k}\sqrt{|dd'|}\Gamma(s_0)\Gamma\left(s_0-\frac{1}{2}\right)\Gamma\left(s_0+\operatorname{sign}(d')\frac{2k+1}{4}\right)}{\Gamma(s_0-k)\Gamma\left(s_0-\operatorname{sign}(d)\frac{2k+1}{4}\right)}\times\\
&\sum_{\psi}\sum_{\substack{n|m,\\(n,N)=1}}n^{k-1}\left(\frac{(-1)^k d}{n}\right)\frac{\overline{b_\psi\left(\frac{m^2d}{n^2}\right)}b_\psi(d')}{\langle\psi,\psi\rangle}\\
&= \frac{6(-1)^{\lfloor k/2\rfloor}2^{1-2s_0-2k}\pi^{\frac{1}{2}-s_0-k}m^{1-s_0-k}\sqrt{|dd'|}\Gamma(2s_0-1)}{\Gamma(s_0-k)}\\
&\sum_{\psi}\sum_{\substack{n|m,\\(n,N)=1}}n^{k-1}\left(\frac{(-1)^k d}{n}\right)\frac{\overline{b_\psi\left(\frac{m^2d}{n^2}\right)}b_\psi(d')}{\langle\psi,\psi\rangle}
\end{aligned}$$

where the sum in φ runs over an orthogonal basis of odd and even Hecke eigenforms of $U_{2k,N}(s_0)$ and ψ runs over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s_0}{2}+\frac{1}{4}\right)$. Since $\operatorname{Tr}_{(-1)^k d,(-1)^k d'}(\varphi) = 0$ if φ is an odd form by Lemma 5.2, we may restrict the sum over φ to even Hecke eigenforms. Summing over m then gives the claim.

In the third case, we additionally use the fact from Proposition 3.8 that $\xi_{2-2k} : U_{2-2k,N}(s_0) \to U_{2k,N}(s_0)$ is an isomorphism and the expression from Proposition 3.9 for the Fourier coefficients of $\xi_{2-2k}\varphi$.

Suppose now that $s_0 = k$ is the spectral parameter of the holomorphic cusp forms, so that $P_m^{2k}(z, s_0)$ is the m-th holomorphic Poincaré series. In the cases $dd' < 0$ and $d, d' < 0$, both sides of the claim are trivially zero, so there is nothing to show. Suppose thus that $d, d' > 0$. From Proposition 4.3 and Proposition 4.14, we see that the trace of $P_m^{2k}(z)$ is a linear combination of Fourier coefficients

$$\langle P_{d'}^{2k,+}, P_{\frac{m^2 d}{n^2}}^{2k} \rangle = \sum_{\psi} \langle P_{d'}^{2k}, \psi \rangle \langle \psi, P_{\frac{m^2 d}{n^2}}^{2k,+} \rangle.$$

A similar calculation as before gives the result in this case as well. □

In Lemma 5.3 we already see the connection between the space $U^+_{k+\frac{1}{2},4N}(s)$ and $U_{2k,N}(2s-1)$ through the operator Shim_d. With an argument in linear algebra, we can then prove that Shim_d is indeed a map between these spaces.

THEOREM 5.4. *Let d be an integer such that $(-1)^k d$ is a fundamental discriminant. The d-th Shimura lift* Shim_d *is a linear map*

$$U^+_{k+\frac{1}{2},4N}(s) \to U_{2k,N}(2s-1).$$

PROOF. Let d' be any integer such that $(-1)^k d' \equiv 0, 1 \pmod 4$. By Lemma 5.3, we see that

$$\sum_{i=1}^{n} \frac{\overline{b_i(d')}}{\langle \psi_i, \psi_i \rangle} \mathrm{Shim}_d(\psi_i) \in U_{2k,N}(2s-1), \tag{5.1}$$

where we wrote $b_i(m) = b_{\psi_i}(m)$ for $m \neq 0$ and $\psi_1, ..., \psi_n$ is an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}(s)$. We will show that, say, $\mathrm{Shim}_d(\psi_1)$ lies in $U_{2k,N}(2s-1)$.

We may find $d'_1, ..., d'_n$ with $(-1)^k d'_j \equiv 0, 1 \pmod 4$ for $j = 1, ..., n$ such that the Matrix $B = (b_i(d'_j))_{i,j=1,...,n}$ is of rank n; if this were not the case, then $\psi_1, ..., \psi_n$ would not be linearly independent. Thus, the linear system

$$\begin{pmatrix} b_1(d'_1) & \dots & b_1(d'_n) \\ b_2(d'_1) & \dots & b_2(d'_n) \\ \vdots & \ddots & \vdots \\ b_n(d'_1) & \cdots & b_n(d'_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution. By (5.1), we deduce

$$\frac{1}{\langle \psi_1, \psi_1 \rangle} \mathrm{Shim}_d(\psi_1) = \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{n} \frac{b_i(d_j)}{\langle \psi_i, \psi_i \rangle} \mathrm{Shim}_d(\psi_i) \in U_{k,N}(2s-1).$$

□

2. The Katok-Sarnak Formula for Maass Cusp Forms of Non-Zero Weights

In this section, we will prove the main result of the first part, namely relating the traces of Maass cusp forms of even weight cusp forms to the Fourier coefficients of half-integral weight cusp forms directly. We already see those two objects appearing in Lemma 5.3 as the scalars of the basis vectors of the vector spaces of Maass cusp forms. An application of the Hecke theory on these spaces will give us the direct relation of these scalar factors.

It is not difficult to prove from the respective definitions that the Hecke operators commute with the operator Shim_d; more precisely, that for a prime p, which does not divide N, we have $\mathrm{Shim}_d \circ T(p^2) = T(p) \circ \mathrm{Shim}_d$. We will exploit this fact in the following generalization of the Katok-Sarnak formula in Theorem 1.2.

THEOREM 5.5. *Let $k \in \mathbf{Z}$, $N \geq 1$ odd, and $\varphi \in U_{2k,N}(s)$ be a normalized even newform. Let d, d' be integers such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$. We have*

$$6(-1)^{\lfloor k/2 \rfloor} \sum_{\mathrm{Shim}_d(\psi)=b_\psi(d)\varphi} \frac{\overline{b_\psi(d')} b_\psi(d)}{\langle \psi, \psi \rangle} = \langle \varphi, \varphi \rangle^{-1} \cdot \begin{cases} 2\sqrt{\pi}\,\mathrm{Tr}_{(-1)^k d, (-1)^k d'}(\varphi), & \text{if } dd' < 0, \\ 2^k \mathrm{Tr}_{(-1)^k d, (-1)^k d'}(\varphi), & \text{if } d, d' > 0, \\ 2^{1-k} \mathrm{Tr}_{(-1)^k d, (-1)^k d'}(\xi_{2k}\varphi), & \text{if } d, d' < 0, \end{cases}$$

where the sum in ψ runs over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s}{2} + \frac{1}{4}\right)$.

PROOF. Pick a basis of normalized newforms $\varphi_1, \ldots, \varphi_n$ of $U^{\mathrm{new}}_{2k,N}(s)$ and denote the p-th Hecke eigenvalue of φ_i under $T(p)$ by $\lambda_i(p)$ for any $i = 1, \ldots, n$ and $(p, N) = 1$. Take $\psi \in U^+_{k+\frac{1}{2},4N}\left(\frac{s}{2} + \frac{1}{4}\right)$ to be a Hecke eigenform. Since the Hecke operators commute with the operator Shim_d, the cusp form $\mathrm{Shim}_d(\psi)$ is still a Hecke eigenform. If $\psi | T(p^2) \neq \lambda_i(p)\psi$ for a $(p, N) = 1$, the cusp form $\mathrm{Shim}_d(\psi)$ is orthogonal to φ_i by the self-adjointness of the Hecke operator $T(p)$. So either $\mathrm{Shim}_d(\psi)$ is an oldform or $\psi | T(p^2) = \lambda_i(p)\psi$ for all $(p, N) = 1$ and some $i = 1, \ldots, n$. In the second case, we must have $\mathrm{Shim}_d(\psi) = b_\psi(d)\varphi_i$ by the multiplicity one property of newforms.

Suppose $dd' < 0$, the other cases are similar. Applying the projection onto $U^{\mathrm{new}}_{2k,N}(s)$ to the equation in Lemma 5.3, we have

$$3(-1)^{\lfloor k/2 \rfloor} \sqrt{|dd'|} \sum_{i=1}^{n} \sum_{\mathrm{Shim}_d(\psi)=b_\psi(d)\varphi_i} \frac{\overline{b_\psi(d')} b_\psi(d)}{\langle \psi, \psi \rangle} \varphi_i(z) = \sqrt{\pi} \sum_{i=1}^{n} \mathrm{Tr}_{(-1)^k d, (-1)^k d'}(\varphi_i) \frac{\varphi_i(z)}{\langle \varphi_i, \varphi_i \rangle},$$

where the sum in ψ runs over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s}{2} + \frac{1}{4}\right)$. The claim follows by linear independence of the φ_i, $i = 1, \ldots, n$. $\square$

3. L-functions of Maass Cusp Forms

The coefficients of half-integral weight cusp forms are closely related to the values of L-functions. In this section, we investigate the central value of the twisted L-function

$$L(\varphi, d, w) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \frac{a_\varphi(n)}{n^w},\ (d, N) = 1, \tag{5.2}$$

for a normalized newform $\varphi \in U_{2k,N}(s)$.

Note that for $k > 0$ and $\varphi \in U_{2-2k,N}(s)$ even and normalized, we have

$$L(\xi_{2-k}\varphi, d, w) = -(4\pi)^{2k-1} \frac{\Gamma(\overline{s} + k/2)}{\Gamma(\overline{s} - k/2)} \overline{L(\varphi, d, \overline{w} + 2k - 1)}$$

by Lemma 3.9 and Corollary 3.17. Hence, we may restrict to Maass cusp forms of non-negative weights in this section, as the twisted L-functions of Maass cusp forms of negative weights reduce to that case.

We can get the L-function for an even normalized newform of weight $2k \geq 0$ from Theorem 5.5 through a Mellin transform. The following integral was first solved by W. Duke, J. Friedlander, and H. Iwaniec [**DFI02**, Lemma 8.2] (there is a typo in the orginial version, see M. Young's paper [**You19**, Sec. 12] for the correct evaluation). We present an alternative proof in our special case, which has the advantage of making the evaluation of the Mellin transform explicit instead of relying on a recursively defined polynomial.

Lemma 5.6. *Let $k \geq 0$ be an even integer and $s \neq \pm k$. The Mellin-transform*

$$\Psi_{2k}(s) = \int_0^\infty \left(W_{k,s-\frac{1}{2}}(y) + (-1)^k \frac{\Gamma(s+k)}{\Gamma(s-k)} W_{-k,s-\frac{1}{2}}(y) \right) \frac{dy}{y} \tag{5.3}$$

takes the values

$$\Psi_{2k}(s) = \frac{1}{2^k\sqrt{\pi}} \Gamma\left(\frac{s}{2} + k\right) \Gamma\left(\frac{1-s}{2} + k\right).$$

Proof. We write

$$\Psi_{2k}^{\pm}(s) = \int_0^\infty \left(W_{k,s-\frac{1}{2}}(y) \pm \frac{\Gamma(s+k)}{\Gamma(s-k)} W_{-k,s-\frac{1}{2}}(y) \right) \frac{dy}{y}.$$

With this notation we have

$$\Psi_{2k}(s) = \begin{cases} \Psi_{2k}^{+}(s), & k \text{ even}, \\ \Psi_{2k}^{-}(s), & k \text{ odd}. \end{cases}$$

Consider the generating series $\sum_{k=0}^{\infty} \Psi_{2k}^{+}(s)\frac{x^k}{k!}$. Using the identities (see [**MO13**, Sec. 7.3.4])

$$e^{\frac{xy}{2(1+x)}} W_{0,s-\frac{1}{2}}\left(\frac{y}{1+x}\right) = \sum_{n=0}^{\infty} W_{n,s-\frac{1}{2}}(y)\frac{x^n}{n!}$$

and

$$e^{-\frac{xy}{2(1+x)}} W_{0,s-\frac{1}{2}}\left(\frac{y}{1+x}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(s-n)} W_{-n,s-\frac{1}{2}}(y)\frac{x^n}{n!},$$

we can transform the generating series to be

$$\begin{aligned}\sum_{k=0}^{\infty} \Psi_{2k}^{+}(s)\frac{x^n}{n!} &= \int_0^{\infty} \sum_{k=0}^{\infty} \left(W_{k,s-\frac{1}{2}}(y) + \frac{\Gamma(s+k)}{\Gamma(s-k)} W_{-k,s-\frac{1}{2}}(y)\right) \frac{x^k}{k!}\frac{dy}{y} \\ &= \int_0^{\infty} W_{0,s-\frac{1}{2}}(y/(1+x)) \left(e^{\frac{xy}{2(1+x)}} + e^{-\frac{xy}{2(1+x)}}\right)\frac{dy}{y}.\end{aligned}$$

This gives

$$\begin{aligned}\sum_{k=0}^{\infty} \Psi_{2k}^{+}(s)\frac{x^k}{k!} &= \int_0^{\infty} W_{0,s-\frac{1}{2}}(y)\left(e^{\frac{xy}{2}} + e^{-\frac{xy}{2}}\right)\frac{dy}{y} \\ &= 2\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!\, 2^{2k}} \int_0^{\infty} y^{2k-1} W_{0,s-\frac{1}{2}}(y)dy.\end{aligned}$$

We also see here that $\Psi_{2k}^{+}(s) = 0$ for k being odd.

The function $W_{0,s-\frac{1}{2}}(y)$ is a modified Bessel-function of second kind $K_{s-\frac{1}{2}}(y)$, namely

$$W_{0,s-\frac{1}{2}}(y) = 2\sqrt{\frac{y}{\pi}} K_{s-\frac{1}{2}}(2y).$$

The integral of the K-Bessel function is well-known and evaluates to

$$\begin{aligned}\int_0^{\infty} y^{2k-1} W_{0,s-\frac{1}{2}}(y)dy &= \frac{2}{\sqrt{\pi}} 2^{-2k-\frac{1}{2}} \int_0^{\infty} y^{2k+\frac{1}{2}-1} K_{s-\frac{1}{2}}(y)dy \\ &= \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{s}{2}+k\right)\Gamma\left(\frac{1-s}{2}+k\right).\end{aligned}$$

Hence, for k even, we get the desired result. The same argument for the generating series $\sum_{k=0}^{\infty} \Psi_{2k}^{-}(s)\frac{x^k}{k!}$ yields the result for the case when k is odd. $\square$

The proof of Lemma 5.6 also shows that

$$\int_0^{\infty} \left(W_{k,s-\frac{1}{2}}(y) - (-1)^k \frac{\Gamma(s+k)}{\Gamma(s-k)} W_{-k,s-\frac{1}{2}}(y)\right)\frac{dy}{y} = 0$$

for all $k \geq 0$.

Applying the Mellin transform in Lemma 5.6 to the traces in Theorem 5.5 for square discriminants will give us an expression for the central value of the twisted L-function as in (5.2). The following theorem is a generalization of a result of M. Baruch and Z. Mao [**BM10**, Thm. 1.4] to higher weights.

THEOREM 5.7. *Let k be a non-negative integer, $\varphi \in U_{2k,N}(s)$ be an even normalized newform with spectral parameter $s \neq k$, and let d be a fundamental discriminant with $(-1)^k d > 0$ and $(d,N) = 1$. Suppose that for all $m \| N$, the eigenvalues w_m of φ under the Atkin-Lehner involution W_m are equal to $\left(\frac{d}{m}\right)$. Then:*

$$6 \sum_{\mathrm{Shim}_d(\psi) = b_\psi(d)\varphi} \frac{|b_\psi(|d|)|^2}{\langle \psi, \psi \rangle} = \frac{2^{\nu(N)} |d|^{k-\frac{1}{2}}}{(4\pi)^k \sqrt{\pi}} \Gamma\left(\frac{s}{2} + k\right) \Gamma\left(\frac{1-s}{2} + k\right) \frac{L(\varphi, d, k)}{\langle \varphi, \varphi \rangle},$$

where the sum in ψ runs over an orthogonal basis of Hecke eigenforms of $U^+_{k+\frac{1}{2},4N}\left(\frac{s}{2} + \frac{1}{4}\right)$ and $\nu(N)$ is the number of distinct prime factors of N.

Recall from Corollary 3.20 that each newform is an eigenfunction for all the Atkin-Lehner involutions, so the statement in Theorem 5.7 makes sense.

PROOF OF THEOREM 5.7. Let $d = d'$ in Theorem 5.5 with $(-1)^k d > 0$. We have

$$6d(-1)^{\lfloor k/2 \rfloor} \sum_{\mathrm{Shim}_d(\psi) = b_\psi(d)\varphi} |b_\psi(|d|)|^2 = 2^k \mathrm{Tr}_{d,d}(\varphi).$$

The trace on the right hand side evaluates to

$$\mathrm{Tr}_{d,d}(\varphi) = \sum_{\mu(d)} \sum_{m \| N} \chi_d(Q_\mu \circ W_m) \int_{C_{Q_\mu \circ W_m}} \varphi(z)(Q_\mu \circ W_m)(z,1)^k dz_{Q_\mu \circ W_m}$$

by Lemma 2.3. Since $\chi_d(Q_\mu \circ W_m) = \left(\frac{d}{\mu}\right)\left(\frac{d}{m}\right)$ by (2.16) we have

$$\begin{aligned} \mathrm{Tr}_{d,d}(\varphi) &= \sum_{\mu(d)} \sum_{m \| N} \left(\frac{d}{\mu}\right)\left(\frac{d}{m}\right) w_m \int_{C_{Q_\mu}} \varphi(z) Q_\mu(z,1)^k dz_{Q_\mu} \\ &= 2^{\nu(N)} \sum_{\mu(d)} \left(\frac{d}{\mu}\right) \int_{C_{Q_\mu}} \varphi(z) Q_\mu(z,1)^k dz_{Q_\mu}. \end{aligned}$$

Putting in the Fourier expansion for φ leads to

$$\begin{aligned}
&\sum_{\mu(d)}\left(\frac{d}{\mu}\right)\int_{C_{Q_\mu}}\varphi(z)Q_\mu(z,1)^k dz_{Q_\mu}\\
&= d^{k-1}|d|i^k\sum_{\mu(d)}\left(\frac{d}{\mu}\right)\int_0^\infty\sum_{n\neq 0}\frac{a_\varphi(n)}{(4\pi|n|)^k}W_{\mathrm{sign}(n)k,s-\frac{1}{2}}(4\pi|n|t)e^{-2\pi in\frac{\mu}{d}}\frac{dt}{t}\\
&= \frac{d^{k-1}|d|}{(4\pi)^k}i^{\frac{k}{2}}\left(\frac{d}{-1}\right)^{1/2}\int_0^\infty\sum_{n\neq 0}\left(\frac{d}{n}\right)\frac{a_\varphi(n)}{n^k}W_{\mathrm{sign}(n)k,s-\frac{1}{2}}(t)\frac{dt}{t}.
\end{aligned}$$

Here we used the evaluation of the Gauss sum

$$\sum_{\mu(d)}\left(\frac{d}{\mu}\right)e\left(-n\frac{\mu}{d}\right)=\sqrt{|d|}\left(\frac{d}{-1}\right)^{1/2}\left(\frac{d}{n}\right).$$

Finally, the the condition $(-1)^k d > 0$ gives $\left(\frac{d}{-n}\right) = (-1)^k\left(\frac{d}{|n|}\right)$ for $n > 0$. From Proposition 3.17 we get

$$\mathrm{Tr}_{d,d}(\varphi) = 2^{\nu(N)}\frac{|d|^{k+\frac{1}{2}}(-1)^{\lfloor k/2\rfloor}}{(4\pi)^k}\Psi_{2k}(s)L(\varphi,d,k)$$

after a standard argument in analytic continuation, where the Mellin transform $\Psi_{2k}(s)$ is defined in (5.3). The evaluation in Lemma 5.6 then finishes the proof. □

W. Kohnen [**Koh85**, Cor. 1] first proved the analog of Theorem 5.7 for holomorphic cusp forms. The proof is the same as the proof we gave for Theorem 5.7 with the exception that there is no distinction between odd and even Maass cusp forms and we use $a_\varphi(n) = 0$ for $n < 0$

Theorem 5.7 can be used to prove the nonnegativity of twisted L-functions of Maass cusp forms at the central value. This is a generalization of Corollary 1.3, which corresponds to the case $d = 1$, $N = 1$ and $k = 0$ in the next corollary.

COROLLARY 5.8. *Let k be a non-negative integer, let $\varphi \in U_{2k,N}(s)$ be an even normalized newform, and let d be a fundamental discriminant such that $(-1)^k d > 0$ and $(d,N) = 1$. Then $L(\varphi,d,k) \geq 0$.*

PROOF. This follows almost immediately from Theorem 5.7; we only need to check that the Γ-factors are positive. If $s \in \mathbf{R}$, i.e. if s is the spectral parameter of an exceptional eigenvalue, then we must have $1-k < s < k$ for $k > 0$ and $0 < s < 1$ for $k = 0$, which ensures that both Γ-factors are positive. If s is not an exceptional eigenvalue, i.e. if $\mathrm{Re}(s) = \frac{1}{2}$, we can use the property $\Gamma(x+iy)\Gamma(x-iy) = |\Gamma(x+iy)|^2$ to show that the Γ-factors are positive. $\square$

Part 2

Cycle Integrals of the Parson Poincaré Series

CHAPTER 6

Hyperbolic Poincaré Series

Let Γ be a Fuchsian group of first kind, i.e. a cofinite discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$; we will furthermore always assume that $-I \in \Gamma$ and that Γ is symmetric, i.e. that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the matrix $\gamma' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ lies in Γ. In this chapter, we will be interested in the space of holomorphic cusp forms $S_{2k}(\Gamma)$ with $k \geq 1$ for the group Γ and construct a generating set of Poincaré series.

In the case of the full modular group $\mathrm{SL}_2(\mathbf{Z})$, a generating set of the space $S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ with $k > 1$ is given by the parabolic Poincaré series. However, the parabolic Poincaré series only exist when Γ contains parabolic elements, i.e. when the surface $\Gamma \setminus \mathbf{H}$ has cusps. Generally, this does not need to be the case for Fuchsian groups of the first kind.

However, it is always the case that Γ contains hyperbolic elements. Parabolic Poincaré series are defined by averaging the exponential function $e(z) = e^{2\pi i z}$ over the cosets $\Gamma_\infty \backslash \Gamma$. Of course, the exponential is invariant under the action of the parabolic elements in the subgroup Γ_∞. Instead of taking functions invariant under the parabolic elements of Γ, we can look at functions which are invariant for a hyperbolic element $\gamma \in \Gamma$, namely the quadratic form $Q_\gamma(z, 1)$. Averaging a suitable power of these rational functions over the cosets $\Gamma_\gamma \setminus \Gamma$ will also give us a cusp form.

More precisely, let $k > 1$, $\gamma \in \Gamma$ be a hyperbolic element, and Q_γ its corresponding binary quadratic form. Let $D_\gamma = \mathrm{disc}(Q_\gamma)$ and $\chi_\gamma = \mathrm{sign}(\mathrm{Tr}(\gamma))$. Define the following Poincaré series

$$P_\gamma^{2k}(z) = -\chi_\gamma \frac{D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{g \in \Gamma_\gamma \setminus \Gamma} \frac{1}{(Q_\gamma \circ g)(z, 1)^k}, \tag{6.1}$$

which converges absolutely and uniformly in z on compact subsets of $\mathbf{H}$. We call series of this type *hyperbolic Poincaré series*. They lie in the space $S_{2k}(\Gamma)$; see [**Kat85**, Thm. 1.i)]. An equivalent formulation to summing over the cosets $\Gamma_\gamma \setminus \Gamma$ in (6.1) would be to sum over the equivalence class of Q_γ under the action of Γ. If γ is conjugacy equivalent to γ^{-1}, i.e. if Q_γ is equivalent to $-Q_\gamma$, the Poincaré series $P_\gamma^{2k}(z)$ vanishes identically for odd k.

More generally, if $\gamma, \sigma \in \Gamma$ lie in the same conjugacy class, the Poincaré series $P_\gamma^{2k}(z)$ and $P_\sigma^{2k}(z)$ are identical. Moreover, for all $P_{\gamma^n}^{2k}(z) = n \cdot \alpha_n^{k-1} P_\gamma^{2k}(z)$ for some $\alpha_n \in \mathbf{R}$, so we may restrict our attention to primitive hyperbolic elements $\gamma \in \Gamma$, since the Poincaré series $P_\gamma^{2k}(z)$ and $P_{\gamma^n}^{2k}(z)$ are linearly dependent.

Given a holomorphic function $f : \mathbf{H} \to \mathbf{C}$, we denote the cycle integral by $C_\gamma^{2k}(f, z_0) = \displaystyle\int_{z_0}^{\gamma.z_0} f(z) Q_\gamma(z,1)^{k-1}\, dz$. The hyperbolic Poincaré series $P_\gamma^{2k}(z)$ are the Riesz representatives for the cycle integrals of cusp forms, namely

$$\langle f, P_\gamma^{2k} \rangle = c_k \cdot C_\gamma^{2k}(f, z_0) \tag{6.2}$$

for all $f \in S_{2k}(\Gamma)$ and where $c_k = 2^{2-k}\binom{2k-2}{k-1}$; see [**Kat85**, Prop. 3]. The right hand side of (6.2) is independent of $z_0 \in \mathbf{H}$.

Lemma 6.1. *Let $k > 0$ and γ be a hyperbolic element. If $f : \mathbf{H} \to \mathbf{C}$ is holomorphic and modular of weight $2k$, then the function $C_\gamma^{2k}(f, z_0)$ is constant in $z_0 \in \mathbf{H}$.*

Proof. The function $C_\gamma^{2k}(f, z_0)$ is differentiable in z_0 and using modularity we get

$$\partial_{z_0} C_\gamma^{2k}(f, z_0) = \left((f|_{2k}\gamma)(z_0) Q_\gamma(z_0, 1)^{k-1} - f(z_0) Q_\gamma(z_0,1)^{k-1} \right) = 0.$$

□

Equation (6.2) also shows that – in the case $\Gamma = \Gamma_0(N)$ – the cusp form $P_{d,d'}^{2k}(z) = \sum_{Q \in \mathcal{Q}_{N,dd'}/\Gamma} \chi_d(Q) P_{\gamma_Q}^{2k}(z)$ with $d, d' > 0$ such that $(-1)^k d$ is a fundamental discriminant and $(-1)^k d' \equiv 0, 1 \pmod 4$ constitute the Fourier coefficients of the kernel function of the d-th holomorphic Shintani lift; see Theorem 1.1.

The right hand side of (6.2) is invariant under the choice of the path by Cauchy's theorem, since the integrand is holomorphic. Moreover, the base point $z_0 \in \mathbf{H}$ can be chosen freely by the modularity of f.

We can also consider the symmetrizations

$$P_\gamma^{2k,+}(z) = \frac{1}{2}\left(P_\gamma^{2k}(z) + P_{\gamma'}^{2k}(z)\right) \text{ and } P_\gamma^{2k,-} = \frac{1}{2i}\left(P_\gamma^{2k}(z) - P_{\gamma'}^{2k}(z)\right) \tag{6.3}$$

of the hyperbolic Poincaré series.[1] The Poincaré series $P_\gamma^{2k,+}(z)$ should not be confused with the projections onto the Kohnen plusspace. Each of the sets $\{P_\gamma^{2k}(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$, $\{P_\gamma^{2k,+}(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$, and $\{P_\gamma^{2k,-}(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$ is a spanning set for the space $S_{2k}(\Gamma)$; see [**Kat85**, Thm. 1].

[1] The inner product of a cusp form $f \in S_{2k}(\Gamma)$ with the symmetrizations corresponds to taking the cycle integral with respect to γ of $f(z) \pm \overline{f(-\overline{z})}$. If the surface $\Gamma \setminus \mathbf{H}$ has cusps, we see from the Fourier expansion of f that $f(z) \pm \overline{f(-\overline{z})}$ is the real and imaginary part of f respectively.

In particular, this means that a cusp form $f \in S_{2k}(\Gamma)$ is completely determined by the values of its cycle integrals.

Since $Q_\gamma(z,1) = -Q_{\gamma'}(-z,1)$, we have $P_{\gamma'}(-z) = (-1)^k P_\gamma(z)$ and in particular that $\mathrm{Im}\langle P_\gamma^{2k}, P_\sigma^{2k}\rangle = \frac{1}{2i}\left(\langle P_\gamma^{2k}, P_\sigma^{2k}\rangle - \langle P_{\gamma'}^{2k}, P_{\sigma'}^{2k}\rangle\right)$. Indeed, pick $z_0 \in S_\gamma$ and chose the geodesic path from z_0 to $\gamma.z_0$. Its image under $z \mapsto -\overline{z}$ corresponds to a geodesic path from $z_1 = -\overline{z_0}$ to $-\overline{\gamma.z_0} = \gamma'.z_1$ on $S_{\gamma'}$. We have

$$\begin{aligned}
&\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz\right)\\
&= \frac{1}{2i}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz - \overline{\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1}\, dz}\right)\\
&= \frac{1}{2i}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz + \int_{z_1}^{\sigma'.z_1} P_\gamma^{2k}(-z) Q_\sigma(-z,1)^{k-1} dz\right)\\
&= \frac{1}{2i}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz - \int_{z_1}^{\sigma'.z_1} P_{\gamma'}^{2k}(z) Q_{\sigma'}(z,1)^{k-1} dz\right).
\end{aligned}$$

So the imaginary part of the cycle integral of $P_\gamma^{2k}(z)$ can be realized as the difference of two cycle integrals. S. Katok [**Kat85**, Thm. 3] used this observation to prove a striking geometric formula for the imaginary part of the cycle integral. As we will shortly see, the proof is an application of the unfolding method, where $\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz\right)$ is transformed to a finite sums of integrals over closed loops in the complex plane.

THEOREM 6.2 (Katok). *Let $k > 1$ be an integer, γ, σ be primitive hyperbolic elements in Γ, D_γ, D_σ their corresponding discriminants and set $\chi_\gamma = \mathrm{sign}(\mathrm{Tr}(\gamma))$ and $\chi_\sigma = \mathrm{sign}(\mathrm{Tr}(\sigma))$. We have*

$$\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^{2k}(z) Q_\sigma(z,1)^{k-1} dz\right) = \chi_\gamma\chi_\sigma (D_\gamma D_\sigma)^{\frac{k-1}{2}} \frac{8}{\binom{2k-2}{k-1}} \sum_p \mu_p^k\, P_{k-1}(\cos\theta_p),$$

where the sum runs over the intersections points p of the geodesics C_γ and C_σ in $\Gamma\backslash\mathbf{H}$, $\theta_p \in [0,\pi]$ is the angle of intersection at p, $\mu_p = \mu_p(\gamma,\sigma)$ is the signed topological intersection number, and P_r denotes the r-th Legendre polynomial.

If Γ is an arithmetic group, the traces of its elements are rational. By a lemma of Wolpert [**Kat85**, Lemma, p. 479], the right hand side in Theorem 6.2 is rational. Since

$$-4i\langle P_\gamma^{2k,+}, P_\sigma^{2k,-}\rangle = (\langle P_\gamma^{2k}, P_\sigma\rangle - \langle P_{\gamma'}^{2k}, P_{\sigma'}^{2k}\rangle) + (\langle P_{\gamma'}^{2k}, P_\sigma\rangle - \langle P_{\gamma'}^{2k}, P_\sigma^{2k}\rangle)$$

for any $\gamma, \sigma \in \Gamma$ hyperbolic, the inner product $\langle P_\gamma^{2k,+}, P_\sigma^{2k,-}\rangle$ is rational as well.

Hence, the $\mathbf{Q}$-vector spaces

$$\begin{aligned} S_{2k}^{+} &= \{f \in S_{2k} : \langle f, P_\gamma^{2k,-}\rangle \in \mathbf{Q},\ \text{for all } \gamma \in \Gamma \text{ primitive hyperbolic}\}, \\ S_{2k}^{-} &= \{f \in S_{2k} : \langle f, P_\gamma^{2k,+}\rangle \in \mathbf{Q},\ \text{for all } \gamma \in \Gamma \text{ primitive hyperbolic}\} \end{aligned} \tag{6.4}$$

define rational structures on the space $S_{2k}(\Gamma)$ for arithmetic subgroups Γ, i.e. $S_{2k}^{\pm} \otimes_{\mathbf{Q}} \mathbf{C} \cong S_{2k}(\Gamma)$ as $\mathbf{C}$-vector spaces, since both $\{P_\gamma^{2k,+}(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$ and $\{P_\gamma^{2k,-}(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$ generate the space $S_{2k}(\Gamma)$. The dimensions of $S_{2k}^{\pm}$ are each equal to $\dim_{\mathbf{C}} S_{2k}(\Gamma)$ and by definition the two spaces S_{2k}^{+} and S_{2k}^{-} form a $\mathbf{Q}$-linear dual pair with the Petersson inner product, i.e. the inner product $\langle\cdot,\cdot\rangle$ constitutes a non-degenerate bilinear map $S_{2k}^{+} \times S_{2k}^{-} \to \mathbf{Q}$.

The two rational structures in (6.4) coincide with the rational structures arising from the Eichler-Shimura isomorphism [**KM87**]. Define the n-th period of a cusp form $f \in S_{2k}(\Gamma)$ by

$$r_n(f) = \int_0^\infty f(it)t^n dt,\ 0 \le n \le 2k-2.$$

Then we have

$$\begin{aligned} S_{2k}^{+} &= \{f \in S_{2k}(\Gamma) : r_n(f) \in \mathbf{Q},\ 0 \le n \le 2k-2,\ n \text{ even}\}, \\ S_{2k}^{-} &= \{f \in S_{2k}(\Gamma) : r_n(f) \in \mathbf{Q},\ 0 \le n \le 2k-2,\ n \text{ odd}\}. \end{aligned}$$

Generally, rational structures on the space of cusp forms contain a lot of arithmetic information, e.g., the rational structure on $S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ given by cusp forms with rational Fourier coefficients, so the rational structures $S_{2k}^{\pm}$ are of significant interest. In the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, W. Kohnen and D. Zagier [**KZ84**, Thm. 4] showed that $P_{1,d}^{2k}$ in fact belongs to S_{2k}^{+}.

THEOREM 6.3 (Kohnen, Zagier). *Let $k > 1$. For all $0 \le n \le 2k-2$ even, we have $r_n(P_{1,d}^{2k}) \in \mathbf{Q}$.*

1. The Space $S_2(\Gamma)$

Since we do not have absolute convergence for $k = 1$ in (6.1), the definition of the hyperbolic Poincaré series of weight 2 is more involved. We define the non-holomorphic hyperbolic Poincaré series of weight 2 associated to a hyperbolic element $\gamma \in \Gamma$ by

$$P_\gamma^2(z,s) = -\chi_\gamma \frac{\sqrt{D_\gamma}}{\pi} \sum_{g \in \Gamma_\gamma \backslash \Gamma} \frac{1}{(Q_\gamma \circ g)(z,1)} \frac{y^s}{|(Q_\gamma \circ g)(z,1)|^s}, \tag{6.5}$$

which is absolutely and uniformly convergent in z on compact subsets of $\mathbf{H}$ for $\mathrm{Re}(s) > 0$. The series $P_\gamma^2(z,s)$ is modular of weight 2, vanishes at the cusps and is square integrable. Similarly to the Selberg Poincaré series, the hyperbolic Poincaré series $P_\gamma^2(z,s)$ satisfies

$$\Delta_2 P_\gamma^2(z,s) = s(1-s)P_\gamma^2(z,s) + D_\gamma s(s+2)P_\gamma^2(z,s+2). \tag{6.6}$$

The hyperbolic Poincaré series admits an analytic continuation through its spectral expansion and is holomorphic at $s=0$; see J. Jorgenson, J. Kramer, and A.-M. von Pippich [**KJvP10**]. The function $P_\gamma^2(z) = P_\gamma^2(z,0)$ is a cusp form of weight 2 for Γ, since

$$0 = \langle P_m^2(z), \Delta_2 P_m^2(z)\rangle = \langle \xi_2 P_\gamma^2, \xi_2 P_\gamma^2(z)\rangle$$

by (6.6) and Stokes' theorem.

As in the higher weight case, we have $P_\gamma^2(z,s) = P_\sigma^2(z,s)$ if γ and σ lie in the same conjugacy class and $P_{\gamma^n}^2(z,s) = nP_\gamma^2(z,s)$ for $n \geq 1$. Some of the Poincaré series $P_\gamma^{2k}(z,s)$ vanish identically, e.g., if γ and γ^{-1} lie in the same conjugacy class. In the remainder of this section, will extend the results for $k > 1$ to $k = 1$ and prove as an application a group-theoretic characterization for the vanishing of the Poincaré series of weight 2.

The hyperbolic Poincaré series of weight 2 are also the Riesz representatives for the cycle integrals.

Proposition 6.4. *Let $\gamma \in \Gamma$ be a hyperbolic element and $f \in S_2(\Gamma)$. We have* $\langle f, P_\gamma^2(z)\rangle = \int_{z_0}^{\gamma.z_0} f(z)dz$.

Proof. This can be done as in W. Kohnen's paper for $\Gamma = \Gamma_0(N)$ [**Koh85**, Prop. 7]. Let $\mathrm{Re}(s) > 0$. By unfolding the inner product, we get

$$\langle f, P_\gamma^{2k}(z,s)\rangle = \int_{\Gamma_\gamma\backslash\mathbf{H}} f(z)\frac{1}{Q_\gamma(\overline{z},1)}\frac{y^s}{|Q_\gamma(z,1)|^s}dxdy.$$

A change of variables gives $\langle f, P_\gamma^{2k}(z,s)\rangle = \frac{1}{\pi}D_\gamma^{-\frac{s}{2}}\int_0^\pi\int_{z_0}^{\gamma.z_0} f(z)(\sin\theta)^s dzd\theta$, where we pick z_0 such that $\theta = \arg(z_0 - w_\gamma) - \arg(z_0 - w_\gamma') \in (0,\pi)$ for the roots $w_\gamma' < w_\gamma$ of $Q_\gamma(z,1) = 0$. With this change of variables, we have $\frac{y^s}{|Q_\gamma(z,1)|^s} = D_\gamma^{-s/2}(\sin\theta)^s$.

By the holomorphy and modularity of $f(z)$, the cycle integral $\int_{z_0}^{\gamma.z_0} f(z)(\sin\theta)^s dz$ is invariant under the choice of path and equal for all choices of $z_0 \in \mathbf{H}$ by Lemma 6.1. Thus, we may write

$$\int_0^\pi\int_{z_0}^{\gamma.z_0} f(z)(\sin\theta)^s dzd\theta = \int_{z_0}^{\gamma.z_0} f(z)dz\int_0^\pi(\sin\theta)^s d\theta.$$

Letting $s \to 0^+$ proves the claim. □

We now show that a cusp form of weight 2 is completely determined by the values of its cycle integrals. The proof of the next theorem is an adaptation of the proof of [**Kat85**, Thm. 2].

THEOREM 6.5. *Let* $f \in S_2(\Gamma)$. *If* $\displaystyle\int_{z_0}^{\gamma.z_0} f(z)dz = 0$ *for all primitive hyperbolic* $\gamma \in \Gamma$, *then* $f = 0$. *In particular, the set* $\{P_\gamma^2(z) : \gamma \in \Gamma \text{ primitive hyperbolic}\}$ *generates* $S_2(\Gamma)$.

PROOF. Define $F(z) = \displaystyle\int_i^z f(w)dw$. The function F is Γ-invariant by assumption (invariance for parabolic and elliptic elements follows trivially from f being a cusp form). Since each cusp form is bounded on a fundamental domain, the function F is Lipschitz-continuous on $\Gamma \setminus \mathbf{H}$. Moreover, we have $R_0 F(z) = 2if(z)$ by definition, where R_0 is the raising operator defined in (3.7).

We proceed to show that the function F is bounded on $\Gamma\setminus\mathbf{H}$, which by the cofiniteness of Γ implies that F is square-integrable. By the continuity of F this is certainly true for each compact subset of $\Gamma \setminus \mathbf{H}$. Moreover, F is bounded at each of the (finitely many) cusps of $\Gamma \setminus \mathbf{H}$, since f is exponentially decaying at each cusp.

So by Stokes' theorem, we have

$$-4\langle f, f\rangle = \langle R_0F, R_0F\rangle = \langle F, L_2R_0F\rangle = 0,$$

since $L_2 f = 0$. So we must have $f = 0$.

The second claim follows from Proposition 6.4. □

Repeating the argument in the proof of Theorem 6.5 with the functions $F^{\pm}(z) = F(z) \pm F(-\overline{z})$ proves that the collection of the symmetrizations $P_\gamma^{2,+}(z)$ and $P_\gamma^{2,-}(z)$, defined as in (6.3), also form spanning sets for $S_2(\Gamma)$.

We now proceed to prove Theorem 6.2 for $k = 1$.

THEOREM 6.6. *Let* γ, σ *be primitive hyperbolic elements in* Γ, D_γ, D_σ *their corresponding discriminants and set* $\chi_\gamma = \operatorname{sign}(\operatorname{Tr}(\gamma))$ *and* $\chi_\sigma = \operatorname{sign}(\operatorname{Tr}(\sigma))$. *We have*

$$\operatorname{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^2(z)dz\right) = \chi_\gamma\chi_\sigma \sum_p \mu_p,$$

where the sum runs over the intersections points p *of the geodesics* C_γ *and* C_σ *in* $\Gamma \setminus \mathbf{H}$ *and* $\mu_p = \mu_p(\gamma, \sigma)$ *is the signed topological intersection number.*

PROOF. The imaginary part of the cycle integral is equal to $\frac{1}{2i}\left(\langle P_\gamma^2, P_\sigma^2\rangle - \langle P_{\gamma'}^2, P_{\sigma'}^2\rangle\right)$. Pick $z_0 \in S_\sigma$. By unfolding, we get for $\mathrm{Re}(s) > 0$:

$$\int_{z_0}^{\sigma.z_0} P_\gamma^2(z,s)dz = -\frac{\chi_\gamma\sqrt{D_\gamma}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{S_\sigma}\frac{1}{(Q_\gamma\circ g)(z,1)}\frac{y^s}{|(Q_\gamma\circ g)(z,1)|^s}dz$$

and

$$\langle P_{\gamma'}^2(\cdot,s), P_{\sigma'}^2\rangle = -\frac{\chi_\gamma\sqrt{D_\gamma}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{S_{\sigma'}}\frac{1}{(Q_{\gamma'}\circ g')(z,1)}\frac{y^s}{|(Q_{\gamma'}\circ g)(z,1)|^s}dz. \tag{6.7}$$

The change of variables $z \to -z$ in (6.7) with $(Q_{\gamma'}\circ g')(-z,1) = -(Q_\gamma\circ g)(z,1)$ gives that $\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^2(z,s)dz\right)$ is equal to

$$-\frac{\chi_\gamma\sqrt{D_\gamma}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{C(\sigma)}\frac{1}{(Q_\gamma\circ g)(z,1)}\frac{y^s}{|(Q_\gamma\circ g)(z,1)|^s}dz,$$

where $C(\sigma)$ is the circle in $\mathbf{C}$ through the roots of $Q_\sigma(z,1) = 0$ oriented according to the orientation of S_γ. Approximating the term $\frac{y^s}{|(Q_\gamma\circ g)(z,1)|^s} = 1 + O(s)$, we see that

$$\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^2(z,s)dz\right) = -\frac{\chi_\gamma\sqrt{D_\gamma}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{C(\sigma)}\frac{1}{(Q_\gamma\circ g)(z,1)}dz + O(s),$$

where by the Residue Theorem the first term on the right hand side is a finite sum over those $g \in \Gamma_\gamma\setminus\Gamma/\Gamma_\sigma$ such that one of the roots of $(Q_\gamma\circ g)(z,1) = 0$ lies in $C(\sigma)$. Letting $s \to 0$, we hence have

$$\mathrm{Im}\left(\int_{z_0}^{\sigma.z_0} P_\gamma^2(z)dz\right) = -\frac{\chi_\gamma\sqrt{D_\gamma}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{C(\sigma)}\frac{1}{(Q_\gamma\circ g)(z,1)}dz.$$

Proceeding as in S. Katok's paper [**Kat85**, p. 477f] or in the proof of Theorem 7.5 below, we get

$$\int_{C(\sigma)}\frac{1}{(Q_\gamma\circ g)(z,1)}dz = -\frac{2\pi i}{\sqrt{D_\gamma}}\chi_\sigma\mu_p,$$

which proves the claim. □

As with higher weight, we can define the rational structures S_2^+ and S_2^-, which are a $\mathbf{Q}$-linear dual pair with the Petersson inner product. Note that it is not necessary here to restrict to arithmetic groups.

1.1. Parabolic Homomorphisms. Define the subspace $H^1_{\mathrm{par}}(\Gamma, \mathbf{C})$ of the homomorphisms $\Gamma \to \mathbf{R}$ by the exact sequence

$$0 \to H^1_{\mathrm{par}}(\Gamma, \mathbf{R}) \to H^1(\Gamma, \mathbf{R}) \to \prod_{\mathfrak{a}} H^1(\Gamma_{\mathfrak{a}}, \mathbf{R}),$$

where $\mathfrak{a}$ denote the cusps of the surface $\Gamma \setminus \mathbf{H}$ and the last map is given by the restriction map. We call the space $H^1_{\mathrm{par}}(\Gamma, \mathbf{R})$ the parabolic homomorphisms, i.e. those homomorphisms $\Gamma \to \mathbf{C}$ which vanish on all parabolic elements.

The cusp forms of weight 2 are essentially given by the parabolic homomorphisms. More precisely, we have the Eichler-Shimura isomorphism; see [**Eich57, Shim59, Shim71, Mol17**].

Theorem 6.7 (Eichler-Shimura). *Let $S_{2,\mathbf{R}}(\Gamma)$ be the real vector space spanned by $P^2_\gamma(z)$. The map*

$$S_{2,\mathbf{R}}(\Gamma) \to H^1_{\mathrm{par}}(\Gamma, \mathbf{R})$$

given by $f \mapsto \left(\gamma \mapsto \mathrm{Re} \displaystyle\int_{z_0}^{\gamma . z_0} f(z)dz\right)$ for hyperbolic $\gamma \in \Gamma$ is an isomorphism.

For all hyperbolic $\gamma, \sigma \in \Gamma$ the property $P^2_\gamma(z) + P^2_\sigma(z) = P^2_{\gamma\sigma}(z)$ follows by Lemma 6.1 and Proposition 6.4, so we see that $\gamma \mapsto \mathrm{Re}\langle f, P^2_\gamma \rangle$ in Theorem 6.7 indeed defines a homomorphism.

We will use the Eichler-Shimura isomorphism to prove a result about the vanishing of the cusp forms $P^2_\gamma(z)$ dependent on group-theoretic conditions of γ in Γ. We see here a crucial distinction between parabolic and hyperbolic Poincaré series, since vanishing theorems for parabolic Poincaré series are very difficult to prove (e.g., Lehmer's conjecture). On the other hand, theorems on the vanishing of Poincaré series built out of rational functions have been previously obtained [**Kra84**].

The Eichler-Shimura isomorphism relates the cusp forms to the parabolic homomorphisms, which implies that a hyperbolic Poincaré series $P^2_\gamma(z)$ vanishes identically if and only if $\phi(\gamma) = 0$ for all $\phi \in H^1_{\mathrm{par}}(\Gamma, \mathbf{R})$. The latter makes it much easier to characterize the primitive hyperbolic $\gamma \in \Gamma$ for which this is true. This is the content of the next lemma.[2]

Lemma 6.8. *Let $\gamma \in \Gamma$ be a primitive hyperbolic element and let Γ' be the subgroup of Γ containing $[\Gamma, \Gamma]$ together with the parabolic and elliptic elements of Γ. We have $\phi(\gamma) = 0$ for each $\phi \in H^1_{\mathrm{par}}(\Gamma, \mathbf{R})$ if and only if there is an $n \geq 1$ such that $\gamma^n \in \Gamma'$.*

[2] I am indebted to Jason DeVito on StackExchange for this proof.

PROOF. Suppose that there is an $n \geq 1$ such that $\gamma^n \in \Gamma'$. It is easy to see that $\phi(\sigma) = 0$ for all $\sigma \in [\Gamma, \Gamma]$. Since the elliptic elements of Γ are of finite order, each homomorphism $\Gamma \to \mathbf{C}$ vanishes on the elliptic elements. By definition, each parabolic homomorphism vanishes on the parabolic elements. Since clearly each homomorphism vanishes on the elements of $[\Gamma, \Gamma]$, each $\phi \in H^1_{\text{par}}(\Gamma, \mathbf{R})$ vanishes on the elements of Γ' and we must have $n \cdot \phi(\gamma) = \phi(\gamma^n) = 0$.

Suppose now that no power of γ lies in the (normal) subgroup Γ'. Since each $\phi \in H^1_{\text{par}}(\Gamma, \mathbf{R})$ vanishes on Γ', we may view ϕ as a homomorphism on the abelian group $G = \Gamma/\Gamma'$. Moreover, ϕ vanishes on the torsion subgroup of G, so we can assume G to be torsion-free. By assumption (the class of) γ in G is not the identity. Since G is abelian and torsion-free, G is a $\mathbf{Z}$-module. Define the $\mathbf{R}$-vector space $V_G = G \otimes_{\mathbf{Z}} \mathbf{R}$, for which the element $v_\gamma = \gamma \otimes 1$ under the inclusion $\iota : \Gamma \to V_G$ is non-zero. Complete v_γ to a basis $\{v_\gamma, v_1, v_2, ...\}$ of V_G and define $f : V_G \to \mathbf{R}$ by $f(v_\gamma) = 1$ and $f(v_i) = 0$ for $i = 1, 2, ...$ The homomorphism $\phi_\gamma = f \circ \iota$ is a homomorphism on G such that $\phi_\gamma(\gamma) \neq 0$. □

With Lemma 6.8 and the Eichler-Shimura isomorphism we are set to show the vanishing result for the set of Poincaré series $\{P^2_\gamma(z) : \gamma \text{ primitive hyperbolic}\}$.

THEOREM 6.9. *Let $\gamma \in \Gamma$ be a primitive hyperbolic element. The Poincaré series $P^2_\gamma(z)$ vanishes identically if and only if a positive power of γ lies in the subgroup generated by $[\Gamma, \Gamma]$ together with the elliptic and the parabolic elements of Γ.*

PROOF. Suppose first that a positive power of γ lies in the subgroup generated by $[\Gamma, \Gamma]$ together with the elliptic and the parabolic elements of Γ. The map $\phi_f(\sigma) = \operatorname{Re} \displaystyle\int_{z_0}^{\sigma.z_0} f(z)dz$ is a parabolic homomorphism $\Gamma \to \mathbf{R}$ for each $f \in S_{2,\mathbf{R}}(\Gamma)$. So we have $\phi_f(\gamma) = 0$ for all $f \in S_{2,\mathbf{R}}(\Gamma)$ by Lemma 6.8. Picking $f = P_\gamma(z)$, we see that $\phi_{P^2_\gamma}(\gamma) = \langle P^2_\gamma, P^2_\gamma \rangle = 0$.

Suppose now that $P^2_\gamma(z)$ vanish identically. By the Eichler-Shimura isomorphism in Theorem 6.7, each $\phi \in H^1_{\text{par}}(\Gamma, \mathbf{R})$ must satisfy $\phi(\gamma) = 0$. Hence, by Lemma 6.8 a positive power of γ lies in the subgroup generated by $[\Gamma, \Gamma]$ together with the elliptic and the parabolic elements of Γ. □

The Theorems 6.5 and 6.9 give a characterization of the existence of cusp forms of weight 2 for each Γ. Let Γ' be defined as in Lemma 6.8. The space $S_2(\Gamma) = \{0\}$ if and only if $\Gamma = \Gamma'$.

CHAPTER 7

The Parson Poincaré Series

Assume that the hyperbolic element $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ lies in the same conjugacy class as γ^{-1}. As we have seen in Chapter 6, each hyperbolic Poincaré series $P_\gamma^{2k}(z)$ vanishes identically for $k > 1$ odd. The right hand side of Theorem 6.2 then also vanishes identically due to the signed topological intersection number μ_p; hence, the theorem has no content in this case. However, what if the signed topological intersection number was not there? Would it still correspond to a cycle integral of a (somewhat) modular function? The goal of this chapter is to give an analog for Theorem 6.2, which is not void of content in the case where $k > 1$ is odd. To this end, we will introduce the Parson Poincaré series, which is not modular, but transforms with a rational cocycle. To make the matter simpler, we will constrain ourselves to the full modular group and use the language of binary quadratic forms in the introduction of this chapter.

For weight $2k$ with $k > 1$ an extension of the Eichler-Shimura isomorphism in Theorem 6.7 gives an isomorphism between the space of cusp forms and the first parabolic cohomology group of Γ with coefficients in a certain space of polynomials of degree at most $2k-2$. For a $f \in S_{2k}(\Gamma)$ the family of polynomial cocycles $\{r_f(g,z) : g \in \Gamma\}$ is given as the periods of the Eichler integrals $F(z) = \displaystyle\int_{z_0}^{z} \frac{(z-w)^{2k-2}}{(2k-2)!} f(w)dw$, i.e. $r_f(g,z) = F(z)|_{2-2k}g - F(z)$ for $g \in \Gamma$. The Eichler integral $F(z)$ is the $(2k-1)$-th primitive of the cusp form f.

Since polynomial parabolic cocycles are given by the cusp forms, a natural generalization is to study the first parabolic cohomology of Γ with coefficients in the rational functions. These rational parabolic cocycles have been extensively studied by M. Knopp [**Kno74, Kno78, Kno81, Kno90**]. Y.J. Choie and D. Zagier [**CZ93**, Thm. 1] gave a comprehensive classification of rational parabolic cocycles, again exploiting a duality between the weights $2k$ and $2-2k$.

It is not easy to see, however, what the corresponding modular integrals to the rational parabolic cocycles are. An early construction of M. Knopp shows that these modular integrals exist [**Kno74**], but the constructed functions are difficult to work with. Possibly the easiest example of a modular integral with rational period function is due to L. A.

Parson [**Par93**]. For an indefinite binary quadratic form Q, we define

$$F_Q^{2k}(z) = -\frac{\mathrm{sign}(\mathrm{Tr}(\gamma_Q)\ \mathrm{disc}(Q)^{k-\frac{1}{2}}}{\pi} \sum_{Q'\sim Q} \frac{\mathrm{sign}(Q')}{(Q'(z,1))^k}, \tag{7.1}$$

which is absolutely and uniformly convergent in z on compact subsets of $\mathbf{H}$ for $k > 1$.[1] The equivalence relation in (7.1) is with respect to the $\mathrm{SL}_2(\mathbf{Z})$-action on the set of binary quadratic forms. We see that $F_Q^{2k}(z)$ is one-periodic, since $\mathrm{sign}(Q' \circ T) = \mathrm{sign}(Q')$, and that

$$\begin{aligned} F_Q^{2k}(z)|_{2k}S - F_Q^{2k}(z) &= -\frac{\mathrm{sign}(\mathrm{Tr}(\gamma_Q)\ \mathrm{disc}(Q)^{k-\frac{1}{2}}}{\pi} \sum_{[A,B,C]\sim Q} \frac{\mathrm{sign}(C) - \mathrm{sign}(A)}{(Az^2+Bz+C)^k} \\ &= \frac{\mathrm{sign}(\mathrm{Tr}(\gamma_Q)\ \mathrm{disc}(Q)^{k-\frac{1}{2}}}{\pi} \sum_{\substack{[A,B,C]\sim Q,\\ AC<0}} \frac{\mathrm{sign}(A)}{(Az^2+Bz+C)^k}. \end{aligned}$$

There are finitely many binary quadratic forms $[A, B, C]$ of discriminant D such that $AC < 0$, since $D = B^2 - 4AC$ implies that $|A|, |C| \leq \frac{D}{4}$ and $|B| \leq \sqrt{D}$. Hence, $F_Q^{2k}(z)$ transforms with a rational period function. An earlier example of an explicit function transforming with rational period function was given by M. Knopp [**Kno90**].

The Parson Poincaré series complete the picture of the hyperbolic Poincaré series in the last chapter. For $k > 1$ and $Q \sim -Q$, the hyperbolic Poincaré series $P_{\gamma_Q}^{2k}(z)$ vanishes identically for k being odd, but not necessarily for k being even. The case is the opposite for the Parson Poincaré series: The Poincaré series $F_Q^{2k}(z)$ vanishes identically for $k > 1$ being even if $Q \sim -Q$, but not necessarily so for k being odd.

To see this, consider the weight 2 analog of Parson's Poincaré series. Let Q be an indefinite binary quadratic form and $D = \mathrm{disc}(Q)$. Define

$$F_Q^2(z) = -\frac{\mathrm{Tr}(\gamma_Q)\sqrt{D}}{\pi} \lim_{s\to 0^+} \sum_{Q'\sim Q} \frac{\mathrm{sign}(Q') y^s}{(Q')(z,1)|Q'(z,1)|^s} - 2\mathrm{Tr}(\gamma_Q) \cdot \log\left(\xi_{\gamma_Q}\right) E_2^*(z), \tag{7.2}$$

where $E_2^*(z) = E_2(z) - \frac{3}{\pi y}$ is the harmonic Eisenstein series of weight 2. We start by collecting various properties of $F_Q^2(z)$, which were proved by T. Matsusaka [**Mat20**, Thm. 2.9].

[1] Our definition of L. A. Parson's modular integral differs from the originial definition by a cusp form of weight $2k$. We chose to give it this form, because the formulas below will then be easier.

THEOREM 7.1 (Matsusaka). *The function $F_Q^2(z)$ is holomorphic, one-periodic, and transforms with the rational cocycle*

$$r_Q(g,z) = F_Q^2(z)|_2 g - F_Q^2(z) = -\frac{\operatorname{sign}(\operatorname{Tr}(\gamma_Q))\sqrt{D}}{\pi} \sum_{\substack{Q_0\sim Q,\\ \operatorname{sign}(Q_0)\operatorname{sign}(Q_0(-d,c))<0}} \frac{\operatorname{sign}(Q_0)}{Q_0(z,1)}, \tag{7.3}$$

where $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

The right hand side of (7.3) is again finite; see [**DIT17**, Rem. 3.1]. Moreover, the condition $\operatorname{sign}(Q_0)\operatorname{sign}(Q_0(-d,c)) < 0$ is equivalent to the two roots $w'_{Q_0} < w_{Q_0}$ of $Q_0(z,1) = 0$ being separated by $-\frac{d}{c}$.

The family of cocycles $\{r_Q(g,z) : g \in \mathrm{SL}_2(\mathbf{Z})\}$ admit unique primitive cocycles $R_Q(g,z) = \int_z^{g.z} F_Q(w)dw$, i.e. weight zero cocycles $R_Q(g,z)$ such that $\partial_z R_Q(g,z) = r_Q(g,z)$. The primitive cocycles are unique, since the difference of any two such cocycles would be a homomorphism $\mathrm{SL}_2(\mathbf{Z}) \to \mathbf{C}$, which must be trivial (the modular group is generated by elliptic elements). In light of (7.3), the primitive cocycle is

$$R_Q(g,z) = -\frac{\operatorname{sign}(\operatorname{Tr}(\gamma_Q))}{\pi} \sum_{\substack{Q_0\sim Q,\\ w'_{Q_0}<-\frac{d}{c}<w_{Q_0}}} \left(\log(z-w_{Q_0}) - \log(z-w'_{Q_0})\right) + \Phi_Q(g)$$

for some $\Phi_Q(g) \in \mathbf{C}$.

Using the cocycle condition $0 = R_Q(\sigma, \sigma^{-1}.z) + R_Q(\sigma^{-1}, z)$ for $g \in \mathrm{SL}_2(\mathbf{Z})$ and letting $z \to i\infty$, we see that

$$\Phi_Q(g) + \Phi_Q(g^{-1}) = \frac{2\cdot\operatorname{sign}(\operatorname{Tr}(\gamma_Q))}{\pi} \sum_{\substack{Q_0\sim Q,\\ w'_{Q_0}<-\frac{d}{c}<w_{Q_0}}} \left(\log(-d/c-w_{Q_0}) - \log(-d/c-w'_{Q_0})\right). \tag{7.4}$$

The imaginary part of $\Phi_Q(\sigma)+\Phi_Q(\sigma^{-1})$ hence is equal to twice the number of intersections of the geodesics $\{S_{Q\circ g} : g \in \Gamma_Q \setminus \mathrm{SL}_2(\mathbf{Z})\}$ with the geodesic $S_{-d/c}$ from $-\frac{d}{c}$ to $i\infty$. As a matter of fact, we already have

$$\operatorname{Im}\Phi_Q(g) = \operatorname{sign}(\operatorname{Tr}(\gamma_Q)) \sum_{\substack{Q_0\sim Q,\\ w'_{Q_0}<-\frac{d}{c}<w_{Q_0}}} 1;$$

see [**DIT17**, Thm. 5.2].

Let $\Psi_Q(\sigma) = \lim_{n\to+\infty} \frac{\operatorname{Im}\Phi_Q(\sigma^n)}{n}$ be the homogenization of $\Phi_Q(\sigma)$. This hyperbolic analog of the classical Rademacher symbol is the intersection number of the two geodesics

S_Q and S_σ projected onto the fundamental domain $\mathrm{SL}_2(\mathbf{Z}) \setminus \mathbf{H}$; see the proof of [**DIT17**, Thm. 6.8]. T. Matsusaka [**Mat20**, Cor. 3.8] gave the hyperbolic Rademacher symbol as a homogenized cycle integral, i.e.

$$\lim_{n\to+\infty} \mathrm{Im}\left(\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_Q^2(z)dz\right) = \chi_{\gamma_Q}\chi_\sigma \sum_{p\in[S_Q]\cap[S_\sigma]} 1, \tag{7.5}$$

where the points p run over the intersection points of the geodesics S_Q and S_σ projected onto $\mathrm{SL}_2(\mathbf{Z}) \setminus \mathbf{H}$ (counted with multiplicity) and $\chi_{\gamma_Q} = \mathrm{sign}(\mathrm{Tr}(\gamma_Q))$, $\chi_\sigma = \mathrm{sign}(\mathrm{Tr}(\sigma))$. One advantage of (7.5) is that we can express the intersection number in terms of continued fractions [**Mat20**, Thm. 4.7].

THEOREM 7.2 (Matsusaka). *Let $\gamma, \sigma \in \mathrm{SL}_2(\mathbf{Z})$ be of the form*

$$\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix}.$$

with $a_i, b_j > 0$ for $i = 0, 1, ..., n$ and $j = 0, 1, ..., m$. Then there exists an explicit function $\psi_\gamma(\sigma) \in \mathbf{Z}$ with $0 \le \psi_\gamma(\sigma) \le 2mn$ such that

$$\sum_{p\in[S_Q]\cap[S_\sigma]} 1 = \sum_{\substack{0\le i<2n,\\ 0\le j<2m}} \min\{a_i, b_j\} - \psi_\gamma(\sigma).$$

Compare (7.5) to Theorem 6.2: Clearly, the two formulas are very similar, the only difference being the topological intersection number μ_p that comes into play in Theorem 6.2.

The goal in this chapter is to generalize the formula (7.5) to the case of the Parson Poincaré series for weight $k > 1$.

1. The Parson Poincaré Series

Let Γ be a Fuchsian group of the first kind; as in the previous chapter, we assume that $-I \in \Gamma$ and that Γ is symmetric. Let $k > 1$, $\gamma \in \Gamma$ be a hyperbolic element, $D_\gamma = \mathrm{disc}(Q_\gamma)$, and $\chi_\gamma = \mathrm{sign}(\mathrm{Tr}(\gamma))$. We define the Parson Poincaré series of weight $2k$ to be

$$F_\gamma^{2k}(z) = -\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{g\in\Gamma_\gamma\setminus\Gamma} \frac{\mathrm{sign}(Q_\gamma \circ g)}{(Q_\gamma \circ g)(z,1)^k}. \tag{7.6}$$

For an indefinite binary quadratic form Q, the function $F_Q^{2k}(z)$ from the introduction of this chapter is equal to $F_{\gamma_Q}^{2k}(z)$ in (7.6). Clearly the sum in (7.6) is absolutely and uniformly convergent in z for compact subsets of $\mathbf{H}$.[2] Thus, $F_\gamma^{2k}(z)$ is a holomorphic function. If γ is in the same conjugacy class as γ^{-1} and k is even, the Parson Poincaré series vanishes identically, as the terms $\frac{\operatorname{sign}(Q_{g\gamma g^{-1}})}{Q_{g\gamma g^{-1}}(z,1)^k}$ and $\frac{\operatorname{sign}(Q_{g^{-1}\gamma^{-1}g})}{Q_{g^{-1}\gamma^{-1}g}(z,1)^k} = -\frac{\operatorname{sign}(Q_{g\gamma g^{-1}})}{Q_{g\gamma g^{-1}}(z,1)^k}$ in the sum (7.6) cancel. The Parson Poincaré series does not trivially vanish if k is odd and γ is in the same conjugacy class as γ^{-1}, since in that case

$$F_\gamma^{2k}(z) = -2 \cdot \frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{\substack{g\in\Gamma_\gamma\backslash\Gamma,\\ \operatorname{sign}(Q_\gamma\circ g)>0}} \frac{1}{(Q_\gamma\circ g)(z,1)^k}.$$

Since $\operatorname{sign}(Q)$ is invariant under the action of parabolic elements on Q, the Parson Poincaré series is invariant under the action of parabolic elements. Generally, $F_\gamma^{2k}(z)$ is not modular, but transforms with a rational period function.

Proposition 7.3. *Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ be a hyperbolic element and let $r_\gamma(\sigma,z) = F_\gamma^{2k}(z)|_{2k}\sigma - F_\gamma^{2k}(z)$. We have*

$$r_\gamma(\sigma,z) = 2 \cdot \frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{\substack{g\in\Gamma_\gamma\backslash\Gamma,\\ w'_{Q_\gamma\circ g}<-\frac{d}{c}<w_{Q_\gamma\circ g}}} \frac{\operatorname{sign}(Q_\gamma\circ g)}{(Q_\gamma\circ g)(z,1)^k},$$

where $w'_Q < w_Q$ denote the real roots of $Q(z,1)=0$.

Proof. First, we calculate:

$$\begin{aligned} F_\gamma^{2k}(z)|_{2k}\sigma - F_\gamma^{2k}(z) &= -\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi}\left(\sum_{g\in\Gamma_\gamma\backslash\Gamma}\frac{\operatorname{sign}(Q_\gamma\circ g)}{(Q\circ g\circ\sigma)(z,1)^k} - \sum_{g\in\Gamma_\gamma\backslash\Gamma}\frac{\operatorname{sign}(Q_\gamma\circ g)}{(Q\circ g)(z,1)^k}\right) \\ &= -\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma}\frac{\operatorname{sign}(Q_\gamma\circ g\circ\sigma^{-1}) - \operatorname{sign}(Q_\gamma\circ g)}{(Q\circ g\circ\sigma)(z,1)^k}. \end{aligned}$$

The terms on the right hand side are equal to zero if and only if $\operatorname{sign}(Q_\gamma\circ g\circ\sigma^{-1}) = -\operatorname{sign}(Q_\gamma\circ g)$. Write $(Q_\gamma\circ g)(z,1) = Az^2+Bz+C$ for $A,B,C\in\mathbf{R}$. We have $\operatorname{sign}(Q_\gamma\circ g\circ\sigma^{-1})\cdot\operatorname{sign}(Q_\gamma\circ g) < 0$ if and only if $A\cdot(A(-d/c)^2+B(-d/c)+C) < 0$, which is equivalent to $w'_{[A,B,C]} < -d/c < w_{[A,B,C]}$. ∎

[2]It might be possible – but tedious – to extend the definition of $F_\gamma^{2k}(z)$ to $k=1$ by the Hecke trick. The difficulty lies in the fact that we are working over a general Fuchsian group of the first kind, so there might not be a Fourier expansion as T. Matsusaka computed in the weight $k=1$ case for the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$; see [**Mat20**, Thm. 1.2]. Moreover, the Parson Poincaré series is not modular, so we cannot use the spectral expansion as we did in Section 1 of Chapter 6.

1.1. The Modular Completion of the Parson Poincaré Series. The Parson Poincaré series can be completed to a locally holomorphic modular form. Let $Q = [A, B, C]$ be an indefinite binary quadratic form. We define

$$Q_z = A|z|^2 + B \cdot \mathrm{Re}(z) + C.$$

The geodesic S_Q is exactly the set of $z \in \mathbf{H}$ for which $Q_z = 0$. Moreover, we have $\frac{Q_{g.z}}{\mathrm{Im}(g.z)} = \frac{(Q \circ g)_z}{y}$. Let $\mathbf{1}_Q(z) = \frac{1}{2}(1 - \mathrm{sign}(Q)\mathrm{sign}(Q_z))$, which is the characteristic function of the part of $\mathbf{H} - S_Q$, which does not contain $i\infty$. The function

$$\tilde{F}_\gamma^{2k}(z) = -\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{g \in \Gamma_\gamma \backslash \Gamma} \frac{\mathrm{sign}(Q \circ g) \mathbf{1}_{Q \circ g}(z)}{(Q \circ g)(z, 1)^k}$$

is locally holomorphic, but admits discontinuities on the net of geodesics $\{S_{Q \circ g} : g \in \Gamma_\gamma \setminus \Gamma\}$. Note that $\mathbf{1}_Q(z) = 0$ if z lies above the net of geodesics associated to the class $[Q]$. So for $\mathrm{Im}(z)$ large enough, we have $\tilde{F}_\gamma^{2k}(z) = 0$. The locally holomorphic Poincaré series $\tilde{F}_\gamma^{2k}(z)$ completes the Parson Poincaré series to a locally holomorphic modular form, namely

$$F_\gamma^{2k}(z) - 2 \cdot \tilde{F}_\gamma^{2k}(z) = -\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi} \sum_{g \in \Gamma_\gamma \backslash \Gamma} \frac{\mathrm{sign}(Q_z)}{(Q \circ g)(z, 1)^k}.$$

It is clear that the right hand side is modular, but it is only locally holomorphic and discontinuous on the net $\{S_{Q \circ g} : g \in \Gamma_\gamma \setminus \Gamma\}$.

1.2. Cycle Integrals of the Parson Poincaré Series. Consider the cycle integral $C_\sigma^{2k}(F_\gamma^{2k}, z_0) = \int_{z_0}^{\sigma.z_0} F_\gamma^{2k}(z) Q_\sigma(z, 1)^{k-1} dz$. Since $F_\gamma^{2k}(z)$ is holomorphic, the integral does not depend on the choice of the path from z_0 to $\sigma.z_0$. However, as $F_\gamma^{2k}(z)$ is not modular, the cycle integral *does* depend on the choice of the base point $z_0 \in \mathbf{H}$. The function $C_\sigma^{2k}(F_\gamma^{2k}, z_0)$ is Lipschitz-continuous on each compact subset K of $\mathbf{H}$: For $z_0, z_1 \in K$, we have

$$\begin{aligned} |C_\sigma^{2k}(F_\gamma^{2k}, z_0) - C_\sigma^{2k}(F_\gamma^{2k}, z_1)| &= \left| \int_{z_0}^{z_1} r_\gamma(\sigma, z) Q_\sigma(z, 1)^{k-1} dz \right| \\ &\leq L_{\gamma,\sigma,k} |z_0 - z_1|, \end{aligned}$$

where the Lipschitz-constant is the maximum of $r_\gamma(\sigma, z) Q_\sigma(z, 1)^{k-1}$ on K.

A similar argument can also be used to prove that the homogenized cycle integrals of $F_\gamma^{2k}(z)$ exist. Assume that γ and σ do not lie in the same conjugacy class. Then $C_\sigma^{2k}(F_\gamma^{2k}, \sigma^n.z_0)$ is readily seen to be a Cauchy sequence in n, since $\sigma^n.z_0 \to w'_\sigma$ or w_σ

by Lemma 1.8 and $r_\gamma(\sigma,z)Q_\sigma(z,1)^{k-1}$ is bounded on neighborhoods of w'_σ and w_σ. It would be interesting to make the dependencies in the Lipschitz constant $L_{\gamma,\sigma,k}$ explicit, something we have not been able to do.

The cycle integrals $C^{2k}_\sigma(F^{2k}_\gamma, z_0)$ can be given explicitly as a functions in z_0.

PROPOSITION 7.4. *Let $k>1$ and $\gamma,\sigma\in\Gamma$ be primitive hyperbolic elements. We have*

$$\int_{z_0}^{\sigma.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz = \int_{i\infty}^{\sigma.i\infty} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz$$
$$+\sum_{n=0}^{2k-2}\sum_{\substack{g\in\Gamma_\gamma\backslash\Gamma,\\ w'_{Q_\gamma\circ g}<\sigma^{-1}.i\infty<w_{Q_\gamma\circ g}}} \rho_{n,g,\gamma,\sigma}(z_0)\ {}_2F_1\left(k, 2k-1-n; 2k; 1-\frac{z_0-w_{Q_\gamma\circ g}}{z_0-w'_{Q_\gamma\circ g}}\right),$$

where $\rho_{n,g,\gamma,\sigma}(z_0)$ is a rational function given by $\rho_{n,g,\gamma,\sigma}(z_0) = \frac{\Gamma(2k-n-1)}{\Gamma(2k)}\frac{\partial_z^n Q_\sigma(z,1)^{k-1}|_{z=z_0}}{(z_0-w'_{Q_\gamma\circ g})^{n-2k-1}}$ and ${}_2F_1(a,b;c;z)$ is the hypergeometric function.

PROOF. We use

$$\int_{z_0}^{\sigma.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz = \int_{i\infty}^{\sigma.i\infty} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz + \int_{i\infty}^{z_0} r_{k,\gamma}(\sigma,z)Q_\sigma(z,1)^{k-1}dz. \tag{7.7}$$

Equation (7.7) can be seen as follows: Differentiating the left and right hand side in z_0 shows that they at most differ by a constant. Letting $z_0\to i\infty$ proves that the constant is zero.

Rewriting the polynomial $Q_\sigma(z,1)^{k-1} = \sum_{n=1}^{2k-2}\frac{\partial_z^n Q_\sigma(z,1)^{k-1}|_{z=z_0}}{\Gamma(n+1)}(z-z_0)^n$ in its Taylor expansion about z_0, we obtain a finite sum of integrals

$$\int_{i\infty}^{z_0} r_{k,\gamma}(\sigma,z)Q_\sigma(z,1)^{k-1}dz = \sum_{n=1}^{2k-2} a_{n,\sigma}(z_0)\sum_{\substack{Q\sim Q_\gamma,\\ w'_Q<\sigma^{-1}.i\infty<w_Q}}\int_{z_0}^{i\infty}\frac{(z-z_0)^n}{(z-w_Q)^k(z-w'_Q)^k}dz,$$

which can be solved. Standard integral transformations give

$$\begin{aligned}&\int_{z_0}^{i\infty}\frac{(z-z_0)^n}{(z-w_Q)^k(z-w'_Q)^k}dz = \\ &\frac{\Gamma(2k-n-1)\Gamma(n+1)}{\Gamma(2k)}(z-w'_Q)^{n-2k+1}\ {}_2F_1\left(k, 2k-n-1, 2k; 1-\frac{z-w_Q}{z-w'_Q}\right).\end{aligned} \tag{7.8}$$

□

That the right-hand side in Proposition 7.4 is symmetric in w'_Q and w_Q also follows from the identity ${}_2F_1(c-a,b,c;z/(z-1)) = (1-z)^b\ {}_2F_1(a,b,c;z)$.

2. Homogenized Cycle Integrals of the Parson Poincaré Series

In this section, we prove the extension of Theorem 6.2 for the homogenized cycle integrals of the Parson Poincaré series. We need to homogenize the cycle integral $C_\sigma(F_\gamma^{2k}, z_0)$ to make it independent of the choice of $z_0 \in \mathbf{H}$. Instead of relying on the explicit evaluation of the cycle integral in Proposition 7.4, we can use the approach of S. Katok [**Kat85**], which we already used in the proof of Theorem 6.6.

THEOREM 7.5. *Let $k > 1$ be an integer, let γ, σ be primitive hyperbolic elements in Γ, and set $\chi_\gamma = \operatorname{sign}(\operatorname{Tr}(\gamma))$ and $\chi_\sigma = \operatorname{sign}(\operatorname{Tr}(\sigma))$. We have*

$$\operatorname{Im}\left(\lim_{n\to\infty}\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz\right) = \chi_\gamma\chi_\sigma(D_\gamma D_\sigma)^{\frac{k-1}{2}}\sum_{p\in[S_\gamma]\cap[S_\sigma]}\mu_p^{k-1}P_{k-1}(\cos\theta_p), \tag{7.9}$$

where the sum runs over the intersections points p of the geodesics S_γ and S_σ projected onto $\Gamma \setminus \mathbf{H}$, $\theta_p \in [0,\pi]$ is the angle of intersection at p, $\mu_p = \mu_p(\gamma,\sigma)$ is the signed topological intersection number, and P_r denotes the r-th Legendre polynomial.

Theorems 6.2 and 7.5 express the cycle integrals of $P_\gamma^{2k}(z)$ resp. $F_\gamma^{2k}(z)$ in a very similar fashion, the only difference being the power with which the signed topological intersection number μ_p appears.

PROOF OF THEOREM 7.5. The main difference to S. Katok's proof [**Kat85**, Thm. 3] is that we cannot unfold the Parson Poincaré series as such.

The imaginary part of the homogenized cycle integral is given by

$$\begin{aligned}
&\operatorname{Im}\left(\lim_{n\to\infty}\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz\right)\\
&= \lim_{n\to\infty}\frac{1}{2i}\left(\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz - \int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(\overline{z})Q_\sigma(\overline{z},1)^{k-1}\overline{dz}\right)\\
&= \lim_{n\to\infty}\frac{1}{2i}\left(\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz + \int_{(\sigma')^n.(-\overline{z_0})}^{(\sigma')^{n+1}.(-\overline{z_0})} F_{k,\gamma'}(z)Q_{\sigma'}(z,1)^{k-1}dz\right)
\end{aligned}$$

after the change of variables $z \to -\overline{z}$.

First assume that γ is not equivalent to σ or σ^{-1}. We rewrite

$$\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz = \int_{z_0}^{\sigma.z_0}(c_nz+d_n)^{-2k}F_{k,\gamma}(\sigma^n.z)Q_\sigma(z,1)^{k-1}dz,$$

where we put $\sigma^n = \begin{pmatrix} * & * \\ c_n & d_n \end{pmatrix}$. Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 1.8 we have

$$\operatorname{sign}(Q_\gamma \circ \sigma^{-n}(1,0)) = \operatorname{sign}(Q_\gamma(-d_n/c_n, 1))$$
$$\to \begin{cases} \operatorname{sign}(Q_\gamma(w'_\sigma, 1)), & \text{if } c(a+d) < 0, \\ \operatorname{sign}(Q_\gamma(w_\sigma, 1)), & \text{if } c(a+d) > 0, \end{cases} \quad \text{as } n \to +\infty. \tag{7.10}$$

If the two geodesics S_γ and S_σ intersect, we have

$$\mu(S_\gamma, S_\sigma) = \chi_\sigma \lim_{n\to+\infty} \operatorname{sign}(Q_\gamma(-d_n/c_n, 1)) \tag{7.11}$$

by (7.10) and Lemma 2.6.

Now the Parson Poincaré series becomes

$$\begin{aligned}(c_n z + d_n)^{-2k} F_{k,\gamma}(\sigma^n.z) &= -\chi_\gamma \frac{D_\gamma^{k-1/2}}{\pi} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \frac{\operatorname{sign}(Q_\gamma \circ g)}{(c_n z + d_n)^{2k}(Q_\gamma \circ g)(\sigma^n.z, 1)^k} \\ &= -\chi_\gamma \frac{D_\gamma^{k-1/2}}{\pi} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \frac{\operatorname{sign}(Q_\gamma \circ g\sigma^{-n})}{(Q_\gamma \circ g)(z,1)^k} \\ &\to -\chi_\gamma \frac{D_\gamma^{k-1/2}}{\pi} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \frac{\operatorname{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z,1)^k}, \text{ as } n \to +\infty,\end{aligned}$$

where in the last line we applied (7.10), assuming $\operatorname{sign}(c(a+d)) < 0$ for the moment. The resulting series is modular of weight $2k$ for the group $\Gamma_\sigma = \{\pm\sigma^n : n \in \mathbf{Z}\}$. Hence, the cycle integral

$$\int_{z_0}^{\sigma.z_0} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \frac{\operatorname{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z,1)^k} Q_\sigma(z,1)^{k-1} dz$$

is independent of the choice of z_0.

Now a typical unfolding argument (using [**Kat85**, Lemma 1]) yields

$$\begin{aligned}&\int_{z_0}^{\sigma.z_0} \sum_{g\in\Gamma_\gamma\backslash\Gamma} \frac{\operatorname{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z,1)^k} Q_\sigma(z,1)^{k-1} dz \\ &= \sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma} \sum_{m\in\mathbf{Z}} \int_{\sigma^m.z_0}^{\sigma^{m+1}.z_0} \frac{\operatorname{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z,1)^k} Q_\sigma(z,1)^{k-1} dz \\ &= \sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma} \int_{S_\sigma} \frac{\operatorname{sign}((Q_\gamma \circ g)(w'_\sigma, 1))}{(Q_\gamma \circ g)(z,1)^k} Q_\sigma(z,1)^{k-1} dz,\end{aligned}$$

where in the last equality we deformed the path connecting w'_σ and w_σ to be the geodesic S_σ.

As similar argument gives

$$\begin{aligned}&\lim_{n\to\infty}\int_{(\sigma')^n.(-\overline{z_0})}^{(\sigma')^{n+1}.(-\overline{z_0})} F_{k,\gamma'}(z)Q_{\sigma'}(z,1)^{k-1}dz\\&=\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{S_{\sigma'}}\frac{\operatorname{sign}((Q_{\gamma'}\circ g')(w_{\sigma'},1))}{(Q_{\gamma'}\circ g')(z,1)^k}Q_{\sigma'}(z,1)^{k-1}dz.\end{aligned}$$

By the change of variables $z\to -z$ and using that $(Q_{\gamma'}\circ g')(-z,1)=-(Q_\gamma\circ g)(z,1)$ and $w_{\sigma'}=-w'_\sigma$, we get

$$\begin{aligned}&\operatorname{Im}\left(\lim_{n\to\infty}\int_{\sigma^n.z_0}^{\sigma^{n+1}.z_0} F_{k,\gamma}(z)Q_\sigma(z,1)^{k-1}dz\right)\\&=-\frac{1}{2i}\chi_\gamma\frac{D_\gamma^{k-\frac{1}{2}}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma/\Gamma_\sigma}\int_{C(\sigma)}\frac{\operatorname{sign}((Q_\gamma\circ g)(w'_\sigma,1))}{(Q_\gamma\circ g)(z,1)^k}Q_\sigma(z,1)^{k-1}dz\end{aligned}\tag{7.12}$$

over the circle $C(\sigma)$ through the roots of $Q_\sigma(z,1)=0$. The integrands are all meromorphic, with poles only at the real roots of $(Q_\gamma\circ g)(z,1)$. Hence, by Cauchy's theorem, they vanish if the geodesic connecting the roots of $(Q_\gamma\circ g)(z,1)$ does not intersect S_σ. The double-cosets in (7.12), for which the integral is non-zero, are in one-to-one correspondence with the intersection points $p\in[S_\gamma]\cap[S_\sigma]$ (counted with multiplicity); see [**Kat85**, p. 477]. We may thus relabel the set of representatives in terms of $g_p\in\Gamma$, which correspond to an intersection point p. We are left with

$$-\frac{1}{2i}\chi_\gamma\chi_\sigma\frac{D_\gamma^{k-1/2}}{\pi}\sum_{p\in[S_\gamma]\cap[S_\sigma]}\mu_p(\gamma,\sigma)\int_{C(\sigma)}\frac{1}{(Q\circ g_p)(z,1)^k}Q_\sigma(z,1)^{k-1}dz$$

by (7.11). This is true independently of $\operatorname{sign}(c(a+d))$ being positive or negative. The orientation of $C(\sigma)$ is positive or negative according to $\operatorname{sign}(Q_\sigma)$ being positive of negative.

We may fix the orientation of $C(\sigma)$ to be clockwise by considering

$$-\operatorname{sign}(Q_\sigma)\chi_\gamma\chi_\sigma\frac{D_\gamma^{k-1/2}}{\pi}\sum_{p\in[S_\gamma]\cap[S_\sigma]}\mu_p(\gamma,\sigma)\int_{C(\sigma)}\frac{1}{(Q\circ g_p)(z,1)^k}Q_\sigma(z,1)^{k-1}dz.$$

Now transform the circle $C(\sigma)$ to the entire imaginary axis $i\mathbf{R}$ oriented from $-i\infty$ to $i\infty$ with $R_\sigma\in\mathrm{SL}_2(\mathbf{R})$ as in Section 2 of Chapter 2. We have $Q_\sigma\circ R_\sigma=[0,\operatorname{sign}(c)\sqrt{D_\sigma},0]$.

Write $[A,B,C]$ for $Q_\gamma \circ R_\sigma$. So

$$\int_{C(\sigma)} \frac{1}{(Q\circ g_p)(z,1)^k} Q_\sigma(z,1)^{k-1} dz = (\mathrm{sign}(Q_\sigma))^{k-1} D_\sigma^{\frac{k-1}{2}} \int_{-i\infty}^{i\infty} \frac{z^{k-1}}{(Az^2+Bz+C)^k} dz.$$

The integral can be solved by [**Kat85**, Lemma 2], which asserts that for $k \in \mathbf{Z}, k \geq 1$, and any $a, b, c \in \mathbf{R}$ with $D = b^2 - 4ac > 0$, we have

$$\int_{-\infty}^{\infty} \frac{t^{k-1}}{(-at^2+bit+c)^k} dt = \begin{cases} 0, & ac > 0, \\ -\mathrm{sign}(a)\, 2\pi i^{1-k} D^{-k/2} P_{k-1}\left(\frac{b}{\sqrt{D}}\right), & ac < 0. \end{cases} \tag{7.13}$$

The intersection angle θ_p between $S_{[A,B,C]}$ and the imaginary axis is given by $\cos\theta_p = \mathrm{sign}(A)\frac{B}{\sqrt{D_\gamma}}$, since $\mathrm{Re}\left(-\frac{B}{2A} + \frac{\sqrt{D_\gamma}}{2|A|} e^{i\theta_p}\right) = 0$. So we get

$$\begin{aligned} &-\frac{1}{2i}\mathrm{sign}(Q_\sigma)\chi_\gamma\chi_\sigma \frac{D_\gamma^{k-1/2}}{\pi} \sum_{p\in[S_\gamma]\cap[S_\sigma]} \mu_p(\gamma,\sigma) \int_{C(\sigma)} \frac{1}{(Q\circ g_p)(z,1)^k} Q_\sigma(z,1)^{k-1} dz \\ &= \chi_\gamma\chi_\sigma (D_\sigma D_\gamma)^{\frac{k-1}{2}} \sum_{p\in[S_\sigma]\cap[S_\gamma]} \mu_p(\gamma,\sigma)\mathrm{sign}(Q_\sigma)^k \mathrm{sign}(Q_\gamma\circ g_p\circ R_\sigma)^k P_{k-1}(\cos\theta_p), \end{aligned}$$

by $P_{k-1}(-x) = (-1)^{k-1}P_{k-1}(x)$. It remains to understand the factor $\mathrm{sign}(Q_\gamma \circ g_p \circ R_\sigma)$. But it is easy to see that the orientation of $S_{Q_\gamma\circ g_p}$ changes under the transformation R_σ if and only if the point w'_σ lies between the roots of $(Q_\gamma \circ g_p)(z,1) = 0$, i.e. if $\mathrm{sign}(Q_\gamma\circ g_p)\mathrm{sign}((Q_\gamma\circ g_p)(w'_\sigma,1)) < 0$. Thus,

$$\mathrm{sign}(Q_\gamma\circ g_p\circ R_\sigma) = -\mathrm{sign}((Q_\gamma\circ g_p)(w'_\sigma,1)).$$

So $\mathrm{sign}(Q_\sigma)\mathrm{sign}(Q_\gamma\circ g_p\circ R_\sigma) = \mu_p(\gamma,\sigma)$ by Lemma 2.6, proving the claim.

In the case where γ is equivalent to σ or σ^{-1}, the proof is the same with minor modifications. Since F_γ^{2k} only depends on the conjugacy class of γ, we may assume that, say, $\gamma = \sigma$. Equation (7.10) only holds for $\gamma \neq \sigma,\ \sigma^{-1}$, as we cannot exchange the limit and $\mathrm{sign}(x)$ as $x \to 0$. But $\mathrm{sign}(Q_\sigma \circ \sigma^{-n}(1,0)) = \mathrm{sign}(Q_\sigma)$ for $n \geq 1$ is constant and hence has a trivial limit.

Keeping the notation from before, we then have

$$(c_n z + d_n)^{-n} F_\sigma^{2k}(\sigma^n.z) \to -\frac{D_\sigma^{k-1/2}}{\pi}\left(\frac{\mathrm{sign}(Q_\sigma)}{Q_\sigma(z,1)^k} + \sum_{\mathrm{Id}\neq g\in\Gamma_\sigma\backslash\Gamma} \frac{\mathrm{sign}(Q_\sigma(w'_\sigma,1))}{(Q_\sigma\circ g)(z,1)^k}\right) \tag{7.14}$$

as $n \to +\infty$. The right-hand side of (7.14) is still modular of weight $2k$ under Γ_σ, as both terms inside the brackets are Γ_σ-invariant.

Unfolding using [**Kat85**, Lemma 3] gives

$$\operatorname{sign}(Q_\sigma)\int_{z_0}^{\sigma.z_0}\frac{dz}{Q_\sigma(z,1)}+\sum_{\mathrm{Id}\neq g\in\Gamma_\sigma\backslash\Gamma/\Gamma_\sigma}\int_{S_\sigma}\frac{\operatorname{sign}((Q_\sigma\circ g)(w'_\sigma,1))}{(Q_\sigma\circ g)(z,1)^k}Q_\sigma(z,1)^{k-1}dz. \tag{7.15}$$

It is easy to see that the first integral in (7.15) is real and hence vanishes when taking the imaginary part. Cauchy's Theorem is still applicable to the other integrals, since we do not integrate over poles. Hence, we can now proceed with the computation in the same way as above. □

3. L-functions of the Parson Poincaré Series

For this section, we let Γ be a non-cocompact Fuchsian group of the first kind. The group Γ then contains parabolic elements.

The stabilizer subgroup of $i\infty$ is given by $\Gamma_\infty=\left\{\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)^n : n\in\mathbf{Z}\right\}$ for some $h>0$. The Parson Poincaré series is h-periodic, since $\operatorname{sign}\left(Q\circ\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)\right)=\operatorname{sign}(Q)$ for any indefinite binary quadratic form Q. It hence admits a Fourier expansion of the form

$$F_\gamma^{2k}(z)=\sum_{\substack{n\in\frac{1}{h}\mathbf{Z},\\ n>0}}a_{2k,\gamma}(n)e\left(nz\right).$$

The Fourier coefficients $a_{2k,\gamma}(n)$ are given in terms of Kloosterman sums and Bessel functions as in [**Par93**, Theorem 3.1] and polynomially bounded in n.[3] We define the L-function of $F_\gamma^{2k}(z)$ twisted by a $\sigma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\Gamma$ by

$$L_{2k,\gamma}\left(s,\sigma\right)=\sum_{\substack{n\in\frac{1}{h}\mathbf{Z},\\ n>0}}\frac{a_{2k,\gamma}(n)e^{-2\pi i\frac{d}{c}n}}{n^s},\quad \operatorname{Re}(s)\gg 1.$$

As with classical L-functions attached to holomorphic cusp forms, the twisted L-function can be realized as the Mellin transform of $F_\gamma^{2k}(z)$, namely

$$L_{2k,\gamma}\left(s,\sigma\right)=\left(\frac{2\pi}{ic}\right)^s\frac{c}{\Gamma(s)}\int_{-\frac{d}{c}}^{i\infty}F_{k,\gamma}(z)(cz+d)^{s-1}dz \tag{7.16}$$

for $\operatorname{Re}(s)\gg 1$ big enough; however, the function $L_{2k,\gamma}(s,\sigma)$ can be analytically continued, so that both sides of (7.16) make sense at the central value $s=k$.

[3]In the case $\Gamma=\mathrm{SL}_2(\mathbf{Z})$ the Fourier coefficients of $F_\gamma^{2k}(z)$ are given as cycle integrals of weakly holomorphic modular forms [**DIT10**, Theorem 3].

Proposition 7.6. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. *The twisted L-function* $L_{2k,\gamma}(s,\sigma)$ *admits an analytic continuation to the the half-plane* $\{s \in \mathbf{C} : \mathrm{Re}(s) > 0\}$.

Proof. The proof is similar as the one for weight 2 [**DIT17**, Thm. 2.1]. Denote the completed L-function by $\Lambda_{2k,\gamma}(s,\sigma) = \left(\frac{c}{2\pi}\right)^s \Gamma(s) L_{2k,\gamma}(s,\sigma)$ and write $z_t = \frac{a}{c} + \frac{i}{tc}$ for $t > 0$, so that $\sigma^{-1}.z_t = -\frac{d}{c} + \frac{it}{c}$. We have

$$\begin{aligned}
\Lambda_{2k,\gamma}(s,\sigma) &= \int_0^\infty F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt \\
&= \int_0^1 F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt + \int_1^\infty F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt \\
&= (-1)^k \int_1^\infty F_\gamma^{2k}(\sigma^{-1}.z_{1/t})(it)^{-2k} t^{-s+2k+1} dt + \int_1^\infty F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt \\
&= (-1)^k \int_1^\infty F_\gamma^{2k}(z_{1/t}) t^{-s+2k-1} dt + (-1)^k \int_1^\infty r_\gamma(\sigma^{-1}, z_{1/t}) t^{-s+2k+1} dt \\
&+ \int_1^\infty F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt.
\end{aligned}$$

The integrals $(-1)^k \int_1^\infty F_\gamma^{2k}(z_{1/t}) t^{-s+2k+1} dt + \int_1^\infty F_\gamma^{2k}(\sigma^{-1}.z_t) t^{s-1} dt$ are entire functions in s, due to the exponential decay of the function $F_\gamma^{2k}(z)$ at $i\infty$.

It remains to analyze the integral $\displaystyle\int_1^\infty r_\gamma\left(\sigma^{-1}, -\frac{d}{c} + \frac{it}{c}\right) t^{-s+2k+1} dt$. The integral converges absolutely and uniformly in s for compact subsets of $\{s \in \mathbf{C} : \mathrm{Re}(s) > 0\}$, since

$$r_\gamma\left(\sigma^{-1}, -\frac{d}{c} + \frac{it}{c}\right) \ll_{\gamma,\sigma,k} t^{-2k}.$$

□

The twisted L-functions also appear in the dual weight primitive cocycle of $r_\gamma(\sigma, z)$. With the integral evaluation (7.8) in the proof of Proposition 7.4, we can show that the weight $2-2k$ cocycle

$$\begin{aligned}
R_\gamma(\sigma, z) = \frac{(2\pi i)^{2k-1} \chi_\gamma D_\gamma^{k-1/2}}{\pi (2k-1)! \binom{2k-2}{k-1}} \sum_{w_Q' < -\frac{d}{c} < w_Q} \frac{1}{|Q(1,0)|(z - w_Q')} \, {}_2F_1\left(k, 1, 2k; 1 - \frac{z - w_Q}{z - w_Q'}\right) \\
+ \frac{\chi_\gamma (2\pi i)^{2k-1} i}{(2k-2)! c^{2k-1}} \sum_{n=0}^{2k-2} \binom{2k-2}{n} i^n \left(\frac{c}{2\pi}\right)^{n+1} \Gamma(n+1) L_{2k,\gamma}(n+1, \sigma^{-1})(cz+d)^n
\end{aligned}$$

for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and Q running over the equivalence class $[Q_\gamma]$ is a $(2k-1)$-th primitive of $r_\gamma(g,z)$, i.e. $\left(\frac{1}{2\pi i}\partial_z\right)^{2k-1} R_\gamma(g,z) = r_\gamma(g,z)$.

The Mellin transform in (7.16) is the cycle integral of the quadratic form $cz+d$ of square discriminant. Thus, there ought to be a natural generalization of Theorem 7.5 to L-functions as cycle integrals of square discriminants (compare with Theorem 5.7). For simplicity, we will assume $k>1$ to be odd, to not deal with signed topological intersection numbers of non-compact geodesics, but a generalization is easily defined.

THEOREM 7.7. *Let $k>1$ be an odd integer, γ, σ be hyperbolic elements in Γ, and $\chi_\gamma = \operatorname{sign}(\operatorname{Tr}(\gamma))$, $\chi_\sigma = \operatorname{sign}(\operatorname{Tr}(\sigma))$. We have*

$$\frac{(k-1)!i^{k-1}}{(2\pi)^k}\operatorname{Re}\left(L_{2k,\gamma}(k,\sigma)\right) = \chi_\gamma\chi_\sigma D_\gamma^{\frac{k-1}{2}} \sum_{p\in[S_\gamma]\,\cap\, S_{\sigma^{-1}.i\infty}} P_{k-1}\left(\cos\theta_p\right),$$

where θ_p denotes the angle of intersection at the point p between the geodesic in the equivalence class of S_γ and the geodesic $S_{\sigma^{-1}.i\infty}$.

PROOF. The proof is a careful application of (7.13). Write $\sigma = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$. The imaginary part of the integral on the right-hand side of (7.16) is equal to

$$\begin{aligned}\operatorname{Im}\int_{-\frac{d}{c}}^{i\infty} F_\gamma^{2k}(z)(cz+d)^{k-1}dz &= c^{k-1}\operatorname{Im}\int_0^{i\infty} F_\gamma^{2k}\left(z-\frac{d}{c}\right)z^{k-1}dz \\ &= -c^{k-1}\frac{\chi_\gamma D_\gamma^{k-\frac{1}{2}}}{\pi}\sum_{g\in\Gamma_\gamma\backslash\Gamma}\operatorname{Im}\left(\int_0^{i\infty}\frac{\operatorname{sign}(Q_\gamma\circ g)z^{k-1}}{(Q_\gamma\circ g)(z-d/c,1)^k}dz\right),\end{aligned}$$

where we can exchange sum and integral by Fubini's theorem, since the integrand is absolutely convergent.

Fix $g\in\Gamma_\gamma\backslash\Gamma$ for the moment and write Az^2+Bz+C for $(Q_\gamma\circ g)(z,1)$. The quadratic form

$$(Q_\gamma\circ g)(z-d/c,1) = Az^2+(B-2Ad/c)\,z+\left(A\,(d/c)^2-Bd/c+C\right) = A'z^2+B'z+C'$$

intersects with the non-compact geodesic $S_{-d/c}$ if and only if

$$A'C' = A\left(A\,(d/c)^2-Bd/c+C\right) < 0.$$

With (7.13), we get

$$\begin{aligned}\operatorname{Im}\left(\int_0^{i\infty}\frac{\operatorname{sign}(A')z^{k-1}}{\left(A'z^2+B'z+C'\right)^k}dz\right) &= \frac{i^{k-1}}{2}\int_{-\infty}^{\infty}\frac{\operatorname{sign}(A')t^{k-1}}{\left(-A't^2+B'it+C'\right)^k}dt \\ &= -\pi D_\gamma^{-k/2}P_{k-1}\left(\frac{B'}{\sqrt{D_\gamma}}\right)\end{aligned}$$

if and only if the geodesic $S_{g\gamma g^{-1}}$ intersects the non-compact geodesic $S_{-d/c}$ associated to the quadratic form $cz+d$. Otherwise, the integral evaluates to zero by (7.13). The intersection angle θ_p between these two geodesics is given by $\cos\theta_p = \operatorname{sign}(A')\frac{B'}{\sqrt{D}}$. Since $k>1$ is odd, the Legendre polyomial $P_{k-1}(x)$ is an even function and the claim follows.[4]

□

4. Periods of the Parson Poincaré Series

In this subsection, we let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$. W. Kohnen and D. Zagier [**KZ84**] studied the periods of the hyperbolic Poincaré series $P_\gamma^{2k}(z)$ defined in (6.1), and showed that certain linear combinations of these periods are rational numbers.

A natural follow-up question to Theorems 7.5 and 7.7 would be to study the periods of the Parson Poincaré series, i.e. $p_n(F_\gamma^{2k}) = \displaystyle\int_0^\infty F_\gamma^{2k}(it)t^n\,dt$ for $0\le n\le 2k-2$. As we did for the hyperbolic Poincaré series, we define the symmetrizations

$$F_\gamma^{2k,+}(z) = F_\gamma^{2k}(z) + F_{\gamma'}^{2k}(z) \text{ and } F_\gamma^{2k,-}(z) = i(F_\gamma^{2k}(z) - F_{\gamma'}^{2k}(z))$$

of the Parson Poincaré series. We split the period polynomial

$$p(F_\gamma^{2k})(x) = \int_0^{i\infty} F_\gamma^{2k}(z)(x-z)^{2k-2}dz = \sum_{n=0}^{2k-2} i^{-n+1}\binom{2k-2}{n}p_n(F_\gamma^{2k})x^{2k-2-n}$$

of $F_\gamma^{2k}(z)$ into its even and odd part as $p(F_\gamma^{2k}) = ip^+(F_\gamma^{2k}) + p^-(F_\gamma^{2k})$ with

$$p^+(F_\gamma^{2k})(x) = \sum_{\substack{0\le n\le 2k-2,\\ n \text{ even}}} (-1)^{n/2}\binom{2k-2}{n}p_n(F_\gamma^{2k})x^{2k-2-n},$$
$$p^-(F_\gamma^{2k})(x) = \sum_{\substack{0< n< 2k-2,\\ n \text{ odd}}} (-1)^{(n-1)/2}\binom{2k-2}{n}p_n(F_\gamma^{2k})x^{2k-2-n}.$$

By closely following the proof of [**KZ84**, Theorem 5], one obtains

$$\begin{aligned} &p^+(F_\gamma^{2k,+})(x) + p^-(F_\gamma^{2k,-})(x) \\ &\doteq -2\sum_{\substack{[a,b,c]\in[Q_\gamma]\\ a<0<c}} (ax^2-bx+c)^{k-1} - \frac{2D^{k-1/2}\zeta_{Q_\gamma}(k)}{\binom{2k-2}{k-1}(2k-1)\zeta(2k)}(x^{2k-2}-1), \end{aligned} \tag{7.17}$$

[4]For $k>1$ even, we get $P_{k-1}\left(\frac{B'}{\sqrt{D_\gamma}}\right) = \operatorname{sign}(A)P_{k-1}(\cos\theta_p)$. This additional sign factor measures whether the geodesic $S_{Q_\gamma\circ g}$ is oriented positively or negatively and can be viewed as an analog of the signed topological intersection number in Section 2.1 of Chapter 2.

where $\zeta_{Q_\gamma}(s)$ is the ζ-function associated with Q_γ as in [**KZ84**, p. 222], $\zeta(s)$ is the Riemann ζ-function, and $\doteq$ means equality up to a non-zero multiplicative constant.[5] In particular, the periods $p_n(F_\gamma^{2k,+})$ for even $0 < n < 2k-2$ and the periods $p_n(F_\gamma^{2k,-})$ for odd $0 < n < 2k-2$ are rational.

We let $F_D^{2k}(z) = \sum_{D_\gamma = D} F_\gamma^{2k}(z)$ where the sum ranges over a system of representatives γ of the conjugacy classes of primitive hyperbolic elements in $\mathrm{SL}_2(\mathbf{Z})$ with discriminant $D_\gamma = D$. From (7.17) it follows that for odd $k > 1$ we have

$$p^+(F_D^{2k}) \doteq -\sum_{\substack{[a,b,c]\in\mathcal{Q}_D\\ a<0<c}} (ax^2+bx+c)^{k-1} - \frac{D^{k-1/2}\zeta(k)L_D(k)}{\binom{2k-2}{k-1}(2k-1)\zeta(2k)}(x^{2k-2}-1), \tag{7.18}$$

where $L_D(s)$ is the Dirichlet L-function associated to the Kronecker symbol $\left(\frac{D}{\cdot}\right)$. Formula (7.18) is the analog of [**KZ84**, Theorem 4]. In particular, the periods $p_n(F_D^{2k})$ for even $0 < n < 2k-2$ are rational and satisfy the symmetry $p_{2k-2-n}(F_D^{2k}) = p_n(F_D^{2k})$.

5. Equidistribution of Intersection Angles

In this section, we sketch a possible application of Theorem 7.5, which might be the content of future work. Let again $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and fix $\gamma \in \Gamma$. Theorem 7.5 might allow us to give another proof the fact that the intersection angles θ_p of a fixed geodesic $[S_\gamma]$ with the net of geodesics S_σ, $\mathrm{disc}(Q_\sigma) = D$, equidistribute to the measure $\frac{1}{2}\sin\theta\, d\theta$ as $D \to +\infty$. This was conjectured by J. Rickards [**Ric21**, Conjecture 4.2] and recently proved by J. Jung and N. Sardari [**JS21**].

To show this, it suffices to prove that $\cos\theta_p$ equidistributes to the Lebesgue measure on $[-1,1]$. The Legendre polynomials P_{k-1} with $k \geq 1$ form a complete orthonormal system of the space of continuous functions on $[-1,1]$. So if we show that for $k > 1$ the Weyl sum

$$\frac{1}{\#\{p \in [S_\gamma]\cap[S_\sigma] : D_\sigma = D\}} \sum_{D_\sigma = D} \sum_{p\in[S_\gamma]\cap[S_\sigma]} P_{k-1}(\cos\theta_p) \to 0, \text{ as } D \to +\infty$$

for D through fundamental discriminants, the equidistribution follows. A Siegel-type bound for the number of intersections can be obtained using Theorem 7.2 and some elementary considerations on continued fractions.

For even $k > 1$ the Weyl sum can be estimated using Theorem 6.2 and the Shimura-theory of cusp forms; see Theorem 1.1. Namely, we can use the non-trivial bounds on

[5]The formulas (7.17) and (7.18) are correct if we normalize F_γ^{2k} as in [**KZ84**]. Since our normalization of F_γ^{2k} is different, we get some simple but unpleasant extra factors.

the Fourier coefficients of half-integral weight cusp forms [**Iwa87**], which will give enough cancellation in the Weyl sum $\sum_{D_\sigma=D}\sum_{p\in[S_\gamma]\cap[S_\sigma]} P_{k-1}(\cos\theta_p)$.

For odd $k>1$ this approach does not work due to the appearance of the μ_p-factor in Theorem 6.2, so the Weyl sum here is given by taking the trace of the homogenized cycle integrals in Theorem 7.5. However, it appears to be difficult to estimate the traces of the homogenized cycle integrals. The correct Weyl sum also appears in the evaluation of

$$\lim_{n\to+\infty}\int_{\gamma^n.z_0}^{\gamma^{n+1}.z_0} F_D^{2k}(z)Q_\gamma(z,1)^{k-1}dz,$$

where $F_D^{2k}(z)$ was defined in Section 4. H. Iwaniec's bound on the Fourier coefficients of half-integral weight cusp forms can be applied to the Fourier coefficients of $F_D^{2k}(z)$, which individually exhibit non-trivial cancellation in D. Yet, due to the homogenization process, one would need some uniformity in D in the convergence of the homogenized cycle integral to use this bound.

Part 3

Hardy Sums

CHAPTER 8

Dedekind Sums

Dedekind sums first appeared in R. Dedekind's commentary [**Ded76**] on a posthumous fragment of B. Riemann on elliptic modular functions. More precisely, these sums determine the multiplier system of the Dedekind η-function

$$\eta(z) = e^{\frac{\pi i}{12} z} \prod_{n=1}^{\infty} (1 - e(nz)), \tag{8.1}$$

which transforms with weight $\frac{1}{2}$ under the action of the modular group $\mathrm{SL}_2(\mathbf{Z})$; since this was R. Dedekind's own contribution to the fragment, the function $\eta(z)$ and the Dedekind sums are named after him.

The Dedekind sums are defined as

$$s(d,c) = \sum_{k=1}^{c-1} \left(\left(\frac{k}{c}\right)\right) \cdot \left(\left(\frac{kd}{c}\right)\right), \ (d,c) = 1, \ c > 0, \tag{8.2}$$

where $((x)) = x - \lfloor x \rfloor - \frac{1}{2}$ for $x \notin \mathbf{Z}$ and $((x)) = 0$ for $x \in \mathbf{Z}$ is the sawtooth-function. It is clear from (8.2) that $s(d+c,c) = s(d,c)$ and $s(-d,c) = -s(d,c)$.

Objects that arise from modular functions typically satisfy many symmetries. The central identity in the theory of Dedekind sums is its reciprocity law, namely

$$s(d,c) + s(c,d) = \frac{1}{4} \frac{d^2 + c^2 + 1}{cd} - \frac{1}{12}, \ (c,d) = 1, \ d, c > 0. \tag{8.3}$$

The reciprocity law can be proved by elementary means; see H. Rademacher and E. Grosswald's monograph [**RG72**, Ch. 1], where four different elementary proofs of (8.3) are presented.

The Dedekind sums have a multitude of often surprising applications: counting lattice points in tetrahedra [**Mor51**], class number formulas for quadratic number fields [**DIT18**], the topology of 4-dimensional manifolds [**Hir71**], and pseudo-random numbers [**Die71**], to name a few.

In this chapter, we will revisit some of their distributional and geometric properties for future reference.

1. The Logarithm of the Dedekind η-function

Let

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)e(nz) \tag{8.4}$$

be the holomorphic Eisenstein series of weight 2, which is the modular integral of the cocycle

$$r(A,z) = \frac{6}{\pi i}\frac{c}{cz+d} = E_2(z)|_2A - E_2(z),\ A = \begin{pmatrix} * & * \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

The cycle integral $R_{E_2}(A,z) = \displaystyle\int_z^{A.z} E_2(w)dw$ is a weight zero cocycle, which is a primitive of $r(A,z)$.

As is well-known, the logarithmic derivative of $\eta(z)$ in (8.1) is equal to the Eisenstein series of weight 2, i.e.

$$\partial_z \log\eta(z) = \frac{\eta'(z)}{\eta(z)} = \frac{\pi i}{12}E_2(z). \tag{8.5}$$

So the cocycle $R_\eta(A,z) = \log\eta(A.z) - \log\eta(z)$ is a weight zero cocycle such that $\partial_z R_\eta(A,z) = \frac{\pi i}{12}r(A,z)$. Since $\frac{\pi i}{12}R_{E_2}(A,z) - R_\eta(A,z)$ is constant in z (differentiate it) and both $R_{E_2}(A,z)$ and $R_\eta(A,z)$ satisfy the weight zero cocycle property, the map $A \mapsto \frac{\pi i}{12}R_{E_2}(A,z) - R_\eta(A,z)$ must be a homomorphism $\mathrm{SL}_2(\mathbf{Z}) \to \mathbf{C}$. But the modular group is generated by torsion elements, so $H^1(\mathrm{SL}_2(\mathbf{Z}),\mathbf{C})$ is trivial and we must have

$$\log\eta(A.z) - \log\eta(z) = \frac{\pi i}{12}\int_z^{A.z} E_2(w)\,dw. \tag{8.6}$$

Through manipulations of the complex logarithm, R. Dedekind found the following transformation law of $\log\eta(z)$:

$$\log\eta(A.z) - \log\eta(z) = \frac{1}{2}\log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) + \frac{\pi i}{12}\Phi(A),\ A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}), \tag{8.7}$$

where

$$\Phi(A) = \begin{cases} b, & \text{if } c = 0, \\ \frac{a+d}{c} - 12\,\mathrm{sign}(c)\ s(d,|c|), & \text{if } c \neq 0. \end{cases} \tag{8.8}$$

We refer to $\Phi(A)$ as the Dedekind symbol. Since $\eta(z)^{24}$ is equal to the modular discriminant function, which is a cusp form of weight 12 with trivial multiplier system, the Dedekind symbol $\Phi(A)$ is integer-valued for all $A \in \mathrm{SL}_2(\mathbf{Z})$.

Both representations $R_\eta(A,z)$ and $R_{E_2}(A,z)$ of the primitive cocycle have their advantages. For instance, the reciprocity law (8.3) follows easily from (8.7) by using $R_\eta(AS,z) = R_\eta(A,S.z) + R_\eta(S,z)$.

By Euclidean division, the properties $s(0,1)=0$, $s(d+c,c)=s(d,c)$, and the reciprocity law are enough to give all values of the Dedekind sums (the latter two properties correspond to the actions of the generators T and S of $\mathrm{SL}_2(\mathbf{Z})$). The Euclidean algorithm is also encoded in the continued fraction expansion of the rational $\frac{d}{c}$. D. Hickerson [**Hic77**] gave the Dedekind sums a representation in terms of the partial quotients of the c.f.e. of $\frac{d}{c}$.

THEOREM 8.1 (Hickerson). *Let $(d,c)=1$, $c>0$ and $a_i \geq 0$ be the partial quotients of the continued fraction expansion of*

$$\frac{d}{c} = [0; a_1, ..., a_{2n+1}] = \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_{2n+1}}}}.$$

The Dedekind sum $s(d,c)$ satisfies the equation

$$s(d,c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a+d}{c} - \sum_{k=1}^{2n+1}(-1)^k a_k\right), \quad 0<a<c,\ ad \equiv 1 \pmod{c}.$$

One can prove Theorem 8.1 by inductive application of (8.3). We will proceed to give a more general proof of Theorem 8.1, which generalizes to the other types of c.f.e.'s that we will work with in Chapter 10. In the following proposition, we express the cycle integral $\displaystyle\int_z^{A.z} E_2(w)\,dw$ more generally in terms of the matrix A as a word in T and V. The proposition is well-known; see [**Lan76, Zag75**]. The proof we present is based on an argument we learned from an unpublished note of W. Duke, which does not use the reciprocity law directly.[1]

PROPOSITION 8.2. *Let $n \geq 1$ be an odd integer and $a_0, ..., a_n$ be integers. Let $A_k = \begin{pmatrix} a_k & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix}$ for $k = 0, ..., n$ and write $\begin{pmatrix} * & * \\ c_k & d_k \end{pmatrix} = A_0 A_1 \cdots A_k$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_0 A_1 \cdots A_n$, we have*

$$\int_z^{A.z} E_2(w)dw = \frac{6}{\pi i}\log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) + \sum_{k=0}^{n}(-1)^k a_k + 3\sum_{k=1}^{n}\mathrm{sign}(c_k d_k). \tag{8.9}$$

[1]This is crucial, since we will later on work with analogs of Dedekind sums, which do not have a reciprocity law.

PROOF. We begin by rewriting the integral as

$$\int_z^{A.z} E_2(w)dw = \frac{6}{\pi i}\int_i^z \frac{c}{cw+d}dw + \int_i^{A.i} E_2(w)dw. \tag{8.10}$$

Equation (8.10) can be seen in the following way: Differentiating both sides in z shows that the left and right hand side at most differ by a constant. Evaluating at $z=i$ then proves that this constant is zero.

Note that $A_k.i = \frac{a_k i - (-1)^k}{(-1)^k i} = i + (-1)^k a_k$. It follows from the Fourier expansion of $E_2(w)$ in (8.4) that

$$\int_i^{A_k.i} E_2(w)\, dw = \int_i^{i+(-1)^k a_k} E_2(w)\, dw = (-1)^k a_k.$$

Now we split the line integral from z to $A.z$ into an integral over $n+1$ paths (this is possible as $E_2(w)$ is a holomorphic function):

$$\begin{aligned}
\int_i^{A.i} E_2(w)\, dw &= \sum_{k=0}^n \int_{A_0\cdots A_{k-1}.i}^{A_0\cdots A_k.i} E_2(w)\, dw \\
&= \sum_{k=0}^n \int_i^{A_k.i} E_2(w)|_2(A_0\cdots A_{k-1})\, dw \\
&= \sum_{k=0}^n \int_i^{A_k.i} E_2(w)\, dw + \frac{6}{\pi i}\sum_{k=1}^n \int_i^{A_k.i} \frac{c_{k-1}}{c_{k-1}w + d_{k-1}}\, dw \\
&= \sum_{k=0}^n (-1)^k a_k + \frac{6}{\pi i}\sum_{k=1}^n \int_i^{A_k.i} \frac{c_{k-1}}{c_{k-1}w + d_{k-1}}\, dw.
\end{aligned}$$

As a primitive of $\frac{c_{k-1}}{c_{k-1}w+d_{k-1}} = \frac{-c_{k-1}}{-c_{k-1}w-d_{k-1}}$, we choose $\log\left(\frac{c_{k-1}w+d_{k-1}}{\operatorname{sign}(c_{k-1})i}\right)$. Since

$$\begin{pmatrix} * & * \\ c_{k-1} & d_{k-1}\end{pmatrix} A_k = \begin{pmatrix} * & * \\ c_{k-1}a_k + (-1)^k d_{k-1} & (-1)^{k+1}c_{k-1}\end{pmatrix} = \begin{pmatrix} * & * \\ c_k & d_k\end{pmatrix},$$

we have $c_{k-1}A_k.i + d_{k-1} = \frac{c_k i + d_k}{(-1)^k i}$. Thus,

$$\sum_{k=1}^n \int_i^{A_k.i} \frac{c_{k-1}}{c_{k-1}w+d_{k-1}}dw = \sum_{k=1}^n \left(\log\left(\frac{c_k i + d_k}{\operatorname{sign}(d_k)}\right) - \log\left(\frac{c_{k-1}i + d_{k-1}}{\operatorname{sign}(c_{k-1})i}\right)\right). \tag{8.11}$$

We may reorder the sum on the right hand side of (8.11) to obtain

$$\frac{6}{\pi i}\log\left(\frac{c_n i+d_n}{\operatorname{sign}(d_n)}\right)-\frac{6}{\pi i}\log\left(\frac{i}{i}\right)+\frac{6}{\pi i}\sum_{k=1}^{n-1}\left(\log\left(\frac{c_k i+d_k}{\operatorname{sign}(c_k)i}\right)-\log\left(\frac{c_k i+d_k}{\operatorname{sign}(d_k)}\right)\right)$$
$$=\frac{6}{\pi i}\log\left(\frac{ci+d}{\operatorname{sign}(d)}\right)+3\sum_{k=1}^{n-1}\operatorname{sign}(c_k d_k).$$

Hence, with (8.10), we arrive at

$$\begin{aligned}\int_z^{A.z}E_2(w)dw &= \frac{6}{\pi i}\log\left(\frac{cz+d}{\operatorname{sign}(c)i}\right)-\frac{6}{\pi i}\log\left(\frac{ci+d}{\operatorname{sign}(c)i}\right)+\sum_{k=0}^{n}(-1)^k a_k\\ &\quad+\frac{6}{\pi i}\log\left(\frac{ci+d}{\operatorname{sign}(d)}\right)+3\sum_{k=1}^{n-1}\operatorname{sign}(c_k d_k)\\ &=\frac{6}{\pi i}\log\left(\frac{cz+d}{\operatorname{sign}(c)i}\right)+\sum_{k=0}^{n}(-1)^k a_k+3\sum_{k=1}^{n}\operatorname{sign}(c_k d_k).\end{aligned}$$

□

In Section 3 of Chapter 2 we saw that the c.f.e.'s are in correspondence with writing a matrix A as a word in T and V, say, $A=T^{a_0}V^{a_1}\cdots T^{a_{n-1}}V^{a_n}$. This may be transformed to

$$A=T^{a_0}SS^{-1}V^{a_1}T^{a_2}\cdots T^{a_{n-1}}SS^{-1}V^{a_n}=\begin{pmatrix}a_0&-1\\1&0\end{pmatrix}\begin{pmatrix}a_1&1\\-1&0\end{pmatrix}\cdots\begin{pmatrix}a_{n-1}&-1\\1&0\end{pmatrix}\begin{pmatrix}a_n&1\\-1&0\end{pmatrix},$$

i.e. to the form of Proposition 8.2. If the integers $a_1,...,a_n$ are all positive, we have $c_k>0$ and $\operatorname{sign}(d_k)=(-1)^{k+1}$ for $k=1,...,n$ in the notation of Proposition 8.2 . Thus, $3\sum_{k=1}^{n}\operatorname{sign}(c_k d_k)=3$. Hence, if we assume that all partial quotients of the c.f.e. of $\frac{a}{c}$ are positive, we get

$$\int_z^{A.z}E_2(w)dw=\frac{6}{\pi i}\log\left(\frac{cz+d}{i}\right)+\sum_{k=0}^{n}(-1)^k a_k+3.$$

Comparing this equation to (8.7) gives Theorem 8.1.

2. Distribution of Dedekind Sums

A particularly interesting application of Theorem 8.1 concerns the value distribution of the Dedekind sums $s(d,c)$. H. Rademacher [**Rad56**] showed that the range of the Dedekind sums in $\mathbf{R}$ was unbounded from below and above. D. Hickerson then proved by means of Theorem 8.1 that the range of the Dedekind sums is dense in $\mathbf{R}$.

Theorem 8.3 (Hickerson). *The set* $\{(d/c,\ s(d,c)) : (d,c) = 1,\ c > 0\}$ *is dense in* $\mathbf{R} \times \mathbf{R}$.

A finer characterization of the values of the Dedekind sums has recently been conjectured by K. Girstmair [**Gir17**].

Theorem 8.3 is not the only distributional result that is known for Dedekind sums. I. Vardi [**Var87**] observed brilliantly that the equidistribution mod 1 of $12 \cdot s(d,c)$ reduces to estimating sums of Kloosterman sums. Indeed, since $\Phi(A) \in \mathbf{Z}$ for all $A \in \mathrm{SL}_2(\mathbf{Z})$, we have

$$\sum_{c=1}^{N} \sum_{d(c)^*} e(12ms(d,c)) = \sum_{c=1}^{N} \sum_{d(c)^*} e\left(-m\Phi\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\right) e\left(\frac{ma+md}{c}\right) = \sum_{c=1}^{N} K(m,m;c),$$

where $1 \leq a < c$ is such that $ad \equiv 1 \pmod{c}$. The Weil bound for the Kloosterman sums $K(m,m;c)$ individually already gives enough non-trivial cancellation to show that $\{12 \cdot s(d,c) : 1 \leq d < c \leq N,\ (d,c) = 1\}$ is equidistributed mod 1 as $N \to +\infty$. The results can be refined by methods of D. Goldfeld and P. Sarnak [**GS83**] to show the sums of Kloosterman sums twisted by multiplier systems exhibit non-trivial cancellation.

Theorem 8.4 (Vardi). *Let* $r > 0$. *The set* $\{12r \cdot s(d,c) : 1 \leq d < c \leq N,\ (d,c) = 1\}$ *is equidistributed modulo* 1 *as* $N \to +\infty$.

Generally, these equidistribution problems of objects that arise from automorphic forms are approached by identifying the Weyl sums as automorphic objects, e.g., inner products of Selberg Poincaré series (as with Dedekind sums) or Fourier coefficients of Eisenstein series (as in Chapter 10 below), to which spectral theory can be applied.

3. The Problem of Counting Lattice Points in a Tetrahedron

In this section, we would like to present a geometric/combinatorial application of Dedekind sums to lattice-counting problems. In fact, the reciprocity law (8.3) can also be proved by counting lattice points in a suitable parallelepiped [**RG72**, Ch. 2.B].

For positive integers u, v, w, let $D(u,v,w)$ be the tetrahedron in $\mathbf{R}^3$ with vertices $(u,0,0)$, $(0,v,0)$, and $(0,0,w)$, i.e.

$$D(u,v,w) = \left\{(x,y,z) \in \mathbf{R}^3 : 0 < \frac{x}{u} + \frac{y}{v} + \frac{z}{w} < 1\right\}.$$

Note that we excluded the vertices in our definition of $D(u,v,w)$.

L. Mordell [**Mor51**] expressed the number of lattice points $\#(D(u,v,w) \cap \mathbf{Z}^3)$ in a pretty formula in terms of Dedekind sums.

THEOREM 8.5 (Mordell). *Let u, v, w be pairwise coprime positive integers. We have*

$$\#(D(u,v,w)\cap \mathbf{Z}^3) = \frac{uvw}{6} + \frac{1}{4}(uv+uw+vw) + \frac{1}{4}(u+v+w) + \frac{1}{12}\left(\frac{uv}{w}+\frac{uw}{v}+\frac{vw}{u}\right) + \frac{1}{12uvw} - 2 - (s(uv,w)+s(uw,v)+s(vw,u)).$$

To prove Theorem 8.5, L. Mordell manipulated the counting function

$$\sum_{\substack{0\le x<u,\\ 0\le y<v,\\ 0\le z<w,\\ (x,y,z)\neq(0,0,0),\\ \frac{x}{u}+\frac{y}{v}+\frac{z}{w}<1}} 1 = \frac{1}{2}\sum_{x,y,z}{}' \left(\left\lfloor \frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right\rfloor - 1\right)\left(\left\lfloor \frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right\rfloor - 2\right), \tag{8.12}$$

where the second sum again goes over $0 \le x < u$, $0 \le y < v$, and $0 \le z < w$ and $\sum'$ means that we omit $x = y = z = 0$. The expression $\left\lfloor \frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right\rfloor$ in (8.12) can only take the values $0, 1, 2$ and we want to count the number of times $\left\lfloor \frac{x}{w}+\frac{y}{v}+\frac{z}{w}\right\rfloor = 0$.

The right hand side in Theorem 8.5 is computationally easier than the left hand side. Indeed, if one were to calculate the left hand side without exploiting symmetries, the computational complexity would be $O(uvw)$. The right hand side, however, is much faster to compute, as Dedekind sums can be computed through the Euclidean algorithm.

CHAPTER 9

Hardy Sums

In this chapter, we will introduce our main objects of study in this part of the thesis: the Hardy sums. These sums are integer-valued analogs of the classical Dedekind sums, which stem from the multiplier systems of certain θ-functions.

Giving another proof of the reciprocity law (8.3) for Dedekind sums, G.H. Hardy [**Har05**] looked at the contour integration of trigonometric functions over a large circle. Changing trigonometric functions in the integrand led him to other finite sums, some of which exhibit reciprocity laws. We will consider two of these "Hardy sums," namely

$$S(d,c) = \sum_{k=1}^{|c|-1} (-1)^{k+1+\left\lfloor \frac{dk}{c} \right\rfloor} \quad \text{for } (d,c)=1,\ c+d \text{ odd}, \tag{9.1}$$

and

$$S_4(d,c) = \sum_{k=1}^{|c|-1} (-1)^{\left\lfloor \frac{dk}{c} \right\rfloor} \quad \text{for } (d,c)=1,\ d \text{ odd}. \tag{9.2}$$

Both sums $S(d,c)$ and $S_4(d,c)$ in (9.1) and (9.2) are integer-valued and it is easy to see that $S(-d,c) = S(d,-c) = -S(d,c)$ and $S(d+2c,c) = S(d,c)$ and that the same identities also hold for $S_4(d,c)$.

The Hardy sums in (9.1) and (9.2) are particularly interesting, as they arise analogously to the Dedekind sums as correction factors in the transformation law of the logarithms of θ-functions under subgroups of $\mathrm{SL}_2(\mathbf{Z})$. These θ-functions are also weight $\frac{1}{2}$ forms, but with different multiplier systems than the function $\eta(z)$ in (8.1).

More precisely, set

$$\theta(z) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z} = \prod_{n=1}^{\infty} (1-e(nz))\left(1+e\left((n-1/2)\,z\right)\right)^2 \tag{9.3}$$

and

$$\theta_4(z) = \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i n^2 z} = \prod_{n=1}^{\infty} (1-e(nz))\left(1-e\left((n-1/2)\,z\right)\right)^2 . \tag{9.4}$$

The functions $\theta(z)$ and $\theta_4(z)$ exhibit modular transformations for the subgroups

$$\begin{aligned}\Gamma_\theta &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d,\ b \equiv c \pmod 2 \right\}, \text{ resp.} \\ \Gamma^0(2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : b \equiv 0 \pmod 2 \right\}\end{aligned} \tag{9.5}$$

instead of the full modular group $\mathrm{SL}_2(\mathbf{Z})$. B. Berndt [**Ber78**] proved that

$$\log\theta(A.z) - \log\theta(z) = \frac{1}{2}\log\left(\frac{cz+d}{i}\right) + \frac{\pi i}{4} S(d,c) \text{ for } A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\theta,\ c>0, \tag{9.6}$$

and

$$\log\theta_4(A.z) - \log\theta_4(z) = \frac{1}{2}\log\left(\frac{cz+d}{i}\right) - \frac{\pi i}{4} S_4(d,c) \text{ for } A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma^0(2),\ c>0, \tag{9.7}$$

so $S(d,c)$ and $S_4(d,c)$ determine the multiplier systems of these θ-functions. The Hardy sums $S(d,c)$ and $S_4(d,c)$ can either be viewed as analogs of the Dedekind symbol or, since they only depend on $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$, as analogs of the Dedekind sums. Both of these viewpoints are true: The distribution of the Hardy sums resembles the distribution of the Dedekind sums, whereas their cohomological properties resemble those of the Dedekind symbol. Moreover, in the wonderful paper of R. Sczech [**Scz95**], where we first learned about these sums, the Hardy sums $S(d,c)$ are given a representation as cotangent sums, which the Dedekind sums also have [**RG72**, Eq. 26].

H. Rademacher [**Rad67**] gave the correction factors of the transformation of $\log\theta$ and $\log\theta_4$ in terms of non-trivial linear combinations of Dedekind sums; see also R. Sitaramachandrarao's elementary approach [**Sit87**]. Explicitly, these linear combinations read as

$$S(d,c) = 8\ s(d,2c) + 8\ s(2d,c) - 20\ s(d,c), \quad \text{for } (d,c)=1,\ c+d \text{ odd},$$

and

$$S_4(d,c) = 8\ s(d,2c) - 4\ s(d,c) \quad \text{for } (d,c)=1,\ d \text{ odd}. \tag{9.8}$$

These equations are well-understood in the context of Eisenstein cohomology [**Ste82**, Ch. 2].

The Hardy sums $S(d,c)$ in (9.1) make sense for c,d both odd or both even, but the value of the sum will then be equal to zero. Also, we have $S_4(d,c)=0$ if we set d to be even in (9.2); see [**Sit87**, Thm. 5.1]. Hence, we always implicitly assume that $c+d$ is odd when talking about the sum $S(d,c)$. Similarly, will always implicitly assume that d is odd in $S_4(d,c)$, i.e. that $\left(\begin{smallmatrix} a & 2b \\ c & d \end{smallmatrix}\right) \in \Gamma^0(2)$ for certain $a, b \in \mathbf{Z}$.

The reciprocity law for Dedekind sums (8.3) arises from the action of the inversion $S \in \mathrm{SL}_2(\mathbf{Z})$. Since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma_\theta$, there is a reciprocity law similar to (8.3) for $S(d,c)$ as well, namely

$$S(d,c) + S(c,d) = \mathrm{sign}(cd), \ (c,d) = 1, \ c + d \text{ odd}. \tag{9.9}$$

R. Sczech [**Scz95**] used the reciprocity law (9.9) to give an alternative proof of the identity

$$\sum_{r=1}^{q} e\left(\frac{r^2}{q}\right) = \prod_{\substack{r=1,\\ r \text{ odd}}}^{q} \left(e\left(\frac{r}{q}\right) - e\left(-\frac{r}{q}\right)\right)$$

for $q \geq 1$ odd, which originally goes back to C.F. Gauss.

Since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \notin \Gamma^0(2)$, we should not expect a reciprocity law for $S_4(d,c)$. Formulas that resemble reciprocity laws were given by J. Meyer [**Mey00**], but expressing the right hand side of the reciprocity law by $-1 + S_4(c^2, cd+1)$. In (10.4), we will give a somewhat unsatisfying reciprocity-type law for $S_4(d,c)$ as well, expressing the right hand side of the reciprocity law as a sum of partial quotients of the $\Gamma^0(2)$-type c.f.e. of $\frac{c}{d}$. Alternatively, we can express the behavior of $S_4(d,c) + S_4(c,d)$ for $d, c > 0$ both odd in geometric terms, namely as the number of lattice points of a certain lattice in a right-angled triangle; see Theorem 11.2 below. Both of these are not reciprocity laws in the sense of (9.9), as we would like to involve the variables d, c more directly.

As for what is known about the distribution of the Hardy sums, the analog of Theorem 8.3 was proved by J. Meyer [**Mey97**].

Theorem 9.1 (Meyer). *The sets* $\{(d/c,\ S(d,c)) : (d,c) = 1,\ c + d \text{ odd}\}$ *and* $\{(d/c,\ S_4(d,c)) : (c,d) = 1,\ d \text{ odd}\}$ *are dense in* $\mathbf{R} \times \mathbf{Z}$.

J. Meyer's proof of Theorem 9.1 used the theory of exponential sums instead of continued fractions, as in the proof of D. Hickerson's Theorem 8.3. In fact, it is possible to obtain an analog of Theorem 8.1 for $S(d,c)$ by means of the reciprocity law (9.9) after defining a suitable c.f.e.. But in absence of a reciprocity law for $S_4(d,c)$, the approach of D. Hickerson does not carry over immediately. In the next chapter, we will define special kinds of c.f.e.'s, for which analogs of Theorem 8.1 exist for the Hardy sums.

To our knowledge, an analog for Vardi's equidistribution result in Theorem 8.4 has not been previously proved, a gap that we will fill in the next chapter.

Finally, we would like to briefly mention the distributional results for the second moments of the Hardy sums by Y. Yi and W. Zhang [**YZ00**], which mirror the results on second moments for the Dedekind sums by J.B. Conrey, E. Fransen, R. Klein, and C. Scott [**CFKS96**]. A finer picture of the even moments of $S(d, c)$ over short intervals has been obtained by Z. Xiu and W. Zhang [**XZ07**]. The mean values of Hardy sums will not further concern us in this thesis.

CHAPTER 10

The Distribution of Hardy Sums

In this chapter, we will prove various distributional results for the Hardy sums $S(d,c)$ and $S_4(d,c)$. We will prove an analog of Theorem 8.1 for the Hardy sums in terms of special continued fraction expansions, which we refer to as expansions of the Γ_θ- and $\Gamma^0(2)$-type. More precisely, we have the following theorem.

THEOREM 10.1. *Let d be an integer and $c > 0$ such that $(d,c) = 1$.*

(1) *For $c+d$ odd, let $\frac{d}{c} = [\![2c_0; 2c_1, ..., 2c_n]\!] = 2c_0 - \frac{1}{2c_1 - \frac{1}{\cdots - \frac{1}{2c_n}}}$ with $c_0 \in \mathbf{Z}$ and $c_1, ..., c_n$ non-zero integers be the negative continued fraction of $\frac{d}{c}$. The Hardy sum $S(d,c)$ takes the form*
$$S(d,c) = -\sum_{k=1}^{n} \operatorname{sign}(c_k).$$

(2) *For d odd, let $\frac{d}{c} = [2a_0; a_1, 2a_2, a_3, ..., 2a_{n-1}, a_n]$ with $a_0 \in \mathbf{Z}$ and $a_1, ..., a_n$ non-zero integers such that $|a_k| > 1$ for $k = 1, 3, ..., n-2$. The Hardy sum $S_4(d,c)$ takes the form*
$$S_4(d,c) = (a_1 + a_3 + ... + a_n) + \sum_{k=1}^{n} (-1)^k \operatorname{sign}(a_k).$$

Note that $S(d,c) + S_4(d,c)$ makes sense if we restrict the coprime integers d, c to be such that d is odd and c is even (hence, the matrix $\begin{pmatrix} a & 2b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ with $a, b \in \mathbf{Z}$ lies both in Γ_θ and $\Gamma^0(2)$). The continued fraction expansion of $\frac{d}{c}$ then can be written as
$$\frac{d}{c} = [\![2c_0, 2c_1, ..., 2c_n]\!] = [2c_0; -2c_1, 2c_2, ..., -2c_n], \quad n \geq 1 \text{ odd},$$
with both continued fraction expansions satisfying the assumptions of both parts of Theorem 10.1. We thus get the following easy corollary to Theorem 10.1.

COROLLARY 10.2. *Let $c > 0$ be an even integer and d an odd integer, $(d,c) = 1$. Let $c_0, c_1, ..., c_n$ be integers with $c_1, ..., c_n$ non-zero and $n \geq 1$ odd such that $\frac{d}{c} =$*

$[\![2c_0, 2c_1, ..., 2c_n]\!]$. *We have*

$$S(d,c) + S_4(d,c) = 2 \sum_{\substack{k=1, \\ k \text{ odd}}}^{c-1} (-1)^{\lfloor \frac{dk}{c} \rfloor} = -2 \cdot (c_1 + c_3 + ... + c_n).$$

With the representation of the Hardy sums in terms of the partial quotients of c.f.e. in Theorem 10.1, we give an alternative proof of J. Meyer's Theorem 9.1 on the density of the range of the Hardy sums. As an application of Corollary 10.2, we can also prove the following density theorem, which appears to be new.

THEOREM 10.3. *The set* $\{(d/c,\ S(d,c) + S_4(d,c)) : (d,c) = 1,\ c > 0 \text{ even},\ d \text{ odd}\}$ *is dense in* $\mathbf{R} \times 2\mathbf{Z}$.

Moreover, we will also prove an analog of I. Vardi's Theorem 8.4 on the equidistribution of the Dedekind sums. Since the Hardy sums are integer-valued, the equidistribution is on $\mathbf{Z}/m\mathbf{Z}$ for $m > 0$ with the counting measure instead of $\mathbf{R}/\mathbf{Z}$ with the Lebesgue measure. Our main tool to show this is the spectral theory of non-holomorphic Eisenstein series with multiplier systems.

THEOREM 10.4. *Let $m > 1$ be an integer.*

(1) *The set*

$$\{(d/c,\ S(d,c) \pmod{m}) : 1 \leq d < c \leq N,\ (d,c) = 1,\ c + d \text{ odd}\}$$

is uniformly distributed on $\mathbf{R}/\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ *as* $N \to +\infty$.

(2) *The set*

$$\{(d/c,\ S_4(d,c) \pmod{m}) : 1 \leq d < c \leq N,\ (d,c) = 1,\ d \text{ odd}\}$$

is uniformly distributed on $\mathbf{R}/\mathbf{Z} \times \mathbf{Z}/m\mathbf{Z}$ *as* $N \to +\infty$.

As an immediate corollary to Theorem 10.4 we get that the sets $\{S(d,c) \pmod{m} : 1 \leq d < c \leq N,\ (d,c) = 1,\ c + d \text{ odd}\}$ and $\{S_4(d,c) \pmod{m} : 1 \leq d < c \leq N,\ (d,c) = 1,\ d \text{ odd}\}$ are both uniformly distributed on $\mathbf{Z}/m\mathbf{Z}$ as $N \to +\infty$.

The distribution in $\mathbf{Z}/m\mathbf{Z}$ of the sequence of the integer-valued Rademacher symbols

$$\Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) - 3 \cdot \operatorname{sign}(c(a+d)),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

ordered by absolute value of the trace is known to be uniform; see J. Ueki's paper [**Uek21**]. Moreover, other distributional results for the Rademacher symbol Ψ have been obtained by J. Mozzochi [**Moz13**], based on an idea of P. Sarnak (see also F. Von Essen's generalization to other multiplier systems [**vEss14**], which is relevant in our case).

1. Continued Fractions of Γ_θ- and $\Gamma^0(2)$-type

In this section, we define non-classical continued fractions and prove the facts that we will later use in our proofs.

1.1. Γ_θ-type c.f.e. Let $c_1, c_2, ...$ be a sequence of non-zero integers and $c_0 \in \mathbf{Z}$. We define the negative continued fraction to be

$$[\![c_0; c_1, c_2, ..., c_n]\!] = c_0 - \cfrac{1}{c_1 - \frac{1}{c_2 - \frac{1}{\cdots - \frac{1}{c_n}}}}.$$

The k-th partial fraction of the sequence is denoted by $\frac{p_k}{q_k} = [\![2c_0; 2c_1, ..., 2c_k]\!]$ with $(p_k, q_k) = 1$. The integers p_k and q_k may be viewed as the entries of a matrix, namely

$$\frac{p_k}{q_k} = (T^{2c_0} ST^{2c_1} ST^{2c_2} S \cdots T^{2c_k} S).i\infty = \begin{pmatrix} p_k & * \\ q_k & * \end{pmatrix}.i\infty.$$

Since Γ_θ is generated by T^2 and S, the matrix $\begin{pmatrix} p_k & * \\ q_k & * \end{pmatrix} = T^{2c_0} ST^{2c_1} ST^{2c_2} S \cdots T^{2c_k} S$ lies in Γ_θ. For this reason we call a c.f.e. of the form $[\![2c_0; 2c_1, ..., 2c_k]\!]$ a Γ_θ-type c.f.e. We shall usually write $[\![2c_1, 2c_2, ..., 2c_n]\!] = [\![0; 2c_1, 2c_2, ..., 2c_n]\!]$ in the case $c_0 = 0$, dropping the semicolon.

To find the other two entries of $\begin{pmatrix} p_k & * \\ q_k & * \end{pmatrix}$, we let the matrix act on the point 0:

$$\begin{pmatrix} p_k & * \\ q_k & * \end{pmatrix}.0 = (T^{2c_0} ST^{2c_1} S \cdots T^{2c_{k-1}} ST^{2c_k} S).0 = (ST^{2c_1} S \cdots T^{2c_{k-1}} S).i\infty = \frac{p_{k-1}}{q_{k-1}}.$$

Hence, $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = T^{2c_0} ST^{2c_1} S \cdots T^{2c_k} S \in \Gamma_\theta$, which implies that $p_k q_{k-1} - p_{k-1} q_k = 1$.

The partial fractions $\frac{p_k}{q_k}$ will be useful in finding a sequence of partial quotients c_i of a Γ_θ-type c.f.e. as an approximation of real numbers. We start by proving two lemmas, relating the resulting sequence of partial fractions to the sequence of partial quotients.

Lemma 10.5. *Let $c_1, c_2, ...$ be a sequence of non-zero integers, $k \geq 1$, and $\frac{p_k}{q_k} = [\![2c_1, ..., 2c_k]\!]$ be the k-th partial fraction. The partial fractions $\frac{p_k}{q_k}$ satisfy $p_k = 2c_k p_{k-1} - p_{k-2}$ and $q_k = 2c_k q_{k-1} - q_{k-2}$ for all $k \geq 3$.*

Proof. By induction. The first three partial fractions are $\frac{p_1}{q_1} = \frac{-1}{2c_1}$, $\frac{p_2}{q_2} = \frac{-2c_2}{4c_1c_2 - 1}$, and $\frac{p_3}{q_3} = \frac{1 - 4c_2c_3}{8c_1c_2c_3 - 2c_1 - 2c_3}$. For $k = 3$, we directly verify the claim.

Suppose the claim for $k-1 \geq 3$. Let $\frac{p_k'}{q_k'} = [\![2c_2, 2c_3, ..., 2c_k]\!]$. Since $\frac{p_k}{q_k} = -\frac{1}{2c_1 - \frac{1}{p_k'/q_k'}} = \frac{-p_k'}{2c_1p_k' - q_k'}$, we have $p_k = -p_k'$ and $q_k = 2c_1p_k' - q_k'$. By the induction hypothesis, we see that

$$\frac{p_k}{q_k} = -\frac{1}{2c_1 - \frac{q_k'}{p_k'}} = -\frac{1}{2c_1 - \frac{2c_kq_{k-1}' - q_{k-2}'}{2c_kp_{k-1}' - p_{k-2}}} = \frac{p_{k-2}' - 2c_kp_{k-1}'}{2c_1c_kp_{k-1}' - 2c_1p_{k-2}' - 2c_kq_{k-1}' + q_{k-2}'}.$$

Hence,

$$p_k = p_{k-2}' - 2c_kp_{k-1}' = 2c_kp_{k-1} - p_{k-2} \text{ and}$$
$$q_k = 2c_k(2c_1p_{k-1}' - q_{k-1}') - (2c_1p_{k-2}' - q_{k-2}') = 2c_kq_{k-1} - q_{k-2}.$$

□

For the partial fractions of the classical continued fraction $\frac{p_k}{q_k} = [a_0; a_1, ..., a_k]$ the analog of Lemma 10.5 states $p_k = a_kp_{k-1} + p_{k-2}$ and $q_k = a_kq_{k-1} + q_{k-2}$ for $k \geq 3$.

With Lemma 10.5 it is then easy to show the following useful fact about the k-th partial fractions.

LEMMA 10.6. *Let $c_1, c_2, ...$ be any sequence of non-zero integers, $k \geq 1$, and $\frac{p_k}{q_k} = [\![2c_1, ..., 2c_k]\!]$ be the k-th partial fraction. Then $|p_{k-1}| < |p_k|$ and $|q_{k-1}| < |q_k|$ for all $k \geq 1$. In particular, $|p_k|, |q_k| \to +\infty$ as $k \to +\infty$.*

PROOF. By induction. The proof is essentially the same for p_k and q_k, so we will only present the one for q_k. We have $q_1 = 2c_1$ and $q_2 = 4c_1c_2 - 1$. Suppose $|q_1| \geq |q_2|$. Then we would have

$$2|c_1| = |q_1| \geq |q_2| \geq 4|c_1| - 1, \tag{10.1}$$

but there is no non-zero integer c_1 satisfying (10.1).

Assume the statement for $k-1 \geq 2$, i.e., that $|q_{k-1}| > |q_{k-2}|$. By Lemma 10.5, we have

$$|q_k| \geq 2|c_k||q_{k-1}| - |q_{k-2}| > |q_{k-1}|,$$

as $|c_k| \geq 1$. □

With Lemma 10.6 we are set to prove the approximation of an irrational number by a Γ_θ-type c.f.e. Its proof also gives an algorithm, by which such an approximation can be computed. The algorithm is analog to the one for classical, $\mathrm{SL}_2(\mathbf{Z})$-type c.f.e.'s in Proposition 2.7.

PROPOSITION 10.7. *Let $x \in (-1, 1)$ be irrational. There is a sequence $c_1, c_2, ...$ of non-zero integers such that $[\![2c_1, 2c_2, ..., 2c_k]\!] \to x$ as $k \to +\infty$.*

PROOF. We construct the sequence $c_1, c_2, \ldots$ as follows. Since $|x| < 1$, we have $\frac{1}{|x|} > 1$. Pick $c_1 \in \mathbf{Z}$ non-zero such that $x_1 = 2c_1 - \frac{1}{x} \in (-1, 1)$. We may write $x = \frac{1}{2c_1 - x_1} = [\![2c_1, x_1^{-1}]\!]$. Then pick $c_2 \in \mathbf{Z}$ non-zero such that $x_2 = 2c_2 - \frac{1}{x_1} \in (-1, 1)$, which again is possible as $\frac{1}{|x_1|} > 1$. We may also write $x = [\![2c_1, 2c_2, x_2^{-1}]\!]$. Continue this process to get a (unique) finite sequence $c_1, \ldots, c_k \in \mathbf{Z} - \{0\}$ and $x_1, \ldots, x_k \in (-1, 1)$ with $x = \frac{1}{2c_1 - \frac{1}{2c_2 - \frac{1}{\cdots - x_k}}} = [\![2c_1, \ldots, 2c_k, x_k^{-1}]\!]$.

This process approximates $x = [\![2c_1, \ldots, 2c_k, x_k^{-1}]\!]$. Indeed, observe that

$$
\begin{aligned}
|[\![2c_1, \ldots, 2c_k, x_k^{-1}]\!] - [\![2c_1, \ldots, 2c_k]\!]| &= \left| \frac{\frac{1}{x_k} p_k - p_{k-1}}{\frac{1}{x_k} q_k - q_{k-1}} - \frac{p_k}{q_k} \right| \\
&= \left| \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k(\frac{1}{x_k} q_k - q_{k-1})} \right| \\
&= \frac{1}{|q_k||\frac{1}{x_k} q_k - q_{k-1}|} \\
&\leq \frac{1}{|q_k|} \to 0, \text{ as } k \to +\infty,
\end{aligned}
$$

where we used Lemma 10.6 and that $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$ is a unimodular matrix. □

Proposition 10.7 gives a *sequence* of integers, which approximates x arbitrarily well. If we were given an $\varepsilon > 0$ and just wanted to find a c.f.e. $[\![2c_1, \ldots, 2c_n]\!]$ in $(x - \varepsilon, x + \varepsilon)$, we could have picked a rational $\frac{a}{c} \in (x - \varepsilon, x + \varepsilon)$ with $c > 0$, $(a, c) = 1$ and $a + c$ odd. Finding $b, d \in \mathbf{Z}$ with $b \equiv c \pmod 2$, $a \equiv d \pmod 2$, and $ad - bc = 1$ yields a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. Writing the matrix A in terms of the generators T^2, S gives $[\![2c_1, \ldots, 2c_n]\!]$. The downside of this argument is that we would still not know how to construct a c.f.e. with the desired properties, while the proof of Proposition 10.7 gives an explicit algorithm. Moreover, the algorithm shows that c.f.e. of Γ_θ-type are unique (this can be proved rigorously through induction on the length of a c.f.e.).

1.2. $\Gamma^0(2)$-type c.f.e. We now turn from the negative continued fractions to the positive continued fraction as in (2.18). Again, if $a, c \in \mathbf{Z}$ with $(a, c) = 1$ and a an odd integer, we may find $d, b \in \mathbf{Z}$ such that $ad - 2bc = 1$, i.e. such that $\begin{pmatrix} a & 2b \\ c & d \end{pmatrix} \in \Gamma^0(2)$. Writing $\begin{pmatrix} a & 2b \\ c & d \end{pmatrix} = \pm T^{2a_0} V^{a_1} \cdots T^{2a_{n-1}} V^{a_n}$ in terms of the generators T^2 and V of $\Gamma^0(2)$

gives rise to a c.f.e.:

$$\frac{a}{c} = \begin{pmatrix} a & 2b \\ c & d \end{pmatrix}.i\infty = \pm T^{2a_0} V^{a_1} \cdots T^{2a_{n-1}} V^{a_n}.i\infty = [2a_0; a_1, ..., 2a_{n-1}, a_n].$$

We call a c.f.e. of this kind a $\Gamma^0(2)$-type c.f.e.

In the remainder of this section, we present an algorithm for finding the $\Gamma^0(2)$-type c.f.e. of a rational number. First, we prove a technical lemma, relating the length of the continued fraction to the parity of the numerator of the rational it is representing. This will make the description of the algorithm cleaner.

LEMMA 10.8. *Let $a_0 \in \mathbf{Z}$ and $a_1, ..., a_n$ be non-zero integers such that a_{2k} is even for all integers $k = 0, 1, ..., \lfloor \frac{n}{2} \rfloor$. Let $c > 0$ and $d \in \mathbf{Z}$ with $(c, d) = 1$ such that $\frac{d}{c} = [a_0; a_1, ..., a_n]$. Then $d \equiv n \pmod 2$.*

PROOF. By induction. The case $n = 0$ is seen from $a_0 \equiv 0 \pmod 2$ and for $n = 1$, we have $\frac{d}{c} = a_0 - \frac{1}{a_1} = \frac{a_0 a_1 - 1}{a_1}$, and it is hence clear that d is odd.

Suppose the claim holds for $n - 2 \geq 0$ and write $\frac{d'}{c'} = [a_2, ..., a_n]$. Then

$$[a_0; a_1, ..., a_n] = a_0 + \frac{1}{a_1 + \frac{c'}{d'}} = \frac{a_0(a_1 + c') + d'}{a_1 + c'} = \frac{d}{c},$$

hence, as a_0 is even, $d \equiv d' \pmod 2$. $\square$

PROPOSITION 10.9. *Let $c > 0$ and d an odd integer such that $(c, d) = 1$. There exists a c.f.e. $\frac{d}{c} = [2a_0; a_1, 2a_2, ..., a_n]$ of odd length $n \geq 1$ with $a_0 \in \mathbf{Z}$, $a_1, ..., a_n \in \mathbf{Z} - \{0\}$ and $|a_1|$, $|a_3|$, ..., $|a_{n-2}| > 1$.*

PROOF. The algorithm is similar to the one given in the proof of Proposition 2.7 or Proposition 10.7. First, pick $a_0 \in \mathbf{Z}$ such that $x_0 = \frac{d}{c} - 2a_0 \in [-1, 1]$. It is not possible that $x_0 = 0$, as then $\frac{d}{c} = [2a_0]$ would be such that $d \equiv 0 \pmod 2$, which we excluded.

Pick $a_1 \notin \{-1, 0, 1\}$ such that $x_1 = \frac{1}{x_0} - a_1 \in (-1, 1)$. If $x_1 = 0$, we are done. If no such a_1 exists, then $x_0 = \pm 1$. Thus, we may pick $a_1 = x_0$ and we are done.

Continue this process to obtain $x_2, x_3, ...$ with $x_{2n} \in [-1, 1]$ and $x_{2n+1} \in (-1, 1)$ until $x_{2n+1} = 0$ for an $n \geq 0$. Note that it is not possible that $x_{2n} = 0$, since by Lemma 10.8 this contradicts the assumption that d is odd.

That the process stops can be seen in the following way. Write $x_1 = \frac{d_1}{c_1}$ with $c_1 > 0$ and $-c_1 < d_1 < c_1$ coprime to c_1. Then $x_2 = \frac{c_1 - 2a_2 d_1}{d_1} = \frac{d_2}{c_2}$ with $c_2 > 0$ and $-c_2 \leq d_2 \leq c_2$ coprime to c_2. Since c_2 divides d_1 and $|d_1| < c_1$, we have $c_2 < c_1$. In fact, the value of the denominator of $x_1, x_2, ...$ strictly decreases with each step. Hence, the process must come to an end. $\square$

Contrary to the Γ_θ-type c.f.e., the continued fraction in Proposition 10.9 is *not* unique. Take, say, $\frac{1}{x_0} = \frac{7}{3}$ in the proof of Proposition 10.9. We may either pick $a_1 = 2$ or $a_1 = 3$ to get $x_1 = \frac{1}{x_0} - a_1 \in (-1, 1)$. Indeed, the continued fractions $\frac{3}{7} = [0; 2, 2, 1] = [0; 3, -2, 2]$ both are of the form as claimed in Proposition 10.9.

Uniqueness can be recovered if we require that the partial quotients a_1, $2a_2$, ..., a_{n-2}, $2a_{n-1}$ are even. In that case, $\frac{d}{c} = [2a_0; a_1, 2a_2, a_3, ..., 2a_{n-1}, a_n]$ the denominator $c > 0$ will then have the same parity as a_n. In particular, if c is even, the $\Gamma^0(2)$-type c.f.e. of $\frac{d}{c}$ can be transformed into a Γ_θ-type c.f.e., namely

$$[2a_0; a_1, 2a_2, ..., 2a_{n-1}, a_n] = [\![2a_0; -a_1, 2a_2, ..., -2a_{n-1}, a_n]\!].$$

Proposition 10.7 is hence also applicable for continued fractions of $\Gamma^0(2)$-type.

2. Cycle Integrals of Modular Integrals

In Propositon 8.2, we saw that the cycle integral of the Eisenstein series $E_2(z)$ gives rise to an expression of the Dedekind symbol as a sum of partial quotients. In this section, we show the analog of (8.6) for $\log\theta$ and $\log\theta_4$ and prove Theorem 10.1.

The space of modular forms of weight 2 is not trivial for the subgroups Γ_θ and $\Gamma^0(2)$. Let $L(z) = 2E_2(z) - E_2\left(\frac{z}{2}\right)$, which is a modular form of weight 2 for $\Gamma^0(2)$. It is easy to see that it is invariant under the $|_2$-operator for $\Gamma^0(2)$ using $\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}.\frac{z}{2} = \frac{1}{2}\begin{pmatrix} a & 2b \\ c & d \end{pmatrix}.z$. The modular form $L(z)$ is the only modular form of weight 2 for $\Gamma^0(2)$ up to scalar multiplication. Its Fourier-expansion is given by

$$L(z) = 1 + 24\sum_{n=1}^{\infty} \sigma_1^{\mathrm{odd}}(n) e^{\pi i n z}, \tag{10.2}$$

where $\sigma_1^{\mathrm{odd}}(n) = \sum_{d|n,\, d \text{ odd}} d$. The space of modular forms of weight 2 for Γ_θ is also one-dimensional and its basis element is given by

$$R(z) = L(z+1) = 1 + 24\sum_{n=1}^{\infty} (-1)^n \sigma_1^{\mathrm{odd}}(n) e^{\pi i n z}.$$

The notation $L(z)$ and $R(z)$ goes back to S. Ramanujan.

Proposition 10.10. *We have*

$$\theta(z) = e\left(\frac{1}{24}\int_\infty^z (E_2(w) - R(w))dw\right) \text{ and } \theta_4(z) = e\left(\frac{1}{24}\int_\infty^z (E_2(w) - L(w))dw\right).$$

Proof. We only show the case of $\theta_4(z)$, the argument for $\theta(z)$ is similar. The integral in the assertion converges, since the zeroth coefficient of the Fourier expansion of $E_2(w)-$

$L(w)$ vanishes. Recall the Jacobi Triple Product Identity in (9.4). Using the Fourier expansion of $E_2(z)$ and $L(z) = 2E_2(z) - E_2\left(\frac{z}{2}\right)$, we see that

$$\begin{aligned}
\frac{1}{24}\int_{\infty}^{z}(E_2(w) - L(w))dw &= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{d|n}\frac{d}{n}e(nz) + \frac{1}{\pi i}\sum_{n=1}^{\infty}\sum_{d|n}\frac{d}{n}e(nz) \\
&\quad - \frac{1}{\pi i}\sum_{n=1}^{\infty}\sum_{d|n}\frac{d}{n}e^{\pi i n z} \\
&= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e(mnz) + \frac{1}{\pi i}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e(mnz) \\
&\quad - \frac{1}{\pi i}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}\left(e(mnz) + e\left(mnz - \frac{mz}{2}\right)\right) \\
&= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e(mnz) - \frac{1}{\pi i}\sum_{m=1}^{\infty}\frac{1}{m}e\left(mz\left(n - \frac{1}{2}\right)\right).
\end{aligned}$$

Set $z = it$ with $t > 0$ and use the Taylor expansion $\log(1-x) = -\sum_{m=1}^{\infty}\frac{1}{m}x^m$ for $0 < x < 1$ to show

$$e\left(\frac{1}{24}\int_{\infty}^{it}(E_2(w) - L(w))dw\right) = \prod_{n=1}^{\infty}(1 - e(nit))\left(1 - e\left((n - 1/2)\,it\right)\right)^2$$

for all $t > 0$. The Identity Theorem finishes the proof. □

Since, for instance, $\frac{\theta_4'(z)}{\theta_4(z)}$ is equal to $\frac{\pi i}{12}\left(E_2(z) - L(z)\right)$ by Proposition 10.10, the next proposition follows immediately.

PROPOSITION 10.11. *For $A \in \Gamma^0(2)$ the logarithm of $\theta_4(z)$ satisfies*

$$\log\theta_4(A.z) - \log\theta_4(z) = \frac{\pi i}{12}\int_{z}^{A.z}(E_2(w) - L(w))dw. \tag{10.3}$$

For $B \in \Gamma_\theta$, the logarithm of $\theta(z)$ satisfies

$$\log\theta(B.z) - \log\theta(z) = \frac{\pi i}{12}\int_{z}^{B.z}(E_2(w) - R(w))dw.$$

We have seen in Section 1 of this chapter that the representation of matrices in the groups $\Gamma^0(2)$ and Γ_θ leads to c.f.e. of a certain type. We may use Proposition 8.2 and Proposition 10.11 to express $\log\theta_4(A.z) - \log\theta_4(z)$ in terms of the matrix representation of $A = \begin{pmatrix} a & 2b \\ c & d \end{pmatrix}$ as a word in the matrices T^2, V in $\Gamma^0(2)$. First, we will express $3\sum_{k=1}^{n}\operatorname{sign}(c_k d_k)$ in Proposition 8.2 in terms of partial quotients of a $\Gamma^0(2)$-type c.f.e.

PROPOSITION 10.12. *Let $n \geq 1$ be an odd integer and $a_0, ..., a_n$ be integers such that $|a_2|, ..., |a_n|$ are at least 2 and $a_1 \neq 0$. Let $A_k = \begin{pmatrix} a_k & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix}$ for $k = 0, ..., n$ and write $\begin{pmatrix} * & * \\ c_k & d_k \end{pmatrix} = A_0 A_1 \cdots A_k$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_0 A_1 \cdots A_n$, we have*

$$\int_z^{A.z} E_2(w)dw = \frac{6}{\pi i} \log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) + \sum_{k=0}^{n} (-1)^k a_k - 3\sum_{k=1}^{n}(-1)^k \mathrm{sign}(a_k).$$

PROOF. We show by induction that $0 < |d_k| \leq |c_k|$, $\mathrm{sign}(c_k) = \mathrm{sign}(a_1 \cdots a_k)$ and $\mathrm{sign}(d_k) = (-1)^{k+1}\mathrm{sign}(a_1 \cdots a_{k-1})$ for all $k = 1, ..., n$, which together with Proposition 8.2 suffices to prove the claim.

Suppose $n = 1$. Then $A_0 A_1 = \begin{pmatrix} * & * \\ a_1 & 1 \end{pmatrix}$, and we see that $|d_1| = 1 \leq |a_1| = |c_1|$, $\mathrm{sign}(c_1) = \mathrm{sign}(a_1)$ and $\mathrm{sign}(d_1) = 1$.

Suppose the claim holds for $k - 1 < n$. Then

$$\begin{pmatrix} * & * \\ c_{k-1} & d_{k-1} \end{pmatrix}\begin{pmatrix} a_k & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ c_{k-1}a_k + (-1)^k d_k & (-1)^{k+1}c_{k-1} \end{pmatrix} = \begin{pmatrix} * & * \\ c_k & d_k \end{pmatrix},$$

which shows that $\mathrm{sign}(d_k) = (-1)^{k+1}\mathrm{sign}(c_{k-1}) = (-1)^{k+1}\mathrm{sign}(a_1 \cdots a_{k-1})$. Also, observing that $c_k = \mathrm{sign}(a_k)\mathrm{sign}(c_{k-1})(|a_k c_{k-1}| \pm d_{k-1})$ gives that $\mathrm{sign}(c_k) = \mathrm{sign}(a_1 \cdots a_k)$, since

$$|a_k c_{k-1}| \pm d_{k-1} \geq 2|c_{k-1}| - |d_{k-1}| \geq |c_{k-1}| = |d_k| > 0.$$

□

Suppose $c > 0$ is an integer and that $-c < d < c$ an odd integer coprime to c. It thus has a c.f.e. of $\Gamma^0(2)$-type $\frac{d}{c} = [0; a_1, 2a_2, ..., 2a_{n-1}, a_n]$ as in Proposition 10.9, i.e., where $a_1, a_2, ..., a_n$ are non-zero integers such that $|a_1|, |a_3|, ..., |a_{n-2}| > 1$. Note that $-\frac{d}{c} = [0; -a_1, -2a_2, ..., -2a_{n-1}, -a_n]$ and hence

$$\begin{pmatrix} d & * \\ -c & * \end{pmatrix} = \pm V^{-a_1} T^{-2a_2} \cdots T^{-2a_{n-1}} V^{-a_n}$$

is a matrix in $\Gamma^0(2)$. Its inverse $\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \pm V^{a_n} T^{2a_{n-1}} \cdots T^{2a_2} V^{a_1}$ then satisfies the assumptions on the partial quotients $a_0, a_1, ..., a_n$ in Proposition 10.12.

PROPOSITION 10.13. (1) *Let n be an odd integer and let*

$$A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} = T^{2a_0} V^{a_1} T^{2a_2} \cdots V^{a_n} \in \Gamma^0(2)$$

with $a_1, ..., a_n$ *being non-zero integers and* $|a_k| > 1$ *for* $k > 1$ *an odd integer. The cocycle* $\displaystyle\int_z^{A.z} (E_2(w) - L(w))dw$ *is equal to*

$$\frac{6}{\pi i} \log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) - 3(a_1 + a_3 + ... + a_n) - 3\sum_{k=1}^{n} (-1)^k \mathrm{sign}(a_k).$$

(2) *Let* n *be an integer and* $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} = T^{2c_0} S T^{2c_1} S \cdots T^{2c_n} S \in \Gamma_\theta$ *with* $c_1, ..., c_n$ *being non-zero integers. The cocycle is given by*

$$\int_z^{A.z} (E_2(w) - R(w))dw = \frac{6}{\pi i} \log\left(\frac{cz+d}{\mathrm{sign}(c)i}\right) - 3\sum_{k=1}^{n} \mathrm{sign}(c_k).$$

Proof. Both cycle integrals

$$\phi(A) = \int_z^{A.z} L(w)\, dw \;\text{ for } A \in \Gamma^0(2)$$

and

$$\psi(B) = \int_z^{B.z} R(w)\, dw \;\text{ for } B \in \Gamma_\theta$$

are constant in z and group homomorphisms $\Gamma^0(2) \to \mathbf{C}$ resp. $\Gamma_\theta \to \mathbf{C}$.[1] We see from the Fourier expansion of $L(z)$ in (10.2) that $\phi(T^2) = 2$. The value for $\phi(V)$ can either be seen from $T^2 V^{-1} T^2 = -V$, so

$$\phi(V) = \phi(-V) = \phi(T^2 V^{-1} T^2) = 2\phi(T^2) - \phi(V),$$

or by specialising to $z = -1 + i$ in the cycle integral, so that $V.(-1+i) = 1+i$. Similarly, it is easy to see that $\psi(T^2) = 2$ and $\psi(S) = 0$, either by using $\psi(S^2) = 0$ or by specialising to $z = i$ in the cycle integral.

For the first part, recall that we may write

$$T^{2a_k} V^{a_{k+1}} = T^{2a_k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} V^{a_{k+1}} = \begin{pmatrix} 2a_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k+1} & 1 \\ -1 & 0 \end{pmatrix}$$

for $k = 0, ..., n-1$. The claim then follows by Proposition 10.12 and Lemma 6.8.

For the second part, write $\pm A = T^{2c_0} S T^{2c_1} S^{-1} \cdots T^{2c_n} S^{(-1)^n}$. We have $T^{2c_k} S^{(-1)^k} = \begin{pmatrix} (-1)^k 2c_k & (-1)^{k+1} \\ (-1)^k & 0 \end{pmatrix}$, i.e.

$$\pm A = \begin{pmatrix} 2c_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2c_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} (-1)^n 2c_n & (-1)^{n+1} \\ (-1)^n & 0 \end{pmatrix}.$$

[1] In fact, they constitute the unique group homomorphisms $\Gamma^0(2) \to \mathbf{C}$ resp. $\Gamma_\theta \to \mathbf{C}$, up to scalar multiplication.

Since $\displaystyle\int_z^{A.z}(E_2(w)-R(w))dw = \int_z^{(-A).z}(E_2(w)-R(w))dw$, we may again use Proposition 10.12 and Lemma 6.8 to prove the claim. □

With Proposition 10.13 we wrote the cocycle of the logarithm of the θ-functions in terms of the matrix representation as words in $T^2,\ S$ resp. $T^2,\ V$, i.e. in c.f.e.s of Γ_θ- resp. $\Gamma^0(2)$-type. Theorem 10.1 immediately follows from the next proposition, which collects our previous results to give the representation of the Hardy sums in terms of the partial quotients of c.f.e. of Γ_θ- resp. $\Gamma^0(2)$-type.

PROPOSITION 10.14. (1) *Let d, c be coprime integers of opposite parity with $c > 0$ and assume that $-c < d < c$. Pick $a, b \in \mathbf{Z}$ such that $ad - bc = 1$. Pick $c_0 \in \mathbf{Z}$ and non-zero integers $c_1, ..., c_n$ with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \pm T^{2c_0}ST^{2c_1}S\cdots T^{2c_n}S$. The Hardy sum $S(d,c)$ can be evaluated as*

$$S(d,c) = -\sum_{k=1}^{n}\operatorname{sign}(c_k).$$

(2) *Let $c > 0$ and d be an odd integer coprime to c such that $-c < d < c$. Pick $a, b \in \mathbf{Z}$ such that $ad - 2bc = 1$. Pick $a_0 \in \mathbf{Z}$ and non-zero integers $a_1, ..., a_n$ such that $|a_k| > 1$ for $k > 1$ an odd integer and $\left(\begin{smallmatrix} a & 2b \\ c & d\end{smallmatrix}\right) = T^{2a_0}V^{a_1}\cdots T^{2a_{n-1}}V^{a_n}$. The Hardy sum $S_4(d,c)$ can be evaluated as*

$$S_4(d,c) = a_1 + a_3 + ... + a_n + \sum_{k=1}^{n}(-1)^k\operatorname{sign}(a_k).$$

PROOF. To prove the first part, note that (9.6), Proposition 10.11, and Proposition 10.13 imply that

$$\begin{aligned}\frac{1}{2}\log\left(\frac{cz+d}{i}\right) + \frac{\pi i}{4}S(d,c) &= \frac{\pi i}{12}\int_z^{A.z}(E_2(w)-R(w))dw \\ &= \frac{1}{2}\log\left(\frac{cz+d}{i}\right) - \frac{\pi i}{4}\sum_{k=1}^{n}\operatorname{sign}(c_k).\end{aligned}$$

To prove the second part, use (9.7), Proposition 10.11, and Proposition 10.13 to obtain

$$\begin{aligned}
&\frac{1}{2}\log\left(\frac{cz+d}{i}\right) - \frac{\pi i}{4}S_4(d,c)\\
&= \frac{\pi i}{12}\int_z^{A.z}(E_2(w) - L(w))dw\\
&= \frac{1}{2}\log\left(\frac{cz+d}{i}\right) - \frac{\pi i}{4}(a_1 + a_3 + ... + a_n) - \frac{\pi i}{4}\sum_{k=1}^{n}(-1)^k \mathrm{sign}(a_k).
\end{aligned}$$

□

A reciprocity-type formula for $S_4(d,c)$ can be given by using Theorem 10.1, namely

$$S_4(d,c) + S_4(c,d) = 2a_0 + 2a_1 + ... + 2a_{n-1} + a_n - 1, \tag{10.4}$$

where $c > d > 0$ are odd, coprime integers and $\frac{c}{d} = [2a_0; 2a_1, ..., 2a_{n-1}, a_n]$ with $n \geq 1$ being odd (there is always a unique continued fraction expansion of that kind with the given conditions on c, d).

3. Meyer's Density Result

In this section, we prove Meyer's Theorem 9.1 that the sets $\{(d/c,\ S(d,c)) : c + d \text{ odd}, c > 0,\ (d,c) = 1\}$ and $\{(d/c,\ S_4(d,c)) : c > 0,\ d \text{ odd},\ (d,c) = 1\}$ are dense in $\mathbf{R} \times \mathbf{Z}$, as well as the new density Theorem 10.3.

The arguments to prove the claim are quite short, but differ between $S(d,c)$ and $S_4(d,c)$. We will present them in separate subsections.

3.1. Proof of the Density of the Hardy Sums $S(d,c)$. Let $(x,m) \in \mathbf{R} \times \mathbf{Z}$ and $\varepsilon > 0$. We need to show that there are coprime integers d, c with $c > 0$ and $d + c$ odd such that

$$\left|x - \frac{d}{c}\right| < \varepsilon \text{ and } S(d,c) = m.$$

Since $S(d + 2c, c) = S(d,c)$, we may assume w.l.o.g. that $x \in (-1,1)$.

By Proposition 10.7, we may find non-zero integers $c_1, c_2, ..., c_n$ such that $\frac{p_n}{q_n} = [\![2c_1, ..., 2c_n]\!]$ lies in $(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$. Pick $M \in \mathbf{Z}$ such that $M + m = S(p_n, q_n)$. If $M = 0$, there is nothing more to show. Thus assume that $M \neq 0$.

Now construct the following rational in $(x - \varepsilon, x + \varepsilon)$: Pick $c_{n+1} \geq 1$ big enough such that

$$\left|[\![2c_1, ..., 2c_n, \mathrm{sign}(M)2c_{n+1}]\!] - \frac{p_n}{q_n}\right| \leq \frac{1}{|q_n|(2|c_{n+1}||q_n| - |q_{n-1}|)} < \frac{\varepsilon}{2|M|},$$

where we made use of Lemma 10.6. Denote $[\![2c_1, ..., 2c_n, \mathrm{sign}(M)2c_{n+1},]\!]$ by $\frac{p_{n+1}}{q_{n+1}}$. Then pick $c_{n+2} \geq 1$ such that

$$\left|[\![2c_1, ..., 2c_n, \mathrm{sign}(M)2c_{n+1}, \mathrm{sign}(M)2c_{n+2}]\!] - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{\varepsilon}{2|M|}, \text{ etc.}$$

Continuing this process $|M|$-times, we get a rational

$$\frac{p_{n+|M|}}{q_{n+|M|}} = [\![2c_1, ..., 2c_n, 2c'_{n+1}, ..., 2c'_{n+|M|}]\!],$$

where $c'_k = \mathrm{sign}(M)c_k$ for $k = n+1, ..., n+|M|$. In particular, $\mathrm{sign}(c'_k) = \mathrm{sign}(M)$.

The rational $\frac{p_{n+|M|}}{q_{n+|M|}}$ lies in $(x-\varepsilon, x+\varepsilon)$, since

$$\left|x - \frac{p_{n+|M|}}{q_{n+|M|}}\right| \leq \left|x - \frac{p_n}{q_n}\right| + \sum_{k=1}^{|M|}\left|\frac{p_{n+k-1}}{q_{n+k-1}} - \frac{p_{n+k}}{q_{n+k}}\right| < \varepsilon$$

by construction. Moreover, it follows from Theorem 10.1 that

$$\begin{aligned} S(p_{n+|M|}, q_{n+|M|}) &= -\sum_{k=1}^{n}\mathrm{sign}(c_k) - \sum_{l=1}^{|M|}\mathrm{sign}(c'_{n+l}) \\ &= m + M - \mathrm{sign}(M)|M| \\ &= m. \end{aligned}$$

3.2. Proof of the Density of the Hardy Sums $S_4(d,c)$ and of $S_4(d,c) + S(d,c)$. To prove Theorem 10.3, we will prove a stronger result than $\{(d/c, S(d,c) + S_4(d,c)) : d \text{ odd}, \ c > 0 \text{ even}\}$ being dense in $\mathbf{R} \times 2\mathbf{Z}$, namely that

$$\{(d/c,\ S(d,c) + S_4(d,c),\ S_4(d,c)) : d \text{ odd},\ c > 0 \text{ even}\} \subset \mathbf{R} \times 2\mathbf{Z} \times (2\mathbf{Z}+1)$$

is a dense subset.

For $S(d,c) + S_4(d,c)$ to be well-defined, we need $\frac{d}{c}$ with $c > 0$ and $(d,c) = 1$ to be such that $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\theta \cap \Gamma^0(2)$. In other words,

$$\frac{d}{c} = [\![2c_0; 2c_1, ..., 2c_n]\!] = [2c_0; -2c_1, 2c_2, -2c_3, ..., -2c_n]$$

should be such that d is odd and $c + d$ is odd, i.e. c is even. By Lemma 10.8, this can only happen if n is odd.

In this case, the Hardy sums are

$$S(d,c) = -\sum_{k=1}^{n} \operatorname{sign}(c_k) \text{ and}$$

$$S_4(d,c) = -2(c_1 + c_3 + ... + c_n) + \sum_{k=1}^{n} \operatorname{sign}(c_k) = -2(c_1 + c_3 + ... + c_n) - S(d,c)$$

by Theorem 10.1. It is clear that $S(d,c) + S_4(d,c) \equiv 0 \pmod 2$. With the next lemma, we can also deduce that $S_4(d,c)$ will have to be odd in this case.

LEMMA 10.15. *Let $c_1, ..., c_n$ be non-zero integers and $\frac{d}{c} = [\![2c_1, ..., 2c_n]\!]$ for $(d,c) = 1$ and $c > 0$. Then*

$$S(d,c) \equiv n \pmod 2.$$

PROOF. Immediately from Theorem 10.1. $\square$

PROPOSITION 10.16. *The set*

$$\{(d/c,\ S(d,c) + S_4(d,c),\ S_4(d,c)) : d \text{ odd},\ c > 0 \text{ even},\ (d,c) = 1\}$$

is dense in $\mathbf{R} \times 2\mathbf{Z} \times (2\mathbf{Z} + 1)$.

PROOF. Let $(x, m_1, m_2) \in \mathbf{R} \times 2\mathbf{Z} \times (2\mathbf{Z} + 1)$ and $\varepsilon > 0$. Again, we need to show that there are coprime integers d, c with $c > 0$ even and d odd such that

$$\left| x - \frac{d}{c} \right| < \varepsilon, \quad S(d,c) + S_4(d,c) = m_1 \text{ and } S_4(d,c) = m_2.$$

As before, we may assume w.l.o.g. that $x \in (-1, 1)$.

Let $\frac{p_n}{q_n} = [\![2c_1, ..., 2c_n]\!]$ be such that $\left| x - \frac{p_n}{q_n} \right| < \frac{\varepsilon}{3}$ and $S(p_n, q_n) = m_1 - m_2$, which is possible by what we proved in Section 3.1 of this chapter. Since $m_1 - m_2$ is odd, Lemma 10.15 tells us that n must be odd as well.

Let $M \in \mathbf{Z}$ be such that $S(p_n, q_n) + S_4(p_n, q_n) = m_1 + 2M$. If $M = 0$, there is nothing to show, since

$$S(p_n, q_n) + S_4(p_n, q_n) = m_1 = m_1 - m_2 + S_4(p_n, q_n).$$

Thus, suppose that $M \neq 0$.

Pick $c_{n+1} > 0$ big enough such that

$$\left| \frac{p_n}{q_n} - [\![2c_1, ..., 2c_n, -\operatorname{sign}(M) 2c_{n+1}]\!] \right| < \frac{1}{2|c_{n+1}||q_n| - |q_{n-1}|} < \frac{\varepsilon}{3}$$

and let $\frac{p_{n+1}}{q_{n+1}} = [\![2c_1, ..., 2c_n, -\mathrm{sign}(M)2c_{n+1}]\!]$. Setting

$$\frac{p_{n+2}}{q_{n+2}} = [\![2c_1, ..., 2c_n, -\mathrm{sign}(M)2c_{n+1}, 2M]\!],$$

we see by Lemma 10.5 that

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}}\right| < \frac{1}{|q_{n+2}|} < \frac{1}{2c_{n+1}|q_{n+1}| - |q_n|} < \frac{\varepsilon}{3}.$$

But hence $|x - \frac{p_{n+2}}{q_{n+2}}| < \varepsilon$ and

$$S(p_{n+2}, q_{n+2}) + S_4(p_{n+2}, q_{n+2}) = m_1 + 2M - 2M = m_1$$

by Corollary 10.2.

Finally, note that $S(p_n, q_n) = S(p_{n+2}, q_{n+2})$, giving that $S_4(p_{n+2}, q_{n+2}) = m_2$. $\square$

To finish the proof that $\{(d/c,\ S_4(d,c)) : d \text{ odd}, c > 0,\ (d,c) = 1\}$ is dense in $\mathbf{R}\times\mathbf{Z}$, we turn to the possibility of $c > 0$ being odd. A continued fraction expansion of $\Gamma^0(2)$-type of the form

$$\frac{d}{c} = [2a_1, 2a_2, ..., 2a_{n-1}, a_n] = [\![-2a_1, 2a_2, ..., 2a_{n-1}, -a_n]\!]$$

is of Γ_θ-type if and only if a_n is even. Hence, assuming that a_n is odd, we see that $c > 0$ ought to be odd. This distinction will be implicit in the proof of the next proposition.

PROPOSITION 10.17. *The set* $\{(d/c,\ S_4(d,c)) : d \text{ odd},\ c > 0,\ (d,c) = 1\}$ *is dense in* $\mathbf{R} \times \mathbf{Z}$.

PROOF. Let $(x, m) \in (-1, 1) \times \mathbf{Z}$ and $\varepsilon > 0$. We need to find coprime integers d, c with $c > 0$ and d odd such that

$$\left|\frac{d}{c} - x\right| < \varepsilon \ \text{ and } \ S_4(d,c) = m. \tag{10.5}$$

If m is odd, there is a $\frac{d}{c}$ satisfying the condition (10.5) by Proposition 10.16. Thus, suppose that m is even.

Again by Proposition 10.16, pick $\frac{p_n}{q_n} = [\![2c_1, ..., 2c_n]\!]$ with n odd such that $S_4(p_n, q_n) = m + 1$ and $|x - \frac{p_n}{q_n}| < \frac{\varepsilon}{3}$.

Choose $c_{n+1} > 0$ big enough such that

$$\left|\frac{p_n}{q_n} - [\![2c_1, ..., 2c_n, 2c_{n+1}]\!]\right| < \frac{1}{2c_{n+1}|q_n| - |q_{n-1}|} < \frac{\varepsilon}{3}$$

and let $\frac{p_{n+1}}{q_{n+1}} = [\![2c_1, ..., 2c_n, 2c_{n+1}]\!]$.

Set

$$\frac{p_{n+2}}{q_{n+2}} = [\![2c_1, ..., 2c_n, 2c_{n+1}, -1]\!] = [-2c_1, 2c_2, ..., -2c_n, 2c_{n+1}, 1].$$

We then have $|x - \frac{p_{n+2}}{q_{n+2}}| < \varepsilon$, since

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}}\right| < \frac{1}{|q_{n+1}|} < \frac{1}{2c_{n+1}|q_n| - |q_{n-1}|} < \frac{\varepsilon}{3}$$

by Lemma 10.5. Also,

$$S_4(p_{n+2}, q_{n+2}) = m + 1 - 1 + 1 - 1 = m$$

by Theorem 10.1. □

4. Uniform Distribution modulo m

4.1. Equidistribution on Compact Abelian Groups. The theory of equidistribution generalizes to the more general setting of compact abelian groups. Let us first consider the case of equidistribution modulo 1. Given a sequence $(a_n)_{n\geq 1}$ in $\mathbf{R}/\mathbf{Z}$, we say that a sequence is equidistributed modulo 1 if the sequence of point measures $\frac{1}{N}\sum_{n=1}^{N}\delta_{a_n}$ converges weakly to the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$. Equivalently, we can use the Weyl criterion for equidistribution modulo 1, which says that a sequence $(a_n)_{n\geq 1}$ is equidistributed modulo 1 if and only if

$$\frac{1}{N}\sum_{n=1}^{N} e(ma_n) \to 0, \text{ as } N \to +\infty \tag{10.6}$$

for all $m \neq 0$. The following definition is the natural extension of the concept of equidistribution to compact abelian groups other than $\mathbf{R}/\mathbf{Z}$.

Definition 10.18. Let G be a compact abelian group. A sequence $(g_n)_{n\geq 1}$ is called equidistributed in G if the measure $\frac{1}{N}\sum_{n=1}^{N}\delta_{g_n}$ converges weakly to the Haar measure μ in G as $N \to +\infty$, i.e., if for every continuous function $f : G \to \mathbf{C}$ we have

$$\frac{1}{N}\sum_{n=1}^{N} f(g_n) \to \int_G f(g)d\mu(g), \text{ as } N \to +\infty.$$

Equidistribution modulo 1 is a special case of Definition 10.18 for the compact abelian group $\mathbf{R}/\mathbf{Z}$ (the Haar measure on $\mathbf{R}/\mathbf{Z}$ is the Lebesgue measure). In our discussion of the integer-valued Hardy sums, we will then no longer consider $\mathbf{R}/\mathbf{Z}$, but the compact abelian group $\mathbf{Z}/m\mathbf{Z}$ for an integer $m > 1$ instead. The Haar measure on $\mathbf{Z}/m\mathbf{Z}$ is the counting measure. The extension of the Weyl criterion (10.6) to the more general Definition 10.18 was proved by B. Eckmann [**Eck43**].

THEOREM 10.19 (Eckmann). *Let $(g_n)_{n\geq 1}$ be a sequence in a compact abelian group G. The sequence $(g_n)_{n\geq 1}$ is equidistributed in G if and only if $\frac{1}{N}\sum_{n=1}^{N}\chi(g_n) \to 0$ as $N \to +\infty$ for all non-trivial characters χ on G.*

The non-trivial characters on $\mathbf{R}/\mathbf{Z}$ are given by $\{e(nx) : n \neq 0\}$, so Theorem 10.19 entails (10.6). In the case $\mathbf{Z}/m\mathbf{Z}$ for $m > 1$, the non-trivial characters are given by $n \mapsto e\left(\frac{j}{m}n\right)$ for $j = 1, ..., m-1$; see [**Niv61, Uch61**].

To show equidistribution of the Hardy sums $S(d,c)$ on $\mathbf{Z}/m\mathbf{Z}$, we must thus show that for all $j = 1, ..., m-1$ we have

$$\frac{1}{\#\Phi_\theta(N)}\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} e\left(\frac{j}{m}S(d,c)\right) \to 0, \text{ as } N \to +\infty, \tag{10.7}$$

where

$$\Phi_\theta(N) = \{d/c : 1 \leq d < c \leq N,\ (d,c) = 1,\ c+d \text{ odd}\}.$$

The first statement of Theorem 10.4 is equivalent to showing that for all $n \in \mathbf{Z}$ and every rational number $0 < r < 1$:

$$\frac{1}{\#\Phi_\theta(N)}\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} e\left(-n\frac{d}{c} + rS(d,c)\right) \to 0, \text{ as } N \to +\infty. \tag{10.8}$$

The second part of Theorem 10.4 is equivalent to the corresponding statement for $S_4(d,c)$. As the methods are virtually the same for $S(d,c)$ and $S_4(d,c)$, we will only focus on the Hardy sum $S(d,c)$ to prove Theorem 10.4. The Dirichlet series

$$Z_r(n,s) = e^{-2\pi i r}\sum_{c=1}^{\infty}\frac{1}{c^{2s}}\sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} e\left(-n\frac{d}{c} + rS(d,c)\right)$$

of the Weyl sums in (10.8) appear in the Fourier expansion of certain Eisenstein series with multiplier systems. The spectral theory of Eisenstein series gives the analytic continuation of these Dirichlet series and information about their poles. Analytically, this encodes information about the growth rate of the Weyl sums (10.8).

4.2. Estimating $\Phi_\theta(N)$. Naturally, to prove equidistribution we need to understand the growth rate of the function

$$\#\Phi_\theta(N) = \#\{d/c : 1 \leq d < c \leq N,\ (d,c) = 1,\ c+d \text{ odd}\}. \tag{10.9}$$

We may also write the function $\#\Phi_\theta(N)$ as a sum of special Euler-totient functions

$$\varphi_\theta(c) = \sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} 1 = \begin{cases} \varphi(n), & \text{if } c \text{ even},\\ \frac{1}{2}\varphi(n), & \text{if } c \text{ odd}. \end{cases}$$

Note that $\varphi_\theta(c) = \sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} e\left(-nd/c\right)$ for $n = 0$. In this notation, $\#\Phi_\theta(N) = \sum_{c=1}^N \varphi_\theta(c)$.

Lemma 10.20. *We have $\Phi_\theta(N) \asymp N^2$ as $N \to +\infty$.*

Proof. First, we notice that

$$\Phi_\theta(N) = \sum_{c=1}^{N} \varphi_\theta(c) = \sum_{c=1}^{\lfloor N/2 \rfloor} \varphi(2c) + \frac{1}{2} \sum_{c=1}^{\lfloor N/2 \rfloor} \varphi(2c-1). \tag{10.10}$$

Let us denote the first sum on the right hand side of (10.10) by $\Lambda_1(N)$ and the second sum by $\Lambda_2(N)$.

We get for $\Lambda_1(N)$:

$$\begin{aligned}
\Lambda_1(N) &= \sum_{k=1}^{\lfloor N/2 \rfloor} \varphi(2k)\\
&= \sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{dd'=2k} d'\mu(d)\\
&= \sum_{\substack{d=1,\\ d \text{ even}}}^{N} \mu(d) \sum_{d'=1}^{\lfloor N/d \rfloor} d' + \sum_{\substack{d=1,\\ d \text{ odd}}}^{\lfloor N/2 \rfloor} \mu(d) \sum_{\substack{d'=1,\\ d' \text{ even}}}^{\lfloor N/d \rfloor} d'\\
&= -\sum_{\substack{d=1,\\ d \text{ odd}}}^{\lfloor N/2 \rfloor} \mu(d) \sum_{d'=1}^{\lfloor N/2d \rfloor} d' + 2\sum_{\substack{d=1,\\ d \text{ odd}}}^{\lfloor N/2 \rfloor} \mu(d) \sum_{d'=1}^{\lfloor N/2d \rfloor} d'\\
&= \sum_{\substack{d=1,\\ d \text{ odd}}}^{\lfloor N/2 \rfloor} \mu(d) \sum_{d'=1}^{\lfloor N/2d \rfloor} d'\\
&= \sum_{\substack{d=1,\\ d \text{ odd}}}^{\lfloor N/2 \rfloor} \mu(d) \frac{1}{2} \left\lfloor \frac{N}{2d} \right\rfloor \left(\left\lfloor \frac{N}{2d} \right\rfloor + 1 \right)\\
&= \frac{N^2}{8} \sum_{\substack{d=1,\\ d \text{ odd}}}^{\infty} \frac{\mu(d)}{d^2} + o(N^2),
\end{aligned}$$

where μ denotes the Möbius-function.

Similarly, one can show that $\Lambda_2(N) = \frac{N^2}{8}\sum_{\substack{d=1,\\ d \text{ odd}}}^{\infty} \frac{\mu(d)}{d^2} + o(N^2)$. $\square$

The proof of Lemma 10.20 also shows that $\Lambda_1(N)$ and $\Lambda_2(N)$ satisfy

$$\Lambda_1(N) - \Lambda_2(N) = o(N^2), \quad \text{as } N \to +\infty, \tag{10.11}$$

since their asymptotic leading terms cancel.

By Lemma 10.20, it thus suffices to show that the Weyl sums in (10.8) are $o(N^2)$ as $N \to +\infty$ to prove Theorem 10.4.

4.3. The case $m = 2$**.** We start by reviewing the proof of the equidistribution of the sequence $\{d/c : 1 \le d < c \le N,\ (d,c) = 1\}$ in $\mathbf{R}/\mathbf{Z}$ as $N \to +\infty$. By the Weyl criterion in (10.6), we need to find non-trivial cancellation in the Ramanujan sums

$$R_c(n) = \sum_{d(c)^*} e\left(n\frac{d}{c}\right) = \sum_{d(c)^*} \cos\left(n\frac{d}{c}\right) \quad \text{for } n \neq 0.$$

We have

$$R_c(n) \le n \text{ for all } n \neq 0, \tag{10.12}$$

which can easily be seen from the von Sterneck formula $R_c(n) = \mu\left(\frac{c}{(c,n)}\right)\frac{\varphi(c)}{\varphi\left(\frac{c}{(c,n)}\right)}$. Hence,

$$\left|\frac{1}{N^2}\sum_{c=1}^{N}\sum_{d(c)^*} e\left(n\frac{d}{c}\right)\right| \le \frac{1}{N^2}Nn \to 0, \text{ as } N \to +\infty \tag{10.13}$$

by (10.12) and thus $\{d/c : 1 \le d < c \le N,\ (d,c) = 1\}$ is equidistributed on $\mathbf{R}/\mathbf{Z}$ as $N \to +\infty$.

Let us now consider the sequence $\{d/c : 1 \le d < c \le N,\ (d,c) = 1,\ c+d \text{ odd}\}$ instead. This sequence is also equidistributed in $\mathbf{R}/\mathbf{Z}$ as $N \to +\infty$. Consider the Weyl sums

$$\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\ c+d \text{ odd}}} e\left(n\frac{d}{c}\right) = \sum_{\substack{c\le N,\\ c \text{ even}}}\sum_{d(c)^*} \cos\left(2\pi n\frac{d}{c}\right) + \sum_{\substack{c\le N,\\ c \text{ odd}}}\sum_{\substack{d(c)^*,\\ d \text{ even}}} \cos\left(2\pi n\frac{d}{c}\right) \tag{10.14}$$

for $n \neq 0$. The first sum on the right hand side of (10.14) is seen to be a sum of Ramanujan sums $R_c(n)$, which is $o(N^2)$ by (10.13). For the second sum, we need to bound the sum

$$\sum_{\substack{d(c)^*,\\ d \text{ even}}} \cos\left(2\pi n\frac{d}{c}\right) = \sum_{\substack{0<d\le\left\lfloor\frac{c}{2}\right\rfloor,\\ (d,c)=1}} \cos\left(4\pi n\frac{d}{c}\right).$$

Since

$$\begin{aligned}R_c(2n) &= \sum_{\substack{0<d\leq\lfloor\frac{c}{2}\rfloor,\\(d,c)=1}} \cos\left(4\pi n\frac{d}{c}\right) + \sum_{\substack{\lfloor c/2\rfloor<d<c,\\(d,c)=1}} \cos\left(4\pi n\frac{d}{c}\right)\\ &= 2\sum_{\substack{0<d\leq\lfloor\frac{c}{2}\rfloor,\\(d,c)=1}} \cos\left(4\pi n\frac{d}{c}\right)\end{aligned}$$

after substituting $d \to -d + c$ in the second sum, we have

$$\sum_{\substack{d(c)^*,\\d\text{ even}}} \cos\left(2\pi n\frac{d}{c}\right) = \frac{R_c(2n)}{2} \ll n.$$

This proves that the Weyl sum $\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\c+d\text{ odd}}} e\left(n\frac{d}{c}\right) = o(N^2)$ as well and thus proves equidistribution of $\{d/c : 1 \leq d < c \leq N,\ (d,c)=1,\ c+d\text{ odd}\}$ on $\mathbf{R}/\mathbf{Z}$ as $N \to +\infty$.

We now give an elementary proof of the equidistribution of $(d/c, S(d,c) \pmod 2)$ on $\mathbf{R}/\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. To this end, we need to bound the Weyl sum

$$\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\c+d\text{ odd}}} e\left(nd/c\right) e^{\pi i S(d,c)}.$$

By Lemma 10.15 the Hardy sum $S(d,c)$ is odd when c is even and $S(d,c)$ is even when c is odd.[2] Thus, we have $e^{\pi i S(d,c)} = (-1)^{c+1}$ and so

$$\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\c+d\text{ odd}}} e\left(nd/c\right) e^{\pi i S(d,c)} = \sum_{\substack{c\leq N,\\c\text{ odd}}}\sum_{\substack{d(c)^*,\\d\text{ even}}} e\left(n\frac{d}{c}\right) - \sum_{\substack{c\leq N,\\c\text{ even}}}\sum_{d(c)^*} e\left(n\frac{d}{c}\right). \tag{10.15}$$

That (10.15) is $o(N^2)$ follows immediately from the bounds we used to bound (10.14) in the case $n \neq 0$.

For $n = 0$, we see that

$$\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\c+d\text{ odd}}} e^{\pi i S(d,c)} = \sum_{c=1}^{N}(-1)^{c+1}\varphi_\theta(c) = \Lambda_2(N) - \Lambda_1(N)$$

in the notation of the proof of Lemma 10.20. The Weyl sum is thus $o(N^2)$ by (10.11) and this proves the first part of Theorem 10.4 for $m = 2$.

[2]Similarly, for the value of $S_4(d,c)$ for $c > 0$, $(d,c) = 1$, and d odd is an odd number if c is even and is an even number if c is odd; see Section 3.2 of this chapter.

Albeit it is probably possible to give an elementary proof of Theorem 10.4 for $m > 2$ as well, we can use the spectral theory of automorphic forms instead. The information of the growth rate of the Weyl sums is encoded in the analytic behavior of certain Eisenstein series. The spectral method could also be used to show the case $m = 2$; but since Eisenstein series of weight 2 are more delicate, we chose to give an elementary proof in that case.

5. Spectral Theory of Eisenstein Series with Multiplier Systems

Let $0 < r < 1$ denote a rational number. Our main tool to prove the uniform distribution of Hardy sums will be the theory of Eisenstein series with a multiplier system. By a multiplier system of weight r on a subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$ we mean a function $\nu : \Gamma \to \mathbf{C}$ such that the cocycle $j(g,z) = cz + d$ with $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ satisfies $\nu(gh)j(gh,z)^r = \nu(g)\nu(h)j(g,h.z)^r j(h,z)^r$ for all $g, h \in \Gamma$. For example, the function $\theta(z)^{8r}$ transforms for $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\theta$ like $\theta(g.z)^{8r} = \nu_r(g)(cz+d)^{4r}\theta(z)^{8r}$, where

$$\nu_r(g) = \begin{cases} e^{2\pi i r(S(d,c) - \mathrm{sign}(c))}, & c \neq 0, \\ e^{2\pi i r(\mathrm{sign}(d)-1)}, & c = 0. \end{cases} \tag{10.16}$$

It is not difficult to see that $\nu_{1/2}$ is a multiplicative homomorphism.

For the group Γ_θ, let $\Gamma_\infty < \Gamma_\theta$ denote the subgroup of its parabolic elements, i.e. the subgroup generated by $\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The multiplier system ν_r is singular with respect to the cusp $i\infty$, i.e. $\nu_r\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = 1$ (see also Propositon 10.11). We may hence view ν_r as a function on the cosets $\Gamma_\infty \setminus \Gamma$.

For a singular multiplier system ν, consider the following Dirichlet series:

$$Z_\nu(n,s) = \sum_{c>0} \frac{1}{c^{2s}} \sum_{\substack{d(c)^*, \\ c+d \text{ odd}}} e\left(-n\frac{d}{c}\right) \nu\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}\right), \tag{10.17}$$

which is clearly absolutely convergent for $\mathrm{Re}(s) > 1$. For our bounds on the Weyl sums (10.8), we will be interested in the special case

$$Z_r(n,s) = Z_{\nu_r}(n,s) = e^{-2\pi i r} \sum_{c=1}^{\infty} \frac{1}{c^{2s}} \sum_{\substack{d(c)^*, \\ c+d \text{ odd}}} e\left(-n\frac{d}{c} + rS(d,c)\right).$$

For a multiplier system ν of weight $4r$, which is singular with respect to $i\infty$, define

$$E_{4r}(z,s;\nu) = \sum_{g\in\Gamma_\infty\backslash\Gamma_\theta} \overline{\nu(g)} j(g,z)^{-4r}\mathrm{Im}(g.z)^{s-2r}, \ \mathrm{Re}(s)>1, \tag{10.18}$$

which is the non-holomorphic Eisenstein series of weight $4r$ and multiplier system ν. The series is absolutely convergent for $\mathrm{Re}(s)>1$ and uniformly convergent in z on compact subsets of $\mathbf{H}$.

PROPOSITION 10.21. *The Fourier expansion of the non-holomorphic Eisenstein series is given by*

$$E_{4r}(z,s;\nu) = y^{s-2r} + \varphi_{4r,\nu}(s)y^{1-s-2r} + \sum_{n\neq 0} y^{-2r}\varphi_{4r,\nu}(n,s)W_{\mathrm{sign}(n)2r,s-1/2}(2\pi|n|y)e^{\pi inx},$$

where

$$\varphi_{4r,\nu}(s) = \pi\frac{2^{2-2s}\Gamma(2s-1)e^{2\pi ir}}{\Gamma(s-2r)\Gamma(s+2r)}Z_\nu(0,s)$$

and

$$\varphi_{4r,\nu}(n,s) = \frac{\pi^s|n|^{s-1}e^{2\pi ir}}{\Gamma(s+\mathrm{sign}(n)2r)}Z_\nu(n,s).$$

PROOF. See, for instance, J. Fay's paper [**Fay77**, Thm. 3.4], where the considered Eisenstein series is $y^{-4r}E_{-4r}(z,s;\nu)$ in our notation. $\square$

Proposition 10.21 is the special case of the Fourier expansion around the cusp $\mathfrak{a} = i\infty$. More generally, for any cusp $\mathfrak{a}$ of Γ_θ, let $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbf{Z})$ be its scaling matrix such that $\sigma_\mathfrak{a}.i\infty = \mathfrak{a}$. The Eisenstein series $E_{4r}(z,s;\nu)$ has a Fourier expansion around $\mathfrak{a}$ given by

$$\begin{aligned} j(\sigma_\mathfrak{a},z)^{4r}E_{4r}(\sigma_\mathfrak{a}^{-1}.z,s;\nu) &= \delta_{\mathfrak{a},\infty}y^{s-2r} + \varphi_{4r,\mathfrak{a},\nu}(s)y^{1-s-2r} \\ &\quad + \sum_{n\neq 0} y^{-2r}\varphi_{4r,\mathfrak{a},\nu}(n,s)W_{\mathrm{sign}(n)2r,s-\frac{1}{2}}(4\pi|n|y)e^{\pi inx} \end{aligned} \tag{10.19}$$

for certain functions $\varphi_{4r,\mathfrak{a},\nu}(s)$ and $\varphi_{4r,\mathfrak{a},\nu}(n,s)$.

It is well-known that the Eisenstein series $E_{4r}(z,s;\nu)$ has an analytic continuation in s to the whole complex plane. From the Fourier expansion in Proposition 10.21, the poles of $E_{4r}(z,s;\nu)$ correspond to poles of the analytically continued Dirichlet series $Z_\nu(n,s)$ or – e.g. in the case $s=\frac{1}{2}$ – to poles of the Γ-factor. For a pole s_0 of $E_{4r}(z,s;\nu)$, we shall be interested in giving an upper bound for $\mathrm{Re}(s_0)$.

PROPOSITION 10.22. *Suppose the Eisenstein series $E_{4r}(z,s;\nu)$ has a pole at $s_0\in\mathbf{C}$ of order $N>0$ with $\mathrm{Re}(s_0)>\frac{1}{2}$. Then the functio $\varphi(z) = \lim_{s\to s_0}(s-s_0)^N E_{4r}(z,s;\nu)$ is a non-zero square-integrable eigenfunction of Δ_{4r} with eigenvalue $(s_0-2r)(1-s_0-2r)$.*

PROOF. Let s_0 be a pole of order $N > 0$. For any cusp $\mathfrak{a}$, we have

$$\begin{aligned} j(\sigma_{\mathfrak{a}}, z)^{4r}\varphi(\sigma_{\mathfrak{a}}^{-1}z) &= \lim_{s\to s_0}(s-s_0)^N j(\sigma_{\mathfrak{a}}, z)^{4r}E_{4r}(\sigma_{\mathfrak{a}}^{-1}.z, s;\nu) \\ &= a_{4r,\mathfrak{a},\nu}(0)y^{1-s_0-2r} + \sum_{n\neq 0} y^{-2r}a_{4r,\mathfrak{a},\nu}(n)W_{\mathrm{sign}(n)2r,s_0-\frac{1}{2}}(4\pi|n|y)e^{\pi inx}, \end{aligned}$$

where $a_{4r,\mathfrak{a},\nu}(n) = \lim_{s\to s_0}(s-s_0)^N\varphi_{4r,\mathfrak{a},\nu}(n,s)$. We see from its Fourier expansion that $\varphi(z)$ is square-integrable if $\mathrm{Re}(s_0) > \frac{1}{2}$ and that it is an eigenform of Δ_{4r} with eigenvalue $(s_0-2r)(1-s_0-2r)$. □

For example, in the case $r = \frac{1}{8}$ the Eisenstein series $E_{1/2}(z, s; \nu_r)$ has a pole at $s = \frac{3}{4}$ and its residue is equal to $\theta(z)$.

PROPOSITION 10.23. *The Eisenstein series $E_{4r}(z, s; \nu)$ is holomorphic on the half-plane* $\mathrm{Re}(s) > \max\{1-2r, 2r\}$.

PROOF. First note that $\max\{1-2r,\ 2r\} \geq \frac{1}{2}$ for $0 < r < 1$. Suppose $E_{4r}(z, s; \nu)$ has a pole at $s_0 \in \mathbf{C}$ of order $N > 0$ with $\mathrm{Re}(s_0) > \frac{1}{2}$. By Proposition 10.22, the function $\varphi(z) = \lim_{s\to s_0}(s-s_0)^N E_{4r}(z, s; \nu)$ is a non-zero square-integrable function with eigenvalue $(s_0-2r)(1-s_0-2r)$ under Δ_{4r}, i.e. a Maass form. By Stokes' Theorem, each Maass form has non-negative eigenvalue $(s_0-2r)(1-s_0-2r) = -s_0^2+s_0+4r^2 \geq 0$.

Since $\mathrm{Re}(s_0) > \frac{1}{2}$, we must have $s_0 \in \left(\frac{1}{2}, 1\right]$. We see that

$$(s-2r)(1-s-2r) = -(s-2r)(s-(1-2r)) < 0 \ \text{ for } s > \max\{2r, 1-2r\},$$

so the pole s_0 cannot lie on the half-plane $\{s \in \mathbf{C} : \mathrm{Re}(s) > \max\{2r, 1-2r\}\}$. □

Generally, the Eisenstein series $E_{4r}(z, s; \nu)$ has a pole at $s = \max\{1-2r, 2r\}$ for $0 < r < \frac{1}{2}$. The Eisenstein series $E_2(z, s; \nu)$ does not have a pole at $s = 1$ and is a harmonic function at that point.

Proposition 10.23 does not give any new information about the poles of the Eisenstein series $E_{4r}(z, s; \nu)$ in the strip $0 < \mathrm{Re}(s) < 1$ for $\frac{1}{2} < r < 1$. However, using lowering operators, we can prove the holomorphy of $E_{4r}(z, s; \nu)$ in the half-plane $\{s \in \mathbf{C} : \mathrm{Re}(s) > 2r-1\}$ through Proposition 10.23.

COROLLARY 10.24. *For $\frac{1}{2} < r < 1$, the Eisenstein series $E_{4r}(z, s; \nu)$ is holomorphic for* $\mathrm{Re}(s) > 2r-1$.

PROOF. Suppose $E_{4r}(z, s; \nu)$ has a pole at s_0 with $\mathrm{Re}(s_0) > \frac{1}{2}$. Using

$$2iy^2\partial_{\bar{z}}E_{4r}(z, s; \nu) = (s-2r)E_{4r-2}(z, s; \nu),$$

we see that s_0 is also a pole for $E_{4r-2}(z, s; \nu)$. By Proposition 10.23, we must have $\mathrm{Re}(s_0) \le \max\{2(r-1/2),\ 1-2(r-1/2)\} = \max\{2r-1,\ -2r\} = 2r-1$. $\square$

From the Fourier expansion of $E_{4r}(z, s; \nu)$ we see that if the Eisenstein series is holomorphic at some point s with $\mathrm{Re}(s) > \frac{1}{2}$, then the Dirichlet series $Z_\nu(n, s)$ are holomorphic at s as well. We rephrase Proposition 10.23 and Corollary 10.24 in a separate corollary in the special case of the Dirichlet series $Z_r(n, s)$, which we will apply in the next section.

COROLLARY 10.25. *The functions $Z_r(n, s)$ are holomorphic for* $\mathrm{Re}(s) > \max\{1-2r, 2r\}$ *for* $0 < r < \frac{1}{2}$ *and for* $\mathrm{Re}(s) > 2r - 1$ *for* $\frac{1}{2} < r < 1$.

The Eisenstein series $E_{4r}(z, s; \nu)$ as such is not square-integrable with respect to the Petersson inner product

$$\langle f, g\rangle = \int_{\Gamma_\theta \backslash \mathbf{H}} f(z)\overline{g(z)} y^{4r} \frac{dx dy}{y^2},$$

the problematic part of $E_{4r}(z, s; \nu)$ being the zeroth Fourier coefficient in (10.19). To make it square-integrable, we consider the truncated Eisenstein series instead.

Let $Y > 0$. For each cusp $\mathfrak{a}$, let $F_{\mathfrak{a}}(Y) = \sigma_{\mathfrak{a}}.F_Y$ be the cuspidal zone around $\mathfrak{a}$, where $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbf{Z})$ is such that $\sigma_{\mathfrak{a}}.i\infty = \mathfrak{a}$ and $F_Y = \{z \in \mathbf{H} : -1 < \mathrm{Re}(s) < 1, \mathrm{Im}(z) > \max\{1, Y\}\}$. We define the truncated Eisenstein series by

$$E^Y_{4r}(z, s; \nu) = \begin{cases} E_{4r}(z, s; \nu) - \delta_{\mathfrak{a},\infty}\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}.z)^{s-2r} - \varphi_{4r,\mathfrak{a},\nu}(s)\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}.z)^{1-s-2r}, & z \in F_{\mathfrak{a}}(Y), \\ E_{4r}(z, s; \nu), & \text{otherwise.} \end{cases}$$

The truncated Eisenstein series $E^Y_{4r}(z, s; \nu)$ is now a square-integrable function on $\Gamma_\theta \setminus \mathbf{H}$ of weight $4r$. Moreover, it satisfies the Maass-Selberg relations.

THEOREM 10.26. *Let s_1 and s_2 with $s_1 \neq s_2$ and $s_1 + s_2 \neq 1$ be regular points of $E_{4r}(z, s; \nu)$ and let $E^Y_{4r}(z, s; \nu)$ be the truncated Eisenstein series. We have*

$$\begin{aligned} \langle E^Y_{4r}(z, s_1; \nu), E^Y_{4r}(z, s_2; \nu)\rangle = {} & (s_2 - s_1)^{-1}\varphi_{4r,\nu}(s_1)Y^{s_2-s_1} + (s_1 - s_2)^{-1}\varphi_{4r,\nu}(s_2)Y^{s_1-s_2} \\ & + (s_1 + s_2 - 1)^{-1}Y^{s_1+s_2-1} \\ & - (s_1 + s_2 - 1)^{-1}\varphi_{4r,\nu}(s_1)\overline{\varphi_{4r,\nu}(s_2)}Y^{1-s_1-s_2}. \end{aligned}$$

PROOF. This is an application of Green's theorem; see, e.g., H. Iwaniec's book [**Iwa21**, Prop. 6.8]. $\square$

The Maass-Selberg relations in the case $\nu = \nu_r$ give a growth estimate for $Z_r(n, \sigma + it)$ with $\frac{1}{2} < \sigma < \frac{3}{2}$ as $t \to \pm\infty$, which we will use for giving a growth estimate for the Weyl sums (10.8).

LEMMA 10.27. *Let* $\frac{1}{2} < \sigma < \frac{3}{2}$ *and* $n \in \mathbf{Z}$. *We have*

$$Z_r(n, \sigma + it) \ll_{\sigma,\, r,\, n} \sqrt{|t|}$$

as $|t| \to +\infty$.

PROOF. Write $s = \sigma + it$ with $\sigma \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $|t| > 1$. From Theorem 10.26 for $\nu = \nu_r$ with $s_1 = s$ and $s_2 = \overline{s}$, we get

$$\begin{aligned}(10.20)\qquad &\int_{\Gamma_\theta\backslash\mathbf{H}} \left|E_{4r}^Y(z, s; \nu_r)\right|^2 y^{4r} d\mu(z) \\ &= \frac{1}{2\sigma - 1}\left(Y^{2\sigma-1} - |\varphi_{4r,\nu_r}(s)|^2 Y^{1-2\sigma}\right) + \frac{1}{2it}\left(-\overline{\varphi_{4r,\nu_r}(s)}Y^{-2it} + \varphi_{4r,\nu_r}(s)Y^{2it}\right),\end{aligned}$$

from which it follows that

$$\frac{1}{2\sigma - 1}\left|\varphi_{4r,\nu_r}(s)\right|^2 Y^{1-2\sigma} = \frac{1}{2\sigma-1}Y^{2\sigma-1} + \operatorname{Re}\left(\frac{\varphi_{4r,\nu_r}(s)Y^{2it}}{it}\right) - \|E_{4r}^Y(z, s; \nu_r)\|_2^2$$

and

$$|\varphi_{4r,\nu_r}(s)|^2 \le Y^{4\sigma-2} + \frac{2\sigma-1}{|t|}Y^{2\sigma-1}\left|\varphi_{4r,\nu_r}(s)\right| \le Y^4 + 2Y^2\left|\varphi_{4r,\nu_r}(s)\right|,$$

i.e. $|\varphi_{4r,\nu_r}(s)| \le (1+\sqrt{2})Y$ uniformly for $|t| > 1$. By Stirling's approximation, we have

$$\frac{\Gamma(2s-1)}{\Gamma(s+2r)\Gamma(s-2r)} \sim \frac{|t|^{-1/2}}{\sqrt{2\pi}}, \text{ as } t \to \pm\infty,$$

hence the claim for $n = 0$ follows by Proposition 10.21. Moreover, from the uniform bound $|\varphi_{4r,\nu_r}(s)| \le (1+\sqrt{2})Y$ for $|t| > 1$, we also get from (10.20) that $\|E_{4r}^Y(z, s; \nu_r)\|_2^2 \le$ is uniformly bounded for $|t| > 1$.

To prove the assertion for $Z_r(n, s)$ with $n \neq 0$, we again use the Maass-Selberg relations to estimate for $Y > 1$:

$$\begin{aligned}&\int_{\Gamma_\theta\backslash\mathbf{H}} |E_{4r}^Y(z, s; \nu_r)|^2 y^{4r}\frac{dxdy}{y^2} \\ &\ge \int_Y^\infty \int_{-1}^1 |E_{4r}^Y(z, s; \nu_r)|^2 y^{4r}\frac{dxdy}{y^2} \\ &= \sum_{m\neq 0}\left|\frac{\pi^s|m|^{s-1}}{\Gamma(s+\operatorname{sign}(m)2r)}\right|^2 |Z_r(m, s)|^2 \int_Y^\infty |W_{\operatorname{sign}(m)2r, s-\frac{1}{2}}(4\pi|m|y)|^2\frac{dy}{y^2} \\ &\ge \left|\frac{\pi^s|n|^{s-1}}{\Gamma(s+\operatorname{sign}(n)2r)}\right|^2 |Z_r(n, s)|^2 \int_Y^{Y+1} |W_{\operatorname{sign}(n)2r, s-\frac{1}{2}}(4\pi|n|y)|^2\frac{dy}{y^2}.\end{aligned}$$

Using the asymptotic formula

$$W_{\operatorname{sign}(n)2r, s-\frac{1}{2}}(4\pi|n|y) \sim \pi^{-1/2}\Gamma(\operatorname{sign}(n)2r + s - 1/2)(\pi|n|y)^{1-s}$$

as $|t| \to +\infty$ (which converges uniformly for y in a compact set), we see that
(10.21)

$$\int_{\Gamma_\theta\backslash\mathbf{H}} |E_{4r}^Y(z,s;\nu_r)|^2 y^{4r}\frac{dxdy}{y^2} \gg \left|\frac{\Gamma(s+\operatorname{sign}(n)2r-1/2)}{\Gamma(s+\operatorname{sign}(n)2r)}\right|^2 |Z_r(n,s)|^2 \int_Y^{Y+1}\frac{1}{y^{2\sigma}}dy.$$

We already noted that $\|E_{4r}^Y(z,s;\nu_r)\|_2^2$ is bounded uniformly for $|t|>1$. The claim then follows from (10.21) and Stirling's approximation for $n \neq 0$. $\square$

6. Uniform Distribution of Hardy Sums

In this section, we will use the results from Section 5 of this chapter to prove Theorem 10.4.

Let $0<r<1$ be a rational number and $n \in \mathbf{Z}$. Since the case $r=\frac{1}{2}$ was already proved in section 4.3, we may assume that $r \neq \frac{1}{2}$. To work out the explicit growth rate of the Weyl sums, we will use Perron's formula, i.e., that for any $\alpha>1$, we have

$$\sum_{c=1}^{N}\sum_{\substack{d(c)^*,\\ c+d\text{ odd}}} e(nd/c+rS(d,c)) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} Z_r(n,s)\frac{N^{2s}}{2s}ds. \tag{10.22}$$

Let $\varepsilon>0$, $T>0$, $\alpha=1+\varepsilon$, and $s_1,\ldots,s_l \in \mathbf{C}$ be the location of the poles of $Z_r(n,s)$ in the strip $\frac{1}{2}<\operatorname{Re}(s)<\frac{3}{2}$. By Lemma 10.27, we have

$$\int_{\alpha-i\infty}^{\alpha+i\infty} Z_r(n,s)\frac{N^{2s}}{2s}ds = \int_{\alpha-iT}^{\alpha+iT} Z_r(n,s)\frac{N^{2s}}{2s}ds + O(N^{2+\varepsilon}/T^{3/2+\varepsilon}).$$

Using Cauchy's theorem, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} Z_r(n,s)\frac{N^{2s}}{2s}ds &= \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} Z_r(n,s)\frac{N^{2s}}{2s}ds + \int_{\frac{1}{2}+\varepsilon+iT}^{\alpha+iT} Z_r(n,s)\frac{N^{2s}}{2s}ds\\ &\quad - \int_{\frac{1}{2}+\varepsilon-iT}^{\alpha-iT} Z_r(n,s)\frac{N^{2s}}{2s}ds + \sum_{j=1}^{n}\operatorname{Res}_{s=s_j} Z_r(n,s)\frac{N^{2s_j}}{2s_j}.\end{aligned}$$

The integrals along the horizontal lines satisfy

$$\begin{aligned}\left|\int_{\alpha-iT}^{\frac{1}{2}+\varepsilon-iT} Z_r(n,s)\frac{N^{2s}}{2s}ds\right| &\le \int_{\alpha-iT}^{\frac{1}{2}+\varepsilon-iT}\left|Z_r(n,s)\frac{N^{2s}}{2s}\right|ds\\ &\le \frac{N^{2\alpha}}{T}\int_{\alpha-iT}^{\frac{1}{2}+\varepsilon-iT}|Z_r(n,s)|\,ds\\ &\ll \frac{N^{2\alpha}}{T^{\frac{1}{2}}}\end{aligned}$$

as $T \to +\infty$ by Lemma 10.27. The integral along the vertical line $\frac{1}{2} + \varepsilon + i\mathbf{R}$ can, again by Lemma 10.27, be bounded by

$$\left| \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} Z_r(n, s) \frac{N^{2s}}{2s} ds \right| \ll_{r,\, \varepsilon,\, n} N^{1+\varepsilon} T^{\frac{1}{2}}. \tag{10.23}$$

Hence, we get

$$\sum_{c=1}^{N} \sum_{\substack{d(c)^*, \\ c+d \text{ odd}}} e(nd/c + rS(d, c)) = \sum_{j=1}^{n} \mathrm{Res}_{s=s_j} Z_r(n, s) \frac{N^{2s_j}}{2s_j} + O\left(N^{2+\varepsilon}/T^{1/2} + N^{1+\varepsilon} T^{1/2}\right)$$

as $T \to +\infty$. Theorem 10.4 then follows by Corollary 10.25 and choosing $T = N$.

CHAPTER 11

Two Geometric Characterizations of Hardy Sums

1. Counting Lattice Points in a Triangle

Problems of counting points with integer coordinates in geometric objects are challenging and well-studied problems and give an interesting connection between number theory and geometry (famous examples include counting integer points on the 3-sphere, the Gauss circle problem or G.H. Hardy and J.E. Littlewood's work on counting points in a right-angled triangle [**Har05**]). These problems are typically very hard. However, in the case of two-dimensional polygons whose vertices have integer coordinates, G. Pick [**Pic99**] found an elegant and simple solution to the problem.

THEOREM 11.1 (Pick). *Let P be a polygon, whose vertices lie in $\mathbf{Z}^2$. We have*

$$\operatorname{area}(P) = \#(P^\circ \cap \mathbf{Z}^2) + \frac{1}{2}\#(\partial P \cap \mathbf{Z}^2) + 1,$$

where P° denotes the interior and ∂P denotes the boundary of P.

We will limit ourselves to the case of triangles. For a pair of positive integers d, c, let $T(d,c)$ be the triangle in $\mathbf{R}^2$ with vertices $(0,0)$, $(d,0)$, and $(0,c)$. The set $T(d,c)$ is the set of points $(x,y) \in \mathbf{R}^2$ such that $0 \le \frac{x}{d} + \frac{y}{c} < 1$ (we exclude the corners with acute angles and the hypotenuse). For simplicity, we shall assume that $(d,c) = 1$.

By Theorem 11.1, we only need to identify the points on the boundary of $T(d,c)$ if we want to find $\#(T(d,c) \cap \mathbf{Z}^2)$, namely

$$\#(T(d,c) \cap \mathbf{Z}^2) = \operatorname{area}(T(d,c)) + \frac{1}{2}\#(\partial T(d,c) \cap \mathbf{Z}^2) - 1. \tag{11.1}$$

As we assumed that d, c are coprime there are no lattice points $(x,y) \in \mathbf{Z}^2$ that satisfy $\frac{x}{d} + \frac{y}{c} = 1$ with both x, y non-zero. Hence, the points on the boundary are given by $(0,0), (1,0), (0,1), \ldots$, i.e. $\#(\partial T(d,c) \cap \mathbf{Z}^2) = c + d + 1$. We then find by equation (11.1):

$$\#(T(d,c) \cap \mathbf{Z}^2) = \frac{1}{2}(c+1)(d+1) - 1. \tag{11.2}$$

The problem of finding the lattice points in $T(d,c)$ becomes more difficult if we consider the lattice $(2\mathbf{Z})^2$ instead of $\mathbf{Z}^2$. Obviously, the problem can be rescaled and solved by Pick's Theorem if both d, c are even. So again, let us assume that d, c are positive coprime integers. In this case, at least one of the vertices of $T(d, c)$ is no longer a lattice point and we cannot use Theorem 11.1.

Here, we shall prove that $\#(T(d, c) \cap (2\mathbf{Z}^2))$ can be expressed in terms of the Hardy sums $S_4(d, c)$ in (9.2). The precise formula for $\#(T(d, c) \cap (2\mathbf{Z})^2)$ for $(d, c) = 1$ in terms of $S_4(d, c)$ is given in the next theorem.[1]

THEOREM 11.2. *Let d, c be positive coprime integers.*

(1) *If d, c are both odd, then*

$$\#(T(d, c) \cap (2\mathbf{Z})^2) = \frac{1}{8}(cd + 2d + 2c + S_4(d, c) + S_4(c, d) - 3).$$

(2) *If d is even, then*

$$\#(T(d, c) \cap (2\mathbf{Z})^2) = \frac{1}{4}\left(c + d + \frac{cd}{2} + S_4(c, d/2)\right).$$

The proof of Theorem 11.2 relies on Mordell's Theorem 8.5. To give a closed expression of $\#(T(d, c) \cap (2\mathbf{Z})^2)$, we consider the problem as a three-dimensional one. Namely, we write

$$\begin{aligned} \#(T(d, c) \cap (2\mathbf{Z})^2) &= \#\left\{(x, y) \in \mathbf{Z}^2_{\geq 0} : \frac{2x}{d} + \frac{2y}{c} < 1\right\} \\ &= \#\left\{(x, y) \in \mathbf{Z}^2_{\geq 0} : \frac{x}{d} + \frac{y}{c} + \frac{1}{2} < 1\right\}. \end{aligned} \tag{11.3}$$

Counting $\#(T(d, c) \cap (2\mathbf{Z})^2)$ is thus equivalent to counting the lattice points in $\mathbf{Z}^3$ of the tetrahedron $D(d, c, 2)$ with vertices $(d, 0, 0)$, $(0, c, 0)$, and $(0, 0, 2)$ intersected with the plane $\{(x, y, z) \in \mathbf{R}^3 : z = 1\}$ by (11.3). Geometrically, this corresponds to projecting $T(d, c) \cap (2\mathbf{Z})^2$ onto the plane $\{(x, y, z) \in \mathbf{R}^3 : z = 1\}$ with base point $(0, 0, 2)$.

1.1. Proof of the First Part of Theorem 11.2. Let d, c be odd and coprime integers. By Theorem 8.5 we have

(11.4)
$$\begin{aligned} \#(D(d, c, 2) \cap \mathbf{Z}^3) = \frac{cd}{3} + \frac{1}{4}(2c + 2d + cd) + \frac{1}{4}(c + d + 2) + \frac{1}{12}\left(\frac{2c}{d} + \frac{2d}{c} + \frac{cd}{2}\right) \\ + \frac{1}{24cd} - 2 - (s(2d, c) + s(2c, d) + s(cd, 2)). \end{aligned}$$

[1] To make the theorem plausible, we can look at the following examples. For the first formula, we have $\#(T(33, 691) \cap (2\mathbf{Z})^2) = 3034$, $S_4(33, 691) = 18$, and $S_4(691, 33) = 0$. For the second formula, we have $\#(T(22, 691) \cap (2\mathbf{Z})^2) = 2079$ and $S_4(691, 11) = 2$.

Since cd is odd, we have $s(cd,2) = s(1,2) = 0$.

We split the number of lattice points in $D(d,c,2)$ into those contained in the plane $\{(x,y,z) \in \mathbf{R}^3 : z = 0\}$ and those contained in the plane $\{(x,y,z) \in \mathbf{R}^3 : z = 1\}$. The latter is, as we already said, equal to $\#(T(d,c) \cap (2\mathbf{Z})^2)$. The former is equal to

$$\#\left\{(x,y) \in \mathbf{Z}^2_{\geq 0} : 0 < \frac{x}{d} + \frac{y}{c} < 1\right\} = \frac{1}{2}(c+1)(d+1) - 2$$

by (11.2) with the exclusion of the point $(0,0)$.

To write $\#(T(d,c) \cap (2\mathbf{Z})^2)$ in terms of Hardy sums, we use the following easy lemma.

Lemma 11.3. *Let d,c be odd and coprime integers. We have*

$$s(2c,d) + s(2d,c) = \frac{4c^2 + 4d^2 + 1}{24cd} - \frac{1}{8}(S_4(d,c) + S_4(c,d)) - \frac{3}{8}.$$

Proof. We use formula (9.8) for the Hardy sum $S_4(d,c)$. A calculation yields

$$\begin{aligned} S_4(d,c) + S_4(c,d) &= 8s(d,2c) - 4s(d,c) + 8s(c,2d) - 4s(c,d) \\ &= 8\left(-s(2c,d) - s(2d,c) + \frac{1}{12}\frac{4c^2+d^2+1}{2cd} + \frac{1}{12}\frac{c^2+4d^2+1}{2cd} - \frac{1}{2}\right) \\ &\quad - 4\left(\frac{1}{12}\frac{c^2+d^2+1}{cd} - \frac{1}{4}\right) \\ &= \frac{4c^2+4d^2+1}{3cd} - 3 - 8\left(s(2c,d) + s(2d,c)\right), \end{aligned}$$

where we used the reciprocity law (8.3) twice. □

Putting (11.4) and Lemma 11.3 together yields the first part of Theorem 11.2.

1.2. Proof of the Second Part of Theorem 11.2. Let d,c be coprime integers and d an even integer. As in the previous section, we would like to apply Mordell's Theorem 8.5 to find $\#(T(d,c) \cap (2\mathbf{Z})^2)$. However, Theorem 8.5 is no longer true if the integers u,v,w are not pairwise coprime. We hence need to adjust the statement.

Proposition 11.4. *Let d,c be coprime integers and suppose that d is even and c is odd. The number of lattice points in the tetrahedron $D(d,c,2)$ is given by*

$$\begin{aligned} \#(D(d,c,2) \cap \mathbf{Z}^3) &= \frac{cd}{3} + \frac{1}{4}(2c+2d+cd) + \frac{1}{4}(c+d+2) \\ &\quad + \frac{1}{12}\left(\frac{2c}{d} + \frac{2d}{c} + \frac{cd}{2}\right) + \frac{1}{6cd} - \frac{5}{2} - (2s(d,c) + s(2c,d)). \end{aligned}$$

Proof. The proof is along the lines of Mordell's proof [**Mor51**]. We only indicate where adjustments need to be made.

There exists exactly one lattice point (x, y, z), $0 \le x < d$, $0 \le y < c$, $0 \le z < 2$, with $\frac{x}{d} + \frac{y}{c} + \frac{z}{2} = 1$, namely $x = \frac{d}{2}$, $y = 0$, and $z = 1$ (this would not be the case if c, d, 2 were pairwise coprime). There exist no lattice points with (x, y, z), $0 \le x < d$, $0 \le y < c$, $0 \le z < 2$, with $\frac{x}{d} + \frac{y}{c} + \frac{z}{2} = 2$.

One starts with the formula

$$\#(D(d,c,2) \cap \mathbf{Z}^3) = \frac{1}{2} \sum_{x,y,z}{}' \left(\left\lfloor \frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right\rfloor - 1 \right) \left(\left\lfloor \frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right\rfloor - 2 \right)$$

as in (8.12), where the sum goes over $0 \le x < d$, $0 \le y < c$, and $0 \le z < 2$ and $\sum'$ means that we omit $x = y = z = 0$.

Now rewrite

$$\#(D(d,c,2) \cap \mathbf{Z}^3) = \frac{1}{2} \sum_{x,y,z}{}' (E - ((E)) - 3/2)(E - ((E)) - 5/2) - \frac{3}{8},$$

where $E = \frac{x}{d} + \frac{y}{c} + \frac{z}{2}$ and the term $-\frac{3}{8}$ is the correction factor for the unique case where $E = 1$.

We split up the sum $\sum'_{x,y,z} (E - ((E)) - 3/2)(E - ((E)) - 5/2) = A + B + C$ with

$$A = \frac{1}{2} \sum_{x,y,z}{}' (E - 3/2)(E - 5/2), \quad B = - \sum_{x,y,z}{}' (E - 2)\, ((E)), \quad C = \frac{1}{2} \sum_{x,y,z}{}' ((E))^2.$$

It is straightforward to calculate A, which is equal to $-\frac{11}{8} + \frac{3c}{d} + \frac{c}{6d} + \frac{3d}{4} + \frac{d}{6c} + \frac{13cd}{24}$ For B, we obtain

(11.5)

$$\begin{aligned}
-B &= \sum_{x,y,z}{}' \frac{x}{d} \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right) \right) + \sum_{x,y,z}{}' \frac{y}{c} \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right) \right) + \sum_{x,y,z}{}' \frac{z}{2} \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right) \right) \\
&\quad - 2 \sum_{x,y,z}{}' \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right) \right) \\
&= \sum_{x=1}^{d-1} \sum_{y=0}^{c-1} \frac{x}{d} \left(\left(\frac{2x}{d} + \frac{2y}{c} \right) \right) + \sum_{y=1}^{c-1} \sum_{x=0}^{d-1} \frac{y}{c} \left(\left(\frac{2x}{d} + \frac{2y}{c} \right) \right) + \sum_{x=0}^{d-1} \sum_{y=0}^{c-1} \frac{1}{2} \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{1}{2} \right) \right) \\
&\quad - 2 \sum_{x=1}^{d-1} \left(\left(\frac{2cx}{d} \right) \right) \\
&= \sum_{x=0}^{d-1} \left(\left(\frac{2cx}{d} \right) \right) \frac{x}{d} + 2 \sum_{y=1}^{c-1} \left(\left(\frac{dy}{c} \right) \right) \frac{y}{c} + \frac{1}{2} \sum_{x} \sum_{y} \left(\left(\left(\frac{2x}{d} + \frac{2y}{c} \right) \right) - \left(\left(\frac{x}{d} + \frac{y}{c} \right) \right) \right) \\
&= s(2c, d) + 2s(d, c),
\end{aligned}$$

where we used the fact that $((x)) + \left(\left(x + \frac{1}{n}\right)\right) + \ldots + \left(\left(x + \frac{n-1}{n}\right)\right) = ((nx))$ for $x \notin \mathbf{Z}$ and $n \geq 1$, as well as $\sum_{\mu(d)} \left(\left(\frac{\mu}{d}\right)\right) = 0$, and that $2y$ runs through a complete set of residues modulo c and that $2x$ runs through a complete set of residues of $d/2$ twice.

To calculate C, we may write

$$C = \frac{1}{2} \sum_{x,y,z}{}' \left(\left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2}\right)\right)^2 = \sum_{x=1}^{cd-1} \left(\left(\frac{x}{cd}\right)\right)^2,$$

It is then easy to calculate $C = -\frac{1}{4} + \frac{1}{6cd} + \frac{cd}{12}$. □

LEMMA 11.5. *Let $d \geq 1$ be an even integer and $c \geq 1$ such that $(d, c) = 1$. We have*

$$2s(d, c) + s(2c, d) = \frac{d^2 + c^2 + 1}{6cd} - \frac{1}{2} - \frac{1}{4} S_4(c, d/2).$$

PROOF. We use formula (9.8) for the Hardy sums $S_4(d, c)$. Applying the reciprocity formula for Dedekind sums, we calculate

$$\begin{aligned} 2s(d, c) + s(2c, d) &= 2 \left(\frac{1}{12} \frac{d^2 + c^2 + 1}{cd} - \frac{1}{4} - s(c, d) \right) + s(c, d/2) \\ &= \frac{d^2 + c^2 + 1}{6cd} - \frac{1}{2} - \frac{1}{4} S_4(c, d/2), \end{aligned}$$

where we used that $s(2c, 2d') = s(c, d')$ for all $d' \in \mathbf{Z}$. □

Proposition 11.4 and Lemma 11.5 prove the second part of Theorem 11.2.

2. Hardy Sums as Signed Intersection Numbers

One of the reasons that make Dedekind and Hardy sums interesting objects of study is that they can be studied from an elementary point of view, but also appear in the rich theory of automorphic forms. In this section, we will realize the Hardy sums as signed intersection numbers, which will allow us to study the Hardy sums by elementary geometric means. The geometric realization we will give for, say, $S(d, c)$ is straightforward to define. Consider the net $\mathcal{N}_\theta$ of geodesics from $\frac{\alpha}{\gamma}$ to $\frac{\beta}{\delta}$, where $(\alpha, \gamma) = 1$, $\gamma > 0$ and $(\beta, \delta) = 1$, $\delta > 0$ with $\alpha\delta - \beta\gamma = \pm 2$, on the hyperbolic upper half plane. We orient a geodesic from $\frac{\alpha}{\gamma}$ to $\frac{\beta}{\delta}$ if $\gamma \leq \delta$ and from $\frac{\beta}{\delta}$ to $\frac{\alpha}{\gamma}$ if $\delta \leq \gamma$ – we allow the geodesic to be doubly-oriented if $\gamma = \delta$; see section 2.1. Then $S(d, c)$ is the signed intersection number of the oriented geodesic from $i\infty$ to $\frac{d}{c}$ with the net $\mathcal{N}_\theta$.

More precisely, let $h_{d/c}$ be the oriented geodesic from $i\infty$ to $\frac{d}{c}$. For any oriented geodesic g, define

$$\varphi_g(x) = \begin{cases} 0, & \text{if } g \cap h_x = \emptyset, \\ +1, & \text{if } g \text{ intersects } h_x \text{ on the right,} \\ -1, & \text{if } g \text{ intersects } h_x \text{ on the left.} \end{cases}$$

By "g intersects h from the right" we mean that if we follow the path of h_x, then the orientation of g at the intersection point is directed to the right. If the geodesic g is doubly-oriented, then we set $\varphi_g(x) = 1 - 1 = 0$.

Theorem 11.6. *Let d, c be integers with $(d,c) = 1$ and $c+d$ odd. The Hardy sum*

$$S(d,c) = -\sum_{g \in \mathcal{N}_\theta} \varphi_g(d/c).$$

Apart from the automorphic interpretation of the Dedekind sums appearing the multiplier system of $\eta(z)$, we may also view the Dedekind sums exclusively from a cohomological perspective. The integer-valued Dedekind symbol Φ in (8.8) satisfies the generalized reciprocity law

$$\Phi(AB) - \Phi(A) - \Phi(B) = 3 \cdot w(A,B), \tag{11.6}$$

where $w(A,B) = -\mathrm{sign}(c_A c_B c_{AB}) \in H^2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z})$ is the Rademacher cocycle for $A, B \in \mathrm{SL}_2(\mathbf{Z})$ and $A = \begin{pmatrix} * & * \\ c_A & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ c_B & * \end{pmatrix}$, $AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix}$. The reciprocity law (8.3) for $s(d,c)$ is then the special case of (11.6) for $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

T. Asai [**Asa70**] constructed the Dedekind symbol as a splitting of the cohomology class in $H^2(\mathrm{SL}_2(\mathbf{R}), \mathbf{R})$ associated with the central extension of the universal covering group. The space $H^2(\mathrm{SL}_2(\mathbf{R}), \mathbf{R})$ is a one-dimensional $\mathbf{R}$-vector space generated by the cocycle $w(A,B)$. The Dedekind symbol Φ can then be defined as the unique "splitting" of $w(A,B)$, i.e. the unique quasimorphism $\Phi : \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{R}$ such that $3 \cdot w(A,B) = \Phi(AB) - \Phi(A) - \Phi(B)$. The uniqueness of Φ follows easily from $H^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{R}) = \{0\}$.

The Dedekind sums $s(a,c)$ in (8.2) are uniquely characterized by the following three properties for $(a,c) = 1$:

(1) $s(0,1) = 0$,
(2) $s(a+c,c) = s(a,c)$,
(3) $s(a,c) - s(-c,a) = \frac{1}{12}\frac{a^2+c^2+1}{ac} - \frac{1}{4}$ for $a, c \neq 0$.

Since for each coprime pair of integers a, c there exists a a matrix $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, we may write $s(A)$ for $s(a,c)$, i.e. view the Dedekind sums as a function $s : \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{Q}$. By definition, the Dedekind sum $s(A)$ is invariant under right-multiplication by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; moreover, we set $s(1,0) = s(I) = 0$. The three defining properties above then correspond to the matrix operations under the generators T and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbf{Z})$, namely for $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ with $a, c \neq 0$:

(1) $s(S) = s(I) = 0$,
(2) $s(T^n A) = s(A)$ for all $n \in \mathbf{Z}$,
(3) $s(A) - s(SA) = \frac{1}{12}\frac{a^2+c^2+1}{cd} - \frac{1}{4}$.

Each matrix in $\mathrm{SL}_2(\mathbf{Z})$ can be written as a word in T, S and therefore we get that the three properties uniquely define the function $s : \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{Q}$.

Similarly, we may view the Hardy sums $S(a,c)$ and $S_4(a,c)$ as functions on their respective groups $\Gamma_\theta = \langle T^2, S\rangle$ and $\Gamma^0(2) = \langle T^2, V\rangle$. To avoid confusion with the matrix S, we denote these functions in gothic print, i.e., $\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$ and $\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Z}$. Put $\mathfrak{S}(A) = S(a,c)$ for $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma_\theta$.[2] The function $\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$ is uniquely determined by the properties:

(1) $\mathfrak{S}(S) = \mathfrak{S}(T) = \mathfrak{S}(I) = 0$,
(2) $\mathfrak{S}(T^{2n} A) = \mathfrak{S}(A)$ for all $n \in \mathbf{Z}$,
(3) $\mathfrak{S}(A) - \mathfrak{S}(SA) = \mathrm{sign}(ac)$ for $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma_\theta$;

and the function $\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Z}$ (similarly defined) is uniquely determined by the properties:

(1) $\mathfrak{S}_4(V) = \mathfrak{S}_4(T) = \mathfrak{S}_4(I) = 0$,
(2) $\mathfrak{S}_4(T^{2n} A) = \mathfrak{S}_4(A)$ for all $n \in \mathbf{Z}$,
(3) $\mathfrak{S}_4(A) + \mathfrak{S}_4(VA) = -\mathrm{sign}(c(a+c))$ for $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma^0(2)$.

In section 2.3, we will be to show the generalized reciprocity law (11.6) for the Hardy sums by an elementary argument, using the geometric realization of the Hardy sums as in Theorem 11.6. The property follows from a close study the behavior of $\mathfrak{S}(A)$ under the action of Γ_θ on the upper half plane by linear fractional transformations.

[2]To make our notations simpler, we consider $S(a,c)$ instead of the usual $S(d,c)$. The change is insubstantial, as $S(d,c) = -\mathfrak{S}(A^{-1}) = \mathfrak{S}(A)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$; see Proposition 11.18.

THEOREM 11.7. (1) *We have* $\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = w(A, B)$ *for* $A, B \in \Gamma_\theta$.
(2) *We have* $\mathfrak{S}_4(AB) - \mathfrak{S}_4(A) - \mathfrak{S}_4(B) = w(A, B)$ *for* $A, B \in \Gamma^0(2)$.

For the subgroups Γ_θ and $\Gamma^0(2)$, the first cohomology group is non-trivial, which makes the splitting of the Rademacher cocycle $w(A, B)$ non-unique.

2.1. Net of Oriented Geodesics. Consider the set of rationals $\frac{\alpha}{\gamma}$ with $(\alpha, \gamma) = 1$, $\gamma \geq 1$, and α, γ both odd and the net of geodesics connecting two such rationals $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ if $\alpha\delta - \beta\gamma = \pm 2$. Note that these geodesics are the image of the geodesic connecting -1 and 1 under Γ_θ.

Now orient the geodesic connecting $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\delta}$ such that the geodesic goes from $\frac{\alpha}{\gamma}$ to $\frac{\beta}{\delta}$ if $\gamma < \delta$ and from $\frac{\beta}{\delta}$ to $\frac{\alpha}{\gamma}$ if $\delta < \gamma$. In the case $\gamma = \delta$, which can only occur if the geodesic connects two integers, we give it two opposing orientations. We will denote the resulting net of oriented geodesics by $\mathcal{N}_\theta$.

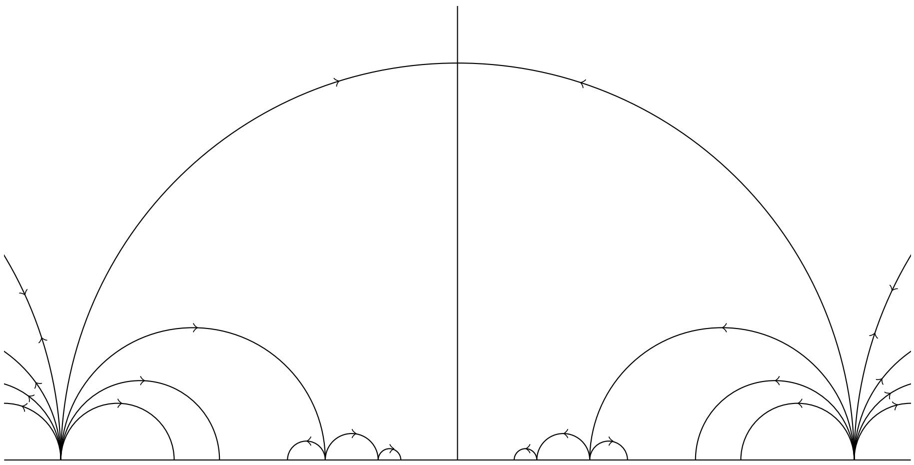

Figure 1: The net $\mathcal{N}_\theta$ for the rationals $\frac{\alpha}{\gamma}$, $(\alpha, \gamma) = 1$, α, γ both odd with $|\gamma| \leq 7$.

LEMMA 11.8. *Let* $\alpha, \beta, \gamma, \delta$ *be odd integers with* $\gamma, \delta \geq 1$ *and* $(\alpha, \gamma) = 1$, $(\beta, \delta) = 1$ *such that* $\alpha\delta - \beta\gamma = \pm 2$. *We have* $\gamma = \delta$ *if and only if* $\gamma = \delta = 1$. *We have* $\alpha = \beta$ *if and only if* $\alpha, \beta = \pm 1$. *If* $\alpha \neq \beta$, *the condition* $\gamma < \delta$ *is satisfied if and only if* $|\alpha| < |\beta|$.

PROOF. If $\alpha = \beta$, then $\alpha\delta - \beta\gamma = \alpha(\delta - \gamma) = \pm 2$, and since α is odd, we have $\alpha = \pm 1$. Analogously, it is showed that $\gamma = \delta$ implies $\gamma = \delta = 1$.

Let us thus suppose that $\gamma, \delta > 1$. Then the integers α, β need to have the same sign to satisfy the equation $\alpha\delta - \beta\gamma = \pm 2$. By multiplying the equation $\alpha\delta - \beta\gamma = \pm 2$ on both sides with -1 if necessary, we may assume w.l.o.g. that $\alpha, \beta > 0$.

Suppose that $\alpha\delta - \beta\gamma = -2$ and that $\alpha \neq \beta$. If $\gamma < \delta$, then $\alpha\delta = \beta\gamma - 2$ implies that $\alpha\delta < \beta\gamma < \beta\delta$, hence $\alpha < \beta$. If $\delta < \gamma$, then $2 = \beta\gamma - \alpha\delta$ implies that $\delta(\beta - \alpha) < 2$ and moreover that $\beta - \alpha < 0$, since $\beta - \alpha$ is an even integer and we excluded $\alpha = \beta$. The case $\alpha\delta - \beta\gamma = 2$ is similar. $\square$

Lemma 11.8 has the following consequence for the oriented net of geodesics $\mathcal{N}_\theta$.

LEMMA 11.9. *The oriented net $\mathcal{N}_\theta$ is invariant under the maps $t_1(z) = -\overline{z}$ and $t_2(z) = z+2$. Moreover, under $t_3(z) = \frac{z}{|z|^2}$ any oriented geodesic $g \in \mathcal{N}_\theta$ whose endpoints are not both of the form $\frac{1}{n}$ for $n \in \mathbf{Z} - \{0, \pm 1\}$ or both integers not equal to ± 1 is mapped to an oriented geodesic in $\mathcal{N}_\theta$.*

PROOF. Suppose we have a geodesic from $\frac{\alpha}{\gamma}$ to $\frac{\beta}{\delta}$ in $\mathcal{N}_\theta$ with $\gamma, \delta > 1$ odd integers, $\delta > \gamma > 1$, $(\alpha, \gamma) = 1$ and $(\beta, \delta) = 1$ with $\alpha\delta - \beta\gamma = \pm 2$. We will only show the case where $\alpha, \beta > 1$.

Under the orientation-reversing map t_1 the geodesic is mapped to the geodesic from $-\frac{\alpha}{\gamma}$ to $-\frac{\beta}{\delta}$, which still lies in $\mathcal{N}_\theta$.

Under the orientation-preserving t_2 the geodesic is mapped to the one from $\frac{\alpha+2\gamma}{\gamma}$ to $\frac{\beta+2\gamma}{\delta}$, which also still lies in $\mathcal{N}_\theta$.

Finally, the orientation-reversing map t_3 sends the geodesic to the geodesic from $\frac{\gamma}{\alpha}$ to $\frac{\delta}{\beta}$, which as a non-oriented geodesic still lies in $\mathcal{N}_\theta$. By Lemma 11.8 the orientation is preserved in $\mathcal{N}_\theta$ under t_3, as $\alpha < \beta$. $\square$

REMARK 11.10. If $g \in \mathcal{N}_\theta$ in Lemma 11.9 connects $\frac{1}{n}$ and $\frac{1}{n+2}$ for some $n \in \mathbf{Z}$, then $t_3(g)$ is a geodesic connecting n and $n+2$, i.e. two integers. But geodesics connecting two integers in $\mathcal{N}_\theta$ are oriented with two opposing orientations – by Lemma 11.8, those are, in fact, the only geodesics in $\mathcal{N}_\theta$ which have opposing orientations. Hence, the oriented geodesic $t_3(g)$ does not have two opposing orientations and does not lie in $\mathcal{N}_\theta$.

REMARK 11.11. More generally, as $t_2(z) = T^2.z$ and $t_1(t_3(z)) = S.z$ and $\Gamma_\theta = \langle T^2, S\rangle$, the proof of Lemma 11.9 shows that the map arising from each matrix $A \in \Gamma_\theta$ preserves geodesics in $\mathcal{N}_\theta$ with the exception of the doubly-oriented geodesics connecting two integers and their preimages under A.

2.2. Intersection Numbers. Let g, h be any two oriented geodesics in $\mathbf{H}$. Define $\varphi_g(h)$ as follows:

$$\varphi_g(h) = \begin{cases} 0, & \text{if } g \cap h = \emptyset, \\ +1, & \text{if } g \text{ intersects } h \text{ on the right,} \\ -1, & \text{if } g \text{ intersects } h \text{ on the left.} \end{cases}$$

By "g intersects h from the right" we mean that if we follow the path of h, then the orientation of g at the intersection point is directed to the right. If the geodesic g is doubly-oriented (as we allowed in section 2.1), then we set $\varphi_g(h) = 1 - 1 = 0$.

We will define the Hardy sums as a signed intersection number with the net $\mathcal{N}_\theta$, which we defined in section 2.1. For a geodesic h, define the signed intersection number

$$I_\theta(h) = \sum_{g \in \mathcal{N}_\theta} \varphi_g(h), \tag{11.7}$$

whenever the sum exists.

We will denote an oriented geodesic h from two points $z_1, z_2 \in \mathbf{H} \cup \mathbf{Q} \cup \{i\infty\}$ by $h(z_1, z_2)$. To simplify notation, we will write $h_x = h(i\infty, x)$.

Let $\mathbf{Q}_\theta = \Gamma_\theta.i\infty$ be the set of rationals $\frac{a}{c}$ with $(a, c) = 1$ and such that $a + c$ is odd. For each $x \in \mathbf{Q}_\theta$, let $I_\theta(x) = I_\theta(h_x)$ be the signed intersection number of h_x with the net $\mathcal{N}_\theta$. We first need to show that the function $I_\theta(x)$ is well-defined.

Lemma 11.12. *For $x \in \mathbf{Q}_\theta$, the geodesic h_x only intersects finitely many geodesics in $\mathcal{N}_\theta$.*

Proof. Since the sign of intersection does not matter to prove the claim, we will ignore the orientation on the geodesics $\mathcal{N}_\theta$ for the remainder of this proof and refer to $\mathcal{N}_\theta$ as a set of non-oriented geodesics.

Suppose we have a geodesic in $\mathcal{N}_\theta$ connecting $0 < \frac{\alpha}{\gamma} < \frac{\beta}{\delta}$, which intersects h_x. The maximal radius of any semi-circular geodesic in $\mathcal{N}_\theta$ is 1, since the radius is given by

$$\frac{1}{2}\left(\frac{\beta}{\delta} - \frac{\alpha}{\gamma}\right) = \frac{1}{2}\frac{\beta\gamma - \alpha\delta}{\gamma\delta} = \frac{1}{\gamma\delta} \leq 1.$$

Consider the triangle Δ with endpoints -1, 1 and ∞. Since Δ is the boundary of a fundamental domain for Γ_θ, the orbit of Δ under Γ_θ gives a triangulation of the upper half plane $\mathbf{H}$. Denote the collection of geodesic boundaries of the triangulation in $\mathbf{H}$ by $\mathcal{N}$. In fact, the set $\mathcal{N}_\theta$ is contained in $\mathcal{N}$. We will prove the stronger claim that the geodesic h_x will intersect only finitely many geodesics of $\mathcal{N}$ in $\mathbf{H}$.

Any geodesic in $\mathcal{N}$ which does intersect h_x is one whose endpoints are such that one is to the left of x and one is to the right of x. Write $x = \frac{a}{c}$ with $(a, c) = 1$ and $a + c$ odd. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ be the unique matrix (up to sign) such that $\Delta_x = A.\Delta$ is a triangle, whose interior is intersected by h_x and has x as a vertex. Explicitly, the matrix $A = \pm\begin{pmatrix} a & c \\ c & d \end{pmatrix} \in \Gamma_\theta$ is determined by $c > 0$ and $-c < d < c$.

Let r_x be the greatest radius of a geodesic bounding Δ_x. Since $\Gamma_\theta.\Delta$ is a triangulation, the radius of a geodesic in $\mathcal{N}$ whose endpoints lie to the left and right of x is bounded from below by r_x: Any geodesic in $\mathcal{N}$, which is such that x lies in between its endpoints, with smaller radius than r_x would have to intersect the triangle Δ_x, which cannot happen.

Thus, for any geodesic from $\frac{\alpha}{\gamma}$ to $\frac{\beta}{\delta}$ in $\mathcal{N}$ intersecting h_x there is a lower bound on its radius $\frac{1}{\gamma\delta}$ (and a fortiori in $\mathcal{N}_\theta$). More precisely, we have

$$1 \leq \gamma\delta \leq \frac{1}{r_x}.$$

Hence, there are only finitely many possible geodesics in $\mathcal{N}$ intersecting h_x. □

Remark 11.13. For instance, the geodesic h_0 from $i\infty$ to 0 intersects only one geodesic in $\mathcal{N}_\theta$, the one connecting -1 and 1. In the notation of the proof of Lemma 11.12, the triangle Δ_0 is given by $S.\Delta$.

Now we prove that the function $I_\theta(x)$ actually satisfies the three properties defining the Hardy sums $S(a, c)$.

Theorem 11.14. *The signed intersection number $I_\theta : \mathbf{Q}_\theta \to \mathbf{Z}$ satisfies the following properties for $x \in \mathbf{Q}_\theta$:*

(1) *$I_\theta(-x) = -I_\theta(x)$,*
(2) *$I_\theta(x+2) = I_\theta(x)$,*
(3) *if $x \neq 0$, then $I_\theta(x) + I_\theta\left(\frac{1}{x}\right) = \mathrm{sign}(x)$.*

Moreover, the function I_θ is uniquely determined by the properties (1)-(3).

As an immediate corollary, we get Theorem 11.6, i.e., that the Hardy sums $\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$ can be realized as a signed intersection number.

Corollary 11.15. *For $A \in \Gamma_\theta$, we have*

$$\mathfrak{S}(A) = I_\theta(A.i\infty).$$

Proof of Theorem 11.14. We first show that $I_\theta(x)$ in (11.7) satisfies these properties. For the first property, note that reflection along the imaginary axis sends the geodesic h_x to h_{-x}. In particular, reflection does not change the orientation of h_x. However, reflection does change the orientation of the geodesics $g \in \mathcal{N}_\theta$ which intersect h_x. Since $\mathcal{N}_\theta$ is invariant under reflections along the imaginary axis by Lemma 11.9, we have $-I_\theta(x) = I_\theta(-x)$.

For the second property, we use that $\mathcal{N}_\theta$ is invariant under translations $z \mapsto z + 2$ – again, by Lemma 11.9 – and that $h_x \mapsto h_{x+2}$ under translation. Hence $I_\theta(x+2) = I_\theta(x)$.

For the third property, we may assume w.l.o.g. that $x \in (-1,1)$ by the second property. Let n_0 be an odd integer, such that $n_0 < \frac{1}{x} < n_0+2$. Let g_0 be the geodesic connecting $\frac{1}{n_0}$ and $\frac{1}{n_0+2}$.

In $\mathcal{N}_\theta$ the geodesic g_0 is oriented from $\frac{1}{n_0}$ to $\frac{1}{n_0+2}$ if n_0 is positive (i.e. $\operatorname{sign}(x) > 0$) and from $\frac{-1}{|n_0|-2}$ to $\frac{-1}{|n_0|}$ if n_0 is negative (i.e. if $\operatorname{sign}(x) < 0$). The geodesic g_1 from n_0 to n_0+2 has two opposing orientations in $\mathcal{N}_\theta$.

We write

$$I_\theta(h_x) = \varphi_{g_0}(h_x) + \sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x) = \operatorname{sign}(x) + \sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x). \tag{11.8}$$

We have $t_3(h_x) = h(0, 1/x)$ and by Lemma 11.9, we see that

$$\sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x) = - \sum_{g \in \mathcal{N}_\theta - \{g_1\}} \varphi_g(h(0,1/x)) = -I_\theta(h(0,1/x)),$$

since t_3 preserves the orientations on $\mathcal{N}_\theta$, except for geodesics connecting points $\frac{1}{n_0}$ and $\frac{1}{n_0+2}$ like g_0; see Remark 11.10. It should also be noted that the doubly-oriented geodesic connecting -1 and 1, which is intersected by both h_x and $h(0,1/x)$ gets mapped to itself under t_3.

Consider the triangle of oriented geodesics from h_0, $h(0,1/x)$ and $h(1/x, i\infty)$. It is clear that $I_\theta(h(1/x,i\infty)) = -I_\theta(h_{1/x})$ and that $I_\theta(h_0)+I_\theta(h(0,1/x))+I_\theta(h(1/x,i\infty)) = 0$. But by Remark 11.13, we have $I_\theta(h_0) = 0$, hence $I_\theta(h(0,1/x)) = I_\theta(h_{1/x})$, which proves the third property.

Let us now show uniqueness: Let $I_1(x)$ and $I_2(x)$ be two functions satisfying the three properties. From the first property, it follows that $I_1(0) = I_2(0) = 0$. By the second property the difference $I_3(x) = I_1(x) - I_2(x)$ is zero at every integer $x \in 2\mathbf{Z}$.

Suppose that $I_3(a/c)$ is zero for all $\frac{a}{c} \in \mathbf{Q}_\theta$, $(a,c) = 1$ with $1 \leq c < c'$. By what we just showed, this is true for $c' = 2$. We now show that the function I_3 also vanishes for $\frac{a'}{c'}$ with $(a',c') = 1$ and $a'+c'$ being odd, from which the claim will follow by induction. By the second property, we may assume w.l.o.g. that $-c' < a' < c'$. Using the third property, we see that $I_3(x) = -I_3\left(\frac{1}{x}\right)$. But

$$I_3\left(\frac{a'}{c'}\right) = -I_3\left(\frac{c'}{a'}\right) = 0$$

by our assumption (since $|a'| < c'$). $\square$

There is an analog for $\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Q}$ which is obtained as an intersection number by shifting the net $\mathcal{N}_\theta$ with $z \mapsto z+1$ and reversing the orientation of the geodesics. This follows from the fact $\mathfrak{S}(x+1) = -\mathfrak{S}_4(x)$.

2.3. The Rademacher Cocycle. We continue our investigation into the geometric nature of the function $\mathfrak{S}$. We will now show that it is in fact *a* splitting of the Rademacher cocycle (11.6) restricted to the subgroup Γ_θ. On the full modular group $\mathrm{SL}_2(\mathbf{Z})$, the Rademacher cocycle $w(A,B)$ in (11.6) splits to the Dedekind symbol (8.8), i.e. for $A, B \in \mathrm{SL}_2(\mathbf{Z})$ with $A = \begin{pmatrix} * & * \\ c_A & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ c_B & * \end{pmatrix}$, and $AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix}$, we have

$$\Phi(AB) - \Phi(A) - \Phi(B) = -3 \cdot \operatorname{sign}(c_A c_B c_{AB}). \tag{11.9}$$

REMARK 11.16. The sign function in (11.9) is defined such that $\operatorname{sign}(0) = 0$.

From property (3) of Theorem 11.14 (the reciprocity formula), it follows that

$$\mathfrak{S}(SA) - \mathfrak{S}(A) = \operatorname{sign}(ac) \tag{11.10}$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$, which, since $\mathfrak{S}(S) = 0$, proves Theorem 11.7 in the case $B = S$.

We now proceed to prove Theorem 11.7 through a series of proposition and lemmas.

LEMMA 11.17. *Let $A, B \in \Gamma_\theta$ and write $A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix}$, $B = \begin{pmatrix} a_B & b_B \\ c_B & d_B \end{pmatrix}$, and $AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix}$. We have*

$$\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = -\operatorname{sign}(c_A c_B c_{AB})$$

if and only if

$$\mathfrak{S}(A'B') - \mathfrak{S}(A') - \mathfrak{S}(B') = -\operatorname{sign}(c_{A'} c_{B'} c_{A'B'}),$$

where $A' = AS = \begin{pmatrix} a_{A'} & b_{A'} \\ c_{A'} & d_{A'} \end{pmatrix}$, $B' = SB = \begin{pmatrix} a_{B'} & b_{B'} \\ c_{B'} & d_{B'} \end{pmatrix}$ and $A'B' = \begin{pmatrix} * & * \\ c_{A'B'} & * \end{pmatrix}$.

PROOF. Due to the symmetry of the claim, it suffices to prove that if $\mathfrak{S}(A'B') - \mathfrak{S}(A') - \mathfrak{S}(B') = -\operatorname{sign}(c_{A'} c_{B'} c_{A'B'})$ holds, then $\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = -\operatorname{sign}(c_{AB} c_A c_B)$ holds as well.

Up to replacing A with $-A$ and B with $-B$ we may assume w.l.o.g. that $c_A, c_B > 0$. For the matrices A' and B' we have $A'B' = -AB$, hence $c_{A'B'} = -c_{AB}$, $c_{A'} = d_A$ and $c_{B'} = a_B$. By the reciprocity formula (11.10), we have

$$\mathfrak{S}(B') = \mathfrak{S}(B) - \operatorname{sign}(a_B c_B)$$

and

$$\mathfrak{S}(A') = \mathfrak{S}(A) - \operatorname{sign}(d_A c_A).$$

It hence follows that

$$\mathfrak{S}(-AB) - \mathfrak{S}(A') - \mathfrak{S}(B') = \mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) + \operatorname{sign}(c_A d_A) + \operatorname{sign}(a_B c_B).$$

If $\operatorname{sign}(c_{A'} c_{B'}) < 0$, then

$$\operatorname{sign}(c_A d_A) + \operatorname{sign}(a_B c_B) = \operatorname{sign}(c_{A'}) + \operatorname{sign}(c_{B'}) = 0,$$

thus

$$\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = \mathfrak{S}(A'B') - \mathfrak{S}(A') - \mathfrak{S}(B') = \operatorname{sign}(c_{A'B'}) = -\operatorname{sign}(c_{AB}).$$

If $\operatorname{sign}(c_{A'} c_{B'}) > 0$, then

$$\operatorname{sign}(c_A d_A) + \operatorname{sign}(a_B c_B) = \operatorname{sign}(c_{A'}) + \operatorname{sign}(c_{B'}) = 2 \operatorname{sign}(c_{A'}) = 2 \operatorname{sign}(c_{B'}).$$

However, as $-c_{A'B'} = c_{AB} = c_{A'} c_A + c_{B'} c_B$, we have $\operatorname{sign}(c_{A'B'}) = -\operatorname{sign}(c_{A'}) = -\operatorname{sign}(c_{B'})$. Thus,

$$\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = -\operatorname{sign}(c_{A'B'}) - 2 \operatorname{sign}(c_{A'}) = \operatorname{sign}(c_{A'B'}) = -\operatorname{sign}(c_{AB}).$$

□

With Lemma 11.17, we may directly prove Theorem 11.7 for the special case $B = A^{-1}$.

PROPOSITION 11.18. *For all $A \in \Gamma_\theta$, we have $\mathfrak{S}(A^{-1}) = -\mathfrak{S}(A)$.*

PROOF. Since $\mathfrak{S}(A)$ is invariant under multiplication on the right by T^2 and under multiplication by $-I$, we may assume w.l.o.g. that $A = T^{2n_1} S T^{2n_2} S \cdots T^{2n_r} S$ for integers $n_1, ..., n_r$ and $r \geq 1$. The proof is by induction on r.

Suppose $r = 1$. Then $\mathfrak{S}(T^{2n_1} S) = \mathfrak{S}(2n_1) = 0$ and $\mathfrak{S}(S^{-1} T^{-2n_1}) = \mathfrak{S}(0) = 0$ by Theorem 11.14, so the claim is true.

Suppose that $r > 1$ and the claim holds for $r - 1$. We have

$$\mathfrak{S}(A^{-1}) = \mathfrak{S}(S T^{-2n_r} \cdots T^{-2n_2} S T^{-2n_1}) = \mathfrak{S}(S T^{-2n_r} \cdots T^{-2n_2} S).$$

By the induction hypothesis we have

$$\mathfrak{S}((T^{2n_2} S \cdots T^{2n_r} S)^{-1}) = -\mathfrak{S}(T^{2n_2} S \cdots T^{2n_r} S),$$

which by Lemma 11.17 implies that

$$\mathfrak{S}((T^{2n_2} S \cdots T^{2n_r} S)^{-1} S) = \mathfrak{S}(S T^{-2n_r} \cdots T^{-2n_2} S) = -\mathfrak{S}(S T^{2n_2} S \cdots T^{2n_r} S).$$

Since $\mathfrak{S}(ST^{2n_2}S\cdots T^{2n_r}S) = \mathfrak{S}(T^{2n_1}ST^{2n_2}S\cdots T^{2n_r}S)$ by Theorem 11.14, the claim follows. $\square$

LEMMA 11.19. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ *and write* $A = \pm T^{2n_0}S\cdots T^{2n_r}ST^{2n_{r+1}}$ *for integers* $n_0, ..., n_{r+1}$ *and* $n_1, n_2, ..., n_r \neq 0$. *We have* $\frac{a}{c} \in (-1, 1)$ *if and only if* $n_0 = 0$.

PROOF. By induction on r. For $r = 0$, we have $\frac{a}{c} = T^{2n_0}S.i\infty = 2n_0$, for which the claim is obviously true.

Suppose $r > 0$ and that the claim holds for $1, ..., r-1$. Let

$$A = \pm T^{2n_0}ST^{2n_1}S\cdots T^{2n_r}ST^{2n_{r+1}} \quad \text{with } n_1, ..., n_r \neq 0;$$

since $S^2 = -I$, there is always such a representation. Let $\frac{a'}{c'} = T^{2n_1}S\cdots T^{2n_r}S.i\infty$, which by the induction hypothesis lies outside $(-1, 1)$. Then

$$\frac{a}{c} = T^{2n_0}ST^{2n_1}\cdots T^{2n_r}S.i\infty = T^{2n_0}S.\frac{a'}{c'} = -\frac{c'}{a'} + 2n_0$$

and, as $-\frac{c'}{a'} \in (-1, 1)$, it is easy to see that $\frac{a}{c} \in (-1, 1)$ if and only if $n_0 = 0$. $\square$

The next lemma proves Theorem 11.7 under certain extra conditions, to which the general case will be reduced.

LEMMA 11.20. *Let* $A, B \in \Gamma_\theta$ *and write* $A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix}$, $B = \begin{pmatrix} a_B & b_B \\ c_B & d_B \end{pmatrix}$, *and* $AB = \begin{pmatrix} a_{AB} & b_{AB} \\ c_{AB} & d_{AB} \end{pmatrix}$. *Suppose that* $-\frac{d_A}{c_A} \in (-1, 1)$, $\alpha = \frac{a_A}{c_A} \in (-1, 1)$, *and* $\beta = \frac{a_B}{c_B} \notin (-1, 1)$. *Then*

$$\mathfrak{S}(AB) - \mathfrak{S}(A) - \mathfrak{S}(B) = -\operatorname{sign}(c_A c_B c_{AB}).$$

PROOF. Upon replacing A by $-A$ and B by $-B$, we may assume w.l.o.g. that $c_A, c_B > 0$. Under these conditions $c_{AB} = c_A a_B + d_A c_B > 0$ if and only if $c_A\beta + d_A > 0$, i.e. $\beta > -\frac{d_A}{c_A}$.

Note that $\frac{a_{AB}}{c_{AB}} = A.\beta = \frac{a_A}{c_A} - \frac{1}{c_A(c_A\beta + d_A)}$. Since $|c_A\beta + d_A| \geq c_A|\beta| - |d_A| > |c_A| - |d_A| \geq 1$ by assumption, it follows that

$$\left|\frac{a_{AB}}{c_{AB}}\right| = \left|\frac{a_A}{c_A} - \frac{1}{c_A(c_A\beta + d_A)}\right| < \left|\frac{a_A}{c_A}\right| + \frac{1}{c_A} \leq 1,$$

since $|a_A| + 1 \leq |c_A|$.

Consider the geodesic $h = h(a_{AB}/c_{AB}, \alpha)$. The geodesic h is the image of $h(\beta, i\infty)$ under application of A. For the oriented triangle consisting of the edges $h(i\infty, a_{AB}/c_{AB})$, h and $h(\alpha, i\infty)$, we have

$$I_\theta(h(i\infty, a_{AB}/c_{AB})) + I_\theta(h) + I_\theta(h(\alpha, i\infty)) = 0.$$

Since $I_\theta(h(i\infty, a_{AB}/c_{AB})) = \mathfrak{S}(AB)$ and $I_\theta(h(\alpha, i\infty)) = -\mathfrak{S}(A)$, it suffices to prove that $I_\theta(h) = -\mathfrak{S}(B) + \operatorname{sign}(c_{AB})$.

By Remark 11.11, the sign of each signed intersection of $h(\beta, i\infty)$ with geodesics in $\mathcal{N}_\theta$ does not change under the application of A with the exception of the single doubly-oriented geodesics that $h(\beta, i\infty)$ intersects (as $h = A.h(\beta, i\infty)$ has both endpoints in $(-1, 1)$, the geodesic $h(\beta, i\infty)$ does not intersect any preimage of a geodesic connecting two integers under A). For the change of the signed intersection due to $h(\beta, i\infty)$ intersecting a doubly-oriented geodesic, we need to introduce a correction factor to $I_\theta(h(\beta, i\infty))$ to find $I_\theta(h)$.

Thus consider the geodesic $\tilde{g}$ from n_0 to $n_0 + 2$ with n_0 an odd integer such that $n_0 < \beta < n_0+2$, which is the only doubly-oriented geodesic in $\mathcal{N}_\theta$ that $h(\beta, i\infty)$ intersects.

Under the matrix A, the geodesic $\tilde{g}$ is mapped to the geodesic in $A.\tilde{g}$ connecting $\frac{a_A n_0 + b_A}{c_A n_0 + d_A}$ and $\frac{a_A(n_0+2)+b_A}{c_A(n_0+2)+d_A}$. To find the orientation of the geodesic $A.\tilde{g}$ in $\mathcal{N}_\theta$, we need to find out whether $|c_A n_0 + d_A| < |c_A n_0 + d_A + 2c_A|$ or the contrary holds. Since $c_A > 0$, this only depends on whether $c_A n_0 + d_A$ is negative or not.

Indeed, the linear function $\mathbf{R} \to \mathbf{R}$, $x \mapsto c_A x + d_A$ switches from negative to positive values at $x = -\frac{d_A}{c_A}$. But this can't happen in the interval $(n_0, n_0 + 2)$, as $-\frac{d_A}{c_A} \in (-1, 1)$ and $\beta \in (n_0, n_0 + 2)$ does not lie in $(-1, 1)$ by assumption.

Thus, the orientation of $A.\tilde{g}$ in $\mathcal{N}_\theta$ only depends on the sign of $c_A\beta + d_A = c_{AB}$. Namely, as $A.(n_0 + 2) - A.n_0 = \frac{1}{(c_A(n_0+2)+d_A)(c_A n_0+d_A)} > 0$, the orientation of $A.\tilde{g}$ is positive, if c_{AB} is negative, and vice versa. Note further that $A.\infty - A.x = \frac{1}{c_A(c_A x+d_A)}$ for any $x \in \mathbf{R}$. Since $c_A n_0 + d_A$ and $c_A(n_0 + 2) + d_A$ have the same sign, the numbers $\alpha - A.n_0$ and $\alpha - A.(n_0 + 2)$ have the same sign as well. In other words, α lies outside the interval bounded by $A.n_0$ and $A.(n_0 + 2)$ or the geodesic h from $\frac{a_{AB}}{c_{AB}}$ to α is oriented outwards of the geodesic $A.\tilde{g}$.

Now if the orientation is positive, then the geodesic h intersects $A.\tilde{g}$ with $A.\tilde{g}$ pointing to the left and if the orientation is negative, then h intersects $A.\tilde{g}$ with $A.\tilde{g}$ pointing to the right, i.e.

$$I_\theta(h) = \operatorname{sign}(c_{AB}) + I_\theta(h(\beta, i\infty)) = \operatorname{sign}(c_{AB}) - \mathfrak{S}(B).$$

$\square$

All cases that we need to consider to prove Theorem 11.7 are covered in the lemmas and propositions above. We are only left with reducing the general case to them.

PROOF OF THEOREM 11.7. Suppose $A \neq B^{-1}$; otherwise the claim follows by Proposition 11.18.

Write

$$A = \pm T^{2n_0} S T^{2n_1} S \cdots T^{2n_r} S T^{2n_{r+1}} \text{ and } B = \pm T^{2m_0} S T^{2m_1} S \cdots T^{2m_t} S T^{2m_{t+1}} \tag{11.11}$$

for $r, t \geq 0$ and $n_1, ..., n_r, m_1, ..., m_t \neq 0$. We may assume that $-\frac{d_A}{c_A} \in (-1, 1)$ and $\frac{a_A}{c_A} \in (-1, 1)$ by Lemma 11.19 (i.e. that $n_0 = n_{r+1} = 0$) upon replacing A with $T^{-2n_0} A T^{-2n_{r+1}}$ and B with $T^{2n_{r+1}} B$. Again, we may assume w.l.o.g. that $c_A, c_B > 0$.

If $|B.i\infty| > 1$ (i.e. $m_0 \neq 0$ in (11.11) by Lemma 11.19), the claim follows from Lemma 11.20. Suppose thus that $|B.i\infty| < 1$, i.e. that $B = \pm S T^{2m_1} S \cdots T^{2m_t} S T^{2m_{t+1}}$. We may now replace A by $A' = AS$ and B by $B' = SB$ as in Lemma 11.17.

Again, by replacing $A' = \pm S T^{2n_1} S \cdots S T^{2n_r}$ by $A' T^{-2n_r}$ and B' by

$$T^{2n_r} B' = \pm T^{2(m_1 + n_r)} S \cdots T^{2m_t} S T^{2m_{t+1}},$$

we may assume that $-\frac{d_{A'}}{c_{A'}} \in (-1, 1)$ and that $c_{A'}, c_{B'} > 0$. If $|B'.i\infty| > 1$, i.e. if $m_1 \neq -n_r$, we may use Lemma 11.20. Otherwise, repeat the argument until we reach $|B'.i\infty| > 1$.

This process will end, for if $r \neq t$, one of the matrices will at some point be equal to the identity, for which the claim is trivial. If $r = t$, we must have $-m_i \neq n_{t+1-i}$ for an $i = 0, ..., t+1$, since we excluded the case $A = B^{-1}$. □

COROLLARY 11.21. *Let $\Phi : \Gamma_\theta \to \mathbf{Z}$ be the classical Dedekind symbol restricted to the subgroup Γ_θ. For the map $\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$ we have*

$$3 \cdot \mathfrak{S} = \Phi - \chi_\theta,$$

where χ_θ is the unique homomorphism $\Gamma_\theta \to \mathbf{Z}$ given by $\chi_\theta(T^2) = 2$ and $\chi_\theta(S) = 0$.

PROOF. We have $\Phi(AB) - \Phi(A) - \Phi(B) = -\mathrm{sign}(c_A c_B c_{AB})$ by (11.9). Hence, $\chi_\theta = \Phi - 3 \cdot \mathfrak{S}$ is a group homomorphism by Theorem 11.7. The values of χ_θ at the generators of Γ_θ are obtained by $\Phi(T^2) = 2$, $\Phi(S) = 0$ and $\mathfrak{S}(T^2) = 0$, $\mathfrak{S}(S) = 0$. □

In fact, Corollary 11.21 is equivalent to Proposition 10.11, but we did not reference automorphic forms at all in our proof.

The same arguments work for the function $\mathfrak{S}_4$ as well. In particular, the function $\mathfrak{S}_4$ satisfies the generalized reciprocity law in Theorem 11.7.

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