Diss. ETH No. 13708

Rigorous bounds for vertex corrections On the conjecture named "Migdal's Theorem"

A dissertation submitted to the

SWISS FEDERAL INSTITUTE OF TECHNOLOGY

ZURICH

for the degree of

Doctor in Mathematics

presented by

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2000

He spent a couple of years on this project
Now he nows how an electron behaves

— The Nits *Mountain Jan* (1981)

Contents

Summary

The basis of the modern physical description of metals is formed by electron-phonon theory, which describes the interaction of the electrons with oscillations of the ion lattice structure, or sound waves. In electron-phonon theory the standard method of computing physical quantities by means of a perturbative expansion in the coupling constant breaks down because the coupling constant is typically not small.

Having to take all higher-order terms into account makes the calculations very difficult. Therefore it is common to make an approximation first proposed by A.B. Migdal in 1958. It consists of leaving out the higher-order contributions to the interaction vertex. This greatly simplifies solving the main equations of theory. Migdal justified this approximation by claiming that, at zero temperature, the higher order contributions vanish linearly in the sound velocity, c. For ordinary metals, this velocity is small relative to the other relevant parameters. This claim has become known as "Migdal's Theorem" although, to our knowledge, no rigorous proof has ever been published. Migdal gave a sketchy argument for the lowest order correction, the "one-loop correction", and then claimed that higher orders would work the same. Other authors have followed him in doing so and have extended the claim to non-zero temperature.

In this thesis the most simple form of electron-phonon theory, the Jellium model, is considered as a statistical quantum field theory at finite temperature in the presence of an ultra-violet cut-off. A rigorous bound is found for the one-loop correction that is indeed $O(c)$ except for a correction term which vanishes along with the temperature. This done using very explicit calculations using a Feynman-trick and repeated integration by parts.

Proper formulation of the theory so that the zero-temperature limit exists requires renormalization of the theory. Here, a so called Fermi-surface renormalization is done, where counter terms are added to the band relation. Using renormalization group techniques and a scale decomposition argument these counter terms are define precisely and it is shown that the zero temperature limit of the renormalized theory is well defined by establishing temperature independent bounds for the values of graphs occurring in the theory.

Finally it is shown that for the renormalized theory, for all $0 < \epsilon < 1$ the vertex corrections of order r are bounded by

$$M_r(\epsilon)\left\{c^{1-\epsilon} + \left(\frac{(\log\beta+1)^2}{\beta}\right)^{1-\epsilon}\right\}$$

for some ϵ-dependent constant $M_r(\epsilon)$, with β the inverse temperature. This is done by combining the Feynman-trick and the integration by parts with the scale decomposition.

Acknowledgements

Before embarking on the road towards Migdal's theorem, I would like to thank a few people who have made this work possible. First of all my advisor, Prof. Manfred Salmhofer, for his patience in explaining things when I was being dense yet again, Profs Knörrer and Mislin for being part of my examining committee, and Dr. Detlef Lehmann and Prof. Eugene Trubowitz for helpful discussions.

I would also like to thank some of my fellow "Gruppe 5" assistants over the years: Andrea, Simon, Philipp, Urs, Etienne, Lorenzo, Meike (I know, but 6 is close to 5), Ralf, Tancredi and honorary member Touraj for helping me pretend to the outer world that I actually still know all my undergraduate maths and for sometimes more than the occasional beer. Keep them computer questions comin'.

A warm word of thanks to my parents for allowing me to study in complete freedom and backing me all the way, to Merijn and Rosa for their love, to my grand mother for here prayers and to my "surprise surprise" extra family in far-away Kootwijkerbroek for their support. Last but not least to my own blond godess, Talitha, for being my 'meisje' and always bringing a smile on my face and for overtaking me just so that I could see how it is done.

Chapter 1

Introduction

1.1 Migdal-Eliashberg theory

1.1.1 The electron-phonon model

Free phonons and polarons

One of the main insights of the physical theories that describe superconductivity is that, at least for ordinary superconductor, the resistanceless current flow at low temperatures arises because of the interplay between the electrons and lattice vibrations, or sound waves. The microscopical, i.e., non-phenomenological, theory that describes these phenomena is called Electron-Phonon theory [Mah81, AGD75, FW71].

The term Electron-Phonon theory is used to refer to a whole class of physical models that have common features. In a metal most electrons in the metal are tightly bound to the nuclei. Together they form positively charged ions. These ions are arranged in a lattice and oscillate only small amounts around their equilibrium positions (at least at low temperatures). A standard computation shows that these deviations from the equilibrium positions can be viewed as a superposition of independent longitudinal density (sound) waves, or *phonons*. These are described by a simple Hamiltonian:

$$H_{\text{ph}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} (b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + \frac{1}{2}) \tag{1.1}$$

where $b^\dagger_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are bosonic creation and annihilation operators for a phonon with wave vector $\mathbf{k}$. $\omega_{\mathbf{k}} = c|\mathbf{k}|$ is the energy of one photon with wave vector $\mathbf{k}$. The proportionality constant c is the speed of sound in the metal. In a finite volume of wavelength of side length L, the components of $\mathbf{k}$ are integer multiple of $\frac{2\pi}{L}$. In the infinite volume limit the sum over the momenta changes into an integral.

Moreover above a certain value of the length of the wave vector the vibrations are inhibited by the crystal structure. Therefore the sum is often considered to be cutoff such that $\omega_{\mathbf{k}} < \omega_D$. The frequency ω_D is called the Debeye frequency. Note that in this model there is no (direct) coupling between the phonons. They form a system of free bosons.

When the ions are all at their equilibrium positions they provide a periodic positively charged background for the remaining more loosely coupled electrons to move in. These interact with each other and with the ions through coulomb forces. However, because of the positively charged background, the inter-electron forces are heavily screened and can be neglected as a first approximation. This simplifies the model extremely, because there is no direct interaction between the electrons and these thus form a system of free fermions. The only interaction between them is indirectly through coupling with the phonons. Such particles are also called 'polarons', however we will most of the time refer to them as electrons or even plain 'fermions'.

Besides neglecting the screened coulomb interactions there is a further simplification we make here in that we assume that also the periodic potential can be neglected. The effect of the ions is assumed to be smeared out evenly over space, and the conduction electrons form a homogeneous gas. This model is called the Jellium model. The kinetic part of the electron Hamiltonian then only contains a Laplacian. It is easily diagonalized in momentum space and given by

$$H_{\text{el}} = \sum_{,k\sigma} e_0(\mathbf{k}) a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \tag{1.2}$$

Here $e_0(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m}$ is the band-relation of the Jellium model, with m the electron mass. Note again that in infinite volume the sum is replaced by an integral. The index σ denotes the spin. As in the rest of the discussion all terms will be diagonal in the spin, we drop the index from the notation (or, equivalently, absorb it in the momentum sum).

A lot of the proofs in this thesis depend directly on the properties of the band relation, such as whether its level surfaces form manifolds, and when yes, how smooth these are. All these questions get relatively simple answers for the Jellium model. In addition as we will see, the rotational symmetry leads to substantial technical simplifications. In more realistic models $e_0(\mathbf{k})$ is replaced by a much more complicated band relation (or even multiple bands).

In addition the periodic potential (and the lattice structure) implies that the momentum space is not $\mathbb{R}^d$ but is a d-dimensional torus. Nevertheless we feel that the Jellium model, apart from being a widely used model in itself, still illustrates the major features of the proof that would be needed for more general models. For an example of what would be involved in such a generalization, see [FST96, FST98, FST99].

So far we have neglected anharmonic effects leading to coupling between phonons, the coulomb repulsion between electrons, and the effect of the periodic potential produced by the ions on the lattice. What remains is the effect of disturbances in the ion-potential produced by deviations of the ions from their equilibrium positions at the lattice sites. This leads to a coupling by coulomb forces between the lattice vibrations, i.e. phonons, and the electrons. The interaction part of the Hamiltonian is given by

$$H_{\text{int}} = \sum_{\mathbf{p},\mathbf{q}} M(\mathbf{q}) a^{\dagger}_{\mathbf{p}+\mathbf{q}} a_{\mathbf{p}} (b_{\mathbf{q}} + b^{\dagger}_{-\mathbf{q}}) \tag{1.3}$$

The interaction matrix element is given by $M(\mathbf{q})^2 = g^2 \omega_{\mathbf{q}}$, where $g^2 = \frac{2\pi^2\eta}{mk_F}$ with η a dimensionless parameter that turns out to be ~ 1 for most metals.

Summarizing, we have for our model the Hamiltonian

$$H = H_0 + H_{\text{int}} \tag{1.4}$$

with

$$H_0 = H_{\text{el}} + H_{\text{ph}} \tag{1.5}$$

describing the free propagation of the particles.

1.1.2 Statistical quantum field theory

Grand canonical averages and Matsubara's method

We are interested in computing statistical mechanical properties of the metal in the above model. That is, for some operator O we are interested in its ensemble average $\langle O \rangle$, also called the expectation value of O. Let N be the total number operator, given by

$$N = \sum_{\mathbf{k}} n(\mathbf{k}) \quad n(\mathbf{k}) = a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \tag{1.6}$$

then in the grand canonical ensemble each state is weighted by the value of $e^{-\beta(H-\mu N)}$ in the state. Here $\beta = \frac{1}{k_B T}$ is the inverse temperature. The chemical potential μ regulates the number of particles in the system. Thus

$$\langle O \rangle = \frac{1}{Z} \operatorname{Tr} e^{-\beta(H-\mu N)} O \tag{1.7}$$

Here $Z = \operatorname{Tr} e^{-\beta(H-\mu N)}$ is the grand canonical partition function and the trace is over all states in the Fock space. Z^{-1} normalizes the average (1.7) such that $\langle 1 \rangle = 1$. Note that both traces are only rigorously defined and finite when the momenta are restricted a bounded set, a so-called ultra-violet cut-off and we will do so below.

These ensemble averages are typically computed using a method due to Matsubara. The electron and phonon field operators in the Heisenberg representation are given by

$$A^\dagger(\mathbf{k}, \tau) = e^{\tau(H-\mu N)} a^\dagger_{\mathbf{k}} e^{-\tau(H-\mu N)} \tag{1.8}$$

$$A(\mathbf{k}, \tau) = e^{\tau(H-\mu N)} a_{\mathbf{k}} e^{-\tau(H-\mu N)} \tag{1.9}$$

$$B^\dagger(\mathbf{k}, \tau) = e^{\tau(H-\mu N)} \sqrt{\frac{\omega(\mathbf{k})}{2}} (b_{\mathbf{k}} + b^\dagger_{-\mathbf{k}}) e^{-\tau(H-\mu N)} \tag{1.10}$$

for $\tau \in [-\beta, \beta]$. Normal time evolution for an operator O would be given by $O(t) = e^{-itH} O e^{itH}$. This is related to the Heisenberg evolution in our case by a substitution $t = i\tau$. Therefore this formalism is often said to be using "imaginary time".

It is customary to introduce the scaling factor $\sqrt{\frac{\omega(\mathbf{k})}{2}}$. This has the advantage that in the interaction term (1.3) the coupling is now simply the constant g.

Two-point functions

A well known result is that all physically interesting statical mechanical quantities can be expressed in terms of correlation functions, in particular the two point correlation functions. In momentum space they are given by (after removing the delta function that sets the two momenta equal which is the result of translation invariance of the model)

$$\mathscr{G}(\mathbf{k}, \tau) = -\langle T_\tau A(\mathbf{k}, \tau) A^\dagger(\mathbf{k}, 0) \rangle \tag{1.11}$$

$$\mathscr{D}(\mathbf{k}, \tau) = -\langle T_\tau B(\mathbf{k}, \tau) B(-\mathbf{k}, 0) \rangle \tag{1.12}$$

Here T_τ is the time ordering operator, which puts any product of operators that are evaluated at τ_1, τ_2, etc. in such an order that the $\tau_{\pi(i)}$'s are decreasing. Here π is the required permutation. In addition if performing the permutation requires interchanging fermionic operators a factor -1 for each pair interchange occurs. These functions are also called Green's functions, propagators or covariances depending on the viewpoint one looks at the theory and we will use all these terms in the rest of the text.

As an example note that from the definition of T_τ it follows that

$$\lim_{\tau\uparrow 0}\mathscr{G}(\mathbf{k},\tau) = \langle A^\dagger(\mathbf{k},0)A(\mathbf{k},0)\rangle \tag{1.13}$$

$$= \langle n(\mathbf{k})\rangle \tag{1.14}$$

Because of the properties of T_τ a little operator algebra shows that for $-\beta \le \tau < 0$

$$\begin{aligned}\mathscr{G}(\mathbf{k},\tau) &= -\mathscr{G}(\mathbf{k},\tau+\beta)\\ \mathscr{D}(\mathbf{k},\tau) &= \mathscr{G}(\mathbf{k},\tau+\beta)\end{aligned} \tag{1.15}$$

When considering the Fourier transform with respect to τ

$$\mathscr{G}(k) = \mathscr{G}(k_0,\mathbf{k}) = \int_0^\beta \mathscr{G}(\mathbf{k},\tau)e^{ik_0\tau}\mathrm{d}\tau \quad \text{with } k_0 = \frac{(2n+1)\pi}{\beta},\ n\in\mathbb{Z} \tag{1.16}$$

$$\mathscr{D}(k) = \mathscr{D}(k_0,\mathbf{k}) = \int_0^\beta \mathscr{D}(\mathbf{k},\tau)e^{ik_0\tau}\mathrm{d}\tau \quad \text{with } k_0 = \frac{n\pi}{\beta},\ n\in\mathbb{Z} \tag{1.17}$$

Thus sums over the frequency argument k_0 either only the *fermionic Matsubara frequencies* $\frac{(2n+1)\pi}{\beta}$ or the *bosonic Matsubara frequencies* $\frac{2n\pi}{\beta}$ occur. This is a direct result of the different signs in (1.15) which reflects the fact that the fermion fields obey anticommutation relations and the boson fields commutation relations.

The interaction representation and the free propagators

The ensemble average $< O >$ is in general very difficult to compute exactly, even in a relatively simple model such as ours. Therefore one often resorts to perturbation theory. Observe that our Hamiltonian is of the form $H = H_0 + H_{\text{int}}$. Define the fields in the interaction representation as

$$\begin{aligned}\psi^\dagger(\mathbf{k},\tau) &= e^{\tau(H_0-\mu N)}a^\dagger_{\mathbf{k}}e^{-\tau(H_0-\mu N)}\\ \psi(\mathbf{k},\tau) &= e^{\tau(H_0-\mu N)}a_{\mathbf{k}}e^{-\tau(H_0-\mu N)}\\ \phi(\mathbf{k},\tau) &= e^{\tau(H_0-\mu N)}\sqrt{\frac{\omega(\mathbf{k})}{2}}(b_{\mathbf{k}}+b^\dagger_{-\mathbf{k}})e^{-\tau(H_0-\mu N)}\end{aligned} \tag{1.18}$$

Definine the free field average $\langle O\rangle_0$ of an operator O as

$$\langle O\rangle_0 = \frac{1}{Z_0}\,\mathrm{Tr}\, e^{-\beta(H_0-\mu N)}O \tag{1.19}$$

with $Z_0 = \mathrm{Tr}\, e^{-\beta(H_0-\mu N)}$. Then the two-point function is given by

$$\mathscr{G}(\mathbf{k},\tau) = -\frac{\langle T_\tau S(\beta)\psi(\mathbf{k},\tau)\psi^\dagger(\mathbf{k},0)\rangle_0}{\langle S(\beta)\rangle_0} \tag{1.20}$$

$$\mathscr{D}(\mathbf{k},\tau) = -\frac{\langle T_\tau S(\beta)\phi(\mathbf{k},\tau)\phi(-\mathbf{k},0)\rangle_0}{\langle S(\beta)\rangle_0} \tag{1.21}$$

Here $S(\tau)$ is the so called S-matrix operator given by

$$S(\tau') = e^{\tau'(H_0-\mu N)}e^{-\tau'(H-\mu N)} \tag{1.22}$$

$$= T_\tau \exp\left(\int_0^{\tau'} H_{\mathrm{int}}(\tau)\mathrm{d}\tau\right) \tag{1.23}$$

where $H_{\mathrm{int}}(\tau) = e^{\tau(H_0-\mu N)}H_{\mathrm{int}}e^{-\tau(H_0-\mu N)}$ and (1.23) can be obtained by solving the propagation equation for $S(\tau)$ iteratively.

Up to now the expressions for the 2-point functions have only been rewritten. They have not been simplified. In the special case $g = 0$ or equivalently $H_{\mathrm{int}} = 0$, $H = H_0$, and $S(\beta) = 1$ and the two-point functions can be computed. Because there is no interaction term between the particles, this is called the free theory. The computation of the 2-point functions in this case is standard. We have

$$C(k) = C(k_0,\mathbf{k}) = -\int_0^\beta \mathrm{d}\tau e^{ik_0\tau}\langle T_\tau\psi(\mathbf{k},\tau)\psi^\dagger(\mathbf{k},0)\rangle_0 \tag{1.24}$$

$$= -\int_0^\beta \mathrm{d}\tau e^{ik_0\tau}\frac{\mathrm{Tr}\, e^{-\beta(H_el-\mu N)}T_\tau\psi(\mathbf{k},\tau)\psi^\dagger(\mathbf{k},0)}{\mathrm{Tr}\, e^{-\beta(H_el-\mu N)}} \tag{1.25}$$

$$= -\int_0^\beta \mathrm{d}\tau e^{ik_0\tau}\frac{\mathrm{Tr}\, e^{-\beta(\sum_{\mathbf{k}} e(\mathbf{k})n(\mathbf{k}))}T_\tau\psi(\mathbf{k},\tau)\psi^\dagger(\mathbf{k},0)}{\mathrm{Tr}\, e^{-\beta(\sum_{\mathbf{k}} e(\mathbf{k})n(\mathbf{k}))}} \tag{1.26}$$

$$= \lim_{t\downarrow 0}\frac{e^{ik_0 t}}{ik_0 - e(\mathbf{k})} \tag{1.27}$$

where we have set $e(\mathbf{k}) = e_0(\mathbf{k}) - \mu = \frac{|\mathbf{k}|^2}{2m} - \mu$. The right hand side of (1.27) is a distribution. The t-limit gives a recipe to compute the inverse fourier transform.

Because we introduce an UV cutoff, this will turn out to be irrelevant in all computations done below, so we do not write it anymore. Similarly

$$D(k) = D(k_0, \omega(\mathbf{k})) = -\int_0^\beta d\tau e^{ik_0\tau} \langle T_\tau \phi(\mathbf{k}, \tau)\phi(-\mathbf{k}, 0)\rangle_0 \tag{1.28}$$

$$= -\frac{\omega(\mathbf{k})^2}{k_0^2 + \omega(\mathbf{k})^2} \tag{1.29}$$

$C(k)$ and $D(k)$ are called the *free propagators*.

At this point it is possible to proceed by expanding the exponent in (1.23) as a (formal) power series in g. This gives a power series expansion for the 2-point functions. They obviously start with the free propagators. In the higher order terms temperature ordered products of multiple fields appear in $\langle\cdot\rangle_0$. These ensemble averages of temperature ordered products are then expanded (in the thermodynamic limit) in sums of integrals over the free propagators using a theorem by Wick and Matsubara. See [AGD75, Mah81, FW71]. Rather than this 'operator formalism' we will follow an approach using functional integrals. This is done mostly because it allows an easier formulation of the renormalization procedure, which is necessary to remove infrared divergences in the expansion of the operator above.

1.1.3 Functional integrals

Some notation

The frequency sums are written as (for a function $f(x)$ of the fermionic frequency x)

$$\int_a^b d_\beta f(x) = \frac{2\pi}{\beta} \sum_{\substack{n\in\mathbb{Z} \\ \frac{\pi}{\beta}(2n+1)\in[a,b]}} f(\frac{\pi}{\beta}(2n+1)) \tag{1.30}$$

This notation emphasizes our interest in the zero temperature limit, $\beta \to \infty$ where

$$\int_{-\infty}^{\infty} d_\beta x f(x) \longrightarrow \int_{-\infty}^{\infty} dx f(x) \tag{1.31}$$

At this point we will take the thermodynamic limit, i.e. $V \to \infty$. In the infinite volume limit the Fourier sums are replaced by integrals

$$\frac{1}{V}\sum_{\mathbf{k}} f(\mathbf{k}) \longrightarrow \int \frac{d\mathbf{k}}{(2\pi)^d} f(\mathbf{k}) \tag{1.32}$$

In a more rigorous introduction of the functional integrals one should keep the volume finite and also discretize coordinate space. However we choose not to do so here because it for our purposes it suffices to introduce them formally and simply assume their algebraic properties.

Generating functional

The main observation of the functional integral approach is to observe that the trace in the definition of expectation value can be evaluated using the following functional integrals. The grand canonical partition function is given by the Feynman-Kac formula

$$Z = \int d\mu_C(\bar{\psi}, \psi) d\mu_D(\phi) e^{-g\mathcal{V}_0} \tag{1.33}$$

Here $d\mu_C$ and $d\mu_D$ are Gaussian "measures". Here the term measure is used loosely because for the fermions the "integral with respect to $d\mu_C$ is simply a notation for a linear functional that has properties similar to integration. Formally they are given by

$$d\mu_C(\bar{\psi}, \psi) = \prod_k d\psi(k) \prod_p d\bar{\psi}(p) e^{-(\bar{\psi}, C^{-1}\psi)} \tag{1.34}$$

$$d\mu_D(\phi) = \prod_k d\phi(k) e^{-\frac{1}{2}(\phi, D^{-1}\phi)} \tag{1.35}$$

$$(f, g) = \int \frac{d_\beta k_0}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} f(k) g(k) \tag{1.36}$$

with C and D the free propagators given by (1.24) and (1.28)

Here the anticommuting (Grassmannian/fermionic) functions $\bar{\psi}$ and ψ and the commuting (bosonic) field ϕ take the role of the field operators $\psi^\dagger$,ψ and ϕ of (1.18). The quadratic parts of the Hamiltonian, i.e. the free theory, have been absorbed in the measures. $\mathcal{V}_0$ represents the interaction. For our model it is given by

$$\mathcal{V}_0(\bar{\psi}, \psi, \phi) = \int \frac{d_\beta q_0}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d_\beta p_0}{2\pi} \int \frac{d\mathbf{p}}{(2\pi)^d} \phi(q) \bar{\psi}(p-q) \psi(-p) \tag{1.37}$$

or equivalently

$$\begin{aligned} &\mathcal{V}_0(\bar{\psi}, \psi, \phi) \\ &= \int \frac{d_\beta q_0}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \int \frac{d_\beta p_0}{2\pi} \int \frac{d\mathbf{p}}{(2\pi)^d} \int \frac{d_\beta k_0}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} \phi(q) \bar{\psi}(k) \psi(p) \delta(k+q+p) \end{aligned} \tag{1.38}$$

For a function $O(\bar{\psi}, \psi, \phi)$ of the fields, the expectation value is given by

$$\langle O \rangle = \int d\mu_C(\bar{\psi}, \psi) d\mu_D(\phi) e^{-g\mathcal{V}_0} O(\bar{\psi}, \psi, \phi) \tag{1.39}$$

Notice the correspondence with the interaction representation. However the time ordering operator that was contained in (1.23) is missing. This is one of the main advantages of this formulation of the theory.

These formula's are proven by expanding the exponentials in the trace using Trotter's product formula. The operator fields turn into functions by projecting them on states. To give a rigorous proof and showing that the measures exist requires more care (such as discretizing the theory and being careful about when to take the continuum limit). We refer to [Sim79, Sal99] for the proof.

For our purposes it will be sufficient to see that the 'characteristic functions' of the gaussian measures are given by

$$\int \mathrm{d}\mu_C(\bar{\psi},\psi) e^{(\bar{\xi},\psi)+(\bar{\psi},\xi)} = e^{(\bar{\xi},C\xi)} \tag{1.40}$$

$$\int \mathrm{d}\mu_D(\phi) e^{(J,\phi)} = e^{\frac{1}{2}(J,DJ)} \tag{1.41}$$

The fermionic fields $\bar{\xi}, \xi$ and the bosonic field J are called source fields. In particular we see that

$$\langle\bar{\psi}(k_1),\psi(k_2)\rangle\big|_{g=o} = \int \mathrm{d}\mu_C \bar{\psi}(k_1)\psi(k_2) \tag{1.42}$$

$$= \int \mathrm{d}\mu_C(\bar{\psi},\psi) \frac{-\delta}{\delta\xi(k_1)} \frac{\delta}{\delta\bar{\xi}(k_2)} e^{(\bar{\xi},\psi)+(\bar{\psi},\xi)}\Big|_{\bar{\xi}=\xi=0} \tag{1.43}$$

$$= \frac{-\delta}{\delta\xi(k_1)} \frac{\delta}{\delta\bar{\xi}(k_2)} e^{(\bar{\xi},C\xi)}\Big|_{\bar{\xi}=\xi=0} \tag{1.44}$$

$$= C(k_1)\delta(k_1 - k_2) \tag{1.45}$$

Thus the covariances C and D are indeed exactly the free propagators given above.

The same argument gives that for

$$Z(\bar{\xi},\xi,J) = \int \mathrm{d}\mu_C(\bar{\psi},\psi)\mathrm{d}\mu_D(\phi) e^{-gV_0} e^{(\bar{\xi},\psi)+(\bar{\psi},\xi)+(J,\phi)} \tag{1.46}$$

the functional $Z(\bar{\xi},\xi,J)/Z(0,0,0)$ is the generating function for expectation values of polynomials of $\bar{\xi}(k_i)$'s, $\xi(k_i)$'s, and $J(k_i)$'s. It therefore contains all the properties of the model. Note that $Z(0,0,0) = Z$.

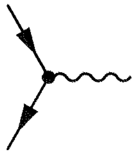
Feynman graphs

$Z(\bar{\xi},\xi,J)$ is evaluated by expanding $e^{-g\mathcal{V}_0}$ as a formal power series in g using the technique mentioned before. Formally

$$Z(\bar{\xi},\xi,J)=\sum_{r=0}^{\infty}\frac{g^r}{r!}\Big(-\mathcal{V}_0(\frac{-\delta}{\delta\xi},\frac{\delta}{\delta\bar{\xi}},\frac{\delta}{\delta J})\Big)^r\int \mathrm{d}\mu_C(\bar{\psi},\psi)\mathrm{d}\mu_D(\phi)e^{(\bar{\xi},\psi)+(\bar{\psi},\xi)+(J,\phi)} \tag{1.47}$$

$$=\sum_{r=0}^{\infty}\frac{g^r}{r!}\Big(-\mathcal{V}_0(\frac{-\delta}{\delta\xi},\frac{\delta}{\delta\bar{\xi}},\frac{\delta}{\delta J})\Big)^r e^{(\bar{\xi},C\xi)}e^{\frac{1}{2}(J,DJ)} \tag{1.48}$$

Evaluating the derivatives of the exponentials on the right of (1.48) requires some book keeping and combinatorics. This is done by means of *Feynman graphs*. The idea is to represent each of the r factors $\mathcal{V}_0$ with a vertex with three attached lines; Because $\mathcal{V}_0$ is given by (1.38) each type of derivative occurs once and is represented by a line attached to the vertex.



As the derivatives act on the exponentials a natural pairing is made between vertices. Taking a derivative $\frac{-\delta}{\delta\xi}$ produces a factor $C(k)\bar{\xi}(k)$. When a derivate with respect to $\bar{\xi}$ from another vertex hits this factor, draw an arrow between the two and associate a factor $C(k)$ with the line. Do the same for derivatives with respect to J but draw a wavy line and associate a phonon propagator D with it. Some lines can remain unconnected and for each such line there is a factor DJ, $C\bar{\psi}$, or $C\psi$. The vertices to which these lines belong are called external vertices. The product of the derivatives can now be expanded into a sum of the terms corresponding to graphs, where the sum is taken over all graphs. Each such term will be a monomial in the source fields. The type of the monomial corresponding to a graph G is given by the external lines of the graph G. The coefficient of a monomial $\bar{\xi}(k_1)\dots J(k_n)$ is of the form $\mathrm{Val}(G)(k_1,\dots,k_{n-1})\delta(k_1+\dots+k_n)$. The function $\mathrm{Val}(G)$ is called the *value of G*. The beauty of this method is that $\mathrm{Val}(G)$ can be constructed from G using a simple recipe called the *Feynman rules*. For details see for instance [Sal99].

It turns out that the effective action

$$G(\bar{\xi},\xi,J,-g\mathcal{V})=\log\int \mathrm{d}\mu_C(\bar{\psi},\psi)\mathrm{d}\mu_D(\phi)e^{-g\mathcal{V}(\bar{\psi}+\bar{\xi},\psi+\xi,\phi+J)} \tag{1.49}$$

is a more convenient object to study. After a shift in the integration variables and using the fact that the integrals are Gaussian

$$G(\bar{\xi}, \xi, J, -g\mathcal{V}_0) = -(\bar{\xi}, C^{-1}\xi) - \frac{1}{2}(J, D^{-1}J) + \log Z(C^{-1}\bar{\xi}, C^{-1}\xi, D^{-1}J) \tag{1.50}$$

G is the generating functional for the amputated connected Green's functions. The term 'amputated' comes from the fact that taking derivatives with respect to the source fields produces factors $C^{-1}(k_i)$ and $D^{-1}(k_i)$. These cancel the pre-factors from the DJ, $C\bar{\psi}$, and $C\psi$ that belonged to the external lines. The term 'connected' comes from the fact that there is also a Feynman graph expansion for G but where only connected graphs occur[Sal99].

To be precise

$$G(\bar{\xi}, \xi, J, -g\mathcal{V}_0) = \sum_{r=0}^{\infty} \frac{g^r}{r!} \sum_{m_F=0}^{\bar{m}_F(r)} \sum_{m_B=0}^{\bar{m}_B(r)} \int \prod_{n=1}^{2m_F+m_B} \frac{d_\beta p_{n,0}}{2\pi} \frac{d\mathbf{p}_n}{(2\pi)^d} \prod_{n=1}^{m_F} \bar{\xi}(p_n)\xi(p_{m_F+n}) \prod_{n=2m_F+1}^{2m_F+m_B} J(p_n)\delta(\sum_{n=1}^{2m_F+m_B} p_n) G_{r,m_F,m_B}(p_1, \ldots, p_{2m_F+m_B-1}) \tag{1.51}$$

where

$$G_{r,m_F,m_B} = \sum_{G \in \mathcal{G}_c(r,m_F,m_F)} \mathrm{Val}(G) \tag{1.52}$$

with $\mathcal{G}_c(r, m_F, m_F)$ the set of all connected graphs with r vertices with a pair of fermion lines and a boson line attached to it and with m_F outgoing fermion lines, m_F ingoing fermion lines and m_B external phonon lines.

For each graph G, $\mathrm{Val}(G)(p_1, \ldots, p_{2m_F+m_B-1})$ is defined by

- Assign momenta $p_1, \ldots p_{m_F}$ to the incoming Fermion lines, momenta $-p_{m_F+1}, \ldots -p_{2m_F}$ to the outgoing Fermion lines and momenta $p_{2m_F+1}, \ldots p_{2m_F+m_B}$ to the external phonon lines. $p_{2m_F+m_B} = -\sum_{n=1}^{2m_F+m_B-1} p_n$.
- Assign momenta to the internal lines preserving momentum at each vertex. This leaves exactly L free momenta $l_1 \ldots l_L$, where L is the number of independent loops in G.
- To each Fermion line assign a propagator

$$C(l) = \frac{1}{il_0 - e(\mathbf{l})} \quad \text{with } l = (l_0, \mathbf{l}), l_0 \in \mathbb{R}, \mathbf{l} \in \mathbb{R}^d \tag{1.53}$$

The dispersion relation $e(\mathbf{l})$ is given by $e(\mathbf{l}) = |\mathbf{l}|^2 - \mu$. l is the momentum flowing in the direction of the line.

- To each boson line with momentum l attach a phonon propagator $-D(l_0, c|\mathbf{l}|)$ where D is given by

$$D(x, y) = \frac{y^2}{x^2 + y^2} \tag{1.54}$$

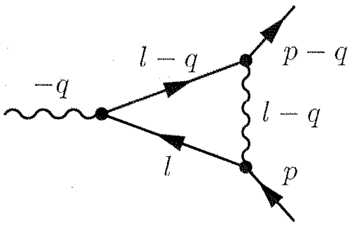
The dispersion relation $c|\mathbf{l}|$ is that given in section 1.1.1.

- Take the product of all the propagators and integrate over all $l_1 \ldots l_L$, with

$$\int \frac{d_\beta l_{n,0}}{2\pi} \int \frac{d\mathbf{l}_n}{(2\pi)^d} \tag{1.55}$$

- Add an overall sign factor $(-1)^{r + \#\text{ Fermion Loops in } G}$.

As example, we compute $G_{311}(p, -(p-q))$. There is only one graph in $\mathcal{G}_c(3, 1, 1)$, namely



Its value according to the Feynman rules is

$$\int_{-\infty}^{\infty} d_\beta l_0 \int d\mathbf{l} C(l) C(l-q) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{1.56}$$

1.1.4 Strong coupling theory

The need for non-perturbative treatment

In 'perturbation theory' one would naively expect that H_{int} only gives small corrections to the solvable theory defined by H_0. In our case we expand around the free field theory. The perturbation is given by $-g\mathcal{V}_0$. The coupling parameter g determines the size of the perturbation.

In Quantum Electrodynamics, which is a relativistic field theory with much the same structure as the theory we consider here, the relevant coupling parameter is indeed small[IZ85]. Good approximations are found by breaking off the expansion (1.51) at small orders of g.

However in Electron-Phonon theory we are not in such a fortunate situation. The coupling parameter g is of order 1 and we cannot simply cut off the series (1.51). Thus we must consider the theory non-perturbatively. This is generally too hard to do. However Migdal [Mig58] (for $T = 0$) and Eliashberg [Eli61] (for $T \neq 0$) found an approximation to the theory not based on perturbation theory that we will consider now [Hol64, AGD75].

Proper Green's functions

We change our viewpoint again by considering the generating function for the proper or one-particle irreducible Feynman diagrams. A graph is called *one-particle irreducible (1PI)* if cutting any of its lines does not make the graph disconnected. It is possible to define this generating function diagrammatically. However the proper way to do so is by means of a Legendre transformation[IZ85].

Consider the generating functional for the connected diagrams

$$G_c(\bar{\xi}, \xi, J) = \log \frac{1}{Z(0,0,0)} Z(\bar{\xi}, \xi, J) \tag{1.57}$$

Write $\mathbf{J} = (\bar{\xi}, \xi, J)$. Define functions

$$\phi(k, \mathbf{J}) = \frac{\delta}{\delta J(k)} G_c \qquad \bar{\psi}(k, \mathbf{J}) = \frac{-\delta}{\delta \xi(k)} G_c \qquad \psi(k, \mathbf{J}) = \frac{\delta}{\delta \bar{\xi}(k)} \tag{1.58}$$

Assume that these relations can be inverted to give $\mathbf{J} = \mathbf{J}(k, \Psi)$ with $\Psi = (\bar{\psi}, \psi, J)$. Then the generating functional for the *proper Green's functions* is defined by

$$\Gamma(\bar{\psi}, \psi, J) = G_c(\bar{\xi}, \xi, J) - (\bar{\xi}, \psi) - (\bar{\psi}, \xi) - (\phi, J)\big|_{\mathbf{J}=\mathbf{J}(k,\Psi)} \tag{1.59}$$

Using the definitions (1.58) and the chain rule for functional derivatives we see that

$$\frac{\delta \Gamma}{\delta \phi(k)} = -J(k, \Psi) \qquad \frac{\delta \Gamma}{\delta \bar{\psi}(k)} = -\xi(k, \Psi) \qquad \frac{-\delta \Gamma}{\delta \psi(k)} = -\bar{\xi}(k, \Psi) \tag{1.60}$$

The Generating functionals can be expanded as a formal power series

$$G_c(\bar{\xi}, \xi, J) = \sum_{m_F} \sum_{m_B} \int \prod_{n=1}^{2m_F+m_B} \frac{\mathrm{d}_\beta p_{n,0}}{2\pi} \frac{\mathrm{d}\mathbf{p}_n}{(2\pi)^d} \prod_{n=1}^{m_F} \bar{\xi}(p_n) \xi(p_{m_F+n}) \prod_{n=2m_F+1}^{2m_F+m_B} J(p_n) \delta\Big(\sum_{n=1}^{2m_F+m_B} p_n\Big) \mathscr{G}_{m_F,m_B}(p_1, \cdots, p_{2m_F+m_B-1}) \tag{1.61}$$

The functions $\mathscr{G}_{m_F,m_B}$ are called the n-point functions of the theory. $\mathscr{G}_{1,0}(k) = \mathscr{G}(k)$ and $\mathscr{G}_{0,2}(k) = \mathscr{D}(k)$ are the *full propagators* of the theory and coincide with those from (1.16) and (1.17). As stated before, the object of the calculations is to compute those two functions. Similarly

$$\Gamma(\bar{\psi},\psi,\phi) = \sum_{m_F}\sum_{m_B}\int\prod_{n=1}^{2m_F+m_B}\frac{\mathrm{d}_\beta p_{n,0}}{2\pi}\frac{\mathrm{d}\mathbf{p}_n}{(2\pi)^d}\prod_{n=1}^{m_F}\bar{\psi}(p_n)\psi(p_{m_F+n}) \prod_{n=2m_F+1}^{2m_F+m_B}\phi(p_n)\delta(\sum_{n=1}^{2m_F+m_B}p_n)\Gamma_{m_F,m_B}(p_1,\ldots,p_{2m_F+m_B-1}) \quad (1.62)$$

To see that the Legendre transform does indeed generate the 1PI graphs, we compute some of the lower order terms. Taking the derivative of the middle equation of (1.58) with respect to $\bar{\psi}(p)$, we get

$$\delta(k-p) = -\frac{\delta}{\delta\bar{\psi}(p)}\frac{\delta}{\delta\xi(k)}G_c \quad (1.63)$$

$$= -\left(\frac{\delta}{\delta\bar{\xi}(p)}\frac{\delta}{\delta\xi(k)}G_c, \frac{\delta\bar{\xi}(p)}{\delta\bar{\psi}(p)}\right) \quad (1.64)$$

using (1.60) this gives

$$\delta(k-p) = \left(\frac{\delta}{\delta\bar{\xi}(p)}\frac{-\delta}{\delta\xi(k)}G_c, \frac{\delta}{\delta\bar{\psi}(p)}\frac{-\delta}{\delta\psi(k)}\Gamma\right) \quad (1.65)$$

where we used that the only nonvanishing second order derivatives of G_c are $\frac{\delta}{\delta\bar{\xi}(p)}\frac{\delta}{\delta\xi(k)}G_c$ and $\frac{\delta}{\delta J(p)}\frac{\delta}{\delta J(k)}G_c$. Setting $\mathbf{J} = \Psi = 0$ gives.

$$\delta(k-p) = \delta(k-p)\mathscr{G}_{1,0}(k)\Gamma_{1,0}(k) \quad (1.66)$$

and thus $\Gamma_{1,0}(k) = \mathscr{G}(k)^{-1}$. Similarly $\Gamma_{0,1}(k) = \mathscr{G}_{0,2}(k)^{-1} = \mathcal{D}(k)^{-1}$.

Taking the derivative of (1.65) with respect to $\phi(q)$ gives

$$0 = \left(\left(\frac{\delta}{\delta J(l)}\frac{\delta}{\delta\bar{\xi}(p)}\frac{-\delta}{\delta\xi(k)}G_c, \frac{\delta}{\delta\bar{\xi}(p)}\frac{-\delta}{\delta\xi(k)}\Gamma\right), \frac{\delta J(l)}{\delta\phi(q)}\right) + \left(\frac{\delta}{\delta\bar{\xi}(p)}\frac{-\delta}{\delta\xi(k)}G_c, \frac{\delta}{\delta\phi(q)}\frac{\delta}{\delta\bar{\psi}(p)}\frac{-\delta}{\delta\psi(k)}\Gamma\right) \quad (1.67)$$

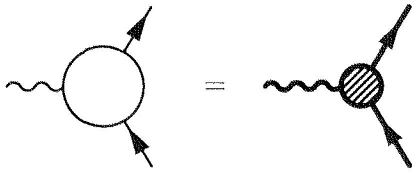
Using (1.60) and setting $\mathbf{J} = \Psi = 0$, we get

$$\mathscr{G}_{1,0}(k)\Gamma_{1,1}(p,k)\delta(p+k+q) = \mathscr{G}_{11}(p,k)\delta(p+k+q)\Gamma_{10}(p)\Gamma_{20}(q) \quad (1.68)$$

Substituting $p \to -p$ gives

$$\Gamma_{1,1}(-p, p-q) = \mathscr{G}_{11}(-p, p-q)\mathscr{G}(p-q)^{-1}\mathscr{G}(-p)^{-1}\mathscr{D}(q)^{-1} \tag{1.69}$$

The function $\Gamma_{1,1}$ is called the proper vertex correction. Its lowest order term in g is simply the vertex factor g. Inspection shows that multiplying with the inverse full propagator takes out exactly those strings of $\mathscr{G}_{11}$ that make it 1-particle reducible. Thus $\Gamma_{1,1}$ indeed expands into 1PI diagrams. If we represent the full propagators by thick lines and coefficients of Γ by filled blobs then this can be represented diagrammatically by



Schwinger-Dyson equations

There exists a set of integral equations relating $\mathscr{G}_{mn}$ and Γ_{pq}. These were discovered by Schwinger and Dyson. These non-perturbative relations contain all the physics of the model. On a functional integral level they can be seen as resulting from the fact that it is possible to do integration by parts (without boundary terms) in the functional integral. In fact this property is one of the main reasons a functional integral approach is natural to the problem. We derive the equations for the Electron-Phonon model analogous to [IZ85]. For derivations in the operator formalism see [Mah81, AGD75].

By integration by parts we have

$$0 = \int \mathrm{d}\mu_C \int \mathrm{d}\mu_D \left(-g\frac{\delta}{\delta \mathcal{V}_0}\bar{\psi}(k) - C^{-1}(k)\phi(k) + \xi(k)\right) e^{-g\mathcal{V}_0} e^{(\bar{\xi},\psi)+(\bar{\psi},\xi)+(J,\phi)} \tag{1.70}$$

$$= \int \mathrm{d}q \int \mathrm{d}p \left(-g\frac{\delta}{\delta J(q)}\frac{\delta}{\delta \bar{\xi}(p)}\delta(q+k+p) - C^{-1}(k)\frac{\delta}{\delta \bar{\xi}(k)} + \xi(k)\right) Z(\bar{\xi},\xi,J) \tag{1.71}$$

where we have written $\int \mathrm{d}l$ for $\int \frac{\mathrm{d}_\beta l_0}{2\pi}\frac{\mathrm{d}\mathbf{l}}{(2\pi)^d}$. The $-C^{-1}(k)\phi(k)$ comes from the measure. Formally it can be seen to arise from the derivative of the Gaussian weight function. Using $Z = e^{G_c}$ we get

$$\xi(k) = g\int \mathrm{d}q \int \mathrm{d}p \frac{\delta}{\delta J(q)}\frac{\delta}{\delta \bar{\xi}(p)} G_c \delta(q+k+p) + C^{-1}(k)\frac{\delta G_c}{\delta \bar{\xi}(k)} \tag{1.72}$$

Taking the derivative with respect to $\xi(l)$ on both sides gives

$$C(k)\delta(k-l) = -gC(k)\int \mathrm{d}q \int \mathrm{d}p \frac{\delta}{\delta J(q)}\frac{-\delta}{\delta \xi(l)}\frac{\delta}{\delta \bar{\xi}(p)} G_c \delta(q+k+p) - \frac{-\delta}{\delta \xi(l)}\frac{\delta}{\delta \bar{\xi}(k)} G_c \tag{1.73}$$

Setting the source fields zero we see

$$\mathscr{G}(k)\delta(k-l) = C(k)\delta(k-l) + gC(k)\int \mathrm{d}q \int \mathrm{d}p \mathscr{G}_{11}(p,k)\delta(q+k+p)\delta(k-l) + \frac{\delta}{\delta\bar{\xi}(k)}\frac{-\delta}{\delta\xi(l)}G_c \tag{1.74}$$

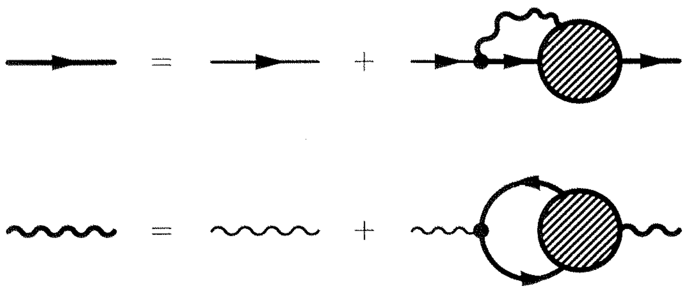
And using (1.68) we thus see

$$\mathscr{G}(k) = C(k) + gC(k)\int \frac{\mathrm{d}_\beta p_0}{2\pi}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d}\mathscr{G}(p)\mathscr{D}(-p-k)\Gamma_{11}(p,k)\mathscr{G}(k) \tag{1.75}$$

A similar calculation shows that

$$\mathscr{D}(k) = D(k) + gD(k)\int \frac{\mathrm{d}_\beta p_0}{2\pi}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d}\mathscr{G}(p)\mathscr{G}(p-k)\Gamma_{11}(-p,p-k)\mathscr{D}(k) \tag{1.76}$$

Graphically these can be represented as



Note that these graphs are *not* amputated.

The Migdal-Eliashberg approximation

The Schwinger-Dyson equations form two coupled integral equations for $\mathscr{G}$ and D. Unfortunately the integrals there also contain a vertex correction. This makes them very hard to solve. The vertex correction must itself be expanded. Moreover the self-consistent expansion of Γ_{11} in terms of $\mathscr{G}$, $\mathscr{D}$ and Γ_{11} contains infinitely many terms.

Migdal observed that there is another parameter (besides g) in the theory that *is* small. He observed that

Approximation 1 (Migdal-Eliashberg). *When c/v_F is small then the corrections to the vertex beyond the zeroth-order term are also small, i.e., we can replace Γ_{11} by its zeroth-order term g.*

The purpose of this thesis is to find rigorous justifications for this approximation. Migdal's original justification (known as "Migdal's Theorem") and our results are discussed in section 1.3.

Eliashberg equations

In the Migdal-Eliashberg approximation the Dyson equations become

$$\mathscr{G}(k) = C(k) + gC(k) \int \frac{\mathrm{d}_\beta p_0}{2\pi} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d} \mathscr{G}(p) \mathscr{D}(-p-k) \mathscr{G}(k) \tag{1.77}$$

$$\mathscr{D}(k) = D(k) + gD(k) \int \frac{\mathrm{d}_\beta p_0}{2\pi} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^d} \mathscr{G}(p) \mathscr{G}(p-k)(-p, p-k) \mathscr{D}(k) \tag{1.78}$$

These equations are called the *Eliashberg Equations*. They are sufficiently simply to make solving them possible [AGD75, Mig58, Eli61].

1.2 Cutoffs, renormalization and finite temperature

1.2.1 An ultra-violet cutoff

Apart from the problems in summing up the perturbation theory expansions in g discussed above, there is the question whether the coefficient functions G_{r,m_F,m_F} are actually well defined.

One problem that can occur is that the functions to be integrated don't decay quickly enough for the integrals to converge. Because large momenta correspond to high energies, this is called the ultra-violet problem. It can be shown that such problems can be eliminated [FT90].

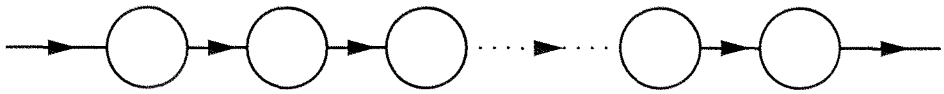
However in models of solid-state physics the physical properties arise from the low energy behavior. The high-energy behaviors is not so interesting. Therefore we assume that in the theory there is a 'cut off' such that all vector momenta occurring are in a compact set $\Omega \subset \mathbb{R}^d$, defined to be the ball of radius $\Lambda/2$ around the origin, for some $\Lambda > 7$. With this particular choice of Ω, $|\mathbf{l} - \mathbf{p}| \leq \Lambda$ for all $\mathbf{l}, \mathbf{p} \in \Omega$.

When dealing with higher order graphs we will use a tighter restriction on the electron momenta that contains the region where the electron propagator is singular. The scale decomposition method used there gives a natural form such a restriction. The region where electron-propagator is bounded gives a trivial Λ dependent contribution.

1.2.2 The Infrared divergences and finite temperature

Divergences in the zero temperature limit

From the construction of the graphs it easy to see that it possible that there exist subgraphs that are chains of $1PI$ 2-legged graphs connected by electron lines, i.e.



By conservation of momentum, all the fermion lines have the same momentum p attached to them. This gives rise to a factor

$$C(p)^n = \frac{1}{(ip_0 - e(\mathbf{p}))^n} \tag{1.79}$$

in the integrand for the value of the graph, for some n. In the zero temperature limit the frequencies are continuous and the factor becomes singular for $p_0 = 0$, $p : e(\mathbf{p}) = 0$. This singularity is not integrable for n big enough. In fact already for $n = 2$ the integral is not absolutely convergent (see (2.3)). Note that the singularity occurs on a submanifold of codimension 1 in $\mathbb{R} \times \Omega \subset \mathbb{R}^{d+1}$ and thus the superficial degree divergence does not depends on the dimensionality of the problem.

Thus the coefficients $G_{rm_F m_B}$ are ill-defined in the limit $\beta = \infty$. This is called an *infrared-divergence* (because it occurs at low energies).

Finite temperature as a regulator

It is possible to eliminate the divergences as in the ultraviolet case by introducing a cutoff for the frequency, the momentum, or both that keeps the values away from the singularity. One would then have to show that the quantities that are computed exist in the limit where the cut-off is removed and computation of the quantities and the removal of the cut-off can be interchanged.

However note that at finite β, the electron propagators are bounded: The frequencies p_0 are of the form $p_0 = \frac{\pi}{\beta}(2n+1)$, and thus

$$\left| \frac{1}{ip_0 - e(\mathbf{p})} \right| \leq \frac{1}{|p_0|} \leq \frac{\beta}{\pi} \tag{1.80}$$

In section 1.4.1 it is shown that similarly the phonon propagator or more importantly its derivatives are also bounded at finite β. This means that all integrals/frequency sums are absolutely convergent when $\beta < \infty$. Finite temperature acts as a natural regulator. The absolute convergence of the integrals is also very convenient as we can exchange integrations at will.

1.2.3 Renormalization

The need for renormalization

Unfortunately the coefficients of the perturbation expansion being finite at finite temperature does not mean that all is well. Although finite,the bounds are not uniform in β and diverge as $\beta \to \infty$. However we want to achieve uniform bounds (at least in the small temperature region) for the n-point functions and in particular for the vertex correction.

The self-energy

As shown in section 1.1.4 $\Gamma_{10} = \mathscr{G}^{-1}$. The zeroth order value approximation to the right hand side is just C^{-1}. Define the contributions to $\Gamma_{10}(p)$ beyond zeroth order as $-\Sigma(p) = \Gamma_{10}(p) - C(p)^{-1}$. $\Sigma(p)$ is called the self energy, because the full propagator is now given by

$$\mathscr{G}(p) = \frac{1}{ip_0 - e(\mathbf{p}) - \Sigma(p)} \tag{1.81}$$

The divergences are artificial

It is now a standard argument to see that the divergences are in fact artificial and result from the fact that we have been expanding around the wrong theory[FST96, Sal99]. Indeed when expanding (1.81) around the free propagator we get

$$\mathscr{G}(p) \text{ ``}=\text{'' } \sum_{r=0}^{\infty} \frac{\Sigma(p)^r}{(ip_0 - e(\mathbf{p}))^r} \tag{1.82}$$

which gives the unwanted powers of the free propagator with non-integrable singularities at $p_0 = e(\mathbf{p}) = 0$. Note however that the left hand side is singular for $p_0 = e(\mathbf{p}) + \Sigma(p) = 0$. Thus switching on the interaction moves the singularity. It is the fact that we try to expand a singular function around a function with a singularity somewhere else that produces the problem.

The counterterm

One solution would be to expand around $ip_0 - e(\mathbf{p}) - \Sigma(p)$, for $\mathscr{G}$ that means not to expand at all. This is what happens in the Dyson-equation formulation above. However this fundamentally changes the formulation of the model as $\Sigma(p)$ depends on the frequency p_0.

Therefore we introduce a function $K(\mathbf{p})$, such that $e(\mathbf{p}) + K(\mathbf{p})$ has the same zero set as $e(\mathbf{p}) + \Sigma(0, \mathbf{p})$. This function is called the *counterterm*. We show that the expansion around $e(\mathbf{p}) + K(\mathbf{p})$ does produce finite coefficients. This procedure is called *Renormalization.*

To complete the analysis one should find a function $e(\mathbf{p})$ such that $e(\mathbf{p}) + K(\mathbf{p})$ gives the physical band relation. This is in general a hard problem [FST00, FST98, FST99, FT91]. We shall not consider it here. Note however that by $O(d)$ symmetry, both e is really only a function of the one variable $|\mathbf{p}|$ and K can be chose such that it is too. Thus the problem is already much simpler in our case, where we do indeed define K in such a way. Our definition for K is given in (3.55). It contains an additional linear term il_0K' where K' is a constant. It amounts to multiplying the propagator with a constant factor $(1 + K')^{-1}$ which can be seen as the result of scaling the fermion fields by $(1 + K')^{-\frac{1}{2}}$.

1.3 Migdal's theorem and variants

1.3.1 Fixing units and notations

Electronic units

A common choice of units when working with the electron model are the so called electronic units. These units are chosen such that

- The electron mass $m = \frac{1}{2}$.
- The Fermi wave vector k_F is 1.

The Fermi velocity k_F is defined such that for $|\mathbf{k}| = k_F$, $e(\mathbf{k}) = 0$. Thus it follows that in these units $\mu = \frac{k_F^2}{2m}$ and the band relation becomes $e(\mathbf{p}) = |p|^2 - 1$.

A smooth cut-off function

Through this text we will need smooth cut-off functions and partitions of unity. Most of these will be formulated in terms of the following

Let $\chi : [0, \infty) \to [0, 1]$, $i = 0, 1$ be a smooth C^∞-function such that

$$\chi(x) = \begin{cases} 0 & |x| \geq 1 \\ 1 & |x| \leq \frac{1}{4} \end{cases} \tag{1.83}$$

and

$$\chi'(x) < 0 \quad \forall x \in (\frac{1}{4}, 1) \tag{1.84}$$

Miscellaneous notation

By $\mathbb{1}(Y(x))$ for some condition $Y(x)$ we denote the indicator function for the set of x's for which $Y(x)$ is true.

The following C^k-norms and restricted forms thereof are used throughout the text: For a d-dimensional multi-index α, $D^\alpha f(\mathbf{p}) = \frac{\partial^{\alpha_1}}{\partial p_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial p_d^{\alpha_d}} f(p_1, \ldots, p_d)$. For $X \subset \Omega$ open, $f \in C^n(X, \mathbb{C})$ and $\delta > 0$.

$$|f|_{n,X} = \sup_{\mathbf{p} \in X} \sum_{|\alpha| \leq n} |D^\alpha f(\mathbf{p})| \tag{1.85}$$

$$|f|_n = |f|_{n,\Omega} \tag{1.86}$$

Denote the n-dimensional measure of of an n-dimensional manifold X by $\mathrm{Vol}(X)$.

1.3.2 Pertubative expansion for the vertex correction

Consider the proper vertex correction $\Gamma(p, q) = \Gamma_{11}(-p, p - q)$ in the electron-phonon model in d-dimensions as described above. Let β be finite. Restrict all vector momenta to be in a compact set Ω as an ultraviolet cut-off. The electron propagator is thus given by

$$C(l) = \frac{1}{il_0 - e(\mathbf{l})} \quad \text{with } l = (l_0, \mathbf{l}), l_0 \in \mathbb{R}, \mathbf{l} \in \Omega \tag{1.87}$$

The dispersion relation $e(\mathbf{l})$ is given by that for the Jellium model, $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. The phonon propagator is $D(l_0, c|\mathbf{l}|)$ where D is given by

$$D(x, y) = \frac{y^2}{x^2 + y^2} \tag{1.88}$$

The coupling constant is given by g.

Expand $\Gamma(p,q)$ as a formal power series in g.

$$\Gamma(p,q) = \sum_{r=1}^{\infty} \Gamma_r(p,q) g^r \tag{1.89}$$

From the topical properties of graphs it follows that $\Gamma_r(p,q) = 0$ for $r \neq 1 + 2L$ where $L \in \mathbb{N} \cup \{0\}$ is the loop order. $\Gamma_1(p,q) = g$ is the coupling constant attached to a simple vertex.

1.3.3 Migal's original statement

First order diagrams

With the definitions given above, the Migdal-Eliashberg approximation discussed in section 1.1.4 is that replacing Γ by its leading order term $\Gamma_1 = g$ gives only a small error. In his paper Migdal made the following claims to support that approximation. These (and their extension to non-zero temperature) are commonly referred to as 'Migdal's theorem'.

$\Gamma_3(p,q)$ is given by

$$\Gamma_3(p,q) = \int_{-\infty}^{\infty} \mathrm{d}_\beta l_0 \int_{\Omega} \mathrm{d}\mathbf{l} C(l) C(l-q) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{1.90}$$

In our notation the claims are

Claim 1 ([Mig58, FW71]). *Let $d = 3$. At $T = 0$ the lowest order correction to $\Gamma \approx \Gamma_1$, $\Gamma_3(p,q)$, is of order c except for some unphysical values of q.*

Migdal gives for the unphysical condition $\mathbf{q} \approx 2, \mathbf{q}_0$. In [FW71] the unphysical condition is given by $|q_0| = 2|\mathbf{q}|$, the factor 2 comes from the derivative of $e(\mathbf{l}) = r^2 - 1$ with respect to $r = |\mathbf{l}|$ at the Fermi surface. Note that the claim the was originally formulated in the zero temperature Green function formalism.

The above result is often extended to hold at non-zero temperature

Claim 2 ([Eli61, Hol64, AGD75]). *Let $d = 3$. At positive temperature the lowest order correction to $\Gamma \approx \Gamma_1$, $\Gamma_3(p,q)$, is of order c except for some unphysical values of q.*

Again the different authors disagree on the exact nature of the unphysical region.

Both [Hol64, FW71] argue that even in the unphysical region there is a logarithmic in the zero temperature limit and that logarithmic divergences do not cause problems because they are integrated over. However although the presence of an additional logarithmic divergence does not change integrability it can dramatically change the value. To us this therefore does not seem a fruitful approach.

Note that Claim 2 clearly does not hold as stated. Because of (1.97) and (1.96), each frequency sum contains one term that does not depend on c at all and thus does not vanish at $c = 0$. In Theorem 2 we prove the claim (accounting for this problem) and extend it all dimensions $d \geq 2$. The only values that give problems will be for $|\mathbf{q}| \approx 2$.

Higher order terms

In addition to the calculations done for the first correction and the claims above there is the following much stronger claim [Mig58, Eli61, Hol64, FW71, AGD75], which we cite in the form stated by Migdal

Claim 3. *It can be shown that [the bounds from the above claims are] not changed when diagrams of a higher order are taken into account [and therefore $\Gamma = g + O(c)$].*

To our knowledge a proof of this claim never appeared in the literature. Most authors suffice to discuss the one-loop correction and cite the above sentence by Migdal.

In this thesis we do prove a bound for the higher order corrections (Theorem 3), but this bound is slightly weaker (mainly for technical reasons).

Using "Migdal's theorem" to validate the approximation

In ordinary metals the velocity of sound relative to the fermi-velocity, c/v_F, is proportional to $\sqrt{\frac{m}{m_{\text{ion}}}}$, or equivalently to ω_D/e_F. This ratio is therefore much smaller than one. Any correction which is of order c/v_F (remember that $v_F = 1$ in our units) is therefore indeed small. Note that in typical metals c is on the order of 10^3 m/s and v_F on the order of 10^6 m/s, which means that $c/v_F \approx 10^{-3}$.

1.3.4 The theorems proven in this work

Theorem 2 (Migdal's 'Theorem' to 1-loop order). *Let $d \geq 2$. Let $\kappa_t = \frac{3}{2}$. Let $e(\mathbf{l}) = |l|^2 - 1$. There exist constants M^*,$\tilde{M}^*$, M_{temp}, and $\tilde{M}_{temp}$, such that for all β, all $p = (p_0, \mathbf{p}) \in$*

$(\frac{\pi}{\beta}+\frac{2\pi}{\beta}\mathbb{Z})\times\Omega$ *and all* $q=(q_0,\mathbf{q})\in\frac{2\pi}{\beta}\mathbb{Z}\times\Omega$ *with* $|\mathbf{q}|\le\kappa_t$*:*

$$|\Gamma_3(p,q)|\le cM^*+M_{temp}\frac{1}{\beta} \tag{1.91}$$

Moreover for $d=3$ *and* $|\mathbf{q}|>\kappa_t$

$$|\Gamma_3(p,q)|\le c(\log c)^2\tilde{M}^*+\tilde{M}_{temp}\frac{(log\beta)^2}{\beta} \tag{1.92}$$

The theorem is proven in Chapter 2 and Appendix A. $M^*=\max\{M_s,M_g\}$ where M_s and M_g are given explicitly in Lemma 15 and Lemma 12 respectively.

We also prove a rigorous bound for the vertex correction to all orders of perturbation theory. However we don't prove the $O(c)$ bound that Migdal conjectured, but the following slightly weaker bound.

Theorem 3. *Let* $d\ge 2$*. Consider the proper vertex correction* $\Gamma(p,q)$ *in renormalized electron-phonon theory with d-dimensions and with all vector momenta in* Ω*. Let the free electron propagator* $C(l)$ *be given as above and let the free phonon propagator be given by*

$$D(l_0,c|\mathbf{l}|)X_d(\mathbf{l}) \tag{1.93}$$

with $X_d=1$ *for* $d\ge 3$ *and* X_2 *a smooth function with* $X_2\le 1$ *and for* $\mathbf{p}\in\operatorname{supp}X_2$, $|\mathbf{p}|\le 1$. *Let* $\Gamma_r(p,q)$ *be given by the formal power series expansion* (1.89).

Let $0<\epsilon<1$*. Then there exist constants* $\{M_r(\epsilon)\}_{r=2}^{\infty}$ *such that for all* $\beta>1$*, all* $p=(p_0,\mathbf{p})\in(\frac{\pi}{\beta}+\frac{2\pi}{\beta}\mathbb{Z})\times\Omega$*, all* $q=(q_0,\mathbf{q})\in\frac{2\pi}{\beta}\mathbb{Z}\times\Omega$ *with* $\mathbf{q}\in\operatorname{supp}X_d$*, and all* $r\ge 2$*:*

$$|\Gamma_r(p,q)|<M_r(\epsilon)\left\{c^{1-\epsilon}+\left(\frac{(\log\beta+1)^2}{\beta}\right)^{1-\epsilon}\right\} \tag{1.94}$$

The exact definition of the renormalized theory is given in section 3.4.2, definition 22.

In the theorem one-loop case we saw that values of $|\mathbf{q}|$ which are close to two are problematic. This is because for bubble diagrams which have a net transfer of momentum $\mathbf{q}$ the integral for the value contains singularities on nearly touching Fermi surfaces. Therefore for such $\mathbf{q}$ a result which contains c up to at most logarithmic factors could only be obtained for $d\ge 3$. This will be discussed in more detail later. The restriction $\mathbf{q}\in\operatorname{supp}X_2$ for $d=2$ is there to ensure that this problematic case does not occur here. Note that when the vertex correction $\Gamma(p,q)$ occurs in values of graphs or in the Dyson equations it always multiplied with a phonon propagator $D(q)$. Thus q only takes values in the support of D. With D given by the special form (1.93) this means that $\mathbf{q}$ will always be in $\sup X_d$ and thus this restriction is a valid one.

As we shall see, the latter theorem depends much more strongly on the special choice of the jellium band relation $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$ than the former. Both use the special form to simplify considerations on the geometry of the Fermi surfaces such as Lemma 5 and Lemma 34. However these either already have been shown to generalize to much more general band relations ([FST98, FST00, FST96]) or are not expected to give significant problems. However in the higher order diagrams we use the $O(d)$ symmetry of the problem to simplify greatly the treatment of strings of 2-particle insertions. Extending the theorem to non $O(d)$ symmetric band relations will therefore require additional ways to bound contributions with those insertions.

1.4 Basic properties of the model

1.4.1 Properties of the phonon propagator

Basic properties

The proof of Theorems 2 and 3 bases on the properties of the boson propagator $D(l_0, c|\mathbf{k}|)$.

$$\int_{\mathbb{R}} \mathrm{d}x D(x,y) = \pi|y| \tag{1.95}$$

For frequency sums at finite β

$$\int \mathrm{d}_\beta x\, D(x,y) = \frac{2\pi}{\beta} \sum_n D(x_n, y) \leq \pi|y| + \frac{2\pi}{\beta} \tag{1.96}$$

Here $x_n = \frac{2\pi}{\beta} n$ are the boson Matsubara-frequencies. To separate the two terms on the right hand side, split up D as

$$D(x,y) = \delta_{x0} + D^*(x,y) \tag{1.97}$$

$$D^*(x,y) = D(x,y)1(x \neq 0) = \begin{cases} \frac{y^2}{x^2+y^2} & \text{when } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{1.98}$$

Because we have cut $x = 0$ explicitly we have for frequency sums over D^*:

$$\int \mathrm{d}_\beta x\, D^*(x,y) = \frac{2\pi}{\beta} \sum_n D^*(x_n, y) \leq \pi|y| \tag{1.99}$$

Proof. The function $D(x,y)$ is a decreasing function of x for $x>0$. Therefore for each $n>0$ the its minimum on the interval $[x_{n-1},x_n]$ is $D(x_n,y)$ and $\sum_{n\in\mathbb{Z}_+} D(x_n,y)$ is a Riemannian undersum for the integral $\int_0^\infty \mathrm{d}x D(x,y)$. Similarly $\sum_{n\in\mathbb{Z}_-} D(x_n,y)$ is a Riemannian undersum for the integral $\int_{-\infty}^0 \mathrm{d}x D(x,y)$. Because $D(0,y)=1$ and $D^*(0,y)=0$, the $n=0$ term gives the remaining term (zero for the latter). □

The derivatives of D have the following properties.

$$|D_y(x,y)| = \left|\frac{\partial}{\partial y} D(x,y)\right| \leq \frac{2D^*(x,y)}{|y|} \tag{1.100}$$

$$|D_{yy}(x,y)| = \left|\frac{\partial^2}{\partial y^2} D(x,y)\right| \leq \frac{8D^*(x,y)}{y^2} \tag{1.101}$$

Note that in particular this implies that for $x=x_n$ discrete bosonic Matsubara-frequencies the derivatives are bounded since $x\neq 0$ implies $|x|\geq \frac{2\pi}{\beta}$ and $y^{-2}D^*(x,y)=(x^2+y^2)^{-1}$.

Integrability of derivatives

The properties of D and its derivative lead to the following bounds which will be used throughout the proofs.

Lemma 4. *Let D defined as above, Let $p\in\Omega$ and $p_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)$ a fermionic Matsubara frequency. Let $Y\subset\mathbb{R}^n$ and $X\subset\Omega$. Let $J: X\to\mathbb{R}$ be a strictly positive C^2 function. Let the map*

$$\begin{aligned} \mathbf{l}: Y &\longrightarrow X \\ (y_1,\ldots,y_n) &\longmapsto \mathbf{l}(y_1,\ldots,y_n) \end{aligned} \tag{1.102}$$

be twice continuously differentiable. Let $v=\sup_{y_1,\ldots,y_n\in Y}|\mathbf{l}-\mathbf{p}|$. Then for all $i,j=1..n$ and finite β

i)

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}_\beta l_0}{2\pi} J(\mathbf{l}) D^*(l_0-p_0, c|\mathbf{l}-\mathbf{p}|) \leq c\frac{1}{2} J(\mathbf{l})|\mathbf{l}-\mathbf{p}| \leq c\frac{1}{2} v|J|_{0,X} \tag{1.103}$$

ii)

$$\left|\frac{\partial}{\partial y_i}\left(J(\mathbf{l})D^*(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)\right)\right|_0 < \infty$$

and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}_\beta l_0}{2\pi}\left|\frac{\partial}{\partial y_i}\left(J(\mathbf{l})D^*(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)\right)\right| \leq cv|J|_{1,X}\left|\frac{\partial \mathbf{l}}{\partial y_i}\right| \tag{1.104}$$

iii)

$$\left| \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \left(J(\mathbf{l}) D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)\right) \right|_0 < \infty \tag{1.105}$$

and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}_\beta l_0}{2\pi} \left| \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \left(J(\mathbf{l}) D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)\right) \right| \\ \leq c \frac{v^2}{|\mathbf{l} - \mathbf{p}|} \left(|J|_{1,X} \left| \frac{\partial^2 \mathbf{l}}{\partial y_i \partial y_j} \right| + 6|J|_{2,X} \left| \frac{\partial \mathbf{l}}{\partial y_i} \right| \left| \frac{\partial \mathbf{l}}{\partial y_j} \right| \right) \tag{1.106}$$

Proof. The proof is a straightforward application of the chain rule and the properties of D. We show iii only. For a function $f(\mathbf{l})$ denote by f'' the matrix with $\frac{\partial}{\partial l_i}\frac{\partial}{\partial l_j}$ as its matrix element on row i and column j. Then

$$\frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} f = \frac{\partial}{\partial y_i} \left(\frac{\partial f}{\partial \mathbf{l}} \cdot \frac{\partial \mathbf{l}}{\partial y_j} \right) = \frac{\partial f}{\partial \mathbf{l}} \cdot \frac{\partial^2 \mathbf{l}}{\partial y_i \partial y_j} + (f'' \frac{\partial \mathbf{l}}{\partial y_i}) \cdot \frac{\partial \mathbf{l}}{\partial y_j} \tag{1.107}$$

and

$$\max_{i,j} \left| \frac{\partial}{\partial l_i} \frac{\partial}{\partial l_j} (J(\mathbf{l}) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)) \right| \leq 12 |J|_2 \frac{v^2}{|\mathbf{l} - \mathbf{p}|^2} D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{1.108}$$

because

$$D(l_0, c|\mathbf{l}|) = \left. \frac{Z}{l_0^2 + Z} \right|_{Z = c^2 |\mathbf{l}|} \tag{1.109}$$

$$\frac{\partial^k}{\partial Z^k} \frac{Z}{l_0^2 + Z} \leq \frac{k!}{Z^k} \frac{Z}{x^2 + Z} \tag{1.110}$$

$$\frac{\partial}{\partial l_i} c^2 |\mathbf{l}|^2 \leq 2c^2 |l_i| \quad \frac{\partial}{\partial l_i} \frac{\partial}{\partial l_j} c^2 |\mathbf{l}| \leq c^2 2\delta_{ij} \tag{1.111}$$

This and a similar calculation for the first half of eqn. (1.107) implies that the integrand is bounded at finite β because

$$\frac{1}{|\mathbf{l} - \mathbf{p}|^2} D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \leq c^2 \frac{\mathbb{1}(l_0 - p_0) \neq 0}{(l_0 - p_0)^2 + c^2 |\mathbf{l} - \mathbf{p}|^2} \leq \frac{c^2}{(\frac{2\pi}{\beta})^2} \tag{1.112}$$

and gives the required inequality. □

Phonon propagators in Feynman Diagrams

The integrability propetries of the phonon propagator and its derivatives lead to bounds for integrals that contain them. They are used directly in Chapter 2. In section 4 their effect is calculated in terms of scale decomposition.

1.4.2 Coordinates and Fermi Surfaces

Polar coordinates

In any discussion of a physical theory involving Fermions, the level surfaces of the band relation $e(\mathbf{l})$ plays an important role. Because of the Pauli principle no two Fermions can occupy the same state. Therefore even if all particles occupy the lowest possible energy levels, there will still be a finite volume of states filled. The boundary of this volume is called the Fermi Surface S. The level at which this occurs is determined by the chemical potential μ. As we have absorbed μ into the definition of the band relation, this level will always be 0. Thus, $S = S(0)$ where $S(\rho) = \{\mathbf{l}|e(\mathbf{l}) = \rho\}$.

Typically the essential behavior of the model is determined by its properties close to the Fermi surface and our case is no different. Therefore we introduce some notation for this region

$$U(S, \mathbf{q}, \eta) = \{\mathbf{l}||e(\mathbf{l} - \mathbf{q})| < \eta\} \tag{1.113}$$

$$U(S, \eta) = U(S, 0, \eta) \tag{1.114}$$

and similar to (1.85))

$$|f|_{n,\delta} = |f|_{n,U(S,\delta)} \tag{1.115}$$

$$|f|_{n,\delta,\mathbf{q}} = |f|_{n,U(S,\delta)\cap U(S,\mathbf{q},\delta)} \tag{1.116}$$

One of the techniques that is used often here and elsewhere is that in this region there exists a natural set of coordinates. These are a generalization of polar coordinates, where the radius is replaced by the energy $e(\mathbf{l})$ and role of the angular variable is played by a projection on the Fermi Surface. In [FST96] existence of such coordinates is proven under very weak conditions on the band relation $e(\mathbf{l})$. In fact is shown there that this change of coordinates can be taken to be C^∞ even if the differentiability of $e(\mathbf{l})$ is much lower.

In our case $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$, and thus the Fermi Surface is a sphere and $e(\mathbf{l})$ is smooth in its neighborhood. The existence of the coordinates (stated in the following Lemma) is then trivial, because the standard polar coordinates can be used. The main purpose of the lemma is therefore to fix notation.

Lemma 5. *Let* $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. *For* $\delta = \frac{1}{5}$ *and* $r_{min} = \frac{1}{2}$

i) $S(\eta) = \{\mathbf{l} \in \mathbb{R}^d | e(\mathbf{l}) = \eta\}$ *is a sphere, hence a strictly convex* C^∞ *manifold, for all* $|\eta| < \delta$, *and*

ii) $U(S,\delta) \subset \{\mathbf{p} \in \Omega || \mathbf{p}| \geq r_{min}\}$ *,and*

iii) There exits an invertible C^k *map* Ψ_0

$$\begin{aligned} \Psi_0 : U(S,\delta) &\longrightarrow (-\delta,\delta) \times S^{d-1} \\ \mathbf{l} &\longmapsto (\rho = e(\mathbf{l}), \theta) \end{aligned} \tag{1.117}$$

Denote $\boldsymbol{\pi}(\rho,\theta) = \Psi_0^{-1}(\rho,\theta)$

iv) Let $\tilde{J}_1(\rho,\theta) = J_1(\boldsymbol{\pi}(\rho,\theta))$ *be the Jacobian of the change of coordinates* $\mathbf{l} = \boldsymbol{\pi}(\rho,\theta)$ *(Note:* $J_1(\mathbf{p})$ *is a function from* $U(S,\delta)$ *to* $\mathbb{R}_+$*). Then*

$$|J_1|_{0,\delta} \leq \frac{1}{2}(1+\delta)^{d-2} \qquad |J_1|_{1,\delta} \leq \frac{d^2}{2}(1+\delta)^{d-2} \tag{1.118}$$

$$|J_1|_{2,\delta} \leq \frac{d^4}{2}(1+\delta)^{d-2} \qquad |J_1^{-1}|_{0,\delta} \leq \frac{1}{r_{min}^{d-2}} \tag{1.119}$$

Remark 6. *The restriction* $\delta \leq \frac{1}{5}$ *is for convenience. It ensures that the function*

$$\mathrm{Log}(ix+y) = \frac{1}{2}\log(x^2+y^2) + i\arctan\frac{y}{x} \tag{1.120}$$

sasisfies for $|x| \leq \frac{1}{2}$ *and* $|y| \leq \delta < e^{-\frac{1}{2}\pi}$

$$|\mathrm{Log}(ix+y)| \leq |\log|y|| + \frac{1}{2}\pi \leq 2|\log|y|| \tag{1.121}$$

Proof. Let

$$\begin{aligned} P : \mathbb{R}^d &\longrightarrow \mathbb{R}_+ \times S^{d-1} \\ \mathbf{l} &\longmapsto (r,\theta) \end{aligned} \tag{1.122}$$

be the polar coordinate map. $P^{-1}(r,\theta) = r\theta$. Then $e(r\theta) = r^2 - 1 = \rho(r)$. Let $r(\rho) = \sqrt{1+\rho}$ be its inverse. We thus have $S(\eta) = \{r(\eta)\theta | \theta \in S^{d-1}\} \simeq S^{d-1}$. The origin is inside $S(\eta)$ for all $|\eta| < \delta$.

On $U(S,\delta)$, $|\rho(r)| \leq \frac{1}{5}$ and thus $r(\rho) \geq \sqrt{1-\frac{1}{5}} \geq r_{\min}$. Moreover $\frac{\partial r(\rho)}{\partial \rho} = \frac{1}{2\sqrt{1+\rho}} = \frac{1}{2r(\rho)} \leq 1$.

Let f be the map $(r,\theta) \mapsto (\rho(r),\theta)$ then

$$\Psi_0 = f \cdot P \tag{1.123}$$

is C^∞ because f and P are.

$$\tilde{J}_1(\rho,\theta) = |\det DP^{-1}| \left|\frac{\partial r(\rho)}{\partial \rho}\right| = \frac{1}{2} r^{d-2} \tag{1.124}$$

$$J_1(\mathbf{l}) = \frac{1}{2}|\mathbf{l}|^{d-2} \tag{1.125}$$

The bounds for the Jacobian and its derivatives now follow trivially. □

Corollary 7. *From the construction of coordinates in the above lemma it follows that for* $\mathbf{l} \in U(S,\delta)$

$$\left|\frac{\partial \mathbf{l}}{\partial \rho}\right| \leq 1 \qquad \left|\frac{\partial}{\partial \theta}\mathbf{l}\right| \leq (1+\delta) \qquad \left|\frac{\partial^2 \mathbf{l}}{\partial \rho^2}\right| \leq 1 \qquad \left|\frac{\partial \tilde{J}_1}{\partial \rho}\right| \leq (d-2)(1+\delta)^{d-3} \tag{1.126}$$

Transversality and interpolating Fermi Surfaces

In the following it is often needed to deal with integrals where the integrand not only has a singularity of on the Fermi surface S but also on a second translated Fermi Surface $S_\mathbf{q} = \{\mathbf{l}|e(\mathbf{l}-\mathbf{q}) = 0\}$. If these two intersect transversally then we can use $e(\mathbf{l}-\mathbf{q})$, or equivalently

$$U(\mathbf{l},\mathbf{q}) = e(\mathbf{l}-\mathbf{q}) - e(\mathbf{l}) \tag{1.127}$$

as a second coordinate besides $\rho = e(\mathbf{l})$ and use the resulting double integral to control the two singularities.

However $\max_{\mathbf{l}\in S}|U(\mathbf{l},\mathbf{q})| \leq \text{const}\,|\mathbf{q}|$ and thus it will not be possible to find bounds for $U(\mathbf{l},\mathbf{q})$ and its derivatives that are uniformly bounded from below in $\mathbf{q}$. This is caused by the fact that for small $\mathbf{q}$ the two surfaces lie very close to another and thus the intersection angle is very sharp.

In fact we can choose a κ_s such that for all $|\mathbf{q}| < \kappa_s$, the shifted Fermi surface $S_\mathbf{q}$ is completely contained in $U(S,\delta)$. This is shown in the following lemma. There it also shown that for small $\mathbf{q}$, the interpolating band relation defined as

$$e(\mathbf{l},\mathbf{q},t) = (1-t)e(\mathbf{l}) + te(\mathbf{l}-\mathbf{q}) \qquad t \in [0,1] \tag{1.128}$$

has properties similar to that of $e(\mathbf{l})$. In particular all intermediate Fermi surfaces $S(t,\mathbf{q}) = \{\mathbf{l}|e(\mathbf{l},\mathbf{q},t) = 0\}$ are indeed also contained in $U(S,\delta)$ and the coordinates (ρ,θ) can be used as "polar coordinates" around these surfaces too.

Last but not least we show that for $\mathbf{q} \neq 0$ the rescaled difference function

$$\tilde{U}(\mathbf{l},\mathbf{q}) = \frac{1}{|\mathbf{q}|}U(\mathbf{l},\mathbf{q}) = \frac{1}{|\mathbf{q}|}(e(\mathbf{l}-\mathbf{q}) - e(\mathbf{l})) \tag{1.129}$$

can be used as coordinate on at least part of $S(t,\mathbf{q})$. $\tilde{U}(\mathbf{l},\mathbf{q}) = 2|\mathbf{l}|\hat{l}\cdot\hat{q}$ thus the part the dependence of $\tilde{U}$ on the angle between $\mathbf{q}$ and $\mathbf{l}$ is essentially quadratic when $\mathbf{q}$ and $\mathbf{l}$ are close to parallel. However for small $\mathbf{q}$ the intersection never lies in this region and $\tilde{U}$ is uniformly bounded from below there.

The case $|\mathbf{q}| \approx 2$ has to be treated differently.

Remark 8. *For $d \geq 3$, there exists a parameterization*

$$\begin{aligned} \Theta_d : [0,\pi] \times S^{d-2} &\longrightarrow S^{d-1} \\ (\theta_1, \tilde{\theta}) &\longmapsto \theta(\theta_1,\theta) = (\cos\theta_1, \sin\theta_1\tilde{\theta}) \end{aligned} \tag{1.130}$$

of S^{d-1} with Jacobian $J_{ang} = (\sin\theta_1)^{d-2}$ bounded by 1 and smooth for $\theta_1 \notin \{0,\pi\}$. Therefore for notational uniformity it is convenient to extend this to $d = 2$ and set $S^0 = \{-1,1\}$. Moreover set $\mathrm{Vol}(S^{d-2}) = 2$, $J_{ang} = 1$, and $\int_{S^0} \mathrm{d}\tilde{\theta} f(\tilde{\theta}) = \sum_{\tilde{\theta}=\pm 1} f(\tilde{\theta})$.

Lemma 9. *Let $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. Let $\delta \leq \frac{1}{5}$, $r_1 = \frac{2}{15}$, $\kappa_s = \frac{1}{80}$, $\eta_1 = \frac{1}{4}\sqrt{2}$, and $\eta_2 = v_0 = \frac{1}{2}$. Then there exist positive functions $\chi_3, \chi_4, \chi_5, \chi_6 \in C^\infty(\mathbb{R}^d)$ such that*

$$\chi_3 + \chi_4 = 1, \quad \chi_5 + \chi_6 = 1 \tag{1.131}$$

and for all $\mathbf{q} \in \mathbb{R}^d$ with $|\mathbf{q}| \leq \kappa_s$

i)

$$\forall \mathbf{l} \in \operatorname{supp}\chi_3 \qquad |e(\mathbf{l})| < r_1 \quad \textit{and} \quad |e(\mathbf{l}-\mathbf{q})| < \frac{5}{4}r_1 \tag{1.132}$$

$$\forall \mathbf{l} \in \operatorname{supp}\chi_4 \qquad |e(\mathbf{l})| > \frac{1}{2}r_1 \quad \textit{and} \quad |e(\mathbf{l}-\mathbf{q})| > \frac{1}{4}r_1 \tag{1.133}$$

ii) For all $\mathbf{l} \in \operatorname{supp}\chi_3\chi_6$, $\mathbf{q} \neq 0$, $|\tilde{U}(\mathbf{l},\mathbf{q})| > \eta_2$, and the coordinate axes can be taken such that on the support of $\chi_3\chi_5$ the the C^∞ parameterization of S^{d-1} with angles $\theta_1, \tilde{\theta}$ described above is such that

$$\left| \frac{\partial}{\partial\theta_1}\tilde{U}(\boldsymbol{\pi}(\rho,\theta(\theta_1,\tilde{\theta})),\mathbf{q}) \right| \geq \eta_1 \tag{1.134}$$

for all $\boldsymbol{\pi}(\rho,\theta(\theta_1,\tilde{\theta})) \in \operatorname{supp}\chi_3\chi_5$. The Jacobian of the change of coordinates to $(\theta_1,\tilde{\theta})$ is then given by $J_{ang}(\theta_1)$ and $J_{ang} > 2^{-\frac{3d}{2}}$ for $\boldsymbol{\pi}(\rho,\theta(\theta_1,\tilde{\theta})) \in \operatorname{supp}\chi_3\chi_5$

iii) $\left|\frac{\partial}{\partial\rho}e(\boldsymbol{\pi}(\rho,\theta),\mathbf{q},t)\right| > v_0$ *for all* $t\in[0,1], \boldsymbol{\pi}(\rho,\theta)\in \operatorname{supp}\chi_3$.

Proof. Let $U(\mathbf{l},\mathbf{q}) = e(\mathbf{l}-\mathbf{q}) - e(\mathbf{l})$ and choose coordinates such that $\mathbf{q} = (q,0,\dots,0)$. Then $U(\mathbf{l},\mathbf{q}) = q^2 - 2ql_1$. On the set where $|e(\mathbf{l})| < \delta$ choose the polar coordinates (ρ,θ) with $\rho = e(\mathbf{l})$ such that θ_1 is the angle between $\mathbf{q}$ and $\boldsymbol{\pi}(\rho,\theta)$. Write $r(\rho)$ for the inverse relation $r(\rho) = \sqrt{\rho+1}$. Then

$$U(\rho,\theta) = q(q-2r(\rho)\cos\theta_1) \tag{1.135}$$

Choose $\chi_4 = 1-\chi(\left(\frac{e(\mathbf{l})}{r_1}\right)^2)$. Then on the support of χ_3

$$|e(\mathbf{l}-\mathbf{q})| \leq |U(\mathbf{l},\mathbf{q})| + |e(\mathbf{l})| \leq r_1 + \frac{r_1}{12}|\frac{r_1}{12}+2r_1| < r_1 + \frac{1}{4}r_1 = \frac{5}{4}r_1 \tag{1.136}$$

On the support of χ_4,

$$|e(\mathbf{l}-\mathbf{q})| \geq |e(\mathbf{l})| - |U(\mathbf{l},\mathbf{q})| > \frac{1}{4}r_1 \tag{1.137}$$

For the second part of the lemma we work with the polar coordinates. We show that although the difference between the two band relations, U, goes to zero with $\mathbf{q}$, $\tilde{U}$, the rescaled difference does not. Choose $\chi_6(\mathbf{l}) = \chi(2(1-(\hat{l}\cdot\hat{q})^2)$. That means that $\chi_6(\boldsymbol{\pi}(\rho,\theta(\theta_1,\hat{\theta}))) = \chi(2(\sin\theta_1)^2)$.

In the polar coordinates we have $\tilde{U}(\rho,\theta_1) = q - 2r(\rho)\cos\theta_1$. On the support of $\chi_6\chi_3$: $\cos\theta_1 > \frac{1}{2}\sqrt{3}$ and thus

$$|\tilde{U}(\rho,\theta)| \geq |2r(\rho)\cos\theta_1| - \kappa_s \geq |\cos\theta_1| - \kappa_s \geq \frac{1}{2}\sqrt{3} - \frac{r_1}{12} > \frac{1}{2} = \eta_2 \tag{1.138}$$

Furthermore, on the support of $\chi_3\chi_5$, $|\sin\theta_1| > \frac{1}{4}\sqrt{2}$ and thus

$$\left|\frac{\partial}{\partial\theta_1}\tilde{U}(\boldsymbol{\pi}(\rho,\theta(\theta_1,\tilde{\theta})),\mathbf{q})\right| = \left|\frac{\partial}{\partial\theta_1}(q-2r(\rho)\cos\theta_1)\right| = |2r(\rho)\sin\theta_1| \geq \frac{1}{4}\sqrt{2} = \eta_1 \tag{1.139}$$

So the coordinates from Remark 8 satisfy Proposition ii of the lemma.

Finally observe that

$$\begin{aligned}\frac{\partial}{\partial\rho}e(\boldsymbol{\pi}(\rho,\theta),\mathbf{q},t) &= \frac{\partial}{\partial\rho}(\rho + tq^2 - 2tqr(\rho)\cos\theta_1)\\ &= 1 - \frac{tq\cos\theta_1}{\sqrt{\rho+1}}\end{aligned} \tag{1.140}$$

and thus

$$\left|\frac{\partial}{\partial\rho}e(\boldsymbol{\pi}(\rho,\theta),\mathbf{q},t)\right| \geq 1-\frac{r_1}{6} \geq \frac{1}{2} \tag{1.141}$$

and so proposition (iii) holds with $v_0 = \frac{1}{2}$. □

By inspecting the bounds above we see that some of them can be extended to much larger values of $\mathbf{q}$.

Corollary 10. *Let $\kappa_t < 2\sqrt{1-r_1}$ and $\mathbf{q} \neq 0$, then there exist constants ξ_1 and ξ_2 such that for all $\mathbf{q} \neq 0$ with $|\mathbf{q}| < \kappa_t$ and all $\boldsymbol{\pi}(\rho,\theta) \in U(S,r_1)$ such that $|\tilde{U}(\boldsymbol{\pi}(\rho,\theta),\mathbf{q})| < \xi_2$*

$$\left|\frac{\partial}{\partial\theta_1}\tilde{U}(\boldsymbol{\pi}(\rho,\theta(\theta_1,\tilde{\theta})),\mathbf{q})\right| > \xi_1 \tag{1.142}$$

1.4.3 The basic argument for factors of c and why they are nontrivial to extract

Pairing with L^1 functions

To see how the bounds in the theorems arise, and where the problems lie, it is instructive to look at a few simpler cases. Let f be a measurable function on $\mathbb{R}\times\Omega$ such that $\sup_{l_0}|f(l_0,\mathbf{k})| \in L^1(\Omega)$. Let D^* be defined as in(1.98). Throughout this section p will be in $(\frac{\pi}{\beta}+\frac{2\pi}{\beta}\mathbb{Z})\times\Omega$ as in Theorem 2. Because D is increasing in its second argument:

$$\left|\int_{\mathbb{R}}\frac{\mathrm{d}_\beta l_0}{2\pi}\int_{\Omega}\mathrm{d}\mathbf{l}f(l_0,\mathbf{l})D^*(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)\right| \leq c\frac{1}{2}\Lambda\|\sup_{l_0}f(l_0,\cdot)\|_1 \tag{1.143}$$

where the L_1 norm is over the region Ω.

The one-loop self-energy

The aditional factors appearing in the integral for the vertex-correction are propagators $C(l)$, which are singular for $l_0 = e(\mathbf{l}) = 0$. Thus when they appear as part of the function f, the right-hand side of (1.143) is not bounded uniformly in β. The problem in proving the theorem is to control these singularities and show that they do not change the qualitative behavior of the integral. To show the methods used we first look at the problem with one aditional singular factor.

Lemma 11. *Let* $p \in (\frac{\pi}{\beta} + \frac{2\pi}{\beta}\mathbb{Z}) \times \Omega$*.For the one1-loop self-energy* $\Sigma_1(p)$*, given by*

$$\Sigma_1(p) = \int_{\mathbb{R}} \frac{d_\beta l_0}{2\pi} \int_{\Omega} \frac{d\mathbf{l}}{(2\pi)^d} \frac{1}{il_0 - e(\mathbf{l})} D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{1.144}$$

$$|\Sigma(p)| \le cM_\Sigma^* + \frac{1}{\beta} M_\Sigma^0 \tag{1.145}$$

with

$$M_\Sigma^* = \frac{\mathrm{Vol}(S^{d+1})}{(2\pi)^d} \Big(\kappa_{\mathbf{p}}(1+\delta)^{d-2} d^2 + (1+\delta)^{d-2} \kappa_{\mathbf{p}} |\log\delta| + 6\Lambda^d(|\log\delta| + \log\Lambda)\Big) \tag{1.146}$$

$$\kappa_{\mathbf{p}} = (1 + \delta + |\mathbf{p}|) \tag{1.147}$$

Sign cancellations in the integral are important to obtain (1.145). This can be seen from the following. Let $c < 1$, $p_0 = 0$ and $\mathbf{p}$ and $\epsilon > 0$ such that $\inf_{\mathbf{l} \in U(S,\delta)} |\mathbf{l} - \mathbf{p}| \ge \epsilon$. Then because D increases monotonically in its second argument

$$\begin{aligned} &\int_{\mathbb{R}} dl_0 \int_{\Omega} d\mathbf{l} \left| \frac{1}{il_0 - e(\mathbf{l})} D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \right| \\ &= \int_{\mathbb{R}} dl_0 \int_{\Omega} d\mathbf{l} \frac{1}{\sqrt{l_0^2 + e(\mathbf{l})^2}} \frac{c^2 |\mathbf{l} - \mathbf{p}|^2}{l_0^2 + c^2 |\mathbf{l} - \mathbf{p}|^2} \end{aligned} \tag{1.148}$$

and restricting to the most singular contribution

$$\ge \int_{-c\delta}^{c\delta} dl_0 \int_{U(S,\delta)} d\mathbf{l} \frac{1}{\sqrt{l_0^2 + e(\mathbf{l})^2}} \frac{c^2\epsilon^2}{l_0^2 + c^2\epsilon^2} \tag{1.149}$$

$$\ge \int_{-c\delta}^{c\delta} dl_0 \int_{-\delta}^{\delta} d\rho \frac{1}{\sqrt{l_0^2 + \rho^2}} d\theta \tilde{J}_1(\rho, \theta) \frac{\epsilon^2}{1 + \epsilon^2} \tag{1.150}$$

$$\ge \text{const}\, \frac{\epsilon^2}{1+\epsilon^2} \int_0^{c\delta} dl_0 \int_{-\delta}^{\delta} d\rho \frac{1}{\sqrt{l_0^2 + \rho^2}} \tag{1.151}$$

$$\ge \text{const}\, \frac{\epsilon^2}{1+\epsilon^2} \int_0^{c\delta} dl_0 \log\left(\frac{\delta}{|l_0|} + \sqrt{\frac{\delta^2}{l_0^2} + 1}\right) \tag{1.152}$$

$$\ge \text{const} \int_0^1 dx\, c\delta \log\left(\frac{1}{c}\left(\frac{1}{|x|} + \sqrt{\frac{1}{x^2} + c^2}\right)\right) \tag{1.153}$$

$$\ge \text{const}\, .\, c|\log c| \tag{1.154}$$

This means that the bound (1.145) does not hold if integrand in (1.144) is replaced by its absolute value.

Proof of Lemma 11. The most convenient way to make use of the cancellations in the integral is to use integration by parts with respect to $\rho = e(\mathbf{l})$.

To prepare for the integration by parts a change to the (ρ, θ) variables is made. To be able to do this the region containing the singularity is cut out. It is also convenient to isolate the c-independent contributions.

$$\Sigma_1(p) = \Sigma_R(p) + \Sigma_S(p) + \Sigma_0(p_0) \tag{1.155}$$

where

$$\Sigma_R(p) = \int_{\mathbb{R}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{1}{il_0 - e(\mathbf{l})} D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \mathbb{1}(|l_0| \geq \frac{1}{2} \vee |e(\mathbf{l})| \geq \delta) \tag{1.156}$$

$$\Sigma_S(p) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{U(S,\delta)} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{1}{il_0 - e(\mathbf{l})} D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{1.157}$$

and

$$\Sigma_0(p_0) = \frac{1}{\beta} \int_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{1}{ip_0 - e(\mathbf{l})} \tag{1.158}$$

is the contribution of the term with $l_0 = p_0$ to the frequency sum. By Lemma 4

$$|\Sigma_R(p)| \leq c \frac{\Lambda \operatorname{Vol}(\Omega)}{2\delta(2\pi)^d} \tag{1.159}$$

In fact we can do better. Define $\tau_y(x)$ for $y > 0$ as

$$\tau_y(x) = \begin{cases} \frac{1}{y} & |x| \leq y \\ \frac{1}{|x|} & |x| > y \end{cases} \tag{1.160}$$

Then using

$$\frac{\mathbb{1}(|l_0| \geq \frac{1}{2} \vee |e(\mathbf{l})| \geq \delta)}{|il_0 - e(\mathbf{l})|} \leq \tau_\delta(e(\mathbf{l})) \leq 3\tau_\delta(|\mathbf{l}| - 1) \tag{1.161}$$

we have

$$|\Sigma_R(p)| \leq c\frac{3}{2} \int_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \tau_\delta(|\mathbf{l}| - 1)|\mathbf{l} - \mathbf{p}| \tag{1.162}$$

By changing to polar coordinates on the right hand side we see that

$$|\Sigma_R(p)| \leq c\frac{3 \operatorname{Vol}(S^{d-1})}{2(2\pi)^d} \Lambda \int_0^{\frac{1}{2}\Lambda} \mathrm{d}r r^{d-1} \tau_\delta(r-1) \leq c\frac{6 \operatorname{Vol}(S^{d-1})\Lambda^d}{(2\pi)^d} (|\log \delta| + \log \Lambda) \tag{1.163}$$

Changing to the variables $(\rho = e(\boldsymbol{\pi}(\rho,\theta)),\theta)$ in $U(S,\delta)$ we have:

$$\Sigma_S(p) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{1}{il_0 - \rho} D^*(l_0 - p_0, c|\boldsymbol{\pi}(\rho,\theta) - \mathbf{p}|) J_1(\boldsymbol{\pi}(\rho,\theta)) \tag{1.164}$$

$$= B_\Sigma - I_\Sigma \tag{1.165}$$

where, by integration by parts in ρ

$$B_\Sigma = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \left[\mathrm{Log}(il_0 - \rho) \int \frac{\mathrm{d}\theta}{(2\pi)^d} D^*(l_0 - p_0, c|\boldsymbol{\pi}(\rho,\theta) - \mathbf{p}|) J_1(\boldsymbol{\pi}(\rho,\theta)) \right]_{\rho=-\delta}^{\rho=\delta} \tag{1.166}$$

$$I_\Sigma = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho\, \mathrm{Log}(il_0 - \rho) \frac{\partial}{\partial\rho} \int \frac{\mathrm{d}\theta}{(2\pi)^d} D^*(l_0 - p_0, c|\boldsymbol{\pi}(\rho,\theta) - \mathbf{p}|) J_1(\boldsymbol{\pi}(\rho,\theta)) \tag{1.167}$$

These two integrals exist at positive temperature because each integrand is bounded by a (β-dependent) constant.

By (1.121) $|\mathrm{Log}(il_0 - \rho)||_{\rho=\pm\delta} \le 2|\log|\rho||$. Using this and Lemma 4:

$$|B_\Sigma| \le c2|\log\delta|\kappa_{\mathbf{p}} \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} |J_1|_{0,\delta} \tag{1.168}$$

Using the same bound on the logarithm

$$|I_\Sigma| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho |2\log|\rho|| \int \frac{\mathrm{d}\theta}{(2\pi)^d} \left| \frac{\partial}{\partial\rho} J_1(\rho,\theta) D(l_0 - p_0, c|\boldsymbol{\pi}(\rho,\theta) - \mathbf{p}|) \right| \tag{1.169}$$

By Lemma 4 this is

$$\le c\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \kappa_{\mathbf{p}} |J_1|_{1,\delta} |\frac{\partial\boldsymbol{\pi}(\rho,\theta)}{\partial\rho}|_{0,\delta} \int_{-\delta}^{\delta} \mathrm{d}\rho(-2\log|\rho|) \tag{1.170}$$

$$\le c2\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \kappa_{\mathbf{p}} |J_1|_{1,\delta} |\frac{\partial\boldsymbol{\pi}(\rho,\theta)}{\partial\rho}|_{0,\delta} \tag{1.171}$$

because

$$\int_{-\delta}^{\delta} \mathrm{d}\rho |\mathrm{Log}|\rho|| \le 2\int_{0}^{\frac{1}{2}} \mathrm{d}\rho |\log\rho| \le 2 \tag{1.172}$$

Applying the bounds from Lemma 5 and Corollary 7 and the definition of Ω the above results give

$$|\Sigma_R(p)| \leq c\frac{6\,\mathrm{Vol}(S^{d-1})\Lambda^d}{(2\pi)^d}(|\log\delta| + \log\Lambda) \tag{1.173}$$

$$|B_\Sigma| \leq c\kappa_{\mathbf{p}}\frac{\Lambda(1+\delta)^{d-1}\,\mathrm{Vol}(S^{d-1})}{(2\pi)^d}|\log\delta| \tag{1.174}$$

$$|I_\Sigma| \leq cd^2\kappa_{\mathbf{p}}\frac{(1+\delta)^{d-2}\,\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \tag{1.175}$$

Thus

$$|\Sigma_R(p) + \Sigma_S(p)| \leq |\Sigma_R(p)| + |B_\Sigma| + |I_\Sigma| \tag{1.176}$$

$$\leq c\,\frac{\mathrm{Vol}(S^{d+1})}{(2\pi)^d}\Big(\kappa_{\mathbf{p}}(1+\delta)^{d-2}d^2 + \kappa_{\mathbf{p}}(1+\delta)^{d-2}|\log\delta| + 6\Lambda^d(|\log\delta| + \log\Lambda)\Big) \tag{1.177}$$

which gives (1.145) and (1.146).

Bounding $\Sigma_0(p_0)$ analogously gives the $\frac{1}{\beta}$ term. See also Appendix A. □

1.4.4 Discussion of previous arguments

Before we start with the proof of Theorem 2, we briefly review the existing arguments given for its validity. Most follow Migdal's origional argument rather directly[Mig58, IOS92, Eli61, Hol64, AGD75, FW71]. Basically they are all variants on the following argument (for $d = 3$)

- First it is remarked that $D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)$ rapidly falls off as $(l_0 - p_0)^2$ when $|l_0 - p_0| \geq c\eta$ for some constant η. This is then used to justify ignoring the contribution of this region.
- In the region $|l_0 - p_0| < c\eta$, the phonon propagator is said to be ≈ 1 and replaced by the constant one. The c-dependence is then purely in the restriction on the l_0 summation/integration region. This reduces the integral to that for the so called particle-hole bubble.

$$\int\limits_{|l_0 - p_0| < c\eta} \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} C(l)C(l-q) \tag{1.178}$$

- In (1.178) the difference $e(\mathbf{l}+\mathbf{q}) - e(\mathbf{l})$ is approximated by its first order Taylor expansion. This is written in the form $|\mathbf{q}|x$ where x contains the angular dependence.

The integrals over x and $E = e(\mathbf{l})$ are then evaluated to get logarithmic factors containing the nominators of the electron propagators. It is then argued that the actual divergences of these logarithm occur only for unphysical parameter values.

In additions to these three main points the different authors apply other minor approximations.

Unfortunately there are various weaknesses in this argument. Although the phonon propagator does indeed decay quadratically in $l_0 - p_0$, large $l_0 - p_0$ regions can still contribute a significant part to $\int \mathrm{d}l_0 D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)$. Moreover, although $D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \approx 1$ for $l_0 - p_0$ small, it follows from (1.101) and Lemma 4 that the second derivative is only integrable (in three dimensions, $d = 2$ requires taking the l_0 sum into account). As there are two integrations done on the rest of the integrand this is important. Integration by parts would mean that for each integration there is a derivate acting on D. Finally one must justify the substitution $e(\mathbf{l} + \mathbf{q}) - e(\mathbf{l}) \approx |\mathbf{q}|x$, which is non-trivial (and for $d = 2$ not even possible everywhere). It is also surprising that none of the authors studying the non-zero temperature model has noticed the presence of the $O(\beta^{-1})$ term (which is typically small so it does not necessary affect the application of the theorem to justify the approximation that higher order corrections can be negclected).

With the benefit of hindsight it is possible to see that our method to prove Theorems 2 and 3 is a combination of a rigorous version of the latter replacement, using a Feynman trick and Lemma 9, and a careful analysis of the behavior of the derivatives of D to replace the former.

1.5 Motivation

Widely used approximation of unsure validity

Our motivation for studying the vertex correction and the Migdal-Eliashberg approximation in detail is twofold. First, the Migdal-Eliashberg approximation is still widely used in the study of the Electron-Phonon model. In fact it forms one of the cornerstones of the microscopic understanding of the theory of superconductivity.

In the light of that it is surprising that a more rigorous study in particular of the higher order contributions was never published. In fact there are even doubts about its validity

[Mah81, Sch64].

New interest from high temperature superconductivity

The discovery of high temperature cuprate superconductors had sparked new interest in the theory of superconductivity. Unfortunately the standard theory does not adequately explain these materials. One of the reasons for that is that the condition for the validity of the Migdal-Eliashberg approximation, i.e., the smallness of $O(c)$, possibly is not satisfied because these materials have an unusually low Fermi velocity. This has caused various authors to look at Migdal's theorem again and study the effects of including contributions beyond Γ_1 [GPS95, DDL97, IOS92].

Other proposed mechanisms for superconductivity in high-temperature superconductors suggest other bosonic excitations as the interaction mechanism. The probable absence of a Migdal-Eliashberg approximation for interactions through phonons has also brought up the question of whether there is a Migdal-theorem like result for these interactions [Mil92, HLBM76].

In this light it seems worthwhile to try to study 'Migdal's Theorem' in more detail and with more rigor, especially given that recent advances in the rigorous treatment of renormalization in fermionic field theories have made available new tools to do so.

Chapter 2

The one-loop vertex correction

2.1 Introduction

2.1.1 Sign cancellations and integration by parts

In this chapter we consider the vertex correction to one-loop order in the presence of an ultra-violet cut-off. The c-dependent contribution is given by

$$\Gamma_3^*(p,q) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int_\Omega \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C(l)C(l-q)D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{2.1}$$

The treatment of the other term in Γ_3,

$$\Gamma_{3,\mathrm{temp}}^*(p_0,q) = \frac{1}{\beta} \int_\Omega \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \, C(l)C(l-q)|_{l_0=p_0} \tag{2.2}$$

is deferred to Appendix A.

As illustrated in section 1.4.3 when discussing the one-loop self-energy contributions, sign cancellations are important. In the one-loop vertex correction, two additional singular factors appear. In fact here the integral is not even absolute convergent at zero temperature. For $\beta = \infty, q = 0$

$$\int \mathrm{d}l |C(l)|^2 \geq \mathrm{const} \int \mathrm{d}l_0 \int_{-\delta}^{\delta} \mathrm{d}\rho \frac{1}{l_0^2 + \rho^2} = \infty \tag{2.3}$$

At finite β the integral does converge absolutely. Below we show that it is bounded uniformly in β and therefore the $\beta \to \infty$ limit does exist. This limit is then taken as the zero temperature theory expression. In the following chapters we shall see that for higher order terms renormalization is needed to define the limit.

Again our main device will be to integrate away the singular factors in the fermion propagators using integration by parts. This introduces some technical difficulties arising from dealing with the derivatives of the other factors. For instance the result of summing the second derivatives of D over the frequency is no longer bounded uniformly in β but only the integral over momenta that term is. In the rest of this chapter we show that it is indeed possible to do the integration by parts and that these technical difficulties can be dealt with, giving a proof of Migdal's theorem to one-loop order.

2.1.2 The three cases

Because there are now two singular factors at least two integrations by parts are needed in the most singular region. The geometry of the singular region depends $\mathbf{q}$. There are three cases:

Small $\mathbf{q}$ The most singular region is at the intersection of the Fermi-surface S and its translate by $\mathbf{q}$, $S_{\mathbf{q}}$. When $\mathbf{q}$ is small these intersect at a very sharp angle. S and $S_{\mathbf{q}}$ almost coincide and the two annuli $U(S, \delta)$ and $U(S, \mathbf{q}, \delta)$ have a large overlap. In fact from Lemma 9 we see that for $r_1 = \frac{2}{3}\delta$ and $\kappa_s = \frac{r_1}{12}$, the annulus $U(S, \mathbf{q}, \frac{1}{4}r_1)$ is completely contained in $U(S, r_1)$.

Touching Fermi surfaces The other case that gives a sharp intersection angle arises when $\mathbf{q}$ is the (almost) the difference between a point on the Fermi-surface and its antipode. The two Fermi-surfaces then (nearly) touch. For $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$, this happens for all $\mathbf{q}$ with $|\mathbf{q}| \approx 2$. In fact we see later on that in this case we do not obtain an $O(c)$ bound. For $d = 3$ we can get $O(c(\log c)^2)$ however.

Transversal Fermi-surfaces Take $\kappa_t = \frac{3}{2}$, (satisfying the precondition of Corollary 10). When $\kappa_s < |\mathbf{q}| < \kappa_t$, the intersection is transversal, and $a = e(\mathbf{l})$ and $b = e(\mathbf{l}-\mathbf{q})$ can be used as independent variables. This means each factor $\frac{1}{il_0 - e(\mathbf{l})}$ and $\frac{1}{i(l_0-q_0)-e(\mathbf{l}-\mathbf{q})}$ can be integrated separately, without producing derivatives of the other Fermion propagator (thus creating more singular factors). Therefore this case is simplest and we will start here.

2.2 Transversally intersecting Fermi-surfaces

2.2.1 Proving the transversal case

In this section we prove the following part of the theorem:

Lemma 12. *There exists constants M_{g1} and M_{g2} such that for all $\mathbf{q}, \mathbf{p}$, all $q_0 \in \frac{2\pi}{\beta}\mathbb{Z}$, and all $p_0 \in \frac{2\pi}{\beta}(\mathbb{Z}+1)$ with*

$$\kappa_s \leq |\mathbf{q}| \leq \kappa_t \tag{2.4}$$

the inequality

$$|\Gamma^*(p,q)_3| \leq c(M_{g1}\kappa_{\mathbf{p}}^2 + M_{g2}\Lambda^{d+1}) \leq cM_g \tag{2.5}$$

holds with $\kappa_{\mathbf{p}} = (1+\delta+\kappa_t+|\mathbf{p}|)$ and $M_g = (M_{g1}\Lambda^2 + M_{g2}\Lambda^{d+1})$.

Proof. This lemma is proven as in the self-energy case by first changing to the appropriate variables in the singular regions and then integrating by parts. The contributions from the regular and singular regions are defined using a partition of unity.

Let $\chi_i : [0,\infty) \to [0,1]$, $i = 0, 1$ be the smooth partition of unity given by $\chi_1 = \chi$ and $\chi_0 = 1-\chi$. Let $\delta_1 = \min\{\frac{\eta_2\kappa_s}{2}, r_1\}$ with η_2, r_1 and κ_2 as in Lemma 9 and

$$C_0(l) = C(l)\chi_0(\delta_1^{-2}(l_0^2 + e(\mathbf{l})^2)) \tag{2.6}$$

$$C_1(l) = C(l)\chi_1(\delta_1^{-2}(l_0^2 + e(\mathbf{l})^2)) \tag{2.7}$$

Now $\chi_0(\delta_1^{-2}(l_0^2 + e(\mathbf{l})^2)) > 0$ implies $l_0^2 + e(\mathbf{l})^2 > \frac{\delta_1^2}{4}$ and then

$$\frac{1}{|il_0 - e(\mathbf{l})|} = \frac{1}{\sqrt{l_0^2 + e(\mathbf{l})^2}} \leq \frac{1}{\sqrt{\frac{\delta_1^2}{4}}} = \frac{2}{\delta_1} \tag{2.8}$$

This implies $C_0(l)$ is C^∞ and $|C_0(l)| \leq \frac{2}{\delta_1}$.

Then

$$\begin{aligned}\Gamma^*(p,q)_3 &= \int_{\mathbb{R}} \frac{d_\beta l_0}{2\pi} \int_{\Omega} \frac{d\mathbf{l}}{(2\pi)^d} \frac{1}{il_0 - e(\mathbf{l})} \frac{1}{i(l_0 - q_0) - e(\mathbf{l}-\mathbf{q})} D^*(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \\ &= A_{00} + A_{10} + A_{01} + A_{11}\end{aligned} \tag{2.9}$$

where

$$A_{jk}(p,q) = \int_{\mathbb{R}} \frac{d_\beta l_0}{2\pi} \int_{\Omega} \frac{d\mathbf{l}}{(2\pi)^d} C_j(l) C_k(l-q) D^*(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \tag{2.10}$$

Because $C_0(l)$ is bounded as

$$|C_0(l)| \le \tau_{\frac{1}{2}\delta_1}(e(\mathbf{l})) \le 6\tau_{\delta_1}(|\mathbf{l}|-1) \tag{2.11}$$

with τ as in (1.160). By Lemma 4

$$|A_{00}| \le \int\limits_{\mathbb{R}} \frac{\mathrm{d}_\beta l_0}{2\pi} 36 \int\limits_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D^*(l_0-p_0, c|\mathbf{l}-\mathbf{p}|)\tau_{\delta_1}(|\mathbf{l}|-1)\tau_{\delta_1}(|\mathbf{l}-q|-1) \le c\frac{216\,\mathrm{Vol}(S^{d-1})\Lambda^d}{\delta_1 2^d \pi^d} \tag{2.12}$$

By directly replacing the propagator by its supremum on its support we get the simpler bound

$$|A_{00}| \le \frac{4}{\delta_1^2}\int\limits_{\mathbb{R}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D^*(l_0-p_0, c|\mathbf{l}-\mathbf{p}|) \le c\frac{1}{\delta_1^2 2^{d-1}\pi^d}\,\mathrm{Vol}(\Omega)\Lambda \tag{2.13}$$

Note that $\chi_1(\delta_1^{-2}(l_0^2+e(\mathbf{l})^2)) > 0$ implies $l_0^2+e(\mathbf{l})^2 \le \delta_1^2$ so in particular $|l_0| \le \delta_1$ and $|e(\mathbf{l})| \le \delta_1$. Therefore

$$A_{10} = \int\limits_{-\delta_1}^{\delta_1} \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_{U(S,\delta_1)} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C_1(l)C_0(l+q)D^*(l_0-p_0, c|\mathbf{l}-\mathbf{p}|) \tag{2.14}$$

$$= \int\limits_{-\delta_1}^{\delta_1} \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_{-\delta_1}^{\delta_1} \mathrm{d}\rho \frac{1}{il_0-\rho} \int\limits_{S^{d-1}} \frac{\mathrm{d}\theta}{(2\pi)^d} K_{10}(l_0, \boldsymbol{\pi}(\rho,\theta))D^*(l_0-p_0, c|\boldsymbol{\pi}(\rho,\theta)-\mathbf{p}|) \tag{2.15}$$

where

$$K_{10}(l) = \chi_1(\delta_1^{-2}(l_0^2+e(\mathbf{l})^2))C_0(l-q)J_1(l) \tag{2.16}$$

Because the indicator function is zero for $\rho = \pm\delta_1$, the boundary terms from integrating by parts vanish, and we get

$$A_{10} = \int\limits_{-\delta_1}^{\delta_1} \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_{-\delta_1}^{\delta_1} \mathrm{d}\rho\, \mathrm{Log}(il_0-\rho) \int\limits_{S^{d-1}} \frac{\mathrm{d}\theta}{(2\pi)^d}\frac{\partial}{\partial\rho}(K_{10}D^*(l_0-p_0, c|\boldsymbol{\pi}(\rho,\theta)-\mathbf{p}|)) \tag{2.17}$$

By Lemma 4 this integral exists at finite β and is bounded as

$$|A_{10}| \le c\kappa_{\mathbf{p}}|K_{10}|_{1,\delta_1}\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d}\left|\frac{\partial\mathbf{l}}{\partial\rho}\right|_{0,\delta}\int\limits_{-\delta_1}^{\delta_1}|2\log|\rho||\mathrm{d}\rho \tag{2.18}$$

where we used $\sup_{\mathbf{l}\in U(S,\delta_1)}|-\mathbf{p}| \le \kappa_{\mathbf{p}}$ Now

$$\begin{aligned} |K_{10}|_{1,\delta_1} \le{}& \frac{2|J_1|_{1,\delta}}{\delta_1} \\ &+ d\frac{2|J_1|_{0,\delta}}{\delta_1}\sup_{U(S,\delta_1))}\left|\frac{\partial}{\partial\mathbf{l}}(\chi_1(\delta_1^{-2}(l_0^2+e(\mathbf{l})^2))\chi_0(\delta_1^{-2}((l_0-q_0)^2+e(\mathbf{l}-\mathbf{q})^2)))\right| \\ &+ d|J_1|_{0,\delta}\frac{4}{\delta_1^2}|e|_1 \end{aligned} \tag{2.19}$$

and

$$\left|\frac{\partial}{\partial y}\chi_0(\delta_1^{-2}(x^2+y^2))\right| \leq 2\frac{|y|}{\delta_1^2}\left|\chi_0'(\delta_1^{-2}(x^2+y^2))\right| \leq 2|\chi|_0\delta_1\frac{1}{\delta_1^2} \leq \frac{2}{\delta_1}|\chi|_1 \tag{2.20}$$

so

$$\left|\frac{\partial}{\partial \mathbf{l}}\chi_0(\delta_1^{-2}((l_0-q_0)^2+e(\mathbf{l}-\mathbf{q})^2))\right| \leq \frac{2|\chi|_1}{\delta_1}\left|\frac{\partial e(\mathbf{l}-\mathbf{q})}{\partial \mathbf{l}}\right| \leq \frac{2|\chi|_1}{\delta_1}|\nabla e(\mathbf{l}-\mathbf{q})|\left|\frac{\partial \mathbf{l}}{\partial \rho}\right| \leq \frac{2|\chi|_1}{\delta_1}|e|_1 \tag{2.21}$$

which gives

$$|K_{10}|_{1,\delta} \leq \frac{2|J_1|_{1,\delta}}{\delta_1} + \frac{12d|e|_1|\chi_1|_1|J_1|_{1,\delta}}{\delta_1^2} \tag{2.22}$$

which with the bounds from Lemma 5 and using $d \geq 2$

$$\leq \frac{d^2}{\delta_1^2}(1+\delta)^{d-2} + \frac{3d^2}{\delta_1^2}(1+\delta)^{d-2}|\chi|_1 \leq \frac{4d^2}{\delta_1^2}|\chi|_1(1+\delta)^{d-2} \tag{2.23}$$

and thus by (1.172)

$$|A_{10}| \leq c\frac{16d^2}{\delta_1^2}\kappa_{\mathbf{p}}(1+\delta)^{d-2}\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d}|\chi|_1 \tag{2.24}$$

Observe that apart from an irrelevant change in integration region A_{01} becomes A_{10} after the change of variables $l_0 \mapsto l_0 + q_0$, $\mathbf{l} \mapsto \mathbf{l} + \mathbf{q}$ and with $\mathbf{p}$ replaced by $\mathbf{p}' = \mathbf{p} - \mathbf{q}$ and likewise p_0 by $p_0' = p_0 - q_0$. Because the bound (2.24) is uniform in $\mathbf{p}$,p_0 and $\mathbf{q}$, A_{01} obeys the same bound.

In A_{11} we need to do integration by parts twice. By our choice of δ_1 and the cut-off function χ on the support of the integrand in A_{11}(For $d = 2$ this support consists of two connected components (see Remark 8). For $\mathbf{q} \neq 0$:

$$|\tilde{U}| = \frac{1}{q}|e(\mathbf{l}-\mathbf{q}) - e(\mathbf{l})| \leq \frac{1}{\kappa_s}(|e(\mathbf{l}-\mathbf{q})| + |e(\mathbf{l})|) \tag{2.25}$$

$$\leq \frac{2\delta_1}{\kappa_s} < \eta_2 \tag{2.26}$$

By Lemma 5

$$A_{11} = \int_{-\delta_1}^{\delta_1}\frac{\mathrm{d}_\beta l_0}{2\pi}\int_{-\delta_1}^{\delta_1}\mathrm{d}\rho\int \mathrm{d}\theta_1\int\frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d}\frac{\chi_1(\delta_1^{-2}l_0^2+\rho^2)}{il_0-\rho}\frac{\chi_1(\delta_1^{-2}((l_0-q_0)^2+e(\boldsymbol{\pi}\mathbf{q})^2))}{i(l_0-q_0)-e(\boldsymbol{\pi}-\mathbf{q})}\Delta \tag{2.27}$$

and

$$\Delta = D(l_0 - p_0, c|\boldsymbol{\pi}(\rho,\theta) - \mathbf{p}|) J_{\mathrm{ang}}(\tilde{\theta}) J_1(\rho,\theta,\tilde{\theta}) \tag{2.28}$$

In each of these it is now possible to change to coordinates

$$\begin{cases} a = e(\mathbf{l}) = \rho \\ b = e(\mathbf{l}+\mathbf{q}) = \rho + U = \rho + q\tilde{U} \end{cases} \tag{2.29}$$

because by Corollary 10

$$|\frac{\partial b}{\partial \theta_1}| = q|\frac{\partial \tilde{U}}{\partial \theta_1}| \geq \kappa_s \eta_1 > 0 \tag{2.30}$$

In these coordinates the integral takes the convenient form

$$A_{11} = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta_1}^{\delta_1} \mathrm{d}a \int_{-\delta_1}^{\delta_1} \mathrm{d}b \frac{1}{il_0 - a} \frac{1}{i(l_0+q_0) - b} \int \frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d} K_{11}(l_0, \mathbf{l}(a,b,\tilde{\theta})) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{2.31}$$

where

$$K_{11}(l) = (\frac{\partial b}{\partial \theta_1})^{-1} J_{\mathrm{ang}} J_1 \chi_1(\delta_1^{-2}(l_0^2 + e(\mathbf{l})^2)) \chi_1(\delta_1^{-2}((l_0 - q_0)^2 + e(\mathbf{l} - \mathbf{q}))) \tag{2.32}$$

We now do an integration by parts with respect to a. Note that this is allowed because by Lemma 4 the integrals are all absolutely convergent at finite β. Again, the boundary terms vanish because the indicator functions (and their derivatives) in K_{11} are zero.

$$A_{11} = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d} \int_{-\delta_1}^{\delta_1} \mathrm{d}a \, \mathrm{Log}(il_0 - a) \frac{1}{i(l_0 - q_0) - b} \frac{\partial}{\partial a} \Big(K_{11} D(l_0 + p_0, \mathbf{l}(a,b,\tilde{\theta}) - \mathbf{p}) \Big) \tag{2.33}$$

After integration by parts with respect to b,

$$A_{11} = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta_1}^{\delta_1} \mathrm{d}a \int_{-\delta_1}^{\delta_1} \mathrm{d}b \int \frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d} \mathrm{Log}(il_0 - a) \, \mathrm{Log}(i(l_0 - q_0) - b)$$
$$\frac{\partial}{\partial b} \frac{\partial}{\partial a} \Big(K_{11} \, D(l_0 + p_0, \mathbf{l}(a,b,\tilde{\theta}) - \mathbf{p}) \Big) \tag{2.34}$$

By Lemma 4 and because $2 < 2|\log\delta_1|$ and on the integration region $|\mathrm{Log}(il_0 + a)| \leq 2|\log a|$,

$$|A_{11}| \leq 4\int \frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d} \int_{-\delta_1}^{\delta_1} \mathrm{d}a|\log|a|| \int_{-\delta_1}^{\delta_1} \mathrm{d}b|\log|b||\, c\frac{\kappa_{\mathbf{p}}^2}{|\mathbf{l}-\mathbf{p}|} \left(|K_{11}|_{1,\delta_1,\mathbf{q}}\left|\frac{\partial \mathbf{l}}{\partial b \partial a}\right| + 6|K_{11}|_{2,\delta_1,\mathbf{q}}\left|\frac{\partial \mathbf{l}}{\partial a}\right|\left|\frac{\partial \mathbf{l}}{\partial b}\right|\right) \tag{2.35}$$

Because $\frac{\partial}{\partial a} = \frac{\partial}{\partial \rho}$

$$\left|\frac{\partial \mathbf{l}}{\partial a}\right| \leq 1 \tag{2.36}$$

Observe that the partial derivative with respect to a is the derivative with respect to $a = e(\mathbf{l})$ along the curve given by $e(\mathbf{l}-\mathbf{q}) = b$ in a plane containing the origin and $\mathbf{q}$, fixed by $\tilde{\theta}$. The partial derivative with respect to b is the derivative with respect to $b = e(\mathbf{l}-\mathbf{q})$ along the curve given by $e(\mathbf{l}) = a$ in a plane containing the origin and $\mathbf{q}$, fixed by $\tilde{\theta}$. Note that a change of variables $\mathbf{l} \to \mathbf{l}+\mathbf{q}$ transforms the latter expression into the former up to $\mathbf{q} \to -\mathbf{q}$. Because the bounds are invariant under these transformations we also have

$$\left|\frac{\partial \mathbf{l}}{\partial b}\right| \leq 1 \quad \text{and} \quad \left|\frac{\partial \mathbf{l}}{\partial a \partial b}\right| \leq \left|\frac{\partial \mathbf{l}}{\partial \theta_1 \partial \rho}\right| \left|\frac{\partial b}{\partial \theta_1}\right|^{-1} \leq (\kappa_s \eta_1)^{-1} \tag{2.37}$$

This gives

$$|A_{11}| \leq 4c\kappa_{\mathbf{p}}^2 \frac{7|K_{11}|_{2,\delta_1,\mathbf{q}}}{(2\pi)^d \kappa_s \eta_1} \int_{-\delta_1}^{\delta_1} \mathrm{d}a|\log|a|| \int_{-\delta_1}^{\delta_1} \mathrm{d}b|\log|b|| \int \mathrm{d}\tilde{\theta}\mathrm{d}\frac{1}{|\mathbf{l}-\mathbf{p}|} \tag{2.38}$$

To see that the remaining integral is bounded we change back to the original $\mathbf{l}$ coordinate.

$$\begin{aligned}\int_{-\delta_1}^{\delta_1} \mathrm{d}a|\log|a|| \int_{-\delta_1}^{\delta_1} \mathrm{d}b|\log|b||\mathrm{d}\tilde{\theta}\frac{1}{|\mathbf{l}-\mathbf{p}|} \\ = \int_{U(S,\delta_1)\cap U(S,\mathbf{q},\delta_1)} \mathrm{d}\mathbf{l}|\log|\mathbf{e}(\mathbf{l})|||\log|\mathbf{e}(\mathbf{l}+\mathbf{q})||\frac{1}{|\mathbf{l}-\mathbf{p}|} J_1^{-1} J_{\mathrm{ang}}^{-1}|\frac{\partial b}{\partial \theta_1}| \\ \leq \frac{4\kappa_t 2^{\frac{3d}{2}}}{r_{\min}^{d-2}} k_{\log}\end{aligned} \tag{2.39}$$

where

$$k_{\log} = \sup_{\mathbf{p},\mathbf{q}} \int_\Omega \mathrm{d}\mathbf{l}|\log|\mathbf{e}(\mathbf{l})|||\log|\mathbf{e}(\mathbf{l}+\mathbf{q})||\frac{1}{|\mathbf{l}-\mathbf{p}|} \tag{2.40}$$

is finite because in two or more dimensions $\frac{1}{|\mathbf{l}-\mathbf{p}|}$ is integrable and the logarithmic factors don't change that. Inserting this gives

$$|A_{11}| \leq 112ck_{\log}\frac{\kappa_{\mathbf{p}}^2 2^{\frac{3}{2}d}}{(2\pi)^d r_{\min}^{d-2}\kappa_s\eta_1}|K_{11}|_{2,\delta_1,\mathbf{q}} \tag{2.41}$$

It remains to compute $|K_{11}|_{2,\delta_1,\mathbf{q}}$. Using the identity $|fg|_2 \leq 2|f|_2|g|_2$ we have

$$|K_{11}|_{2,\delta_1,\mathbf{q}} \leq 2^4|(\frac{\partial b}{\partial\theta_1})^{-1}J_{\arg}|_{2,\delta_1,\mathbf{q}}|J_1|_{2,\delta_1}|\chi_1(\delta_1^{-2}(l_0^2+e(\mathbf{l})^2))|_{2,\delta_1}^2 \tag{2.42}$$

Lemma 5 gives $|J_1|_{2,\delta_1}$ and we have

$$|\chi_1(\delta_1^{-2}(l_0^2+e(\mathbf{l})^2))|_{2,\delta_1} < \frac{8|\chi|_2|e|_2}{\delta_1^2} \tag{2.43}$$

Finally

$$\left|(\frac{\partial b}{\partial\theta_1})^{-1}J_{\arg}\right|_2 = \frac{1}{q}\left|(\frac{\partial\tilde{U}}{\partial\theta_1})^{-1}J_{\arg}\right|_2 \tag{2.44}$$

From the construction of the coordinate θ_1 in Lemma 9 we see that

$$\left|(\frac{\partial\tilde{U}}{\partial\theta_1})^{-1}J_{\arg}\right|_2 = \left|\frac{1}{q}\frac{\sin^{d-3}\theta_1}{2|\mathbf{l}|}\right|_2 \leq \frac{d^2}{r_{\min}^3}(1+\frac{(d-3)2^{\frac{17}{2}}}{r_{\min}^2}) \tag{2.45}$$

Inserting these bounds and those from Lemma 5 gives

$$|A_{11}| \leq c\frac{2^{16+\frac{1}{2}d}k_{\log}d^6\kappa_{\mathbf{p}}^2(1+\delta)^2|\chi|_2^2|e|_2^2}{\pi^d r_{\min}^{d+1}\kappa_s^2\eta_1\delta_1^4}(1+\frac{(d-3)2^{\frac{17}{2}}}{r_{\min}^2}) \tag{2.46}$$

Taking all the bounds together and using that $\delta_1 = \frac{\eta_2\kappa_s}{2}$ it follows that the lemma is satisfied with

$$M_{g1} = \frac{8(1+\delta)^{d-2}|\chi|_2^2}{\kappa_s^2\eta_2^2}\left(\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d}16d^2 + \frac{2^{17+\frac{1}{2}d}k_{\log}d^6|e|_2}{\eta_1\eta_2^2\pi^d r_{\min}^{d+1}\kappa_s^4}(1+\frac{(d-3)2^{\frac{17}{2}}}{r_{\min}^2})\right) \tag{2.47}$$

$$M_{g2} = \frac{8|\chi|_2^2}{\kappa_s^2\eta_2^2}\frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \tag{2.48}$$

□

2.3 Small q

2.3.1 The method

The main idea of this proof is again to exploit sign cancellations in the $C(l)C(l-q)$ factor by using integration by parts. The basic observation in dealing with the integral at small $\mathbf{q}$

is that for $q = (q_0, \mathbf{q}) = 0$ the singular factor has the form

$$\frac{1}{(il_0 - \rho)^2} \tag{2.49}$$

which is easy to integrate twice with respect to ρ, hence a double integration by parts with respect to ρ works similarly to the integration by parts employed in the previous section.

However in general the two nominators appearing in $C(l)C(l-q)$ are different. In order to do the integration by parts it is convenient to bring this factor in a form, similar to (2.49), that is easier to integrate. Following [Sal99], Section 4.5.5, this is done by using an interpolation, otherwise know as 'Feynman Trick'; It is based on the observation that

$$\frac{1}{a(x)b(x)} = \frac{1}{a(x)-b(x)}\left(\frac{1}{b(x)} - \frac{1}{a(x)}\right) = \int_0^1 \frac{1}{((1-t)a(x)+tb(x))^2} \tag{2.50}$$

when the integral on the right hand side exists. It is effectively expressing $\frac{1}{b}$ by Taylor expansion around $\frac{1}{a}$. This works best when the difference is small, which is why it is the appropriate method to use here.

The result is only a single function in the denominator at the expense of it being t-dependent. The latter point is relevant here; In order to apply this method the t- and x-integration must be exchanged and this exchange must be justified.

It is instructive to do this first for the special limits $(q_0 = 0, \mathbf{q} \neq 0)$ and $(q_0 \neq 0, \mathbf{q} = 0)$ and then do the general case which is dealt with in basically the same way but contains some technical complications.

2.3.2 The condition on q

In this section the first-order vertex correction is considered at small transfer momentum $\mathbf{q}$. In Section 2.1.2 this was defined as

$$|\mathbf{q}| \leq \kappa_s$$

where κ_s is given by Lemma 4 which implies that for $\delta' = \frac{1}{2}r_1$,

$$U(S, \mathbf{q}, \delta') \subset U(S, \delta) \quad \forall \mathbf{q}, |\mathbf{q}| \leq \kappa_s \tag{2.51}$$

The coordinates $a = e(\mathbf{l}), b = e(\mathbf{l} - \mathbf{q})$ used in section 2.2.1 are inappropriate in this case which can be seen for example from the fact that the Jacobian diverges for $|\mathbf{q}| \to 0$ (See eqn. (2.30)).

The main idea is now is not to use coordinates that are based on Fermi-Surface S, but coordinates based on the interpolating Fermi-surface $S(t, \mathbf{q})$ defined after (1.128).

2.3.3 Taylor expansion for q / Feynman trick

In this section we look at the limit where $(q_0 = 0, \mathbf{q} \neq 0)$.

$$\Gamma_3^*(p, (0, \mathbf{q})) = I_\mathbf{q}(\mathbb{R}, 1) \tag{2.52}$$

where

$$I_\mathbf{q}(Y, X) = \int\limits_Y \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) Z_\mathbf{q} X(\mathbf{l}) \tag{2.53}$$

$$\begin{aligned} Z_\mathbf{q} &= \frac{1}{il_0 - e(\mathbf{l})} \frac{1}{il_0 - (e(\mathbf{l}) + U(\mathbf{l}, \mathbf{q}))} \\ &= \frac{-1}{U(\mathbf{l}, \mathbf{q})} \left(\frac{1}{il_0 - e(\mathbf{l})} - \frac{1}{il_0 - (e(\mathbf{l}) + U(\mathbf{l}, \mathbf{q}))} \right) \end{aligned} \tag{2.54}$$

First the contribution to the integral from the region outside the anulus $U(S, r_1)$ is cut out using the smooth partition of unity defined in Lemma 9. This contribution is easy to bound: Because $U(S, \mathbf{q}, \frac{1}{2}r_1) \subset U(S, \delta)$, for $\mathbf{l} \in \sup \chi_4$, $|e(\mathbf{l} - \mathbf{q})| \geq \frac{1}{4}r_1$ and $|e(\mathbf{l})| \geq \frac{1}{2}r_1$. So

$$|Z_\mathbf{q}| \leq \frac{8}{r_1^2} \quad \text{for } \mathbf{l} \in \sup \chi_4 \text{ or } |l_0| \geq \frac{1}{2} \tag{2.55}$$

Thus bounding the nonsingular part of the integral

$$|I_\mathbf{q}(\mathbb{R}, 1)| \leq |I_\mathbf{q}(\mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}], 1)| + +|I_\mathbf{q}([-\frac{1}{2}, \frac{1}{2}], \chi_4)| + |I_\mathbf{q}([-\frac{1}{2}, \frac{1}{2}], \chi_3)| \tag{2.56}$$

$$\leq \frac{16 \operatorname{Vol}(\Omega)\Lambda}{r_1^2 (2\pi)^d} c + |I_\mathbf{q}([-\frac{1}{2}, \frac{1}{2}], \chi_4)| \tag{2.57}$$

In $I_\mathbf{q}$ we do a Taylor expansion around $\mathbf{q} = 0$: At finite β, the Fermi-frequency $|l_0| \geq \frac{\pi}{\beta}$. Therefore for all $t \in [0, 1]$

$$\left| \frac{1}{(il_0 - (e(\mathbf{l}) + t(U(\mathbf{l}, \mathbf{q}))))^2} \right| \leq \frac{\pi^2}{\beta^2} \tag{2.58}$$

the integral

$$\int\limits_0^1 \mathrm{d}t \frac{1}{(il_0 - (e(\mathbf{l}) + t(U(\mathbf{l}, \mathbf{q}))))^2} \tag{2.59}$$

is absolutely convergent and equals $Z_{\mathbf{q}}$. Inserting this in the integral

$$I_{\mathbf{q}}([-\frac{1}{2},\frac{1}{2}],\chi_3) = I_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \int_0^1 \mathrm{d}t \frac{\chi_3(\mathbf{l})}{(il_0 - (e(\mathbf{l}) + t(U(\mathbf{l},\mathbf{q}))))^2} \tag{2.60}$$

Because of the absolute convergence at finite temperature the order of integrations can be exchanged by Fubini's theorem, so that t-integral is taken last.

$$I_1 = \int_0^1 \mathrm{d}t \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{D(l_0 - p_0, |\mathbf{l}-\mathbf{p}|)}{(il_0 - e(\mathbf{l},\mathbf{q},t))^2} \chi_3(\mathbf{l}) \tag{2.61}$$

where $e(\mathbf{l},\mathbf{q},t)$ is the interpolating dispersion relation defined in (1.128). I_1 is bounded by finding a t-independent bound for the inner two integrals.

Inside this region we change to the coordinates $(\rho_t = e(\mathbf{l},\mathbf{q},t),\theta)$. This change of coordinates is the composition of the coordinate map $\mathbf{l} \mapsto (\rho,\theta)$ with the map $\rho \mapsto e(\mathbf{l}(\rho,\theta),\mathbf{q},t)$. By Lemma 5 the former exists on the support of χ_3 and is smooth. By Lemma 9, $\frac{\partial}{\partial\rho} e(\mathbf{l}(\rho,\theta),\mathbf{q},t) > v_0$ and thus the latter is also regular and smooth. Therefore the composition is too. Let the Jacobian be denoted by J_3;

$$J_3 = J_1 \Big(\frac{\partial}{\partial \rho} e(\mathbf{l}(\rho,\theta),\mathbf{q},t) \Big)^{-1} \tag{2.62}$$

This gives

$$I_1 = \int_0^1 \mathrm{d}t \int_{|l_0|\le\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\theta}{(2\pi)^d} \int \mathrm{d}\rho_t \frac{D^* J_3 \chi_3}{(il_0 - \rho_t)^2} \tag{2.63}$$

We now proceed as in the self energy case by repeated integration by parts:

$$I_1 = -\int_0^1 \mathrm{d}t \int_{|l_0|\le\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\theta}{(2\pi)^d} \int \mathrm{d}\rho_t \, \mathrm{Log}(il_0 - \rho_t) \frac{\partial^2}{\partial {\rho_t}^2} (D^* J_3 \chi_3) \tag{2.64}$$

Here $\mathrm{Log}(il_0 - \rho_t)$ denotes the principal branch of the logarithm. The intermediate and the resulting integrals exist because by Lemma 4 the integrands are bounded in absolute value at finite β. The boundary terms vanish because χ_3 vanishes for large ρ_t.

Like in the transversal case bounding the integral requires some care, because the second derivatives appearing are no longer bounded (uniformly in β). However they still are integrable for $d \ge 2$ and a factor c can still be extracted uniformly in β.

After exchanging the finite sum over l_0 with the integrals and again using $|\mathrm{Log}(il_0-\rho_t)| \leq 2|\mathrm{Log}|\rho_t||$ one sees that

$$|I_1| \leq 2\int_0^1 \mathrm{d}t \int \frac{\mathrm{d}\theta}{(2\pi)^d} \int \mathrm{d}\rho_t |\mathrm{Log}(|\rho_t|)| \int_{|l_0|\leq\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \left| \frac{\partial^2 D J_3 \chi_3}{\partial \rho_t{}^2} \right| \tag{2.65}$$

Applying Lemma 5 and noting that the inequalities corollary 7 remain true for ρ_t

$$\leq 28c\kappa_{\mathbf{p}}^2 \frac{|J_3|_{2,\delta}|\chi_3|_2}{(2\pi)^d} \sup_t \int \mathrm{d}\theta \int \mathrm{d}\rho_t |\mathrm{Log}(|\rho_t|)| \frac{1}{|\mathbf{l}-\mathbf{p}|} \tag{2.66}$$

For example note that $\left|\frac{\partial \mathbf{l}}{\partial \rho_1}\right| \leq \left|\frac{\partial \mathbf{l}}{\partial r}\right| \left|\frac{\partial \rho_t}{\partial r}\right|^{-1}$ and

$$\left|\frac{\partial \rho_t}{\partial r}\right| = |2r - 2q\cos\theta_1 t| \geq 2r_{\min} - 2k_s > 1 \tag{2.67}$$

Changing back to to integration variable $\mathbf{l}$ in the remaining integral

$$\int \mathrm{d}\theta \int \mathrm{d}\rho_t |\mathrm{Log}(|\rho_t|)| \frac{1}{|\mathbf{l}-\mathbf{p}|} = \int_{U(S,r_1)} \mathrm{d}\mathbf{l} \frac{|\log|e(t,\mathbf{l},\mathbf{q})||}{|\mathbf{l}-\mathbf{p}|} J_3(t,\mathbf{q},\mathbf{l})^{-1} \tag{2.68}$$

$$\leq \frac{|e|_1}{r_{\min}^{d-1}} \bar{k}_{\log} \tag{2.69}$$

where

$$\bar{k}_{\log} = \sup_{t\in[0,1]} \sup_{\mathbf{p}} \int_\Omega \mathrm{d}\mathbf{l}(1 + |\log|(1-t)e(\mathbf{l}) + te(\mathbf{l}+\mathbf{q})||) \frac{1}{|\mathbf{l}-\mathbf{p}|} \tag{2.70}$$

The constant $\bar{k}_{\log}$ is finite because $\frac{1}{|\mathbf{l}-\mathbf{p}|}$ is integrable in two or more dimensions and the logarithm does not change that.

2.3.4 Taylor expansion for q_0

In this section we look at the case where $q_0 \neq 0$ and $\mathbf{q} = 0$).

$$\Gamma_3^*(p, (q_0, 0)) = I_{q_0}(\mathbb{R}, 1) \tag{2.71}$$

where

$$I_{q_0}(Y, X) = \int_Y \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D^*(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) Z_{\mathbf{q}} X(\mathbf{l}) \tag{2.72}$$

$$\begin{aligned} Z_{q_0} &= \frac{1}{il_0 - e(\mathbf{l})} \frac{1}{i(l_0 - q_0) - e(\mathbf{l})} \\ &= \frac{1}{iq_0}\left(\frac{1}{il_0 - e(\mathbf{l})} - \frac{1}{i(l_0-q_0) - e(\mathbf{l})}\right) \end{aligned} \tag{2.73}$$

Let $\mathcal{A}(q_0) = \{x||x| < \frac{1}{2} \wedge |x - q_0| < \frac{1}{2}\}$. For simplicity we only consider the part where $l_0 \in \mathcal{A}(q_0)$ here.

The integral I_{q_0} is split up again in different parts to allow integration by parts

$$I_{q_0}(\mathcal{A}(q_0), 1) = I_{q_0}(\mathcal{A}(q_0), \chi_3) + I_{q_0}(\mathcal{A}(q_0), \chi_4) \tag{2.74}$$

These terms are bounded individually. One is trivial

$$I_{q_0}(\mathcal{A}(q_0), \chi_4) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{|e(\mathbf{l})| > \frac{1}{2} r_1} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} |Z_{q_0}| D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{2.75}$$

From the form (2.73) for Z_{q_0} one sees direct that when $|e(\mathbf{l})| > \frac{1}{2} r_1$ then $|Z_{q_0}| \leq \frac{4}{r_1^2}$. Therefore by Lemma 4

$$|I_{q_0}(\mathcal{A}(q_0), \chi_4)| \leq \frac{2c\Lambda}{r_1^2 (2\pi)^d} \mathrm{Vol}(\Omega) \tag{2.76}$$

In the singular regions one could be tempted to proceed as in the previous section and use a taylor expension in Z_{q_0}. However the integral

$$\int \mathrm{d}t \frac{1}{(i(l_0 - tq_0) - e(\mathbf{l}))^2} \tag{2.77}$$

does not always converge. Although at finite β, l_0 and q_0 are bounded away from zero, the linear combination $l_0 + tq_0$ can still become zero. When the expression is inserted in the integral/sum it does not lead to an absolutely convergent integral and one cannot apply Fubini to take the t-integral last. However the integral is sufficiently simple to do at least one integration by parts.

$$\begin{aligned} & I_{q_0}(\mathcal{A}(q_0), \chi_3) \\ &= \int_{\mathcal{A}(q_0)} \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \frac{1}{iq_0} \left(\frac{1}{il_0 - e(\mathbf{l})} - \frac{1}{i(l_0 + q_0) - e(\mathbf{l})} \right) \chi_3(\mathbf{l}) \end{aligned} \tag{2.78}$$

$$= \int_{\mathcal{A}(q_0)} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho \int \frac{\mathrm{d}\theta}{(2\pi)^d} D^*(l_0 - p_0, c|\mathbf{l}(\rho, \phi) - \mathbf{p}|) J_1(\mathbf{l}(\rho, \phi)) \frac{\chi_3}{-iq_0} \left(\frac{1}{il_0 - \rho} - \frac{1}{i(l_0 - q_0) - \rho} \right) \tag{2.79}$$

Now observe again that $\frac{1}{il_0-\rho} = -\frac{\partial}{\partial\rho}\,\mathrm{Log}(il_0-\rho)$. One integration by parts gives

$$= \int\limits_{\mathcal{A}(q_0)} \mathrm{d}_\beta l_0 \int\limits_{-\delta}^{\delta} \mathrm{d}\rho \frac{1}{-iq_0}\left(\mathrm{Log}(il_0-\rho) - \mathrm{Log}(i(l_0-q_0)-\rho)\right) \int \frac{\mathrm{d}\theta}{(2\pi)^d}\frac{\partial}{\partial\rho}\Big(J_1\chi_3 D^*\Big) \tag{2.80}$$

At this point it is possible to do a Taylor expansion in the frequencies because we can control its convergence. Observe that if $\rho \neq 0$ then

$$\frac{1}{-iq_0}\left(\mathrm{Log}(i(l_0-q_0)-\rho) - \mathrm{Log}(il_0-\rho)\right) = \int\limits_0^1 \mathrm{d}t \frac{1}{i(l_0-tq_0)-\rho} \tag{2.81}$$

In the integrand above, ρ can be zero, so the integral in (2.81) doesn't always converge. However at finite temperature the integral

$$I_4 = \int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int \mathrm{d}t \frac{1}{i(l_0-tq_0)-\rho} \int \mathrm{d}\theta \frac{\partial}{\partial\rho}\Big(J_1\chi_3 D^*\Big) \tag{2.82}$$

is absolute convergent at each l_0. By Lemma 4, $\frac{\partial}{\partial\rho}\Big(J_1\chi_3 D^*\Big)$ is bounded at finite β and

$$\left|\int \mathrm{d}\phi \frac{\partial}{\partial\rho}\Big(J_1\chi_3 D^*\Big)\right| \le \mathrm{const} \tag{2.83}$$

at fixed β. Thus

$$\int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int \mathrm{d}t \left|\frac{1}{i(l_0-tq_0)-\rho} \int \mathrm{d}\theta \frac{\partial}{\partial\rho}\Big(J_1 D^*\Big)\right| \le \mathrm{const} \int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int\limits_0^1 \mathrm{d}t \frac{1}{|i(l_0-tq_0)-\rho|} \tag{2.84}$$

The remaining integral is bounded by recognising that it it just an integral over the absolute value of a single fermion propagator. Because $q_0 \neq 0$, it is possible to change to the integration variable $x = l_0 - tq_0$:

$$\int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int\limits_0^1 \mathrm{d}t \frac{1}{|i(l_0-tq_0)-\rho|} = \frac{1}{q_0}\int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int\limits_{l_0}^{l_0-q_0} \mathrm{d}x \frac{1}{|ix-\rho|} \tag{2.85}$$

$$\le \frac{1}{q_0}\int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int\limits_{l_0}^{l_0-q_0} \frac{1}{10} + \frac{1}{q_0}\int\limits_{-\delta}^{\delta} \mathrm{d}\rho \int\limits_{l_0}^{l_0-q_0} \mathrm{d}x \frac{1}{\sqrt{x^2+\rho^2}} \mathbb{1}(|ix-\rho| \le 10) \tag{2.86}$$

$$< \infty \tag{2.87}$$

Because the integral exists for each l_0 and the sum over l_0 is finite we can take the sum inside integral. By Fubini's theorem and (2.81) therefore

$$I_{q_0}(\mathcal{A}(q_0),\chi_3) = \int_0^1 \mathrm{d}t \int_{\mathcal{A}(q_0)} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho \frac{1}{i(l_0 - tq_0) - \rho} \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{\partial}{\partial \rho}\Big(J_1\chi_3 D^*\Big) = I_5 \tag{2.88}$$

Here the right hand side is the result of another integration by parts with respect to ρ:

$$I_5 = \int_0^1 \mathrm{d}t \int_{\mathcal{A}(q_0)} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho \,\mathrm{Log}(i(l_0 - tq_0) - \rho) \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{\partial^2}{\partial \rho^2}\Big(J_1\chi_3 D^*\Big) \tag{2.89}$$

For $l_0 \in \mathcal{A}(q_0)$, $|l_0 - q_0| \le (1-t)|l_0| + t|l_0 - q_0| < \frac{1}{2}$ and thus $|\mathrm{Log}(i(l_0 - tq_0) - \rho)| \le 2|\log|\rho||$,

$$I_5 \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}_\beta l_0}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}\rho 2|\mathrm{Log}|\rho|| \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{\partial^2}{\partial \rho^2}\Big(J_1\chi_3 D^*\Big) \tag{2.90}$$

by Lemmas 4 and 7

$$I_5 \le c4\kappa_{\mathbf{p}}^2 \left(6|J_1\chi_3|_{2,\delta} + |J_1\chi_3|_{1,\delta}\right) \int_{-\delta}^{\delta} \mathrm{d}\rho |\mathrm{Log}|\rho|| \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{1}{|1-\mathbf{p}|} \tag{2.91}$$

By applying (2.68)–(2.70) in the case $t = 0$

$$|I_{q_0}(\mathcal{A}(q_0),\chi_3)| \le c\frac{56\kappa_{\mathbf{p}}^2 |e|_1}{r_{\min}^{d-2}} \bar{k}_{\log} |J_1|_{2,\delta} |\chi_3|_2 \tag{2.92}$$

2.3.5 General small $(q_0, \mathbf{q})$

The 'Feynman trick' and integration by parts

For the general form we want to combine the methods of the two previous sections. Note that the introduction of the interpolation and interchange of integrals needed to be done at different stages. The same occurs here. Observe that

$$C(l)C(l-q) = F_1(l,q)z_1(l,q) + F_2(l,q)z_2(l,q) \tag{2.93}$$

where

$$\begin{aligned} z_1(l,q) &= \frac{1}{i(l_0 - q_0) - e(\mathbf{l})} \frac{1}{i(l_0 - q_0) - e(\mathbf{l} - \mathbf{q})} \\ z_2(l,q) &= \frac{1}{il_0 - e(\mathbf{l})} \frac{1}{i(l_0 - q_0) - e(\mathbf{l})} \end{aligned} \tag{2.94}$$

and

$$F_1(l,q) = \frac{|\mathbf{q}|\tilde{U}(\mathbf{l},\mathbf{q})}{iq_0 + |\mathbf{q}|\tilde{U}(\mathbf{l},\mathbf{q})} \quad F_2(l,q) = 1 - F_1(l,q) = \frac{iq_0}{iq_0 + |\mathbf{q}|\tilde{U}(\mathbf{l},\mathbf{q})} \tag{2.95}$$

which is the application of the left-hand side of (2.50) to the frequency and vector momentum parts of the propagator separately.

Each of the terms of terms on the right hand side of (2.93) is now similar to $I_{\mathbf{q}}$ or I_{q_0}. The main difference is the appearance of the factors $F_1(l,q)$ and $F_2(l,q)$. Performing integration by parts to remove singularities from z_1 and z_2 can produce derivatives acting on them. The main addition of this section will therefore be to deal with the technicalities involved in either avoiding such derivatives by an appropriate change of coordinates or in bounding the derivatives when we cannot do so.

As we have seen in the previous sections, the method used is to combine a Feynman trick and integration by parts to deal with the singularities from the Fermion propagators. After that the factor D^* and its derivatives are dealt with. Therefore it is convenient to capture the effect of the first step in a Lemma.

We start by formulating what happens when there is only one integration by parts done. This is both illustrative and useful as we will need this later on.

Lemma 13 (Integration by parts). *Let* $|\mathbf{q}| \leq \kappa_s$, $\omega(l_0, \mathbf{l})$ *be a differentiable function with* $\sup \omega \subset [-\frac{1}{2}, \frac{1}{2}] \times U_\delta(S) \cap U_\delta(S_{\mathbf{q}})$ *and let*

$$I = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C(l)C(l-q)\omega(l_0,\mathbf{l}) \tag{2.96}$$

then

$$|I| \leq \text{const} \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} (|Z_1(l,q,t)| + |Z_2(l,q,t)|)(|\omega(l_0,\mathbf{l})| + |\nabla\omega(l_0,\mathbf{l})|) \tag{2.97}$$

where const *is independent of* q *and*

$$Z_1(l,q,t) = \frac{1}{i(l_0 - q_0) - (1-t)e(\mathbf{l}) + te(\mathbf{l}-\mathbf{q})} \tag{2.98}$$

$$Z_2(l,q,t) = \frac{1}{i(l_0 - tq_0) + e(\mathbf{l})} \tag{2.99}$$

Proof. From Lemma 9 we see that when $\mathbf{q}$ is sufficiently small then we can choose annuli around S and $S_{\mathbf{q}}$ (independently of $\mathbf{q}$) such that one is contained in the other and that outside of the larger both $e(\mathbf{l})$ and $e(\mathbf{l}-\mathbf{q})$ are bounded away from zero. Decomposing the integration region in outside and inside the larger annulus (or rather applying a smooth approximation of that decomposition) we get

$$I = I_4 + I_3 \tag{2.100}$$

where

$$I_j = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C(l)C(l-q)\chi_j(\mathbf{l})\omega(l_0,\mathbf{l}) \tag{2.101}$$

By Lemma 9 on the support of χ_4, $|C(l)| < \frac{2}{r_1}$ and $|C(l-q)| < \frac{4}{r_1}$ and thus

$$|I_4| \leq \frac{8}{r_1^2} \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} |\omega(l_0,\mathbf{l})| \tag{2.102}$$

Note that in most applications of the lemma we will consider ω is such that $\sup \omega \cap (\mathbb{R} \times \sup \chi_4) = 0$ and thus $I_4 = 0$.

Inserting (2.93) gives the following split

$$I_3 = I_{31} + I_{32} \tag{2.103}$$

where

$$I_{3j} = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, F_i(l,q) z_i(l,q)\chi_3(\mathbf{l})\omega(l_0,\mathbf{l}) \quad i = 1,2 \tag{2.104}$$

To bound I_{3j}, complete the Feynman Trick using

$$z_1(l,q) = \int_0^1 \mathrm{d}t\, \tilde{z}_1(l,q,t) \tag{2.105}$$

where

$$\tilde{z}_1(l,q,t) = \frac{1}{\left(i(l_0 - q_0) - e(\mathbf{l},\mathbf{q},t)\right)^2} \tag{2.106}$$

and insert this expression into (2.104) ($j = 1$). Note that at finite β we have $|l_0 - q_0| \geq \frac{\pi}{\beta}$ which implies $|\tilde{z}_1| \leq \frac{\beta^2}{\pi^2}$ and thus the integral is absolutely bounded and it is allowed to change the integration order to take the t integral last. This results in

$$I_{31} = \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \tilde{z}_1(l,q) F_1(l,q)\chi_3(\mathbf{l})\omega(l_0,\mathbf{l}) \tag{2.107}$$

We will now perform an integration by parts with respect to ρ integration $\tilde{z}_1$ and therefore we change to the coordinates (ρ, θ). However the integration by parts will produce terms where F_1 is differentiated and these will have to be controlled. This is done again in two ways in different parts of the integration region (See Lemma 9): On the support of χ_6, i.e. for small angles between $\mathbf{l}$ and $\mathbf{q}$, the (scaled) difference $\tilde{U}$ will be bounded away from zero and the derivatives of F_1 will be bounded. On the region where $\tilde{U}$ is small, Lemma 9 shows that the Jacobian of the change of integration variables $\theta_1 \mapsto \tilde{U}$ is bounded and thus we can use coordinates $(\rho, \tilde{U}, \tilde{\theta})$ there and integrate with respect to $\tilde{U}$. So the integral is split as (using notation from Lemma 9)

$$I_{31} = I_{315} + I_{316} \tag{2.108}$$

where for $m = 5, 6$

$$I_{31m} = \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho \, \tilde{z}_1(l,q) F_1(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l}) \tag{2.109}$$

with

$$\int \mathrm{d}X_m = \begin{cases} \int \frac{\mathrm{d}\tilde{\theta}}{(2\pi)^d} \int \mathrm{d}\tilde{U} & m = 5 \\ \int \frac{\mathrm{d}\theta}{(2\pi)^d} & m = 6 \end{cases} \tag{2.110}$$

$$J_m = \begin{cases} J(\rho, \theta(\theta_1(\tilde{U}), \tilde{\theta})) J_{\mathrm{ang}}(\theta_1(\tilde{U}), \tilde{\theta}) \frac{1}{|\frac{\partial \tilde{U}}{\partial \theta_1}|} & m = 5 \\ J(\rho, \theta) & m = 6 \end{cases} \tag{2.111}$$

Note that $\mathbf{l}$ was used as a shorthand notation for

$$\mathbf{l} = \begin{cases} \pi(\rho, \theta(\theta_1(\tilde{U}), \tilde{\theta})) & m = 5 \\ \pi(\rho, \theta) & m = 6 \end{cases} \tag{2.112}$$

After this preparation the integration by parts is simple. For $m = 5, 6$

$$I_{31m} = -\int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho \, Z_1(l,q) \frac{\partial}{\partial \rho} \left(\frac{F_1(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l})}{(\frac{\partial}{\partial \rho}\mathbf{p}) \cdot \nabla e(\mathbf{l}, \mathbf{q}, t)} \right) \tag{2.113}$$

with Z_1 as defined in the lemma.

For I_{32} a similar expression can be found. However in this case it is *not* possible to do the interpolation and then take the t integral last to do the integration by parts. The difficulty is that although at finite β both l_0 and q_0 are bounded away from 0, but $l_0 - tq_0$ need not be. Thus the absolute convergence is not guaranteed. However it turns out that the z_2 is simple enough to integrate directly. Observe first that for $q_0 = 0$, $F_2 \equiv 0$ and thus $I_{32} = 0$. We can thus WLOG assume $q_0 \neq 0$ in the following.

$$z_2 = \frac{1}{-iq_0}\left(\frac{1}{il_0 - e(\mathbf{q})} - \frac{1}{i(l_0 - q_0) - e(\mathbf{q})}\right) \tag{2.114}$$

Note in particular that the pre-factor does not depend on l. This allows direct integration by parts.

$$I_{32} = I_{325} + I_{326} \tag{2.115}$$

where for $m = 5, 6$

$$\begin{aligned} I_{32m} &= -\int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho\, z_2(l,q) F_2(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l}) \\ &= \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho\, \tilde{Z}_2(l,q) \frac{\partial}{\partial \rho} \left(F_2(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l})\right) \end{aligned} \tag{2.116}$$

where

$$\tilde{Z}_2(l,q) = \frac{-1}{iq_0}\left(\mathrm{Log}(il_0 - e(\mathbf{l})) - \mathrm{Log}(i(l_0 - q_0) - e(\mathbf{l}))\right) \tag{2.117}$$

where Log denotes the main branch of the logarithm.

Although interpolation was not necessary here to do the ρ-integral we now introduce it afterwards to bring the expression in the same form as for I_{31}. Observe that $\tilde{Z}_2$ can be written as

$$\tilde{Z}_2(l,q) = \int_0^1 \mathrm{d}t\, Z_2(l,q,t) \tag{2.118}$$

with

$$Z_2(l,q,t) = \frac{-1}{i(l_0 - tq_0) - e(\mathbf{l})} \tag{2.119}$$

For any bounded positive function $g(l_0, \mathbf{l}, t)$ the integral

$$\int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \int_0^1 \mathrm{d}t |Z_2(l,q,t)| g(l_0, \mathbf{q}, t) \tag{2.120}$$

is convergent and thus we can insert (2.118) in (2.116) and change the order of integration to obtain

$$I_{32m} = \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho \, Z_2(l,q) \frac{\partial}{\partial \rho}\Big(F_2(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0,\mathbf{l}) \Big) \tag{2.121}$$

It remains to show that $\left| \frac{\partial}{\partial \rho} \left(\frac{F_1(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0,\mathbf{l})}{(\frac{\partial}{\partial \rho}\mathbf{l}) \cdot \nabla e(\mathbf{l},\mathbf{q},t)} \right) \right| \leq \mathrm{const}\,(|\omega| + |\nabla \omega|)$ and similarly $\left| \frac{\partial}{\partial \rho} \Big(F_1(l,q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0,\mathbf{l}) \Big) \right| \leq \mathrm{const}\,(|\omega| + |\nabla \omega|)$

By continuity, compactness of the support of the integrand and Lemma 9 we have

$$|J_m \chi_m \chi_3|_1 \leq \mathrm{const} \qquad m = 5, 6 \tag{2.122}$$

and

$$\left| \frac{1}{(\frac{\partial}{\partial \rho}\mathbf{l}) \cdot \nabla e(\mathbf{l},\mathbf{q},t)} \right|_1 \leq \frac{\mathrm{const}}{v_0^2} \tag{2.123}$$

so the lemma follows if we can show $|\frac{\partial}{\partial \rho}\mathbf{p} \cdot \nabla F_k| \leq \mathrm{const}$ on the support of χ_m for $m = 5, 6$ and $k = 1, 2$.

Note that this expression has an implicit dependence on m. When $m = 5$ the partial derivative keeps $(\tilde{U}, \tilde{\theta})$ constant. Observing that $F_k = F_k(q_0, \tilde{U})$ we see that

$$\frac{\partial}{\partial \rho}\mathbf{l} \cdot \nabla F_k = \left. \frac{\partial F_k}{\partial \rho} \right|_{\tilde{U},\tilde{\theta}} = 0 \tag{2.124}$$

For $m = 6$ the partial derivative is $\left. \frac{\partial}{\partial \rho} \right|_{\theta_1,\tilde{\theta}}$ and the value of $\frac{\partial}{\partial \rho}\mathbf{l} \cdot \nabla F_k$ is nontrivial. However on the support of χ_6

$$|\nabla F_k| \leq \mathrm{const}\,|F_k| \left| \frac{|\mathbf{q}|}{iq_0 - |\mathbf{q}|\tilde{U}} \right| \leq \frac{\mathrm{const}}{|\tilde{U}|} < \frac{\mathrm{const}}{\eta_2} \tag{2.125}$$

and thus the lemma follows. □

Double integration by parts

Although doing a single integration by parts is sufficient to extract the weaker bounds proven in the following Chapters, for a bound proportional to c is necessary to exploit the sign cancellations even more by doing another one. This is captured by the following lemma.

Lemma 14 (Double integration by parts). *Let* $|\mathbf{q}| \le \kappa_s$. *Let* $\mathcal{A}(q_0)$ *be given by* $\mathcal{A}(q_0) = \{l_0 | |l_0| < \frac{1}{2}, |l_0 - q_0| < \frac{1}{2}\}$. *Let* $\omega(l_0, \mathbf{q})$ *be differentiable function of* $\mathbf{l}$ *that vanishes for* $l_0 \notin \mathcal{A}(q_0)$. *Let* I_3 *be given by*

$$I_3 = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C(l) C(l-q) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l}) \tag{2.126}$$

Then there exits a constant M_2 *such that*

$$|I| \le M_2 \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} (1 + |\mathrm{Log}|e(\mathbf{l})|| + |\mathrm{Log}|e(\mathbf{l}, \mathbf{q}, t)||) \sum_{|\alpha| \le 2} |\delta^\alpha \omega| \tag{2.127}$$

Proof. Proceed exactly as in the proof of the previous Lemma, up to the moment where absolute values were taken. By applying an extra integration of parts to (2.113)

$$I_{31m} = \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho \, \mathrm{Log}(i(l_0 - q_0) - e(\mathbf{l}, \mathbf{q}, t))$$
$$\frac{\partial}{\partial \rho}\Big(\frac{1}{\frac{\partial}{\partial \mathbf{l}} \cdot \nabla e(\mathbf{l}, \mathbf{q}, t)} \frac{\partial}{\partial \rho}\Big(\frac{F_1(l, q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l})}{(\frac{\partial}{\partial \rho}\mathbf{p}) \cdot \nabla e(\mathbf{l}, \mathbf{q}, t)} \Big)\Big) \tag{2.128}$$

For $l_0 \in \mathcal{A}(q_0)$, $|l_0| < \frac{1}{2}$ and thus $|\mathrm{Log}(i(l_0 - q_0) - e(\mathbf{l}, \mathbf{q}, t))|2 \le |\mathrm{Log}|e(\mathbf{l}, \mathbf{q}, t)||$. Splitting the factors containing ω and its derivatives, taking supremum-norms of the others and finally changing back to cartesian coordinates gives:

$$|I_{31}| \le M_{31} \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} |\mathrm{Log}|e(\mathbf{l}, \mathbf{q}, t)|| \sum_{|\alpha| < 2} |\partial^\alpha \omega| \tag{2.129}$$

The constant M_{31} is computed later.

A similar extra integration by parts in (2.121) gives

$$I_{32m} = -\int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \mathrm{d}X_m \int \mathrm{d}\rho \, \mathrm{Log}(i(l_0 - tq_0) - e(\mathbf{l})) \frac{\partial^2}{\partial \rho^2} \Big(F_2(l, q) J_m \chi_m(\mathbf{l}) \chi_3(\mathbf{l}) \omega(l_0, \mathbf{l}) \Big) \tag{2.130}$$

For $l_0 \in \mathcal{A}(q_0)$, $|l_0 - tq_0| \le (1-t)|l_0| + t|l_0 - q_0| \le \frac{1}{2}$ and thus $|\mathrm{Log}(i(l_0 - tq_0) - e(\mathbf{l}))| \le 2|\mathrm{Log}|e(\mathbf{l})||$. Again factoring out the ω dependent factors and taking supnorms for the others gives

$$|I_{31}| \le M_{32} \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} |\mathrm{Log}|e(\mathbf{l})|| \sum_{|\alpha| < 2} |\partial^\alpha \omega| \tag{2.131}$$

where nothing t-dependent is actually left in the integrand.

Thus the Lemma holds with $M_2 = \max\{M_{31}, M_{32}\}$. What remains is to compute the constants. We must bound

$$K_{mj} = \left| \frac{\partial}{\partial\rho}\Big|_m \left(\frac{1}{\frac{\partial}{\partial\rho}\Big|_m e(\mathbf{l},\mathbf{q},t_j)} \frac{\partial}{\partial\rho}\Big|_m \left(\frac{1}{\frac{\partial}{\partial\rho}\Big|_m e(\mathbf{l},\mathbf{q},t_j)} F_j J_m \chi_m \chi_3 \omega(l_0,\mathbf{l}) \right)\right)\right| J_m^{-1} \tag{2.132}$$

with $j = 1, 2$ and $t_2 = 0$. Here

$$\frac{\partial}{\partial\rho}\Big|_m = \begin{cases} \frac{\partial}{\partial\rho}\Big|_{\tilde{U},\tilde{\theta}} & m = 5 \\ \frac{\partial}{\partial\rho}\Big|_{\theta_1,\tilde{\theta}} & m = 6 \end{cases} \tag{2.133}$$

Note that $\frac{\partial}{\partial\rho}\Big|_m e(\mathbf{l},\mathbf{q},0) = \frac{\partial}{\partial\rho}\rho = 1$. Moreover $e(\mathbf{l},\mathbf{q},t) = \rho + tq\tilde{U}$ and thus

$$\frac{\partial}{\partial\rho}\Big|_5 e(\mathbf{l},\mathbf{q},t_j) = \frac{\partial}{\partial\rho}\Big|_{\tilde{U},\tilde{\theta}} (\rho + tq\tilde{U}) = 1 \tag{2.134}$$

and (as in the previous lemma) by construction

$$\frac{\partial}{\partial\rho}\Big|_5 F_j = 0 \tag{2.135}$$

Therefore

$$K_{5j} \le \left| \left(\frac{\partial}{\partial\rho}\Big|_{\tilde{U}} \right)^2 J_5 \chi_5 \chi_3 \omega \right| J_5^{-1} \tag{2.136}$$

Using the identity

$$\left| \frac{\partial^2}{\partial\rho^2} f(\mathbf{l}) \right| \le \Big(\sum_{\alpha\le 2} |\partial^\alpha f|\Big)\Big(\|\frac{\partial\mathbf{l}}{\partial\rho}\|^2 + |\frac{\partial^2\mathbf{l}}{\partial\rho^2}|\Big) \tag{2.137}$$

we can split the bound in individual factors:

$$K_{5j} \le 2^5 |J_1^{-1}|_{0,U_5} |J_1|_{2,U_5} |J_{\text{ang}}(\frac{\partial\tilde{U}}{\partial\theta_1})^{-1}|_{2,U_5} |\chi_5|_2 |\chi_3|_2 \Big(\|\frac{\partial\mathbf{l}}{\partial\rho}\|^2 + |\frac{\partial^2\mathbf{l}}{\partial\rho^2}|\Big)\Big(\sum_{\alpha\le 2} |\partial^\alpha\omega|\Big) \tag{2.138}$$

where $U_m = \operatorname{supp}\chi_3\chi_m$.

We now continue by computing the various factors. Observe that

$$\frac{\partial}{\partial\rho}\Big|_{\tilde{U},\tilde{\theta}} \mathbf{l} = \frac{\partial}{\partial\rho}\Big|_{\theta} \mathbf{l} + \frac{\partial\mathbf{l}}{\partial\theta_1}\frac{\partial\theta_1}{\partial\rho}\Big|_{\tilde{U}} \tag{2.139}$$

and

$$0 = \frac{\partial}{\partial\rho}\Big|_{\tilde{U}} \tilde{U} = -\frac{\cos\theta_1}{\sqrt{\rho+1}} + 2\sqrt{\rho+1}\sin\theta_1 \frac{\partial\theta_1}{\partial\rho}\Big|_{\tilde{U}} \tag{2.140}$$

The latter implies $\frac{\partial\theta_1}{\partial\rho}\Big|_{\tilde{U}} = \frac{1}{2\tan\theta_1(\rho_1+1)}$. Using this we have

$$(\|\frac{\partial \mathbf{l}}{\partial\rho}\|^2 + |\frac{\partial^2 \mathbf{l}}{\partial\rho^2}|) \leq (2 + \frac{10\sqrt{2}}{r_{\min}^3}) \tag{2.141}$$

Continuing with the next factor

$$|J_5^{-1}|_{0,U_5} \leq |J_1^{-1}|_{0,U_5}|J_{\text{ang}}^{-1}|_{0,U_5}|\frac{\partial\tilde{U}}{\partial\theta_1}|_{0,U_5} \leq \frac{2^{1+\frac{3}{2}d}(1+\delta)}{r_{\min}^{d-2}} \tag{2.142}$$

Using Lemma 5 and (2.45) and gathering up the factors gives

$$K_{5j} \leq \Big(\sum_{\alpha\leq 2}|\partial^\alpha\omega|\Big)\frac{2^{7+\frac{5}{2}}d^4(1+\delta)^{d-1}}{r_{\min}^{d+1}}(1+\frac{5\sqrt{2}}{r_{\min}^2})((1+\frac{(d-3)2^{\frac{17}{2}}}{r_{\min}^2})|\chi_5|_2|\chi_3|_2 \tag{2.143}$$

Let $w(\rho) = \frac{\partial}{\partial\rho}\big|_6 e(\mathbf{l},\mathbf{q},t_j)$. $|w(\rho)| > v_0$ by Lemma 9. Because

$$\begin{aligned}&\left|\frac{\partial}{\partial\rho}\left(\frac{1}{w(\rho)}\frac{\partial}{\partial\rho}\left(\frac{1}{w(\rho)}f(\mathbf{l})\right)\right)\right|\\ &\qquad\leq (\sum_{\alpha\leq 2}|\partial^\alpha f|)\frac{1}{w(\rho)}(1+3|m'|+3|m'|^2+|m''|) \leq \frac{1+24\kappa_s}{v_0^4 r_{\min}^3}(\sum_{\alpha\leq 2}|\partial^\alpha f|)\end{aligned} \tag{2.144}$$

we have

$$\kappa_{6j} \leq \frac{1+24\kappa_s 2^5}{v_0^4 r_{\min}^3}\Big(\sum_{\alpha\leq 2}|\partial^\alpha\omega|\Big)|J_1^{-1}|_{0,U_6}|J_1|_{2,U_5}|\chi_6|_2|\chi_3|_2|F_j|_{2,U_6} \tag{2.145}$$

$$\leq \Big(\sum_{\alpha\leq 2}|\partial^\alpha\omega|\Big)\frac{2^8(1+24\kappa_s)d^4(1+\delta)^{d-2}(2+\delta)}{v_0^4 r_{\min}^{d+1}\eta_2^2} \tag{2.146}$$

where we have used (as in the previous lemma) that on U_6 the derivatives of F_j are bounded uniformly in q:

$$|F_2|_{2,U_6} \leq \frac{8(2+\delta)}{\eta_2^2} \tag{2.147}$$

Summarizing this means that the bound is satisfied for

$$M_2 = \frac{2^8 d^4|\chi_3|_2|\chi_5|_2(1+\delta)^{d-2}(2+\delta)}{r_{\min}^{d+1}}\max\left\{2^{\frac{5}{2}d-2}(1+\frac{5\sqrt{2}}{r_{\min}^2})(1+\frac{(d-3)2^{\frac{17}{2}}}{r_{\min}^2}),\ \frac{1+24\kappa_s}{v_0^4\eta_2}\right\} \tag{2.148}$$

□

Extracting the factor c

Having shown how to deal with the sign cancellations in the previous sections and with the technicalities hidden in Lemma 14, it is now quite straight forward to bound the first order vertex correction.

Lemma 15. *There exists a constant M_s such that for $\mathbf{p} \in \Omega$ and all $|\mathbf{q}| < \kappa_s$.*

$$|\Gamma_3^*(p,q)| \leq cM_s \tag{2.149}$$

with $\kappa_{\mathbf{p}}$ as in Lemma 12. Moreover we can take $M_s = \sup_{\mathbf{p}\in\Omega} \frac{\mathrm{Vol}\, S^{d-1}}{(2\pi)^d}(\frac{8}{\Lambda^{d+1} r_1^2} + 20d\kappa_{\mathbf{p}}(1+\delta)^{d-2}\Lambda|\chi_3|_1) + 12d^2\kappa_{\mathbf{p}}^2 M_2 \frac{\bar{k}_{log}}{(2\pi)^d}$.

Proof. As stated before, the essentials of the argument are contained in the Lemma's of the previous section. What remains is to deal with the simpler cases that were left out their for clarity, bring the integral in the required form and finally to extract the factor c from the resulting integral.

Denote by $I(f(l_0,\mathbf{q}))$

$$I(f(l_0,\mathbf{q})) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int_\Omega \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C(l)(C-q)D^*(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|)f(l_0,\mathbf{l}) \tag{2.150}$$

Let $H(l_0)$ be $H(l_0) = \mathbb{1}(|l_0| < \frac{1}{2})$ and write $H_s(l_0) = H(l_0)H(l_0 - q_0)$. Note that $\operatorname{supp} H_s = \mathcal{A}(q_0)$.

Using the partitions of unity $\chi_3 + \chi_4 = 1$, $H(l_0) + (1 - H(l_0)) = 1$, and $H(l_0 - q_0) + (1 - H(l_0 - q_0)) = 1$, gives

$$\begin{aligned}\Gamma_3^*(p,q) &= I(\chi_4 + \chi_3(1-H(l_0))(1-H(l_0-q_0)))\\ &\quad + I(\chi_3 H(l_0)(1-H(l_0-q_0))) + I(\chi_3 H(l_0-q_0)(1-H(l_0))) + I(\chi_3 H_s(l_0))\end{aligned} \tag{2.151}$$

By Lemma 9 on the support of χ_4 and $(1-H(l_0))(1-H(l_0-q_0))$, $C(l)C(l-q)$ is bounded. In particular $|C(l)| < \frac{2}{r_1}$ and $|C(l-q)| < \frac{4}{r_1}$ and

$$|I(\chi_4 + \chi_3(1-H(l_0))(1-H(l_0-q_0)))| \leq \frac{16}{r_1^2} \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} D(l_0 - p_0, |\mathbf{l}-\mathbf{p}|) \tag{2.152}$$

and thus by Lemma 4

$$|I(\chi_4 + \chi_3(1-H(l_0))(1-H(l_0-q_0)))| \leq c\frac{8\Lambda\,\mathrm{Vol}(\Omega)}{r_1^2(2\pi)^d} \tag{2.153}$$

We now turn to the next two terms in the decomposition. It suffices to concentrate on $I(\chi_3 H(l_0)(1-H(l_0-q_0)))$; Apart from a shift q in the integration variable $I(\chi_3 H(l_0-q_0)(1-H(l_0)))$ behaves identically and thus obeys the same bound as the difference between χ_3 and its translate by $\mathbf{q}$ is irrelevant in the computation (remember $|\mathbf{q}| < \kappa_s$).

On the support of $(1-H(l_0-q_0))$, $|q_0| > \frac{1}{2}$ and thus $C(l-q)$ is a bounded function. As $\sup \chi_3 \subset U(S,\delta)$, we can change to the coordinates ρ, θ

$$I(\chi_3 H(l_0)(1-H(l_0-q_0))) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\theta}{(2\pi)^d} \int \mathrm{d}\rho \frac{1}{il_0-\rho} \Delta(l_0, \mathbf{l}) \tag{2.154}$$

with

$$\Delta(l_0,\mathbf{l}) = J_1 \chi_3 H(l_0) \frac{\mathbb{1}|l_0-q_0| \geq \frac{1}{2}}{i(l_0-q_0)-e(\mathbf{l}-\mathbf{q})} D(l_0-p_0, |\mathbf{l}-\mathbf{p}|) \tag{2.155}$$

A single integration by parts gives

$$I(\chi_3 H(l_0)(1-H(l_0-q_0))) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\theta}{(2\pi)^d} \int \mathrm{d}\rho \, \mathrm{Log}(il_0-\rho) \frac{\partial}{\partial \rho} \Delta(l_0, \pi(\rho,\theta)) \tag{2.156}$$

On the support of the integrand $|\mathrm{Log}(il_0-\rho)| \leq 2|\log|\rho||$. Taking absolute values of the factors in the integrand and applying this inequality and Lemma 4 we get

$$|I(\chi_3 H(l_0)(1-H(l_0-q_0)))| \leq c 2\kappa_{\mathbf{p}} |J_1|_{1,\delta} |\chi_3|_1 \left| \frac{\mathbb{1}|l_0-q_0| \geq \frac{1}{2}}{i(l_0-q_0)-e(\mathbf{l}-\mathbf{q})} \right|_{1,\delta} \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d} \int_{-\delta}^{\delta} |\log|\rho|| \mathrm{d}\rho \tag{2.157}$$

Using inequality (1.172), Lemma 5 and

$$\left| \frac{\mathbb{1}|l_0-q_0| \geq \frac{1}{2}}{i(l_0-q_0)-e(\mathbf{l}-\mathbf{q})} \right|_{1,\delta} \leq 2 + 4|e|_1 \leq 5d\Lambda \tag{2.158}$$

we finally have

$$|I(\chi_3 H(l_0)(1-H(l_0-q_0)))| \leq c10 \frac{d\Lambda(1+\delta)^{d-2} \kappa_{\mathbf{p}} \,\mathrm{Vol}\, S^{d-1}}{(2\pi)^d} |\chi_3|_1 \tag{2.159}$$

The remaining term $I(\chi_3 H_s(l_0))$ can be bounded using Lemma 14 where $\omega(l_0,\mathbf{l}) = D(l_0-p_0, |\mathbf{l}-\mathbf{p}|) H_s(l_0)$.

$$|I(\chi_3 H_s(l_0))| \leq \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} (1 + |\mathrm{Log}|e(\mathbf{l})|| + |\mathrm{Log}|e(\mathbf{l},\mathbf{q},t)||) \sum_{|\alpha|\leq 2} |\delta^\alpha D| \tag{2.160}$$

By Lemma 4 this is finite and bounded by

$$|I(\chi_3 H_s(l_0))| \leq c6d^2\kappa_{\mathbf{p}}^2 M_2 \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{1 + |\mathrm{Log}|e(\mathbf{l})|| + |\mathrm{Log}|e(\mathbf{l},\mathbf{q},t)||}{|\mathbf{l}-\mathbf{p}|} \tag{2.161}$$

$$\leq c12d^2\kappa_{\mathbf{p}}^2 M_2 \frac{\bar{k}_{\log}}{(2\pi)^d} \tag{2.162}$$

where

$$\bar{k}_{\log} = \sup_{t\in[0,1]} \sup_{\mathbf{q},\mathbf{p}} \int \mathrm{d}\mathbf{l} \frac{(1 + |\mathrm{Log}|e(\mathbf{l},\mathbf{q},t)||)}{|\mathbf{l}-\mathbf{p}|} \tag{2.163}$$

□

2.4 Almost tangent Fermi surfaces in three dimensions

In this section the vertex correction is considered in the case where the Fermi surface and its translate by $\mathbf{q}$ touch or $\mathbf{q}$ is in a region close to the value where the surfaces touch. Here the behavior is strongly dependent on the dimension d. Here the behavior is considered for $d = 3$. For simplicity we also restrict ourselves to the standard spherical band relation.

Proposition 16. *Let $d = 3$ and $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. Let $\mathbf{q} \in \mathbb{R}^d$. Let ϵ be given by $|\mathbf{q}| = q = 2 - \epsilon$. Then for $|\epsilon| \leq \frac{1}{2}$ there exists a constant γ such that*

$$|\Gamma_3^*((q_0,\mathbf{q}),(p_0,\mathbf{p}))| \leq \gamma c |\mathrm{Log}\, c|^2 \tag{2.164}$$

We start by isolating the region where the two Surfaces meet. Let

$$\begin{gathered} \chi_0(x) + \chi_1(x) = 1\,, \quad \forall x \in \mathbb{R}\,;\ \chi_0, \chi_1 \in C^\infty \\ \operatorname{supp}\chi_0 \subset [-\frac{1}{3}, \frac{1}{3}] \text{ and } \chi_0 \equiv 1 \text{ on } [-\frac{1}{4}, \frac{1}{4}] \end{gathered} \tag{2.165}$$

and set

$$\chi(\mathbf{k}) = \chi_0(e(\mathbf{k}))\chi_0(e(\mathbf{k}-\mathbf{q})) \tag{2.166}$$

Furthermore, let

$$Z = \{x \in \mathbb{R} \,|\, |x| < \frac{1}{2} \wedge |x - q_0| < \frac{1}{2}\} \tag{2.167}$$

Then we can write

$$\Gamma_3^*((q_0,\mathbf{q}),(p_0,\mathbf{p})) = \tilde{I} + \hat{I} \tag{2.168}$$

where

$$\tilde{I} = \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} C((l_0,\mathbf{l}))C((l_0+p_0,\mathbf{l}-\mathbf{q}))D(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)\chi(\mathbf{l}) \tag{2.169}$$

By construction on the support of $(1-\chi(\mathbf{l})\mathbf{1}(|l_0|<\frac{1}{2}))$, $|(il_0-e(\mathbf{l}))^{-1}| < 4$ it is straightforward to see that

$$|\hat{I}| \leq \hat{\gamma} c \tag{2.170}$$

where we can take

$$\hat{\gamma} = 102\frac{\Lambda}{u_0}\mathrm{Vol}(S^{d-1})|J_1|_{1,\frac{1}{3}} \tag{2.171}$$

2.4.1 Parameter representation

In order to exploit sign cancellations occurring because of the changing sign of the fermion covariances, we introduce a parameter representation for the fermion covariance.

Denoting $X = is(l_0)s(l_0-q_0)D^*(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)\chi(\mathbf{l})$

$$\tilde{I} = B_{1>2} + B_{1<2} \tag{2.172}$$

where

$$\begin{aligned} B_{1>2} &= \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \int_0^\infty \mathrm{d}t_2 \int_{t_2}^\infty \mathrm{d}t_1 e^{-|l_0|t_1 - is(l_0)e(\mathbf{l})t_1} e^{-|l_0-q_0|t_1 - is(l_0-q_0)e(\mathbf{l}-\mathbf{q})t_2} X \\ B_{1<2} &= \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \int_0^\infty \mathrm{d}t_1 \int_{t_1}^\infty \mathrm{d}t_2 e^{-|l_0|t_1 - is(l_0)e(\mathbf{l})t_1} e^{-|l_0+q_0)|t_1 - is(l_0+q_0)e(\mathbf{l}-\mathbf{q})t_2} X \end{aligned} \tag{2.173}$$

We'll show the bound here only for $B_{1>2}$ as the other is almost identical after shift of integration variables.

As we want to integrate by parts only for large values of the parameters we introduce another set of cutoff functions.Let $h_0, h_1 \in C^\infty(\mathbb{R})$ and even, such that $h_0 + h_1 \equiv 1$ and $h_1(x) = 1$ for $|x| \geq 2$, $h_1(x) = 0$ for $|x| \leq 1$ and $h_1' > 0$ for $1 < x < 2$.

The following lemma estimates the effect of decay in the parameters to get bounds in l_0.

Lemma 17. *Let* $\alpha > \frac{1}{2}$, $\zeta > 0$, $g(\rho)$ *and* $f(\rho)$ *be* C^1*-functions on an open set containing* $[-\zeta, \zeta]$, *with* $g(0) = 0$, $f(\rho) > 0$ *and* $\operatorname{supp} f \subset [-\zeta, \zeta]$. *Let* $m_{jkr}(l_0, q_0, \rho)$ *be given by*

$$m_{jkr}(l_0, q_0, \rho) = \int_0^\infty \mathrm{d}t_2 \int_{\alpha t_2}^\infty \mathrm{d}t_1 \frac{h_j(t_2)}{t_2^j} \left| \left(\frac{\partial}{\partial\rho}\right)^r \frac{h_k(s(l_0)t_1 - s(l_0 - q_0)t_2 + s(l_0 - q_0)g(\rho)t_2)}{(s(l_0)t_1 - s(l_0 - q_0)t_2 + s(l_0 - q_0)g(\rho)t_2)^k} \right| e^{-|l_0|t_1 - |l_0 - q_0|\alpha t_2} \tag{2.174}$$

for $j, k, r \in \{0, 1\}, r \le k$. *Suppose there exist* $0 < \delta_1 < 1$ *and* $\delta_2 > 0$ *such that*

$$|g(x)| \le \delta_1 \quad \textit{and} \quad |g'(x)| > \delta_2 \qquad \forall x \in [-\zeta, \zeta]. \tag{2.175}$$

then there exists a constant M *such that*

i) $|m_{jk0}(l_0, q_0, \rho)| \le M(1 + k|\mathrm{Log}|l_0||)(1 + j|\mathrm{Log}|l_0 - q_0||) \quad \forall \rho \in [-\zeta, \zeta],\ j, k \in \{0, 1\}$

ii)

$$\int_{-\zeta}^{\zeta} m_{j11}(l_0, q_0, \rho) f(\rho) \mathrm{d}\rho$$

$$\le M(1 + k|\mathrm{Log}|l_0||)(1 + j|\mathrm{Log}|l_0 - q_0||) \left(\sup_\rho |f(\rho)| + \int_{-\zeta}^{\zeta} \left| \frac{\partial f}{\partial \rho} \right| \mathrm{d}\rho \right)$$

Proof. We start by rewriting the parameterization, writing $s(l_0, q_0)$ for $s(l_0)s(l_0 - q_0)$

$$m_{110}(l_0, q_0, \rho) = \int_\alpha^\infty \mathrm{d}t_1 \int_1^{t_1/\alpha} \mathrm{d}t_2 \frac{h_1(t_2)}{t_2} \frac{h_1(t_1 - s(l_0, q_0)(1 - g(\rho))t_2)}{|t_1 - s(l_0, q_0)(1 - g(\rho))t_2|} e^{-|l_0|t_1 - |l_0 - q_0|t_2} \tag{2.176}$$

As $t_1 > t_2 > 0$

$$\frac{1}{t_2} \frac{1}{|t_1 - s(l_0, q_0)(1 - g(\rho))t_2|} \le \frac{1}{t_1 + s(l_0, q_0)g(\rho)t_2} \left(\frac{1}{t_2} + \frac{1}{|t_1 - s(l_0, q_0)(1 - g(\rho))t_2|} \right) \tag{2.177}$$

$$\le \frac{1}{(1 - \delta_1)t_1} \left(\frac{1}{t_2} + \frac{1}{|t_1 - s(l_0, q_0)(1 - g(\rho))t_2|} \right) \tag{2.178}$$

Inserting this identity in (2.176) gives

$$m_{110}(l_0,q_0,\rho) \leq \frac{1}{1-\delta_1}\int_\alpha^\infty \mathrm{d}t_1 \int_1^{t_1/\alpha} \mathrm{d}t_2 \,\frac{1}{t_1}\frac{1}{t_2} e^{-|l_0|t_1 - |l_0-q_0|\alpha t_2}$$
$$+ \frac{1}{1-\delta_1}\int_\alpha^\infty \mathrm{d}t_1 \int_1^{t_1/\alpha} \mathrm{d}t_2 \,\frac{1}{t_1}\frac{h_1(t_1 - s(l_0,q_0)(1-g(\rho))t_2)}{|t_1 - s(l_0,q_0)(1-g(\rho))t_2|} e^{-|l_0|t_1 - |l_0-q_0|t_2} \quad (2.179)$$

Treating each term individually

$$\int_\alpha^\infty \mathrm{d}t_1 \int_1^{t_1/\alpha} \mathrm{d}t_2 \,\frac{1}{t_1}\frac{1}{t_2} e^{-|l_0|t_1 - |l_0-q_0|t_2\alpha} \leq \int_{\frac{1}{2}}^\infty \mathrm{d}t_1 \int_1^\infty \mathrm{d}t_2 \,\frac{1}{t_1}\frac{1}{t_2} e^{-|l_0|t_1 - |l_0-q_0|t_2} \quad (2.180)$$
$$\leq (1+|\mathrm{Log}|l_0||)(1+|\mathrm{Log}|l_0-q_0||)(1+|\mathrm{Log}\,\alpha|) \quad (2.181)$$

Consider

$$U = \int_1^{t_1/\alpha} \mathrm{d}t_2 \,\frac{h_1(t_1 - s(l_0,q_0)(1-g(\rho))t_2)}{|t_1 - s(l_0,q_0)(1-g(\rho))t_2|} e^{-|l_0-q_0|\alpha t_2} \quad (2.182)$$

Assuming for simplicity that $|l_0 - q_0|\alpha \leq 1$, then

$$U \leq \int_1^\infty \mathrm{d}t_2 \,\frac{\mathbf{1}(1 \leq |t_1 - s(l_0,q_0)(1-g(\rho))t_2| \leq \frac{1}{|l_0-q_0|})}{|t_1 - s(l_0,q_0)(1-g(\rho))t_2|} e^{-|l_0-q_0|\alpha t_2}$$
$$+ \int_1^\infty \mathrm{d}t_2 \,\frac{\mathbf{1}(|t_1 - s(l_0,q_0)(1-g(\rho))t_2| \geq \frac{1}{|l_0-q_0|})}{|t_1 - s(l_0,q_0)(1-g(\rho))t_2|} e^{-|l_0-q_0|\alpha t_2} \quad (2.183)$$

$$U \leq \int_1^\infty \mathrm{d}t_2 \,\frac{\mathbf{1}(1 \leq |t_1 - s(l_0,q_0)(1-g(\rho))t_2| \leq \frac{1}{|l_0-q_0|})}{|t_1 - s(l_0,q_0)(1-g(\rho))\alpha t_2|}$$
$$+ \int_1^\infty \mathrm{d}t_2 \, |l_0 - q_0| e^{-|l_0-q_0|\alpha t_2} \quad (2.184)$$

$$U \leq 1 + \frac{1}{1-\delta_1} \int_1^{\frac{1}{\alpha|l_0-q_0|}} \frac{1}{u}\,\mathrm{d}u \quad (2.185)$$

$$U \leq \frac{1}{1-\delta_1}(1+|\mathrm{Log}|l_0-q_0||)(1+|\mathrm{Log}\,\alpha|) \quad (2.186)$$

And thus

$$\frac{1}{1-\delta_1}\int\limits_{\alpha}^{\infty} \mathrm{d}t_1 \int\limits_{1}^{t_1/\alpha} \mathrm{d}t_2 \, \frac{1}{t_1}\frac{h_1(t_1 - s(l_0,q_0)(1-g(\rho))t_2)}{|t_1 - s(l_0,q_0)(1-g(\rho))t_2|} e^{-|l_0|t_1 - |l_0-q_0|\alpha t_2}$$
$$\leq \frac{(1+|\mathrm{Log}\,\alpha|)}{(1-\delta_1)^2}(1+|\mathrm{Log}|l_0||)(1+|\mathrm{Log}|l_0-q_0||) \quad (2.187)$$

and

$$m_{110}(l_0,q_0,\rho) \leq \frac{2(1+|\mathrm{Log}\,\alpha|)}{(1-\delta_1)^2}(1+|\mathrm{Log}|l_0||)(1+|\mathrm{Log}|l_0-q_0||) \quad (2.188)$$

The bounds on m_{jkl0} for other values of j and k follow similarly using $\int_0^\infty \mathrm{d}t h_0(t)u(t) \leq 2\|u\|_\infty$.

The second half of the Lemma is proven by reducing it to the first case using integration by parts with respect to ρ. However some care is needed to deal with the fact that we have taken absolute values. We make use of the following identity (which can be shown by splitting up the integral into parts where the sign of $w(x)$ is constant).

Remark 18. *Let $w(x) = W'(x)$. Then*

$$\int\limits_{-\zeta}^{\zeta} \mathrm{d}x|w|f(x) \leq \int\limits_{-\zeta}^{\zeta} \mathrm{d}x|W||f'(x)| + 2(\textit{number of zeros of } w \textit{ in } [-\zeta,\zeta]\cup \mathrm{supp}\,W)\,\mathrm{supp}_{[-\zeta,\zeta]}|W||f| \quad (2.189)$$

Let n_h be the number of zeroes of $\frac{\partial}{\partial x}\frac{h_1(x)}{x}$ on $|x| \geq 1$. N.B. It is possible to choose h_1 such that $n_h = 4$. Then

$$\int\limits_{-\zeta}^{\zeta} m_{j11}(l_0,q_0,\rho)f(\rho)\mathrm{d}\rho \leq \left(2n_h \sup_\rho m_{j10}(l_0,q_0,\rho)|f(\rho)| + \int\limits_{-\zeta}^{\zeta} m_{j10}(l_0,q_0,\rho)\left|\frac{\partial f}{\partial \rho}\right|\mathrm{d}\rho\right) \quad (2.190)$$

and the bound follows by applying the first part.

Summarizing we can take

$$M = \frac{4n_h(1+|\mathrm{Log}\,\alpha|)}{(1-\delta)^2} \quad (2.191)$$

□

2.4.2 Integration by parts

The basic idea now is to convert sign cancellations in the original integral into decay in the parameters. As usual this is done using integration by parts in the appropriate variables.

Let $(\rho, \hat{\theta}, \theta_1)$ be usual the polar coordinates with $\rho = e(\mathbf{l})$. In order to capture the volume factor from the dimensionality and the oscillating behavior in the parameters we change to coordinates $(\rho, \hat{\theta}, \phi)$ where $2\sqrt{\rho+1}(1-\cos\theta_1) = \phi^2$. This implies $\sin\theta_1 \mathrm{d}\theta_1 = \phi(\rho+1)^{-\frac{1}{2}}\mathrm{d}\phi$ and the Jacobian for $d=3$ with the spherical band relation is therefore of the form $J(\rho)\phi$.

Let $\alpha = \frac{1}{q-1}$, $a = q\alpha = \frac{q}{q-1}$. In these coordinates $\alpha e(\mathbf{l}-\mathbf{q})$ takes the form $\alpha e(\mathbf{l}(\rho,\hat{\theta},\phi)+\mathbf{q}) = -\rho + \xi(\rho) + a\phi^2$. Here $\xi(\rho) = q^2 + \alpha q(\rho - 2\sqrt{\rho+1})$. We see that $\xi'(0) = 0$ and $\xi''(\rho) \neq 0$ on the support of χ. In addition because $|(|q-2) < \frac{1}{2}$, for $\rho \in [-\frac{1}{3}, \frac{1}{3}]$, $|\chi'| < \frac{9}{10}$.

In these coordinates

$$B_{1>2} = \alpha \int_Z \mathrm{d}_\beta l_0 \int_0^\infty \mathrm{d}t_2 \int_{\alpha t_2}^\infty \mathrm{d}t_1 \int \mathrm{d}\hat{\theta} \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho e^{-|l_0|t_1 - is(l_0)\rho t_1}$$
$$e^{-|l_0-q_0|\alpha t_2 + is(l_0-q_0)(\rho-\xi(\rho))t_2} e^{-is(l_0-q_0)a\phi^2 t_2} \phi J X \quad (2.192)$$

We can then split of the integral as

$$B_{1>2} = I_{00} + I_{01} + I_{10} + I_{11} \quad (2.193)$$

where

$$I_{jk} = \alpha \int_Z \mathrm{d}_\beta l_0 \int_0^\infty \mathrm{d}t_2 \int_{\alpha t_2}^\infty \mathrm{d}t_1 \int \mathrm{d}\hat{\theta} \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho e^{-|l_0|t_1 - |l_0-q_0|\alpha t_2} e^{-iw} h_k(w') e^{-is(l_0-q_0)a\phi^2 t_2} \phi h_j(t_2) J X$$
$$(2.194)$$

$$\begin{aligned} w &= s(l_0)\rho t_1 - s(l_0-q_0)(\rho - \xi(\rho))t_2 \\ w' &= s(l_0)t_1 - s(l_0-q_0)(1-\xi'(\rho))t_2 \end{aligned} \quad (2.195)$$

Using

$$\int_0^{10} \mathrm{d}\phi e^{-is(l_0-q_0)a\phi^2 t_2} \phi h_1(t_2) J X$$
$$= \frac{is(l_0-q_0)}{a}\frac{h_1(t_2)}{t_2}(JX)|_{\phi=0} - \frac{is(l_0-q_0)}{a}\int_0^{10} \mathrm{d}\phi \frac{h_1(t_2)}{t_2} e^{-is(l_0-q_0)a\phi^2 t_2} \frac{\partial JX}{\partial \phi} \quad (2.196)$$

and

$$\int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho e^{-iw} h_1(w') JX = \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho e^{-iw} \frac{h_1(w')}{iw'} \frac{\partial JX}{\partial \rho} + \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho e^{-iw} \left(\frac{\partial}{\partial \rho} \frac{h_1(w')}{iw'} \right) JX \qquad (2.197)$$

we integrate by parts j-times with respect to ϕ and k-times with respect to ρ to obtain for $j, k = 0, 1$:

$$\begin{aligned} |I_{jk}| \leq & \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, m_{jk0}(\rho, l_0, q_0) \left| \left(\frac{\partial}{\partial \rho} \right)^k \left(\frac{\partial}{\partial \phi} \right)^j (JX) \right| \\ & + \delta_{k1} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, m_{j11}(\rho, l_0, q_0) \left| \left(\frac{\partial}{\partial \phi} \right)^j (JX) \right| \\ & + \delta_{j1} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, m_{1k0}(\rho, l_0, q_0) \left| \left(\frac{\partial}{\partial \rho} \right)^k (JX) \right|_{\phi=0} \\ & + \delta_{k1}\delta_{j1} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, m_{111}(\rho, l_0, q_0) \, |(JX)|_{\phi=0} \qquad (2.198) \end{aligned}$$

where m_{jkr} are as defined in Lemma 17 with $g(\rho) = \xi'(\rho)$.

The last half of (2.198) is bounded using Lemma 17 and $2|\mathrm{Log}|l_0|| > (1 + |\mathrm{Log}|l_0||)$ on Z to give

$$\begin{aligned} & \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, m_{1k0}(\rho, l_0, q_0) \left| \left(\frac{\partial}{\partial \rho} \right)^k (JX) \right|_{\phi=0} \\ & \leq \mathrm{const} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat{\theta} \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, |\mathrm{Log}|l_0|| |\mathrm{Log}|l_0 + q_0|| \left| \frac{1}{|\mathbf{l} - \mathbf{p}|} D(l_0 - p_0, |\mathbf{l} - \mathbf{p}|) \right|_{\phi=0} \\ & \leq \mathrm{const} \int \mathrm{d}\hat{\theta} \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \, c|\mathrm{Log}\, c|^2 |\mathrm{Log}\, |\mathbf{l} - \mathbf{p}||^2 \leq \mathrm{const} \, .c|\mathrm{Log}\, c|^2 \qquad (2.199) \end{aligned}$$

Because $\int \mathrm{d}\rho |\mathrm{Log}|\mathbf{l}(\rho, \hat{\theta}, 0) - \mathbf{p}||^2 \leq \mathrm{const}$.

As

$$\int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta |\mathrm{Log}|l_0|||\mathrm{Log}|l_0+q_0|| \sup_\rho |(JX)|_{\phi=0}$$
$$\le \mathrm{const} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta |\mathrm{Log}|l_0|||\mathrm{Log}|l_0+q_0||D(l_0-p_0,c\Lambda)$$
$$\le \mathrm{const}\,|\mathrm{Log}\,c|^2 c\,\Lambda(\mathrm{Log}(\Lambda)^2 \le \mathrm{const}\,.c|\mathrm{Log}\,c|^2 \quad (2.200)$$

we have using Lemma 17 and $JX>0$.

$$\int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, m_{111}(\rho,l_0,q_0)\,|(JX)|_{\phi=0}$$
$$\le \mathrm{const} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, m_{110}(\rho,l_0,q_0) \left|\frac{\partial}{\partial\rho}(JX)\right|_{\phi=0}$$
$$+\,\mathrm{const} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta |\mathrm{Log}|l_0|||\mathrm{Log}|l_0+q_0|| \sup_\rho |(JX)|_{\phi=0} \le \mathrm{const}\; c(\mathrm{Log}\,c)^2 \quad (2.201)$$

For the other two terms we apply the lemma again to get

$$\int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, m_{jk0}(\rho,l_0,q_0) \left|\left(\frac{\partial}{\partial\rho}\right)^k \left(\frac{\partial}{\partial\phi}\right)^j (JX)\right|$$
$$\le \mathrm{const} \int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, |\mathrm{Log}|l_0|||\mathrm{Log}|l_0+l_0|| \frac{1}{|\mathbf{l}-\mathbf{p}|^2} D(l_0-p_0,c|\mathbf{l}-\mathbf{p}|)$$
$$\le \mathrm{const}\,.c(\log c)^2 \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, \frac{(\log|\mathbf{l}-\mathbf{p}|)^2}{|\mathbf{l}-\mathbf{p}|}\chi(\mathbf{l}) \le \mathrm{const}\,.c(\log c)^2 \quad (2.202)$$

where we have used

Remark 19. *Let the coordinates $(\rho,\hat\theta,\phi)$ be as introduced above, then at fixed $\hat\theta$,*

$$\int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, \frac{(\log|\mathbf{l}-\mathbf{p}|)^2}{|\mathbf{l}-\mathbf{p}|}\chi(\mathbf{l}) \le \mathrm{const} \quad (2.203)$$

Proof. Note first that on the support of $(1-\mathbf{1}(||\mathbf{l}-\mathbf{p}||<\frac{1}{4}))$ the integrand is bounded by a constant times a logarithm squared and thus the contribution from this region is trivially finite.

Let $\mathbf{k}(\hat\theta)$ be the orthogonal projection of $\mathbf{p}$ on the plane given by extending $\{\mathbf{l}(\rho,\hat\theta,\phi)||\rho| < \frac{1}{3}, \phi = 0\ldots 10\}$. Let $\mathbf{k}_2(\hat\theta,\phi)$ be the orthogonal projection of $\mathbf{k}(\hat\theta)$ on the ray through the origin and $\mathbf{l}$. Then $|\mathbf{l}-\mathbf{p}| \geq |\mathbf{l}(\rho,\hat\theta,\phi) - \mathbf{k}(\hat\theta)| = \sqrt{|\mathbf{l}(\hat\theta,\rho,\phi) - \mathbf{k}_2(\hat\theta,\phi)|^2 + |\mathbf{k}_2(\hat\theta,\phi) - \mathbf{k}(\hat\theta)|^2}$.

Let $\hat k_3$ be such that $\{\hat l, \hat k_3\}$ forms a righthanded set of coordinate vectors for the plane and let

$$z = (\mathbf{l} - \mathbf{k}_2)\cdot \hat l \quad w = (\mathbf{k}_2 - \mathbf{k})\cdot \hat k_3 \tag{2.204}$$

then

$$\sqrt{|\mathbf{l}(\hat\theta,\rho,\phi) - \mathbf{k}_2(\hat\theta,\phi)|^2 + |\mathbf{k}_2(\hat\theta,\phi) - \mathbf{k}(\hat\theta)|^2} = \sqrt{z^2+w^2} \tag{2.205}$$

Geometrically z and w are the signed lengths of $\mathbf{l}-\mathbf{k}_2$ and $\mathbf{k}_2 - \mathbf{k}$ respectively.

It remains to show that on the support of $\chi(\mathbf{l})\mathbf{1}(||\mathbf{l}-\mathbf{p}|| < \frac{1}{4})$, $\left|\frac{\partial z}{\partial \rho}\right| > \text{const}$ and $\left|\frac{\partial w}{\partial \phi}\right| > \text{const}$. Then we can change to these variables to see that

$$\begin{aligned}
&\int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho \frac{(\log|\mathbf{l}-\mathbf{p}|)^2}{|\mathbf{l}-\mathbf{p}|}\chi(\mathbf{l})\mathbf{1}(||\mathbf{l}-\mathbf{p}|| < \frac{1}{4}) \\
&\qquad \leq \text{const} \int_0^{z_{\max}} \mathrm{d}z \int_0^{w_{\max}} \mathrm{d}w \frac{(\log|\mathbf{l}-\mathbf{p}|)^2}{\sqrt{z^2+w^2}}\chi(\mathbf{l})\mathbf{1}(||\mathbf{l}-\mathbf{p}|| < \frac{1}{4}) \leq \text{const}
\end{aligned} \tag{2.206}$$

The derivative of z is simple to bound as $z = r(\mathbf{l}) - r(\mathbf{k}_2)$ and $\frac{\partial r}{\partial \rho} > \text{const}$. Remembering that we denoted by θ_1 (from the polar coordinates) the angle in the constant $\hat\theta$ plane, then we can see explicitly

$$w = |\mathbf{k}(\hat\theta)|\sin(\theta_1(\mathbf{l}(\rho,\hat\theta,\phi)) - \theta_1(\mathbf{k}(\hat\theta))) \tag{2.207}$$

$$\begin{aligned}
\left|\frac{\partial w}{\partial \phi}\right| &= |\mathbf{k}(\hat\theta)|\cos(\theta_1(\mathbf{l}(\rho,\hat\theta,\phi)) - \theta_1(\mathbf{k}(\hat\theta)))\frac{\partial \theta_1}{\partial \phi} \\
&\geq \text{const}
\end{aligned} \tag{2.208}$$

by construction. □

In bounding the term

$$\delta_{k1}\int_Z \mathrm{d}_\beta l_0 \int \mathrm{d}\hat\theta \int_0^{10} \mathrm{d}\phi \int_{-\frac{1}{3}}^{\frac{1}{3}} \mathrm{d}\rho\, m_{j11}(\rho, l_0, q_0)\left|\left(\frac{\partial}{\partial \phi}\right)^j (JX)\right| \tag{2.209}$$

by applying Lemma 4 it is required to be more careful with taking the supremum and integrating. This is done in the following lemma

Remark 20. *Let* $f_{kl}(x, y_1, y_2) = |\mathrm{Log}|x + y_1||^k |\mathrm{Log}|x + y_2||^l$. *Then*

i) For

$$I_{kl} = \int_{-\infty}^{\infty} \mathrm{d}x f_{kl}(x, y_1, y_2) \int_{0}^{10} \mathrm{d}\phi \sup_{\rho} \frac{1}{|\mathbf{l} - \mathbf{p}|} \frac{|\mathbf{l} - \mathbf{p}|^2}{x^2 + |\mathbf{l} - \mathbf{p}|^2} \tag{2.210}$$

$$|I_{kl}| \leq \text{const} \quad \textit{uniform in } u_1, y_2, \mathbf{p}. \tag{2.211}$$

ii) For

$$I^* = \int_{Z} \mathrm{d}l_0 f_{11}(l_0, 0, q_0) \int_{0}^{10} \mathrm{d}\phi \sup_{\rho} \frac{1}{|\mathbf{l} - \mathbf{p}|} \frac{c^2 |\mathbf{l} - \mathbf{p}|^2}{(l_0 - p_0)^2 + c^2 |\mathbf{l} - \mathbf{p}|^2} \tag{2.212}$$

$$|I^*| \leq \text{const } c(\log c)^2 \tag{2.213}$$

Proof. i. First note that the result is obvious for the integration over regions where $|\mathbf{l} - \mathbf{p}| > \frac{1}{4}$ (or $|l_0 - p_0| > \frac{1}{2}$) so the below will silently assume that on the integration region the opposite holds. Let $\mathbf{k}(\hat{\theta})$, $\mathbf{k}_2(\hat{\theta}, \phi)$, ϕ and w be defined as in the previous remark. Let $v(\hat{\theta})$ be the signed length of $\mathbf{k}(\hat{\theta}) - \mathbf{p}$, i.e. $|v(\hat{\theta})| = |\mathbf{k}(\hat{\theta}) - \mathbf{p}|$. Then by construction

$$|\mathbf{l} - \mathbf{p}| = \sqrt{v(\hat{\theta})^2 + w(\hat{\theta}, \phi)^2 + z(\hat{\theta}, \phi, \rho)^2} \tag{2.214}$$

Let $u(\hat{\theta}, \phi) = \sqrt{v(\hat{\theta})^2 + w(\hat{\theta}, \phi)^2}$

$$|\mathbf{l} - \mathbf{p}| = \sqrt{u(\hat{\theta}, \phi)^2 + z(\hat{\theta}, \phi, \rho)^2} \tag{2.215}$$

The integral thus can be rewritten as

$$I_{kl} = \int_{-\infty}^{\infty} \mathrm{d}x f_{kl}(x, y_1, y_2) \int_{0}^{10} \mathrm{d}\phi \sup_{\rho} F \tag{2.216}$$

where

$$F = \frac{\sqrt{u(\hat{\theta}, \phi)^2 + z(\hat{\theta}, \phi, \rho)^2}}{x^2 + u(\hat{\theta}, \phi)^2 + z(\hat{\theta}, \phi, \rho)^2} \tag{2.217}$$

The idea is now that although the function F is maximal for $x^2 = u(\hat{\theta}, \phi)^2 + z(\hat{\theta}, \phi, \rho)^2$ (and then equal $\frac{1}{x}$) this value is not always in the domain of integration/taking the supremum. So we get

$$\sup_\rho F \leq \sup_\rho F\mathbf{1}(x^2 < u^2) + \sup_\rho F\mathbf{1}(u^2 \leq x^2 \leq \frac{1}{16}) + \sup_\rho F\mathbf{1}(\frac{1}{16} < x^2) \tag{2.218}$$

$$\leq \mathbf{1}(x^2 < u^2)\frac{|u(\hat{\theta},\phi)|}{x^2 + u(\hat{\theta},\phi)^2} + \mathbf{1}(u^2 \leq x^2 \leq \frac{1}{16})\frac{1}{x} + \frac{1}{4x^2 + \frac{1}{4}} \tag{2.219}$$

(where the last bound comes from the restriction $|\mathbf{l} - \mathbf{p}| \leq \frac{1}{4}$). The restrictions on the integration range now make the integral finite. For instance

$$\int_0^{10} \mathrm{d}\phi \int_{|u(\phi,\hat{\theta}|}^{\frac{1}{4}} \frac{1}{x} \leq \text{const} + \int_0^{10} \mathrm{d}\phi \,\mathrm{Log}\,|u(\hat{\theta},\phi)| \leq \text{const} \int_0^{10} \mathrm{d}\phi \,\mathrm{Log}\,|w(\hat{\theta},\phi)| \leq \text{const} \tag{2.220}$$

as $|\frac{\partial w}{\partial \phi}| > \text{const}$ for small w and thus ϕ. The additional logarithmic divergences in f_{kl} change nothing but the value of the constant.

i $\Longrightarrow$ ii. This implication follows by rescaling:

$$I^* = \int_{Z+p_0} \mathrm{d}l_0 f_{11}(l_0, p, p_0 + q_0) \int_0^{10} \mathrm{d}\sup_\rho \frac{|\mathbf{l}-\mathbf{p}|}{(\frac{l_0}{c})^2 + |\mathbf{l}-\mathbf{p}|^2} \tag{2.221}$$

$$\begin{aligned}
&\leq \int_{Z+p_0} \mathrm{d}l_0 (\log c)^2 \int_0^{10} \mathrm{d}\sup_\rho \frac{|\mathbf{l}-\mathbf{p}|}{(\frac{l_0}{c})^2 + |\mathbf{l}-\mathbf{p}|^2} \\
&\quad + |\log c| \int_{Z+p_0} \mathrm{d}l_0 f_{10}(\frac{l_0}{c}, \frac{p_0}{c}, 0) \int_0^{10} \mathrm{d}\phi \sup_\rho \frac{|\mathbf{l}-\mathbf{p}|}{(\frac{l_0}{c})^2 + |\mathbf{l}-\mathbf{p}|^2} \\
&\quad + |\log c| \int_{Z+p_0} \mathrm{d}l_0 f_{01}(\frac{l_0}{c}, 0, \frac{p_0+q_0}{c}) \int_0^{10} \mathrm{d}\phi \sup_\rho \frac{|\mathbf{l}-\mathbf{p}|}{(\frac{l_0}{c})^2 + |\mathbf{l}-\mathbf{p}|^2} \\
&\quad + \int_{Z+p_0} \mathrm{d}l_0 f_{11}(\frac{l_0}{c}, \frac{p_0}{c}, \frac{l_0-p_0}{c}) \int_0^{10} \mathrm{d}\phi \sup_\rho \frac{|\mathbf{l}-\mathbf{p}|}{(\frac{l_0}{c})^2 + |\mathbf{l}-\mathbf{p}|^2}
\end{aligned} \tag{2.222}$$

□

Chapter 3

Power counting and renormalization

3.1 Going to higher order

3.1.1 Restricting to a simpler case

In this and the following the chapter the higher order terms in the renormalized perturbation expansion are considered. Unfortunately controlling such terms requires quite some technicalities. As stated in the introduction we will restrict to the weaker bound proportional to $c^{1-\epsilon}$. Combined with the restriction to the spherical symmetric Jellium band relation this removes some of the technical problems.

3.1.2 Some notation

Univeral constant const

We will focus on the existence of constants and not their value. For that reason, we use the general notation 'const' as a placeholder for a constant that may change from inequality to inequality, but is independent of the important parameters of the problem. In particular c, q and β, however it is allowed to depend on ϵ.

Properties of graphs

Let G be a graph. Then we denote by $L(G)$ its set of lines and by $V(G)$ its set of vertices. $V_n(G)$ is the set of vertices of G with coordination number n. Denote by $L_F(G)$ and $L_B(G)$ the set of Fermion and Boson lines respectively. For a line l of G we will sometimes write P_l for its propagator and p_l for the momentum flowing through l.

Let $\{U_i\}_{i=1}^n$ be a set of pairwise disjunct subgraphs of G. Then the quotient graph $G/\{U_i\}_{i=1}^n$ is the graph obtained from G by replacing each U_i by a vertex v_i that is connected to the rest of G by exactly the same number and type of lines as U_i. If $n = 1$ we often write G/U for $G/\{U\}$.

3.2 Wick ordering

Another basis

A first useful method to attack the perturbation expansion is to do the expansion not in the basis of the normal monomials, but in the so called 'Wick ordered' or Wick-monomials. Note that a Wick-monomial of degree $2n$ is actually a polynomial degree $2n$. Expansion in terms of Wick-polynomials amounts to a partial resummation of the perturbation series.

Definition

Recall that it is possible to write each monomial of the fields $\bar{\psi}$ and ψ by means of the generating function

$$\bar{\psi}(k)\psi(p_1)\dots\bar{\psi}(k_n)\bar{\psi}(p_n) = \frac{-\delta}{\delta\chi(k_1)}\frac{\delta}{\delta\bar{\chi}(p_1)}\cdots\frac{-\delta}{\delta\chi(k_n)}\frac{\delta}{\delta\bar{\chi}(p_n)}e^{(\bar{\chi},\psi)+(\bar{\psi},\chi)}\bigg|_{\bar{\chi}=\chi=0} \tag{3.1}$$

This relation is used to define the Wick-monomials with to respect to covariance C, denoted using $:\cdot:_C$, as

$$:\bar{\psi}(k)\psi(p_1)\dots\bar{\psi}(k_n)\bar{\psi}(p_n):_C = \frac{-\delta}{\delta\chi(k_1)}\frac{\delta}{\delta\bar{\chi}(p_1)}\cdots\frac{-\delta}{\delta\chi(k_n)}\frac{\delta}{\delta\bar{\chi}(p_n)}\mathcal{W}_C\bigg|_{\bar{\chi}=\chi=0} \tag{3.2}$$

where

$$\mathcal{W}_C = e^{(\bar{\chi},\psi)+(\bar{\psi},\chi)-(\bar{\chi},C\chi)} \tag{3.3}$$

The Wick-monomials are useful because they are in a orthogonal with respect to the measure $\mathrm{d}\mu_C$. Because

$$\int \mathrm{d}\mu_C(\bar{\psi},\psi)\mathcal{W}_C(\bar{\psi},\psi) = 1 \tag{3.4}$$

we have for instance

$$\int d\mu_C(\bar{\psi},\psi) :\bar{\psi}(k)\psi(p_1)\ldots\bar{\psi}(k_n)\bar{\psi}(p_n):_C = \delta_{n0} \tag{3.5}$$

It is useful here to keep the covariance with respect to which Wick-ordering was done in the notation as in our case these will differ most of the time from the covariance with which the integration is done.

As an example the Wick-monomial of degree 2 is given by

$$:\bar{\psi}(k)\psi(p):_C = \bar{\psi}(k)\psi(p) + C(k)\delta(k-p) \tag{3.6}$$

Note that the extra term in the Wick-ordering is obtained by taking the pair of fields and 'contracting' it. That is replacing it by a propagator with the proper momentum dependence. In fact Wick-ordering is often defined recursively by (3.6) and extending to larger degrees by contracting all possible pairs.

Wick ordering, graphs and Wick lines

The expectation values of monomials in the fields are computed by expanding them into a sum over Feynman-graphs. If we instead compute the expectation value for (products) of Wick-monomials it is a well known result (e.g., [Sal99]) that this gives also a Feynman-graph expansion, but the contractions such as in (3.6) cancel exactly those graphs where there are lines that connect a vertex to itself. Such lines are called *Wick lines*. As a result graphs containing Wick-lines do not occur.

The Wick-ordered electron-phonon vertex

Below we assume the bare interaction vertex function $\mathcal{V}_0$ to a be Wick-ordered monomial with respect to $C_{<0}$. This is a common thing to do when studying field theory. In this specific case this assumption is even trivially satisfied. By momentum conversion the contraction of the Eletron-phonon vertex contains a factor $D(0) = 0$ and therefore vanishes.

3.3 The localization operator

3.3.1 Using localization to define the counterterm

As already stated in the introduction the aim of renormalization is to show that there exists a function $k(l_0, \mathbf{p})$ such that when $\frac{1}{il_0 - e(\mathbf{l}) - k(l_0,\mathbf{l})}$ is used as the free propagator it has the same singularity as the full propagator $\frac{1}{il_0 - e(\mathbf{l}) - k(l_0,\mathbf{l}) - \Sigma(l_0,\mathbf{l})}$. In particular we want the bare and the physical Fermi surfaces (i.e. the zero sets of $il_0 - e(\mathbf{l}) - k(l_0, \mathbf{l})$ and $il_0 - e(\mathbf{l}) - k(l_0, \mathbf{l}) - \Sigma(l_0, \mathbf{l})$ to be the same. In addition we want to take the function k as simple as possible, in particular we want it to have no, or very simple l_0 dependence.

Definition 21. *Let $t \in [0, 1]$ then P_t is the contraction defined by*

$$P_t\mathbf{l} = \mathbf{l}(t\rho(\mathbf{l}), \theta(\mathbf{l})) \tag{3.7}$$

We have $P_1 = \mathrm{id}$.

For the space of functions on $\mathbb{R} \times \mathbb{R}^d$ define the projection ℓ as

$$(\ell f)(p_0, \mathbf{p}) = f(0, P_0\mathbf{p}) \tag{3.8}$$

Then the condition on the zero set can be formulated as

$$\ell(il_0 - e(\mathbf{l}) - k(l_0, \mathbf{l}) - \Sigma(l_0, \mathbf{l})) = 0 \tag{3.9}$$

Because ℓ is a projection it easy to see that if we can choose k such that $k = -\ell\Sigma$ then the condition is satisfied.

3.3.2 A stricter localization

Definition

Note however that we have considerable freedom in choosing the function k. It turns out that in the case of a spherically symmetric band relation $e(\mathbf{l})$ it is convenient to require more of the counterterm. This done by replacing the projection ℓ by a new projection L defined by

$$(Lf)(p_0, \mathbf{p}) = f(0, P_0\mathbf{p}) + p_0(\frac{\partial}{\partial p_0} f)(0, P_0\mathbf{p}) + \frac{\partial}{\partial t} f(0, P_t\mathbf{p})\Big|_{t=0} \tag{3.10}$$

Properties of the localization operator

When the function T is given in the coordinates l_0, ρ, θ the projections become in these coordinates (for a function $T(l_0, \rho, \theta)$)

$$(\ell T)(l_0, \rho, \theta) = T(0, 0, \theta) \tag{3.11}$$

$$(LT)(l_0, \rho, \theta) = T(0, 0, \theta) + l_0(\frac{\partial}{\partial l_0}T)(0, 0, \theta) + \rho(\frac{\partial}{\partial \rho}T)(0, 0, \theta) \tag{3.12}$$

It is obvious from this form that $\ell^2 = \ell$ and $L^2 = L$, and that the projections map all functions on the zeroth- and first-order Taylor polynomials in l_0 and ρ. Moreover, if the function T is radially symmetric in $\mathbb{R}^d$, i.e. it doesn't depend on θ, then this also holds for lT and LT. In fact, for a radially symmetric T, lT is a constant and LT a first-order polynomial in l_0 and ρ.

For later it is good to have an expression for $(1-L)T$, the remainder of the second-order Taylor expansion:

$$(1 - L)T = \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s(\mathcal{D}T)(sl_0, s\rho, \theta) \tag{3.13}$$

where $\mathcal{D}$ is given by

$$\mathcal{D} = l_0^2 \frac{\partial^2}{\partial {l_0}^2} + 2l_0\rho \frac{\partial}{\partial l_0}\frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \rho^2} \tag{3.14}$$

We also need just the first-order expansion

$$\begin{aligned}(1 - L)T &= (1 - \ell)T - l_0(\frac{\partial}{\partial l_0}T)(0, 0, \theta) + \rho(\frac{\partial}{\partial \rho}T)(0, 0, \theta) \\ &= \int_0^1 \mathrm{d}t \left\{ l_0 \left((\frac{\partial T}{\partial l_0})(tl_0, t\rho, \theta) - (\frac{\partial T}{\partial l_0})(0, 0, \theta) \right) - \rho \left((\frac{\partial T}{\partial \rho})(tl_0, t\rho, \theta) - (\frac{\partial T}{\partial \rho})(0, 0, \theta) \right) \right\}\end{aligned} \tag{3.15}$$

When T is radially symmetric

$$\frac{\partial}{\partial l_0}\ell T = \frac{\partial}{\partial \rho}\ell T = 0 \tag{3.16}$$

and

$$\frac{\partial}{\partial l_0}LT = (\frac{\partial}{\partial l_0}T)(0, 0, \cdot) \quad \frac{\partial}{\partial \rho}LT = (\frac{\partial}{\partial \rho}T)(0, 0, \cdot) \tag{3.17}$$

$$\frac{\partial^2}{\partial {l_0}^2}LT = \frac{\partial}{\partial l_0}\frac{\partial}{\partial \rho}LT = \frac{\partial^2}{\partial \rho^2}LT = 0 \tag{3.18}$$

3.3.3 Counterterm

Changing the propagator means changing the measure in the functional integral. However, for the analysis of the theory it turns out to be convenient to keep the same measure for both renormalized and the unrenormalized theory. Observe that $(\bar{\psi}, C^{-1}\psi)$ appears as the quadric part in the exponential. Adding the function $K(p)$ thus has the effect of adding an extra quadratic term. So instead of absorbing this term in the covariance it is also possible to view it as an extra so called 'counterterm' in the interaction that is quadratic in the fermion fields.

That means we will use as an interaction $\mathcal{V}$ in our renormalized theory

$$\mathcal{V}(\bar{\psi}, \psi, \phi) = -g\mathcal{V}_0(\bar{\psi}, \psi, \phi) + \mathcal{K}(\bar{\psi}, \psi) \tag{3.19}$$

where $\mathcal{K}(\bar{\psi}, \psi)$ is a Wick-monomial of degree 2. Again here the Wick-ordering is of no consequence as Wick lines on a 2-vertex produce disconnected graphs.

3.3.4 Extending the localization operator

In order to make it convenient to implement the localization condition on the generating functionals, we extend the localization operator to a projection $\mathbf{L}$ on formal power series of the fields. First define $\mathbf{L}$ on Wick-monomials. If $Q(\bar{\psi}, \psi)$ is a Wick monomial of degree $2m \neq 2$ in the fields then $\mathbf{L}Q = 0$. If Q is of the form

$$Q(\bar{\psi}, \psi) = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} Q_2(l) : \bar{\psi}(l)\psi(l) :_C \tag{3.20}$$

the localization $\mathbf{L}$ acts as

$$\mathbf{L}Q(\bar{\psi}, \psi) = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} L(Q_2)(l) : \bar{\psi}(l)\psi(l) :_C \tag{3.21}$$

$$\tag{3.22}$$

Then extend $\mathbf{L}$ by linearity to all formal power series in the fields that have coefficients containing an overall momentum-conserving δ-function. All polynomials and formal power series that occur here are of this form, because of translation invariance.

3.4 Multiscale Analysis

The Multiscale Analysis techniques used here are based on [Sal98, Riv91, Gal85, GN85a, GN85b, FT90, FT91, FST96, FST98, FST99, Leh94, FLKT95, FKLT99]

3.4.1 The flow of the effective action

A decomposition of the Fermion propagator

The guiding principle of (multi-)scale analysis is to weigh the rate of increase of the propagator around the singularity by decomposing the space around the singularity in small shells that come exponentionally closer. The index of such a shell is called the scale. The contribution of a shell to an integral containing the propagator is bounded by the supremum of the absolute value of the propagator times the volume of the shell. As the shell is closer to the singularity the supremum increases while the volume falls. If this decay is quick enough to compensate for the increase in the supremum then the product, i.e., the contribution of this shell, is finite. If, additionally, the contributions form a summable series, the integral is finite. Therefore, (multi-)scale analysis can be seen as a powerful and exact version of power counting.

This method was originally developed to deal with ultra-violet divergences [Riv91, Gal85, GN85a, GN85b]. In that case the decay of the propagator is weighed against an increasing volume of shells, but the technique is otherwise the same.

The division of space into shells is done using a smooth partition of 1 on the support of C. Let $M > \max\{4, \frac{1}{r_1}\}$. Let $a \in C^\infty(\mathbb{R}_0^+, [0,1])$ such that

$$a(x) = \begin{cases} 0 & \text{for } x \leq M^{-4} \\ 1 & \text{for } x \geq M^{-2} \end{cases} \tag{3.23}$$

and $a'(x) > 0$ for $x \in (M^{-4}, M^{-2})$. Then define for $j \in \mathbb{Z}_-$

$$f_j(x) = a(M^{-2j}x) - a(M^{-2(j-1)}x) \tag{3.24}$$

which gives

$$1 - a(x) = \sum_{j=\infty}^{-1} f_j(x) \tag{3.25}$$

$$\operatorname{supp} f_j = [M^{2j-4}, M^{2j}] \tag{3.26}$$

Note the standard convention that the *scale* j is a negative integer. This is because we are studying the infra-red end of the model. The positive integers are typically used in studies of the ultraviolet contributions. If we define $\mathbb{1}_j(x) = \mathbb{1}(|x| < M^j)$ then $|f_j|(x^2) \leq \mathbb{1}_j(x)$.

We can now insert the partition (3.25) into to the propagator. Define the *(hard) propagator of scale* j, C_j as

$$\begin{aligned} C_j(l) &= C(l) f_j(l_0^2 + e(1)^2) = \frac{f_j(l_0^2 + e(1)^2)}{il_0 - e(1)} \quad \text{for } j < 0 \\ C_0(l) &= C(l) a(l_0^2 + e(1)^2) = \frac{a(l_0^2 + e(1)^2)}{il_0 - e(1)} \end{aligned} \tag{3.27}$$

Because of the special choice of the scale zero propagator, the full propagator C decomposes as

$$C(l) = C_{\leq 0}(l) = \sum_{j \leq 0} C_j(l) \tag{3.28}$$

The name $C_{\leq 0}$ for the complete propagator will be used to emphasize the presence of the sum over all scales when appropriate.

Later we also need the *soft propagator of scale* j, $C_{<j}$, defined as

$$C_{<j}(l) = \sum_{k<j} C_k(l) \tag{3.29}$$

Note that on the support of $C_j(l)$, by construction , $|il_0 - e(1)| > M^{j-2}$ and thus C_j is no longer singular. For any multi-index α we have that

$$|\partial^\alpha C_j(l)| \leq \text{const}\, M^{-(1+|\alpha|)j} \mathbb{1}_j(l_0) \mathbb{1}_j(e(1)) \tag{3.30}$$

for all $j < 0$.

After a change to the coordinates (ρ, θ) of Lemma 5 one easily sees that

$$\int \mathrm{d}1\, \mathbb{1}_j(e(1)) \leq \text{const}\, M^j \tag{3.31}$$

$$\int \mathrm{d}_\beta l_0\, \mathbb{1}_j(l_0) < \text{const}\, M^j \tag{3.32}$$

Using these three inequalities we have for $j < 0$

$$\int \mathrm{d}_\beta l_0 \int \mathrm{d}1 |C_j(l_0, 1)| \leq \text{const}\, M^j \tag{3.33}$$

and for $j \le 0$

$$\int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} |C_{<j}(l_0, \mathbf{l})| \le \text{const}\, M^j \tag{3.34}$$

On the support of C_0 the momenta are not restricted to a compact set and therefore it has to be treated differently. The UV-cutoff will still restrict the vector momentum, but the frequency sum contains infinitely many terms.

It is in fact a good warm-up exercise in working with scales to show that (3.33) implies (3.34). Inserting the definition of the soft propagator

$$I = \int \mathrm{d}l_0 \int \mathrm{d}\mathbf{l} |C_{<j}(l_0, \mathbf{l})| \le \int \mathrm{d}l_0 \int \mathrm{d}\mathbf{l} \sum_{k<j} |C_k(l_0, \mathbf{l})| \tag{3.35}$$

At finite β, $|l_0|$ is bounded from below and thus the sum on the right is actually finite. Therefore it can be exchanged with the integral. Using (3.33) we have

$$I \le \sum_{k<j} \int \mathrm{d}l_0 \int \mathrm{d}\mathbf{l} |C_k(l_0, \mathbf{l})| \tag{3.36}$$

$$\le \text{const} \sum_{k<j} M^k \tag{3.37}$$

$$= \text{const} \sum_{k<j} M^{k-j} M^j = \text{const}\, M^j \sum_{k<0} M^k \le \text{const}\, M^j \tag{3.38}$$

Given a graph G it is possible to insert (3.28) in the integral for $\mathrm{Val}(G)$. This gives a scale decomposition for each fermion line in the graph. For this reason the method is called *multiscale analysis*. As in the preceding trivial example it is possible to take the sums outside of the integrals. Each fermion line l in the graphs then has a propagator C_{j_l} attached to it. Bounding $\mathrm{Val}(G)$ then amounts to using (3.30) and (3.33) and showing that the factors that occur are summable. Thus bounding the graphs has been reduced to a conceptually simple, albeit sometimes technically challenging book-keeping exercise. Before we embark on this, we first show that such graphs in fact occur naturally as part of a discrete renormalization group flow.

Semigroup properties of the effective action

Recall the effective action defined in (1.49), which was given as:

$$G(\bar{\xi}, \xi, \eta, \mathcal{V}) = \log \int \mathrm{d}\mu_{C_{\le 0}}(\bar{\psi}, \psi) \mathrm{d}\mu_D(\phi) e^{\mathcal{V}(\bar{\psi}+\bar{\xi}, \psi+\xi, \chi+\eta)} \tag{3.39}$$

Define $\mathcal{V}_{\text{eff}}(\bar{\xi}, \xi, \eta)$ as

$$\mathcal{V}_{\text{eff}}(\bar{\xi}, \xi, \eta) = \log \int \mathrm{d}\mu_D(\phi) e^{\mathcal{V}(\bar{\xi}, \xi, x+\eta)} \tag{3.40}$$

Then the total effective action is given by

$$G(\bar{\xi}, \xi, \eta, -g\mathcal{V}) = \log \int \mathrm{d}\mu_{C_{\leq 0}}(\bar{\psi}, \psi) e^{\mathcal{V}_{\text{eff}}(\bar{\psi}+\bar{\xi}, \psi+\xi, \eta)} \tag{3.41}$$

We concentrate on the those contributions to the generating functions with no external phonon sources. Thus we can set η to zero on both sides. Dropping the $\mathcal{V}$-dependency from the notation this gives

$$G(\bar{\xi}, \xi) = \log \int \mathrm{d}\mu_{C_{\leq 0}}(\bar{\psi}, \psi) e^{\mathcal{V}_{\text{eff}}(\bar{\psi}+\bar{\xi}, \psi+\xi)} \tag{3.42}$$

Note that the phonon part of the theory is now completely contained in $\mathcal{V}_{\text{eff}}$ which appears instead of $\mathcal{V}$ in the interaction. This can be interpreted as a new field theory with 'effective interaction' $\mathcal{V}_{\text{eff}}$. This is one reason for calling G the effective action.

The renormalization procedure we follow now is a further application of this idea. It is originally due to Wilson. Our current formulation is taken from [FT91, FST96] and companion papers. See also [Sal99] for a more detailed introduction and a more elegant approach. Let the general effective action be given by

$$G_X(\bar{\xi}, \xi, \mathcal{U}) = \log \int \mathrm{d}\mu_X(\bar{\psi}, \psi) e^{\mathcal{U}(\bar{\psi}+\bar{\xi}, \psi+\xi)} \tag{3.43}$$

If the covariance $X = X_1 + X_2$ is a sum of two covariances then the Gaussian measure is simply the product measure of the measures $\mathrm{d}\mu_{X_1}$ and $\mathrm{d}\mu_{X_2}$. This can be used to write

$$G_{X_1+X_2}(\bar{\xi}, \xi, \mathcal{U}) = \log \int \mathrm{d}\mu_{X_1}(\bar{\psi}, \psi) e^{\mathcal{G}_{X_2}(\bar{\psi}+\bar{\xi}, \psi+\xi)} = G_{X_1}(\bar{\xi}, \xi, G_{X_2}(\cdot, \mathcal{U})) \tag{3.44}$$

Because of this property the method is often called the *Renormalization Group Approach*, although of course (3.44) is only a semigroup representation.

Integrating out fluctuations scale by scale

Proceed by using the semigroup property on the decomposition (3.28). As we already saw in the example above, this decomposition is effectively a finite sum at finite β. This will be implicit most of the time, but to make it explicit observe that there is some $I = I(\beta)$ such that.

$$G(\cdot, \mathcal{V}_{\text{eff}}) = G_{C_{<I}}(\cdot, G_I(\cdot, \mathcal{V}_{\text{eff}})) = G_I(\cdot, \mathcal{V}_{\text{eff}}) \tag{3.45}$$

where $G_k(\bar{\chi}, \chi; \mathcal{U}) = G_{\sum_{j=k}^{0} C_j}(\mathcal{U})$ is called the effective interaction of scale k. This identity can be seen from the characteristic function of the measure $\mathrm{d}\mu_{C_{<I}}$ and the fact that I can be chosen such that the because of support properties $(\bar{\chi}, C_{<I}\chi) = 0$. Alternatively observe that in a Feynman graph expansion the right-hand side is the leading term, and the only one that does not involve integration over lines with $C_{<I}$.

Inserting the finite decomposition of the propagator we get the recursive expression

$$G_j = G_{C_j}(\cdot, G_{j+1}) \quad \forall I \leq j < -1 \tag{3.46}$$

On a physical level this means that at scale j the theory can be seen as an effective theory field theory where the propagators are of scale j and the higher-scale contributions appear as effective interactions. The higher scale fluctuations are said to have been "integrated out".

It is convenient to rewrite (3.46) as

$$G_j = G_0 + \sum_{j \leq k < 0} (G_k - G_{k+1}) \tag{3.47}$$

3.4.2 Renormalization

Introducing the counterterm scale by scale

Recall that $\mathcal{V}(\bar{\chi}, \chi, \eta) = -g\mathcal{V}_0(\bar{\chi}, \chi, \eta) + \mathcal{K}(\bar{\chi}, \chi)$ with $\mathcal{K}$ a Wick monomial of degree 2. Thus

$$\mathcal{V}_{\text{eff}} = V_0 + \mathcal{K} \tag{3.48}$$

with

$$V_0 = \log \int \mathrm{d}\mu_D(\phi) e^{-g\mathcal{V}_0(\bar{\xi}, \xi, \phi+\eta)} \tag{3.49}$$

If we now also define

$$\mathcal{E}^j = \begin{cases} G_j - G_{j+1} & j < 0 \\ G_0 - \mathcal{V}_{\text{eff}} & j = 0 \end{cases} \tag{3.50}$$

$$\tag{3.51}$$

we can rewrite (3.47) as

$$G_j = V_0 + \mathcal{K} + \sum_{j \le k \le 0} \mathcal{E}^k \tag{3.52}$$

Note that at finite beta $G = G_I$ and $\mathcal{E}^k = 0$ for $k < I$. Therefore for $j = I$, (3.52) can be written as

$$G = V_0 + \mathcal{K} + \sum_{k \le 0} \mathcal{E}^k \tag{3.53}$$

Remembering that the object of counterterm is to regularize the theory such that $\mathbf{L}G = 0$ and that $\mathbf{L}$ projects on the wick-polynomials of degree 2, we get the following equation for the counterterm.

$$\begin{aligned} 0 = \mathbf{L}G &= \mathbf{L}V_0 + \mathbf{L}\mathcal{K} + \mathbf{L}\sum_{k \le 0} \mathcal{E}^k \\ &= \mathbf{L}\mathcal{K} + \mathbf{L}\sum_{k \le 0} \mathcal{E}^k \end{aligned} \tag{3.54}$$

Using $\mathbf{L}^2 = \mathbf{L}$, one sees that (3.54) is 'solved' by

$$\mathcal{K}(\bar{\chi}, \chi) = \sum_{k \le 0} -\mathbf{L}\mathcal{E}^k(\bar{\chi}, \chi) \tag{3.55}$$

Note that this is actually an equation for $\mathcal{K}$ as the $\mathcal{E}^j$'s still depend on $\mathcal{K}$. However when $\mathcal{K}$ is expanded as a formal power series in g, then for the contributions of order r only terms of order $r' < r$ appear on the right hand side by virtue of the construction of $\mathcal{E}^j$. Thus on this level (3.55) can be interpreted as a recursive definition of its own solution as a formal power series.

The renormalized theory

Now that we have an expression for the counter term, it possible to define the renormalized theory.

Definition 22. *The renormalized theory is given by the effective action* (3.39) *where* $\mathcal{V} = -g\mathcal{V}_0 + \mathcal{K}$ *with* $\mathcal{K}$ *given by* (3.55).

3.4.3 Gallavotti-Nicolò trees

Tree expansion

Above the multiscale analysis was used to define the counterterm. However as we shall see now, it also a great help in finding bounds for the coefficients in the power series expansion of G. The main reason for this is that the sum over all graphs gets subdivided into sums over graphs with a particular structure of subgraphs (which dictates the structure of the divergences). Again we follow [FT91] and [FST96].

After inserting the definition for the counter term (3.52) can be arranged in the convenient form

$$G_j = V_0 + \sum_{j \leq k}(1-\mathbf{L})\mathcal{E}^k + \sum_{k<j} -\mathbf{L}\mathcal{E}^k \tag{3.56}$$

The objective is to expand the coefficients G_j in a series of Feynman-Graphs. As was stated in the introduction these occur when expanding the exponential of the interaction. This can be made more explicit [Sal99]: Let $\mathcal{U}$ be a polynomial in the fields. Define the evaluation operators $\mathcal{E}_n$ as

$$\mathcal{E}_n(C, \mathcal{U}_1, \ldots, \mathcal{U}_n) = \prod_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \log \int d\mu_C e^{\sum_{i=1}^{n} \lambda_i \mathcal{U}_1} \Bigg|_{\lambda_1 = \cdots = \lambda_n = 0} \tag{3.57}$$

With this definition we have for $k < 0$

$$\mathcal{E}^k = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}_n(C_j, \underbrace{G_{j+1}, \ldots, G_{j+1}}_{n \text{ times}}) \tag{3.58}$$

This sum starts at $n = 1$ because we have explicitly subtracted the $n = 0$ contribution in the definition of $\mathcal{E}^k$. In fact if $\mathcal{U}$ is a Wick ordered monomial with respect to C, then $\mathcal{E}_1(C, \mathcal{U}) = 0$. Denote by $P_{C,n}$ is the orthogonal projection on the wick-ordered polynomials of degree n.

Insert (3.58) in (3.56) and expand each term as a Wick-monomial with respect to $C_{<j}$. The resulting sum can be kept finite by writing everything as formal power series g and doing this order by order. The equation (3.56) is a recursive definition expressing G_j in terms of the other G_k's with appear as arguments to $\mathcal{E}_n$'s. Note that $\mathcal{E}_n$ is multi-linear. Therefore the sum over Wick-monomial contributions to (3.56) can be taken outside.

These sums in the expression of G can be represented as a sum over *Gallavotti-Nicolò trees* (GN-trees, or often just 'trees') [Gal85, GN85a, GN85b]. A GN-tree is a triple (t, E_f, P_f), here t is a tree rooted at root ϕ. The number of branches above a fork f is denoted n_f. E_f is a map from the forks of τ to the positive even integers. P_f is a map from the forks f of t to $\{1, (1 - \mathbf{L}), -\mathbf{L}\}$ such that $P_f = 1$ when $E_f \geq 4$ and $P_f \neq 1$ when $E_f = 2$. A fork with $P_f = 1 - \mathbf{L}$ is called an *R-fork*. A fork with $P_f = -\mathbf{L}$ is called a *C-fork*. Most of the time we will simply write t for the triple (t, E_f, P_f). Denote by $\pi(f)$ the fork just below a fork f.

A particular contribution to G is given by a GN-tree (t, E_f, P_f) and a labeling j_f of the forks and leaves of t as follows. Starting with the root, take a fork f and:

- If f is a leaf, then $j_f = 0$ and take the term V_0.
- If f is a fork and $E_f \geq 4$ then $j_f > j_{\pi(f)}$. Take the term $P_{C_{<j_f}, E_f} \frac{1}{n_f!} \mathcal{E}_{n_f}^{j_f}$. Each of the n_f arguments to $\mathcal{E}_{n_f}^{j_f}$ is given by the term corresponding a sibling of f. These are determined recursively.
- If f is a fork, $E_f = 2$, and f is an R-fork then $j_f > j_{\pi(f)}$. Take the term $(1 - \mathbf{L}) P_{C_{<j_f}, E_f} \frac{1}{n_f!} \mathcal{E}_{n_f}^{j_f}$. The arguments are determined as above.
- If f is a fork, $E_f = 2$, and f is an C-fork then $j_f < j_{\pi(f)}$. Take the term $(-\mathbf{L}) \frac{1}{n_f!} \mathcal{E}_{n_f}^{j_f}$. The arguments are determined as above.

A labeling with scales j_f with the properties $j_v = 1$ for a leaf, $j_f < j_{\pi(f)}$ for a C-fork, and $j_f > j_{\pi(f)}$ otherwise is called *compatible to t*. Note that because of the Wick-ordering only trees with $n_f \geq 2$ for all f with $j_f < 0$ contribute.

Labeled graphs

To complete the expansion in to graphs we use the following standard result (See, e.g., [Sal99]): If for each $i = 1, \ldots, n$, $\mathcal{U}_i$ is a Wick-ordered monomial of degree $2m_i$ with respect to $C_{\leq j} = C_j + C_{<j}$ and the order r' contribution to the coefficient is given by $U_{i,r'}$ times a momentum preserving delta-function, then the coefficient of order r of $P_{C_{<j}, m} \mathcal{E}_n(C_j, \mathcal{U}_1, \ldots, \mathcal{U}_1)$ is a up to a delta function given by $g^r E_{2m,r,j,n}$ with

$$E_{2m,r,j,n} = \sum_{\sum_{i=1}^n r_i = r} \sum_G \sum_s \mathrm{Val}(G, s)(C_j, C_{<j}, U_{1,r_1}, \ldots, U_{n,r_n}) \tag{3.59}$$

Here G is the sum over all (connected) graphs with $2m$ external legs and n vertices where vertex i has $2m_i$ legs and where no line connects a vertex to itself when $j < 0$.[1] The sum over s is over all maps $s : L(G) \to \{0, 1\}$ such that G is connected by just the lines with $s(l) = 0$.

1. At j=0 it possible that Wick-lines occur because V_0 is not necessarily a wick-ordered polynomial.

The lines with $s(l) = 0$ are called *hard lines* and the lines with $s(l) = 1$ are called *soft lines*. $\mathrm{Val}(G, s)$ is defined like the normal Feynman rules, but lines $s(l) = 0$ have C_j as a propagator and lines with $s(l) = 1$ have $C_{<j}$ as a propagator. The vertex with index i has U_{i,r_i} as a vertex function. With each line there is thus associated a labeling (j_l, s_l) with $j_l = j$ and $s_l = s(l)$.

The connectedness by hard lines condition comes from the fact that the integration is with respect to $d\mu_{C_j}$ and the logarithm is taken. Graphs with lines connecting a vertex to itself do not appear because the vertex functions are Wick-ordered monomials. Finally the extra soft lines come from the contractions with $C_{<j}$ that result from expanding unordered monomials in terms of those that are Wick-ordered with respect to $C_{<j}$.

To compute $G_{2m,r}$ in terms of Feynman-Graphs, apply the above result to the contribution of a tree t rooted at ϕ and a labeling with scales j_f compatible to t. If the tree does not just consist of a single leaf this expands $P_{C<j_\phi,E_\phi}\mathcal{E}^{j_\phi}_{n_f}$ in terms of a graphs with labels (j_l, s_l) with $j_l = j_\phi$. However the vertex functions are either the scale zero vertices V_0 or they are of the form $P_f P_{C_{<j_f},E_f}\mathcal{E}^{j_f}_{n_f}$ and thus can be expanded in a set of graphs with labelings (j_l, j_s) with $j_l = j_f$. If we take the sum over these graphs outside then each of these graphs appears as subgraphs replacing the vertices. Repeat this process recursively until the leaves are reached. Note that to each fork there is subgraph G_f and these subgraphs are partially ordered by inclusion such that $G_f \subset G_{f'}$ iff $f > f'$ in t.

Exchanging the sum over scale assignments to the tree with the sums over subgraphs and combining those in one big sum over graphs gives

$$G_{2m,2r} = \sum_{j\leq 0}\sum_{t}\prod_{f\in t}\frac{1}{n_f!}\sum_{G}\sum_{J\in\mathcal{J}(t,G,j)}\mathrm{Val}(G^J) \tag{3.60}$$

Here $\mathcal{J}(t, G, j)$ is the set of all labelings J of G that have root scale j and are compatible with t. A *labeling* J of G is assignment the lines of G with labels (j_l, s_l). A graph G with a labeling J is denoted by G^J and is called a *labeled graph*. A labeling J is *compatible* to t if there is a mapping $f \mapsto G_f$ from the tree t to the subgraphs of G such that

- $G_\phi = G$, $G_f \subset G_{f'}$ iff $f > f'$,
- When f and f' are not ordered then G_f and $G_{f'}$ are disjunct.
- for each f all lines l of the quotient graph $\tilde{G}(f) = G_f/\{G_{f'}\}_{\pi(f')=f}$ have the same scale, say j_f,
- $\tilde{G}_f$ has n_f vertices.

- $\tilde{G}_f$ is connected by its hard lines, and
- The labeling of t defined by the j_f's and assigning scale zero to the leafs is compatible to t.

If $\mathcal{J}(t, G, j) \neq \emptyset$ for some j then G is called *compatible* to t or vice versa. A labeled graph is said to have *root scale* j if $j_\phi = j$.

Each vertex V_0 or of order 2 and corresponds to a leaf of t. The sum over trees t is thus a sum over all trees t with r leafs. As $n_f \geq 2$ for all f with $j_f < 0$ the number of such trees is finite. A bound due to Felder for the number of such trees with given incidence numbers is given in appendix F of [GN85a]. However we do not require it here.

For a graph with none of the $E_f = 2$, $\mathrm{Val}(G^J)$ is simply given by the usual Feynman-rules with hard or soft propagator of scale j_l associated with a line l. If G contains an G_f with $E_f = 2$, $\mathrm{Val}(G^J)$ is defined by applying P_f to the value of $G_f^{J_f}$ and inserting that as a vertex function in the containing graph. Here J_f is $J|_{G_f}$. A more explicit expression is given in Section 5.3.2.

3.4.4 Powercounting for labeled graphs

The decomposition of (3.60) is convenient because the subgraph structure induced by the tree characterizes the power counting. This is captured in the following *Power Counting Lemma.*
Lemma 23 (Power Counting [FT91, FST96]). *Let $j \leq 0$, t a tree with $E_f \geq 4$, G a graph and $J \in \mathcal{J}(t, G, j)$. Then*

$$|\mathrm{Val}\, G^J|_0 \leq \mathrm{const} \prod_{v\ leaf} |\mathrm{Val}(v)|_0\, M^{D_\phi j_\phi} \prod_{f > \phi} M^{D_f (j_f - j_{\pi(f)})} \tag{3.61}$$

Here $D_f = \frac{1}{2}(4 - E_f) - |V_2(G_f)|$.

This lemma actually holds more generally. For instance; For the unrenormalized theory, i.e. $\mathbf{L} = 0$, the restriction $E_f \geq 4$ can be dropped.

Proof. We give an inductive proof that exploits the recursiveness of the labeled graphs that occurs naturally because of the tree structure [Leh94]. The induction is on the height of the tree, i.e., the maximal number of forks between a leaf of the tree and the root.

A graph belonging to a tree of height zero consists of just a vertex function, so the lemma holds trivially. If the tree has nonzero height then consider $\tilde{G}(\phi)$. Denote by t_f the subtree of t rooted at f, and by J_f the labeling of G_f induced by J (N.B. $J_f \in \mathcal{J}(t_f, G_f, j_f)$). Then

$$|\mathrm{Val}(G^J)|_0 \leq |\mathrm{Val}'(\tilde{G}^J(\phi))|_0 \prod_{\pi(v)=\phi} |\mathrm{Val}(v)|_0 \prod_{\pi(f)=\phi} |\mathrm{Val}\, G_f^{J_f}|_0 \tag{3.62}$$

Where $\mathrm{Val}'(\tilde{G}^J(\phi))$ means the value of the graph where all vertex functions have been replaced by 1 and all propagators by their absolute values.

Note that all lines of $\tilde{G}^J(\phi)$ have scale j_ϕ. Start with the case $j_\phi < 0$. By construction it is connected by hard lines. It is possible to choose a spanning tree T of consisting of hard lines only. By (3.30) for each line $l \in T$ the propagator $|C_l|$ is bounded by const $M^{-j_l} =$ const M^{-j_ϕ}. Each line $l \in L(\tilde{G}(\phi) \setminus T)$ forms part of an independent loop containing l and lines of T. We can bound the integral corresponding to this loop using (3.33) and (3.34). Note that only C_{j_l,s_l} appears in the integral because we have already taken suprema of the propagators of the lines in the tree. Taking all these together gives

$$|\mathrm{Val}'(\tilde{G}^J(\phi))|_0 \leq \text{const}\, M^{-j_\phi |L(T)|} M^{j_\phi |L(\tilde{G}(\phi) \setminus T)|} \tag{3.63}$$

This power counting bound can be rearranged using the topological properties of the graph and the spanning tree:

$$|L(\tilde{G}(\phi) \setminus T)| = |L(\tilde{G}(\phi))| - |L(T)| \tag{3.64}$$

$$|L(T)| = |V(T)| - 1 = |V(\tilde{G}(\phi))| - 1 \tag{3.65}$$

$$|L(\tilde{G}(\phi))| = \frac{1}{2}\left(\sum_{\pi(v)=\phi} E_v + \sum_{\pi(f)=\phi} E_f - E_\phi \right) \tag{3.66}$$

$$|V(\tilde{G}(\phi))| = \sum_{\pi(v)=\phi} 1 + \sum_{\pi(f)=\phi} 1 \tag{3.67}$$

All these taken together gives

$$|L(\tilde{G}(\phi) \setminus T)| - |L(T)| = 2 - \frac{1}{2}E_\phi + \sum_{\pi(v)=\phi} \frac{1}{2}E_v - 2 + \sum_{\pi(f)=\phi} \frac{1}{2}E_f - 2 \tag{3.68}$$

Note that $\frac{1}{2}E_v - 2 = 0$ for $E_v = 4$ and $\frac{1}{2}E_v - 2 = -1$ for $E_v = 2$. Because the graphs G_f for $\pi(f) = \phi$ are disjunct we have $V_2(G_\phi \setminus \{G_f\}_{\pi(f)=\phi}) = V_2(G_\phi) - \sum_{\pi(f)=\phi} V_2(G_f)$. Together this gives

$$|\mathrm{Val}(G^J)|_0 \leq \text{const}\, M^{j_\phi(2 - \frac{1}{2}E_\phi - |V_2(G_\phi)|)} \prod_{\pi(v)=\phi} |\mathrm{Val}(v)|_0 \prod_{\pi(f)=\phi} |\mathrm{Val}\, G_f^{J_f}|_0 M^{-j_\phi(2 - \frac{1}{2}E_f - |V_2(G_f)|)} \tag{3.69}$$

and the lemma follows by applying the induction hypothesis to $|\mathrm{Val}\, G_f^{J_f}|_0$.

It remains to treat the case $j_\phi = 0$. Note that if root scale is zero, there can be no forks above the root. In addition when $j_\phi = 0$, $M^{D_\phi j_\phi} = 1$, and the Lemma reduces to showing that the value of the graph is bounded. Because the scale zero propagator is bounded it suffices to show this for graphs that are 1PI.

Consider the graphs with the phonon lines replaced by effective 4-vertices V_0. The scale zero graph can contain Wick-lines. As the original electron-phonon vertex was Wick-ordered these can only occur if the scale zero line closes a loop that contains the phonon line. Therefore We for all Wick-lines we can replace the two-legged subgraph containing the Wick-line and the vertex by a two-legged effective vertex. If the Wick-line was a hard line then this vertex has value

$$w_{21}(p) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_\Omega \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C_0(l) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{3.70}$$

which is bounded because Ω is bounded and $\int \mathrm{d}_\beta l_0 D(l_0,)$ is finite (See also Sections 1.4.3 and 4). If the Wick line was a soft line then it has value

$$w_{22}(p) = \int \frac{\mathrm{d}_\beta l_0}{2\pi} \int\limits_\Omega \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} C_{<0}(l) D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{3.71}$$

which is finite by (3.34).

What remains is to bound the value of an 1PI graph containing scale zero lines two-legged and four-legged vertices that have bounded vertex functions and without Wick-lines. By (3.34) it suffices to show this for graphs containing only hard propagators C_0. Because of the UV-cutoff the vector momenta integrals are all over bounded sets and thus it remains to show that the frequency integrals are finite. We will do this using a power counting and scale decomposition argument just as before. However here it is much easier as the frequency integrals are 1-dimensional. Note that

$$|C_0(l)| \le \tau_{M^{-1}}(l_0) \tag{3.72}$$

Therefore

$$|\mathrm{Val}(G^J)|_0 \le \mathrm{const}\, |\mathrm{Val}_\tau(G)|_\infty \tag{3.73}$$

where Val_τ is given by the integral constructed from G by using $\tau_{M^{-1}}(p_{l,0})$ as the propagator on each line l and for each line l' that closes a loop an "integral" $\int \mathrm{d}_\beta p_{l',0}$ is taken.

Inserting the decomposition

$$\tau_{M^{-1}}(x) = \sum_{j\geq 0} \bar{\tau}_j \tag{3.74}$$

with

$$\bar{\tau}_j = \begin{cases} \frac{\mathbb{1}(M^{j-2}<|x|\leq M^{j-1})}{|x|} & j>0 \\ M\mathbb{1}(|x|<M^{-1}) & j=0 \end{cases} \tag{3.75}$$

gives a new assignment of (positive) scales to lines of the graph. As G now contains no Wick-lines, each loop contains at least two lines. Therefore it possible to take a spanning tree T for G such that the line with maximum scale k is in it. Bounding the value of each such labeled graph using

$$\int \mathrm{d}_\beta l_0 \bar{\tau}_j(l_0) \leq \text{const} \tag{3.76}$$

we have

$$|\mathrm{Val}_\tau(G)|_\infty \leq \text{const} \sum_{k\geq 0} \sum_{\substack{j_l\geq 0 \; l\in L(G) \\ \max\{j_l\}=k}} \prod_{l\in T} M^{-j_l} \tag{3.77}$$

$$\leq \text{const} \sum_{k\geq 0} k^{|L(G\setminus T)|} M^{-k} \leq \text{const} \tag{3.78}$$

This is of course a very crude bound but sufficient for our purposes. □

In the following we shall also explicitly need the spanning trees defined above to compute the value of the graph. Concatenating the spanning trees constructed recursively above for each $\tilde{G}(f)$ it follows directly that

Corollary 24. *For each graph G, tree t and labeling J such that J is compatible to t and G it is possible to choose a spanning tree T for G such that for all $f\in t$, $\tilde{G}(f)\cap T$ is a spanning tree for $\tilde{G}(f)$.*

Such a tree is called *compatible* to the scales J.

3.5 Improved power counting

3.5.1 Why the power counting can be improved.

In Lemma 23 the power counting was obtained by taking the C^{∞} norm of the propagators on the tree and ignoring their support properties. However we do can better if a propagator on tree is contained in multiple loops. First we characterize the graphs where this occurs. Then afterwards we will prove an improved power counting lemma similar to [FST98, Sal99].

3.5.2 Overlapping Graphs and Gallavotti-Nicolò trees

The structure of non-overlapping graphs

In this section we review some properties of graphs that will be needed later on. Most of the results and proofs are from [FST96]. For a more detailed discussion and more examples please refer to that publication. Most of these properties deal with the consequences of overlappingness.

Definition 25. *A graph G is called* overlapping *if there exists a line of G that is part of two independent loops.*

Some properties of overlapping and non-overlapping graphs that are used below are:

Remark 26.

i) If S is a subgraph of G and S is overlapping, then G is overlapping.

ii) If S is a connected subgraph of G and G is non-overlapping, then G/S is non-overlapping.

iii) If G consists of two vertices v with 3 or more lines of G connecting them, then G is overlapping.

These are proven in[FST96].

Below it is shown that being non-overlapping has strong consequences for the structure of the graph. First we define some terminology for these structures. These definitions are given recursively.

Definition 27. *If G is a two legged graph then it is called a* Generalized Self-contracted Two-legged *(GST) graph if the two external lines are connected to a single vertex and the other legs of this vertex are pairwise connected by strings of two-legged graphs that are themselves GST graphs.*

A four-legged graph G is called a Dressed Bubble Chain *(DBC) of length $n \geq 0$ when* *there are $n+1$ vertices $\{v_0, \ldots, v_n\}$ with $E(v_r) \geq 4 \forall r$, such that: Two of the external lines are connected to v_0 and the other two to v_n, for all $r = 0, \ldots, n-1$ v_r and v_{r+1} are connected by two strings of GST graphs, and for each v_r, $r = 0, \ldots, n$ the remaining legs of v_r are pairwise connected by strings of GST graphs.*

Lemma 28. *Let G be a non-overlapping 1-particle irreducible graph containing only even-legged vertices. Then*

i) If $E(G) = 2$, G is a GST graph.

ii) If $E(G) = 4$, G is a DBC.

This is an easy consequence of the following two propositions

Proposition 29. *Let G be a an even legged 1PI graph containing only vertices with even incidence number such that for each pair (ℓ_o, ℓ_i) of external legs the graph $\hat{G}$ obtained by connecting (ℓ_o, ℓ_i) is overlapping. Then G itself is overlapping.*

Proof. Assume G is not overlapping. Let ℓ_o and ℓ_i be two external lines. Because $\hat{G}$ produced by connecting them is overlapping, but G is not, the new loop created must thus be one of those that overlap. The loop containing these two lines in $\hat{G}$ shares a line ℓ with another loop L. Because $L \subset G$, $\ell \in G$. Thus the path connecting ℓ_i and ℓ_o in G contains at least the line ℓ. This implies that ℓ_i and ℓ_o are connected to different vertices, say v_i and v_o. By repeating this argument for all pairs containing v_i, it follows that ℓ_i is in fact the only external line connected to v_i.

Because v_i has an even number of legs, there are an odd number r of lines connecting v_i to the other part of the graph, $G - v_i$. Let $G - v_i = \bigcup_{j=1}^{m} S_j$ be the decomposition of $G - v_i$ in its connected components. Let r_j be the number of lines connecting S_j and v_i. Because $\sum_{j=1}^{m} r_j = r$ is odd, at least of the r_j's is odd. We can WLOG assume it is r_1. Because G is non overlapping, the subgraph W consisting of S_1, v_i and the r_1 lines connecting them is nonoverlapping. As S_1 is connected, W/S_1 is nonoverlapping too. However W/S_1 consists of two vertices connected by r_1 lines and thus $r_1 < 3$. However this implies that S_1 is connected to the rest of the graph by $r = 1$ lines and which is in contradiction with the fact that the graph is $1PI$. Thus, G is overlapping. □

We can even show that the inverse of this argument holds:

Proposition 30. *Let G be a $1PI$ graph, let ℓ_o and ℓ_i be external lines of G connected to two different external vertices. Then the graph $\hat{G}$ produced by connecting them is overlapping.*

Proof. Choose a spanning tree of G. As the external vertices of ℓ_o and ℓ_i are different, there is a line ℓ in the spanning tree on the path between these two. Because G is 1PI, there is a loop in G containing ℓ. This loop is also contained in $\hat{G}$. Because G is connected, connecting ℓ_o and ℓ_i produces another loop in $\hat{G}$. This loop contains the path in the spanning tree connecting the two vertices and therefore ℓ. Thus ℓ is contained in two different loops in $\hat{G}$. □

Proof of Lemma 28. Let G be 2-legged, 1PI and non-overlapping. The result is trivial if G contains just one vertex. Assume as induction hypothesis that the lemma has been proven for all graphs with less vertices then G.

By prop 29 connecting the two external lines will not make the graph overlap. By prop 30 the two external lines are therefore connected to a single external vertex v. Let C be a connected component of $G - v$. Consider the graph $\hat{C}$ containing C,v and the r lines connecting them. $\hat{C}/C$ is a graph containing two vertices connected by r lines. If $r \leq 1$ then $\hat{C}$ and therefore G would not be 1PI. If $r \geq 3$, $\hat{C}/C$ would be overlapping and therefore $\hat{C}$ and G would be too. Thus $r = 2$. C is a two legged graph connected to a pair of lines of v. C is two legged and thus a string of two legged 1PI-graphs. Each of those is non-overlapping and thus by the induction hypothesis a GST graph. Note that C can in particular be a single line, it would then form a normal self contraction.

Turning to the case where G is a four-legged graph observe that by prop 29 there exists a pair of lines (ℓ_o, ℓ_i) such that connecting them by a line ℓ does not produce an overlapping graph. Note that $G+\ell$ is two legged and non-overlapping and therefore by the first half of the lemma a GST graph. G is obtained by cutting the line ℓ of this graph. Inspection shows that cutting a line in a GST graph produces a DBC with possible chains of two-legged vertices connected to the external legs. However they cannot occur here because G is 1PI. □

The relation between GN-trees, spanning trees and overlap

Below we state some more properties of overlapping graphs that we will use. In particular they state how the notion of overlapping and the multiscale analysis interrelate. Most of these are intuitively clear, but just require a bit of care in manipulating graphs to proof. For the proofs as well as pictorial examples we refer to [FST96].

The notion of the quotient graph $\tilde{G}$ can be generalized. Let t be G a tree and a graph that are compatible. Let τ be a subtree of t with the same root. Let the boundary $B(\tau, t)$ be given by

$$B(\tau,t) = \{f \in t \| f \notin \tau, \pi(f) \in \tau\} \tag{3.79}$$

Then we can define the quotient graph $\tilde{G}(\tau)$ by

$$\tilde{G}(\tau) = G/\{G_f\}_{f\in B(\tau,t)} \tag{3.80}$$

As a special case be allow $\tau = \emptyset$, then $B(\emptyset, t) = \phi)$ and $\tilde{G}(\emptyset) = G/G_\phi = G/G$ is a single vertex.

Lemma 31. *Let G be a graph, t a tree compatible to G. Then there is a unique maximal subtree τ rooted at the root of t, such that $\tilde{G}(\tau)$ is non-overlapping, but for each tree $\tau \subsetneq \tau' \subset t$, $\tilde{G}(\tau')$ is overlapping. If $\tilde{G}(\phi)$ is overlapping $t = \emptyset$.*

For the proof see [FST96].

Lemma 32. *Let G be a graph, t a compatible tree. Let τ be the maximal non-overlapping subtree as given in Lemma 31. Let J be a labeling compatible to t and G. Let T be the spanning defined in Corollary 24. Then there exists a line $l^* \in T$ with scale $j^* = \min\{j_f | f \in B(\tau, t)\}$ and lines $l_1, l_2 \in L(G) \setminus L(T)$ such that l^* is contained in the loops generate by these two lines.*

Proof: See [FST96].

The value of a GST graph

Let G be a two legged, 1PI graph. Let t be a compatible tree and $\tau \subset t$ be the maximal non-overlapping subtree. Let $\tau \neq \emptyset$. Then $\tilde{G}(\tau)$ is a non-overlapping and by Lemma 28, $\tilde{G}(\tau)$ is a GST graph.

The recursive structure of GST graphs is reflected in the expression for their value ([FST96], Remark 2.41). First, we introduce some notation. Let G be a $E(G) = 2m$ legged graph. Then $\mathrm{Val}(G)$ is a function of $2m-1$ momenta. Let $\mathcal{M}(G)$ be the function of m momenta obtained from $\mathrm{Val}(G)(p_1, \dots, p_{2m-1})$ as

$$\mathcal{M}(G)(p_1, \dots, p_m) = \mathrm{Val}(G)(p_1, \dots, p_m, p_2, \dots, p_m) \tag{3.81}$$

Remark 33. *Let G, t, τ be as above and let $J \in \mathcal{J}(t, G, j)$ then $\mathrm{Val}\, G^J(q)$ has the structure*

$$\mathrm{Val}\, G^J(q) = \prod_{r=2}^{m'} \left(\int \mathrm{d}_\beta l_{r,0} \int \mathrm{dl}_r\, S_r(l_r) \right) \mathcal{M}(G_f^{J_f})(q, l_2 \dots, l_{m'}) \tag{3.82}$$

where G_f is the generalized external vertex of the GST graph $\tilde{G}^J(\tau)$, i.e., $f \in B(\tau, t)$ or f a vertex and G_f contains the external vertices of G, which is $E_f = 2m'$ legged.

The S_r's are strings of propagators and two legged subdiagrams given by

$$S_r(p) = \left(\prod_{k=1}^{w_r - 1} C_{j_{r,k}}(p) \mathcal{P}_{r,k} T_{r,k}(p) \right) C_{j_{r,w_r}}(p) \tag{3.83}$$

with

- $\mathcal{P}_{r,k} = (1 - L)$ *when* $T_{r,k} = \mathrm{Val}(G_{f_{r,k}})$ *with* $f_{r,k}$ *an R-fork.*
- $\mathcal{P}_{r,k} = L$ *when* $T_{r,k} = \mathrm{Val}(G_{f_{r,k}})$ *with* $f_{r,k}$ *a C-fork.*
- $\mathcal{P}_{r,k} = 1$ *when* $T_{r,k}$ *is the value of a two legged subgraph that is itself a 1PI GST graph.* $T_{r,k}$ *is then again of the form* (3.82).

In particular $j_r < j_f$ *for all* $r = 2 \ldots m'$ *and* $\min\{j_2, \ldots, j_r\} = j$.

3.5.3 Volume Improvement bounds

Below we prove a variant of the "volume improvement lemma" introduced in [FST96, FST98, Sal99]. This is later used to show that for a large subset of the graphs the power counting is improved over standard power counting. Scale decomposition restricts the momentum at each line to values in a small annulus around the Fermi-surface. The integrals to be bounded will then be a constant times the intersection volume of such annuli with the Fermi-surface for each line shifted by some constant vector.

Most of the time it will be sufficient to bound the value of the intersection by the volume of the thinnest annulus. When the annulus has thickness ϵ, that volume is proportional to ϵ. However it is possible to do better by observing that when two annuli of thicknesses ϵ_2 and ϵ_3 intersect transversally their intersection volume is $\sim \epsilon_2 \epsilon_3$. However this is not the case when either the centers of the two annuli are very close such that the tangent spaces to the Fermi-surfaces are at small angles everywhere or when the arrangement is such that the Fermi-surfaces (nearly) touch at the intersection point. The essential observation is that values of the relative shift $\mathbf{k}$ where these problems occur themselves occupy a very small region. When $\mathbf{k}$ contains another momentum that is integrated over this in fact generates an extra factor $\sim \epsilon_3 \log \epsilon_3$ in the bound for the outer integral.

In [FST98] this bound is proven for a large class of band relations $e(\mathbf{l})$ whose level surfaces are C^2-manifolds with strictly positive curvature. In [FST96] a weaker bound is proven for an even more general class of Fermi surfaces. In this section the bound is generalized to the case where the three band relations and thus Fermi surfaces are not identical, but are in fact elements of the set of functions given by

$$e(\mathbf{l}, \mathbf{q}, t) = (1-t)e(\mathbf{l}) + te(\mathbf{l}+\mathbf{q}) \tag{3.84}$$

where $t \in [0,1]$ and $\mathbf{q}$ a small given vector (See section 1.4.2), and $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. As we have seen in section 2.3 the interpolating Fermi surfaces arise when combining a product of a propagator and its translate by $\mathbf{q}$ using a Feynman-trick.

For $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$, $\{\mathbf{l}|e(\mathbf{l}) = 0\}$ and $\{\mathbf{l}||e(\mathbf{l}+\mathbf{q})| = 0\}$ are spheres. More importantly (Assuming WLOG $\mathbf{q} = (-q, 0, \ldots, 0)$) for all $t \in [0,1]$

$$e(\mathbf{l}, \mathbf{q}, t) = \sum_{i=1}^{d} l_i^2 - 1 + tq^2 - 2l_1 qt \tag{3.85}$$

$$= (l_1 - tq)^2 + \sum_{i=2}^{d} l_i^2 - 1 + (t - t^2)q^2 \tag{3.86}$$

and thus in this case the level surfaces of $e(\mathbf{l}, \mathbf{q}, t)$ are also spheres (but with a t and $\mathbf{q}$ dependent radius).

This implies that we can choose (for some $\delta > 0$), as a parameterization of the level surface $\rho = e(\mathbf{l}, \mathbf{q}, t)$ the map $\mathbf{p}_{t,\mathbf{q}}(\rho, \theta)$ given by

$$\begin{aligned} (-\delta, \delta) \times S^{d-1} &\longrightarrow \mathbb{R}^d \\ (\rho, \theta) &\longmapsto \mathbf{p}_{t,\mathbf{q}}(\rho, \theta) = \mathbf{p}_\circ(t, \mathbf{q}) + \tilde{p}(r(\rho, t, \mathbf{q}), \theta) \end{aligned} \tag{3.87}$$

where $(r, \theta) \mapsto \tilde{p}(r, \theta) = r\theta$ is the standard polarcoordinate map, $\mathbf{p}_\circ(t, \mathbf{q})$ is the center of the sphere and $r(\rho, t, \mathbf{q}) = \sqrt{\rho + 1 - (t - t^2)q^2}$ is the radius of the the level surface of level ρ. This constellation simplifies the proof greatly as the tangentspace to the level surface only depends on θ, not on ρ.

It is convenient to have a notation for the set of points contained in annuli around the Fermi surfaces. Denote

$$\mathcal{A}(t, \mathbf{q}, \epsilon) = \{\mathbf{l}|\,|e(\mathbf{l}, q, t)| < \epsilon\} \tag{3.88}$$

With these definitions we prove:

Lemma 34. *Let $K \subset \mathbb{R}^d$ be compact, let $\kappa_s \le \frac{1}{2}$. Then there exist constants $\bar{\delta} < \delta$ and $M(\bar{\delta})$ such that for all $v_1, v_2 \in \{-1, 1\}$ all $0 < \epsilon_1, \epsilon_2, \epsilon_3 < \bar{\delta}$, all $t_1, t_2, t_3 \in [0,1], |\mathbf{q}| \le \kappa_s$, all $\mathbf{Q} \in K$*

and $t' = 1$

$$I(\epsilon_1, \epsilon_2, \epsilon_3) = \int\limits_{\mathcal{A}(t_1,\mathbf{q},\epsilon_1)} \frac{d^d \mathbf{l}_1}{(2\pi)^d} \int\limits_{\mathcal{A}(t_2,\mathbf{q},\epsilon_2)} \frac{d^d \mathbf{l}_2}{(2\pi)^d} \mathbb{1}(|e(v_1 P_{t'} \mathbf{l}_1 + v_2 \mathbf{l}_2 + \mathbf{Q}, \mathbf{q}, t_3)| < \epsilon_3) \tag{3.89}$$

$$\leq M(\bar{\delta})\epsilon_1\epsilon_2\epsilon_3 |\log(\max\{\epsilon_1, \epsilon_2, \epsilon_3\})| \tag{3.90}$$

Moreover when $\epsilon_1 \leq \epsilon_3$ *and* $\epsilon_2 \leq \epsilon_3$ *this bound also holds for* $t' \in [0, 1]$.

Proof. For $t' = 1$, exchanging $\mathbf{l}_1$,$\mathbf{l}_2$ and $\mathbf{l}_1 + v_2\mathbf{l}_2 + \mathbf{Q}$ corresponds to a simple change of integration values and we can without loss of generality assume $\epsilon_1 \leq \epsilon_3$ and $\epsilon_2 \leq \epsilon_3$. By choice of constant we can also assume $\epsilon_3 < \frac{1}{20}$. Let $\mathbf{p}_i(\rho, \theta) = \mathbf{p}_{t_i,\mathbf{q}}(\rho, \theta)$, $i = 1, 2, 3$, be the shifted polar coordinates defined in (3.87), i.e. $e(\mathbf{p}_i(\rho, \theta), \mathbf{q}, t_i) = \rho$. Then because this change of coordinates is C^∞ and has Jacobian bounded by $r^{d-1}|\frac{\partial \rho}{\partial r}|^{-1} \leq \frac{1}{2}(1+\rho)^{\frac{d-2}{2}}$

$$I(\epsilon_1, \epsilon_2, \epsilon_3) \leq \frac{(1+\bar{\delta})^{d-2}}{4\pi^2} \int_{-\epsilon_1}^{\epsilon_1} d\rho_1 \int_{-\epsilon_2}^{\epsilon_2} d\rho_1 W(t'\rho_1, \rho_2, \epsilon_3) \tag{3.91}$$

where

$$W(t'\rho_1, \rho_2, \epsilon_3) =$$
$$\max_{v_1,v_2=\pm 1} \sup_{\mathbf{Q}\in K} \int\limits_{S^{d-1}} \frac{d\theta_1}{(2\pi)^{d-1}} \int\limits_{S^{d-1}} \frac{d\theta_2}{(2\pi)^{d-1}} \mathbb{1}(|e(v_1\mathbf{p}_1(t'\rho_1, \theta_1) + v_2\mathbf{p}_2(\rho_2, \theta_2) + \mathbf{Q}, \mathbf{q}, t_3)| < \epsilon_3) \tag{3.92}$$

By the mean value theorem

$$|e(v_1\mathbf{p}_1(t'\rho_1, \theta_1) + v_2\mathbf{p}_2(\rho_2, \theta_2) + \mathbf{Q}, \mathbf{q}, t_3)| < \epsilon_3 \tag{3.93}$$

implies

$$|e(v_1\mathbf{p}_1(0, \theta_1) + v_2\mathbf{p}_2(0, \theta_2) + \mathbf{Q}, \mathbf{q}, t_3)| < \epsilon_3 + \frac{|e|_1}{u_0}(\epsilon_1 + \epsilon_2) \tag{3.94}$$

$$< (1 + 2\frac{|e|_1}{u_0})\epsilon_3 \tag{3.95}$$

and thus

$$W(t'\rho_1, \rho_2, \epsilon_3) \leq W(0, 0, (1 + 2\frac{|e|_1}{u_0})\epsilon_3) \tag{3.96}$$

Therefore it remains to show that

$$W(0, 0, \epsilon) \leq \text{const}\, \epsilon \log \epsilon \tag{3.97}$$

□

Proof of inequality (3.97). Below this inequality is proven for the spherical band relation $e(\mathbf{p}) = |\mathbf{p}|^2 - 1$. In this simple case it is possible to use very explicit coordinates and bounds. The proof is along the lines of Appendix B.8 of [Sal99]. First we prove the a bound where only a single integral is taken (called the "1-loop bound", as this occurs when bounding a loop involving two fermion lines). This will produce a bound almost $\propto \epsilon$, except for a few singular regions. Integration over this bound due to the second integral then gives the bound with the singularity producing the logarithmic factor.

The expression that occurs for the inner integral is given by

$$\mathcal{T}(\epsilon, \mathbf{k}, v_2) = \int\limits_{S^{d-1}} \frac{d\theta}{(2\pi)^{d-1}} \mathbb{1}(|e(v_2\mathbf{p}_2(0,\theta) + \mathbf{k} - v_2\mathbf{p}_{\circ,2} + \mathbf{p}_{\circ,3}, \mathbf{q}, t_3)| < \epsilon_3) \tag{3.98}$$

and we can bound $W(0,0,\epsilon)$ in terms of this as

$$W(0,0,\epsilon) \leq \max_{v_1,v_2=\pm 1} \sup_{\mathbf{Q}\in K} \int\limits_{S^{d-1}} d\theta \mathcal{T}(\epsilon, \tilde{p}(r(0,t_1,\mathbf{q}),\theta) + v_1\mathbf{p}_{\circ,1} + v_2\mathbf{p}_{\circ,2} - \mathbf{p}_{\circ,3} + \mathbf{Q}, v_2) \tag{3.99}$$

$$\leq \max_{v_2=\pm 1} \sup_{\mathbf{Q}'\in K'} \int\limits_{S^{d-1}} d\theta \mathcal{T}(\epsilon, \tilde{p}(r(0,t_1,\mathbf{q}),\theta) + \mathbf{Q}', v_2) \tag{3.100}$$

where $K' = \{\mathbf{l} \in \mathbb{R}^d | d(\mathbf{l}, K) \leq 3\kappa_s\}$ because $|\mathbf{p}_\circ(t,\mathbf{q})| \leq \kappa_s$.

Proposition 35. *There exists a constant $\mathcal{Q}$ independent of* $\mathbf{k}$*, such that for all $v_2 = \pm 1$, all $|\mathbf{q}| < \kappa_s \leq \frac{1}{2}$ and all $0 < \epsilon < \frac{1}{4}$*

$$\mathcal{T}(\epsilon, \mathbf{k}, v_2) \leq \mathcal{Q} \begin{cases} 1 & \text{if } |\mathbf{k}| \leq 2\epsilon \\ \frac{\epsilon}{\sqrt{|\mathbf{k}|}} \left(\frac{1}{\sqrt{|g_+(|\mathbf{k}|)|}} + \frac{1}{\sqrt{|g_-(|\mathbf{k}|)|}} \right) & \text{if } 2\epsilon < |\mathbf{k}| < \frac{5}{2} \text{ and } \forall s \in \{\pm\}\ |g_s(|\mathbf{k}|)| > 2\epsilon \\ \left(\frac{\epsilon}{|\mathbf{k}|}\right)^{\frac{d'-1}{2}} & \text{if } 2\epsilon < |\mathbf{k}| < \frac{5}{2} \text{ and } \exists s \in \{\pm\}\ |g_s(|\mathbf{k}|)| \leq 2\epsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.101}$$

where $g_\pm(k) = \hat{T} + k(k \pm 2R_2)$, $T_i = (t_i - t_i^2)q^2$, $\hat{T} = T_3 - T_2$, $R_2 = \sqrt{1 - T_2}$, *and* $d' = \min\{3, d\}$.

Note that when $\hat{T} \approx 0$ and $\mathbf{k} \approx 0$, $g_\pm(|\mathbf{k}|) \sim |\mathbf{k}|$ and thus the bound is $\sim |\mathbf{k}|^{-1}$. Similarly when $|\mathbf{k}| \approx 2R_2$, $g_-(|\mathbf{k}|) \sim |\mathbf{k}| - 2R_2$ and thus the bound is proportional to $||\mathbf{k}| - 2R_2|^{-\frac{1}{2}}$. This is indeed the behavior obtained previously for $\hat{T} = 0$[Sal99, FST98].

Proof. Setting ϕ to be the angle between $v_2\mathbf{p}_2(0,\theta)$ and $\mathbf{k}$ and denoting $k = |\mathbf{k}|$ as usual

$$e(v_2\mathbf{p}_2(0,\theta) + \mathbf{k} - v_2\mathbf{p}_{0,2} + \mathbf{p}_{0,3}, \mathbf{q}, t_3) = |v_2\tilde{p}(R_2,\theta) + \mathbf{k}|^2 - 1 + T_3 \tag{3.102}$$

$$= |\tilde{p}(R_2,\theta)|^2 + k^2 + 2|\tilde{p}(R_2,\theta)|k\cos\phi - 1 + T_3 \tag{3.103}$$

$$= T_3 - T_2 + k(k + 2R_2\cos\phi) \tag{3.104}$$

$$= \hat{T} + k(k + 2R_2\cos\phi) \tag{3.105}$$

Observing that for $|\mathbf{q}| < \kappa_s \le \frac{1}{2}$, $|T_i| \le \frac{1}{16}$, $|\hat{T}| \le \frac{1}{8}$ and $\frac{7}{8} < R_2 < \frac{9}{8}$.

Changing to coordinates $(\phi, \tilde{\theta})$ with ϕ as above and $\tilde{\theta} \in S^{d-2}$, then

$$\mathcal{T}(\epsilon, \mathbf{k}, v_2) = \frac{1}{(2\pi)^d} \int\limits_{S^{d-2}} \mathrm{d}\tilde{\theta} \int\limits_0^\pi (\sin\phi)^{d-2} \mathbb{1}(|\hat{T} + k(k + 2R_2\cos\phi)| < \epsilon)\mathrm{d}\phi \tag{3.106}$$

$$\le \mathcal{T}'(k,\epsilon) = \frac{1}{\pi 2^{d-2}} \int\limits_0^\pi (\sin\phi)^{d'-2} \mathbb{1}(|\hat{T} + k(k + 2R_2\cos\phi)| < \epsilon)\mathrm{d}\phi \tag{3.107}$$

When $k \ge \frac{5}{2}$, $k + 2R_2\cos\phi > k - 2R_2 > \frac{1}{4}$. Thus $|\hat{T} + k(k + 2R_2\cos\phi)| > \frac{1}{4}k - |\hat{T}| > \frac{1}{2} - |\hat{T}| > \frac{1}{4} > \epsilon$ so the integrand vanishes identically on this region.

When $k \le 2\epsilon$, the integrand is bounded by 1 and thus $\mathcal{T}(\epsilon, \mathbf{k}, v_2) \le 2^{2-d}$.

Let $k > 2\epsilon$. Let $v = \cos\frac{\phi}{2}$. Then $\hat{T} + k(k + 2R_2\cos\phi) = g_-(k) + 4R_2kv^2 = g_+ + 4R_2k(v^2 - 1)$, and $\sin\phi = 2v\sqrt{1 - v^2}$:

$$\mathcal{T}'(k,\epsilon) = \frac{1}{\pi} \int\limits_0^1 v^{d'-2}(1 - v^2)^{\frac{d'-3}{2}} \mathbb{1}(|g_-(k) + 4kR_2v^2| < \epsilon)\mathrm{d}v \tag{3.108}$$

When $|g_-| \le 2\epsilon$, $|g_- + 4R_2kv^2| < \epsilon$ implies $v^2 < \frac{3\epsilon}{4kR_2}$. On the one hand this means that $v^2 < \frac{3}{8R_2} < \frac{3}{7} < \frac{1}{2} < 1$ and $(1 - v^2)^{-\frac{1}{2}} < 2$. On the other hand we can use this restriction to replace the integration bounds and then drop the indicator function. Together this gives

$$\mathcal{T}'(k,\epsilon) \le \frac{2}{\pi} \int\limits_0^{\sqrt{\frac{3\epsilon}{4kR_2}}} v^{d'-2}\mathrm{d}v = \frac{2}{\pi(d'-1)} \left(\frac{3\epsilon}{4kR_2}\right)^{\frac{d'-1}{2}} \le \frac{2}{\pi}\left(\frac{\epsilon}{k}\right)^{\frac{d'-1}{2}} \tag{3.109}$$

When $|g_+| \le 2\epsilon$, change to the variable $u = \sqrt{1 - v^2}$. This means that $v = \sqrt{1 - u^2}$ and thus using $\mathrm{d}v = (1 - u^2)^{-\frac{1}{2}}u\mathrm{d}u$

$$\mathcal{T}'(k,\epsilon) = \frac{1}{\pi} \int\limits_0^1 u^{d'-2}(1 - u^2)^{\frac{d'-3}{2}} \mathbb{1}(|g_+(k) - 4kR_2u^2| < \epsilon)\mathrm{d}u \le \frac{2}{\pi}\left(\frac{\epsilon}{k}\right)^{\frac{d'-1}{2}} \tag{3.110}$$

just as before because the sign in the indicator function is irrelevant.

When $|g_-| > 2\epsilon$ and $|g_+| > 2\epsilon$ we use a combination of the observations above. Note that $v = 0$ or $v = 1$ can no longer be trivially excluded. Therefore, we split up the integration region into two parts.

$$\mathcal{T}'(k,\epsilon) = \mathcal{T}_0'(k,\epsilon) + \mathcal{T}_1'(k,\epsilon) \tag{3.111}$$

with

$$\mathcal{T}_0'(k,\epsilon) = \frac{1}{\pi}\int_0^{\frac{1}{2}} v^{d'-2}(1-v^2)^{\frac{d'-3}{2}}\mathbb{1}(|g_-(k) + 4kR_2v^2| < \epsilon)\mathrm{d}v \tag{3.112}$$

$$\mathcal{T}_1'(k,\epsilon) = \frac{1}{\pi}\int_{\frac{1}{2}}^{1} v^{d'-2}(1-v^2)^{\frac{d'-3}{2}}\mathbb{1}(|g_-(k) + 4kR_2v^2| < \epsilon)\mathrm{d}v \tag{3.113}$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2}} u^{d'-2}(1-u^2)^{\frac{d'-3}{2}}\mathbb{1}(|g_+(k) - 4kR_2u^2| < \epsilon)\mathrm{d}u \tag{3.114}$$

where in the last equation the change of variables $u = \sqrt{1-v^2}$ was used again. As $\mathcal{T}_0'(k,\epsilon)$ and $\mathcal{T}_1'(k,\epsilon)_1$ differ only by the appearance of $-g_+$ instead of g_-, it suffices to bound the former.

If $g_-(k) > 2\epsilon > 0$, $g_-(k) + 4kR_2v^2 > 2\epsilon$ and $\mathcal{T}_0'(k,\epsilon) = 0$. If $g_-(k) < 0$, set $a = \sqrt{\frac{|g_-(k)|}{4kR_2}} > 0$. Then observing that on the integration region $(1-v^2)^{-\frac{1}{2}} < \frac{4}{3}$, we have

$$\mathcal{T}_0'(k,\epsilon) \leq \int_0^{\frac{1}{2}} \mathrm{d}v\,\mathbb{1}(|v^2 - a^2| < \frac{\epsilon}{4kR_2}) \tag{3.115}$$

and with $|v^2 - a^2| = |v-a||v+a| \geq |v-a|a$

$$\leq \int_0^{\frac{1}{2}} \mathrm{d}v\,\mathbb{1}(|v-a| < \frac{\epsilon}{4kR_2a}) \leq \frac{2\epsilon}{4kR_2a} \leq \frac{\epsilon}{\sqrt{kR_2|g_-(k)|}} \tag{3.116}$$

Thus the proposition holds with $Q = 2$.

For $k > \frac{5}{2}$, $|\hat{T} + k(k + 2R_2\cos\phi)| > \frac{1}{2}$ so the integrand vanishes identically always. □

Note that when $d \geq 3$, $d' = 3$ and $v\mathrm{d}v = \frac{1}{2}\mathrm{d}w$, with $w = v^2$ appears in the integral. A change of coordinates thus gives

$$\mathcal{T}_0'(k,\epsilon) \le \frac{2}{3\pi}\int_0^{\frac{1}{4}} \mathbb{1}(|g_-(k) + 4kR_2 w| < \epsilon)\mathrm{d}w < \frac{\epsilon}{k}. \tag{3.117}$$

Combining this with the observation that when the $\mathbf{q}$ is zero $\hat{T} = 0$, we arrive at the following:

Corollary 36 (Tranversal one-loop bound). *Let $d \ge 3$, $e(\mathbf{p}) = |\mathbf{p}|^2 - 1$ and $\eta > 0$. Then there exists an $\mathcal{M}(\eta)$ such that for all $\epsilon_1, \epsilon_2 < \eta$, and all $\mathbf{k}$ with $|\mathbf{k}| > 2\eta$,*

$$\int \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d}\mathbb{1}(|e(\mathbf{l})| < \epsilon_1)\mathbb{1}(|e(\mathbf{l}+\mathbf{k})| < \epsilon_2) < \mathcal{M}(\eta)\frac{\epsilon_1\epsilon_2}{|\mathbf{k}|} \tag{3.118}$$

Moreover, for $d = 2$ this inequality holds for $2\eta < |\mathbf{k}| < 2 - 2\eta$.

Now apply Proposition 35 to (3.100). Observe that the bound given there only depends on $|\tilde{p}(r(o,t_1,\mathbf{q}),\theta) + \mathbf{Q}'|$. If we decompose the coordinates $\theta = (\phi, \tilde{\theta})$ such that ϕ is the angle between $\tilde{p}(r(o,t_1,\mathbf{q}),\theta)$ and $\mathbf{Q}'$. Then

$$\xi(\phi) \equiv |\tilde{p}(r(o,t_1,\mathbf{q}),\theta) + \mathbf{Q}'| = \sqrt{(Q - R_1)^2 + 2QR_1(1 - \cos\phi)} \tag{3.119}$$

where $Q = |\mathbf{Q}'|$, and $R_1 = r(o,t_1,\mathbf{q}) = \sqrt{1 - T_1}$. Therefore Proposition 35 implies $W(0,0,\epsilon) \le \sup_{Q\ge 0} \mathcal{W}(\epsilon, Q)$, where

$$\mathcal{W}(\epsilon, Q) = 2\int_0^{\pi} |\sin\phi|^{d-2}\tilde{\mathcal{T}}(\epsilon, \xi(\phi))\mathrm{d}\phi \tag{3.120}$$

and

$$\tilde{\mathcal{T}}(\epsilon,\xi) = \begin{cases} 1 & \text{if } \xi \le 2\epsilon \\ \frac{\epsilon}{\sqrt{\xi}}\left(\frac{1}{\sqrt{|g_+(\xi)|}} + \frac{1}{\sqrt{|g_-(\xi)|}}\right) & \text{if } 2\epsilon < \xi < \frac{1}{4} \text{ and } \forall s \in \{\pm\}\ |g_s(\xi)| > 2\epsilon \\ \epsilon\left(\frac{1}{\sqrt{|g_+(\xi)|}} + \frac{1}{\sqrt{|g_-(\xi)|}}\right) & \text{if } 2R_2 - \frac{1}{4} < \xi < \frac{5}{2} \text{ and } \forall s \in \{\pm\}\ |g_s(\xi)| > 2\epsilon \\ \left(\frac{\epsilon}{\xi}\right)^{\frac{d'-1}{2}} & \text{if } 2\epsilon < \xi < \frac{1}{4} \text{ and } \exists s \in \{\pm\}\ |g_s(\xi)| \le 2\epsilon \\ \epsilon^{\frac{d'-1}{2}} & \text{if } 2R_2 - \frac{1}{4} < \xi < \frac{5}{2} \text{ and } \exists s \in \{\pm\}\ |g_s(\xi)| \le 2\epsilon \\ 8\sqrt{2}\epsilon & \frac{1}{4} \le \xi \le 2R_2 - \frac{1}{4} \\ 0 & \textit{otherwise} \end{cases} \tag{3.121}$$

where we have used $Q \leq 1$ and for $\frac{1}{4} \leq \xi \leq 2R_2 - \frac{1}{4}$, $g_{\pm}(\xi) \geq \frac{1}{8}$. The region $\frac{1}{4} \leq \xi \leq 2R_2 - \frac{1}{4}$ has been split off because when $\xi < \frac{1}{4}$ or $\xi > 2R_2 - \frac{1}{4}$, $|\frac{\mathrm{d}}{\mathrm{d}\xi} g_{\pm}(\xi)| > \frac{1}{2}$.

The rest of the proof is now to show that in the regions where (3.121) does not give the bound immediately the coordinate change $\phi \mapsto \xi(\phi)$ is sufficiently regular. Observe that $\xi(\phi) \geq |R_1 - Q|$. Thus when $Q \notin A = [R_1 - \frac{1}{4}, R_1 + \frac{1}{4}] \cup [R_1 + 2R_2 - \frac{1}{4}, R_1 + \frac{5}{2}]$, $\frac{1}{4} < \chi < 2R_1 - \frac{1}{4}$ $\chi > \frac{5}{2}$, and therefore for such Q, $\mathcal{W}(\epsilon, Q) \leq 8\pi\sqrt{2}\epsilon$.

Now assume $Q \in A$, in particular therefore $Q > R_1 - \frac{1}{4} > \frac{1}{2}$. Substituting $v = \sqrt{2QR_1}\sin(\frac{1}{2}\phi)$, using $\tilde{\mathcal{T}}(\epsilon, \chi(v)) \leq 1$, and denoting $\chi(v) = \xi(\phi(v))$

$$\mathcal{W}(\epsilon, Q) \leq 2 \int_0^{\sqrt{2QR_1}} (2QR_1 - v^2)^{-\frac{1}{2}} \tilde{\mathcal{T}}(\epsilon, \chi(v)) \mathrm{d}v \tag{3.122}$$

$$\leq \frac{2\epsilon}{R_1} + 2 \int_{\epsilon}^{\sqrt{2QR_1}-\epsilon} (2QR_1 - v^2)^{-\frac{1}{2}} \tilde{\mathcal{T}}(\epsilon, \chi(v)) \mathrm{d}v \tag{3.123}$$

$$\leq (8\sqrt{2} + \frac{1}{R_1})\epsilon + I_0 + I_1 \tag{3.124}$$

with

$$I_0 = 2 \int_{\epsilon}^{\sqrt{2QR_1}-\epsilon} (2QR_1 - v^2)^{-\frac{1}{2}} \tilde{\mathcal{T}}(\epsilon, \chi(v)) \mathbb{1}(\chi(v) < \frac{1}{4}) \mathrm{d}v \tag{3.125}$$

$$I_1 = 2 \int_{\epsilon}^{\sqrt{2QR_1}-\epsilon} (2QR_1 - v^2)^{-\frac{1}{2}} \tilde{\mathcal{T}}(\epsilon, \chi(v)) \mathbb{1}(2R_2 - \frac{1}{4} < \chi(v) < \frac{5}{2}) \mathrm{d}v \tag{3.126}$$

Because $\chi(v) = \sqrt{(Q - R_1)^2 + v^2}$ we see that when $Q > R_1 + \frac{1}{4}$, the integral I_0 vanishes. Moreover

$$\left| \frac{\mathrm{d}\chi(v)}{\mathrm{d}v} \right|^{-1} = \frac{\chi(v)}{v} = \frac{\chi}{\sqrt{\chi^2 - (Q - R_1)^2}} \leq \frac{\sqrt{\chi}}{\sqrt{\chi - |Q - R_1|}} \tag{3.127}$$

On the support of the integrand in I_0, $2QR_1 - v^2 = Q^2 + R_1^2 - \chi^2 > R_1^2$, $|Q - R_1| + \chi \leq \frac{1}{2}$ and $v = \sqrt{\chi^2 - (Q - R_1)^2} > \epsilon$ implies $\chi - |Q - R_1| > 4\epsilon^2$. Making the substitution $\chi = \chi(v)$ in I_0

$$I_0 \leq \frac{2}{R_1} \int_{|Q-R_1|}^{\frac{1}{4}} \frac{\sqrt{\chi}}{\sqrt{\chi - |Q - R_1|}} \tilde{\mathcal{T}}(\epsilon, \chi) \mathbb{1}(|\chi - |Q - R_1|| > 4\epsilon^2) \mathrm{d}\chi \tag{3.128}$$

Inserting (3.121) gives

$$I_0 \leq \frac{2\epsilon}{R_1} \int\limits_{|Q-R_1|}^{\frac{1}{4}} \frac{\mathbb{1}(|\chi - |Q-R_1|| > 4\epsilon^2)}{\sqrt{\chi - |Q-R_1|}} \Big(\sum_{s=\pm;j=0,1} A_j(g_s(\chi))\Big)\mathrm{d}\chi \tag{3.129}$$

where $A_0(x) = x^{-\frac{1}{2}}\mathbb{1}(|x| > 2\epsilon)$ and $A_1(x) = \epsilon^{-\frac{1}{2}}\mathbb{1}(|x| \leq 2\epsilon)$. Taking the sum out of the integral and applying the Cauchy-Schwartz inequality

$$I_0 \leq \frac{2\epsilon}{R_1} \sum_{s=\pm;j=0,1} \Big(\int\limits_{|Q-R_1|}^{\frac{1}{4}} \frac{\mathbb{1}(|\chi - |Q-R_1|| > 4\epsilon^2)}{\chi - |Q-R_1|}\mathrm{d}\chi\Big)^{\frac{1}{2}} \Big(\int\limits_{|Q-R_1|}^{\frac{1}{4}} A_j(g_s(\chi))^2 \mathrm{d}\chi\Big)^{\frac{1}{2}} \tag{3.130}$$

The bound for the left integral is now easy and

$$I_0 \leq \frac{4\epsilon}{R_1}\sqrt{|\log\epsilon|} \sum_{s=\pm;j=0,1} \Big(\int\limits_{|Q-R_1|}^{\frac{1}{4}} A_j(g_s(\chi))^2 \mathrm{d}\chi\Big)^{\frac{1}{2}} \tag{3.131}$$

As observed before, when $\chi < \frac{1}{4}$, $|\frac{\mathrm{d}}{\mathrm{d}\chi} g_s(\chi)| > \frac{1}{2}$ for both $s = \pm$. Substituting $u = g_s(\chi)$ gives

$$I_0 \leq \frac{8\epsilon}{R_1}\sqrt{|\log\epsilon|} \sum_{j=0,1} \Big(\int\limits_{-1}^{1} A_j(u)^2 \mathrm{d}u\Big)^{\frac{1}{2}} \leq \frac{32}{R_1}\epsilon|\log\epsilon| \tag{3.132}$$

because $\int_{-1}^{1} A_1^2(u)\mathrm{d}u = \epsilon^{-1}\int_{-2\epsilon}^{2\epsilon}\mathrm{d}u = 4$ and $\int_{-1}^{1} A_0^2(u)\mathrm{d}u = 2\int_{2\epsilon}^{1}\frac{1}{u}\mathrm{d}u \leq 4|\log\epsilon|$.

Changing to the coordinate $\chi = \chi(v)$ in I_1 and applying (3.121), we have

$$I_1 \leq \epsilon\sqrt{10} \sum_{i=0,1}\sum_{j=0,1}\sum_{s=\pm} \int\limits_{B_i} \frac{\mathbb{1}(|\chi - |Q-R_1|| > 2\epsilon^2)}{\sqrt{Q^2+R_1^2-\chi^2}\sqrt{\chi - |Q-R_1|}} A_j(g_s(\chi))\mathrm{d}\chi \tag{3.133}$$

with $B_0 = \{\chi \| Q-R_1| < \chi < \sqrt{Q_1^2+R_1^2-QR_1}\}$ and $B_1 = \{\chi|\chi \geq \sqrt{Q_1^2+R_1^2-QR_1}, v(\chi) \leq \sqrt{2QR_1}-\epsilon\}$. If we can show that

$$C_i = \int\limits_{B_i} \frac{\mathbb{1}(|\chi - |Q-R_1|| > 2\epsilon^2)}{|Q^2+R_1^2-\chi^2||\chi - |Q-R_1||} \leq \text{const}\,|\log\epsilon| \tag{3.134}$$

then the bound follows as above after applying the Cauchy-Schwartz inequality. For $\chi \in B_0$, $|Q^2 - R_1^2 - \chi^2| > QR_1$ and we easily see $C_0 \leq \frac{4}{R_1}|\log\epsilon|$. For $\chi \in B_1$, $|\chi - |Q-R_1|| > \sqrt{QR_1}$ and thus

$$C_1 \leq \frac{2}{R_1}\int \frac{\mathbb{1}(\sqrt{Q^2+R_1^2-QR_1} \leq \chi)\mathbb{1}(|\sqrt{Q^2+R_1^2}-\chi| > \alpha\epsilon)}{|\sqrt{Q^2+R_1^2}-\chi|} \tag{3.135}$$

with $\alpha = \frac{1}{40}$ and thus

$$C_1 \leq \frac{12}{R_1}|\log\epsilon| \tag{3.136}$$

□

3.5.4 Improved power counting

At this point we can prove that when a labeled graph is overlapping, there is a sharper power counting bound than that of Lemma 23. First we show a few direct corollaries of the proof of Lemma 23.

Corollary 37. *Let $j < 0$, t a tree with $E_f \geq 4$, G a graph and $J \in \mathcal{J}(t, G, j)$. Let T be a spanning tree of G compatible to the scales (Corollary 24). If there exists a line $l^* \in T$ of non-zero scale, such that l^* is not contained in any loop in G, then*

$$|\mathrm{Val}\, G^J|_0 \leq \mathrm{const} \prod_{v\ leaf} |\mathrm{Val}(v)|_0\, M^{D_\phi j_\phi} \prod_{f>\phi} M^{D_f(j-j_{\pi(f)})} z^*(p^*) \tag{3.137}$$

where

$$z^*(l) = \mathbb{1}_{j^*}(|l_0|)\mathbb{1}_{j^*}(\mathrm{e}(\mathbf{l})) \tag{3.138}$$

$j^* = j_{l^*}$ and p^* is a linear combination of the external momenta.

Proof. As l^* is not contained in any loop, the momentum p^* does not depend on any loop momenta. By inspection of the proof of Lemma 23 it is obvious we can keep the indicator function. □

Lemma 38. *Let $j < 0$, t a tree with $E_f \geq 4$, G a graph and $J \in \mathcal{J}(t, G, j)$. Let T be a spanning tree of G compatible to the scales (Corollary 24). If there exists a line $l^* \in T$ of non-zero scale, such that l^* is just contained in one loop of G, then*

$$|\mathrm{Val}\, G^J|_0 \leq \mathrm{const} \prod_{v\ leaf} |\mathrm{Val}(v)|_0\, M^{D_\phi j_\phi} \prod_{f>\phi} M^{D_f(j-j_{\pi(f)})} M^{-j_2} \int \mathrm{d}_\beta p_0 \int \mathrm{d}\mathbf{p} |C_{j_2,s_2}(p)| z^*(Q + v_2 p) \tag{3.139}$$

where $v_2 = \pm 1$, $s_2 = \pm 1$ indicates whether the propagator is soft or hard, Q is some linear combination of the external momenta, z^* as above and $j_2 = j_{f_2}$. Here f_2 is the highest fork of t such that l^* is in a loop in G_{f_2}.

Proof. Because of the recursiveness of Lemma 23, we can WLOG assume $f_2 = \phi$. Consider $\tilde{G}(\phi)$. Apply Corollary 37 to the fork f, with $\pi(f) = \phi$ and $l^* \in G_f$, if this fork exists. Apply Lemma 23 to the other forks above ϕ. If such a fork f does not exists, that implies that l^* and the both appear at scale j_ϕ and the result is direct.

Because l^* is only contained in the loop generated by l_2, p^* is of the form $p^* = Q + v_2 p$ with p the momentum flowing through l_2 and does not depend on any other momenta. Therefore we can choose to not perform the integral over p. Thus there is exactly a gain const M^{j_2} missing from the powercounting. □

Lemma 39. *Let $j < 0$, t a tree with $E_f \geq 4$, G a graph and $J \in \mathcal{J}(t, G, j)$. Let T a spanning tree of G compatible to the scales (Corollary 24). If there exists a line $l^* \in T$ of non-zero scale, such that l^* is contained in two loops of G, then*

$$|\mathrm{Val}\, G^J|_0 \leq \mathrm{const} \prod_{v\ leaf} |\mathrm{Val}(v)|_0\, M^{D_\phi j_\phi} \prod_{f>\phi} M^{D_f(j-j_{\pi(f)})} I \tag{3.140}$$

with

$$I = M^{-j_1-j_2} \int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k} \int \mathrm{d}_\beta p_0 \int \mathrm{d}\mathbf{p} |C_{j_1,s_1}(k)||C_{j_2,s_2}(p)| z^*(Q + v_1 + v_2 p) \tag{3.141}$$

where $v_1, v_2 = \pm 1$, $s_1, s_2 = \pm 1$ indicates whether the propagator is soft or hard, Q is some linear combination of the external momenta, z^* as above and $j_i = j_{f_i}$. Here f_2 is the highest fork of t such that l^* is in a loop in G_{f_2}. f_1 is the highest fork of t such that l^* is in two loops in G_{f_1}.

Proof. This an obvious variant of the proof of the of the previous lemma. □

Theorem 40 (Improved Power Counting, [FST98, FST96]). *Let t be a tree, G a graph, $j \leq 0$, and $J \in \mathcal{J}(t, G, j)$. Let τ be the maximal non-overlapping subtree. When $B(\tau, t) \neq \emptyset$ let $j^* = \min\{j_f | f \in B(\tau, t)\}$. When $B(\tau, t) = \emptyset$ let $j^* = 0$. Then*

$$|\mathrm{Val}\, G^J|_0 \leq \mathrm{const} \prod_{v\ leaf} |\mathrm{Val}(v)|_0\, (|j^*| + 1) M^{j^*} M^{D_\phi j_\phi} \prod_{f>\phi} M^{D_f(j-j_{\pi(f)})} \tag{3.142}$$

Proof. The result is trivial when $j^* = 0$. Assume $j^* < 0$. Choose a spanning tree T according to Corollary 24. By Lemma 32 there exists a line $l^* \in T$ with scale j^* and the preconditions of Lemma 39 are met. Thus (3.140) holds and it remains to bound I. We do the case $s_1 = s_2 = 0$ here. The extension to soft lines is trivial.

Drop the frequency restriction from z^*, use (3.30) and perform the frequency sums. This gives

$$I \leq \mathrm{const}\, M^{-j_1-j_2} \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p} \mathbb{1}_{j_1}(e(\mathbf{k})) \mathbb{1}_{j_2}(e(\mathbf{p})) \mathbb{1}_{j^*}(\mathbf{Q} + v_1 \mathbf{k} + v_2 \mathbf{p}) \tag{3.143}$$

By Lemma 34

$$I \leq \text{const}\, M^{-j_1-j_2}(|j^*|+1)M^{j_1+j_2+j^*} \tag{3.144}$$

and the theorem follows. □

Chapter 4

Phonon lines in loops

4.1 Phonon lines and derivatives

In the previous chapter a scale decomposition was introduced for the fermion propagators only. The phonon lines were absorbed in the scale zero effective vertices. As the phonon propagators are bounded functions this viewpoint is almost always sufficient for power counting purposes. However sometimes it is necessary to take them into account. In this chapter we introduce the techniques needed to deal with these cases. There will be three primary reasons for taking phonon lines in account:

The first is obvious. It is our aim to extract a factor c and thus our bounds must at some point involve a phonon propagator. Second, exploiting sign cancellations through integration by parts and the renormalization procedure introduces derivatives and the derivatives of the phonon operator are not bounded but only integrable. This will be dealt with by introducing a scale decomposition for the phonon propagator when there is derivative acting on it. Third, the localization operator and the renormalization procedure contain derivatives with respect to the frequency. Unfortunately bounds analogous to Lemma 4 for (integrals over) the derivatives which give correct factors c only exist for derivatives with respect to the vector momentum. Each frequency derivative gives a factor c^{-1}.

4.2 Derivatives of two legged graphs

4.2.1 Rerouting the external momenta

As we shall see in Chapter 5, the frequency derivatives are only taken for graphs that only have two external fermion legs. This is fortunate in the light of the following lemma, which holds for graphs with an arbitrary number of additional external phonon legs.

Lemma 41. *Let G be a graph in electron-phonon theory, such that G has 2 external fermion legs (and an arbitrary number of external phonon legs). Let v_1 and v_2 be the external vertices connected to the electron phonon lines. Then there exists a spanning tree T rooted at v_1 such that the path in T from v_1 to v_2 contains only fermion lines.*

Proof. The lemma is an easy consequence of charge conservation at the electron-phonon vertices. Note that the result is trivial when $v_1 = v_2$. The proof is by induction on $V(G)$. If $V(G) = 1$, then $v_1 = v_2$ and the result holds. Let $n > 1$ and assume the lemma holds for $V(G) = n - 1$. Without loss of generality we can assume $v_1 \neq v_2$. Let G' be the graph that consists of G without v_2 and the lines that connect it to the rest of G. These must one phonon line and one fermion line l. Therefore G' is a graph with two external fermion lines, one additional external phonon lines, and with $V(G') = n - 1$. By the induction hypothesis there is a spanning tree $T' \subset G'$ with the desired property. It is now easy to see that $T = T' \cup l$ is the desired tree. □

Corollary 42. *Let G be a graph in electron-phonon theory, such that G has 2 external fermion legs and no external phonon legs. Then it is possible to fix momenta in G such that the external momentum only flows through fermion lines.*

This implies that for a graph G that has 2 external fermion legs and no external phonon legs, it possible to given an expression for $\mathrm{Val}(G)(p)$ where the only factors in the integrand that depend on p are fermion propagators. In other words the derivatives of $\mathrm{Val}(G)(p)$ can be computed as a sum of terms that only involve derivatives of propagators belonging to fermion lines.

4.2.2 Some notation for derivatives of graphs

When taking derivatives of the value $\mathrm{Val}(G)$ of a graph a sum occurs because of the Leibnitz rule. Each term in the sum contains the derivative acting on one of the propagators. This can be represented as the value of a graph where the line the propagator belongs to is associated with the derivated propagator. This will the main form wherein derivatives of graphs will be studied in the following. Therefore we introduce some notation for them.

Let G be a Feynman graph, let $m = 0, \ldots, d$, and let l be a line of G. Then denote by $\partial_{lm} G$ the graph such that $\mathrm{Val}(\partial_{lm} G_{jl})$ contains $(\frac{\partial}{\partial p_m} P_l)(p_l)$ instead of the propagator $P_l(p_l)$ for the line l, but is otherwise identical. Let $s \geq 1$ and $l_1, \ldots l_s \in L(G)$ and $m_1, \ldots, m_s = 0, \ldots, d$ then we denote $\partial^s_{l_1 m_1 \ldots l_s m_s} G$ for $\partial_{l_1 m_1} \partial_{l_2 m_2} \cdots \partial_{l_s m_s} G$.

In this notation we can write down the form in which we will use Corollary 42 (which is again a trivial corollary).

Corollary 43. *Let G be a graph in electron-phonon theory, such that G has 2 external fermion legs and no external phonon legs. Let $m_1, m_2 \in \{0, \ldots, d\}$. Then there exists some subset L of $L_F(G)$ such that*

$$\left| \frac{\partial}{\partial p_{m_1}} \mathrm{Val}(G)(p) \right| \leq \sum_{l \in L} |\mathrm{Val}(\partial_{l m_1} G)(p)| \tag{4.1}$$

$$\left| \frac{\partial}{\partial p_{m_1}} \frac{\partial}{\partial p_{m_2}} \mathrm{Val}(G)(p) \right| \leq \sum_{l_1, l_2 \in L} |\mathrm{Val}(\partial^2_{l_1 m_1 l_2 m_2} G)(p)| \tag{4.2}$$

4.2.3 Derivatives acting on soft lines

The result from the previous section does not imply that there is no need to deal with derivatives acting on phonon lines. One of the reasons for this is that if G is a labeled graph, the tree T given by Lemma 41 is not necessarily compatible with the GN-tree structure. In particular it is possible for T to contain soft lines. Therefore it is possible that the corresponding soft propagators can get derivatives acting on them. The power counting is not sufficient to control these derivatives.

The following lemmas show that these derivatives can be eliminated by means of integration by parts. First we define a generalization of the absolute value of a soft propagator. For $s = 0, 1$ and $j \leq 0$ define

$$C_{<j,s}(l) = \sum_{k<j} \frac{\mathbb{1}(M^{k-2} < |il_0 - e(\mathbf{l})| < M^k)}{|il_0 - e(\mathbf{l})|^{1+s}} \tag{4.3}$$

Lemma 44. *Let $j \leq 0$. Let $r = 0, 1, \ldots, d$. Let $g(l_0, \mathbf{l})$ be a function with integrable derivative.*

Then

$$\left|\int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \left(\frac{\partial}{\partial l_r} C_{<j}(l)\right) g(l_0, \mathbf{l})\right|$$
$$\leq \text{const} \sum_{s=0}^{1} \sum_{m=1}^{d} M^{-j(1-s)} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} C_{<j,0}(l) \left| \left(\frac{\partial}{\partial l_m}\right)^s g(l_0, \mathbf{l})\right| \quad (4.4)$$

Proof. The important point of this lemma is that on the right side $m \neq 0$ even if $r = 0$. Therefore consider the case $r = 0$ first (the other derivatives are similar, but easier). Note that $C_j(l) = (il_0 - e(\mathbf{l}))^{-1} a(M^{2j}(l_0^2 + e(\mathbf{l})^2))$. The derivative can act on the propagator or on the cut-off function. If it acts on the cut-off function we use

$$\left|\frac{\partial}{\partial l_r} a(M^{2j}(l_0^2 + e(\mathbf{l})^2))\right| \leq \text{const}\, M^{-j} \mathbb{1}(M^{j-2} < |il_0 - e(\mathbf{l})| < M^j) \quad (4.5)$$

to see that this gives a contribution to the $s = 0$ term. Inserting the other half of the derivative in the integral gives for $r = 0$.

$$I = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \frac{-ia(M^{2j}(l_0^2 + e(\mathbf{l})^2))}{(il_0 - e(\mathbf{l}))^2} g(l_0, \mathbf{l}) \quad (4.6)$$

which in the coordinates of Lemma 5 is given by

$$= \int \mathrm{d}_\beta l_0 \int \mathrm{d}\theta \int \mathrm{d}\rho \frac{-ia(M^{2j}(l_0^2 + \rho^2))}{(il_0 - \rho)^2} g(l_0, \boldsymbol{\pi}(\rho, \theta)) J_1 \quad (4.7)$$

An integration by parts in the ρ integral gives

$$I = i \int \mathrm{d}_\beta l_0 \int \mathrm{d}\theta \int \mathrm{d}\rho \frac{1}{il_0 - \rho} a(M^{2j}(l_0^2 + \rho^2)) \frac{\partial}{\partial \rho}(gJ_1)$$
$$+ i \int \mathrm{d}_\beta l_0 \int \mathrm{d}\theta \int \mathrm{d}\rho \frac{gJ_1}{il_0 - \rho} \frac{\partial}{\partial \rho} a(M^{2j}(l_0^2 + \rho^2)) \quad (4.8)$$

The lemma now follows by (4.3), (4.5) and the fact that on the support of the integrand J_1, its derivative and its inverse are bounded by constant.

If $r > 0$ the lemma follows more directly by doing an integration by parts in l_r. □

By doing a second integration by parts if required the following lemma about second order derivatives follows:

Lemma 45. *Let $j \leq 0$, $r_1, r_2 = 0, 1, \ldots d$. Let $g(l)$ and $h(l)$ be functions whose second order* *and first order derivatives respectively are integrable. Then*

$$\left|\int \mathrm{d}_\beta l_0 \int \mathrm{dl} \left(\frac{\partial}{\partial l_{r_2}} \frac{\partial}{\partial l_{r_1}} C_{<j}(l)\right) g(l_0, \mathrm{l}) h(l_0, \mathrm{l})\right|$$

$$\leq \text{const} \sum_{\substack{s_1+s_2+s_3+s_4=2-z \\ 0\leq z\leq s_2\leq 1}} \sum_{m_1,m_2=1}^{d} M^{-js_1}$$

$$\int \mathrm{d}_\beta l_0 \int \mathrm{dl} \mathcal{C}_{<j,z}(l) \left|(\frac{\partial}{\partial l_{m_1}})^{s_2} (\frac{\partial}{\partial l_{m_2}})^{s_3} g(l)\right| \left|(\frac{\partial}{\partial l_{m_1}})^{s_4} h(l)\right| \quad (4.9)$$

Moreover, let $j_1 \leq j_2 \leq 0$, $g(l,h)$ *a function with integrable second derivative and* $h_1(l)$, $h_2(l)$, $h_3(l,p)$ *functions with integrable first derivative*

$$\left|\int \mathrm{d}_\beta k_0 \int \mathrm{dk} \int \mathrm{d}_\beta l_0 \int \mathrm{dl} \left(\frac{\partial}{\partial l_{r_1}} C_{<j}(l)\right) \left(\frac{\partial}{\partial k_{r_2}} C_{<j}(k)\right) h_1(l) h_2(k) g(l,k) h_3(l,k))\right|$$

$$\leq \text{const} \sum_{\substack{s_1+s_2+s_3+s_4=1 \\ t_1+t_2+t_3+t_4=1-s_3}} \sum_{m_1,m_2=1}^{d} M^{-j_1 s_1} M^{-j_2 t_1}$$

$$\int \mathrm{d}_\beta l_0 \int \mathrm{dl} \mathcal{C}_{<j_1,0}(l) \left|(\frac{\partial}{\partial l_{m_1}})^{s_4} h_1(l)\right| \int \mathrm{d}_\beta k_0 \int \mathrm{dk} \mathcal{C}_{<j_2,s_3}(k) \left|(\frac{\partial}{\partial k_{m_2}})^{t_4} h_2(k)\right|$$

$$\left|(\frac{\partial}{\partial l_{m_1}})^{s_2} (\frac{\partial}{\partial k_{m_2}})^{t_2} g(l,k)\right| \left|(\frac{\partial}{\partial l_{m_1}})^{s_3} (\frac{\partial}{\partial k_{m_2}})^{t_3} h_3(l,k)\right| \quad (4.10)$$

Although these formula are look complicated their content is again simple. We can re-express all derivatives on soft propagators in terms of derivatives with respect to the vector momentum of the other part of the integral. In addition it is possible to avoid having two derivatives acting on the functions h and h_3 at the cost of leaving one of the derivatives on the soft propagator.

The functions g and h above will typically be products of propagators and these can contain phonon propagators. Thus derivatives of the phonon propagator arise in this way. *However these are all derivatives with respect to components of the vector momentum only.* The rest of this chapter will discuss how such derivatives can be bounded.

4.3 Volume improvements for point singularities

When bounding graphs where derivatives of the phonon propagator appear we will bound the singularities in the derivative using a scale decomposition. Because these are point singularities there is a much better volume behavior. The following lemma captures the volume bounds when taken alone and when combined with a factor coming from the fermion-propagator.

Lemma 46. *Let $d \geq 2$. For all ϵ_1,ϵ_2 there exists constants W_1 and W_2 independent of ϵ such that for all $\mathbf{p} \in \mathbb{R}^d$ and all $t' \in [0,1]$.*

$$\int \mathrm{d}\mathbf{k} \mathbb{1}(|\mathbf{k}-\mathbf{p}| < \epsilon_1)\mathbb{1}(|e(\mathbf{k})| < \epsilon_2) \leq W_1 \epsilon_1^d \tag{4.11}$$

$$\int \mathrm{d}\mathbf{k} \mathbb{1}(|P_{t'}\mathbf{k}-\mathbf{p}| < \epsilon_1)\mathbb{1}(|e(\mathbf{k})| < \epsilon_2) \leq W_2 \epsilon_1^{d-1}\epsilon_2 \tag{4.12}$$

Proof. Identity (4.11) is trivial. Just drop the second factor from the integral. What remains is just the volume of a d-dimensional sphere of radius ϵ_1. In the other integral we change to the coordinates (ρ, θ). By adjustment of the constant we can assume $\epsilon_1 < \frac{1}{4}$.

$$\int \mathrm{d}\mathbf{k} \mathbb{1}(|P_{t'}\mathbf{k}-\mathbf{p}| < \epsilon_1)\mathbb{1}(|e(\mathbf{k})| < \epsilon_2) \tag{4.13}$$

$$\leq \mathrm{const} \int \mathrm{d}\rho \int \mathrm{d}\theta \mathbb{1}(|\boldsymbol{\pi}(t'\rho,\theta)-\mathbf{p}| < \epsilon_1)\mathbb{1})\mathbb{1}(|\rho| < \epsilon_2) \tag{4.14}$$

$$\leq \mathrm{const}\, \epsilon_2 \sup_{|\rho| \leq \epsilon_2} \int_{S^{d-1}} \mathrm{d}\theta \mathbb{1}(|\boldsymbol{\pi}(t'\rho,\theta)-\mathbf{p}| < \epsilon_1) \tag{4.15}$$

Taking θ_1 as the angle between $\mathbf{p}$ and $\pi(t'\rho, \theta)$

$$\leq \mathrm{const} \int_0^{\frac{\pi}{4}} \mathrm{d}\theta_1 \mathbb{1}(|\sin_{\theta_1}| < \epsilon_1) \sin^{d-2}\theta_1 \tag{4.16}$$

$$\leq \mathrm{const}\, \epsilon_1^{d-1}\epsilon_2 \tag{4.17}$$

□

4.4 Controlling derivatives of D

4.4.1 Splitting the phonon propagator

The following lemmas capture the effect of bounding a loop-integral with a phonon line in the corresponding loop. This phonon line may have derivatives acting on it. This is controlled by observing that the singularity in the derivative of D is a point singularity and thus leads to a better volume improvement. The results in this section correspond to those of Lemma 4.

First define a convenience expression for the typical pre-factor that is extracted. Define λ_s as

$$\lambda_s(c,\beta,\epsilon) = c^{1-\epsilon} + \left(\frac{(1+\log\beta)^s}{\beta}\right)^{1-\epsilon} \tag{4.18}$$

The c and β dependent bounds come from different parts of the phonon propagator. As we have seen in equations (1.96)– (1.99) the β dependent distribution comes from the term in the sum where the frequency argument is close to zero. To make this explicit we write

$$D(l_0, c|\mathbf{l}-\mathbf{p}|) = D_0(l_0, c|\mathbf{l}-\mathbf{p}|) + D_1(l_0, c|\mathbf{l}-\mathbf{p}|) \tag{4.19}$$

where

$$\begin{aligned} D_1(l_0, c|\mathbf{l}-\mathbf{p}|) &= D(l_0, c|\mathbf{l}-\mathbf{p}|)\mathbb{1}(-\frac{\pi}{\beta} < l_0 \leq \frac{\pi}{\beta}) \\ D_0(l_0, c|\mathbf{l}-\mathbf{p}|) &= D(l_0, c|\mathbf{l}-\mathbf{p}|)(1 - \mathbb{1}(-\frac{\pi}{\beta} < l_0 \leq \frac{\pi}{\beta})) \end{aligned} \tag{4.20}$$

The cutoff function has been chosen such that $\int_{X^*} \mathrm{d}_\beta l_0 D_0(l_0, M) \leq \int \mathrm{d}l_0 D(l_0, M)$. Note that if M is some set of Matsubara frequencies and $q_0 \in \mathbf{R}$, i.e. is not necessarily a Matsubara frequency, then $|\{l_0 \in M||D_1(l_0, c|\mathbf{l}-\mathbf{p}|) \neq 0\}| = 1$. Non-Matsubara frequency arguments to the propagator occur through the localization and interpolation done in the renormalization procedure. For instance taking $L\,\mathrm{Val}(G)$ for a two-legged graph sets the external momentum to zero, which is not a fermionic Matsubara frequency.

4.4.2 No derivatives

When there is no derivative acting on the phonon line the frequency part of the integral is enough:

Lemma 47. *Let $p \in \mathbb{R}^{d+1}$. Let $j \leq 0$. Then there is a constant* 'const' *such that for all $\epsilon \in [0,1]$*

$$\left| \int \mathrm{d}_\beta l_0 \, \mathbb{1}(|l_0| < M^j) D(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \right| \leq \mathrm{const}\, \lambda_0(c, \beta, \epsilon) M^{j\epsilon} \tag{4.21}$$

Proof. Observe that the integral can be bounded in two ways. First as

$$\begin{aligned} \left| \int \mathrm{d}_\beta l_0 \, D(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \right| &\leq \left| \int \mathrm{d}_\beta l_0 \, D_0(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \right| + \left| \int \mathrm{d}_\beta l_0 \, D_1(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \right| \\ &\leq \left| \int \mathrm{d}l_0 D(l_0 - p_0, c|\mathbf{l}-\mathbf{p}|) \right| + \frac{2\pi}{\beta} \leq \pi c|\mathbf{l}-\mathbf{p}| + \frac{2\pi}{\beta} \end{aligned} \tag{4.22}$$

and secondly

$$\int \mathrm{d}_\beta l_0 \, \mathbb{1}(|l_0| < M^j) \leq \int \mathrm{d}l_0 \, \mathbb{1}(|l_0| < M^j) \leq \mathrm{const}\, M^j \tag{4.23}$$

The inequality (4.21) is simply the ϵ-weighted mean of the two bounds (with $|\mathbf{l}-\mathbf{p}| \leq \Lambda$). □

4.4.3 A single derivative in a loop

The above result needs to be combined with volume effects in the momentum integrals when there are derivatives present.

Lemma 48. *Let* $s = 0, 1$, $\epsilon \in (0, 1]$, $p \in \mathbb{R}^{d+1}$, *and* $I' < 0$. *Let* W_j *be given by one of the following:*

i) $W_j(l) = |C_j(l)|$ *with* $j < 0$ *or* $W_j(l) = M^{-j}\mathbb{1}_j(|l_0|)\mathbb{1}_j(|e(\mathbf{l})|)$

ii) $W_j(l) = \mathcal{C}_{<j,0}(l)$

iii) $W_j(l) = \dfrac{\mathbb{1}(l_0^2 + e(\mathbf{l}, \mathbf{q}, t)^2 < 4M^{2j})}{|il_0 + e(\mathbf{l}, \mathbf{q}, t)|}$ *for some* t.

iv) $W_j(l) = \dfrac{\mathbb{1}(M^{2I'} < (l_0 + tq_0)^2 + e(\mathbf{l})^2 < 4M^{2j})}{|i(l_0 + tq_0) + e(\mathbf{l})|}$ *for some* t.

Then there exists a constant 'const' *independent of* c, t, t', I', *and* p, *but possibly dependent on* ϵ, *such that for all* I_{js} *given by*

$$I_{js} = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, W_j(l) |\nabla^s D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)| \tag{4.24}$$

$$|I_{js}| \le \text{const}\, \lambda_0(c, \beta, \epsilon)(|j| + 1)^s M^{\epsilon j} = \text{const}\, \lambda_0(c, \beta, \epsilon) M^j (|j| + 1)^s M^{(\epsilon - 1)j} \tag{4.25}$$

Proof. Noting that the derivative of D is bounded as

$$|\nabla D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)| \le \frac{D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)}{|\mathbf{l} - \mathbf{p}|} \tag{4.26}$$

we introduce a scale for $|\mathbf{l} - \mathbf{p}|$ by inserting

$$1 = \sum_{m<0} f_m(|\mathbf{l} - \mathbf{p}|^2) + a(|\mathbf{l} - \mathbf{p}|^2) \tag{4.27}$$

in the integral. The essential argument of the proof is now that the extra volume gain as given in Lemma 46 is sufficient to control the factor $|\mathbf{l} - \mathbf{p}|^{-1} \sim M^{-m}$.

After making the scale sum in the soft line of case ii explicit this leads to terms of the form

$$H_{km} = \int \mathrm{d}l_0 \int \mathrm{d}\mathbf{l}\, |\mathcal{C}_k(l)| \frac{f_m(|\mathbf{l} - \mathbf{p}|^2)}{|P_{t'}\mathbf{l} - \mathbf{p}|^s} D(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{4.28}$$

$$\begin{aligned} &\le \text{const}\, M^{-k-ms} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} \\ &\quad \mathbb{1}(|l_0| < M^k)\mathbb{1}(|e(\mathbf{l})| < M^k)\mathbb{1}(|P_{t'}\mathbf{l} - \mathbf{p}| < M^m) D(l_0 - p_0, \hat{c}|\mathbf{l} - \mathbf{p}|) \end{aligned} \tag{4.29}$$

with $\mathcal{C}_j(l) = \mathcal{C}_{j,0}(l)$ given by

$$\mathcal{C}_{j,s}(l) = \frac{\mathbb{1}(M^{j-2} < |il_0 - e(\mathbf{l})| < M^j)}{|il_0 - e(\mathbf{l})|^{1+s}} \tag{4.30}$$

When $k \leq m$ we use an ϵ-weighted mean of

$$\int_{\mathbb{R}} \mathrm{d}_\beta l_0 D(l_0, c|\mathbf{l}|) \leq \pi c|\mathbf{l}| + \frac{2\pi}{\beta} < \text{const}\,\lambda_0(c,\beta,0) \tag{4.31}$$

and $\int_{\mathbb{R}} \mathrm{d}l_0 \mathbb{1}(|l_0| < M^k) \leq 2M^k$ to give

$$H_{km} \leq \text{const}\,\lambda_0(c,\beta,\epsilon) M^{-k-ms} M^{\epsilon k} \int \mathrm{d}\mathbf{l}\,\mathbb{1}(|e(\mathbf{l})| < M^k)\mathbb{1}(|\mathbf{l}-\mathbf{p}| < M^m) \tag{4.32}$$

using Lemma 46

$$\leq \text{const}\,\lambda_0(c,\beta,\epsilon) M^{-k-ms} M^{\epsilon k} \begin{cases} M^k M^m & k \leq m \\ M^{2m} & m < k \end{cases} \tag{4.33}$$

We don't need the extra factor $M^{(1-\epsilon)m}$ that would come from the integral over D_0 and thus we can handle both parts of D together.

For case i where $j < 0$

$$I_{js} \leq \text{const}\,\Big(\sum_{j \leq m < 0} H_{jm} + \sum_{m<j} H_{jm} + H_{j0} \Big) \tag{4.34}$$

where

$$H_{j0} = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, |W_j(l)| \frac{a(|\mathbf{l}-\mathbf{p}|^2)}{|\mathbf{l}-\mathbf{p}|^s} D(l_0 - p_0, \hat{c}|\mathbf{l}-\mathbf{p}|) \tag{4.35}$$

and because on the support of the integrand $|\mathbf{l}-\mathbf{p}| > \frac{1}{2}$, it is trivial to see that for all four cases.

$$H_{j0} \leq \text{const}\,\lambda_0(c,\beta,\epsilon) M^{\epsilon j} \tag{4.36}$$

For the sum over $m < j$ we use the above bounds for H_{jm}:

$$\sum_{m<j} H_{jm} \leq \text{const}\,\lambda_0(c,\beta,\epsilon) M^{(\epsilon-1)j} \sum_{m<j} M^{(2-s)m} \tag{4.37}$$

$$\leq \text{const}\,\lambda_0(c,\beta,\epsilon) M^{(\epsilon-1)j} M^{(2-s)j} = \text{const}\,\lambda_0(c,\beta,\epsilon) M^{\epsilon j} \tag{4.38}$$

For the other sum

$$\sum_{m \geq j} H_{jm} \leq \text{const}\,\lambda_0(c,\beta,\epsilon) \sum_{j \leq m < 0} M^{\epsilon j} M^{(1-s)m} \leq \text{const}\,\lambda_0(c,\beta,\epsilon)(|j|+1)^s M^{\epsilon j} \tag{4.39}$$

In case ii the bounds are the same but for an additional sum.

$$I_{js} \leq \sum_{k<j} \sum_{k \leq m<0} H_{km} + \sum_{k<j} \sum_{m<k} H_{km} + \sum_{k<j} H_{k0} \tag{4.40}$$

$$\leq \text{const}\, \lambda_0(c, \beta, \epsilon) \left(\sum_{k<j} (|k|+1)^s M^{\epsilon k} + \sum_{k<j} M^{(2-s)k} + \sum_{k<j} M^{\epsilon j} \right) \tag{4.41}$$

$$\leq \text{const}_\epsilon\, \lambda_0(c, \beta, \epsilon)(|j|+1)^s M^{\epsilon j} \tag{4.42}$$

where the constant is now allowed to depend on ϵ.

The two remaining cases are completely analogous to the case where W_j is the absolute value of the soft propagator. Interpolation does not change the power counting. As $l_0 - tq_0$ is not necessarily a Matsubara frequency there is no natural lower scale $I = I(\beta)$ on the scale sums in case iv. This is replaced by the explicit scale I'. Note that the constants in the bound depend on neither, because

$$\sum_{I<j} |j|^n M^{\epsilon j} \leq \sum_{-\infty<j} |j|^n M^{\epsilon j} \leq \text{const}\,(n, \epsilon) \tag{4.43}$$

. □

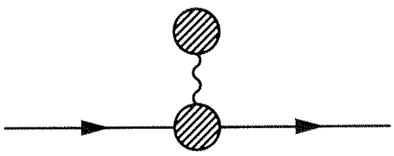
Remark 49. *In other words, Lemma 48 implies that, when a phonon line of a graph with root scale j is differentiated, this gives a factor $\lambda_0(c, \beta, \epsilon)(|j|+1)M^{(\epsilon-1)j}$ compared to normal power counting, provided this phonon line is contained in a loop. In contrast when a fermion line gets differentiated there is an additional factor M^{-j}.*

Note that in the above lemma a volume gain in the momentum part of the integral was exploited. In particular this means that other volume improvements on lines involving the integration variable can not be used. We will show that indeed when a derivative of a phonon propagator appears these volume improvements will not be necessary.

4.4.4 The position of phonon lines in two legged graphs

The following lemma implies that for the important class of graphs where we take derivatives with respect to external momenta, namely the two legged graphs, the phonon line is always in a loop.

Lemma 50. *Let G be a graph with two external fermion lines. Let G contain a phonon line not contained in a loop. Then G is a tadpole graph (when the phonon lines are not condensed into effective 4-vertices) with this phonon line connecting the two parts, i.e. G is of the form*



Moreover any derivative with respect to the external momenta of the value of this graph will never hit this line.

Proof. Let l be this phonon line. As l is not contained in any loop $G \setminus l$ is disconnected, $G \setminus l = G_1 \dot{\cup} G_2$. As l is a phonon line the two external fermion lines have to both connect to G_1 or both to G_2. This means that G has the required form.

Because G is a tadpole graph, $\mathrm{Val}(G)$ contains $P_l(0)$ where P_l is the propagator on this line. In particular this does not depend on the external momentum. □

4.4.5 Controlling second order derivatives

In the renormalization procedure second order derivatives of the phonon propagator can appear too. However we must thread with care. First of all it is again important that the derivatives are with respect to the components of the vector momentum only. Secondly it is here that the split from (4.19) becomes important. We can only control two derivatives acting on D_0. The bound uses that such second order derivatives can only occur when the phonon line is contained in two loops.

Lemma 51. *Let $t' \in (0,1]$ and $p, r \in \mathbb{R}^{d+1}$. Let $I' < 0$. Let $W_{j_1}(l)$ and $\bar{W}_{j_2}(k)$ each one of the four possibilities of Lemma 48. Let $v_1, v_2 \in -1, 1$. Let $a, b \in \{1, \dots, d\}$. Let I and $\bar{I}$ be* *given by*

$$I = \int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k}\, \bar{W}_{j_2}(k) \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, W_{j_1}(l) \left| (\frac{\partial}{\partial l_a}\frac{\partial}{\partial l_b} D_0)(t' l_0 + v_1 k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \right| \tag{4.44}$$

$$\bar{I} = \int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k}\, \bar{W}_{j_2}(k) \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, W_{j_1}(l) \left| (\frac{\partial}{\partial l_a} D_0)(t' l_0 + v_1 k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \right| \left| (\frac{\partial}{\partial l_b} D_0)(t' l_0 + v_2 k_0 - r_0, c|P_{t'}\mathbf{l} + v_2\mathbf{k} - \mathbf{r}|) \right| \tag{4.45}$$

then for $t' = 1$

$$\max_{v_1, v_2 = \pm 1} |I| \leq \mathrm{const}\, c(|\underline{j}| + 1)^2 M^{\underline{j}} \tag{4.46}$$

$$\max_{v_1, v_2 = \pm 1} |\bar{I}| \leq \mathrm{const}\, c(|\underline{j}| + 1)^2 M^{\underline{j}} \tag{4.47}$$

where $\underline{j} = \min\{j_1, j_2\}$. *Moreover, when* $j_1 \leq j_2$ *and* $t' < 1$ *then for all* $\epsilon \in (0,1)$

$$\max_{v_1, v_2 = \pm 1} |I| \leq \mathrm{const}\, (\frac{c}{t'})^{1-\epsilon} (|j_1| + 1)^2 M^{j_1} \tag{4.48}$$

$$\max_{v_1, v_2 = \pm 1} |\bar{I}| \leq \mathrm{const}\, (\frac{c}{t'})^{1-\epsilon} (|j_1| + 1)^2 M^{j_1} \tag{4.49}$$

Proof. For simplicity we assume $W_{j_1} = \mathcal{C}_{<j_1}$ and $\bar{W}_{j_2} = \mathcal{C}_{<j_2}$. Note that

$$\left| (\frac{\partial}{\partial l_a}\frac{\partial}{\partial l_b} D_0)(l_0 + v_1 k_0 - p_0, c|\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \right| \leq \frac{\mathrm{const}}{|\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|^2} D_0(l_0 + v_1 k_0 - p_0, c|\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \tag{4.50}$$

Writing the scale sums in the soft propagators explicitly and introducing a scale sum for the phonon momentum/momenta as in (4.27) we have

$$I \leq \mathrm{const} \left(I_0 + I_{\leq\leq} + I_{>\leq} + I_{>>} \right) \tag{4.51}$$

with

$$
\begin{aligned}
I_0 = &\int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k}\, \bar{W}_{j_2}(k) \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, W_{j_1}(l) \\
&\frac{\mathbb{1}(|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}| > \frac{1}{2})}{|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|^2} D(t'l_0 + v_1k_0 - p_0, c|\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|)
\end{aligned} \tag{4.52}
$$

$$
\begin{aligned}
I_{\leq\leq} &= \sum_{s_1<j_1} \sum_{s_1\leq s_2<j_2} \sum_{s_1\leq m} I_{s_1s_2m} \leq \sum_{s_1<j} \sum_{s_1\leq s_2} \sum_{s_1\leq m} I_{s_1s_2m} \\
I_{>\leq} &= \sum_{s_2<j_2} \sum_{s_2<s_1<j_1} \sum_{s_2\leq m} I_{s_1s_2m} \leq \sum_{s_2<j} \sum_{s_2<s_1} \sum_{s_2\leq m} I_{s_1s_2m} \\
I_{>>} &= \sum_{m<0} \sum_{m<s_1<j_1} \sum_{m<s_2<j_2} I_{s_1s_2m} \leq \sum_{m<j} \sum_{m<s_1} \sum_{m<s_1} I_{s_1s_2m}
\end{aligned} \tag{4.53}
$$

with

$$
\begin{aligned}
I_{s_1s_2m} = &\int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, \mathcal{C}_{s_1}(l)\mathcal{C}_{s_2}(k) \\
&\frac{\mathbb{1}_m(|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|)}{|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|^2} D_0(t'l_0 + v_1k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \\
\leq &\ \mathrm{const}\, M^{-s_1-s_2-2m} \\
&\int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, \mathbb{1}(|l_0| < M^{s_1})\mathbb{1}(|k_0| < M^{s_2}) \\
&\mathbb{1}(|e(\mathbf{l})| < M^{s_1})\mathbb{1}(|e(\mathbf{k})| < M^{s_2})\mathbb{1}(|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}| < M^m) \\
&D_0(tl_0 + v_1k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|)
\end{aligned} \tag{4.54}
$$

We bound this last expression in different ways depending on which scale is lowest. Extra care is needed in case $t' < 1$ because the integral is then no longer invariant under interchanges of s_1 and s_2. We start off with $I_{\leq\leq}$.

After dropping the indicator function for restriction on k_0, the k-integral in $I_{s_1s_2m}$ becomes

$$
\int \mathrm{d}_\beta k_0 \int \mathrm{d}\mathbf{k} D_0(tl_0 + v_1k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|)\mathbb{1}(|e(\mathbf{k})| < M^{s_2})\mathbb{1}(|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}| < M^m) \tag{4.55}
$$

On the support of the integrand in the $\mathbf{k}$-integral the frequency part is bounded as

$$
\int \mathrm{d}_\beta k_0 D_0(tl_0 + v_1k_0 - p_0, c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}|) \leq \pi c|P_{t'}\mathbf{l} + v_1\mathbf{k} - \mathbf{p}| \leq \pi c M^m. \tag{4.56}
$$

Note that he presence of the factor M^m is essential, which is why bounding D_1 in this would fail. Applying Lemma 46 to the vector part and the standard bounds for the l-integral (Note that the t' dependence plays no role here), we arrive at the following bound for $I_{s_1 s_2 m}$

$$I_{s_1 s_2 m} \le \text{const}\, M^{-s_1-s_2-2m} M^{2s_1} c M^m M^{s_2+m} \tag{4.57}$$
$$= \text{const}\, c M^{s_1} \tag{4.58}$$

This implies

$$I_{\le\le} \le \sum_{s_1<\underline{j}} \sum_{s_1\le s_2} \sum_{s_1\le m} \text{const}\, c M^{s_1} \tag{4.59}$$
$$\le \sum_{s_1<\underline{j}} \text{const}\, c(|s_1|+1)^2 M^{s_1} \tag{4.60}$$
$$\le \text{const}\, c(|\underline{j}|+1)^2 M^{\underline{j}} \tag{4.61}$$

When $t'=1$, $I_{s_1 s_2 m}$ is symmetric under exchange of the two integration variables and thus the same bound follows for $I_{>\le}$ there.

Observing that for $m < s_1 \le s_2$

$$I_{s_1 s_2 m} \le \text{const}\, M^{-s_1-s_2-2m} c M^{3m} M^{2s_1} \le \text{const}\, c M^m M^{s_1-s_2} \le c M^m \tag{4.62}$$

Again by symmetry, the same bound follows when $t'=1$, $m < s_2 < s_1$. Consequently, for $t'=1$

$$I_{>>} \le \sum_{m<\underline{j}} \sum_{m<s_1<0} \sum_{m<s_2<0} \text{const}\, c M^m \le \text{const}\, c(|\underline{j}|+1)^2 M^{\underline{j}} \tag{4.63}$$

The I_0 contribution is easily seen to obey the bound. The lemma has now been proven for $t'=1$ and $\epsilon=1$. It remains to handle the case $t'<1$.

Let $c' = \frac{c}{t'}$ before. Applying Hölder's Lemma to the l_0-integral and Lemma 46 we see that

$$I_{s_1 s_2 m} \le \text{const}\, M^{-s_1-s_2-2m} (c')^{1-\epsilon} M^{(1-\epsilon)m} M^{\epsilon s_1} M^{s_1+m} M^{2s_2} \tag{4.64}$$
$$\le \text{const}\, (c')^{1-\epsilon} M^{s_2+\epsilon s_1-\epsilon m} \tag{4.65}$$

As $j_1 \le j_2$,

$$I_{>\le} = \sum_{s_2<j_2} \sum_{s_2<s_1<j_1} \sum_{s_2 \le m} I_{s_1 s_2 m} \tag{4.66}$$

$$\le \sum_{s_1<j_1} \sum_{s_2<s_1} \sum_{s_2<m} I_{s_1 s_2 m} \tag{4.67}$$

$$= \text{const}\,(c')^{1-\epsilon} \sum_{s_1<j_1} M^{\epsilon s_1} \sum_{s_2<s_1} \sum_{s_2<m} M^{s_1-\epsilon m} \tag{4.68}$$

$$= \text{const}\,(c')^{1-\epsilon} \sum_{s_1<j_1} M^{\epsilon s_1} \sum_{s_2<s_1} (|s_2|+1) M^{(1-\epsilon)s_2} \tag{4.69}$$

using $\epsilon < 1$

$$\le \text{const}\,(c')^{1-\epsilon} \sum_{s_1<j_1} M^{\epsilon s_1} (|s_1|+1) M^{(1-\epsilon)s_1} \le \text{const}\,(c')^{1-\epsilon}(|j_1|+1) M^{j_1} \tag{4.70}$$

$$\tag{4.71}$$

Similarly

$$I_{>>} \le \sum_{m<s_2<s_1<j_1} I_{s_1 s_2 m} + \sum_{m<j_1} \sum_{s_1>m} \sum_{s_2 \ge s_1} I_{s_1 s_2 m} \tag{4.72}$$

and after inserting bound (4.62) one sees

$$\le \sum_{m<s_2<s_1<j_1} I_{s_1 s_2 m} + \sum_{m<j_1} \sum_{s_1>m} \sum_{s_2 \ge s_1} \text{const}\, c M^m \tag{4.73}$$

$$\le \text{const}\, c(|j_1|+1)^2 M^{j_1} + \text{const} \sum_{m<s_2<s_1<j_1} (c')^{1-\epsilon} M^{(1-\epsilon)m} M^{\epsilon s_1} \tag{4.74}$$

$$\le \text{const}\, c(|j_1|+1)^2 M^{j_1} + \text{const} \sum_{m<s_1<j_1} (c')^{1-\epsilon} (|m|+1) M^{(1-\epsilon)m} M^{\epsilon s_1} \tag{4.75}$$

$$\le \text{const}\, c(|j_1|+1)^2 M^{j_1} + \text{const} \sum_{s_1<j_1} (c')^{1-\epsilon} (|s_1|+1) M^{(1-\epsilon)s_1} M^{\epsilon s_1} \tag{4.76}$$

$$\le \text{const}\,(c')^{1-\epsilon}(|j_1|+1)^2 M^{j_1} \tag{4.77}$$

The proof for $\bar{I}$ starts by introducing scales for both phonon momenta. Then one keeps only that factor D that corresponds to the lowest scale momentum. This then trivially gives the same power counting as above. □

Remark 52. *The power counting without the phonon factors would have been* const $M^{j_1+j_2}$. *Thus we see that from the above lemma with* $\epsilon = 1$ *that apart from polynomial factors that* *can be controlled, the two derivatives give rise to one factor M^{j^*} when they hit phonon lines. Here j^* is the highest scale where the phonon line(s) are contained in a loop.*

Finally we discuss what happens with D_1 terms. The second order derivatives do not give integrable terms because there is no gain from the frequency sum. For that reason we avoid second derivatives of this term altogether. However as Lemma 44 shows, that means that sometimes we must leave a derivative on a soft line. The following corollary is the single-derivative bound adapted to this case.

Corollary 53. *Let $m \in \{1, \ldots, d\}$, $j \leq 0$, $p_0 \in \mathbb{R}^d$, and let*

$$H = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} C_{<j,1} \left| \frac{\partial}{\partial l_m} D_1 \right| (l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{4.78}$$

Then

$$|H| \leq \text{const} \, \frac{(\log \beta + 1)^2}{\beta} \tag{4.79}$$

Proof. Let $I(\beta)$ be the lowest scale such that $\frac{\pi}{\beta} < M^{I(\beta)-2}$. Then

$$H = \sum_{k=I(\beta)}^{j-1} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} C_{k,1} (\frac{\partial}{\partial l_m} D_1)(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{4.80}$$

Define $H^1_{km} = H_{km}|_{D \to D_1}$ as in Section 4.4.3. Then

$$H \leq \text{const} \sum_{k=I(\beta)}^{j-1} M^{-k} (\sum_{k \leq m < 0} H^1_{km} + \sum_{m<k} H^1_{km} + H^1_{k0}) \tag{4.81}$$

proceeding as in Section 4.4.3 with $c = 0$ and $\epsilon = 1$ we get

$$\leq \text{const} \, \beta^{-1} \sum_{k=I(\beta)}^{j-1} M^{-k} ((|k| + 1) M^k + 2M^k) \tag{4.82}$$

$$\leq \text{const} \, \beta^{-1} \sum_{k=I(\beta)}^{j-1} (|k| + 1) \tag{4.83}$$

$$\leq \text{const} \, \frac{(|I(\beta)| + 1)^2}{\beta} \leq \text{const} \, \frac{(\log \beta + 1)^2}{\beta} \tag{4.84}$$

□

4.4.6 Scale zero graphs

The lemmas of the previous sections do not directly carry over to integrals containing scale zero hard propagators C_0 because the frequency integrals do no converge with just one propagator present per integral. However most of the content of the previous lemmas is to show that the one integral is sufficient to control both the divergences of the electron propagator and those of the derivatives of D by balancing the volume gains in the frequency and momentum integrals against each other . As the propagator C_0 is not singular at all this not needed in this case.

Recall that in bounding the scale zero contribution we used $|C_0(l)| \leq \sum_{k\geq 0} \bar{\tau}_k(l_0)$ and bounded the frequency integrals simply by the volume of Ω. The identity

$$\int \mathrm{d}_\beta l_0 \bar{\tau}_j(l_0) \leq \text{const} \tag{4.85}$$

was used to bound the integrals.

As D is an integral function of the frequency the presence of D in the integral only improves the decay. The fact that the vector momentum integrals are up to this point completely unused gives sufficient lee-way to control the derivatives. That is the content of the following lemma

Lemma 54. *Let $a = 0, 1$. Then there exists a constant such that for all $p \in \mathbb{R} \times \Omega$, all $s = 0, \ldots, 2-a$, all $m_1, m_2 = 1, \ldots, d$. and for all $k \geq 0$*

$$I = \int_\Omega \mathrm{d}\mathbf{l} \int \mathrm{d}_\beta l_0 \bar{\tau}_k(l_0) \left| \left(\prod_{r=1}^{s} \frac{\partial}{\partial l_{m_r}} D_a \right)(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \right| \leq \text{const} \begin{cases} c & a = 0 \\ \beta^{-1} & a = 1 \end{cases} \tag{4.86}$$

Proof. By simply using $|\tau_k| < M$ it can be dropped from the integral to get

$$I \leq \text{const} \int_\Omega \mathrm{d}\mathbf{l} \frac{1}{|\mathbf{l} - \mathbf{p}|^s} \int \mathrm{d}_\beta l_0 D_a(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|) \tag{4.87}$$

$$\leq \text{const} \int_\Omega \mathrm{d}\mathbf{l} \frac{1}{|\mathbf{l} - \mathbf{p}|^{s-1+a}} \begin{cases} c & a = 0 \\ \beta^{-1} & a = 1 \end{cases} \tag{4.88}$$

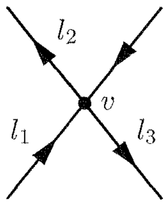
and the Lemma follows because $s - 1 + a < 2$. □

Therefore even when the loop that contains D with derivatives acting on it is at scale 0 the power counting effect is as in Remarks 49 and 52 or better.

4.5 Phonon lines in overlapping graphs

Lemma 55. *Let G be an overlapping graph. Then G contains a phonon line k and a loop $\mathfrak{L}$ such that $k \in \mathfrak{L}$.*

Proof. By definition G contains two loops $\mathfrak{L}$ and $\mathfrak{L}'$ such that $L_F(\mathfrak{L}) \cap L_F(\mathfrak{L}') \neq \emptyset$, $L_F(\mathfrak{L}) \setminus L_F(\mathfrak{L}') \neq \emptyset$ and $L_F(\mathfrak{L}') \setminus L_F(\mathfrak{L}) \neq \emptyset$. In fact there exists lines $l_1 \in L_F(\mathfrak{L}) \cap L_F(\mathfrak{L}')$, $l_2 \in L_F(\mathfrak{L}) \setminus L_F(\mathfrak{L}')$, $l_3 \in L_F(\mathfrak{L}') \setminus L_F(\mathfrak{L})$ such that l_1, l_2, l_3 are all connected to the same (effective) 4-vertex v.



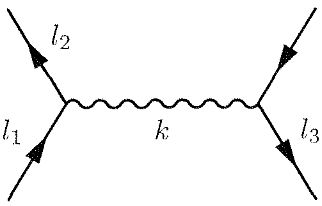
To see this take any $l_0 \in L_F(\mathfrak{L}) \cap L_F(\mathfrak{L}')$. Start by taking $l = l_0$. Let l' be the next line of $L_F(\mathfrak{L})$ according to the direction of the fermion line. Then there are two cases:

i) $l' \in L_F(\mathfrak{L}')$. Then $l' \in L_F(\mathfrak{L}) \cap L_F(\mathfrak{L}')$. Set l to l' and repeat.

ii) $l' \notin L_F(\mathfrak{L}')$. Then take $l_1 = l$, $l_2 = l'$ and v the vertex connecting them. Take l_3 the next fermion line after l along $\mathfrak{L}'$. l_3 is connected to v because l is.

It remains to show that case ii must occur. If it doesn't then this implies that case i occurs always. As $|L_F(\mathfrak{L})| < \infty$ this means that after some amount of steps $l = l_0$ again. However then $L_F(\mathfrak{L}) = L_F(\mathfrak{L}) \cap L_F(\mathfrak{L}')$. Which is a contradiction.

Now expand the graph back to the full electron-phonon graph. Let k be the phonon line corresponding to v. If $k \in \mathfrak{L}$ then we are finished. However if $k \notin \mathfrak{L}$ then l_1 and l_3 connect to 3-vertices on different sides of k, i.e. G contains the subgraph



and thus $k \in \mathfrak{L}'$.

□

Chapter 5

Power counting for Electron-Phonon Graphs

5.1 Introduction

In this chapter and following the (improved) power counting for the renormalized theory is considered. It will then be applied to prove that the theory is renormalizable. In [FST96] improved power counting bounds are obtained for a general class of interactions with bounded derivatives of the interaction propagator, using the localization operator ℓ as defined in (3.8). In our case an analogous bound is possible requiring more attention to derivatives acting on the phonon propagator.

In the proofs of [FST96] where the localization operator ℓ is used to act on the values of 2-legged graphs, bounding graphs containing strings of 2-legged insertions sometimes requires more than just the renormalization gain. Analysis of sign cancellations in those strings is required. For the localization operator L the same bounds apply. However all types of 2-legged insertions give rise to an extra power counting improvement and the use of sign cancellations in the product of propagators is replaced by the improved renormalization bound.

5.2 Bounds on Strings

5.2.1 Some norms of Graphs

Below we will often need to bound norms of graphs that are mixtures of of L_1, C^k and *sup*-norms. The following definition introduces a notation for them.

Definition 56. *Let* $\Theta \in C^k(\mathbb{R}\times\Omega)$ *for some* k. *Let* $t' \in (0,1]$ *and* $g(l_0,1)$ *a bounded function.*

Then define

$$\mathcal{N}_s(\Theta, g, t') = \sum_{i_r=0\dots d} \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l}\, g(l_0, \mathbf{l}) \left| \prod_{r=1}^{s} \left(\frac{\partial}{\partial l_{i_r}} \right) \Theta(t' l_0, P_{t'} \mathbf{l}) \right| \quad 0 \le s \le k \tag{5.1}$$

$$\mathcal{N}_{j,s}(\Theta) = \mathcal{N}_{j,s}(\Theta, \mathbb{1}_j(|l_0|)\mathbb{1}_j(|e(\mathbf{l})|), 1) \tag{5.2}$$

Furthermore, let $n \in \mathbb{N}$ and $\Theta \in C^k((\mathbb{R} \times \Omega)^n)$. Let $A_n = \{1, \dots, n\} \times \{0, \dots, d\}$. Then for $z = 0, 1, 2$, $\mathcal{N}^{n-z,z}_{j_{z+1}\dots j_n,0}$ is defined as

$$\mathcal{N}^{n-z,z}_{j_z\dots j_n,s}(\Theta, t') = \Big(\prod_{1\le i \le z} \sup_{l_{i,0},\mathbf{l}_i} \Big) \sum_{i_r \in A_{n-z}} \prod_{m=z+1}^{n} \left(\int \mathrm{d}_\beta l_{m,0} \int \mathrm{d}\mathbf{l}_m\, \mathbb{1}_{j_m}(|l_{m,0}|)\mathbb{1}_{j_m}(|e(\mathbf{l}_m)|) \right)$$
$$\left| \prod_{r=1}^{s} \left(\frac{\partial}{\partial l_{i_r}} \right) \Theta((t l_{0,0}, P_{t'} \mathbf{l}_0), l_1, \dots, l_n) \right| \quad 0 \le s \le k \tag{5.3}$$

Note that we take the supremum over the first z momenta. In particular for $z = 0$ we integrate over all momenta.

We also write

$$\mathcal{N}^{n-z,z}_{j_{z+1}\dots j_n,s}(\Theta) = \mathcal{N}^{n-z,z}_{j_{z+1}\dots j_n,s}(\Theta, 1) \tag{5.4}$$

Note that when $|\Theta|_s < \infty$, $i \in \{0, 1, 2\}$

$$\mathcal{N}^{n-i,i}_{j_1\dots j_{n-i},s}(\Theta) \le \text{const}\, M^{\sum_{r=1}^{n-i} 2 j_r} |\Theta|_s \tag{5.5}$$

5.2.2 Bounds on Strings and Bubbles

In this section we consider 2-legged graphs, which consists of a string of propagators and 1PI 2-legged subgraphs as drawn in Section 1.2.2. It is shown that the presence of the counter terms does indeed lead to improved bounds.

Lemma 57 (Simple String Lemma). *Let $j < 0$, $n \ge 1$, $k \in \{0, \dots, n-1\}$, Let $k' \in \{k, \dots, n-1\}$, $m \in \{1, \dots, n\}$. Let $T_1, \dots T_{n-1} \in C^2(\mathbb{R} \times \Omega)$ and $g \in C^0(\mathbb{R} \times \Omega)$. Let*

$$S_j(l) = C_j(l)^m C_{j+1}(l)^{n-m} \prod_{w=1}^{k} (1 - L) T_w(l) \prod_{w=k+1}^{k'} L T_w(l) \prod_{w=k'+1}^{n-1} T_w(l) \tag{5.6}$$

$$I_j = \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} |g(l)| |S_j(l)| \tag{5.7}$$

then there exists a constant U_1 such that

$$|I_j| \le U_1 \prod_{w=k'+1}^{n-1} (|T_w|_0 M^{-j}) \prod_{w=k+1}^{k'} (|T_w|_0 M^{-j} + |T_w|_1) M^{-j} \mathcal{N}_{j,0}(g) \prod_{w=1}^{k} |T_w|_1 \tag{5.8}$$

Moreover, when $k \geq 1$ then

$$|I_j| \leq U_1 \prod_{w=k'+1}^{n-1} (|T_w|_0 M^{-j}) \prod_{w=k+1}^{k'} (|T_w|_0 M^{-j} + |T_w|_1)|g|_{0,j} \prod_{w=2}^{k} |T_w|_1 \int_0^1 \mathrm{d}t' \mathcal{N}_{j,2}(T_1, t') \quad (5.9)$$

when $\int_0^1 \mathrm{d}t' \mathcal{N}_{j,2}(T_1, t') < \infty$.

Proof. Assume for simplicity that $k' = n - 1$. The extension to $k' < n - 1$ is trivial. As the signs of the factors in the integrand are not relevant for these bounds we start by taking the absolute values of each. Using that both $|C_j|_0 <$ const M^{-j} and $|C_{j+1}|_0 <$ const M^{-j} we can bound $|C_j(l)^m C_{j+1}(l)^{n-m}| \leq M^{-nj} \mathbb{1}_j(|l_0|) \mathbb{1}_j(|e(\mathbf{l})|)$. Which gives

$$|I_j| \leq \text{const}\, M^{-nj} \mathcal{N}_{j,0}(g \prod_{w=1}^{k} (1 - L)T_w \prod_{w=k+1}^{n-1} LT_w) \quad (5.10)$$

Noting that the indicator functions in $\mathcal{N}$ restrict the region where the values of the other factors need to be bounded

$$|I_j| \leq \text{const}\, M^{-nj} \prod_{w=1}^{k} |(1 - L)T_w|_{0,j} \prod_{w=k+1}^{n-1} |LT_w|_{0,j} \mathcal{N}_{j,0}(g) \quad (5.11)$$

or alternatively when $k > 0$

$$|I_j| \leq |g|_{0,j}\, \text{const}\, M^{-nj} \prod_{w=2}^{k} |(1 - L)T_w|_{0,j} \prod_{w=k+1}^{n-1} |LT_w|_{0,j} \mathcal{N}_{j,0}((1 - L)T_1) \quad (5.12)$$

As $|LT| \leq |T|_0 + |l_0| |\frac{\partial T}{\partial l_0}|_0 + |\rho| |\frac{\partial T}{\partial \rho}|_0$,

$$|LT_w|_{0,j} \leq |T_w|_0 + M^j |T_w|_1 = M^j (|T_w|_0 M^{-j} + |T_w|_1) \quad (5.13)$$

Using expression (3.15) for $(1 - L)T_w$ it is easy to see that

$$|(1 - L)T_w|_{0,j} \leq 2M^j |T_w|_1 \quad (5.14)$$

Inserting these bounds into (5.11) (and excluding factors coming from $\mathcal{N}_{j,0}(g)$) gives a total power counting factor of $M^{kj} M^{(n-k-1)j} M^{-nj} = M^{-j}$ and thus (5.8) follows.

Turning to equation (5.12) we want to find a simpler bound for $\mathcal{N}_{j,0}((1 - L)T_1)$ not involving L. Using the Taylor expansion (3.13)

$$\mathcal{N}_{j,0}((1 - L)T_1) = \mathcal{N}_{j,0}(\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s (\mathcal{D}T_1)(sl_0, s\rho, \theta)) \leq \text{const}\, M^{2j} \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \mathcal{N}_{j,2}(T_1, s) \quad (5.15)$$

which implies (5.9) after insertion in (5.12) as the power counting factor is now

$$M^{(k-1)j}M^{(n-k-1)j}M^{2j}M^{-nj} = 1. \tag{5.16}$$

Note that the factors $\mathcal{N}_{j,0}(g)$ and $\mathcal{N}_{j,2}(T_1, s)$ normally produce an other factor M^{2j} and thus the complete power counting is M^j and M^{2j} respectively. □

Corollary 58 (Simple Bubble). *Let $j < 0$, and for all $i = 1, 2$ $n_i \geq 1$, $k_i \in \{0, \ldots, n_i - 1\}$, $k'_i \in \{k, \ldots, n-1\}$, $m_i \in \{1, \ldots, n_i\}$. Let $T_{1,i}, \ldots T_{n_i-1,i} \in C^2(\mathbb{R} \times \Omega)$ and $g_i \in C^0(\mathbb{R} \times \Omega)$. Let $q \in \mathbb{R} \times \Omega$ for both i. Let $k_1 + k_2 > 0$, and*

$$S_{j,i}(l) = C_j(l)^{m_i} C_{j+1}(l)^{n_i - m_i} \prod_{w=1}^{k_i} (1-L)T_{w,i}(l) \prod_{w=k_i+1}^{k'_i} LT_{w,i}(l) \prod_{w=k'_i+1}^{n_i-1} T_{w,i}(l) \tag{5.17}$$

$$I_j = \int \mathrm{d}_\beta l_0 \int \mathrm{d}l g_1(l) g_2(l) S_{j,1}(l) S_{j,2}(l+q) \tag{5.18}$$

then there exists a constant $\bar{U}_1$ such that

$$|I_j| \leq \bar{U}_1 M^{-j} \prod_{\substack{(w,i)\\ k_i < w \leq k'_i}} (|T_{w,i}|_0 M^{-j} + |T_{w,i}|_1) \prod_{\substack{(w,i)\\ k'_i < w < n}} (|T_{w,i}|_0 M^{-j})$$
$$|g_1|_{0,j} |g_2|_{0,j} \prod_{\substack{(w,i) \neq (1,s)\\ w \leq k_i}} |T_{w,i}|_1 \int_0^1 \mathrm{d}t' \mathcal{N}_{j,2}(T_{1,s}, t') \tag{5.19}$$

where $s = 1$ if $k_1 > 0$ and $s = 2$ otherwise, when $\int_0^1 \mathrm{d}t' \mathcal{N}_{j,2}(T_{1,s}, t') < \infty$.

Proof. The proof of the corollary is completely analogous to that of the previous lemma but for the fact that as there now are two strings there is another propagator not paired with a corresponding T_w and thus there remains another factor M^{-j} in the power counting. □

In the following there is also a need to bound strings with derivatives acting on them. The following shows that the power counting just contains the expected factor M^{-j} for each derivative of a string at scale j.

Lemma 59. *Let $j < 0$, $n \geq 1$, $k \in \{0, \ldots, n-1\}$, Let $k' \in \{k, \ldots, n-1\}$, $m \in \{1, \ldots, n\}$. Let $s = 1$ or $s = 2$. Let $a_1, a_2 \in \{0, \ldots, d\}$. Let $T_1, \ldots T_{n-1} \in C^2(\mathbb{R} \times \Omega)$ and $g \in C^0(\mathbb{R} \times \Omega)$. Let S_j be given by (5.6).*

$$I_j = \int \mathrm{d}_\beta l_0 \int \mathrm{d}l |g(l)| \left| \left(\prod_{i=1}^{s} \frac{\partial}{\partial l_{a_i}}\right) S_j(l) \right| \tag{5.20}$$

then there exists a constant U_2 such that for $s = 1$

$$|I_j| \leq U_2 M^{-j} M^{-j} \mathcal{N}_{j,0}(g) \sum_{r=0}^{n-1} \eta(\{r\}) \chi_0(r) \tag{5.21}$$

where for some set B of numbers $0, \ldots, n-1$

$$\eta(B) = \prod_{\substack{w=k'+1 \\ w \notin B}}^{n-1} (|T_w|_0 M^{-j}) \prod_{\substack{w=k+1 \\ w \notin B}}^{k'} (|T_w|_0 M^{-j} + |T_w|_1) \prod_{\substack{w=1 \\ w \notin B}}^{k} |T_w|_1 \tag{5.22}$$

$$\chi_0(r) = \begin{cases} |T_r|_1 & r > 0 \\ 1 & r = 0 \end{cases} \tag{5.23}$$

Moreover, there exists a constant U_2 such that

$$|I_j| \leq U_2 M^{-sj} \sum_{r_1, r_2 \in A_s} (\chi_1(r_1)\chi_0(r_2)(1 - \delta_{r_1 r_2}) + \chi_2(r_1)\delta_{r_1 r_2})\, \eta(B_{r_1 r_2}) \tag{5.24}$$

where

$$\chi_1(r) = \begin{cases} |g|_{0,j} \int_0^1 \mathrm{d}t' \mathcal{N}_{2,j}(T_r, t') & 0 < r \leq k \\ M^{-j} \mathcal{N}_{j,0}(g) \chi_0(r) & \text{otherwise} \end{cases} \tag{5.25}$$

$$\chi_2(r) = \begin{cases} |g|_{0,j} \int_0^1 \mathrm{d}t' \mathcal{N}_{2,j}(T_1, t') & r = 0 \wedge k \geq 1 \\ M^{-j} \mathcal{N}_{j,0}(g) & r = 0 \wedge k = 0 \\ 0 & k < r \leq k' \\ |g|_{0,j} \mathcal{N}_{2,j}(T_r) & 0 < r \leq k \vee r > k' \end{cases} \tag{5.26}$$

Here A_s is given by

$$A_1 = \{(r_1, 0) | r_1 = 0, \ldots, n-1\} \tag{5.27}$$

$$A_2 = \{(r_1, r_2) | r_1, r_2 = 0, \ldots, n-1\} \tag{5.28}$$

and $B(r_1, r_2) = \{r_1, r_2\}$ when $k = 0$,$r_1 \neq 0$ or $r_2 \neq 0$. When $k \geq 1$, then $B(0,0) = \{1\}$.

Note that here again the expression is more complex than its contents. Although the derivatives change the explicit form of the factors we will see later that these have essentially the same power counting. Therefore the only change on the level of power counting is the appearance of the factor M^{-sj}.

Proof. Distribute the derivatives using Leibnitz's rule. The derivative with respect to l_{a_i} either hits the subgraph r_i or the propagators. In the latter case, take $r_i = 0$ and it the derivative gives a factor M^{-j} trivially. We will show that when $r_i \neq 0$ the corresponding factor gives power counting similar to that used in the proof of Lemma 57 but also with a factor M^{-j}. When $r > k'$ then we simply have

$$|\frac{\partial}{\partial l_{a_i}} T_{r_i}| \leq |T_{r_i}|_1 \leq M^{-j}(M^j |T_{r_i}|_1) \tag{5.29}$$

The momentum dependence of L is trivial and thus for $k < r_1 \leq k'$ a similar bound holds

$$|\frac{\partial}{\partial l_{a_i}} LT_{r_i}| \leq |T_{r_i}|_1 \leq M^{-j}(M^j |T_{r_i}|_1) \tag{5.30}$$

For $0 < r_i \leq k$ the bound is even simpler than without the derivative because the derivative gives a direct gain.

$$|\frac{\partial}{\partial l_{a_i}} (1 - LT_{r_i})| \leq |T_{r_i}|_1 \leq M^{-j}(M^j |T_{r_i}|_1) \tag{5.31}$$

The second derivative of LT with respect to ρ and l_0 vanishes. Therefore for $0 < r_1 \leq k$ and $k' < r$ we have

$$\mathcal{N}_{j,0}(|\frac{\partial}{\partial l_{a_1}} \frac{\partial}{\partial l_{a_2}} (1 - \mathbb{1}(r \leq k) LT_{r_i})|) \leq \mathcal{N}_{j,2}(T_r) \leq M^{-2j}(M^{2j} \mathcal{N}_{j,2}(T_r)) \tag{5.32}$$

Note that the derivatives have given the factor M^{2j} directly and interpolation is not needed.

The most difficult term is when there is only a single derivative acting on $(1 - L)T_r$ but we still want to extract the improved renormalization factor M^{2j}. Here we use the representation (3.15) for $(1 - L)T_r$. For simplicity we consider only the $\frac{\partial}{\partial l_0}$ derivative

$$\begin{aligned} &\mathcal{N}_{j,0}(\frac{\partial}{\partial l_0}(1 - T)T_r) \\ &\qquad \leq \int_0^1 \mathrm{d}t \mathcal{N}_{j,0}((\frac{\partial T_r}{\partial l_0})(tl_0, P_t \mathbf{l}) - (\frac{\partial T}{\partial l_0})(0, P_0 \mathbf{l})) \\ &\qquad + M^j \int_0^1 \mathrm{d}t \left(\mathcal{N}_{j,0}((\frac{\partial^2 T_r}{\partial {l_0}^2}), t) + \mathcal{N}_{j,0}(\frac{\partial}{\partial l_0} \frac{\partial T_r}{\partial \rho}, t) \right) \\ &\qquad \leq \text{const}\, M^j \int_0^t \mathrm{d}t \mathcal{N}_{2,j}(T_r, t) = \text{const}\, M^{-j}(M^{2j} \int_0^t \mathrm{d}t \mathcal{N}_{2,j}(T_r, t)) \end{aligned} \tag{5.33}$$

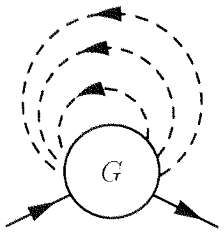
Which is indeed the bound in (5.15) but for another factor M^{-j}. □

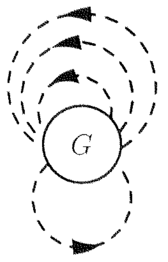
5.3 Power counting with the L-operator

5.3.1 Graphical representation of the norms

We now have bounds for overlapping graphs and for strings, with power counting improvements through extra volume effects renormalization. By Remark 33 these are in fact all the tools we need.

It is good to have a graphical interpretation of the norms defined in the previous section and of the definition of $\mathcal{M}$ in (3.81). Let $\tilde{\mathcal{M}}(G)$ be the graph obtained from G by connecting the incoming line with momentum p_i to the outgoing line with momentum p_{m+i-1} for $i = 2 \dots m$. Similarly define $\bar{\mathcal{M}}(G)$ as the graph where also the remaining two external lines are connected. Below these two graphs are drawn for $m = 4$. The new lines are drawn dashed





$\tilde{\mathcal{M}}(G)$ $\bar{\mathcal{M}}(G)$

The graph $\tilde{\mathcal{M}}(G)$ is two legged. When the added Fermion line is thought of as having propagator 1 then $\mathrm{Val}(\tilde{\mathcal{M}}(G))$ is given by (cf. equation (5.3)):

$$\prod_{r=2}^{m} \left(\int \mathrm{d}_\beta l_{r,0} \int \mathrm{dl}_r \right) \mathcal{M}(G)(l_1, \dots, l_m) \tag{5.34}$$

The graph $\tilde{\mathcal{M}}(G)$ (and related graphs) occur naturally in Theorem 60, because of the recursive structure for two legged non-overlapping graphs given by Remark 33. We will also use the two following trivial properties of $\tilde{\mathcal{M}}(G)$:

- When G is overlapping, then $\tilde{\mathcal{M}}(G)$ is overlapping.
- When $\tilde{\mathcal{M}}(G)$ is overlapping and l is part of two overlapping loops then $l \in G$.

5.3.2 Another representation for labeled graphs

Let t be a tree rooted at ϕ and G a graph compatible to it. We now define a tree τ as follows. Take the maximal subtree $t' \subset t$ such that for all $f \in t'$ with f not a leaf $E(f) > 2$. Thus the leaves of t' are either the leaves of t or forks with $E(f) = 2$. τ is created from t' by iterating over all $f \in t'$ with $E(f) = 2$ and following the following procedure recursively:

- If G_f is 1PI then stop.
- Each 1-particle reducible two-legged fork G_f is a string of 1PI two legged graphs. If these are sub forks of f connect them to $\pi(f)$ directly. If they are graphs of scale j_f (these are called same scale insertions (SSI)) then create a new fork f' for each of them in the tree. These have the special property that the scale $j_{f'}$ isn't summed over but always $j_{f'} = j_{\pi(f')} + 1$. Note that by momentum conservation for a 1-particle reducible fork f the only term contributing to the scale sum is $j_f = j_{\pi(f)} + 1$ and thus eliminating the fork from the tree (and from the scale sums) is allowed. After these new forks have been inserted replace them by vertices as above.

We also construct a corresponding graph G' similar to the construction in e.g. [FST96], but only replacing the two legged subgraphs. Apply the replacement procedure above to t. G' is created from G by replacing each two-legged subfork f such that G_f is 1PI and for all $\phi < f' < f$, $E(f') > 2$ or $G_{f'}$ is 1P-reducible by a two legged vertex v_f with scale j_f and value

$$\mathrm{Val}(v_f) = \sum_{J \in \mathcal{J}(t_f, G_f, j_f)} \mathcal{P}_{v_f} \, \mathrm{Val}(G_f^J) \tag{5.35}$$

with

$$\mathcal{P}_{v_f} = \begin{cases} L & \text{if } f \text{ is a C fork} \\ (1-L) & \text{if } f \text{ is an R fork} \\ 1 & \text{if } f \text{ is a SSI} \end{cases} \tag{5.36}$$

Correspondingly replacing t_f in t by f gives τ. We will often identify v_f and f.

Such 2-legged vertices/subgraphs act in many ways like fermion lines. For instance if for a leaf v_f of τ we define $\partial_{fm} G$ as the graph that is constructed from G by replacing G_f by a vertex with the value $\frac{\partial}{\partial l_m}(\mathcal{P}_f G_f)(l)$. It is obvious that Corollary 43 trivially extends to graphs that have 2-legged vertices in this way. As 2-legged vertices thus behave like lines in the above sense we define $F_2(t, G)$, a set containing both lines and vertices, as follows

$$F_2(t, G) = L_F(G') \cup V_2(G') \tag{5.37}$$

5.3.3 The power counting bound

Theorem 60. *Let $\epsilon \in (0,1]$. Let $t' \in (0,1]$ for $\epsilon < 1$ and $t' = 1$ for $\epsilon = 1$. Let (t, G) be a pair consisting of a graph G compatible with a tree t for our theory with $e(\mathbf{l}) = |\mathbf{l}|^2 - 1$. Let ϕ be the root of t. Let $s \leq 1$ when $E(G) = 2m = 2$ and $s' = 0$ when $E(G) = 2m \geq 4$. . Let $j \leq 0$. If $E(G) = 2$ then let G be 1PI. Let $l_1, l_2 \in F_2(t, G)$. Let $a_1, a_2 \in \{0, \ldots, d\}$.*

Then there is a polynomial Pol *such that*

i) When either

(a) $E(G) = 2$ and when $\epsilon < 1$ $s' = 0$, or

(b) $s' = 0$ and $\tilde{G}(\phi)$ is overlapping, then

$$\sum_{J \in \mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_{s'} \leq \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|j|) M^{j(2-\frac{1}{2}E(G)-s'+\epsilon)} \tag{5.38}$$

(c) If $E(G) \geq 4$ and $\tilde{G}(\phi)$ is non overlapping

$$\sum_{J \in \mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_0 \leq \mathrm{Pol}(|j|) M^{j(2-\frac{1}{2}E(G))} \tag{5.39}$$

ii) When $\tilde{\mathcal{M}}(\tilde{G}(\phi))$ is overlapping and $E(G) = 2m \geq 4$, then for all scales $j_2, \ldots, j_m < 0$ and for all scales $\bar{\jmath} \geq \max\{j_2, \ldots, j_m\}$)

$$\sum_{J \in \mathcal{J}(t,G,\bar{\jmath})} \mathcal{N}^{m-1,1}_{j_2 \cdots j_m,0}(\mathcal{M}(\partial_{l_1 a_1} G^J)) \leq \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{j}|) M^{\sum_{r=2}^m 2j_r + \bar{\jmath}(2-m) + \underline{j}(\epsilon-1)} \tag{5.40}$$

where $\underline{j} = \min\{j_2, \ldots, j_m\}$. Moreover for $E(G) = 2m \geq 4$ and $\tilde{G}(\phi)$ not necessarily overlapping

$$\sum_{J \in \mathcal{J}(t,G,\bar{\jmath})} \mathcal{N}^{m-2,2}_{j_3 \cdots j_m,0}(\mathcal{M}(\partial_{l_1 a_1} G^J)) \leq \mathrm{Pol}(|\underline{j}|) M^{\sum_{r=1}^{m-2} 2j_r + \bar{\jmath}(2-m) - \underline{j}} \tag{5.41}$$

where $\underline{j} = \min\{j_3, \ldots, j_m\}$ if $m \geq 3$, $\underline{j} = \bar{\jmath}$ for $m = 2$.

iii) When $E(G) = 2m \geq 4$ and $\tilde{\mathcal{M}}(G(\phi))$ is overlapping, then for all $j_1, \ldots, j_m < 0$ such that $\underline{j} = \min\{j_1, \ldots, j_m\} = j_1$ when $t' < 1$ and all $\bar{\jmath} \geq \max\{j_1, \ldots, j_m\}$).

$$\sum_{J \in \mathcal{J}(t,G,\bar{\jmath})} \mathcal{N}^{m,0}_{j_1 \cdots j_m,0}(\mathcal{M}(\partial^2_{l_1 a_1 l_2 a_2} G^J), t')$$

$$\leq (t')^{\epsilon-1} \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{j}|) M^{\sum_{r=1}^m 2j_r + \bar{\jmath}(2-m) - j^* + \underline{j}(\epsilon-1)} \tag{5.42}$$

$$\leq (t')^{\epsilon-1} \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{j}|) M^{\sum_{r=1}^m 2j_r + \bar{\jmath}(2-m) + \underline{j}(\epsilon-2)} \tag{5.43}$$

where $\underline{j} = \min\{j_1, \ldots, j_m\}$, $j^* = \min\{j_1, \ldots, j_m\} \setminus \{j_i\}$, with i chosen such that $j_i = \underline{j}$.

iv) When $E(G) = 2m = 2$, then for all $j_1 < 0$ and all $\bar{\jmath} \geq j_1$.

$$\sum_{J \in \mathcal{J}(t,G,\bar{\jmath})} \mathcal{N}^{1,0}_{j_1,2}(\mathrm{Val}(G^J), t') \leq (t')^{\epsilon-1} \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j_1|) M^{2j_1+\bar{\jmath}-\bar{\jmath}+j_1(\epsilon-1)} \tag{5.44}$$

$$\leq (t')^{\epsilon-1} \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j_1|) M^{2j_1+\bar{\jmath}+j_1(\epsilon-2)} \tag{5.45}$$

Proof. We prove this by induction on the depth P of the pair (t, G) defined by

$$P = \max\{k | \exists f_1 > \ldots > f_k > \phi \text{ with } E(G_{f_i}) = 2 \text{ for } 1 \leq i \leq k\}. \tag{5.46}$$

This proof will be analogous to that of Theorem 2.46 of [FST96] with the addition that the vertex functions no longer have bounded derivatives and that the derivatives must rerouted to avoid phonon lines. As we are not interested in having very strong bounds for the constants involved, no special care is taken in bounding overlapping four legged graphs. For that reason four-legged graphs do not appear in the definition of the depth P used here.

We start off the induction at $P = 0$. Then proceed on a case by case basis. We will denote by s the number of derivatives acting on the graph, i.e. $s = s'$ in (5.38), $s = 0$ in (5.39), $s = 1$ in (5.40) and (5.41), and $s = 2$ in (5.43) and (5.44).

Graphs of depth zero

Case A: $P = 0, s = 0$ and either $\tilde{G}(\phi)$ is overlapping, i.e. G is overlapping at root scale, or $E(G) = 2$ and $j = 0$. Using the improved power counting bounds we have

$$\sum_{J \in \mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_0 \leq \mathrm{const}\, |j| M^{(1+D_\phi)j} \sum_{J \in \mathcal{J}(t,G,j)} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \tag{5.47}$$

by bounding all phonon propagators by 1.

However by Lemma 55 there is at least one phonon line k contained a loop when a graph is overlapping. Likewise by Lemma 50 there is a phonon line in a loop for $j = 0$. Let j^* be the highest scale where D is contained a loop. Then by Lemma 47 (or Lemma 54 if $j = 0$) the l_0 integral in this loop can be bounded by $\mathrm{const}\, \lambda_2(c,\beta,\epsilon) M^{\epsilon j^*}$ instead of $\mathrm{const}\, M^{j^*}$. Using $M^{\epsilon j^*} = M^{(\epsilon-1)j^*} M^{j^*}$ we thus get the bound

$$\sum_{J \in \mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_0 \leq \mathrm{const}\, \lambda_2(c,\beta,\epsilon) |j| M^{(1+D_\phi)j} M^{(\epsilon-1)j^*} \sum_{J \in \mathcal{J}(t,G,j)} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \tag{5.48}$$

using $(\epsilon - 1) \leq 0$ and $j \leq j^*$.

$$\leq \text{const}\, \lambda_2(c,\beta,\epsilon)|j| M^{D_\phi j} M^{\epsilon j} \sum_{J\in\mathcal{J}(t,G,j)} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \tag{5.49}$$

Remember that $D_f = 2 - \frac{1}{2}(E(G_f^J))$. As $P = 0$, $D_f \leq 0$ for all $f > \phi$ and thus

$$\sum_{J\in\mathcal{J}(t,G,j)} |\text{Val}(G^J)|_0 \leq \lambda_2(c,\beta,\epsilon)\,\text{Pol}(|j|) M^{(2-\frac{1}{2}E(G)+\epsilon)j} \tag{5.50}$$

which gives part ib and ia for the case $\tilde{G}(\phi)$ is overlapping or $j = 0$. The case $E(G) = 2$, $\tilde{G}(\phi)$ overlapping and $j < 0$ cannot occur at depth $P = 0$ because G is 1PI it consists of a single (effective) vertex with generalized self contractions. As $P = 0$ these strings cannot contain two-legged insertions and therefore they must be single wick lines. However because of Wick ordering these cannot occur at scales $j < 0$.

Case B: $P = 0$,$s = 0$, $E(G_f) \geq 4$ and $\tilde{G}(\phi)$ is non-overlapping. Proposition ic follows directly from Lemma 23 because there are two-legged subgraphs.

Case C: P=0,s=1, $E(G) = 2$ and $\tilde{G}(\phi)$ is overlapping or $j = 0$. By the remark above this is all of item ia at depth $P = 0$. As Using Corollary 43 we see that for all $a \in \{0,\ldots d\}$

$$|\frac{\partial}{\partial p_a}\text{Val}(G^J)(p)| \leq \sum_{l_1\in F_2(t,G)} |\text{Val}(\partial_{l_1 a} G^J)| \tag{5.51}$$

and treat this case together with Case D (see the remark at the end of Case D).

Case D: $P = 0$,$s = 1$ and $E(G) \geq 4$. Fix a spanning tree T for G^J compatible to the scales. Because $P = 0$, l_1 is either a hard line or a soft line. If l_1 is hard line set $l = l_1$ and $\hat{G} = G$, and proceed with (5.53). If l is a soft line then we need to another integration by parts using Lemma 44.

Denote by $A(G)$ a graph with the same structure as G but such that at each line of G the absolute value of the propagator is taken when computing the value. This does not change the power counting behavior of the graph. Because l_1 is a soft line there is a loop L (consisting of hard fermion and boson lines) in G that contains l_1. Then from Lemma 44 we have:

$$|\text{Val}(\partial_{l_1 a} G^J)| \leq M^{-j_{l_1}} |\text{Val}(A(\tilde{G}^J))| + \sum_{a'=1}^{d} \sum_{l\in L\setminus l_1} |\text{Val}(A(\partial_{la'}\hat{G}^J))| \tag{5.52}$$

where $\tilde{G}$ is the graph G, apart from the the propagator on line l_1 which has been replaced by $\mathcal{C}_{<j_{l_1},0}$. Again this does not change the power counting of the graph.

Proceed by bounding each of $\mathcal{N}^{m-z,z}_{j_{z+1}\cdots j_m,0}(\mathcal{M}(A(\partial_{la}\hat{G}^J)))$ separately using power counting. Starting with $z=1$

$$\mathcal{N}^{m-1,1}_{j_{z+1}\cdots j_m,0}(\mathcal{M}(A(\partial_{la}\hat{G}^J))) = \mathrm{Val}(\bar{G}_l)\prod_{i=z+1}^{m} M^{j_i} \tag{5.53}$$

where the two-legged labeled graph, $\bar{G}_l$, is obtained from $A(\partial_{la}\hat{G}^J)$ by constructing the graph $\tilde{\mathcal{M}}(A(\partial_{la}\hat{G}^J))$ as described above and attaching $M^{-j_i}\mathbb{1}_{j_i}(|p_{i,0}|)\mathbb{1}_{j_i}(|e(\mathbf{p}_i)|)$ to the line added with momentum p_i ($i=z+1\dots m$). The line now has scale j_i.

There are two cases:

l **is a Fermion line** . Then the derivative gives rise to a factor const M^{-j_l}, where j_l is the scale assigned to l. As there are no C-forks allowed ($P=0$), $M^{-j_l}\le M^{-\bar{\jmath}}$. If $j<0$ by assumption $\tilde{\mathcal{M}}(\tilde{G}(\phi))$ is overlapping and thus $\bar{G}_l$ is overlapping at the root scale of G^J, i.e. scale $\bar{\jmath}$. Moreover by Lemma 55, there exits at least one phonon line k in $\bar{G}_l$ that is in a loop. Denote by j^* the lowest scale at which k is in a loop. Note that j^* can be one of the $j_2\dots j_m$.

We can apply the improved power counting at scale $\bar{\jmath}$ and Lemma 47 together because the former only uses the momentum part of the integral and the latter only the frequency part. Taken together they give

$$\begin{aligned}\mathcal{N}^{m-1,1}_{j_2\cdots j_{m-1},0}(A(\partial_{la}\hat{G}^J)) &\le \mathrm{Val}(\bar{G}_l)\prod_{i=2}^{m} M^{j_i}\\ &\le \text{const}\,(|\bar{\jmath}|+1)M^{\bar{\jmath}}M^{-j_l}\lambda_2(c,\beta,\epsilon)M^{(\epsilon-1)j^*}\prod_{i=2}^{m} M^{2j_i}M^{D_\phi\bar{\jmath}}\prod_{f>\phi} M^{D_f(j_f-j_{\pi f})} &(5.54)\\ &\le \text{const}\,(|\bar{\jmath}|+1)\lambda_2(c,\beta,\epsilon)M^{(\epsilon-1)\underline{j}}\prod_{i=1}^{m} M^{2j_i}M^{D_\phi\bar{\jmath}}\prod_{f>\phi} M^{D_f(j_f-j_{\pi f})} &(5.55)\end{aligned}$$

By assumption $D_f\le 0$ for all f thus the sum over all labelings $J\in\mathcal{J}(\tau,G,\bar{\jmath})$ converges and (5.40) follows because $D_\phi=2-m$ and $(\epsilon-1)=(\epsilon-s)$. The first term in (5.52) is bounded in the same way as the factor $M^{-j_{l_1}}$ is the same as would have come from a derivative on a hard line of scale j_{l_1}.

For $z=2$ not enough momenta are integrated over to make the graph overlapping. Likewise there is no guarantee that there is a phonon line in a loop. However that does not matter as for (5.41) no improved power counting is needed.

l is a phonon line The derivatives of $D(p_{l,0}, c|\mathbf{p}_l|)$ are not bounded. We will bound them using Remark 49/Lemma 48. Recall that this case comes from (5.52) and thus $a \geq 1$ and l is contained in a loop in G.

Let j^* be the highest scale such that l is contained in a loop. As before this can be one of $j_2 \ldots j_m$. Then by Remark 49 there is an additional factor const $\lambda_2(c,\beta,\epsilon)(|j^*|+1)M^{(\epsilon-1)j^*}$ with respect to the power counting of the graph without a derivative acting on l. That is, one loop is used up to get the statement of Remark 49. Note again that this loop could have been one of the overlapping loops and thus we can no longer get the volume improvement from this. However the fact that the extra factors in this case are less divergent is sufficient to get the required result.

$$\mathcal{N}^{m-1,1}_{j_2\cdots j_m,0}(A(\partial_{l_a}\hat{G}^J)) \leq \mathrm{Val}(\bar{G}_j)\prod_{i=2}^{m} M^{j_i}$$

$$\leq \text{const}\,\lambda_2(c,\beta,\epsilon)(|j^*|+1)M^{(\epsilon-1)j^*}\prod_{i=2}^{m} M^{2j_i} M^{D_\phi \bar{\jmath}}\prod_{f>\phi} M^{D_f(j_f - j_{\pi f})} \tag{5.56}$$

$$\leq \text{const}\,\lambda_2(c,\beta,\epsilon)(|\underline{j}|+1)\prod_{i=1}^{m} M^{2j_i} M^{(2-m)\bar{\jmath}} M^{(\epsilon-s)\underline{j}}\prod_{f>\phi} M^{D_f(j_f - j_{\pi f})} \tag{5.57}$$

which gives (5.40) when summed over all J.

When $z=2$ then we use the same bound with $\epsilon = \frac{1}{2}$ and throw away a factor of $M^{\frac{1}{2}j^*} \leq 1$.

Note that $E(G)=2$ is just a pathological version of this. There is no integration over the external vertex, but the graph is already two legged and overlapping. Insertion the bound in (5.51) gives Case C item ia.

Case E: $P=0$, $s=2$, $E(G)=2m=2$. Because $P=0$, $\tilde{G}(\phi)$ is overlapping or $j=0$ as discussed at the end of Case A. As G is a two legged graph we can use Corollary 43 to bound

$$\mathcal{N}^{1,0}_{j_1,2}(\mathcal{M}(G^J)) \leq \sum_{a_1,a_2=0}^{d}\ \sum_{l_1,l_2\in F_2(t,G)} \mathcal{N}^{1,0}_{j_1,0}(\mathcal{M}(\partial^2_{l_1 a_1 l_2 a_2} G^J)) \tag{5.58}$$

Proceed by bounding the terms in the sum on the right together with

Case F: $P=0$,$s=2$, $E(G_f)=2m\geq 2$ and $\tilde{\mathcal{M}}(\tilde{G}(\phi))$ is overlapping.

As before we have to deal with the case where l_1 or l_2 is a soft line first. If only one of the lines is a soft line then we can integrate the derivative away using Lemma 44 just as in (5.52). If both lines are soft lines more care is needed as we want to make sure no second derivative is taken of D_1. Let $\mathcal{K}(G)$ be the set of all labelings $K : L_B(G) \to \{0,1\}$ of the phonon lines of G. Then define $G^{J,K}$ as the graph where each phonon line l has $D_{K(l)}$ as the propagator. Choose a spanning tree compatible with the scales in the usual way and fix momenta accordingly. When $l_1 \neq l_2$ both l_1 and l_2 generate a loop (L_1 and L_2 respectively) in G that can contain both Fermion and Boson lines. When $l_1 = l_2$ then set $L_1 = L_2$ to the loop generated. We apply Lemma 45 in these loops/ this loop. If $l_1 = l_2$ we apply the first half where g contains all the propagators from the Fermion lines in $L_1 \setminus \{l_1\}$ and all the propagators from Boson lines $l \in L_1$ with $K(l) = 0$, and h is the product of all Boson lines $l \in L_1$ with $K(l) = 1$. If $l_1 \neq l_2$ we apply the second half with g the product of the propagators belonging to the Fermion lines in $(L_1 \cup L_2) \setminus \{l_1, l_2\}$ and all the boson lines $l \in L_1 \cup L_2$ with $K(l) = 0$; For $i = 1,2$ take h_i the product of propagators of the boson lines $l_1 \in (L_1 \cup L_2) \cap (L_i \setminus (L_1 \cap L_2))$ with $K(l) = 1$. Finally take h_3 the product of the boson propagators of $l \in L_1 \cap L_2$ with $K(l) = 0$.

Lemma 45 makes sure that none of the propagators in h,h_1,h_2, or h_3 has more than one derivative acting on it. In addition the only derivatives acting on phonon propagators are with respect to components of the vector momentum. Thus we have to bound either

$$I_{l_1=l_2} = \mathcal{N}^{m,0}_{j_1 \cdots j_m,2}(\mathcal{M}(A(\partial^{s_1}_{l'_1 m'_1} \partial^{s_2}_{l'_2 m'_2} G^{J,K}\Big|_{P_{l_1}=M^{-j_{l_1}(s_3+s_4)}C_{<j_{l_1}},s_5}))) \tag{5.59}$$

or

$$I_{l_1 \neq l_2} = \mathcal{N}^{m,0}_{j_1 \cdots j_m,2}(\mathcal{M}(A(\partial^{s_1}_{l'_1 m'_1} \partial^{s_2}_{l'_2 m'_2} G^{J,K}\Big|_{P_{l_1}=M^{-j_{l_1}s_3}C_{<j_{l_1}},0, P_{l_2}=M^{-j_{l_2}s_4}C_{<j_{l_2}},s_5}))) \tag{5.60}$$

Here P_l denotes the propagator on line l, l'_1 and l'_2 are hard Fermion lines or phonon lines, $s_i \leq 1$, $s_1 + s_2 + s_3 + s_4 + s_5 = 2$, $s_1 + s_3 \leq 1$, $s_2 + s_4 \leq 1$, $m'_i \neq 0$ when l'_i a phonon line. $s_5 = 1$ only when $s = 1$, l'_1 is a phonon line and $K(l'_1) = 1$. The cases where one of the lines, say l_2, is a hard line produce similar terms where we can set $l'_2 = l_2$, $m'_2 = m_2$ and possibly leave off the replacement of the propagator P_{l_2} on line l_2.

In the following we write I for all these terms. It is sufficient to show the required bound follows when each I is summed over J because the sum over the labelings K, and over the l'_i, s_i's, and m'_i's where appropriate, is finite. For the purposes of power counting it is also again useful to view the resulting expression as the value of a graph obtained by extending G.

$$I = \mathrm{Val}(\bar{G}) \prod_{i=1}^{m} M^{j_i} \tag{5.61}$$

where $\bar{G}$ is the graph $\bar{\mathcal{M}}(A(\partial^{s_1}_{l'_1 m'_1}\partial^{s_2}_{l'_2 m'_2} G^{J,K})$ with replacement of the propagators of l_1 and l_2 if applicable, and in computing each value each new (dashed) line is taken to have the propagator $M^{-j_i}\mathbb{1}_{j_i}(|p_{i,0}|)\mathbb{1}_{j_i}(|e(\mathbf{p}_i)|)$ when p_i is the (formerly external) momentum flowing through the line. Note that $\bar{G}$ has no external legs. The method to bound $\mathrm{Val}(G)$ depends on the type of the lines l'_1 and l'_2.

At least one of l'_1,l'_2 is a Fermion line or $s_3 + s_4 \geq 1$, $s_5 = 0$. Assume WLOG l'_1 is a Fermion line (in case $s_3 + s_4 \geq 1$ the power counting behaves as if a hard line of scale j_{l_1} or j_{l_2} was differentiated). Then the derivative at l'_1 gives rise to a factor const $M^{-j_{l'_1}}$. Note also that

$$\mathcal{N}^{m,0}_{j_1\cdots j_m,0}(\mathcal{M}(G^J)) \leq M^{2j_1}\mathcal{N}^{m-1,1}_{j_2\cdots j_m,0}(\mathcal{M}(G^J)) \tag{5.62}$$

when the right hand side is finite. This is the trivial bound for the effect of the extra integration over the remaining external fermion line. Thus the power counting for I is the same as for const $M^{2j_1}M^{-j_{l'_1}}\mathcal{N}^{m-1,1}_{j_2\cdots j_m,0}(\mathcal{M}(A(\partial^{s_1}_{l'_1 m'_1}\partial^{s_2}_{l'_2 m'_2} G^{J,K}))$. Note that $l'_1 \in G$ and thus const $M^{2j_1}M^{-j_{l'_1}} \leq$ const $M^{2j_1}M^{-\bar{\jmath}} \leq$ const $M^{2j_1 - j^*}$ which is exactly the extra factor in (5.43) relative to (5.40).

Both lines are phonon lines, $s_5 = 0$ This is the new aspect of this case; there are two derivatives on phonon lines. The effect of this will be bounded using Lemma 51 and Remark 52. However this is not always needed:

Pick a spanning tree T compatible to the scales, fix momenta and bound all propagators on the tree except those for l'_1 and l'_2 by their suprema.

Consider the case there are two loops $\mathfrak{L}_1, \mathfrak{L}_2 \in \bar{G}$, such that $\mathfrak{L}_1 \neq \mathfrak{L}_2$ and $l'_i \in \mathfrak{L}_i$ for $i = 1, 2$ and $l'_2 \notin \mathfrak{L}_1$ or vice versa. Without loss of generality we can assume the former. Note that this implies that $p_{l'_2}$ does not depend on the loop momentum of $\mathfrak{L}_1$. Thus we can apply lemma 44 once with $\epsilon = \frac{1}{2}$ and once with ϵ as given by the preconditions of the theorem. This gives a factor

$$\lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|\underline{k}|)M^{(\epsilon-1)\underline{k}}M^{-\frac{1}{2}\bar{k}} \leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|\underline{k}|)M^{(\epsilon-1)\underline{k}}M^{-\bar{k}} \tag{5.63}$$

in addition to the ordinary power counting without the derivatives. Here $\underline{k}$ and $\bar{k}$ are defined as follows. Let k_i the lowest scale in $\mathfrak{L}_i$ or equivalently the highest scale where $\mathfrak{L}_i$ appears. Then $\underline{k} = \min\{k_1, k_2\}$ and $\bar{k} = \max\{k_1, k_2\} \geq j^*$.

If there are no such two loops then either

a) l_1' is not contained in two loops.

b) There exists loops $\mathfrak{L}_1, \mathfrak{L}_2 \in \bar{G}$ with $l_1' \in \mathfrak{L}_1 \cap \mathfrak{L}_2$, but only such that $l_2' \in \mathfrak{L}_1 \cap \mathfrak{L}_2$ too.

First we show that case a cannot occur. Observe that by construction $l_1' \in L_1$. Set $\mathfrak{L}_1 = L_1$. Assume that there exists no other loop $\mathfrak{L}_2 \in \bar{G}$ that contains l_1'. $\mathfrak{L}_1 \setminus T = l_1$. The graph $H = \bar{G} \setminus l_1$ contains l_1' but not in a loop. However H is two legged and thus by Lemma 50 a tadpole graph that is disconnected if l_1' is cut. This implies that l_1' does not depend on p_{l_1}, which is a contradiction.

Thus we must be in case b. Note that this implies that $p_{l_1'} = \pm p_{l_2'}$. Therefore the integral is zero when $K(l_1') \neq K(l_2')$ because $\operatorname{supp} D_0 \cap \operatorname{supp} D_1 = \emptyset$. By construction $K(l_1') = K(l_2') = 1$ cannot occur in case b. Therefore $K(l_1') = K(l_2') = 0$ and we can apply Lemma 51 and get:

$$I \leq \text{const}\,(|\underline{k}| + 1)^2 (\frac{c}{t'})^{1-\epsilon} M^{(\epsilon-1)\underline{k}-\bar{k}} \prod_{i=1}^{m} M^{2j_i} M^{D_\phi \bar{\jmath}} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \tag{5.64}$$

$$\leq \text{const}\,(|\underline{j}| + 1)^2 (\frac{c}{t'}) M^{(\epsilon-1)\underline{j}} M^{-j^*} M^{\sum_{i=1}^{m} 2j_i} M^{(2-m)\bar{\jmath}} \prod_{f>\phi} M^{D_f(j_f - j_{\pi(f)})} \tag{5.65}$$

and (5.43) follows after summing over the scales, l_1', l_2' and the type of derivatives.

Otherwise, i.e, $s_5 = 1$ $s_5 = 1$ implies that l_1' is a phonon line, $s1 = K(l_1') = 1$. No second derivative occurs as $s_2, s_3, s_4 = 0$. Bound the propagators on all lines of spanning tree except l_1' by their suprema. Apply Corollary 53 to the integral over p_{l_2} and ordinary power counting to the remaining integrals. In ordinary power counting the p_{l_2} integral would have produced a factor $M^{j_{l_2}}$. Thus the difference is given by

$$\text{const}\,\frac{(\log\beta + 1)^2}{\beta} M^{-j_{l_2}} \leq \text{const}\,\lambda_2(c, \beta, \epsilon) M^{(\epsilon-1)\underline{j}} M^{-j^*} \tag{5.66}$$

And thus the bound follows after carrying out the remain summations over the scales, K, l_1', and m_1'.

Overlapping higher depth graphs

Now turn to the cases with $P > 0$. Assume as the induction Hypothesis that the propositions of the theorem have all been proven for graphs with depth $P' < P$.

Construct the graph G' as above. By construction G' has depth $P = 0$ and the new vertices have depth $P' < P$ if not a SSI and can be bounded using the induction hypothesis. Moreover

$$\sum_{J\in\mathcal{J}(t,G,j)} \mathrm{Val}(G^J) = \sum_{J\in\mathcal{J}(\tau,G',j)} \mathrm{Val}((G')^J) \tag{5.67}$$

Case G: $P > 0$, $s = 0$ and $\tilde{G}(\phi)$ is overlapping or $j = 0$. $G'(\phi)$ is also overlapping. The power counting for $G'^{,J}$ is as in Case A, with extra factors coming from $M^{-j_{\pi(f)}}\,\mathrm{Val}(v_f)$ for each two legged vertex. The effects from these on the scale sum are bounded using the induction hypothesis with $\epsilon = \epsilon' = \frac{1}{2}$ to show that they are bounded by at most a polynomial in the scale:

- If v_f is from a C fork the extra factor is

$$M^{-j_{\pi(f)}} \sum_{j_f<j_{\pi(f)}} \sum_{J\in\mathcal{J}(t_f,G_f,j_f)} |L\,\mathrm{Val}(G_f^J)|_{0,j_{\pi(f)}} \tag{5.68}$$

$$\le \sum_{j_f<j_{\pi(f)}} \sum_{J\in\mathcal{J}(t_f,G_f,j_f)} M^{-j_{\pi(f)}}|\mathrm{Val}(G_f^J)|_0 + |\mathrm{Val}(G_f^J)|_1 \tag{5.69}$$

where equation (5.13) was used. Applying the IH for $\epsilon = \epsilon'$

$$\le \sum_{j_f<j_{\pi(f)}} \lambda_2(c,\beta,\epsilon')\,\mathrm{Pol}(|j_f|)(M^{(1+\epsilon')j_f-j_{\pi(f)}} + M^{\epsilon' j_f}) \tag{5.70}$$

$$= \lambda_2(c,\beta,\epsilon') \sum_{j_f<j_{\pi(f)}} \mathrm{Pol}(|j_f|)M^{\epsilon' j_f} \tag{5.71}$$

and setting $\epsilon' = \frac{1}{2}$

$$\le \lambda_2(c,\beta,\frac{1}{2})\,\mathrm{Pol}(|j_{\pi(f)}|) \tag{5.72}$$

- If v_f is from an R fork the extra factor is bound using (5.14) as follows

$$M^{-j_{\pi(f)}} \sum_{j_f>j_{\pi(f)}} \sum_{J\in\mathcal{J}(t_f,G_f,j_f)} \left|(1-L)\,\mathrm{Val}(G_f^J)(p)\right|_{0,j_{\pi(f)}}$$

$$\le \sum_{j_f>j_{\pi(f)}} \sum_{J\in\mathcal{J}(t_f,G_f,j_f)} |\mathrm{Val}(G_f^J)|_1 \le \lambda_2(c,\beta,\epsilon') \sum_{j_f>j_{\pi(f)}} \mathrm{Pol}(|j_f|)M^{\epsilon' j_f}$$

$$\le \mathrm{const}\,\lambda_2(c,\beta,\epsilon') \tag{5.73}$$

where we have set $\epsilon' = 1$ in the last line.

- If v_f is from a SSI then apply normal power counting to the graph G_f to see that the extra factor is

$$M^{-j_{\pi(f)}} M^{j_f} \leq \text{const} \tag{5.74}$$

because for these SSI's $j_f = j_{\pi(f)} + 1$.

Case H: $P > 0$, $s = 1$ and $E(G) \geq 4$ or $E(G) = 2$ and $G(\phi)$ is overlapping or $j = 0$. As above the power counting for G' reduces to Case D. The only new factors are either as above, or when instead of a line l one of the vertices v_f is differentiated. Note that because of the spherical symmetry $|LG_f|_1 \leq |G_f|_1$, i.e. no second derivative appears. This gives the following bounds

- If v_f is from a C fork then we use the IH with $\epsilon = 1$ (this is allowed because $s = 1$)

$$M^{-j_{\pi(f)}} \sum_{j_f < j_{\pi(f)}} \sum_{J \in \mathcal{J}(t_f, G_f, j_f)} |\text{Val}(G_f^J)|_1 \\ \leq M^{-j_{\pi(f)}} \sum_{j_f < j_{\pi(f)}} \text{Pol}(|j_f|) M^{j_f} \leq \text{const}\, \text{Pol}(|j_{\pi(f)}|) \tag{5.75}$$

- If v_f is from an R fork the extra factor is

$$M^{-j_{\pi(f)}} \sum_{j_f > j_{\pi(f)}} \sum_{J \in \mathcal{J}(t_f, G_f, j_f)} |\text{Val}(G_f^J)|_1 \leq \text{const}\, \text{Pol}(|j_{\pi(f)}|) M^{-j_{\pi(f)}} \tag{5.76}$$

  Thus the effect (apart from polynomial factors) is the same as if one of the fermion lines of $G_{\pi(f)}$ was differentiated.

- If v_f is from a SSI then apply the lemma to the smaller graph G_f to see that the extra factor is

$$M^{-j_{\pi(f)}} M^{j_f} \leq \text{const} \tag{5.77}$$

because for these SSI's $j_f = j_{\pi(f)} + 1$.

Case I: $P > 0$, $s = 2$, $E(G) = 4$ or $E(G) = 2$ and $\tilde{G}(\phi)$ overlapping. What happens when one of the derivatives acts on a new vertex v_f that is not hit by the other is discussed above. If any of the new vertex gets differentiated twice, observe that because G' is 1PI each two legged vertex only occurs in a loop in G'.

Note that by the spherical symmetry all second derivatives of $L\,\text{Val}(v_f)$ are zero. Thus if v_f corresponds to a C fork the integral vanishes when both derivatives act on v_f. Moreover if v_f

corresponds to an R-fork

$$\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}(1-L)\,\mathrm{Val}(G_f)=\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\,\mathrm{Val}(G_f) \tag{5.78}$$

Thus only the full $\mathrm{Val}(G_f)$ appears in the bound and not the Taylor error term. This means that interpolated values do not appear. The IH applies with $t'=1$ and can thus be used with $\epsilon=1$. For a SSI the pre-factor was 1 from the start.

If v_f corresponds to an R fork, observe that $\mathrm{Val}(A(\partial_{v_f a_1 v_f a_2}G^J))$ contains the subintegral

$$M^{-j_\pi(f)}\int \mathrm{d}_\beta^{d+1}p|C_{j^*,s^*}(p)||\frac{\partial}{\partial p_{a_1}}\frac{\partial}{\partial p_{a_2}}\,\mathrm{Val}(v_f)| \tag{5.79}$$

for some line j^* of G' which if $s^*=0$ is bounded by

$$M^{-j_\pi(f)}M^{-j^*}\sum_{J\in\mathcal{J}(t_f,G_f,j_f)}\mathcal{N}^{1,0}_{j^*,2}(\mathrm{Val}(G_f^J)) \tag{5.80}$$

which by the induction hypothesis is bounded by

$$\lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j^*|)M^{\epsilon j^*}M^{j_f}M^{-j_\pi(f)} \tag{5.81}$$

which is an extra factor $\lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j^*|)M^{(\epsilon-1)j^*}M^{-j_\pi(f)}$ compared to the normal power counting and thus this term obeys the bound. When $s^*=1$ is bound is analogous. A SSI is treated in exactly the same way.

Non overlapping graphs

Case J: $P>0$,$s=0$, $E(G_f)\geq 4$ and $\tilde{G}(\phi)$ is non-overlapping. As in Case G the power counting is as in the $P=0$ case because the two-legged subgraphs give at most polynomials in their root scale.

Note that below only graphs with root scale $j<0$ occur because scale zero graphs can always be treated as if they were overlapping because $M^{\epsilon j}=1$ and no gain is needed.

Case K: $P>0$,$s=0$ and $E(G)=2$, G is 1PI, $j<0$ and $\tilde{G}(\phi)$ is non overlapping. Note that in this case $\epsilon<1$. G is a GST graph and thus $\mathrm{Val}(G^J)$ has the structure (3.82). At least one of the strings has scale $j_i=j$.

G_f in (3.82) is either a single vertex or G_f is such that expanding to the root scale j_f of G_f would make G overlapping. If the strings of two legged diagrams in G are replaced by dashed lines, this gives $\tilde{\mathcal{M}}(G_f)$. This then implies that $\tilde{\mathcal{M}}(\tilde{G}_f(f))$ is overlapping (Note that f is by definition the root of G_f). However because of the power counting gains that the L operator gives to the strings we will not really need this detailed structure here. In the following we write $u = f$ and $U = G_f$.

Applying the first half of Lemma 57 to each of the strings gives the bound

$$|\mathrm{Val}\, G^J(q)|_0 \leq \mathrm{const}\, \mathcal{N}^{m'-1,1}_{j_2\cdots j_{m'},0}(\mathcal{M}(U)) \prod_{r=2}^{m'} M^{-j_r} \prod_{k=1}^{w_r-1} Q_{r,k} \tag{5.82}$$

with

$$Q_{r,k} = \begin{cases} |T_{r,k}|_1 & \text{when } f_{r,k} \text{ is an R-fork} \\ M^{-j_r}|T_{r,k}|_0 + |T_{r,k}|_1 & \text{when } f_{r,k} \text{ is a } C\text{-fork} \\ M^{-j_r}|T_{r,k}|_0 & \text{when } T_{r,f} \text{ is a SSI} \end{cases} \tag{5.83}$$

The sum over all scale assignments to (G, t) factors into the sum over all scale assignment to subgraphs of U, all scale assignments to $G_{r,k}$ and sums over the j_r's. Applying this to (5.82) gives

$$\sum_{J\in\mathcal{J}(t,G,j)} |\mathrm{Val}\, G^J(q)|_0$$

$$\leq \mathrm{const} \sum_{\substack{\{j_r\}_{r=1}^{m'} \\ \min\{j_r\}=j}} \sum_{j_u>\max\{j_r\}} \sum_{J_u\in\mathcal{J}(t_u,U,j_u)} \mathcal{N}^{m'-1,1}_{j_2\cdots j_{m'},0}(\mathcal{M}(U)) \prod_{r=2}^{m'} M^{-j_r} \prod_{k=1}^{w_r-1} \mathcal{Q}_{r,k} \tag{5.84}$$

with

$$\mathcal{Q}_{r,k} = \begin{cases} \sum_{j_{r,k}>j_r} \sum_{J_{r,k}\in\mathcal{J}(t_{r,k},G_{r,k},j_{r,k})} |T_{r,k}|_1 & \text{when } f_{r,k} \text{ is an R-fork} \\ \sum_{j_{r,k}<j_r} \sum_{J_{r,k}\in\mathcal{J}(t_{r,k},G_{r,k},j_{r,k})} (M^{-j_r}|T_{r,k}|_0 + |T_{r,k}|_1) & \text{when } f_{r,k} \text{ is a C-fork} \\ M^{-j_r} \sum_{J_{r,k}\in\mathcal{J}(t_{r,k},G_{r,k},j_r)} |T_{r,k}|_0 & \text{when } T_{r,f} \text{ is a SSI} \end{cases} \tag{5.85}$$

As before using the induction hypothesis and ordinary power counting for the SSI's we see that

$$\mathcal{Q}_{r,k} \leq \mathrm{Pol}(|j_r|) \tag{5.86}$$

However inserting this in (5.84) does not give the desired inequality (5.38). The bound on $\mathcal{N}^{m'-1,1}_{j_2\cdots j_{m'},0}(\mathcal{M}(U))$ gives at most an improvement $\lambda_2(c,\beta,\epsilon)M^{\epsilon j_u}$. However $j < j_u$ and we need to extract the factor elsewhere.

The scale j does exist in the string S_i (because i was chosen such that $j_i = j$). If S_i contains a C fork or a SSI (say corresponding to $T_{i,z}$) then apply the first half of Lemma 57 to (3.82) to obtain

$$\sum_{J\in\mathcal{J}(t,G,j),j_i=j} |\mathrm{Val}\, G^J(q)|_0 \tag{5.87}$$

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=2}^{m'} \\ j_r\geq j_i=j}} \sum_{j_u>\max\{j_r\}} \sum_{J_u\in\mathcal{J}(t_u,U,j_u)} \mathcal{N}^{m'-1,1}_{j_2\cdots j_{m'},0}(\mathcal{M}(U)) \prod_{r=2}^{m'} M^{-j_r} \prod_{k=1}^{w_r-1} \mathcal{Q}_{r,k} \tag{5.88}$$

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=1}^{m'} \\ j_r\geq j_i=j}} \sum_{j_u>\max\{j_r\}} \mathrm{Pol}(|j|) M^{\sum_{r=2}^{m'} j_r + j_u(2-m')} \mathcal{Q}_{i,z} \tag{5.89}$$

applying the IH to $\mathcal{Q}_{i,z}$ if it is a C fork or this case recursively for a SSI

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=2}^{m'} \\ j_r\geq j_i=j}} \sum_{j_u>\max\{j_r\}} \lambda_2(c,\beta,\epsilon)\, \mathrm{Pol}(|j|) M^{\sum_{r=2}^{m'} j_r + j_u(2-m')+\epsilon j} \tag{5.90}$$

$$\leq \lambda_2(c,\beta,\epsilon)\, \mathrm{Pol}(|j|) M^{(\epsilon+1)j} \tag{5.91}$$

When S_i is a string of R-forks, apply the second half of Lemma 57 to obtain (where we have assumed for notational simplicity that $i = 2$).

$$\sum_{J\in\mathcal{J}(t,G,j),j_2=j} |\mathrm{Val}\, G^J(q)|_0 \tag{5.92}$$

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=3}^{m'} \\ j_r\geq j_2=j}} \sum_{j_u>\max\{j_r\}} \sum_{J_u\in\mathcal{J}(t_u,U,j_u)} \prod_{r=3}^{m'} \left(M^{j_r} \prod_{k=1}^{w_r-1} \mathcal{Q}_{r,k} \right) |\mathcal{M}(U)|_{0,j}$$
$$\left(\prod_{k=2}^{w_2-1} \mathcal{Q}_{2,k} \int_0^1 \mathrm{d}t \sum_{j'>j} \sum_{J'\in\mathcal{J}(t_{2,1},G_{2,1},j')} \mathcal{N}^{1,0}_{j,2}(G^{J'}_{2,1},t) \right) \tag{5.93}$$

using the previous bounds and the IH

$$\leq \lambda_2(c,\beta,\epsilon)\, \mathrm{Pol}(|j|) \sum_{\substack{\{j_r\}_{r=3}^{m'} \\ j_r\geq j_2=j}} \sum_{j_u>\max\{j_r\}} M^{\sum_{r=3}^{m'} j_r + j_u(m'-2)} M^{2j+(\epsilon-1)j} \int_0^1 \mathrm{d}t\, t^{\epsilon-1} \tag{5.94}$$

$$\leq \lambda_2(c,\beta,\epsilon)\, \mathrm{Pol}(|j|) M^{(1+\epsilon)j} \tag{5.95}$$

Case L: $P > 0, s = 1$ and $E(G) = 2$, G is 1PI, $j < 0$ and $\tilde{G}(\phi)$ is nonoverlapping. We use the structure (3.82) of the graph. Using Corollary 43 we see that the derivative can written as a sum over terms where it either acts on a fermion line/2-vertex in U or on one of the strings.

$$\left|\frac{\partial}{\partial q_a} \operatorname{Val} G^J(q)\right| \leq I_1 + \sum_{i=2}^{m'} I_{2,i} \tag{5.96}$$

where

$$I_1 = \sum_{k \in F_2(t_u, U^J)} \prod_{r=2}^{m'} \left(\int \mathrm{d}_\beta l_{r,0} \int \mathrm{dl}_r \, |S_r(l_r)|\right) |\mathcal{M}(\partial_{ka} U^J)(q, l_2 \ldots, l_{m'})| \tag{5.97}$$

$$I_{2,i} = \prod_{\substack{r=2 \\ r \neq i}}^{m'} \left(\int \mathrm{d}_\beta l_{r,0} \int \mathrm{dl}_r \, |S_r(l_r)|\right) \int \mathrm{d}_\beta l_{r,0} \int \mathrm{dl}_i \left|\frac{\partial}{\partial l_{i,a}} S_r(l_i)\right| |\mathcal{M}(U)(q, l_2 \ldots, l_{m'})| \tag{5.98}$$

To bound I_1 use (5.84) with $U \to \partial_{ka} U$ and (5.86), which gives

$$\sum_{J \in \mathcal{J}(t,G,j)} I_1 \tag{5.99}$$

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=1}^{m'} \\ \min\{j_r\}=j}} \sum_{j_u > \max\{j_r\}} \sum_{J_u \in \mathcal{J}(t_u, U, j_u)} \mathcal{N}^{m'-1,1}_{j_2 \ldots j_{m'},0}(\mathcal{M}(\partial_{ka} U)) \prod_{r=2}^{m'} M^{-j_r} \prod_{k=1}^{w_r - 1} \mathcal{Q}_{r,k} \tag{5.100}$$

$$\leq \text{const} \sum_{\substack{\{j_r\}_{r=1}^{m'} \\ \min\{j_r\}=j}} \sum_{j_u > \max\{j_r\}} \sum_{J_u \in \mathcal{J}(t_u, U, j_u)} \operatorname{Pol}(|j|) \mathcal{N}^{m'-1,1}_{j_2 \ldots j_{m'},0}(\mathcal{M}(\partial_{ka} U)) \prod_{r=2}^{m'} M^{-j_r} \tag{5.101}$$

and using the IH

$$\leq \text{const}\, \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(|j|) M^{\epsilon j} \tag{5.102}$$

To bound the contribution $I_{2,i}$, apply Lemma 59 to string S_i. Again assuming for simplicity that $i = 2$ we get

$$\sum_{J \in \mathcal{J}(t,G,j)} I_{2,2} \leq \text{const} \sum_{\substack{\{j_r\}_{r=3}^{m'} \\ j_r \geq j}} \sum_{j_u > \max\{j_r\}} \sum_{J_u \in \mathcal{J}(t_u, U, j_u)} \prod_{r=3}^{m'} \left(M^{j_r} \prod_{k=1}^{w_r - 1} \mathcal{Q}_{r,k}\right) |\mathcal{M}(U)|_{0,j}$$
$$M^{-j_2} \sum_{r_1=0}^{w_2 - 1} v(r_1) \prod_{\substack{k=1 \\ k \neq r_1}}^{w_2 - 1} \mathcal{Q}_{2,k} \tag{5.103}$$

where

$$v(r_1) = \begin{cases} \int_0^1 \mathrm{d}t \sum_{j'>j_2} \sum_{J' \in \mathcal{J}(t_{2,1}, G_{2,r_1}, j')} \mathcal{N}^{1,0}_{j,2}(G^{J'}_{2,r_1}, t) & f_{2,r_1} \text{ is an } R\text{-fork.} \\ M^{j_2} \sum_{j'<j_2} \sum_{J' \in \mathcal{J}(t_{2,1}, G_{2,r_1}, j')} |\mathrm{Val}(G_{2,r_1})|_1 & f_{2,r_1} \text{ is a } C\text{-fork.} \\ M^{j_2} \sum_{J' \in \mathcal{J}(t_{2,1}, G_{2,r_1}, j_2)} |\mathrm{Val}(G_{2,r_1})|_1 & f_{2,r_1} \text{ is a SSI.} \end{cases} \tag{5.104}$$

By the induction hypothesis

$$M^{\sum_{r=3}^{m'} j_r} \sum_{J \in (t_u, U, j_u)} |\mathcal{M}(U)|_{0,j} \le M^{-j_2} M^{\sum_{r=2}^{m'} j_r} \sum_{J \in (t_u, U, j_u)} |\mathcal{M}(U)|_{0,j} \le \mathrm{Pol}(|j|) M^{j-j_2} \tag{5.105}$$

$$\sum_{J \in \mathcal{J}(t,G,j)} I_{2,2} \le M^{j-2j_2} \mathrm{Pol}(|j|) \sum_{r_1=0}^{w_2-1} v(r_1) \tag{5.106}$$

$$\le \mathrm{Pol}(|j|) M^{j+j_2(\epsilon-1)} \le \mathrm{Pol}(|j|) M^{\epsilon j} \tag{5.107}$$

Case M: $P > 0, s = 2$ and $E(G) = 2$, G is 1PI, $j < 0$ and $\tilde{G}(\phi)$ is nonoverlapping. This bound is analogous to the previous case. The extra derivative gives rise to another sum over lines in $F_2(t, G)$. If one of the lines, say l_2 is in $U = G_f$, then the bound is simply a repeat of the above with U replaced by $\partial_{l_2 a_2} U$. The extra derivative generates an additional factor $M^{-j_u} \le M^{-\bar{\jmath}}$ and the bound follows.

The new case is where there are two strings, say S_α and S_β that are differentiated, or one string S_α is differentiated twice ($\beta = \alpha$). Assume WLOG that $j_\beta \ge j_\alpha$. Applying Lemma 59 to both (where we take (5.21) for $\beta \ne \alpha$) gives analogous to above

$$\mathrm{Pol}(|j|) M^{j_1} \prod_{\substack{r=1 \\ r \notin \{\alpha,\beta\}}}^{m'} M^{j_r} \sum_{J \in (t_u, U, j_u)} |\mathcal{M}(U)|_{0,j} M^{-j_\alpha - j_\beta} \lambda_2(c,\beta,\epsilon) M^{j_\alpha(1+\epsilon)} M^{(1-\delta_{\alpha\beta}) j_\beta}$$

$$\le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j_1|) M^{2j_1} M^{j_\alpha(\epsilon-2)} \le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j_1|) M^{2j_1} M^{j_1(\epsilon-1)} \tag{5.108}$$

□

5.4 Renormalizability

With the power counting theorem for the renormalized theory from the previous section, the perturbative renormalizability of the theory can be shown easily. As an example we show this for graphs with two external fermion legs. Extending this to more general graphs means taking the appropriate norms.

Theorem 61 (Renormalizability of two legged Graphs). *Let $\Sigma_{2r}(p)$ be the contribution of order $2r$ in g to the Fermion self-energy in electron-phonon theory renormalized according to Definition 22. Then there exists a constant K_{2r} such that*

$$|\Sigma_{2r}|_0 \leq K_{2r}\lambda_2(c,\beta,\frac{1}{2}) \tag{5.109}$$

In particular is K_{2r} independent of β.

Proof. By (3.60), $\Sigma_{2r}(p)$ is given by

$$\Sigma_{2r}(p) = \sum_{j<0}\sum_{t}\prod_{f\in t}\frac{1}{n_f!}\sum_{G \text{ 1PI}}\sum_{J\in\mathcal{J}(t,G,j)} \mathrm{Val}(G^J) \tag{5.110}$$

where the sum is over trees t with $2r$ leaves. Using Proposition i of Theorem 60:

$$|\Sigma_{2r}(p)|_0 \leq \text{const}\sum_{j<0}\sum_{t}\prod_{f\in t}\frac{1}{n_f!}\sum_{G \text{ 1PI}, G\sim t}\lambda_2(c,\beta,\epsilon)M^{j(1+\epsilon)} \tag{5.111}$$

for all $0<\epsilon<1$. As the number of trees with $2r$ leaves and coordination number greater than two at each vertex is bounded and the number of graphs compatible to such a tree is also bounded, we have

$$|\Sigma_{2r}(p)|_0 \leq \text{const}\sum_{j<0}\lambda_2(c,\beta,\epsilon)M^{j(1+\epsilon)} \tag{5.112}$$

Note that there exists a constant κ such for all $c<1$ and $\beta>1$, $\lambda_2(c,\beta,\frac{1}{2})<\kappa$. Therefore

$$|\Sigma_{2r}(p)|_0 \leq \text{const}\,\lambda_2(c,\beta,\frac{1}{2})\sum_{j<0}M^{\frac{3}{2}j} = K_{2r}\lambda_2(c,\beta,\frac{1}{2}) \tag{5.113}$$

□

As the functions Σ_{2r} are uniformly bounded in a neighborhood of $\beta=\infty$, this implies by the Lebesgue dominated convergence theorem that the limit $\lim_{\beta\to\infty}\Sigma_{2r}$ exists and is well defined.

Chapter 6

Bounding vertex corrections

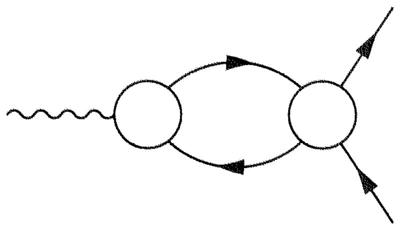
6.1 Strategy

6.1.1 From vertex corrections to four-legged graphs

In the previous chapter the power counting for electron-phonon theory was considered. In Theorem 60 bounds are given for graphs that have only external Fermion lines. Here we will argue that these bounds can still be used to give bounds for the electron-phonon vertex correction.

Like all other graphs, graphs contributing to the vertex correction can contain strings of two legged subgraphs and must be renormalized. Therefore the formalism of the previous is chapter is needed to deal with the counter terms.

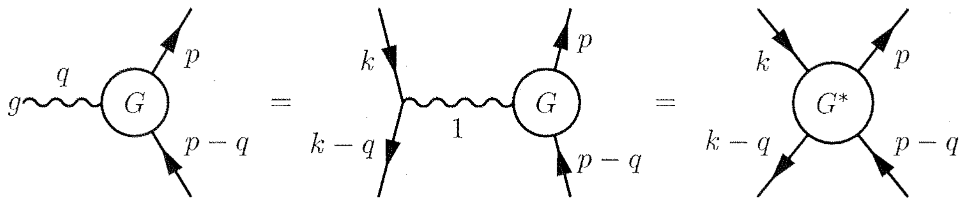
In addition for a large class of electron-phonon vertex-correction graphs, the graph structure is that of a one-loop graph where the electron-phonon vertex has been replaced by a higher order electron-phonon vertex-correction and the phonon line has been replaced by a four legged graph:



We will see below that the graphs that are not of this form are relatively easy to deal with. In the case of the one-loop correction we extract the sign cancellations in the loop by means of integration by parts and then use the properties of the phonon propagator and its derivatives. The more general graphs above are bounded in the same way and therefore the properties of the derivatives of the values of four-legged graphs and of electron-phonon vertex corrections are needed. Theorem 60 provides such bounds for a large class of four-legged graphs.

The purpose of this chapter is to extend those results to other four-legged graphs and to electron-phonon vertex corrections. This will be done by means of an inductive argument. First we note that the distinction between the two types of graphs under consideration can be eliminated by means of a notational trick. Let G any graph contributing to Γ_r, with value $\mathrm{Val}(G)(p,q)$. Let G' be the graph constructed from G adding another vertex connected to the original external phonon line by a phonon propagator. G' has four external fermion legs (and no external phonon legs). The value of G' is given by $\mathrm{Val}(G')(k,k-q,p-q)$ where k is the momentum flowing into the new external vertex. By the structure of the graph and by conservation of momentum $\mathrm{Val}(G')(k,k-q,p-q) = gD(q)\,\mathrm{Val}(G)(p,q)$, in particular the value does not depend on k at all. The factor $D(q)$ comes from the new phonon line l^* connecting the new vertex with G.

Let G^* be the four-legged graph constructed in the same way, but with $P_{l^*} = 1$. Then we have $\mathrm{Val}(G^*)(k,k-q,p-q) = g\,\mathrm{Val}(G)(p,q)$, or graphically



In the following we can therefore focus on four-legged graphs provided we allow graphs of the type G^*. Properties such as overlappingness trivially extend to such graphs. Note that by construction the line with propagator 1 does not appear in a loop in G^* and we will only use below that the propagator is bounded by 1.

6.1.2 Overlapping graphs and factors c.

In the rest of the chapter we will bound four legged graphs (possibly constructed from an electron-phonon vertex correction in the above way) using scale decomposition. Let G be such a graph and t a tree rooted at ϕ compatible to it. We will show a bound of the type

$$\sum_{j\leq 0}\sum_{J\in\mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_0 \leq \mathrm{const}\,\lambda_2(c,\beta,\epsilon) \quad 0<\epsilon<1 \tag{6.1}$$

From Theorem 60 it follows that this is direct when $\tilde{G}(\phi)$ is overlapping, i.e. G is overlapping at root scale.

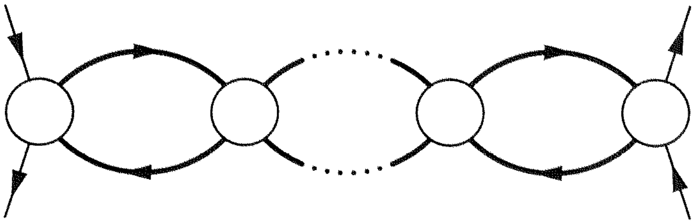
$$\sum_{j<0} \sum_{J\in\mathcal{J}(t,G,j)} |\mathrm{Val}(G^J)|_0 \le \mathrm{const}\,\lambda_2(c,\beta,\epsilon) \sum_{j\le 0} \mathrm{Pol}(|j|)M^{\epsilon j} \le \mathrm{const}\,\lambda_2(c,\beta,\epsilon) \tag{6.2}$$

The problem therefore lies in dealing with labeled graphs that do *not* overlap at root scale.

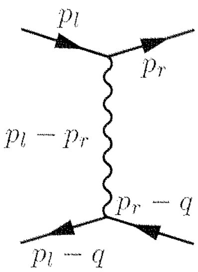
6.2 Some simple non-overlapping graphs

6.2.1 General form

In order to see how to treat the graphs that are not overlapping at root scale, it is good to do some simple cases 'by hand' first. By Lemma 28 a non overlapping four-legged graph is a Dressed Bubble Chain, i.e. it has the form



Here, the 4-vertices in the diagram are either effective vertexes created by higher scale lines such that expanding the root scale of the corresponding graphs will make the total graph overlapping or they are single vertices, i.e. boson lines if these are reintroduced in the graph. If the latter then a 4-vertex is the artificial propagator with value 1 if it is the leftmost 4-vertex of a graph G^* constructed from the vertex correction and a single phonon line for the other graphs. This single phonon line is arranged such that the momentum of the loop(s) connecting to the 4 vertex flows trough it (otherwise the original graph would be 1-particle reducible by cutting this phonon line). Graphically this looks like



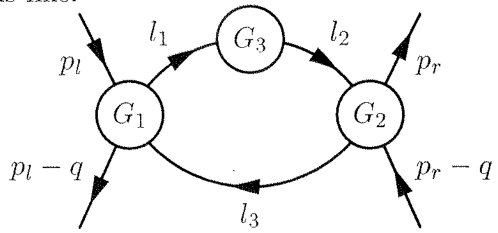
Note that although the 4-vertices as shown above must contain higher scale lines it is possible for them to contain lines at the root scale. In that case they are composed of $4+2n$ effective vertices of strictly higher scale where the $2n$ pairs of additional external lines are connected by strings of root scale. Because of the Wick-ordering these cannot be single propagators. Each such string must thus contain a 2-legged insertion (which can be a SSI).

The thick lines indicate the full Fermion propagator, i.e. they are strings of two particle insertions. The simplest examples of such graphs beyond the bubble are those which are bubble-like diagrams but with a two-legged insertion on one of the lines and the graphs that consist of two consecutive bubbles. To illustrate the method used, we apply it to each of these simple cases first. After that the general method for an arbitrary chain of bubbles with an arbitrary number of two-legged insertions is given.

6.2.2 Bubbles with a single two-legged insertion

General structure

Consider a labeled graph with root scale j such that the maximal non-overlapping expansion looks like:



Here $G_1, \ldots, G_3$ are 1PI graphs. Denote the scale of the line l_i by j_i. Denote the root scale of G_i by h_i. When root scale of the total graph G is j at least two of the j_i's are equal to j because of Wick Ordering. Each of the lines has a hard/soft label s_i. At most one of the lines can be a soft line ($\sum_{i=1}^{3} s_i \leq 1$) and this line must have scale j. Note that if either l_1 or l_2 is the soft line then the propagator of the of the soft lines is bounded on the support of the integrand because the momentum is the same as that flowing through the hard line.

C-forks and SSIs as insertions

The easiest case is when the fork f_3 corresponding to the subgraph G_3 is a C-fork. Then (because of Wick ordering) $h_3 = j_{f_3}$ is the lowest scale in the graph ($h_3 < \min\{j_1, j_2\} = j$).

We use Theorem 60 to extract the factor $c^{1-\epsilon}$ and the improved power counting factor $M^{h_3\epsilon}$. The scale sum over all scales that give rise to graphs of this type is given by

$$I = \sum_{j\leq 0}\sum_{r=1}^{3} \sum_{\substack{s_1+s_2+s_3\leq 1\\ s_r=0}} \sum_{\substack{j-\delta_{r1}<j_r<0\\ j_i=j\ i\neq r}} \sum_{h_3<j} \sum_{h_1>\max\{j_1,j_3\}} \sum_{h_2>\max\{j_2,j_3\}} f(p_l,p_r,q,\{j_i\},\{s_i\},\{h_i\}) \tag{6.3}$$

with

$$f(l,p,q,\{j_i\},\{s_i\},\{h_i\}) = \sum_{J_1\in\mathcal{J}(t_{f_1},G_1,h_1)} \sum_{J_2\in\mathcal{J}(t_{f_2},G_2,h_2)} \sum_{J_1\in\mathcal{J}(t_{f_3},G_3,h_3)}$$
$$\int \mathrm{d}_\beta^{d+1} l C_{j_1,s_1}(l) C_{j_2,s_2}(l) C_{j_3,s_3}(l-q)(L\,\mathrm{Val}(G_3^{J_3}))(l)\,\mathrm{Val}(G_1^{J_1})(p_l,l-q,l)\,\mathrm{Val}(G_2^{J_2})(l,p_r-q,p_3) \tag{6.4}$$

Note that because of the momentum conservation and the support properties of the propagator the only two terms contributing to the j_r sum for $r=1,2$ are $j_r=j$ and $j_r=j+1$.

Using the standard bound (5.13) and the improved power counting bound from Theorem 60 to the graph G_3 gives (noting that $h_3\leq j$):

$$\begin{aligned} &|L\,\mathrm{Val}(G_3^{J_3})|_{0,j} \\ &\qquad \leq \mathrm{const}\, M^j\lambda_2(c,\beta,\epsilon)(\mathrm{Pol}(|h_3|)M^{(1+\epsilon)h_3}M^{-j}+\mathrm{Pol}(|h_3|)M^{\epsilon h_3}) \\ &\qquad\qquad \leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|h_3|)M^{\epsilon h_3}M^j \end{aligned} \tag{6.5}$$

and applying normal power counting from Theorem 60 each of the graphs G_1,G_2 we get the bound:

$$|f(l,p,q,\{j_i\},\{s_i\},\{h_i\})|$$
$$\leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|)M^{\epsilon h_3}M^j\int \mathrm{d}_\beta^{d+1}l|C_{j_1,s_1}(l)||C_{j_2,s_2}(l)||C_{j_3,s_3}(l-q)| \tag{6.6}$$
$$\leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|)M^{\epsilon h_3}M^{-j_1-j_2-j_3}M^{3j} \tag{6.7}$$

where we used $j=\min\{j_1,j_2,j_3\}$ and for $j<0$ used the standard bounds for the propagators. When $j=0$ the presence of three propagators in the integral in (6.6) gives sufficient decay to make the frequency part convergent. The restriction on the scale of the C-fork gives the required bound:

$$\sum_{h_3<j} |f(l,p,q,\{j_i\},\{s_i\},\{h_i\})| \le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{-j_1-j_2-j_3} M^{3j} \sum_{h_3<j} M^{\epsilon h_3} \tag{6.8}$$

$$\le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{-j_1-j_2-j_3} M^{(3+\epsilon)j} \tag{6.9}$$

$$\le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{\epsilon j} \tag{6.10}$$

and thus

$$I \le \sum_{j\le 0} \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{\epsilon j} \le \text{const}\ \lambda_2(c,\beta,\epsilon) \tag{6.11}$$

When G_3 does not belong to a subfork but to a same scale insertion then the bound is the same. The only differences are: In (6.8) there is no sum over h_3 but the factor $M^{\epsilon j}$ appears immediately as $h_3 = j$. Moreover the projection L is absent and thus (6.5) holds immediately.

R-forks as insertions

We now turn to the case where G_3 is a graph corresponding to an R-fork. This is where we profit most from the more complicated projection L. The extra improvement the this gives to the bounds of values of R-forks is enough to extract a factor c by power counting only. Without it a partial integration argument similar to that of the next section would have been necessary to exploit the sign cancellations.

Denote $V_1(l) = \mathrm{Val}(G_1^{J_1})(p_l, l-q, p_l-q)$, $V_2(l) = \mathrm{Val}(G_2^{J_2})(l, p_r-q, p_3)$ and $V_3(l) = ((1-L)\,\mathrm{Val}(G_3^{J_3})))(l)$. The scale sum over all scales of G where G has the required form at root scales is given by

$$I = \sum_{j<0} \sum_{r=1}^{3} \sum_{\substack{s_1+s_2+s_3\le 1\\ s_r=0}} \sum_{\substack{j-\delta_{r1}<j_r<0\\ j_i=j\ i\ne r}} \sum_{\substack{h_1>\max\{j_1,j_3\}\\ h_2>\max\{j_2,j_3\}\\ h_3>\max\{j_1,\mathrm{J}_2\}}} g(p_l,p_r,q,\{j_i\},\{s_i\},\{h_i\}) \tag{6.12}$$

with

$$g(l,p,q,\{j_i\},\{s_i\},\{h_i\}) = \sum_{J_1\in\mathcal{J}(t_{f_1},G_1,h_1)} \sum_{J_2\in\mathcal{J}(t_{f_2},G_2,h_2)} \sum_{J_1\in\mathcal{J}(t_{f_3},G_3,h_3)} \int \mathrm{d}_\beta^{d+1} l\, C_{j_1,s_1}(l) C_{j_2,s_2}(l) C_{j_3,s_3}(l-q) V_1(l) V_2(l) V_3(l) \tag{6.13}$$

Note that j cannot be zero if there is an R-fork present in the graph at scale j.

By expanding the scale sum in the soft line and observing that because identical momentum flows through l_1 and l_2 their scales can only differ by at most 1, the sum can be rearranged as

$$I = I_1 + I_2 \tag{6.14}$$

with

$$I_s = \sum_{j<0} \sum_{(j_1,j_2,j_3)\in A_{s,j}} \sum_{\substack{h_1>\max\{j_1,j_3\}\\ h_2>\max\{j_2,j_3\}\\ h_3>\max\{j_1,j_2\}}} g(p_l, p_r, q, \{j_i\}, \{s_i = 0\}, \{h_i\}) \tag{6.15}$$

with

$$\begin{aligned} A_{0,j} &= \{(j,j),(j-1,j),(j,j-1)\} \times \{j,\dots,-1\} \setminus \{(j,j,j)\} \\ A_{1,j} &= \{(j,j),(j+1,j),(j,j+1)\} \times \{-\infty,\dots,j\} \quad \forall j < -1 \\ A_{1,-1} &= \{(j,j)\} \times \{-\infty,\dots,j\} \end{aligned} \tag{6.16}$$

Note that this situation is in fact atypical. If there had been an insertion in the lower string of the bubble too then the all scales sums from soft lines would have been restricted by conservation of momentum. Here the sum over j_3 is in fact only finite because of finite β. When $(j_1, j_2, j_3) \in A_{1,j}$ then j_3 is the lowest scale in the graph. We must combine the gain from the Tailor expansion in the upper string with the volume gain in the lower string. That requires a small extension of Theorem 60:

Lemma 62. *Let G be two legged and 1PI. Let $\mathbf{q} \in \mathbb{R} \times \Omega$. Let t be a tree compatible to t and $j_1 \le \bar{\jmath} < 0$. Let $t' \in (0,1]$. Let $\epsilon \in (0,1)$. Then*

$$\sum_{J\in\mathcal{J}(t,G,\bar{\jmath})} \mathcal{N}_2(\mathrm{Val}(G^J), \mathbb{1}_{j_1}(|l_0 - q_0|)\mathbb{1}_{j_1}(e(\mathbf{l}-\mathbf{q})), t') \le (t')^{\epsilon-1}\lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j_1|)M^{2j_1+j_1(\epsilon-1)} \tag{6.17}$$

Proof. This is a trivial corollary of the proof of (5.44), which is the $q = 0$ case. The indicator functions are used in analogy to a soft propagator, and the shift is irrelevant for such bounds. □

Using ordinary power counting for $\mathrm{Val}(G_1^{J_1})$ and $\mathrm{Val}(G_2^{J_2})$ and carrying out the scale sums over J_1 and J_2. we see that for $(j_1, j_1, j_3) \in A_s$,

$$|g(p_l, p_r, q, \{j_i\}, \{s_i = 0\}, \{h_i\})| \tag{6.18}$$

$$\leq \mathrm{Pol}(|j|) M^{-j_1-j_2-j_3} M^{2j} \int_0^1 \mathrm{d}t \sum_{J \in \mathcal{J}(t_{f_3}, G_3, h_3)} \mathcal{N}_2(\mathrm{Val}(G_3^J), \mathbb{1}_{j_{1+2s}}(|l_0 - sq_0|) \mathbb{1}_{j_{1+2s}}(e(\mathbf{l} - s\mathbf{q})), t) \tag{6.19}$$

$$\leq \lambda_2(c, \beta, \epsilon)\, \mathrm{Pol}(|j_{1+2s}|) M^{-j_1-j_2-j_3} M^{2j} M^{(1+\epsilon) j_{1+2s}} \tag{6.20}$$

$$\leq \lambda_2(c, \beta, \epsilon)\, \mathrm{Pol}(|j_{1+2s}|) M^{\epsilon j_{1+2s}} \tag{6.21}$$

Inserting these bounds and carrying out the sums over the scales h_1,h_2,and h_3 gives

$$|I_0| \leq \lambda_2(c, \beta, \epsilon) \sum_{j<0} \sum_{j_3 > j} \mathrm{Pol}(|j|) M^{\epsilon j} \leq \mathrm{const}\, \lambda_2(c, \beta, \epsilon) \tag{6.22}$$

$$|I_1| \leq \lambda_2(c, \beta, \epsilon) \sum_{j<0} \sum_{j_3 \leq j} \mathrm{Pol}(|j_3|) M^{\epsilon j_3} \leq \lambda_2(c, \beta, \epsilon) \sum_{j<0} \mathrm{Pol}(|j|) M^{\epsilon j} \leq \mathrm{const}\, \lambda_2(c, \beta, \epsilon) \tag{6.23}$$

6.2.3 Two bubbles

Derivatives acting D_1

In the previous chapters we have seen that derivatives acting on D_1 can give rise to complications and these were circumvented by making sure no D_1 term was derivated twice. The reason that was necessary is that in taking the Taylor expansion of the value of two legged graphs, these are evaluated at a frequency in between the Matsubara frequencies. However for phonon lines that are not in a two-legged insertion the frequency is always a boson Matsubara frequency and there is a much simpler argument.

By construction $\mathrm{supp}\, D_1(\cdot, c|\mathbf{l}|) \cap \frac{2\pi}{\beta}\mathbb{Z} = 0$. Therefore,

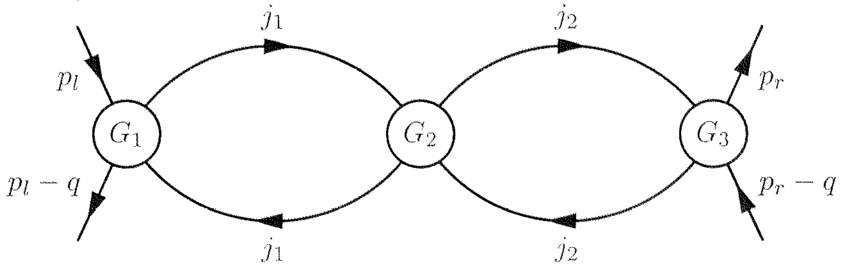
$$D_1(l_0, c|\mathbf{l}|) = \begin{cases} 1 & l_0 = 0 \\ 0 & l_0 \in \frac{2\pi}{\beta}\mathbb{Z}^* \end{cases} \tag{6.24}$$

Which implies that for all $1 \leq a < d$, and all $l_0 \in \frac{2\pi}{\beta}\mathbb{Z}$

$$\frac{\partial}{\partial l_a} D_1(l_0, c|\mathbf{l}|) = 0 \tag{6.25}$$

A re-summation of the scales

In the previous section bounds were given for when G was a generalized bubble graph with insertions on the bubble lines. To see that happens when the graph is a sequence of bubbles, consider a two bubble sequence (without two legged insertions in the bubble lines). The corresponding graph has the form:



The scale assignments to G_1, G_2 and G_3 are such that expanding the root scale of each subgraph will make the total graph overlapping or such that there is a 2-legged insertion at root scale. For simplicity we only consider the former case in this example.

In addition as the scale zero propagator C_0 has a different power counting, a simpler but different method is required to bound the bubble if they appear. Again to avoid distracting from the main technique we assume they do not occur here. For the same reasons we only consider the case $|\mathbf{q}| < \kappa_s$ here.

$$g(j_1, j_2, J_1, J_2, J_3) =$$
$$\sum_{s_1+s_2\leq 1} \sum_{r_1+r_2\leq 1} \int \mathrm{d}_\beta^{d+1} p_1 \int \mathrm{d}_\beta^{d+1} p_2 C_{j_1,s_1}(p_1) C_{j_1,s_2}(p_1 - q) C_{j_2,r_1}(p_2) C_{j_2,r_2}(p_2 - q)$$
$$V_1(p_1)V_2(p_1, p_2)V_3(p_2) \quad (6.26)$$

where for all $h_1, h_2, h_3 \leq 0$, $j_1 < \underline{h}_1 = \min\{h_1, h_2\}$, $j_2 < \underline{h}_2 = \min\{h_2, h_3\}$, $J_i \in \mathcal{J}(t_{f_i}, G_i, h_i)$; Furthermore $V_1(p_1) = \mathrm{Val}(G_1^{J_1})(p_l, p_1 - q, p_l - q)$,$V_2(p_1, p_2) = \mathrm{Val}(G_2^{J_2})(p_1, p_2 - q, p_1)$ and $V_3(p_2) = \mathrm{Val}(G_3^{J_3})(p_2, p_r - q, p_2 - q)$.

The difficulty is that neither of the h_i's is the smallest scale in the graph ($j < h_i$, $\forall i$). However any improvement in the power counting (for $\mathbf{q} \approx 0$) will come only from the overlap in the graph at scale h_1,h_2 or h_3 (or from the fact that they are just single vertices created by one phone-line and thus have scale 0). This difficulty is overcome by resummation of all scales lower than the scales where the overlap occurs.

$$f(J_1, J_2, J_3) = \sum_{j_1 < \underline{h}_1} \sum_{j_2 < \underline{h}_2} g(j_1, j_2, J_1, J_2, J_3) = \int \mathrm{d}_\beta^{d+1} p_1 \int \mathrm{d}_\beta^{d+1} p_2 C(p_1) C(p_1 - q) C(p_2) C(p_2 - q) S_{\underline{h}_1}(p_1) S_{\underline{h}_2}(p_2) V_1(p_1) V_2(p_1, p_2) V_3(p_2) \quad (6.27)$$

where $S_h(l) = a(M^{-2(h-1)}(l_0^2 + e(\mathbf{l})^2)) a(M^{-2(h-1)}((l_0 - q_0)^2 + e(\mathbf{l} - \mathbf{q})^2))$. However this means that the integrand is now no longer bounded, or more precisely: It is still bounded by the fact that the l_0 "integral" is really a finite sum not including the l_0 value, however it is no longer bounded uniformly in β.

However the important point is that due to sign cancellations the integral is not. The sign cancellations are exploited by doing an integration by parts in each of the two momentum integrals. Before we can do that we combine the propagators in $C(p_i)C(p_i - q)$ using a Feynman trick, i.e. interpolating between the two values of the argument. This was already done in section 2.3.5 in Lemma 13.

Applying this lemma to (6.27) we see that

$$|f(J_1, J_2, J_3)| \leq \sum_{a_1+b_1+c_1 \leq 1} \sum_{a_2+b_2+c_2 \leq 1} \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} w(a_i, b_i, c_i, r_i, J_i) \quad (6.28)$$

and thus

$$\left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i}, G_i, h_i)} \right) |f(J_1, J_2, J_3)| \leq 64 \max_{\substack{a_1+b_1+c_1 \leq 1 \\ a_2+b_2+c_2 \leq 1 \\ r_1, r_2 = 0,1}} \left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i}, G_i, h_i)} \right) w(a_i, b_i, c_i, r_i, J_i) \quad (6.29)$$

where $w = w(a_i, b_i, c_i, r_i, J_i)$ is given by

$$w(a_i, b_i, c_i, r_i, J_i) = \int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \int \mathrm{d}_\beta^{d+1} p_1 \int \mathrm{d}_\beta^{d+1} p_2 \, |Z_{r_1}(p_1, t_1)| |Z_{r_2}(p_2, t_2)| |(\nabla)^{a_1} S_{\underline{h}_1}(p_1)| |(\nabla)^{a_2} S_{\underline{h}_2}(p_2)| \, |(\nabla)^{b_1} V_1(p_1)| |(\nabla_{\mathbf{p}_1})^{c_1} (\nabla_{\mathbf{p}_2})^{b_2} V_2(p_1, p_2)| |(\nabla)^{c_2} V_3(p_3)| \quad (6.30)$$

Decoupled derivatives

When $c_1 + b_2 < 2$, i.e. when only one of the derivatives acts on V_2 then the two integrals can be disentangled

$$w(a_i, b_i, c_i, r_i, J_i) \leq \\ |V_\xi|_0 \left(\sup_{p_2} M^{-a_1 \underline{h}_1} \int_0^1 \mathrm{d}t_1 \int \mathrm{d}_\beta^{d+1} p_1 \, |Z_{r_1}(p_1, t_1)| S_{\underline{h}_1}(p_1)| (\nabla)^{b_1+c_1} V_{\eta_1}(p_1)| \right) \\ \left(\sup_{p_1} M^{-a_2 \underline{h}_2} \int_0^1 \mathrm{d}t_2 \int \mathrm{d}_\beta^{d+1} p_2 \, |Z_{r_2}(p_2, t_2)| S_{\underline{h}_2}(p_2)| (\nabla)^{b_2+c_2} V_{\eta_2}(p_2)| \right) \tag{6.31}$$

where

$$\eta_1 = \begin{cases} 1 & b_1 = 1 \text{ or } a_1 = 1, h_1 < h_2 \\ 2 & \text{otherwise} \end{cases} \qquad \eta_2 = \begin{cases} 2 & b_2 = 1 \text{ or } a_2 = 2, h_2 < h_3, \eta_1 \neq 2 \\ 3 & \text{otherwise} \end{cases} \tag{6.32}$$

$$\xi \in \{1, 2, 3\} \setminus \{\eta_1, \eta_2\} \tag{6.33}$$

In other words: When a derivative from a loop acts on a subgraph then keep that in the integral. If there is a derivative acting in both loops, take the sup-norm of the remaining subgraph.If there is a derivative acting on the cutoff functions in a loop then keep the lowest scale subgraph in the loop if that is not taken by the previous rule. Otherwise always take the rightmost subgraph in the loop. Take the sup norm over the remaining subgraph. This procedure takes care that if G_1 consists of only the added line with propagator 1 then the integral is taken over the other one which is guaranteed to contain a phonon line. Moreover it ensures that the scale improvement comes from the lowest possible scale.

Each of the integrals in (6.31) can be bounded separately. Observe that the integration is required as the derivatives of V_{η_1} and V_{η_2} are a priori not uniformly bounded in β. Moreover we need to extract an improved power counting gain from these factors. This is trivial if G_{η_1} and G_{η_2} are overlapping, however if they are not we only know that they make the total graph overlapping when expanded. The following Lemma shows that there remains just enough structure from the original graph to extract a gain power from this overlap.

Consider a graph G and a tree $t \sim G$. Construct the graph G' as in section 5.3.2. It contains only two and four-legged vertices. Each four-legged vertex consists of one phonon line. Define $L_2(t,G)$ as

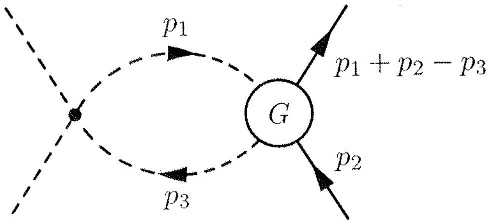
$$L_2(t,G) = L_F(t,G') \cup L_B(t,G') \cup V_2(t,G') \tag{6.34}$$

For a line or two vertex $l \in L_2(t,G)$, $m \in \{1,\ldots,d\}$, a labeling J and $s = 0,1$, define $\partial_{lms}G^J$ as the graph obtained by G by replacing the propagator of line l by

$$P_l(p) = \begin{cases} \frac{\partial}{\partial p_m} C(p) & s = 0,\ l \text{ a hard fermion line} \\ 0 & s = 1, \text{ or } l \text{ a soft line} \\ \frac{\partial}{\partial p_m} D_s(p_0, c|\mathbf{p}|) & l \text{ a phonon line} \end{cases} \tag{6.35}$$

Lemma 63. *Let G be a four-legged graph with value $\mathrm{Val}(G)(p_1,p_2,p_3)$ and t a tree of subgraphs compatible to G. Let ϕ be the root of t. Let $0 < \epsilon \le 1$. Let $s = 0,1$.*

Let $\tilde{W}_1(G)$ be the graph constructed from G by attaching two of its external legs to a 4-vertex v, i.e. $\tilde{W}_1(G)$ is



Let $j \le j' \le 0$. Let J be in $\mathcal{J}(t,G,j')$. Let W_j be one of the functions from Lemma 48 Let $\mathscr{W}_{1,s}(\mathrm{Val}(G^J),j,q,p)$ be given by

$$\int \mathrm{d}_\beta^{d+1} l W_j(l) | \left(\frac{\partial}{\partial \mathbf{l}}\right)^s \mathrm{Val}(G^J)(l, p-q, l-q)| \tag{6.36}$$

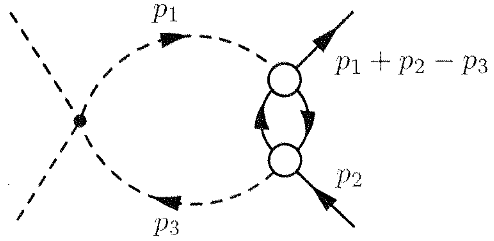
Let $l \in L_2(t,G)$ be such that l is contained in a loop in $\tilde{W}_1(G)$. Let $m = 1,\ldots,d$ and $s = 0,1$. Then if either $\tilde{W}_1(\tilde{G}(\phi))$ is overlapping, i.e. $\tilde{W}_1(G^J)$ is overlapping at the root scale j' of G^J, or G is a single effective 4-vertex build up from a phonon line such that the momentum on this line depends on $\mathbf{l}$:

$$\sum_{J \in \mathcal{J}(t,G,j')} \mathscr{W}_{1,0}(\mathrm{Val}(\partial_{lms})G^J),j,q,p) \le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{\epsilon j} M^{(1-s)j'} \tag{6.37}$$

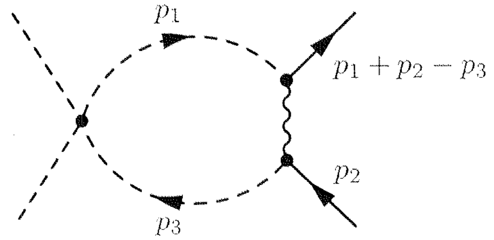
and

$$\sum_{J \in \mathcal{J}(t,G,j')} \mathscr{W}_{1,s}(\mathrm{Val}(G^J),j,q,p) \le \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|j|) M^{\epsilon j} M^{(1-s)j'} \tag{6.38}$$

Note that in the above two cases the graph $\tilde{W}(\tilde{G}(\phi))$ either has the structure



where the single bubble can be replaced by a whole chain, or in case G is a single vertex the graph looks like



Proof. In $\mathscr{W}_{1,1}(\mathrm{Val}(G^J), j, q, p)$ the derivative is with respect to vector momentum only. By taking a spanning tree compatible to J for G and computing the derivative by taking the corresponding expression for $\mathrm{Val}(G^J)$, we see that (6.38) reduces to a sum over terms of the form (6.37). Thus it remains to show (6.37).

The proof of the lemma is similar to Case D and Case H of Theorem 60, the only real difference being that $\mathscr{W}_{1,0}(\mathrm{Val}(\partial_{lms})G^J), j, q, p)$ is now bounded by M^j times value of the four-legged graph graph $\tilde{W}_1(A(\partial_{lms} G^J))$. Here v is taken to have a value 1 and the new lines to have propagators $W_j(p)$ and M^{-j} respectively. The two new lines thus have scale j. The line with propagator M^{-j} can always be taken into the spanning tree and thus the lack of indicator functions restricting the momentum on this line is not a problem for the power counting. Repeating the arguments given in the proof of Theorem 60 thus gives that this value is bounded as

$$\sum_{J \in \mathcal{J}(t,G,j')} \mathrm{Val}(\tilde{W}_1(A(\partial_{lmn} G^J))) \leq \lambda_2(c, \beta, \epsilon)\, \mathrm{Pol}(|j|) M^{-(1-\epsilon)j} M^{(1-s)j'} \tag{6.39}$$

which gives the required bounded when summed over the finite number of l and combined with the factor M^j.

Note that the presence of a possibly interpolated band relation in $W_j(p)$ does not matter as we have shown that the volume improvement also holds in that case. □

Note that expanding graph G_2 makes the total graph overlapping: This implies that either the graph containing just G_1 and G_2 and the loop connecting them is overlapping when G_2 is expanded or that this holds for the other subgraph containg G_2 and G_3. The following remark shows that this implies that then both subgraphs have this property.

Remark 64. *Let G be a graph with four external legs. Let G be such that when two of its external legs are connected to another 4-vertex the total graph is overlapping. Then the graph created by connecting the other two legs to a 4-vertex is also overlapping.*

Proof. This trivial when G is itself overlapping. If G is not overlapping, then it must be a dressed bubble chain of length at least one, and the two connected external lines must be on different ends of the chain. Then at each the end there is one other external line and thus connecting those also gives an overlapping graph. □

Observing that

$$|Z_1(p,q,t)|S_h(p) \leq \frac{\mathbb{1}(l_0^2 + e(\mathbf{l},\mathbf{q},t)^2 < 4M^{2j})}{|il_0 + e(\mathbf{l},\mathbf{q},t)|} \tag{6.40}$$

$$|Z_2(p,q,t)|S_h(p) \leq \frac{\mathbb{1}((l_0+tq_0)^2 + e(\mathbf{l})^2 < 4M^{2j})}{|i(l_0+tq_0) + e(\mathbf{l})|} \tag{6.41}$$

$$\leq \lim_{I'\to-\infty} \frac{\mathbb{1}(M^{2I'}(l_0+tq_0)^2 + e(\mathbf{l})^2 < 4M^{2j})}{|i(l_0+tq_0) + e(\mathbf{l})|} \tag{6.42}$$

we can apply Lemma 63 (and its mirror version for the integral over the other external legs) in each of the integrals. Let $\underline{h}_m = \min\{\underline{h}_1, \underline{h}_2\} = \min\{h_1, h_2, h_3\}$, this is the lowest scale in the graph after resummation. Let $\underline{h}_M = \max\{\underline{h}_1, \underline{h}_2\}$. Applying the lemma at finite I' for p_i with $\epsilon = \frac{1}{2}$ when $\underline{h}_i \neq \underline{h}_m$ gives:

$$\left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i}, G_i, h_i)}\right) w(a_i, b_i, c_i, r_i, J_i) \tag{6.43}$$

$$\leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|\underline{h}_m|) M^{-a_1\underline{h}_1} M^{-a_2\underline{h}_2} M^{\epsilon\underline{h}_m} M^{\frac{1}{2}\underline{h}_M} M^{(1-(b_1+c_1))h_{\eta_1}} M^{(1-(b_2+c_2))h_{\eta_2}} \tag{6.44}$$

$$\leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|\underline{h}_m|) M^{\epsilon\underline{h}_m} \tag{6.45}$$

The bound is independent of I'. By the dominated convergence theorem the exchange of the I' limit and the integrals over t_1 and t_2 is allowed and the limiting function (6.31) obeys the same bound.

Two derivatives acting on the same graph

The remaining case is $c_1 + b_2 = 2$, i.e. the middle subgraph is differentiated once from the left and once from the right. Again this bounded using a generalization of Theorem 60. Because we will need the fact any lines that are hit by derivatives, must be in the loop generated by connecting the external lines as in the larger graph we use the following indicator function to do that.

Definition 65. *Consider a four legged graph G with ingoing lines with momenta p_1, p_2, and one outgoing momentum p_3, such that its value is given by $\mathrm{Val}(G)(p_1, p_2, p_3)$. Let $L_2(G)$ be the set of all fermion lines, boson lines and two legged 1PI subgraphs in G that are not contained in other two legged 1PI subgraphs. Let $J \in \mathcal{J}(t, G, j)$ for some j and t.. Let $k_1, k_2 \in L_2(G)$. Denote by e_1,e_2,e_3,e_4 the external lines corresponding to the momenta p_1,p_2,p_3,$p_4 = p_1+p_2-p_3$ respectively. Let E_1, E_2 be disjunct pairs of in and outgoing external lines (note that E_1 specifies E_2). Then define*

$$\zeta_1(k_1, G, J, E_1) = \begin{cases} 1 & l \in L_2(t, G) \text{ and there exists a spanning tree } T(J) \subset G \text{ compatible to } J \text{ such that } k_1 \text{ is on the path in } T(J) \text{ connecting the pair of external lines } E_1 \\ 0 & \text{otherwise} \end{cases} \tag{6.46}$$

$$\zeta_2(k_1, k_2, G, J, E_1) = \zeta_1(k_1, G, J, E_1)\zeta_1(k_2, G, J, E_2) \tag{6.47}$$

This definition is useful because of the following: Let G be a four-legged graph with external momenta as above. Let t be a tree and J set of scales compatible to T with root scale j. Let $k_1 \in L_2(t, G)$. Let T be a spanning tree compatible to the scales and fix the momenta p_l according to this tree and such that $p_1 = l$, $p_2 = p - q$ and $p_3 = l - q$. Then for $m = 1 \dots d$

$$|\frac{\partial}{\partial l_m} p_{k_1}| \leq \zeta_1(l_1, G, J, (e_1, e_3)) \tag{6.48}$$

This is an inequality because there can be multiple spanning trees compatible to the scales.

Using this we have

$$\sum_{J\in\mathcal{J}(t,G,j)} \left|\frac{\partial}{\partial \mathrm{l}_m} \mathrm{Val}(G)(l,p-q,l-q)\right|$$

$$\leq \sum_{J\in\mathcal{J}(t,G,j)} \sum_{k_1\in L_2(t,G)} \sum_{s=0}^{1} |\mathrm{Val}(\partial_{k_1 ms}G)||\frac{\partial}{\partial l_m} p_{k_1}| \tag{6.49}$$

$$\leq \sum_{J\in\mathcal{J}(t,G,j)} \sum_{k_1\in L_2(t,G)} \sum_{s=0}^{1} \zeta_1(l_1, G, J, (e_1, e_3))|\mathrm{Val}(\partial_{k_1 ms}G)| \tag{6.50}$$

$$\leq \sum_{k_1\in L_2(G)} \sum_{s=0}^{1} \sum_{J\in\mathcal{J}(t,G,j)} \zeta_1(l_1, G, J, (e_1, e_3))|\mathrm{Val}(\partial_{k_1 ms}G)| \tag{6.51}$$

The indicator function ζ_2 is non-zero exactly then when the two derivatives acting on the graph can be controlled.

Lemma 66. *Let G be a four-legged graph with value $\mathrm{Val}(G)(p_1,p_2,p_3)$ and t a tree of subgraphs compatible to G. Let ϕ be the root of t. Let $0<\epsilon\leq 1$. Let $j_1, j_2 \leq 0$. Let $\max\{j_1,j_2\}\leq j' \leq 0$. Let $k_1,k_2\in L_2(G)$. Let $s_1,s_2=0,1$. Let $m_1,m_2=1,\dots d$. Let W_j and W'_{j_1} each be one of the functions from Lemma 48 Let $\mathscr{W}_2(\mathrm{Val}(\partial_{k_1m_1s_1}\partial_{k_2m_2s_2}G^J), j_1, j_2, q)$ be given by*

$$\int \mathrm{d}^{d+1}_\beta l_1 \int \mathrm{d}^{d+1}_\beta l_2 W_{j_1}(l_1) W'_{j_2}(l_2) |\mathrm{Val}(\partial_{k_1m_1s_1}\partial_{k_2m_2s_2}G^J)(l_1, l_2-q, l_1-q)| \tag{6.52}$$

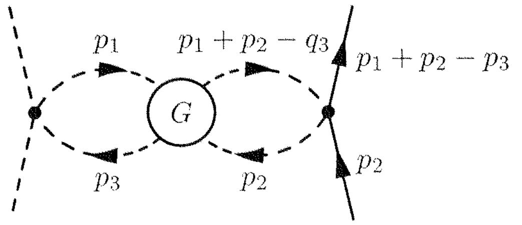
Then if either $\tilde{W}_1(\tilde{G}(\phi))$ is overlapping, i.e. $\tilde{W}_1(G^J)$ is overlapping at the root scale j' of G^J, or G is a single effective 4-vertex build up from a phonon line such that the momentum on this line depends on l*:*

$$\sum_{J\in\mathcal{J}(t,G,j')} \zeta_2(k_1,k_2,G,J,(e_1,e_2))\mathscr{W}_2(\mathrm{Val}(\partial_{k_1m_1s_1}\partial_{k_2m_2s_2}G^J), j_1, j_2, q)$$

$$\leq \lambda_2(c,\beta,\epsilon)\,\mathrm{Pol}(|\underline{j}|)M^{\epsilon\underline{j}} \tag{6.53}$$

where $\underline{j}=\min\{j_1,j_2\}$.

Proof. This proof follows that of Lemma 63 and Theorem 60 in representing the value to be bounded, $\mathscr{W}_2(\mathrm{Val}(\partial_{k_1m_1s_1}\partial_{k_2m_2s_2}G^J), j_1, j_2, q)$, as the value of a construct four-legged graph that is overlapping times a some scale factors. Let $\tilde{W}_2(G)$ be the graph constructed from G by attaching e_1 and e_3 to a 4-vertex v_1 and e_2, e_4 to a vertex v_2, i.e. $\tilde{W}_2(G)$ is



If the two new internal lines connecting v_1 to G are taken to have scale j_1 with propagators $W_{j_1}(p_1)$ and M^{-j_1} respectively and the same is done for the lines connecting v_2, thus having scale j_2 with propagators $W'_{j_2}(p_1)$ and M^{-j_2} respectively then $\mathscr{W}_2(\mathrm{Val}(G^J), j_1, j_2, q)$ is bounded by $M^{j_1+j_2}$ times the value of the graph $\tilde{W}_2(A(\partial_{k_1 m_1 s_1}\partial_{k_2 m_2 s_2} G^J))$. The bound then follows from the arguments given in the proof for Theorem 60, with k_1 and k_2 replacing l'_1 and l'_2. Note that $\zeta_2(k_1, k_2, G, J, s_1) \neq 0$ implies that k_1 and k_2 are both in loops in $\tilde{W}_2(G)$, are either hard lines, 2-legged 1PI subgraphs or phonon lines. Moreover if $s_1 = 1$ or $s_2 = 1$ then as remarked earlier the value of the graph vanishes identically.

To illustrate the argument consider the worst case behavior which occurs when both derivatives act on the same phonon line k that is not in any loop of G. k is contained, however, in both loops created by integrating over l_1 and l_2. Thus there is a factor const $M^{(\epsilon-1)\underline{j}} M^{-\bar{\jmath}}$ compared to the normal power counting which just gives $\mathrm{Pol}(|\underline{j}|)$ ($\bar{\jmath} = \max\{j_1, j_2\}$). Combined with the pre-factor $M^{j_1+j_2} = M^{\underline{j}+\bar{\jmath}}$ this gives the required bound for this term. □

Applying the lemma and Theorem 60 to w for $c_1 + b_2 = 2$ (noting that by our simplifying assumption only the $s_1 = s_2 = 0$ occurs).

$$\left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i}, G_i; h_i)}\right) w(a_i, b_i, c_i, r_i, J_i) \tag{6.54}$$

$$\leq \left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i}, G_i; h_i)}\right) |V_1|_0 |V_3|_0$$
$$\int_0^1 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 \int \mathrm{d}_\beta^{d+1} p_1 \int \mathrm{d}_\beta^{d+1} p_2\, |Z_{r_1}(p_1, t_1)| S_{\underline{h}_1}(p_1) |Z_{r_2}(p_2, t_2)| \tag{6.55}$$
$$S_{\underline{h}_1}(p_1) S_{\underline{h}_2}(p_2) |\nabla_{\mathbf{p}_1} \nabla_{\mathbf{p}_2} V_2|$$

$$\leq \mathrm{const}\, \lambda_2(c, \beta, \epsilon)\, \mathrm{Pol}(|\underline{h}_m|) M^{c\underline{h}_m} \tag{6.56}$$

Taken together

$$|I| \leq \sum_{h_1,h_2,h_3} \left(\prod_{i=1}^{3} \sum_{J_i \in \mathcal{J}(t_{f_i},G_i,h_i)} \right) |f(J_1,J_2,J_3)| \tag{6.57}$$

$$\leq \lambda_2(c,\beta,\epsilon) \sum_{h_1,h_2,h_3} \mathrm{Pol}(|\underline{h}_m|) M^{\epsilon \underline{h}_m} \tag{6.58}$$

$$\leq \lambda_2(c,\beta,\epsilon) \sum_{\underline{h}_m<0} \mathrm{Pol}(|\underline{h}_m|) M^{\epsilon \underline{h}_m} \tag{6.59}$$

$$\leq \mathrm{const}\, \lambda_2(c,\beta,\epsilon) \tag{6.60}$$

6.2.4 Scale zero lines

Before turning to the treatment of the general case we briefly consider the effect of scale zero propagators C_0 in the graph. Note that these can only occur if $\underline{h}_1 = 0$ and/or $\underline{h}_2 = 0$. The scale sums can then include $j_1 = 0$ or $j_2 = 0$. Such a scale sums then gives

$$\sum_{j \leq 0} \sum_{s_1+s_2 \leq 1} C_{j,s_1}(l) C_{j,s_2}(l-q)$$

$$= C(l)C(l-q)S_0(l) + C_{<0}(l)C_0(l-q) + C_0(l)C_{<0}(l-q) + C_0(l)C_0(l-q) \tag{6.61}$$

Because $|C_0|$ is bounded by a constant and $|C_{<0}|$ is integrable, the integral

$$\int \mathrm{d}_\beta l_0 \int \mathrm{dl} |C_{<0}(l)||C_0(l-q)| \tag{6.62}$$

is finite. Because two propagators give sufficient decay to make the frequency integral finite, the integral

$$\int \mathrm{d}_\beta l_0 \int_\Omega \mathrm{dl} |C_0(l)||C_0(l-q)| \tag{6.63}$$

is also finite. Therefore the extra factors are simpler do deal with, it just remains to show the factor $\lambda_2(c,\beta,\epsilon)$ can be extracted. For that reason define $\mathscr{W}_0(G)$ for four-legged graphs G with value $\mathrm{Val}(G)(p_1,p_2,p_3)$ as

$$\mathscr{W}_0(G) = \int \mathrm{d}_\beta l_0 \int_\Omega \mathrm{dl}\, \tau_{M^{-1}}(l_0) \tau_{M^{-1}}(l_0 - q_0) |\mathrm{Val}(G)(l, p-q, l-q)| \tag{6.64}$$

Note that when $|\mathrm{Val}(G)|_0 < \infty$,

$$\mathscr{W}_0(G) \leq \mathrm{const}\, |\mathrm{Val}(G)|_0 \int \mathrm{d}_\beta l_0 \tau_{M^{-1}}(l_0) \tau_{M^{-1}}(l_0 - q_0) \leq \mathrm{const}\, |\mathrm{Val}(G)|_0 \tag{6.65}$$

Finally note that if j_1 or j_2 is zero, then the subgraphs in that bubble must be simple vertices.

6.3 Full proof

6.3.1 A decomposition for four legged graphs

Let G be a four-legged graph and t a tree with root ϕ such that $\tilde{G}(\phi)$ is non-overlapping. Then there exists a subtree t' of t routed at ϕ, such that $\tilde{G}(t')$ (i.e. the graph where all forks above t' have been replaced by effective vertices) is non-overlapping and does not contain 2-legged non-leaves. In fact t' can be chosen maximal in the sense that for any other tree $t'' \subset t$ with this property $t'' \subset t'$.

As shown in the previous sections, in the presence the extended localisation operator L there is also a power counting gain if the graph contains a 2-legged (effective) vertex at root scale. This allows the the set of graphs that has to be treated specially to be restricted further; For any pair (G, t) define t_N as the unique maximal subgraph of t such that $\tilde{G}(t_N)$ is not overlapping and for all forks and leaves $f \in t_N$, $E(G_f) > 2$. This implies that when $\tilde{G}(\phi)$ is overlapping or contains a two legged subgraph $t_N = \phi$. Moreover let the notation $t' \precsim G$ denote that t' is a tree of subgraphs of G where the leaves of t' are allowed to be non-trivial subgraphs of G. Thus $t' \precsim G$ iff there exists a tree $t'' \sim G$ with $t' \subset t''$ as trees. For such trees also define a new notation for quotient graphs: Let $t\prime \precsim G$, then

$$G_{t'} = \tilde{G}(\tau) \quad \tau \subset t' \text{ such that } t' = \tau \cup B(\tau, t') \tag{6.66}$$

This notations allows keeping track which subgraphs were collapsed to produce the quotient graphs. These are exactly the leaves of t'.

We will factor the sum over the trees compatible to G in the sum over trees for the non-overlapping part and sums over the trees for the subgraphs. However we must take care in the latter sums as not all trees compatible to the subgraph can occur: For instance let $t \precsim G$ with a leaf f such that $E(G_f) = 4$ and such that there exists a tree $t_2 \sim G_f$ with root ϕ_2 such that G_{f,ϕ_2} is a Dressed Bubble Chain of length at least one. By Lemma 28 G_t is a DBC of length n. Now it is possible that G_f is oriented such that then when G_f is expanded to ϕ_2 scale in G_t the total graph becomes a DBC of length at least $n+1$. Thus $t_N = t$, but for the concatenation $t_3 = t + t_2$, the maximal non-overlapping subgraph $t_{3,N} \not\supset t$. Therefore the sum over the trees of G_f only contains those trees that make G overlapping when the root scale G_f is expanded or produce 2-legged subgraphs.

We now make that restriction more formal: Let $V_{E,\text{in}}(G_f)$ and $V_{E,\text{out}}(G_f)$ be the set of ingoing and outgoing external lines of G_f respectively. As G is a DBC, each external line of G_f is connected to exactly one other by a path in G. [1] This introduces a natural pairing on the external lines of G_f: Let $\mathfrak{p}_\mathfrak{f}$ be defined as

$$\mathfrak{p}(G, G_f) = \{(l_o, l_i) \in V_{E,\text{out}}(G_f) \times V_{E,\text{in}}(G_f) \| \ l_0 \text{ and } l_i \text{ connected in } G/G_f\} \tag{6.67}$$

For $(l_1, l_2) \in V_{E,\text{out}}(G_f) \times V_{E,\text{in}}(G_f)$ define $\tilde{W}_{(l_1,l_2)}(G)$ to be the graph produced by connecting the two lines l_1 and l_2 to a new 4-vertex v (c.f. $\tilde{W}_{j,1}$). If t_f is a tree compatible to G_f with root ϕ_f then replacing the effective vertex for G_f in G_{t_N} by the root scale graph G_{ϕ_f} results in an overlapping graph if and only if there exists $(l_o, l_i) \in \mathfrak{p}_f$ such that $\tilde{W}_{(l_o,l_i)}(\tilde{G}(\phi_f))$ is overlapping. Thus we can define for a non-trivial graph G and a pairing $\mathfrak{p} \subset V_{E,\text{out}}(G_f) \times V_{E,\text{in}}(G_f)$

$$\mathcal{T}(G, \mathfrak{p}) = \{t \sim T \| \exists f : \pi(f) = \phi, E(G_f) = 2 \vee \exists (l_o, l_i) \in \mathfrak{p} : \tilde{W}_{(l_o,l_i)}(\tilde{G}(\phi)) \text{ overlapping}\} \tag{6.68}$$

$\mathcal{T}(G_f, \mathfrak{p}(G, G_f))$ are exactly those subtrees that appear under the tree t_N. Note that we also allow $\mathfrak{p}$ to be empty and then

$$\mathcal{T}(G, \emptyset) = \{t \sim T \| \exists f : \pi(f) = \phi, E(G_f) = 2 \vee \ \tilde{G}(\phi) \text{ overlapping}\} \tag{6.69}$$

In addition for a G the trivial graph containing of just one vertex v define $\mathcal{T}(G, \mathfrak{p}) = \{v\}$.

This leads to the decomposition

$$\text{Val}(G) = \sum_{j \le 0} \sum_{t \sim G} \sum_{J \in \mathcal{J}(G,t,j)} \text{Val}(G^J) \tag{6.70}$$

$$= \sum_{j \le 0} \sum_{\substack{t \precsim G \\ t_N = t}} \sum_{J_N \in \mathcal{J}(G_t,t,j)} \prod_{f \text{ leaf of } t} \left(\sum_{t_f \in \mathcal{T}(G_f, p(G,G_f))} \sum_{J_f \in \mathcal{J}(G_f,t_f,j_f)} \right) \text{Val}(G^{J=(J_N,\{J_f\})}) \tag{6.71}$$

where have used the convention that for a single vertex V, $\sum_{J_f \in \mathcal{J}(G_f,t_f,j_f)} = \delta_{j_f 0}$. In other words we decompose the scale sum in a sum over all partial assignments of scales that make the

1. It can also be seen to be an immediate consequence of nonoverlappedness. If there is a path not through G_f connecting two external lines of G_f, then this is a loop in G_{t_N}/G_f. If there exist another external line connecting to one of the two, then this means G_{t_N}/G_f has another loop sharing at least one line and thus G_{t_N}/G_f is overlapping

graph non-overlapping (and not containing any 2-legged insertions) at the highest assigned scale and then sum over assignments of higher scales that do make the graph overlap (or contain 2-legged vertices) at some scale. Note that j_f is fixed by choice of J_N.

The sum contains, pathologically, the case where the graph is already overlapping at root scale. In that case $t = \phi$, $G_t = G/G_\phi$ which is just a single four-legged vertex and the set $\mathcal{J}(G_t, t, j)$ contains only one element where $j_f = j_\phi = j$. Thus $G_f = G$ is the whole graph and j_f is the root scale. In that case $\mathfrak{p}(G, G_f) = \emptyset$ and so the right sum appears by virtue of definition (6.69).

We want to treat the non-overlapping graphs by resummation of scales where there is no overlap. In order for this to be practical the layout of the maximal non-overlapping graph must stay fixed, i.e. we want to fix G_t and then sum over all t that produce that, and over all scale assignments j and J_N at fixed j_f. Note that G_t the quotient graph of G/G_f where is indexed by the leaves f of t. Thus keeping G_t fixed is the same as keeping the G_f's fixed. Define $\mathcal{A}_n$ as

$$\mathcal{A}_n(G) = \Big\{ \{G_i\}_{i=1}^n, G_i \subset G\,\forall i \Big| G_i\text{'s disjoint}, \bigcup_{i=1}^n V(G_i) = V(G),$$
$$G/\{G_i\}_{i=1}^n \text{ is not-overlapping and does not contain 2-legged vertices.}\Big\} \qquad (6.72)$$

Note that for $\{G_i\}_{i=1}^n \in \mathcal{A}_n(G)$ and $t \sim G/\{G_i\}_{i=1}^n$, $t_N = t$ if and only if $E(G_f) > 2$ for all $f \in t$.

Taking all this together we can reorder the sum as

$$\mathrm{Val}(G) = \sum_{n\geq 1} \sum_{\{G_i\}_{i=1}^n \in \mathcal{A}_n(G)} \sum_{j\leq 0} \sum_{\substack{t\sim G/\{G_i\}_{i=1}^n \\ E(G_f)>2\forall f\in t}} \sum_{J_N\in\mathcal{J}(\{G_i\}_{i=1}^n, t, j)}$$
$$\prod_{i=1}^n \left(\sum_{t_i\in\mathcal{T}(G_i,\mathfrak{p}(G,G_i))} \sum_{J_i\in\mathcal{J}(G_i,t_i,j_i)} \right) \mathrm{Val}(G^{J=(J_N,\{J_i\})}) \qquad (6.73)$$

In order to be able to re-sum the scales on $G/\{G_i\}_{i=1}^n$ the range of scales for the sum needs to be fixed. However we also want the G_i's to be the higher scale graphs in the scale decomposition. This can be a achieved by keeping the scales of the leaves of t fixed as we sum over scale assignments. For a graph G with n vertices, $\{v_i\}_{i=1}^n$, a tree $t \sim G$, and a set $\{h_i\}_{i=1}^n$ of scales define

$$\mathcal{J}(G, t, j, \{h_i\}_{i=1}^n) = \{J \in \mathcal{J}(G, t, j) | j_{v_i} = h_i\ \forall i = 1, \ldots, n\} \qquad (6.74)$$

And thus we have, finally

$$\mathrm{Val}(G) = \sum_{n\geq 1} \sum_{\substack{\{h_i\}_{i=1}^n \\ h_i\leq 0\,\forall i}} \sum_{\{G_i\}_{i=1}^n \in \mathcal{A}_n(G)} \sum_{j\leq 0} \sum_{\substack{t\sim G/\{G_i\}_{i=1}^n \\ E(G_f)>2\,\forall f\in t}} \sum_{J_N\in\mathcal{J}(\{G_i\}_{i=1}^n,t,j,\{h_i\}_{i=1}^n)} \prod_{i=1}^n \left(\sum_{t_i\in\mathcal{T}(G_i,\mathfrak{p}(G,G_i))} \sum_{J_i\in\mathcal{J}(G_i,t_i,h_i)} \right) \mathrm{Val}(G^{J=(J_N,\{J_i\})}) \tag{6.75}$$

$$= \sum_{n\geq 1} \sum_{\{G_i\}_{i=1}^n \in \mathcal{A}_n(G)} \sum_{\substack{\{h_i\}_{i=1}^n \\ h_i\leq 0\,\forall i}} \sum_{j\leq 0} \sum_{\substack{t\sim G/\{G_i\}_{i=1}^n \\ E(G_f)>2\,\forall f\in t}} \sum_{J_N\in\mathcal{J}(\{G_i\}_{i=1}^n,t,j,\{h_i\}_{i=1}^n)} \mathrm{Val}((G/\{G_i\}_{i=1}^n)^{J_N}, \{ \sum_{t_i\in\mathcal{T}(G_i,\mathfrak{p}(G,G_i))} \sum_{J_i\in\mathcal{J}(G_i,t_i,h_i)} \mathrm{Val}(G_i^{J_i})\}_{i=1}^n) \tag{6.76}$$

Before proceeding with bounding the graphs observe that in our case G is 4-legged and therefore by Lemma 28 the non-overlapping graph $G/\{G_i\}_{i=1}^n$ is a dressed bubble chain. Because in addition $G/\{G_i\}_{i=1}^n$ does not contain 2-legged subgraphs and because of Wick ordering the graph at scale $j < 0$ it must be a normal bubble chain of length $n-1$, consisting of n 4-legged graphs connected by pairs of single propagators. If $j = 0$ there is no Wick-ordering but the graphs must be of the same form as there are no vertices of more than 4 legs at scale zero. Thus the only sets of graphs that occur in $\mathcal{A}_n(G)$ are sets of 4-legged graphs. From the structure of that graph it is easy to see that there are no subgraphs of $G/\{G_i\}_{i=1}^n$ that are 2-legged and thus the restriction $E(G_f) > 2$ in the sum above is superfluous.

6.3.2 Bounding bubble chains

Formulating the bound

After all these preliminaries are done we come to the core argument of this thesis which is captured in the following theorem:

Theorem 67. *Let G be a 4-legged 1PI graph with the 4-vertices coming from the phonon lines oriented such that after cutting any phonon line the graph is still connected. Let $n \geq 1$. Let $\{G_i\}_{i=1}^n \in \mathcal{A}_n(G)$. Let $\mathfrak{p}_i = \mathfrak{p}(G, G_i)$ for all $2 \leq i < n$. Let either $\mathfrak{p}_1 = \mathfrak{p}(G, G_1)$ or $\mathfrak{p}_1 = \mathfrak{p}(G, G_1) \cup (l_o, l_i)$ where (l_o, l_i) is a pair of external lines of G. Let l and p be fermion 4-momenta and q a boson 4-momentum. Let*

$$U_i(h_i) = \sum_{t_i\in\mathcal{T}(G_i,\mathfrak{p}_i)} \sum_{J_i\in\mathcal{J}(G_i,t_i,h_i)} \mathrm{Val}(G_i^{J_i}) \tag{6.77}$$

and

$$\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q) = \mathcal{U}(\{U_i(h_i)\}_{i=1}^n; l, p, q) \tag{6.78}$$

where

$$\begin{aligned}&\mathcal{U}(\{X_i\}_{i=1}^n; l, p, q)\\ &= \sum_{j\le 0} \sum_{t\sim G/\{G_i\}_{i=1}^n} \sum_{\substack{J_N \in \\ \mathcal{J}(\{G_i\}_{i=1}^n, t, j, \{h_i\}_{i=1}^n)}} \mathrm{Val}((G/\{G_i\}_{i=1}^n)^{J_N}, \{X_i\}_{i=1}^n)(l, p-q, l-q)\end{aligned} \tag{6.79}$$

Moreover assume without loss of generality that the G_i's are numbered such that G_1 is the external vertex in $G/\{G_i\}_{i=1}^n$ connected to the external lines with momenta l and $l-q$, and G_n the other external vertex. Then for all $0 < \epsilon < 1$

i) There exists a constant 'const' such that for all l, p, q and with $\mathfrak{p}_1 = \mathfrak{p}(G, G_1)$

$$I_1 = |\mathcal{U}(\{h_i\}_{i=1}^n)| \le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{h}|) M^{\epsilon \underline{h}} \tag{6.80}$$

where $\underline{h} = \min\{h_i\}_{i=1}^n$

ii) Let $\mathfrak{p}_1 = \mathfrak{p}(G, G_1) \cup (l_o, l_i)$, with (l_o, l_i) the external lines with momentum l flowing through them. Let $k_1 \in L_2(G_1)$. Let $s_1 = 0, 1$. Let $m_1 = 1, \ldots, d$.

$$\bar{U}_1(h_1, k_1, m_1, s_1) = \sum_{t_1 \in \mathcal{T}(G_1, \mathfrak{p}_1)} \sum_{J_1 \in \mathcal{J}(G_1, t_1, h_1)} \zeta_1(k_1, G_1, J_1, (l_o, l_i))\,\mathrm{Val}(\partial_{k_1 m_1 s_1} G_1^{J_1}) \tag{6.81}$$

and

$$\bar{\mathcal{U}}(\{h_i\}_{i=1}^n; k_1, m_1, s_1; l, p, q) = \mathcal{U}(\{\bar{U}_1(h_1, k_1, m_1, s_1)\} \cup \{U_i(h_i)\}_{i=2}^n; l, p, q) \tag{6.82}$$

Then there exists a polynomial 'Pol' such that for all $0 \ge h_1 \ge h'$, $|\mathbf{q}| < \kappa_s$ and $s = 0, 1$:

$$I_2 \le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|h_1|) M^{\epsilon \underline{h}'} M^{(1-s)h_1} \tag{6.83}$$

$$I_2 = \begin{cases} \mathscr{W}_{1,0}(\bar{\mathcal{U}}(\{h_i\}_{i=1}^n; k_1, m_1, s_1); h', q, p) & s = 1 \\ \mathscr{W}_{1,0}(\mathcal{U}(\{h_i\}_{i=1}^n); h', q, p) & s = 0 \end{cases} \tag{6.84}$$

where $\underline{h}' = \min\{h_i\}_{i=2}^n \cup \{h'\}$

iii) When $n = 1$, let $k_1, k_2 \in L_2(G_1)$, $m_1, m_2 = 1, \ldots, d$, $s_1, s_2 = 0, 1$. Denote by

$$\begin{aligned}&\hat{U}_1(h_1, \{k_i, m_i, s_i\}_{i=1}^2)\\ &= \sum_{t_1 \in \mathcal{T}(G_1, \mathfrak{p}_1)} \sum_{J_1 \in \mathcal{J}(G_1, t_1, h_1)} \zeta_2(k_1, k_2, G_1, J_1, (l_o, l_i)) \operatorname{Val}(\partial_{k_1 m_1 s_1} \partial_{k_2 m_2 s_2} G_1^{J_1})\end{aligned} \tag{6.85}$$

There exists a polynomial 'Pol' such that for all $h_1' \leq h_1 \leq 0$, $h_2' \leq h_1 \leq 0$, and $|\mathbf{q}| < \kappa_s$:

$$I_3 = \mathscr{W}_2(\hat{U}_1(h_1, \{k_i, m_i, s_i\}_{i=1}^2), h_1', h_2', q) \leq \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(|\underline{h}'|) M^{\epsilon \underline{h}''} \tag{6.86}$$

where $\underline{h}'' = \min\{h_1', h_2', \{h_i\}_{i=2}^{n-1}\}$.

Proof. As noted above, Lemma 28 implies that $G/\{G_i\}_{i=1}^n$ is a dressed bubble chain of length $n-1$ containing only four legged graphs. For small $\mathbf{q}$, the proof of lemma is by induction on the length of the chain, i.e. n.

Let $n = 1$. Here $G = G_1$, the sums in (6.77) contain only the pathological scale assignment $j = h_1$, making h_1 the root scale and thus $\mathcal{U}(h_1; l, p, q) = U_1(h_1, l, p-q, l-q)$. Applying definition (6.69) gives

$$U_1 = \sum_{t \in \mathcal{T}(G, \emptyset)} \sum_{J \in \mathcal{J}(G, t, h_1)} \operatorname{Val}(G^J) \tag{6.87}$$

$$= \sum_{\substack{t \sim G \\ \tilde{G}(\phi) \text{ overlapping}}} \sum_{J \in \mathcal{J}(G,t,h_1)} \operatorname{Val}(G^J) + \sum_{\substack{t \sim G \\ \tilde{G}(\phi) \text{ non-overlapping} \\ \exists f : \pi(f) = \phi, E(G_f) = 2}} \sum_{J \in \mathcal{J}(G,t,h_1)} \operatorname{Val}(G^J) \tag{6.88}$$

For the overlapping graphs we can use the standard bound from Theorem 60, Proposition i to see that

$$\sum_{\substack{t \sim G \\ \tilde{G}(\phi) \text{ overlapping}}} \sum_{J \in \mathcal{J}(G,t,h_1)} |\operatorname{Val}(G^J)| \leq \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(|h_1|) M^{h_1 \epsilon} \tag{6.89}$$

When $\tilde{G}(\phi)$ is non-overlapping but contains a two legged graph we can not use Theorem 60 directly, but the result is still a direct corollary of its proof. Construct τ and G' as in the $P > 0$ case of that proof. Identity (5.67) holds and thus it suffices to find a bound for $\operatorname{Val}((G')^J)$. G' is a four legged graph and τ is of depth $P = 0$. Thus

$$\sum_{J \in \mathcal{J}(\tau, G', h_1)} |\operatorname{Val}((G')^J)| \leq \text{const} \prod_{\text{vertices } v_f} M^{-j_{\pi(f)}} \sum_{j_f} \sum_{J_f \in \mathcal{J}(\tau, G_f, j_f)} |P_{v_f} \operatorname{Val}(G_f^{J_f})| \tag{6.90}$$

(The restriction on the sum $\sum_{j_f}$ depends on the type of fork.) Note that at least one of the 2-legged vertices v_f, say v_g, has $\pi(g) = \phi$ and thus $j_{\pi(g)} = h_1$. For all others the scale sums are bounded using inequalities (5.72), (5.73), and (5.74). For v_g an R- or a C-fork we can apply (5.73) and (5.71) respectively for $\epsilon' = \epsilon$ to see that

$$M^{-j_{\pi(g)}} \sum_{j_g} \sum_{J_g \in \mathcal{J}(\tau, G_g, j_g)} |P_{v_g} \operatorname{Val}(G_f^{J_g})| \leq \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(|h_1|) M^{\epsilon h_1} \tag{6.91}$$

When G_g is a SSI we use (5.74) and apply the same technique inductively to G_g to see that the same bound holds because as a result of wick ordering a SSI can only occur at scale $j_g < 0$ if it contains a C or an R graph. Thus also here

$$\sum_{J \in \mathcal{J}(G,t,h_1)} |\operatorname{Val}(G^J)| \leq \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(|h_1|) M^{h_1 \epsilon} \tag{6.92}$$

and the lemma follows because the sums over trees just give finite constants.

To obtain the bounds for I_2 and I_3 make a decomposition analogous to (6.88):

$$U_1 = \sum_{t \in \mathcal{T}(G, \{(l_o, l_i)\})} \sum_{J \in \mathcal{J}(G,t,h_1)} \operatorname{Val}(G^J) \tag{6.93}$$

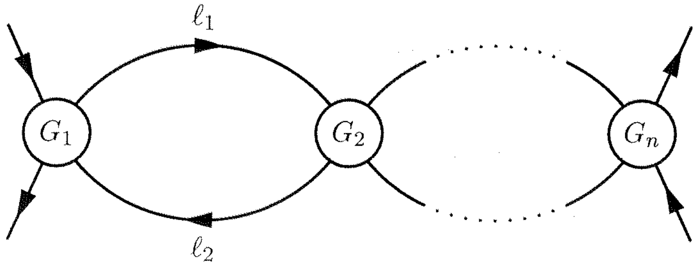
$$= \sum_{\substack{t \sim G \\ \tilde{W}_{(l_o,l_i)}(\tilde{G}(\phi)) \text{ overlapping}}} \sum_{J \in \mathcal{J}(G,t,h_1)} \operatorname{Val}(G^J) + \sum_{\substack{t \sim G \\ \tilde{W}_{(l_o,l_i)}(\tilde{G}(\phi)) \text{ non-ov.l.} \\ \exists f : \pi(f) = \phi, E(G_f) = 2}} \sum_{J \in \mathcal{J}(G,t,h_1)} \operatorname{Val}(G^J) \tag{6.94}$$

Observe that because of Remark 64 the lemma reduces to Lemma 63 and Lemma 66 respectively when $\tilde{W}_{(l_o,l_i)}(\tilde{G}(\phi))$ is overlapping. When $\tilde{G}(\phi)$ contains a two-legged vertex the bound is completely analogous to above. N.B. $\tilde{G}(\phi)$ is a DBC and thus the derivatives only act on the four-legged graphs at the end of the chain.

Longer chains at small $\mathbf{q}$

Assume as induction hypothesis that the lemma has been proven for all $1 \leq n' < n$. The proof proceeds in 2 ways depending on the value of $\mathbf{q}$.

When $|\mathbf{q}| < \kappa_s$ observe that by Lemma 28, $G/\{G_i\}_{i=1}^n$ is a bubble chain and in particular G_1 is connected to the rest of the graph by a pair of lines ℓ_1 and ℓ_2 attached to G_2.



Start with the case where $h_1 \neq 0$ and $h_2 \neq 0$. Denote by (j_i, s_i) the scale and the hard/soft-label of the line ℓ_i for $i = 1, 2$. The sum over trees and compatible scale assignments to $G/\{G_i\}_{i=1}^n$ is simply the sum over all scales that produce such a structure and thus includes all (j_1, s_1) and (j_2, s_2) such that $s_1 + s_2 \leq 1$ and $\max\{j_1, j_2\} < \underline{h}_1 = \min\{h_1, h_2\}$. Denote $G^* = G \setminus (G_1 \cup \{\ell_1, \ell_2\})$. Then

$$\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q) = \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C(x) C(x-q) S_{\underline{h}_1}(x) U_1(h_1; l, x, q) g^*(\{h_i\}_{i=2}^n; x, p, q) \quad (6.95)$$

where

$$g^*(\{h_i\}_{i=2}^n; x, p, q) = \sum_{j \leq 0} \sum_{t \sim G^*/\{G_i\}_{i=2}^n} \sum_{\substack{J_N \in \\ \mathcal{J}(G^*/\{G_i\}_{i=2}^n, t, j, \{h_i\}_{i=2}^n)}} \mathrm{Val}((G^*/\{G_i\}_{i=2}^n)^{J_N}; U_2', \{U_i\}_{i=3}^n)(x, p-q, x-q) \quad (6.96)$$

where we have written $U_2' = U_2$ with $\mathfrak{p}_2 = \mathfrak{p}(G, G_2) = \mathfrak{p}(G^*, G_2) \cup \{(\ell_1.\ell_2)\}$ to emphasize the fact that expanding G_2 makes the total graph overlap. Note however that because of Remark 64 this is redundant when $n > 3$ because then G^* is itself a non-trivial chain.

The expression (6.95) is now in the right form to apply interpolation and integration by parts. Applying Lemma 13 gives

$$|\mathcal{U}(\{h_i\}_{i=1}^n; p, l, q)| \leq \text{const} \sum_{\substack{a_1+a_2+a_3 \leq 0 \\ b=1,2}} u_{a_1 a_2 a_3 b} \quad (6.97)$$

with

$$u_{a_1 a_2 a_3 b} = \int_0^1 \mathrm{d}t \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} |Z_b(x, q, t)| |\nabla^{a_3} S_{\underline{h}_1}| |\nabla^{a_1} U_1| |\nabla^{a_2} g^*| \quad (6.98)$$

Using the support properties of S and its derivatives

$$u_{a_1 a_2 a_3 b} \leq \text{const}\, M^{-a_3 \underline{h}_1} \int_0^1 \mathrm{d}t \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} W^b_{\underline{h}_1}(x) |\nabla^{a_1} U_1| |\nabla^{a_2} g^*| \quad (6.99)$$

with

$$W_j^b(x) = \lim_{I' \to -\infty} W_j^b(x, I') \tag{6.100}$$

$$W_j^b(x, I') = \begin{cases} \frac{\mathbb{1}((l_0-q_0)^2+e(\mathbf{x},\mathbf{q},t)^2<4M^{2j})}{|i(l_0-q_0)+e(\mathbf{x},\mathbf{q},t)|} & \text{when } b=1 \\ \frac{\mathbb{1}(M^{2I'}<(l_0-tq_0)^2+e(\mathbf{x},\mathbf{q},t)^2<4M^{2j})}{|i(l_0+tq_0)+e(\mathbf{x})|} & \text{when } b=2. \end{cases} \tag{6.101}$$

As in the example case with two bubbles the finite I' expression

$$u_{a_1a_2a_3bI'} = \text{const}\, M^{-a_3\underline{h}_1} \int_0^1 \mathrm{d}t \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} W^b_{\underline{h}_1}(x, I') |\nabla^{a_1} U_1||\nabla^{a_2} g^*| \tag{6.102}$$

is bounded first. As we will see the bound will be independent of I' and therefore interchanging the limit and the t-integral is allowed, which implies the required bound for $u_{a_1a_2a_3b}$.

The x-integration is used to extract the factor coming from overlap and/or control the derivatives. When $a_3 = 1$, a factor $M^{\underline{h}_1}$ is needed to compensate and thus we keep the subgraph that has the lowest scale. If both are equal keep g^*, this takes care of the cases where G_1 is the single artificial vertex coming from the line with boson propagator 1.

When $a_1 = 1$ or $a_2 = 0$ and $h_1 < h_2$, bound $|g^*| \leq |g^*|_0$ and use expression (6.51) for the derivative. This gives

$$u_{a_1a_2a_3bI'} \leq \text{const}\, M^{-a_3\underline{h}_1} |g^*|_0 \psi_L(a_1) \tag{6.103}$$

$$\psi_L(a_1) = \begin{cases} \mathscr{W}^b_{1,0}(U_1|_{p\leftrightarrow l}, \underline{h}_1, q, l) & a_1 = 0 \\ \sum_{k_1\in L_2(G_1)} \sum_{m=1}^d \sum_{s_1=1}^2 \mathscr{W}^b_{1,0}(\bar{U}_1(h_1, k_1, m_1, s_1)|_{p\leftrightarrow l}, \underline{h}_1, q, l) & a_1 = 1 \end{cases} \tag{6.104}$$

Here the label b on $\mathscr{W}^b_{1,0}$ indicates that W^b_j appears. When $n = 2$, then G^* is a four legged graph with scales $\geq h_2$ and thus $||g^*|| \leq \text{Pol}(|h_2|)$ by theorem 60. Inserting this and using the $n = 1$ case of the induction hypothesis

$$u_{a_1a_2a_3bI'} \leq \lambda_2(c, \beta, \epsilon) \,\text{Pol}(|\underline{h}_1|) M^{\epsilon \underline{h}_1} M^{(1-a_1-a_3)h_1} \leq \lambda_2(c, \beta, \epsilon) \,\text{Pol}(|\underline{h}_1|) M^{\epsilon \underline{h}_1} \tag{6.105}$$

When $n > 2$ the bound is analogous: Note that then g^* is a nontrivial bubble chain and thus the value doesn't change if $\mathfrak{p})(G, G_2)$ is replaced by $\mathfrak{p}_2$. g^* is therefore exactly of the form I_1 with n replaced by $n - 1 < n$ and we can apply the induction hypothesis. By the IH we have $|g^*| \leq \text{const Pol}(|\underline{h} \geq 2|) M^{\epsilon \underline{h}_{\geq 2}}$, with $\underline{h}_{\geq 2} = \min\{h_i\}_{i=2}^n$, which when inserted gives

$$u_{a_1a_2a_3bI'} \leq \lambda_2(c, \beta, \epsilon) \,\text{Pol}(|\underline{h}_1|) M^{\epsilon \underline{h}_1} \,\text{Pol}(|\underline{h}_{\geq 2}|) M^{\epsilon \underline{h}_{\geq 2}} \leq \lambda_2(c, \beta, \epsilon) \,\text{Pol}(|\underline{h}'|) M^{\epsilon \underline{h}'} \tag{6.106}$$

Likewise, when $a_2 = 1$ or $a_1 = 0$ and $h_2 \leq h_1$

$$u_{a_1 a_2 a_3 b I'} \leq \text{const}\, M^{-a_3 \underline{h}_1} |U_1|_0 \psi_R(a_2) \tag{6.107}$$

$$\psi_R(a_2) = \begin{cases} \mathscr{W}^b_{1,0}(g^*, \underline{h}_1, q, p) & a_2 = 0 \\ \sum_{k_1 \in L_2(G_1)} \sum_{m=1}^{d} \sum_{s_1=1}^{2} \mathscr{W}^b_{1,0}(\mathcal{U}_2, q, p) & a_2 = 1 \end{cases} \tag{6.108}$$

with

$$\mathcal{U}_2 = \mathcal{U}(\{\bar{U}_2(h_2, k_1, m_1, s_1)\} \cup \{U_i(h_i)\}_{i=2}^{n}, \underline{h}_1) \tag{6.109}$$

Here we used that because of the structure of g^* any derivative with respect to $\mathbf{x}$ acts only on lines of G_2.

After applying the induction hypothesis to $\psi_R(a_1)$, which is of the form I_2 and using standard power counting to bound $|U_1|_0 \leq \text{Pol}(|h_1|)$, we get (denoting $m = \min\{\underline{h}_1\} \cup \{h_i\}_{i=2}^{n}$):

$$u_{a_1 a_2 a_3 b I'} \leq \lambda_2(c, \beta, \epsilon)\, \text{Pol}(|h_1|) M^{(1-a_3-a_2)h_2}\, \text{Pol}(|m|) M^{\epsilon m} \leq \lambda_2(c, \beta, \epsilon)\, \text{Pol}(|\underline{h}|) M^{\epsilon |\underline{h}|} \tag{6.110}$$

Inserting these bounds into (6.97) and taking the limit $I' \to \infty$ gives the bound of I_1.

Now we turn to I_2. We concentrate $s = 1$, the $s = 0$ case is similar but simpler. By the structure of the graph we once more have

$$Y_2 = \bar{\mathcal{U}}(\{h_i\}_{i=1}^{n}; l_1, m_1, s_1; l, p, q) =$$
$$\int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C(x) C(x-q) S_{\underline{h}_1}(x) \bar{U}_1(h_1, k_1, m_1, s_1; l, x, q) g^*(\{h_i\}_{i=2}^{n}; x, p, q) \tag{6.111}$$

and This implies

$$I_2 = \mathscr{W}_{1,0}(Y_2, h', q, p) \tag{6.112}$$

Apply Lemma 13. This gives

$$I_2 \leq \text{const} \lim_{I' \to -\infty} \sum_{\substack{a_1+a_2+a_3 \leq 0 \\ b=1,2}} v_{a_1 a_2 a_3 b I'} \tag{6.113}$$

with

$$v_{a_1 a_2 a_3 b I'} = \int_0^1 \mathrm{d}t \int \mathrm{d}_\beta l_0 \int \mathrm{d}\mathbf{l} W_{h'}(l) \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} |Z_b(x, q, t)| |\nabla^{a_3}_{\mathbf{x}} S_{\underline{h}_1}| |\nabla^{a_1}_{\mathbf{x}} \bar{U}_1| |\nabla^{a_2}_{\mathbf{x}} g^*| \tag{6.114}$$

When $a_2 = 1$ or $a_1 = 0$ and $h_2 \le h_1$ applying the same bounds as before gives

$$v_{a_1a_2a_3bI'} \le \text{const}\, M^{-a_3\underline{h}_1}\psi_R(a_2)\sup_x \int \mathrm{d}_\beta l_0 \int \mathrm{dl} W_{h'}(l)|\check{U}_1(k_1, s_1, 1)| \tag{6.115}$$

$$\le \text{const}\, M^{-a_3\underline{h}_1}\psi_R a_2 \sup_x \mathscr{W}_{1,0}(\bar{U}_1, h', q, x) \tag{6.116}$$

$$\le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|m|)M^{\epsilon m}M^{(1-a_2-a_3)h_2}M^{\epsilon h'}M^{(1-s)h_1} \tag{6.117}$$

$$\le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{h}'|)M^{\epsilon\underline{h}'}M^{(1-s)h_1} \tag{6.118}$$

when $a = 1$ or $a_2 = 0$ and $h_1 < h_2$, then

$$v_{a_1a_2a_3bI'} \le \text{const}\, |g^*|_0 M^{-a_3h_1}\Psi_L(a_1) \tag{6.119}$$

where

$$\Psi(a_1) = \begin{cases} \int \mathrm{d}_\beta x_0 \int \mathrm{dx} W^n_{\underline{h}_1}(x, I')\mathscr{W}_{1,0}(\bar{U}_1(h_1, k_1, m_1, s_1), h', q, x) & a_1 = 0 \\ \sum_{k_2 \in L_2(G_1)} \sum_{m_2=1}^d \sum_{s_2=0}^1 \mathscr{W}_2(\hat{U}_1(h_1, \{k_i, m_i, s_i\}_{i=1}2), h', \underline{h}_1, q) & a_1 = 1 \end{cases} \tag{6.120}$$

and denoting $m' = \min\{h', \underline{h}_1\}$, the induction hypothesis gives

$$v_{a_1a_2a_3bI'} \le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|m'|)|g^*|_0 \begin{cases} M^{\epsilon m'} & a_1 = 1 \\ M^{\underline{h}_1 - a_3h_1}M^{\epsilon h'} & a_1 = 0 \end{cases} \tag{6.121}$$

using $h' \le \underline{h}_1 \implies M^{h'+\epsilon\underline{h}_1} < M^{\epsilon\underline{h}_1}M^{h_1}$

$$\le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|m'|)|g^*|_0 M^{\epsilon m'}M^{(1-s)h_1} \tag{6.122}$$

$$\le \lambda_2(c, \beta, \epsilon)\,\mathrm{Pol}(|\underline{h}'|)M^{\epsilon\underline{h}'}M^{(1-s)h_1} \tag{6.123}$$

Inserting this in (6.113) gives the bound for I_2.

Chains at larger $\mathbf{q}$

Consider the case $|\mathbf{q}| > \kappa_s$ with $\underline{h}_1 < 0$. Similar to Section 2.2.1, there is a volume gain coming from transversality of the Fermi surface and its translate by $\mathbf{q}$. This eliminates the need for integration by parts. Recall that by the structure of the graph $G/\{G_i\}_{i=1}^n$, we have (c.f. (6.95)).

$$\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q) = \int \mathrm{d}_\beta x_0 \int \mathrm{dx} C(x)C(x-q)S_{\underline{h}_1}(x)U_1(h_1; l, x, q)g^*(\{h_i\}_{i=2}^n; x, p, q) \tag{6.124}$$

Reintroducing the scale decomposition for the propagators we have that

$$|\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q)| \leq \text{const} \sum_{j_1 \leq \underline{h}_1} \sum_{j_2 \leq \underline{h}_1} I_{j_1 j_2} \tag{6.125}$$

with

$$\begin{aligned} I_{j_1 j_2} = M^{-j_1 j_2} \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} \mathbb{1}_{j_1}(|x_0|) \mathbb{1}_{j_1}(e(|\mathbf{x}|)) \\ \mathbb{1}_{j_2}(|x_0 - q_0|) \mathbb{1}_{j_2}(e(|\mathbf{x} - \mathbf{q}|)) |U_1(h_1; l, x, q)| g^*(\{h_i\}_{i=2}^n; x, p, q) \end{aligned} \tag{6.126}$$

Apply the normal power counting bound for the four-legged graph U_1, $|U_1| \leq \text{Pol}(|h_1|)$. As $|\mathbf{q}| > \kappa_s$ and for $d = 2$, $|\mathbf{q}| \leq 1 < 2 - \kappa_s$, Corollary 36 can be applied to see that

$$I_{j_1 j_2} \leq \text{Pol}(h_1) \sup_{\mathbf{x} \in U(S,\delta)} \int \mathrm{d}_\beta x_0 \mathbb{1}_{j_1}(|x_0|) \mathbb{1}_{j_2}(|x_0 - q_0|) |g^*(\{h_i\}_{i=2}^n; x, p, q)| \tag{6.127}$$

When $n > 2$, we can apply the induction hypothesis to to see that $|g*|(\{h_i\}_{i=2}^n; x, p, q) \leq \lambda_2(c, \beta, \epsilon) \text{Pol}(|\underline{h}_{\geq 2}|) M^{\epsilon \underline{h}_{\geq 2}}$, where $\underline{h}_{\geq 2} = \min\{h_2, \ldots, h_n\}$. Then it follows that

$$I_{j_1 j_2} \leq \lambda_2(c, \beta, \epsilon) \text{Pol}(\underline{h}) M^{\epsilon \underline{h}} M^{\underline{j}} \tag{6.128}$$

where $\underline{j} = \min\{j_1, j_2\}$. When $n = 2$, $g^* = U_2$. Denote by (l_o, l_i) the pair of external lines of G_2 that connect to the rest of the graph, then

$$U_2 = \sum_{t \in \mathcal{T}(G_2, (l_o, l_i))} \sum_{J \in \mathcal{J}(G_2, t, h_2)} \text{Val}(G_2^J) \tag{6.129}$$

$$= \sum_{t \in \mathcal{T}(G_2, \emptyset)} \sum_{J \in \mathcal{J}(G_2, t, h_2)} \text{Val}(G_2^J) + \sum_{t \in N(G_2)} \sum_{J \in \mathcal{J}(G_2, t, h_2)} \text{Val}(G_2^J) \tag{6.130}$$

where $N(G_2)$ the set of all trees $t \sim G_2$ such that $\tilde{G}_2(\phi)$ non-overlapping, $\tilde{W}_{(l_o, l_i)}(G_2)$ overlapping, and such that there exist no fork f with $\pi(f) = \phi, E(G_f) = 2\}$. By the induction hypothesis the sum on the left-hand side is bounded by $\lambda(c, \beta, \epsilon) \text{Pol}(|h_2|) M^{\epsilon h_2}$ as above. For $t \in N(G)$ there must be a phonon propagator in $\text{Val}(G^J)(x, p - q, x - q)$ that depends on x_0 when the momenta are fixed compatible to the scales. Therefore by applying Lemma 47 in the x_0 integral.

$$\sum_{J \in \sqcup, \mathcal{G}_\in, \langle_\in} \int \mathrm{d}_\beta x_0 \mathbb{1}_{j_1}(|x_0|) \mathbb{1}_{j_2}(|x_0 - q_0|) \text{Val}(G_2^J)(x, p - q, x - q) \leq \lambda_2(c, \beta, \epsilon) \text{Pol}(|h_2|) M^{\epsilon \underline{j}} \tag{6.131}$$

Summarizing

$$|\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q)| \leq \lambda_2(c, \beta, \epsilon) \text{Pol}(\underline{h}) \sum_{j_1 \leq \underline{h}_1} \sum_{j_2 \leq \underline{h}_1} M^{\epsilon \underline{j}} M^{\epsilon \underline{h}_{\geq 2}} \leq \lambda_2(c, \beta, \epsilon) \text{Pol}(\underline{h}) M^{\epsilon \underline{h}} \tag{6.132}$$

Scale zero bubble

In the above we have made the restriction $\underline{h}_1 < 0$. In this section we study the case where $\underline{h}_1 = 1$. Summing the scales using (6.61) gives

$$\mathcal{U}(\{h_i\}_{i=1}^n; l, p, q) = \begin{cases} U_{01} + U_{02} + U_{03} + U_{04} & G_1 \text{ and } G_2 \text{ single phonon lines} \\ U_{01} & \text{otherwise} \end{cases} \tag{6.133}$$

where

$$\begin{aligned} U_{01} &= \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C(x) C(x-q) S_0(x) U_1(0; l, x, q) g^*(\{h_i\}_{i=2}^n; x, p, q) \\ U_{02} &= \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C_{<0}(x) C_0(x-q) U_1(0; l, x, q) g^*(\{h_i\}_{i=2}^n; x, p, q) \\ U_{03} &= \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C_0(x) C_{<0}(x-q) U_1(0; l, x, q) g^*(\{h_i\}_{i=2}^n; x, p, q) \\ U_{02} &= \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} C_0(x) C_0(x-q) U_1(0; l, x, q) g^*(\{h_i\}_{i=2}^n; x, p, q) \end{aligned} \tag{6.134}$$

The term U_{01} is the result of summing over the scales $j_1, j_2 < \underline{h}_1 = 0$. If the graphs G_1 and G_2 are not single phonon lines, then they must contain scale 0 fermion lines and therefor the bubble cannot contain them. Thus U_{01} is the only contribution. U_{01} is of exactly the same form as before, so it can bounded analogously.

In the new terms U_{02} and U_{03} we use the fact that $|C_0| < 2M$ which means that the integral behaves just as in the larger $\mathbf{q}$ case with gain $M^0 = 1 = M^{\underline{h}_1}$. Thus we have using $|U_1| \leq 1$ and the induction hypothesis

$$|U_{02}| \leq \text{const} \int \mathrm{d}_\beta x_0 \int \mathrm{d}\mathbf{x} |C_{<0}(x)||g^*(\{h_i\}_{i=2}^n; x, p, q)| \tag{6.135}$$

$$\leq \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(\underline{h}_{\geq 2}) M^{\epsilon \underline{h}_{\geq 2}} = \lambda_2(c, \beta, \epsilon) \operatorname{Pol}(\underline{h}) M^{\epsilon \underline{h}} \tag{6.136}$$

and a similar bound for U_{03}.

By definition

$$|U_{04}| \leq \mathscr{W}_0(g^*(\{h_i\}_{i=2}^n; x, p, q)) \tag{6.137}$$

When $n > 2$, g^* is the value of a nontrivial bubble chain and the result follows from (6.65) and the induction hypothesis. If $n = 2$, then $g^*(x, p, q) = \operatorname{Val}(G_2)(x, p, q) = D(x_0 - p_0, c|\mathbf{x} - \mathbf{p}|)$, and using Lemma 54

$$|U_{04}| \leq \text{const}\, \lambda_2(c, \beta, \epsilon) = \text{const}\, \lambda_2(c, \beta, \epsilon) M^{\underline{h}_1} = \text{const}\, \lambda_2(c, \beta, \epsilon) M^{\underline{h}} \tag{6.138}$$

In case there is a derivative acting from the left, it must hit U_1 and we keep that in the integral, otherwise that case is completely analogous. □

6.4 The final result

Having done all the leg-work in the previous sections the proof of the Theorem we set out show is now a direct result.

Theorem 3. For a graph G contributing to $\Gamma(p,q)$ construct the corresponding 4-legged graph G^*. Apply decomposition (6.76) and proposition i of Theorem 67.

$$|\mathrm{Val}(G)(p,q)| \leq |\mathrm{Val}(G^*)(p,q)| \leq \sum_{n\geq 1} \sum_{\{G_i\}_{i=1}^n \in \mathcal{A}_n(G^*)} \sum_{\substack{\{h_i\}_{i=1}^n \\ h_i \leq 0 \, \forall i}} \lambda_2(c,\beta,\epsilon) \, \mathrm{Pol}(|\underline{h}|) M^{\epsilon \underline{h}} \tag{6.139}$$

where $\underline{h} = \min\{h_i\}_{i=1}^n$.

$$\leq \sum_{\underline{h} \leq 0} \lambda_2(c,\beta,\epsilon) \, \mathrm{Pol}(|\underline{h}|) M^{\epsilon \underline{h}} \tag{6.140}$$

$$\leq \lambda_2(c,\beta,\epsilon) \tag{6.141}$$

which is the required result. □

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Appendix A

The one-loop β-Dependent Term

A.1 The self-energy

The main point of this appendix is to argue that the $O(\beta^{-1})$ dependent term can be bounded in complete analogy to contribution that is proportional to c. To illustrate this we do the computation for the self energy in detail.

Continuation of the proof of Lemma 11. In section 1.4.3 the additional contribution was identified as.

$$\Sigma_0(p) = \frac{1}{\beta} \int_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{1}{ip_0 - e(\mathbf{l})} \tag{A.1}$$

As stated we will proceed exactly as in section 1.4.3. First split of the singular region

$$\Sigma_0(p_0) = \sigma_R + \sigma_S \tag{A.2}$$

where

$$\begin{aligned} \sigma_R &= \frac{1}{\beta} \int_{\Omega} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{\mathbb{1}(|p_0| \geq \frac{1}{2} \wedge |e(\mathbf{l})| \geq \delta)}{ip_0 - e(\mathbf{l})} \\ \sigma_S &= \frac{1}{\beta} \int_{U(S,\delta)} \frac{\mathrm{d}\mathbf{l}}{(2\pi)^d} \frac{\mathbb{1}(|p_0| < \frac{1}{2})}{ip_0 - e(\mathbf{l})} \end{aligned} \tag{A.3}$$

The regular contribution σ_R is trivially bounded: $|\sigma_R| \leq \frac{1}{\beta} \frac{\mathrm{vol}(\Omega)}{\delta (2\pi)^d}$. When $|p_0| \geq \frac{1}{2}$ then $\sigma_S = 0$. So can we assume $|p_0| < \frac{1}{2}$ in the following and drop the indicator function from the notation.

A change to polar coordinates followed by an integration by parts gives

$$\sigma_S = \frac{1}{\beta}\int_{-\delta}^{\delta} \mathrm{d}\rho \int \frac{\mathrm{d}\theta}{(2\pi)^d} \frac{J_1(\boldsymbol{\pi}(\rho,\theta))}{ip_0 - \rho} \tag{A.4}$$

$$= B_0 - I_0 \tag{A.5}$$

where

$$B_0 = \frac{-1}{\beta}\left[\mathrm{Log}(ip_0 - \rho)\int \frac{\mathrm{d}\theta}{(2\pi)^d} J_1(\boldsymbol{\pi}(\rho,\theta))\right]_{\rho=-\delta}^{\rho=\delta} \tag{A.6}$$

$$I_\Sigma = \frac{-1}{\beta}\int_{-\delta}^{\delta} \mathrm{d}\rho\, \mathrm{Log}(ip_0 - \rho)\int \frac{\mathrm{d}\theta}{(2\pi)^d}\frac{\partial}{\partial\rho} J_1(\boldsymbol{\pi}(\rho,\theta)) \tag{A.7}$$

Because $|p_0| < \frac{1}{2}$ and $|\rho| \le \frac{1}{2}$ we have as before $|\mathrm{Log}(ip_0 - \rho)| < 2|\log|\rho||$. Therefore the two terms are bounded as

$$|B_0| \le \frac{1}{\beta}\frac{2|\log\delta|\,\mathrm{Vol}(S^{d-1})(1+\delta)^{d-2}}{(2\pi)^d}) \tag{A.8}$$

and

$$|I_0| \le \frac{2}{\beta}\int_{-\delta}^{\delta} \mathrm{d}\rho|\log|\rho||\int \frac{\mathrm{d}\theta}{(2\pi)^d}\left|\frac{\partial}{\partial\rho} J_1(\boldsymbol{\pi}(\rho,\theta))\right|$$

$$\le \frac{1}{\beta}\frac{d^2\,\mathrm{vol}(S^{d-1})(1+\delta)^{d-2}}{(2\pi)^d}\int_{-\delta}^{\delta}\mathrm{d}\rho(-\log|\rho|) \le \frac{1}{\beta}\frac{2d^2\,\mathrm{vol}(S^{d-1})(1+\delta)^{d-2}}{(2\pi)^d} \tag{A.9}$$

Therefore the Lemma holds with

$$M^0_\Sigma = \frac{\mathrm{vol}(S^{d-1})}{(2\pi)^d}\left(\frac{\Lambda^d}{\delta} + 2(1+\delta)^{d-2}|\log\delta| + 2d^2(1+\delta)^{d-2}\right) \tag{A.10}$$

□

A.2 The vertex correction

Note that the proof given above is in fact identical to that of the $O(c)$ term, after the replacement of D^* by a Kronecker-delta that sets $l_0 = p_0$. p_0 is still a Fermion frequency, i.e., non-zero at fine β. The l_0 sum is taken last and only computed after an l_0 independent bound has been found for the other factors in the integral.

Inspection of the proofs in Chapter 2 shows that the same is the case there. A replacement of $D^*(l_0 - p_0, c|\mathbf{l} - \mathbf{p}|)$ by $\delta_{p_0 l_0}$ gives no problems and leads to bounds where c is replaced by β^{-1}. In particular note the following

- The integral of the frequencies does not produce a factor $|\mathbf{l} - \mathbf{p}|$. However these were only needed to control such factors coming from the derivative of D. In case of the Kronecker-delta there are no such terms.
- Although the derivatives of $\delta_{p_0 l_0}$ with respect to $\mathbf{p}$ vanish, integration by parts is still necessary because of the presence of cut-off functions and Jacobians.
- The presence of a frequency integral/sum is not needed to do interpolation in the frequency argument. As can be seen from (2.85), the t integral plays the role of the frequency integral there.
- For $|\mathbf{q}| > \kappa_t$ the bound for the other factors is no longer independent of l_0 at the time the l_0 sum is evaluated. However the logarithmic divergency leads to logarithmic factors in β in exactly the same way as for c.

Apart from these points the calculations are simply a repeat of those in Chapter 2, albeit somewhat simpler at times, so we refrain from doing them in detail here.

Curriculum Vitae

A sun of Johan and Miep Vroonhof-Smit, Jan (Johannes Gerardus Maria) Vroonhof was born April 28th, 1972 in Leiderdorp, the Netherlands. From 1985 until 1990 he attended the VWO Gymnasium at Bonaventura College in Leiden (NL). In September 1990 he started studying Physics and Mathematics in parallel at the University of Leiden and obtained the combined Mathematics-Physics propaedeuse in August 1991.

As a part of the physics course he did a semester internship in Experimental Physics on High-temperature superconductors at the Kamerlingh Onnes Laboratorium in Leiden. His master thesis in theoretical physics was done beteen February 1994 and April 1995 at the Lorentz Institute of Theoretical physics. The work, done under the supervision of Dr. P.J.M. Bongaarts concerned the connections between the Antifield formalism for (quantum) gauge theories and the mathematical theory of 'Variatioal Bi-complex'.

Afterwards, from May until September 1995, he wrote a Master thesis in Mathematics at the university's Institute of Mathematics under the supervision of Prof. Dr. G. van Dijk. The work handled about explicit decompositions of L_2-spaces over the classical homogeneous spaces. He graduated from both courses in September 1995.

In October 1995 he moved to Zürich and started working towards his Ph.D. under supervision of Prof. Dr. Manfred Salmhofer. Until April 2000 he worked as a research and teaching assistant at the Mathematics department of the Federal Institute of Technology in Zürich.

Zusammenfassung

Eine der Grundlagen der modernen physikalischen Beschreibung von Metallen ist die Elektron-Phonon-Theorie. Sie beschreibt die Wechselwirkung der Elektronen mit Schwingungen des Ionen-Gitter (Schalwellen). In der Elektron-Phonon-Theorie funktioniert die Standard–Berechnungsmethode physikalischer Größen mit Hilfe einer Störungsentwicklung in der Kopplungskonstante nicht mehr, weil diese Konstante normalerweise nicht klein ist.

Alle Glieder höherer Ordnung in Betracht ziehen zu müssen macht die Berechnungen sehr schwierig. Deshalb ist es üblich, eine Näherung zu machen, die 1953 erstmals von Migdal vorgeschlagen wurde. Sie besteht daraus, die Beiträge höherer Ordnung an den Wechselwirkungsvertex zu vernachlässigen. Dies bringt eine sehr starke Vereinfachung der Hauptgleichungen der Theorie hervor. Migdal begründete diese Näherung damit, dass (bei Temperatur Null) die Korrekturen höherer Ordnung linear in der Schallgeschwindigkeit c verschwinden würden. Für gewöhnliche Metalle ist c tatsächlich klein im Vergleich zu den anderen relevanten Parametern. Diese Aussage ist unter dem Name „Das Migdalsche Theorem“ bekannt worden, obwohl nach unseren Kenntnissen, nie ein rigoroser Beweis veröffentlicht wurde. Für das Korrekturglied tiefster Ordnung gab Migdal ein skizzenhaftes Argument und behauptete weiter, für höhere Ordnungen würde es nicht anders gehen. Darin sind ihm spätere Autoren gefolgt, die auch die Aussage auf Temperaturen ungleich Null ausdehnten.

In dieser Arbeit wird die Elektron-Phonon-Theorie in einer ihrer einfachsten Varianten, dem Jellium-Modell, betrachtet in Form einer statistischen Quantenfeld-Theorie bei nicht verschwindender Temperatur und mit einem Ultraviolett-Cut-Off. Es wird rigoros gezeigt, dass das Korrekturglied tiefster Ordnung tatsächlich von der Ordnung $O(c)$ ist bis auf ein mit der Temperatur verschwindendes Glied. Dies wird mit Hilfe eines Feynman-Tricks und wiederholter Partieller Integration gemacht.

Eine Formulierung der Theorie in welcher der Temperatur-Null Limes existiert, erfor-

dert eine Renormierung. In dieser Arbeit wird eine so genannte Fermi-Flächen-Renormierung vorgenommen, wobei zur Elektron-Energiefunktion neue Glieder hinzugefügt werden. Mit Hilfe der Renormierungsgruppe und einer Skalenzerlegung werden diese Glieder genau definiert und es wird durch bestimmte temperaturunabhängige Schranken für den in der Theorie auftretende Graphen gezeigt, dass der Temperatur-Null-Limes existiert.

Schliesslich, wird gezeigt, dass für die renormierte Theorie gilt, dass für alle $0 < \epsilon < 1$ die Vertexkorrekturen der Ordnung r beschränkt sind durch

$$M_r(\epsilon) \left\{ c^{1-\epsilon} + \left(\frac{(\log \beta + 1)^2}{\beta} \right)^{1-\epsilon} \right\}$$

für eine ϵ-abhängige Konstante $M_r(\epsilon)$, wobei β die inverse Temperatur ist. Dabei wird die Methode des Feynman-Tricks und der Partiellen Integration mit der Skalenzerlegung kombiniert.