Diss. Nr. 4835

Optimales Rangieren nach Turnieren

ABHANDLUNG

zur Erlangung der Würde eines Doktors der Mathematik
der
EIDGENÖSSISCHEN TECHNISCHEN HOCHSCHULE
ZÜRICH

vorgelegt von

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Angenommen auf Antrag von
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Juris Druck + Verlag Zürich
1972

Summary

The thesis grew out of the attempt to show the optimality (in a decision theoretical sense) of a ranking procedure after knockout tournaments which J. A. Hartigan proposed 1966 in his paper "Probability completion of a knockout tournament" (Ann. Math. Statist. 37, 495 - 503). (He ranks the players according to the mean of the ranking vectors $r = (r_1, r_2, \ldots, r_n)$ consistent with the acyclic tournament outcome, where $r_i = k$ means, that player number i has been assigned to the k-th place in the chosen ranking.)

The mathematical model therefore required a more rigorous definition of things like "game", "tournament plan" and "outcome of a tournament" than it has been done so far. Using graph theoretic language, a "tournament plan" $\mathcal{P}$ is interpreted as a mapping of the set $\mathcal{T}$ of all complete digraphs T (with the n players as vertices) into the set $\mathcal{D}$ of all digraphs D of order n, such that a) $\mathcal{P}(T) \subset T$ and b) $\mathcal{P}(T) \subset T^* \Longrightarrow \mathcal{P}(T^*) = \mathcal{P}(T)$ for all $T, T^* \in \mathcal{T}$. The image $\mathcal{P}(T)$ of $\mathcal{P}$ applied on T is called a "tournament outcome of $\mathcal{P}$ ". A plan $\overline{\mathcal{P}}$ is called "equivalent" to the given plan $\mathcal{P}$, if there exists a permutation of the players (vertices) which transforms all outcomes $\mathcal{P}(T)$, $T \in \mathcal{T}$, into the outcomes $\overline{\mathcal{P}}(T^*)$, $T^* \in \mathcal{T}$. A plan $\mathcal{P}$ is called "simple" if all possible outcomes of $\mathcal{P}$ are isomorphic; they then are acyclic too. (E. g. symmetric knockout plans on 2^m players are equivalent and simple.) There exist relations between the number of different plans equivalent to a simple plan $\mathcal{P}$ and the number of orders consistent with an outcome of $\mathcal{P}$.

The model describing the ranking problem is the following: One assumes a) that out of a class K of equivalent plans with only acyclic outcomes a plan $\mathcal{P}$ is chosen at random, b) that the relative "strength" of the players is given by an unknown ranking vector r and finally c) that the outcome $F = \mathcal{P}(T_r)$, where T_r is the transitive complete digraph representing r, can be observed.

The decision problem then is determined by:

$S_n := \{r\}$, space of all ranking vectors r : parameter space,

$\overline{\mathcal{F}} := \{ F = \mathcal{P}(T_r) \; ; \; \mathcal{P} \in K, \; r \in S_n \}$, space of the outcomes : sample space,

$S_n =: \{d\}$: decision space and the loss function $L(d,r)$, $d,r \in S_n$, with $L : S_n \times S_n \longrightarrow \mathbb{R}_1^+ \cup \{0\}$.

Given a prior distribution $p := (p_r \; ; \; r \in S_n)$ and an outcome F with $R(F)$ the set of all ranking vectors consistent with F , a ranking vector d* is a Bayes solution iff

$$\sum_{r \in R(F)} L(d^*,r)\, p_r \leq \sum_{r \in R(F)} L(d,r)\, p_r \quad \text{for all } d \in S_n .$$

Making two obvious assumptions on L concerning monotony and invariance, one can show that $d^* \in R(F)$. If L furthermore is separable i. e. $L : (d,r) \longmapsto \sum_{i=1}^{n} f(d_i, r_i)$ then there exist "Branch and Bound" procedures for the search of d* and finally if the matrix (f) is of a special form, d* can be found by means of a generalisation of Hartigan's procedure (e. g. if $f(i,j) := v(i)(i - j)$, v monotonically increasing, then d* is the "Hartigan solution" itself.)

A class of loss functions, different frome the separable one, leads to a generalisation and unification of procedures which are already well known in order to rank the players in Round Robin tournaments as e. g. the "row sum"-procedure and Slater's principle of "minimum inconsistencies".

Finally an approach is made to find an optimum "seeding" of the players (i. e. an optimum choice of a plan $\mathcal{P}$ out of a given class K) if prior information on the players' strength is available and a ranking procedure is prescribed.