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Hofer geometry for Lagrangian loops, a Legendrian knot and a travelling wave

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Abstract

We examine three invariants of exact loops of Lagrangian submanifolds that are modelled on invariants introduced by Polterovich for loops of Hamiltonian symplectomorphisms. One of these is the minimal Hofer length in a given Hamiltonian isotopy class. We determine the exact values of these invariants for loops of projective Lagrangian planes. The proof uses the Gromov invariants of an associated symplectic fibration over the 2-disc with a Lagrangian subbundle over the boundary.

The last two chapters concern different topics and can be read completely independently.

Zusammenfassung

Wir untersuchen drei Invarianten einer exakten 1-parametrigen periodischen Schar von Lagrange'schen Untermannigfaltigkeiten, die modelliert sind nach den von L.Polterovich eingeführten Invarianten für 1-parametrige periodische Scharen von Hamilton'schen Symplektomorphismen. Eine davon ist die minimale Hofer Länge in einer gegebenen Hamilton'schen Isotopieklasse. Wir bestimmen die genauen Werte dieser Invarianten für Scharen von projektiven Lagrange'schen Ebenen. Der Beweis verwendet die Gromov Invarianten einer dazugehörigen symplektischen Faserung über der 2-Scheibe mit vorgegebenem Lagrange'schen Unterbündel auf dem Rand.

Die letzten zwei Kapitel betreffen andere Themen und können völlig unabhängig gelesen werden.

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Chapter 1

Introduction

Most of this work concerns the study of Hofer geometry for exact loops of Lagrangian submanifolds of a symplectic manifold (M, ω). Think of such a loop as a submanifold $\Lambda \subset S^1 \times M$ such that the projection $\Lambda \to S^1$ is a submersion and

$$\Lambda_t := \{z \in M \quad (e^{2\pi it}, z) \in \Lambda\}$$

is a Lagrangian submanifold of M for every t. The loop is called exact if there exists a Hamiltonian isotopy ψ_t of M such that $\psi_t(\Lambda_0) = \Lambda_t$ for every t. By definition, the Hamiltonian isotopy is the flow defined by a time dependent Hamiltonian function H_t via

$$\frac{d}{dt}\psi_t = X_{H_t} \circ \psi_t \text{ and } \psi_0 = \text{id}.$$

Note that the isotopy defines the Hamiltonian function uniquely up to a time dependent function and hence the Hofer length of an exact Lagrangian loop Λ is well defined and given by

$$\ell(\Lambda) := \int_0^1 \left(\max_{\Lambda_t} H_t - \min_{\Lambda_t} H_t\right) dt,$$

where the Hamiltonian functions $H_t : M \to \mathbb{R}$ are chosen such that the corresponding Hamiltonian isotopy $\psi_t : M \to M$ satisfies $\psi_t(\Lambda_0) = \Lambda_t$. It is interesting to minimize the Hofer length over the Hamiltonian isotopy class of Λ. Here two exact Lagrangian loops $t \mapsto \Lambda_t$ and $t \mapsto \Lambda'_t$ are called Hamiltonian isotopic if there exists a smooth function $[0, 1] \times \mathbb{R} \to \mathcal{L} : (s, t) \mapsto \Lambda_{s,t}$ such that

$$\Lambda_{0,t} = \Lambda_t, \qquad \Lambda_{1,t} = \Lambda'_t,$$

the map $t \mapsto \Lambda_{s,t}$ is an exact Lagrangian loop for every s, and the 1-form $\partial_s \Lambda_{s,t} \in \Omega^1(\Lambda_{s,t})$ is exact for all s and t This infimum is one of the central notions of this thesis and will be denoted by

$$\nu(\Lambda) = \nu(\Lambda; M, \omega) := \inf_{\Lambda \sim \Lambda'} \ell(\Lambda').$$

As an explicit example consider the space $\mathcal{L} = \mathcal{L}(\mathbb{C}P^n, \mathbb{R}P^n)$ of Lagrangian submanifolds of CP" that are diffeomorphic to RP". It contains the finite dimensional manifold $\mathrm{PL}(n+1)$ of projective Lagrangian planes. The space $\mathrm{PL}(n+1)$ is the orbit of $\mathbb{R}P^n$ under the action of $\mathrm{PU}(n+1)$ and its fundamental group is isomorphic to $\mathbb{Z}_{n+1}$. Consider the loop $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ defined by

$$\Lambda^k := \bigcup_{t \in \mathbb{R}} \{e^{2\pi i t}\} \times \phi_{kt}(\mathbb{R}P^n), \tag{1.1}$$

where $\phi_t([z_0 : \dots : z_n]) := [e^{\pi i t} z_0 : z_1 : \dots : z_n]$ and $k \in \mathbb{Z}$. The loops Λ^j and Λ^k are homotopic in $\mathrm{PL}(n+1)$ (as based loops) if and only if they are Hamiltonian isotopic (as free loops) if and only if $k - j$ is divisible by $n+1$. If $k - j$ is not divisible by $n+1$ then Λ^j and Λ^k can be distinguished by the Maslov index. More precisely, every Lagrangian loop $\Lambda \subset S^1 \times \mathbb{C}P^n$, with fibres Λ_t Lagrangian isotopic to RP", has a well defined Maslov index $\mu(\Lambda) \in \mathbb{Z}_{n+1}$. It is defined as the Maslov index of a smooth map $u : D = \{z \in \mathbb{C} \mid |z| \leq 1\} \to M$ such that $u(e^{2\pi i t}) \in \Lambda_t$. Such maps u always exist and the Maslov indices of any two such maps differ by an integer multiple of $n+1$. It turns out that

$$\mu(\Lambda^k) \equiv k \bmod n+1. \tag{1.2}$$

In the case $n = 1$ the loop Λ^1 is obtained by rotating a great circle on the 2-sphere through 180 degrees around an axis that passes through the circle. The result is an embedding of the Klein bottle into $S^1 \times S^2$.

The image of this embedding is a Lagrangian submanifold of $D \times S^2$ with respect to a suitable symplectic form. In contrast Λ^0 is a Lagrangian torus in $D \times S^2$. In general, the cases where n is even and where n is odd are topologically different. If n is even, then Λ^k is diffeomorphic to S^1 x RP" for every k. If n is odd then Λ^j is diffeomorphic to Λ^k if and only if $k - j$ is even, and Λ^k is orientable if and only if k is even. In particular, observe that Λ^k is diffeomorphic to $\Lambda^0 = S^1 \times \mathbb{R}P^n$ whenever k is even.

Fix $k \in \{1, , n\}$ and consider the exact Lagrangian loop

$$\Lambda := \bigcup_{t \in \mathbb{R}} \{e^{2\pi i t}\} \times \psi_t(\mathbb{R}P^n),$$

where

$$\psi_t([z_0 : : z_n]) := ([z_0 : e^{\pi i t} z_1 : \cdots : e^{\pi i t} z_k : z_{k+1} : \cdots : z_n])$$

This loop is Hamiltonian isotopic to Λ^k and it has Hofer length $1/2$, whereas Λ^k has Hofer length $k/2$. The next theorem asserts that Λ minimizes the Hofer length in its Hamiltonian isotopy class and hence is a geodesic for the Hofer metric.

Theorem A *Let $\omega \in \Omega^2(\mathbb{C}P^n)$ denote the Fubini-Studyform that satisfies rhe normalization condition $\int_{\mathbb{C}P^n} \omega^n = 1$. Then*

$$\nu(\Lambda^k; \mathbb{C}P^n, \omega) = \frac{1}{2}$$

for $k = 1, \ldots, n$ and $\nu(\Lambda^0) = 0$.

This is a Lagrangian analogue of a theorem by Polterovich [Pl] conceming loops of Hamiltonian symplectomorphisms of complex projective space. In order to prove Theorem A we follow the strategy of [Pl] and introduce two other invariants of exact Lagrangian loops $\Lambda \subset S^1 \times M$ that can be expressed in terms of Hamiltonian connection 2-forms τ on the trivial bundle $D \times M$ that vanish over Λ. Let 7(A) $\subset \Omega^2(D \times$ M) denote the space of such connection 2-forms. The relative K-area $\chi(\Lambda)$ is obtained by minimizing the Hofer norm of the curvature Ω_τ over 7'(A). The third invariant is related to the relative cohomology classes $[\tau] \in H^2(D \times M, \Lambda; \mathbb{Z})$ of $\tau \in$ 7(A). These form a l-dimensional affine space parallel to the subspace generated by the integral cohomology class $\sigma :=$ $[dx \wedge dy/\pi]$. For τ_0, $\tau_1 \in \mathcal{T}(\Lambda)$ define $s(\tau_1, \tau_0) \in \mathbb{R}$ by $s(\tau_1, \tau_0)\sigma =$ $[\tau_1] - [\tau_0]$. The invariant $\varepsilon(\Lambda)$ is defined by

$$\text{E(A)} := \varepsilon^+(\tau_0, \Lambda) - \varepsilon^-(\tau_0, \Lambda),$$

for $\tau_0 \in \mathcal{T}(\Lambda)$, where

$$\varepsilon^+(\tau_0, \text{A}) := \inf\{s(\tau, \tau_0) \mid \tau \in 7(\text{A}),\ \tau^{n+1} > 0\},$$

$$\varepsilon^-(\tau_0, \Lambda) := \sup\{s(\tau, \tau_0)\ \ \tau \in \mathcal{T}(\Lambda),\ \tau^{n+1} < 0\}.$$

Theorem B *For every exact* ***Lagrangian loop*** $\Lambda \subset S^1 \times M$

$$\varepsilon(\Lambda) \le \chi(\Lambda) = \nu(\Lambda).$$

A lower bound for $\varepsilon(\Lambda)$ can sometimes be obtained by studying pseudoholomorphic sections of $D \times M$ with boundary values in Λ. We assume that the pair (M, An) is monotone and fix a relative homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ that satisfies

$$n \pm \mu_\Lambda(A) \le N - 2,$$

where $n =$ dim An $=$ dim $M/2$, N denotes the minimal Maslov number of the pair (M, Λ_0), and μ_Λ denotes the Maslov class. Under these assumptions we define Gromov invariants

$$\text{Gr}_A^{\pm}(\Lambda) \in H_{n\pm\mu_\Lambda(A)}(\Lambda_0; \mathbb{Z}_2)$$

A connection 2-form $\tau \in$ 7(A) and an ω-compatible almost complex structure J on M determine an almost complex structure $\tilde{J} = \tilde{J}(\tau,$ J) on $D \times M$. Under our assumptions the moduli space of $\tilde{J}(\tau,$ f/)-holomorphic sections of $D \times M$ is, for a generic τ, a compact smooth manifold of dimension $n \pm \mu_\Lambda(A)$. The Gromov invariant is defined as the image of the mod-2 fundamental class under the evaluation map $u \mapsto$ u(1). Now let $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ be given by (1.1) with $1 \le k \le n$. Let $A^\pm \in$ $H_2(D \times \mathbb{C}P^n, \Lambda^k; \mathbb{Z})$ be the homology classes of the constant sections $u^+(x,$ y) $\equiv$ [1 : 0 : : 0] and $u^-(x,$ y) $\equiv$ [0 : : 0 : 1].

Theorem C $\mathrm{Gr}^{\pm}_{A^{\pm}}(\Lambda^k) \neq 0$.

Theorem C can be interpreted as an existence result for pseudoholomorphic sections and we shall use this to prove that $\varepsilon(\Lambda^k) \geq 1/2$. On the other hand the Hamiltonian isotopy class of Λ^k contains a loop of length equal to $1/2$. Hence Theorem A follows from Theorem B.

We expect that the same techniques can be used to obtain similar results for general symplectic quotients of $\mathbb{C}^n$ by subgroups of U(n). These quotients will not, in general, satisfy our assumption of monotonicity for the definition of the Gromov invariants. However, it should be possible to derive the same conclusions by using the invariants introduced in Cieliebak–Gaio–Salamon [CGS] instead. This Programme will be carried out elsewhere.

We conclude the introduction by describing the background of this problem. In [P1, P2, P3, P4] Polterovich studied the Hofer length of loops

$$\psi_t = \psi_{t+1} : M \to M$$

of Hamiltonian symplectomorphisms. Let $P \to S^2$ denote the Hamiltonian fibration associated to the Hamiltonian loop. Polterovich introduced invariants $\nu^{\pm}(P)$, $\chi^{\pm}(P)$, and $\varepsilon^{\pm}(P)$ on which our invariants are modelled. Here $\nu^{+}(P)$ is obtained by minimizing the positive part of the Hofer length in a given Hamiltonian isotopy class, the K-area $\chi^{+}(P)$ is a symplectic analogue of an invariant introduced by Gromov [G2], and the invariant $\varepsilon^{+}(P)$ is based on the coupling construction of Guillemin–Lerman–Sternberg [GLS]. In [P1, P2] Polterovich proves that these invariants are equal:

$$\varepsilon^{\pm}(P) = \chi^{\pm}(P) = \nu^{\pm}(P).$$

We adopt the convention $\pm\nu^{\pm}(P) \geq 0$. Let us denote by $\nu(P)$, $\chi(P)$, and $\varepsilon(P)$ the Hamiltonian analogues of our invariants of Lagrangian loops. These were also considered by Polterovich and he noted that

$$\varepsilon(P) = \varepsilon^{+}(P) - \varepsilon^{-}(P) = \nu^{+}(P) - \nu^{-}(P) \leq \nu(P)$$

This is the Hamiltonian analogue of Theorem B. Now consider the Lagrangian loop $\Lambda \subset S^1 \times \overline{M} \times M$ given by

$$\Lambda_t = \mathrm{graph}(\psi_t).$$

The invariants introduced by Polterovich are related to our invariants by

$$\nu(\Lambda) \leq \nu(P), \qquad \varepsilon(\Lambda) \leq \varepsilon(P).$$

The Gromov invariants of the fibration P associated to a Hamiltonian loop were independently studied by Seidel [S2, S3, S4] and his results were used by Lalonde-McDuff-Polterovich [LMP] to prove that Hamiltonian loops act trivially on homology. Our results on the Gromov invariants can be viewed as Lagrangian analogues of results in [Pl, S2] on the Gromov invariants of symplectic fibrations.

The present work is organized as follows. In Chapter 2 we give a brief introduction into symplectic geometry. The main definitions and some of the main theorems are stated. In Chapter 3 we discuss background material about the Hofer metric. The space of Lagrangian submanifolds is naturally foliated by Hamiltonian isotopy classes and the Hofer metric is defined on each leaf of this foliation. In Chapter 4 we introduce the invariants u(A), $\chi(\Lambda)$, and $\varepsilon(\Lambda)$ of exact Lagrangian loops and give a proof of Theorem B. In the 2-dimensional case the invariant $\nu(\Lambda)$ can sometimes be computed explicitly. This is done in Chapter 5 for the 2-torus. We also prove a result about Hamiltonian isotopy on Riemann surfaces, that is needed in this chapter. In Chapter 6 we introduce the Gromov invariants and in Chapter 7 we prove Theorems A and C. Chapters 3 till 7 have appeared as a joint paper [AS]. The short Chapter 8 is of a different flavour and presents an example of a non-trivial Legendrian submanifold of the 1-jet bundle of $\mathbb{R}^2$. Finally, in Chapter 9 a completely different problem is addressed. We study a fourth order nonlinear partial differential equation and search for travelling wave solutions. This chapter has appeared as a joint paper [AH]. The appendices concern symplectic connections, the Maslov index and an argument due to Taubes needed to go from the $\mathcal{C}^\ell$ category to the $\mathcal{C}^\infty$ category. They are added for the convenience of the reader.

Chapter 2

Preliminaries in symplectic geometry

In this chapter we will give a brief introduction into symplectic geometry. We restrict ourselves to the definitions and theorems needed to read the following chapters. For a more elaborate introduction into the subject see for example [MS 1] or [P5].

2.1 Symplectic manifolds

Let M be a smooth manifold (throughout this work all manifolds are assumed to be smooth unless otherwise stated). A symplectic structure on M is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. If M admits such a structure **we call (M, ω) a symplectic manifold.** Note that symplectic manifolds are orientable and even dimensional. The first example of a symplectic manifold is $\mathbb{C}^n = \mathbb{R}^{2n}$ with the Standard symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, where the coordinates on $\mathbb{C}^n$ are given by $(x_1 + iy_1, , x_n + iy_n)$. As a second example we observe that every oriented Riemann surface is a symplectic manifold. Its symplectic form is given by an area form.

An important class of examples of symplectic manifolds is given by $\mathbb{C}P^n$ - the space of complex lines in $\mathbb{C}^{n+1}$. A point in $\mathbb{C}P^n$ is the equivalence class of a nonzero complex $(n+1)$-vector [z] = [zu : : z,] under the equivalence relation $[z_0 : : z_n] \equiv [\lambda z_0 : : \lambda z_n]$ for $\lambda \neq 0$. One can check that the following 2-form defines a symplectic form

$$\tau_0 = \frac{i}{2(\sum_{l=0}^{n} \bar{z}_l z_l)^2} \sum_{k=0}^{n} \sum_{j \neq k} (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k).$$

This symplectic form is known as the Fubini-Study form.

Another important class of examples of symplectic manifolds is the cotangent bundle. Given a manifold N, consider its cotangt bundle T^*N. This manifold carries a canonical symplectic form $\omega_{\text{can}} = -d\lambda_{\text{can}}$. Here λ_{can}, locally known as $\lambda_{\text{can}} = \sum_{i=1}^{n} p_i dq_i$, is detined in the following lemma.

Lemma 2.1.1 *The 1-form $\lambda_{\text{can}} \in \Omega^1(T^*N)$ is uniquely characterised by the property that*

$$\sigma^* \lambda_{\text{can}} = \sigma$$

for every 1-form $\sigma : N \to T^*N$.

One of the first main results in symplectic geometry is the following theorem

Theorem 2.1.2 (Darboux's Theorem) *Every symplectic form ω on M is locally diffeomorphic to the standardform ω_0 on $\mathbb{C}^n$.*

This means in particular that, unlike in Riemannian geometry, symplectic manifolds do not admit local symplectic invariants. The type of invariants that are studied in symplectic geometry are therefore completely different than the classical Riemannian invariants.

2.2 Symplectic and Hamiltonian diffeomorphisms

A diffeomorphism $\phi : M \to M$ that preserves the symplectic form, i.e. $\phi^*\omega = \omega$, is called a **symplectomorphism.** The space of such diffeomorphisms forms a group and is denoted by Symp(M, w). As in Riemannian geometry there is a one-to-one correspondence between vector fields and 1-forms on M via

$$X \mapsto L(X)\ \omega.$$

Here X is a vector field X : $M \to TM$ and ι denotes the interior product so that $\iota(X)\omega(\cdot) = \omega(X, \cdot)$. Given a smooth function $H : M \to \mathbb{R}$, *we* define the **Hamiltonian vector field** X_H associated to the **Hamiltonian function** H by

$$\iota(X_H)\,\omega = dH.$$

If M is closed, the vector field X_H generates a smooth 1-parameter group of diffeomorphisms ϕ_H^t satisfying

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \qquad \phi_H^0 = \mathrm{id}.$$

A smooth map $[0,1] \times M \to M$; $(t, q) \mapsto \psi_t(q)$ with $\psi_t \in \mathrm{Symp}(M, \omega)$ for all t and $\psi_0 = \mathrm{id}$ is called a **symplectic isotopy.** Any such isotopy is generated by a unique family of vector fields $X_t : M \to TM$ by

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t$$

The l-forms $\iota(X_t)\,\omega$ are closed. If they are exact then there exists a family of Hamiltonian functions $H_t : M \to \mathbb{R}$ such that

$$\iota(X_t)\,\omega = dH_t,$$

for all t. In this case H_t is called the **time-dependent Hamiltonian** and ψ_t is called a **Hamiltonian isotopy.** A given symplectomorphism $\psi \in \mathrm{Symp}(M, \omega)$ is called a **Hamiltonian diffeomorphism** if there exists a Hamiltonian isotopy $\psi_t \in \mathrm{Symp}(M, \omega)$ connecting $\psi_0 = \mathrm{id}$ to $\psi_1 = \psi$. The space of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ is connected and forms a subgroup of $\mathrm{Symp}(M, \omega)$. The group of Hamiltonian diffeomorphisms is a much studied object in symplectic topology see e.g. [P5]. Observe that for simply connected manifolds it coincides with $\mathrm{Symp}(M, \omega)$.

2.3 Lagrangian submanifolds

Let $L \subset M$ be a submanifold. We call L a **Lagrangian** submanifold if $\dim L = 1/2 \dim M$ and $\omega|_{TL} \equiv 0$. Going back to the examples of symplectic manifolds, we immediately obtain some examples of Lagrangian submanifolds. $\mathbb{R}^n \subset (\mathbb{C}^n, \omega_0)$ is a Lagrangian submanifold. Here we have identified $\mathbb{R}^n$ with $y_1 = = y_n = 0$. On a Riemann surface every embedded curve is a Lagrangian submanifold.

An important role in this work is played by the Lagrangian submanifold $RP^n \subset \mathbb{C}P^n$. As a last example we remark that the zero section Nu in T^*N is Lagrangian.

Analogous to Darboux's theorem we have the Lagrangian neighbourhood theorem which is due to A. Weinstein.

Theorem 2.3.1 (Weinstein neighhourhood Theorem) *Given a symplectic manifold (M, ω) and let $L \subset M$ be a compact Lagrangian submanifold. Then there exist a neighbourhood $U \subset T^*L$ of the zero section L_0, a neighbourhood $V \subset M$ of L and a diffeomorphism $\phi : U \to V$ such that*

$$\phi^*\omega = -d\lambda,$$

which identifies L_0 in a canonical way with L. Here $\lambda = \lambda_{can}$ is the canonical 1-form on T^*L.

As before we see that there are no local symplectic invariants of Lagrangian submanifolds. This work is concemed with the study of global invariants of Lagrangian submanifolds.

2.4 Almost complex structures

An **almost complex structure** on a manifold M is a complex structure J on the tangent bundle TM. That is, J associates smoothly with every $x \in M$ a linear map $J = J_x : T_xM \to T_xM$ satisfying $J^2 = -\mathrm{id}$. If (M, ω) is a symplectic manifold then there exists an almost complex structure J on M and a Riemannian metric $\langle \cdot, \cdot \rangle$ on M such that

$$\omega_x(u, Jv) = \langle u, v \rangle_x, \qquad \text{for all } u, v \in T_xM.$$

Moreover, if ∇H denotes the gradient of a function H with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$ then we find the following representation for the Hamiltonian vector field

$$X_H(x) = J\nabla H(x) \in T_xM.$$

Given a symplectic manifold (M, ω) we say that an almost complex structure J is **compatible** with ω if the 2-form $\omega(\cdot, J\,\cdot)$ defines a Riemannian metric. Gromov [G1] showed that the space of compatible almost complex structures $\mathcal{J}(M, \omega)$ is contractible.

Chapter 3

The Hofer metric for Lagrangian submanifolds

Let (M, ω) be a $2n$-dimensional symplectic manifold and L be a compact connected n-manifold without boundary. Denote by

$$\mathcal{X} = \{\iota \in \mathrm{Emb}(L, M) \quad \iota^*\omega = 0\}$$

the space of Lagrangian embeddings of L into M. The group of diffeomorphisms of L, $\mathcal{G} = \mathrm{Diff}(L)$, acts on this space by $\iota \mapsto \iota \circ \phi$ for $\phi \in \mathcal{G}$. Two Lagrangian embeddings ι_0, $\iota_1 \in \mathcal{X}$ lie in the same $\mathcal{G}$-orbit if and only if they have the same image $\Lambda = \iota_0(L) = \iota_1(L)$. Hence the quotient space

$$\mathcal{L} := \mathcal{X}/\mathcal{G}$$

can be naturally identified with the set of Lagrangian submanifolds of M that are diffeomorphic to L. A function $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ is called **smooth** if there exists a smooth function $\mathbb{R} \times L \to M : (t, q) \mapsto \iota_t(q)$ such that $\iota_t(L) = \Lambda_t$ for all t. One can think of $\mathcal{L}$ as an infinite dimensional manifold, see e.g. [D].

Lemma 3.0.1 *The tangent space of $\mathcal{L}$ at a point $\Lambda \in \mathcal{L}$ can be naturally identified with the space of closed 1-forms on Λ:*

$$T_\Lambda\mathcal{L} = \{\beta \in \Omega^1(\Lambda) \mid d\beta = 0\}$$

Proof: Let $\mathbb{R} \times L \to M : (t, q) \mapsto \iota_t(q)$ be a smooth function such that $\iota_t \in \mathcal{X}$ for all t and define

$$\alpha_t := \omega(v_t, d\iota_t\cdot) \in \Omega^1(L), \qquad v_t := \partial_t\iota_t \in \mathcal{C}^\infty(L, \iota_t{}^*TM), \tag{3.1}$$

where, as usual, $\partial_t\iota_t(q) = \frac{d}{dt}\iota_t(q)$. Then, since $\iota_t{}^*\omega = 0$ for all t,

$$0 = \partial_t(\iota_t{}^*\omega) = \iota_t^*(d(\iota(v_t)\omega)) = d(\iota_t^*\omega(v_t, \cdot)) = d(\omega(v_t, d\iota_t\cdot)) = d\alpha,$$

and hence the tangent space of $\mathcal{X}$ at ι is given by

$$T_\iota\mathcal{X} = \left\{v \in \mathcal{C}^\infty(L, \iota^*TM) \mid \omega(v, d\iota\cdot) \in \Omega^1(L) \text{ is closed}\right\}$$

On the other hand, given $\phi_t \in \mathrm{Diff}(L)$, we see that

$$\partial_t(\iota \circ \phi_t) = d\iota(\partial_t\phi_t),$$

where we denote the tangent map $T\iota$ by $d\iota$. Hence the tangent space to the $\mathcal{G}$-orbit consists of all vector fields of the form $v = d\iota \circ \xi$, sometimes also written as $v = d\iota(\xi)$, where $\xi \in \mathrm{Vect}(L)$. [1] Note that $\omega(d\iota(\xi), d\iota\cdot) = 0$ since $\iota(L)$ is a Lagrangian submanifold. Therefore the map $w \mapsto \omega(w, d\iota\cdot)$ from $T_\iota\mathcal{X}$ into the space of closed 1-forms identilies the quotient space $T_\iota\,\mathcal{X}/\,T_\iota(\iota\,\mathcal{G})$ with the space of closed 1-forms on L.

If $\iota_t, \iota_t' \in \mathcal{X}$ are two smooth paths in $\mathcal{X}$ that satisfy $\iota_t' = \iota_t \circ \phi_t$ for some path $\phi_t \in \mathcal{G}$ then the vector fields $v_t := \partial_t\iota_t$ and $v_t' := \partial_t\iota_t'$ are related by

$$\begin{aligned} v_t' &= \partial_t(\iota_t \circ \phi_t) \\ &= \partial_t\iota_t \circ \phi_t + d\iota_t \circ \partial_t\phi_t \\ &= \partial_t\iota_t \circ \phi_t + d\iota_t \circ \xi_t \circ \phi_t \\ &= v_t \circ \phi_t + d\iota_t \circ \xi_t \circ \phi_t \end{aligned}$$

where $\xi_t \in \mathrm{Vect}(L)$ generates the diffeomorphism ϕ_t via $\partial_t\phi_t = \xi_t \circ \phi_t$. Hence the 1-forms $\alpha_t := \omega(v_t, d\iota_t\cdot)$ and $\alpha_t' := \omega(v_t', d\iota_t'\cdot)$ are related by

$$\begin{aligned} \alpha_t'(\cdot) &= \omega(v_t', d\iota_t'\cdot) \\ &= \omega(v_t \circ \phi_t + d\iota_t \circ \xi_t \circ \phi_t, d\iota_t \circ d\phi_t\cdot) \\ &= \omega(v_t \circ \phi_t, d\iota_t \circ d\phi_t\cdot) \\ &= \phi_t^*\omega(v_t, d\iota_t\cdot) \\ &= \phi_t^*\alpha_t(\cdot). \end{aligned}$$

[1] Here $\mathrm{Vect}(L)$ denotes the space of vector fields on L.

Hence two closed 1-forms $\alpha, \alpha' \in \Omega^1(L)$ corresponding to two Lagrangian embeddings ι and $\iota' = \iota \circ \phi$ represent the same tangent vector of $\mathcal{L}$ if and only if $\alpha' = \phi^*\alpha$ or, equivalently, $\iota_*\alpha = \iota'_*\alpha'$. Here, in abuse of notation, we have for all $q \in L$

$$\iota_*\alpha(q)(Y(\iota(q))) := \alpha(q)\big((d\iota(q))^{-1}Y(\iota(q))\big) \qquad (3.2)$$

for all $Y(\iota(q)) \in \mathrm{im}\ d\iota(q) \subset T_{\iota(q)}M$. This proves the lemma. □

Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds. We define the derivative of this path at time t by

$$\partial_t \Lambda_t := \iota_{t*}\alpha_t \in \Omega^1(\Lambda_t),$$

where the path $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ is chosen such that $\iota_t(L) = \Lambda_t$ for every t and α_t is defined by (3.1). The proof of Lemma 3.0.1 shows that the 1-form $\beta_t = \iota_{t*}\alpha_t \in \Omega^1(\Lambda_t)$ is closed and is independent of the choice of the lift $t \mapsto \iota_t$ used to define it, as we have just seen.

We wish to study Hamiltonian isotopies of Lagrangian submanifolds. We will see that this corresponds to paths in $\mathcal{L} = \mathcal{X}/\mathcal{G}$ that are tangent to the subbundle

$$\mathcal{H} = \left\{(\Lambda, \beta) \in T\mathcal{L} \mid \Lambda \in \mathcal{L}, \beta \in \Omega^1(\Lambda) \text{ is exact}\right\}$$

where we have used the identification of the tangent space with the space of closed I-forms on Λ. Abstractly, one can think of $\mathcal{H}$ as a distribution on $\mathcal{L}$. As a side remark we point out that this distribution is integrable, see [W]. We shall see that the leaf through Ao $\in \mathcal{L}$ consists of all Lagrangian submanifolds of M that are Hamiltonian isotopic to Λ_0 To be more precise, let $\mathbb{R} \times M \to \mathbb{R} : (t, z) \mapsto H_t(z)$ be a smooth Hamiltonian function and denote by

$$\mathbb{R} \times M \to M : (t, z) \mapsto \psi_t(z)$$

the Hamiltonian isotopy generated by H via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \iota(X_t)\omega = dH_t, \qquad \psi_0 = \mathrm{id}. \qquad (3.3)$$

Lemma 3.0.3 and lemma 3.0.4 prove the following proposition.

Proposition *3.0.2 A path in $\mathcal{L}$ is tangent to $\mathcal{H}$ if and only if it is generated by a Hamiltonian isotopy.*

Lemma *3.0.3 Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds and ψ_t be a Hamiltonian isotopy on M generated by the Hamiltonian functions $H_t : M \to \mathbb{R}$ via (3.3). Then $\Lambda_t = \psi_t(\Lambda_0)$ for every t if and only if*

$$\partial_t \Lambda_t = dH_t|_{\Lambda_t}$$

for every t.

Proof: Choose a smooth path $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ such that $\iota_t(L) = \Lambda_t$ for every t and let $\alpha_t \in \Omega^1(L)$ be defined by (3.1). Recall that $\partial_t \Lambda_t := \iota_{t*}\alpha_t$. Then $\partial_t \Lambda_t = dH_t|_{\Lambda_t}$ if and only if $d(H_t \circ \iota_t) = \alpha_t$. It follows from definition (3.1) that this is equivalent to

$$\omega(X_t(\iota_t(q)) - \partial_t \iota_t(q), d\iota_t(q)\xi) = 0$$

for all $\xi \in T_q L$, $q \in L$ and t. Hence, since a Lagrangian submanifold is maximally isotropic,

$$X_t(\iota_t(q)) - \partial_t \iota_t(q) \in \operatorname{im} d\iota_t(q)$$

for all t and all q. This means that there exists a smooth family of vector fields $\xi_t \in \mathrm{Vect}(L)$ such that

$$X_t \circ \iota_t = \partial_t \iota_t + d\iota_t \circ \xi_t.$$

Equivalently, for the Aows we get $\psi_t \circ \iota_0 = \iota_t \circ \phi_t$, where ψ_t is the flow of X_t and where the isotopy $\phi_t \in \mathrm{Diff}(L)$ is generated by ξ_t via $\partial_t \phi_t = \xi_t \circ \phi_t$ and $\phi_0 = \mathrm{id}$. This proves the lemma. □

The previous lemma shows that every path in $\mathcal{L}$ that is generated by a Hamiltonian isotopy is tangent to $\mathcal{H}$. The converse is proved next.

Lemma *3.0.4 A smooth path $[0, 1] \to \mathcal{L} : t \mapsto \Lambda_t$ is tangent to $\mathcal{H}$ if and only if there exists a Hamiltonian isotopy $t \mapsto \psi_t$ such that $\psi_t(\Lambda_0) = \Lambda_t$ for every t.*

Proof: The "if" part was proved in Lemma 3.0.3. Suppose that the path $t \mapsto \Lambda_t$ is tangent to $\mathcal{H}$. Choose a smooth function

$$[0,1] \to \mathcal{X} : t \mapsto \iota_t$$

such that $\iota_t(L) = \Lambda_t$ for every t and let $\alpha_t \in \Omega^1(L)$ be defined by (3.1). By assumption, α_t is exact for every t. Fix a smooth path $q_t \in L$ and, for every t, choose $h_t : L \to \mathbb{R}$ such that

$$dh_t = \alpha_t, \qquad h_t(q_t) = 0.$$

Then the function $\mathbb{R} \times L \to \mathbb{R} : (t, q) \mapsto h_t(q)$ is smooth. We construct a smooth function $[0,1] \times M \to \mathbb{R} : (t, z) \mapsto H_t(z)$ such that

$$H_t \circ \iota_t = h_t. \tag{3.4}$$

Choose an almost complex structure J on M that is compatible with ω. Let $\varepsilon > 0$ be so small that, for every $t \in [0,1]$, the map

$$T\Lambda_t \to M : (z, v) \mapsto \exp_z(Jv)$$

restricts to a diffeomorphism from the ε-neighbourhood of the zero section in $T\Lambda_t$ onto the open neighbourhood

$$U_t := \left\{\exp_z(Jv) \mid z \in \Lambda_t,\ v \in T_z\Lambda_t,\ |v| < \varepsilon\right\}$$

of Λ_t in M. Choose a cutoff function $\rho : [0, \varepsilon] \to [0,1]$ such that $\rho(r) = 1$ for $r < \varepsilon/3$ and $\rho(r) = 0$ for $r > 2\varepsilon/3$. Define $H_t : M \to \mathbb{R}$ by

$$H_t(\exp_z(Jv)) := \rho(|v|) h_t \circ \iota_t^{-1}(z)$$

for $z \in \Lambda_t$ and $v \in T_z\Lambda_t$ with $|v| < \varepsilon$, and by $H_t(z) := 0$ for $z \in M \setminus U_t$. Then H_t satisfies (3.4) and hence

$$dH_t|_{\Lambda_t} = \iota_{t*} dh_t = \iota_{t*}\alpha_t = \partial_t \Lambda_t$$

By Lemma 3.0.3, the Hamiltonian isotopy ψ_t generated by H_t satisfies $\psi_t(\Lambda_0) = \Lambda_t$ for every t. This proves the lemma. □

Remark3.0.5 The Hamiltonian functions, which are constructed in Lemma 3.0.4, satisfy

$$\max H_t = \max h_t, \qquad \min H_t = \min h_t \tag{3.5}$$

for every t. With a slightly more sophisticated argument one can show that different Hamiltonian functions can be chosen such that the Hamiltonian vector fields X_t satisfy $\partial_t \iota_t = X_t \circ \iota_t$ and hence the resulting Hamiltonian isotopy satisfies

$$\psi_t \circ \iota_0 = \iota_t. \tag{3.6}$$

However, in general there does not exist a Hamiltonian isotopy that satisfies both (3.5) and (3.6).

Lemma 3.0.6 *Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds. Let $\mathbb{R} \to \mathrm{Symp}(M, \omega) : t \mapsto \psi_t$ be a symplectic isotopy and define $\beta_t \in \Omega^1(M)$ by $\beta_t := \iota(Y_t)\omega$, where $\partial_t \psi_t = Y_t \circ \psi_t$. Then β_t is closed and the path $\Lambda'_t := {\psi_t}^{-1}(\Lambda_t)$ satisfies*

$$\partial_t \Lambda'_t = {\psi_t}^* \left(\partial_t \Lambda_t - \beta_t|_{\Lambda_t}\right).$$

Proof: That ß is closed follows from $0 = \mathcal{L}_{Y_t}\omega = d(\iota(Y_t)\omega)$. Now choose a lift $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ of $t \mapsto \Lambda_t$ and introduce the embeddings and 1-forms

$$\iota'_t := {\psi_t}^{-1} \circ \iota_t, \qquad \alpha_t := \omega(\partial_t \iota_t, d\iota_t \cdot), \qquad \alpha'_t := \omega(\partial_t \iota'_t, d\iota'_t \cdot)$$

Then

$$\begin{aligned}
\alpha'_t(\cdot) &= \omega(\partial_t \iota'_t,\ d\iota'_t \cdot) \\
&= \omega(\partial_t \psi_t^{-1} \circ \iota_t + d\psi_t^{-1} \circ \partial_t \iota_t, d\psi_t^{-1} \circ d\iota_t \cdot) \\
&= \omega(-d\psi_t^{-1} \circ Y_t \circ \iota_t, d\psi_t^{-1} \circ d\iota_t \cdot) + \omega(\partial_t \iota_t, d\iota_t \cdot) \\
&= -{\iota_t}^* \beta_t(\cdot) + \alpha_t(\cdot)
\end{aligned}$$

and hence

$$\partial_t \Lambda'_t = \iota'_{t*}\alpha'_t = {\psi_t}^* \iota_{t*} \alpha'_t = {\psi_t}^* (\iota_{t*}\alpha_t - \beta_t) = {\psi_t}^* (\partial_t \Lambda_t - \beta_t)$$

as claimed. □

The subbundle $\mathcal{H}$ c TG carries a natural norm. As in Hofer [H] we define the norm of an exact 1-form $\alpha = dh \in \Omega^1(\Lambda)$ by

$$\|dh\| := \max_{\Lambda} h - \min_{\Lambda} h.$$

This norm gives rise to a distance function on each leaf of the foliation determined by $\mathcal{H}$. Indeed, let $\mathcal{L}_0$ be such a leaf. By Proposition 3.0.2, the submanifold $\mathcal{L}_0 \subset \mathcal{L}$ is the Hamiltonian isotopy class of any given Lagrangian submanifold Λ of M in $\mathcal{L}_0$. Let $[0, 1] \to \mathcal{L}_0 : t \mapsto \Lambda_t$ be a smooth path in $\mathcal{L}_0$. The length of this path is defined by

$$\ell(\{\Lambda_t\}) := \int_0^1 \|\partial_t \Lambda_t\| \, dt, \tag{3.7}$$

with the 1-form $\partial_t \Lambda_t = \iota_{t*}\alpha_t \in \Omega^1(\Lambda_t)$. Proposition 3.0.2 together with the Statement (3.5) in Remark 3.0.5 show that

$$\ell(\{\Lambda_t\}) = \inf_{\psi_t(\Lambda_0)=\Lambda_t} \ell(\{\psi_t\}), \tag{3.8}$$

where the infimum runs over all Hamiltonian isotopies $t \mapsto \psi_t$ that satisfy ψ_t (An) $= \Lambda_t$ for all t and $\ell(\{\psi_t\})$ denotes the Hofer length (cf. [H]) which is defined by

$$\ell(\{\psi_t\}) := \int_0^1 (\max_M H_t - \min_M H_t) dt.$$

Here ψ_t is generated by the Hamiltonian function H_t via (3.3). Here we have used that h_t in (3.5) is delined in terms oft $\mapsto \Lambda_t$. Now let $\Lambda, \Lambda' \in \mathcal{L}_0$ and denote by $\mathcal{P}(\Lambda$, A') the space of all smooth paths $[0, 1] \to \mathcal{L}_0$: $t \mapsto$ A, that connect $\Lambda_0 = \Lambda$ with At = A'. The distance between Λ and A' (belonging to the same leaf of $\mathcal{L}_0$) is defined by

$$d(\Lambda, \text{A'}) := \inf_{\{\Lambda_t\} \in \mathcal{P}(\Lambda, \Lambda')} \ell(\{\Lambda_t\}) \tag{3.9}$$

It follows immediately from (3.8) that

$$d(\Lambda, \Lambda') = \inf_{\psi(\Lambda)=\Lambda'} d(\text{id}, \psi) \tag{3.10}$$

where the infimum runs over all Hamiltonian symplectomorphisms ψ of M that satisfy $\psi(\Lambda) = A'$. The distance d(id, ψ) denotes the Hofer distance (cf. [H]) defined by

$$d(\mathrm{id}, \psi) := \inf_{F \in \mathcal{F}(\psi)} \int_0^1 \max F_t - \min F_t \, dt,$$

where $\mathcal{F}(\psi)$ is the set of all time dependent Hamiltonian functions generating ψ at time one. The function (3.9) is obviously nonnegative, symmetric, and satisfies the triangle inequality. That it defines a metric is a deep theorem due to Chekanov [C].

Theorem 3.0.7 (Chekanov) *If* $\Lambda \neq A'$ *then* $d(\Lambda, A') > 0$.

Remark 3.0.8 (Naturality) The distance function (3.9) satisfies

$$d(\phi(\Lambda), \phi(\Lambda')) = d(\Lambda, \Lambda')$$

for every symplectomorphism ϕ (not necessarily Hamiltonian) and any two Lagrangian submanifolds Λ, Λ' that are Hamiltonian isotopic. This follows from (3.10) and the identity $d(\mathrm{id}, \psi) = d(\mathrm{id}, \phi \circ \psi \circ \phi^{-1})$ for every Hamiltonian symplectomorphism ψ.

Remark 3.0.9 In [Mi] Milinković studied geodesics in the space of Lagrangian submanifolds with respect to the above metric. In this paper he generalizes a result by Bialy and Polterovich [BP] and proves that the distance of two exact Lagrangian submanifolds $\Lambda = \mathrm{graph}(dS)$ and $\Lambda' = \mathrm{graph}(dS')$ of the cotangent bundle T^*L is given by

$$d(\Lambda, A') = \|d(S - S')\|$$

Chapter 4

Invariants of Lagrangian loops

In this chapter we shall consider exact loops of Lagrangian submanifolds. In the terminology of the previous chapter we work with loops contained in any single leaf $\mathcal{L}_0$ of the foliation of $\mathcal{L}$ determined by $\mathcal{H}$. We shall construct three new invariants of Hamiltonian isotopy classes of such loops and study the relations between them.

4.1 The minimal length

We shall use the notation introduced in Chapter 3. A Lagrangian loop in the symplectic manifold (M, ω) is a smooth function $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ such that

$$\Lambda_{t+1} = \Lambda_t$$

for all $t \in \mathbb{R}$. Such a loop determines a subset $\Lambda \subset S^1 \times M$ defined by

$$\Lambda := \left\{ (e^{2\pi i t}, z) \mid t \in \mathbb{R}, z \in \Lambda_t \right\} \tag{4.1}$$

Note that the Lagrangian loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ is smooth if and only if this set Λ is a smooth submanifold of $S^1 \times$ M. We shall frequently identify the loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ with the corresponding submanifold $\Lambda \subset S^1 \times$ M.

A Lagrangian loop $t \mapsto \Lambda_t$ is called exact if it is tangent to the subbundle $\mathcal{H}$, this means that $\partial_t \Lambda_t \in \Omega^1(\Lambda_t)$ is exact for every t. Two exact

Lagrangian loops $t \mapsto \Lambda_t$ and $t \mapsto \Lambda'_t$ are called **Hamiltonian isotopic** if there exists a smooth function $[0, 1] \times \mathbb{R} \to \mathcal{L} : (s, t) \mapsto \Lambda_{s,t}$ such that

$$\Lambda_{0,t} = \Lambda_t, \qquad \Lambda_{1,t} = \Lambda'_t,$$

the map $t \mapsto \Lambda_{s,t}$ is an exact Lagrangian loop for every s, and the 1-form $\partial_s \Lambda_{s,t} \in \Omega^1(\Lambda_{s,t})$ is exact for all s and t. Here the function $[0, 1] \times \mathbb{R} \to \mathcal{L} : (s, t) \mapsto \Lambda_{s,t}$ is called smooth if there exists a smooth map $[0, 1] \times \mathbb{R} \times L \to M : (s, t, q) \mapsto \iota_{s,t}(q)$ such that $\iota_{s,t}(L) = \Lambda_{s,t}$ for all s and t. Let $\Lambda, \Lambda' \subset S^1 \times M$ be two exact Lagrangian loops. If Λ is Hamiltonian isotopic to Λ' we denote this by

$$\Lambda \sim \Lambda'.$$

A Hamiltonian isotopy class corresponds to a component in the free loop space of a leaf $\mathcal{L}_0 \subset \mathcal{L}$ of the foliation determined by $\mathcal{H}$. To every such Hamiltonian isotopy class we assign the real number

$$\nu(\Lambda) := \inf_{\Lambda' \sim \Lambda} \ell(\Lambda'), \tag{4.2}$$

where the length $\ell(\Lambda) = \ell(\{\Lambda_t\})$ of a loop $\Lambda : t \mapsto \Lambda_t$ is defined in (3.7). So $\nu(\Lambda)$ is obtained by minimizing the Hofer length over all exact Lagrangian loops that are Hamiltonian isotopic to Λ.

Example: Consider the 2-sphere as a symplectic manifold and let Λ_0 be a great circle passing through the north pole, see Figure 4.1. Rotating this great circle through 180 degrees yields a loop Λ_t, $0 \leq t \leq 1$, of Lagrangian submanifolds. This example shows that the images $\Lambda_t = \iota_t(S^1)$ form a loop, while the embedding itself is not a loop, that is $\iota_t \neq \iota_{t+1}$.

This example will be the red thread throughout this work and should be kept in mind.

4.2 Relative K-area

Following Polterovich [P1], [P2] we introduce the concept of relative K-area. This invariant is defined in terms of Hamiltonian connections on the trivial symplectic fibre bundle $D \times M \to D$. We restrict our attention

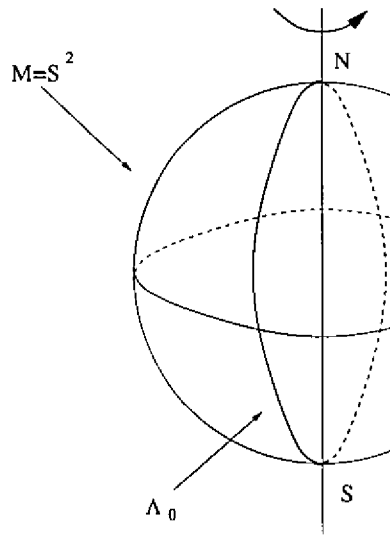

Figure 4.1: A Lagrangian loop on the 2-sphere

to those connections that preserve the subbundle $\Lambda \subset D \times M$ defined by (4.1). Here $D = \{x + iy \mid x^2 + y^2 \leq 1\} \subset \mathbb{C}$ denotes the closed unit disc, That the set of these connections is not empty will be proved in Lemma 4.2.2. The basic notions of symplectic connections and curvature are put together in Appendix A. We will always think of a connection on $D \times M$ as a horizontal distribution. Any such connection is determined by a **connection 2-form** on $D \times M$ of the form

$$\tau = \omega + \alpha \wedge dx + \beta \wedge dy + f dx \wedge dy$$

where $\alpha = \alpha_{x,y} \in \Omega^1(M)$, $\beta = \beta_{x,y} \in \Omega^1(M)$, and $f = f_{x,y} \in \Omega^0(M)$ depend smoothly on $x + iy \in D$. The horizontal subspace is the τ-orthogonal complement of the vertical subspace. The tangent space at the point (x, y, z) splits naturally

$$T_{(x,y,z)}(D \times M) = T_{x,y}D \oplus T_z M,$$

where $T_z M$ is the vertical subspace. More explicitly, the horizontal lifts of $\partial/\partial x$ and $\partial/\partial y$ [1] at (x, y, z) $\in D \times M$ are the vectors $(\partial_x, X_{„,}(z))$ and

[1]Here $\partial_x = (\partial_x)_{x,y}$ and $\partial_y = (\partial_y)_{x,y}$ are shorthand notation for $\partial/\partial x$ and $\partial/\partial y$ respectively

$(\partial_y, Y_{x,y}(z))$, respectively. Here the vector fields $X = X_{x,y}$, $Y = Y_{x,y} \in \mathrm{Vect}(M)$ satisfy by definition

$$\tau((\partial_x, X_{x,y}(z)), \xi) = 0 \qquad \tau((\partial_y, Y_{x,y}(z)), \xi) = 0$$

for all $\xi \in T_zM$. Hence the vector fields X, Y $\in \mathrm{Vect}(M)$ are given by

$$\iota(X)\omega = \alpha, \qquad \iota(Y)\omega = \beta.$$

Observe that the horizontal lifts of ∂_x and ∂_y are independent of the choice of f and hence the connection associated to τ is independent of f. It is called **symplectic** if $\alpha_{x,y}$ and $\beta_{x,y}$ are closed for all $(x, y) \in D$, and **Hamiltonian** if the 1-forms on M, $\alpha_{x,y}$ and $\beta_{x,y}$ are exact for all $(x, y) \in D$ and τ is closed.[2] Thus a Hamiltonian connection 2-form has the form

$$\tau = \omega + dF \wedge dx + dG \wedge dy + (\partial_x G - \partial_y F + c) dx \wedge dy, \tag{4.3}$$

where $F, G : D \times M \to \mathbb{R}$ and $c : D \to \mathbb{R}$ are smooth maps such that the following functions $F_{x,y} = F(x, y, \cdot)$ and $G_{x,y} = G(x, y, \cdot)$ on M have mean value zero:

$$\int_M F_{x,y}\omega^n = \int_M G_{x,y}\omega^n = 0,$$

for all $(x, y) \in D$. **Caveat:** in (4.3) the d in dF denotes the differential on M and NOT the differential on $D \times M$, i.e. dF denotes the smooth family $(x, y) \mapsto dF_{x,y}$ of 1-forms on M, and similarly for dG. We shall only consider Hamiltonian connections with the property that parallel transport along the boundary preserves A. Given a Hamiltonian connection 2-form on $D \times M$ of the form (4.3). We can define the family of functions $H_t : M \to \mathbb{R}$ as follows.

$$H_t := -2\pi \sin(2\pi t) F_{\cos(2\pi t), \sin(2\pi t)} + 2\pi \cos(2\pi t) G_{\cos(2\pi t), \sin(2\pi t)}. \tag{4.4}$$

The Parameter t parametrizes the boundary of the disc as follows $(\cos 2\pi t, \sin 2\pi t)$, $0 \leq t \leq 1$.

[2] In [MSI] a connection is called Hamiltonian if parallel transport along every loop in the base is a Hamiltonian symplectomorphism. In the case of a simply connected base this is equivalent to the existence of a closed 2-form τ that represents this connection. In contrast, we call a connection Hamiltonian if parallel transport along every path is a Hamiltonian symplectomorphism. This notion only makes sense when the structure group of the bundle in question is the group of Hamiltonian symplectomorphisms.

Lemma 4.2.1 *Let τ be a Hamiltonian connection 2-form on $D \times M$ of the form (4.3) and let H_t be the associared functions defined in (4.4). Recall that $\mathcal{L}$ is the set of Lagrangian embeddings of L into M. Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop and let the associared set of Lagrangian loops $\Lambda \subset D \times M$ be defined by (4.1). Choose a smooth parametrization $\iota : \mathbb{R} \times L \to D \times M$ such that $\iota(t, q) = (e^{2\pi i t}, \iota_t(q))$ and $\iota_t(L) = \Lambda_t$. Then the following statements are equivalent.*

(i) *The parallel transport of τ along a path γ in the boundary ∂D preserves the family Λ_t.*

(ii) $\iota^* \tau = 0$.

(iii) $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$ *for every* $t \in \mathbb{R}$.

Proof: We first prove that (i) is equivalent to (iii). Given a curve $t \mapsto x(t) + iy(t)$, where $x(t)^2 + y(t)^* = 1$. The horizontal lift of its tangent vector is given by

$$(\dot{x}(t) + i\dot{y}(t),\ \dot{x}(t)X_{F_{x(t),y(t)}} + \dot{y}(t)X_{G_{x(t),y(t)}}).$$

Here X_F, X_G are the Hamiltonian vector fields on M determined by the functions F and G via (3.3). So parallel transport of τ along a path $x(t) + iy(t)$ is determined by the Hamiltonian functions

$$H_t = \dot{x}(t)F_{x(t),y(t)} + \dot{y}(t)G_{x(t),y(t)}.$$

The functions H_t on M in (4.4) correspond to the path $t \mapsto e^{2\pi i t}$. We know from Lemma 3.0.3 that the Hamiltonian isotopy ψ_t determined by H_t preserves Λ if and only if $\psi_t(\Lambda_0) = \Lambda_t$ for all t if and only if $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$ for every t. This shows that (i) is equivalent to (iii).

It remains to prove the equivalence of (ii) and (iii). Given two tangent vectors (a, ξ), $(a', \xi') \in T_{t,q}(\mathbb{R} \times L)$. Then using complex notation and recalling the formula (4.3),

$$\begin{aligned}
\iota^*\tau((a,\xi),(a',\xi')) =& \tau((a2\pi i e^{2\pi i t}, a\partial_t\iota_t + d\iota_t\circ\xi),\\
& (a'2\pi i e^{2\pi i t}, a'\partial_t\iota_t + d\iota_t\circ\xi'))\\
=&\omega(a\partial_t\iota_t + d\iota_t\circ\xi, a'\partial_t\iota_t + d\iota_t\circ\xi') +\\
& 2\pi\sin(2\pi t)a(dF)(a'\partial_t\iota_t + d\iota_t\circ\xi') -\\
& 2\pi\sin(2\pi t)a'(dF)(a\partial_t\iota_t + d\iota_t\circ\xi) -\\
& 2\pi\cos(2\pi t)a(dG)(a'\partial_t\iota_t + d\iota_t\circ\xi') +\\
& 2\pi\cos(2\pi t)a'(dG)(a\partial_t\iota_t + d\iota_t\circ\xi)\\
=& dt\wedge\omega(\partial_t\iota_t - X_t\circ\iota_t, d\iota_t\cdot)((a,\xi),(a',\xi'))\\
=& dt\wedge(\alpha_t - \iota_t^* dH_t)[(a,\xi),(a',\xi')]
\end{aligned}$$

where $X_t \in \mathrm{Vect}(M)$ denotes the Hamiltonian vector field of H, as defined in (3.3). Now $\iota^*\tau = 0$ if and only if $\alpha_t - \iota_t^* dH_t = 0$ if and only if $dH_t|_{\Lambda_t} = \iota_{t*}\alpha_t = \partial_t\Lambda_t$. This completes the proof of the lemma. □

For every exact Lagrangian loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ let us denote the set of Hamiltonian connections that preserve Λ by

$$\mathcal{T}(\Lambda) = \left\{\tau \in \Omega^2(D\times M) \mid \tau \text{ has the form (4.3), } \tau|_{T\Lambda} = 0\right\}$$

Here we have used Statement (ii) of Lemma (4.2.1). We shall prove in Lemma 4.2.2 below that this set is nonempty. Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda'_t$ be another exact Lagrangian loop. In abuse of the usual notation, a diffeomorphism of pairs, mapping the subset Λ onto Λ',

$$\Psi : (D\times M, \Lambda) \to (D\times M, \Lambda')$$

is called a **fibrewise (Hamiltonian) symplectomorphism** if it has the form $\Psi(x+iy, z) = (x+iy, \psi_{x,y}(z))$, where $\psi_{x,y} : M \to M$ is a (Hamiltonian) symplectomorphism for all x, y. In the case $\Lambda = \Lambda'$ we denote by $\mathcal{G}(\Lambda)$ the group of fibrewise Hamiltonian symplectomorphisms of $(D\times M, \Lambda)$. This group acts on $\mathcal{T}(\Lambda)$ by $\tau \mapsto \Psi^*\tau$, see Proposition 4.2.3. Recall that the **curvature** of a connection is a 2-form with values in the Lie-algebra of the structure group of the bundle. In our case the latter is the space of time dependent Hamiltonian vector fields which can be identified with the space of time dependent Hamiltonian functions and we will denote the curvature of the connection 2-form τ of the form (4.3) by $\Omega_\tau(x, y, z) dx \wedge dy$. Here $\Omega_\tau : D \times M \to \mathbb{R}$ is a function given by

$$\Omega_\tau(x, y, z) := \{F_{x,y}, G_{x,y}\}(z) + \partial_y F_{x,y}(z) - \partial_x G_{x,y}(z) \tag{4.5}$$

for $x + iy \in D$ and $z \in M$. To see this let ∂_x and ∂_y be two tangent vectors to the discs. Their horizontal lifts to the tangent space $T_{(x,y,z)}(D \times M)$ are given by $\partial_x + X_{F_{x,y}}$ and $\partial_y + X_{G_{x,y}}$. Denote by 'vert' the projection onto the vertical tangent space. The curvature evaluated at ∂_x and ∂_y is by definition given by [3]

$$\begin{aligned}
&[\partial_x + X_F, \partial_y + X_G]^{\text{vert}} = \\
&\quad = \left(\nabla_{\partial_y + X_G}(\partial_x + X_F) - \nabla_{\partial_x + X_F}(\partial_y + X_G)\right)^{\text{vert}} = \\
&\quad = X_{\partial_y F} + \nabla_{X_G} X_F - X_{\partial_x G} \quad \nabla_{X_F} X_G = \\
&\quad = X_{\partial_y F - \partial_x G + \{F, G\}}.
\end{aligned}$$

Be careful, when one changes the 2-form $dx \wedge dy$ then the function Ω_τ changes accordingly.

Lemma 4.2.2 (i) *For every exact Lagrangian loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ the set $\mathcal{T}(\Lambda)$ is nonempty.*

(ii) *Two exact Lagrangian loops Λ und Λ' are Hamiltonian isotopic if and only if the corresponding pairs $(D \times M, \Lambda)$ and $(D \times M, A')$ are jibrewise Hamiltonian symplectomorphic.*

(iii) *If τ is a Hamiltonian connection 2-form on $D \times M$ and Ψ is a fibrewise Hamiltonian symplectomorphism of $D \times M$ then the coefficient functions Ω_τ of the curvature transform as follows*

$$\Omega_{\Psi^*\tau} = \Omega_\tau \circ \Psi.$$

[3] Here we have used the convention of the sign of the Lie bracket as in [MS1] Remark 3.3, that is $[X, Y] = -\mathcal{L}_X Y$.

Proof: We prove (i). Choose the maps $\iota_t : L \to M$ parametrising $\Lambda_t = \iota_t(L)$ fort $\in \mathbb{R}$. Let $\phi_t \in \mathrm{Diff}(L)$ be defined by

$$\iota_{t+1} \circ \phi_t = \iota_t. \tag{4.6}$$

We claim that since L is connected there exists a smooth path $\mathbb{R} \to L : t \mapsto q_t$ such that, for every $t \in \mathbb{R}$,

$$q_{t+1} = \phi_t(q_t). \tag{4.7}$$

To prove the claim choose q_t in the interval $0 \le t \le 1$ such that $q_t = q_1$ for $1 - \varepsilon \le t \le 1$ and $q_t = \phi_t{}^{-1}(q_1)$ for $0 \le t \le \varepsilon$. Then define q_t for $t \in \mathbb{R}$ such that (4.7) is satisfied. Since the loop is an exact Lagrangian loop, there exists a unique function $h_t : L \to \mathbb{R}$

$$dh_t = \alpha_t := \omega(\partial_t \iota_t, d\iota_t \cdot), \qquad h_t(q_t) = 0.$$

From (4.6) and (4.7) we get

$$\iota_{t+1}(q_{t+1}) = \iota_t \circ \phi_t^{-1}(\phi_t(q_t)) = \iota_t(q_t)$$

so that the function $t \mapsto \iota_t\,(q_t)$ is 1 -periodic in t. The computation in the proof of Lemma 3.0.1 yields

$$\iota_{t+1*}\alpha_{t+1} = (\iota_t \circ \phi_t^{-1})_* \phi_t^{-1*}\alpha_t = \iota_{t*}\alpha_t,$$

which shows that the 1-forms $\iota_{t*}\alpha_t$ are also 1-periodic in t. Hence, by the product rule of the d Operator, the functions $h_t \circ \iota_t{}^{-1} : \Lambda_t \to \mathbb{R}$ satisfy

$$h_{t+1} \circ \iota_{t+1}^{-1} = h_t \circ \iota_t^{-1} + c(t)$$

for some function $c : \mathbb{R} \to \mathbb{R}$. Evaluating these functions at the point $\iota_{t+1}\,(q_{t+1})$ we conclude in view of (4.6) and (4.7) that the function c vanishes and hence the functions $h_t \circ \iota_t^{-1}$ are 1-periodic in t and hence, so are the functions $H_t : M \to \mathbb{R}$ defined in the proof of Lemma 3.0.4 by the formula (3.4), namely $H_t \circ \iota_t = h_t$. *Now* define $\widetilde{H}_t : M \to \mathbb{R}$ by

$$\widetilde{H}_t(z) := H_t(z) - \frac{\int_M H_t\,\omega^n}{\int_M \omega^n}$$

Choose a smooth cutoff function $\rho : [0, 1] \to [0, 1]$ satisfying $\rho(r) = 0$ for $r < \varepsilon$ and $\rho(r) = 1$ for $r > 1 - \varepsilon$ and define the 2-form τ on $(D \setminus \{0\}) \times M$ by

$$\Phi^*\tau = \omega + \rho(r)d\widetilde{H}_t dt + \dot{\rho}(r)\widetilde{H}_t dr \wedge dt, \tag{4.8}$$

where the map $\Phi : [0, 1] \times [0, 1] \times M \to D \times M$ is given by $\Phi(r, t, z) = (re^{2\pi it}, z)$. Explicitly, τ has the form (4.3) where $F, G : D \times M \to \mathbb{R}$ are given by

$$F_{x,y} = \frac{-\sin(2\pi t)\rho(r)}{2\pi r}\widetilde{H}_t, \qquad G_{x,y} = \frac{\cos(2\pi t)\rho(r)}{2\pi r}\widetilde{H}_t, \tag{4.9}$$

for $x + iy = re^{2\pi it}$ These functions have mean value zero and satisfy (4.4) with H_t replaced by $\widetilde{H}_t$. Since $H_t \circ \iota_t = h_t$ it follows as in the proof of Lemma 3.0.4 that

$$d\widetilde{H}_t|_{\Lambda_t} = dH_t|_{\Lambda_t} = \partial_t\Lambda_t.$$

By Lemma 4.2.1, the parallel transport of τ along the boundary preserves Λ. Hence τ is an element of $\mathcal{T}(\Lambda)$. This proves (i).

We prove (ii). Assume first that there exists a fibrewise Hamiltonian symplectomorphism of the form $\Psi(x + iy, z) = (x + iy, \psi_{x+iy}(z))$ such that

$$\psi_{e^{2\pi it}}(\Lambda_t) = \Lambda'_t$$

for every t Define

$$\psi_{s,t} := \psi_{se^{2\pi it}}, \qquad \Lambda_{s,t} := \psi_{s,t}(\Lambda_t)$$

for $0 \le s \le 1$ and $t \in \mathbb{R}$. Then $t \mapsto \Lambda_{s,t}$ is an exact Lagrangian loop for every s and $\partial_s \Lambda_{s,t} \in \Omega^1(\Lambda_{s,t})$ is exact for all s and t. Hence the Lagrangian loop $\Lambda_{1,t} = \Lambda'_t$ is Hamiltonian isotopic to $\Lambda_{0,t} = \psi_0(\Lambda_t)$. Since ψ_0 is a Hamiltonian symplectomorphism, the loop $t \mapsto \psi_0(\Lambda_t)$ is Hamiltonian isotopic to $t \mapsto \Lambda_t$. Conversely, suppose that $t \mapsto \Lambda_t$ and $t \mapsto \Lambda'_t$ are two exact Lagrangian loops that are Hamiltonian isotopic. Choose an exact isotopy $(s, t) \mapsto \Lambda_{s,t}$ such that $\Lambda_{0,t} = \Lambda_t$, $\Lambda_{1,t} = \Lambda'_t$, and $\partial_s\Lambda_{s,t} = 0$ for $s \le 1/2$. As in the proof of (i), one can construct a smooth family of Hamiltonian functions $H_{s,t} : M \to \mathbb{R}$ such that

$$H_{s,t+1} = H_{s,t}, \qquad dH_{s,t}|_{\Lambda_{s,t}} = \partial_s\Lambda_{s,t}.$$

Denote by $X_{s,t}$ the associated Hamiltonian vector fields defined by $\iota(X_{s,t})\omega = dH_{s,t}$ and denote by $\psi_{s,t} : M \to M$ the flow in the Parameters defined by

$$\partial_s \psi_{s,t} = X_{s,t} \circ \psi_{s,t}, \qquad \psi_{0,t} = \mathrm{id}.$$

Then $\psi_{s,t} = \mathrm{id}$ for $s \leq 1/2$ and the required fibrewise Hamiltonian symplectomorphism is defined by $\Psi(se^{2\pi it}, z) := (se^{2\pi it}, \psi_{s,t}(z))$.

We prove (iii). Let τ be given by (4.3) and suppose that

$$\Psi(x + iy, z) = (x + iy, \psi_{x,y}(z))$$

is a fibrewise Hamiltonian symplectomorphism. Choose the associated parametrised Hamiltonian functions $A, B : D \times M \to \mathbb{R}$ such that the functions $A_{x,y} := A(x + iy, \cdot)$ and $B_{x,y} := B(x + iy, \cdot)$ have mean value zero and the Hamiltonian vector fields $X_{A_{x,y}} =: X_A$ and $X_{B_{x,y}} =: X_B$ satisfy

$$\partial_x \psi = X_A \circ \psi, \qquad \partial_y \psi = X_B \circ \psi. \tag{4.10}$$

It follows from Proposition 4.2.3 below that

$$\Psi^*\tau = \omega + d\widetilde{F} \wedge dx + d\widetilde{G} \wedge dy + (\partial_x \widetilde{G} - \partial_y \widetilde{F} + c) dx \wedge dy,$$

where

$$\widetilde{F} = (F - A) \circ \Psi, \qquad \widetilde{G} = (G - B) \circ \Psi.$$

Recall that d refers to the manifold M only. Hence, recalling the definition (4.5),

$$\begin{aligned} \Omega_{\Psi^*\tau} &= \partial_x \widetilde{G} \quad \partial_y \widetilde{F} - \{\widetilde{F}, \widetilde{G}\} \\ &= \partial_x (G - B) \circ \Psi + d(G - B) \circ X_A \circ \Psi \\ &\quad - \partial_y (F - A) \circ \Psi - d(F - A) \circ X_B \circ \Psi \\ &\quad - \{(F - A), (G - B)\} \circ \Psi \\ &= (\partial_x G - \partial_y F - \{F, G\}) \circ \Psi \\ &\quad - (\partial_x B - \partial_y A - \{A, B\}) \circ \Psi \\ &= \Omega_\tau \circ \Psi. \end{aligned}$$

The last equality follows from the definition of A and B in (4.10). This proves the lemma. ❑

Proposition 4.2.3 *Let $\Psi : D \times M \to D \times M$ be a fibrewise Hamiltonian symplectomorphism and let τ be given by (4.3) then*

$$\Psi^*\tau = \omega + d\widetilde{F} \wedge dx + d\widetilde{G} \wedge dy + (\partial_x \widetilde{G} - \partial_y \widetilde{F}) dx \wedge dy,$$

with the functions $\widetilde{F} = (F - A) \circ \Psi$ and $\widetilde{G} = (G - B) \circ \Psi$, where the functions A and B are defined in (4.10)..

Proof: We will prove that

- $\Psi^*\omega = \omega - d(A \circ \Psi) \wedge dx - d(B \circ \Psi) \wedge dy + \{A, B\} \circ \Psi dx \wedge dy$
- $\Psi^*(dF \wedge dx) = d(F \circ \Psi) \wedge dx + \{B, F\} \circ \Psi dx \wedge dy$
- $\Psi^*(dG \wedge dy) = d(G \circ \Psi) \wedge dy - (A, G) \circ \Psi dx \wedge dy.$

This implies that

$$\begin{aligned}\Phi^*\tau &= \omega + d((F - A) \circ \Psi) \wedge dx + d((G - B) \circ \Psi) \wedge dy \\ &\quad (\partial_x G - \partial_y F - \{A, G\} + \{B, F\} + \{A, B\}) \circ \Psi dx \wedge dy.\end{aligned}$$

So we can set $\widetilde{F} = (F - A) \circ \Psi$ and $\widetilde{G} = (G - B) \circ \Psi$. It remains to check that the last term is indeed $\partial_x \widetilde{G} - \partial_y \widetilde{F}$. We compute

$$\begin{aligned}\partial_x \widetilde{G} &= \partial_x ((G - B) \circ \Psi) \\ &= (\partial_x G - \partial_x B) \circ \Psi + (dG - dB)(X_A \circ \Psi) \\ &= (\partial_x G - \partial_x B - (A, G\} + \{A, B\}) \circ \Psi.\end{aligned}$$

Analogously we find,

$$\partial_y \widetilde{F} = (\partial_y F - \partial_y A - (B, F\} - \{A, B\}) \circ \Psi.$$

So we conclude indeed that

$$\begin{aligned}\partial_x \widetilde{G} - \partial_y \widetilde{F} &= (\partial_x G - \partial_y F - \{A, G\} + \{B, F\} \\ &\qquad + 2\{A, B) - \partial_x B + \partial_y A) \circ \Psi \\ &= (\partial_x G - \partial_y F - \{A, G\} + \{B, F\} + \{A, B\}) \circ \Psi.\end{aligned}$$

The last step follows in view of

$$\partial_x B - \partial_y A - (A, B\} = 0.$$

This is true because

$$\begin{aligned} X_{\partial_x B - \partial_y A - \{A,B\}} \circ \psi &= X_{\partial_x B} \circ \psi - X_{\partial_y A} \circ \psi - [X_A, X_B] \circ \psi \\ &= X_{\partial_x B} \circ \psi + (\nabla_{X_A} X_B) \circ \psi - X_{\partial_y A} \circ \psi \\ &\quad - (\nabla_{X_B} X_A) \circ \psi \\ &= \nabla_x (X_B \circ \psi) - \nabla_y (X_A \circ \psi) \\ &= \nabla_x \partial_y \psi - \nabla_y \partial_x \psi = 0. \end{aligned}$$

It remains to prove the three Statements we started the proof with. Note that the second and third are of the same nature and can be proved in one step. But we will first prove the first assertion. Let (a, b, v), (a', b', v') be tangent vectors of $D \times M$ in a given point. By this short hand notation we mean the vector $a\partial_x + b\partial_y + v$ where v is a tangent vector to M. Note that $a, b, a', b' \in \mathbb{R}$. Then we get

$$\begin{aligned} \Psi^*\omega\big((a, b, v), (u', b', v')\big) &= \omega(aX_A \circ \psi + bX_B \circ \psi + d\psi(v), \\ &\qquad a'X_A \circ \psi + b'X_B \circ \psi + d\psi(v')) \\ &= \omega(v, U') + a\,\omega(X_A \circ \psi, d\psi(v')) \\ &\quad - a'\omega(X_A \circ \psi, d\psi(v)) \\ &\quad + b\,\omega(X_B \circ \psi, d\psi(v')) \\ &\quad - b'\omega(X_B \circ \psi, d\psi(v)) \\ &\quad + (ab' - a'b)\omega(X_A \circ \psi, X_B \circ \psi). \end{aligned}$$

This proves that

$$\Psi^*\omega = \omega - d(A \circ \Psi) \wedge dx - d(B \circ \Psi) \wedge dy + \{A, B\} \circ \Psi dx \wedge dy.$$

We also have

$$\begin{aligned} \Psi^* dF &= dF \circ d\psi + dF(X_A \circ \psi)dx + dF(X_B \circ \psi)dy \\ &= d(F \circ \psi) - (A, F\} \circ \psi dx - \{B, F\} \circ \psi dy, \end{aligned}$$

which implies that

$$\Psi^*(dF \wedge dx) = d(F \circ \Psi) \wedge dx + \{B, F\} \circ \Psi\, dx \wedge dy,$$

and similarly for $\Psi^*(dG \wedge dy)$.

□

As in [P1], [P2], after all those definitions, we are ready to introduce the new concept of the relative K-area of an exact Lagrangian loop Λ. It is defined by the formula

$$\chi(\Lambda) := \inf_{\tau \in \mathcal{T}(\Lambda)} \|\Omega_\tau\|,$$

where

$$\|\Omega_\tau\| := \int_D \left(\max_{z\in M} \Omega_\tau(x,y,z) - \min_{z \in M} \Omega_\tau(x,y,z) \right) dxdy.$$

Note that by Lemma 4.2.2(iii) the Hofer norm of the curvature is invariant under fibrewise Hamiltonian symplectomorphisms. Hence the relative K-area of an exact Lagrangian loop Λ is an invariant of Hamiltonian isotopy classes of loops of exact Lagrangian submanifolds. For the reader's convenience we recall the definition of the real number $\nu(\Lambda)$ from (4.2)

$$\nu(\Lambda) := \inf_{\Lambda' \sim \Lambda} \ell(\Lambda').$$

One of our main results is the following theorem.

Theorem 4.2.4 *For every exact Lagrangian loop* $\Lambda \subset S^1 \times M$

$$\chi(\Lambda) = \nu(\Lambda).$$

Proof: Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop. Let $\tau \in \Omega^2(D \times M)$ be the connection 2-form defined by (4.8) in the proof of Lemma 4.2.2, where the cutoff function $\rho : [0,1] \to [0,1]$ is chosen to be nondecreasing. Then

$$\Phi^*(Fdx + Gdy) = \rho \widetilde{H}_t dt,$$

where $F, G : D \times M \to \mathbb{R}$ are given by (4.9) and $\Phi(r, t, z) = (re^{2\pi i t}, z)$. Taking the differential of this 1-form on $[0,1]^2 \times M$ we find

$$\Phi^*((\partial_x G - \partial_y F) dx \wedge dy) = \dot\rho \widetilde{H}_t dr \wedge dt.$$

Since $\{F, G\} = 0$ and $\Phi^*(dx \wedge dy) = 2\pi r dr \wedge dr$ we obtain

$$\Omega_\tau(re^{2\pi it}, z) = -\frac{\dot\rho(r)}{2\pi r}\widetilde{H}_t(z).$$

Moreover,

$$\left\|\widetilde{H}_t\right\| = \max_M \widetilde{H}_t - \min_M \widetilde{H}_t = \max_{\Lambda_t} \widetilde{H}_t - \min_{\Lambda_t} \widetilde{H}_t,$$

and hence

$$\|\Omega_\tau\| = \int_0^1 \int_0^1 \dot\rho(r) \left\|\widetilde{H}_t\right\| \, drdt = \int_0^1 \left\|\widetilde{H}_t\right\| \, dt = \ell(\Lambda).$$

This implies $\chi(\Lambda) \le \ell(\Lambda)$. If Λ and Λ' are Hamiltonian isotopic then, by Lemma4.2.2 (ii), there exists a fibrewise Hamiltonian symplectomorphism Ψ of $D \times M$ such that $\Psi(\Lambda) = \Lambda'$. Hence $\tau \in \mathcal{T}(\Lambda')$ if and only if $\Psi^*\tau \in \mathcal{T}(\Lambda)$. By Lemma 4.2.2 (iii), $\chi(\Lambda) = \chi(\Lambda') \le \ell(\Lambda')$. Hence $\chi(\Lambda) \le \nu(\Lambda)$.

We prove that $\nu(\Lambda) \le \chi(\Lambda)$. Let $\tau \in$ T(A). We shall construct an exact Lagrangian loop Λ' that is Hamiltonian isotopic to Λ and satisfies

$$\ell(\Lambda') \le \|\Omega_\tau\|. \tag{4.11}$$

Suppose that τ has the form (4.3). Since the function c in (4.3) has no effect on the curvature we may assume, without loss of generality, that $c \equiv 0$. Define $H = H_{r,t} : M \to \mathbb{R}$ and $K =$ K,, $: M \to \mathbb{R}$ by the formula

$$\Phi^*\tau = \omega + dK \wedge dr + dH \wedge dt + (\partial_r H - \partial_t K) dr \wedge dt.$$

Explicitly,

$$K_{r,t} = \cos(2\pi t) F_{re^{2\pi it}} + \sin(2\pi t) G_{re^{2\pi it}},$$

$$H_{r,t} = 2\pi r \cos(2\pi t) G_{re^{2\pi it}} - 2\pi r \sin(2\pi t) F_{re^{2\pi it}}.$$

Define the Hamiltonian symplectomorphisms $\psi_{r,t} : M \to M$ by

$$\partial_r \psi_{r,t} = X_{K_{r,t}} \circ \psi_{r,t}, \qquad \psi_{0,t} = \mathrm{id}.$$

Then the loop

$$\Lambda'_t = \psi_{1,t}{}^{-1}(\Lambda_t)$$

is evidently Hamiltonian isotopic to Λ. We shall prove that it satisfies (4.11). To see this, denote by Ψ the fibrewise Hamiltonian symplectomorphism of $[0, 1]^2 \times M$ given by

$$\Psi(r, t, z) = (r, t, \psi_{r,t}(z)).$$

Then, as in the proof of Lemma 4.2.2, we obtain

$$\Psi^*\Phi^*\tau = \omega + dH' \wedge dt + \partial_r H' dr \wedge dt,$$

where $H'_{r,t} = (H_{r,t} - B_{r,t}) \circ \psi_{r,t}$ and $B_{r,t} : M \to \mathbb{R}$ is defined by $\partial_t \psi_{r,t} = X_{B_{r,t}} \circ \psi_{r,t}$. These functions satisfy

$$\|\Omega_\tau\| = \int_0^1 \int_0^1 \|\partial_r H'_{r,t}\| \, dr dt, \qquad H'_{0,t} = 0.$$

Moreover, by Lemma 3.0.6, we have

$$\begin{aligned}
\partial_t \Lambda'_t &= \psi_{1,t}{}^* \left(\partial_t \Lambda_t - dB_{1,t}|_{\Lambda_t}\right) \\
&= \psi_{1,t}{}^*(dH_{1,t}|_{\Lambda_t}) - d(B_{1,t} \circ \psi_{1,t})|_{\Lambda'_t} \\
&= dH'_{1,t}|_{\Lambda'_t}.
\end{aligned}$$

Hence the length of Λ' is given by

$$\begin{aligned}
\ell(\Lambda') &= \int_0^1 \left(\max_{\Lambda'_t} H'_1, - \min_{\Lambda'_t} H'_{1,t}\right) dt \\
&\le \int_0^1 \left(\max_M H'_{1,t} - \min_M H'_{1,t}\right) dt \\
&= \int_0^1 \left(\max_M \left(\int_0^1 \partial_r H'_{r,t} \, dr\right) - \min_M \left(\int_0^1 \partial_r H'_{r,t} \, dr\right)\right) dt \\
&\le \int_0^1 \int_0^1 \left(\max_M \partial_r H'_{r,t} - \min_M \partial_r H'_{r,t}\right) dr dt \\
&= \int_0^1 \int_0^1 \left\|\partial_r H'_{r,t}\right\| \, dr dt \\
&= \|\Omega_\tau\|.
\end{aligned}$$

Thus we have proved that for every $\tau \in \mathcal{T}(\Lambda)$ there exists an exact Lagrangian loop Λ' that is Hamiltonian isotopic to Λ and satisfies $\ell(\Lambda') \leq \|\Omega_\tau\|$. Hence $\chi(\Lambda) \leq \nu(\Lambda)$ and this proves the theorem. □

4.3 The non-symplectic interval

Let $\Lambda \subset D \times M$ be an exact Lagrangian loop and $\tau \in \mathcal{T}(\Lambda)$ be a Hamiltonian connection 2-form. Since τ is closed and vanishes on Λ (see Lemma 4.2.1) it determines a relative cohomology class (see [BT] Chapter 1.6)

$$[\tau] \in H^2(D \times M, \Lambda; \mathbb{R}).$$

We will show that this relative cohomology class is not unique. However, two different Hamiltonian connection 2-forms yield relative cohomology classes that only differ by a multiple of the pull back of the generator of $H^2(D, \partial D; \mathbb{R})$.

Let Σ be a compact oriented Riemann surface with (possibly empty) boundary $\partial\Sigma$. A smooth map $v : (\Sigma, \partial\Sigma) \to (D \times M, \Lambda)$ determines a 2-dimensional relative homology class

$$[v] := v_*[\Sigma] \in H_2(D \times M, \Lambda; \mathbb{Z}).$$

The pairing of this class with $[\tau]$ is given by

$$\langle [\tau], [v] \rangle = \int_\Sigma v^*\tau.$$

Since every 2-dimensional relative integral homology class of the pair $(D \times M, \Lambda)$ can be represented by a smooth map v as above (see [T] or [S], Remark after Lemma 1.45) , the cohomology class $[\tau]$ is uniquely determined by these pairings. Define $\sigma \in H^2(D \times M, \Lambda; \mathbb{R})$ by

$$\langle \sigma, [v] \rangle = \deg(\pi \circ v) \tag{4.12}$$

for every $v : (\Sigma, \partial\Sigma) \to (D \times M, \Lambda)$, where

$$\pi : (D \times M, \Lambda) \to (D, \partial D)$$

denotes the obvious projection. In (4.12) the degree of a smooth map $v_0 : (\Sigma, \partial \Sigma) \to (D, \partial D)$ is understood as the degree of its restriction to the boundary. It agrees with the number of preimages of an interior regular value, counted with appropriate signs (cf. Milnor [M], Chapter 5). Note that

$$\sigma = \frac{1}{\pi}[dx \wedge dy]$$

and hence σ agrees with the pull back of the positive integral generator of $H^2(D, \partial D; \mathbb{R})$ under the projection π.

Lemma 4.3.1 *Let $\tau_0, \tau_1 \in \mathcal{T}(\Lambda)$ be Hamiltonian connection forms. Then there exists a constant $s = s(\tau_1, \tau_0) \in \mathbb{R}$ such that*

$$[\tau_1] - [\tau_0] = s\sigma.$$

Proof: Let τ_i be given by (4.3) with F, G, c replaced by F_i, G_i, c_i for $i = 0, 1$. Denote

$$F := F_1 - F_0, \qquad G := G_1 - G_0, \qquad c := c_1 - c_0,$$

and let $H_t : M \to \mathbb{R}$ be defined by (4.4). Since $\tau_0, \tau_1 \in \mathcal{T}(\Lambda)$ it follows from Lemma 4.2.1

$$dH_{1t}|_{\Lambda_t} = dH_{0t}|_{\Lambda_t} = \partial \Lambda_t$$

Hence $dH_t|_{\Lambda_t} = 0$ where $H := H_1 - H_0$ so there exists a function $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

$$H_t|_{\Lambda_t} \equiv h(t)$$

for every $t \in \mathbb{R}$. We shall prove that the required identity holds with

$$s := \int_0^1 h(t)\, dt + \int_D c\, dx dy.$$

To see this note that, by (4.4),

$$(F dx + G dy)|_{\Lambda} = \pi^* \alpha_h. \tag{4.13}$$

where $\alpha_h \in \Omega^1$ (St) denotes the push forward of the 1-form

$$h dt \in \Omega^1(\mathbb{R}/\mathbb{Z})$$

under the diffeomorphisms $\mathbb{R}/\mathbb{Z} \to S^1 : [t] \mapsto e^{2\pi i t}$. Let Σ be a compact oriented Riemann surface and $v : \Sigma \to D \times M$ be a smooth function such that $v(\partial\Sigma) \subset \Lambda$. Denote $v_0 := \pi \circ v : (\Sigma, \partial\Sigma) \to (D, \partial D)$. Then

$$\begin{aligned}
\int_\Sigma v^*(\tau_1 - \tau_0) &= \int_\Sigma v^*(dF \wedge dx + dG \wedge dy \\
&\qquad + (\partial_x G - \partial_y F + c) dx \wedge dy) \\
&= \int_{\partial\Sigma} v^*(Fdx + Gdy) + \int_\Sigma v_0{}^*(cdx \wedge dy) \\
&= \int_{\partial\Sigma} v_0{}^*\alpha_h + \int_\Sigma v_0{}^*(cdx \wedge dy) \\
&= s \ \deg(v_0).
\end{aligned}$$

The penultimate equality follows from (4.13) and the last from the identities

$$\int_{\partial\Sigma} v_0{}^*\alpha_h = \deg(v_0) \int_{S^1} \alpha_h \tag{4.14}$$

and

$$\int_\Sigma v_0{}^*(cdx \wedge dy) = \deg(v_0) \int_D cdx \wedge dy. \tag{4.15}$$

Here (4.14) is the degree theorem for maps between compact one dimensional manifolds and (4.15) is the degree theorem for maps between 2-manifolds with boundary. More precisely, if the function $c : D \to \mathbb{R}$ has mean value zero then there exists a 1-form $\alpha \in$ R'(D) such that $d\alpha = cdx \wedge dy$ and $\alpha|_{TS^1} = 0$. To see this recall that if $\int_D cdxdy = 0$ then there exists a function $f : D \to \mathbb{R}$ such that

$$\begin{cases} \Delta f = c \\ \frac{\partial f}{\partial \nu}|_{\partial D} = 0. \end{cases}$$

Define the functions $a, b : D \to \mathbb{R}$ by $a := -\frac{\partial f}{\partial y}$ and $b := \frac{\partial f}{\partial x}$ and the 1-form

$$\alpha := adx + bdy.$$

Clearly $d\alpha = c dx \wedge dy$. To see that α vanishes on the boundary, observe that

$$\begin{aligned} 0 = \frac{\partial f}{\partial \nu}|_{\partial D} &= df(\nu) \\ &= (bdx - ady)(\cos 2\pi t, \sin 2\pi t) \\ &= b\cos 2\pi t - a \sin 2\pi t \\ &= (adx + bdy)(-\sin 2\pi t, \cos 2\pi t) \\ &= adx + bdy|_{TS^1}. \end{aligned}$$

This implies that the left hand side of (4.15) vanishes according to the following computation

$$\begin{aligned} \int_\Sigma v_0^*(cdx \wedge dy) = \int_\Sigma v_0^* d\alpha &= \int_{\partial\Sigma} v_0^*\alpha \\ &= \deg(v_0) \int_{\partial D} \alpha = \deg(v_0)\int_D d\alpha \\ &= \deg(v_0)\int_D cdx \wedge dy = 0. \end{aligned}$$

If c does not have mean value zero then there exists a constant $k \in \mathbb{R}$ such that $c - k$ has mean value zero so $\int_\Sigma v_0^*((c-k)dx \wedge dy) = 0$ and hence it suffices to establish (4.15) for constant functions c and, by Stokes, this reduces to (4.14). This proves the lemma. □

Let $\tau_0 \in \mathcal{T}(\Lambda)$. We shall now address the question which cohomology classes $[\tau_0] + s\sigma$ can be represented by nondegenerate Hamiltonian connection 2-forms. Such a 2-form is a symplectic form on $D \times M$ with respect to which Λ is a Lagrangian submanifold. Denote

$$\mathcal{T}^{\pm}(\Lambda) := \left\{\tau \in \mathcal{T}(\Lambda) \mid \pm\tau^{n+1} > 0\right\}$$

Here the inequality $\tau^{n+1} > 0$ means that $\tau^{n+1} = f\, dx \wedge dy \wedge \omega^n$, where $f : D \times M \to \mathbb{R}$ is a positive function. For $\tau_0 \in \mathcal{T}(\Lambda)$ we define

$$\varepsilon^+(\tau_0, \Lambda) := \inf\left\{s(\tau, \tau_0) \quad \tau \in \mathcal{T}^+(\Lambda)\right\},$$

$$\varepsilon^-(\tau_0, \Lambda) := \sup\left\{s(\tau, \tau_0) \mid \tau \in \mathcal{T}^-(\Lambda)\right\}$$

The proof of Theorem 4.3.3 below shows that the class $[\tau_0] + s\sigma$ can be represented by a symplectic form $\tau \in \mathcal{T}^{\pm}(\Lambda)$ for $\pm s$ sufficiently large and hence ● !c&*(ru, A) $< \infty$. Evidently, $\varepsilon^{\pm}(\tau_1, \Lambda) - \varepsilon^{\pm}(\tau_0, \Lambda) = s(\tau_1, \tau_0)$. Hence the number

$$\varepsilon(\Lambda) := \varepsilon^{+}(\tau_0, \Lambda) - \varepsilon^{-}(\tau_0, \Lambda)$$

is independent of the connection 2-form $\tau_0 \in \mathcal{T}(\Lambda)$ used to define it. This number is called the **width of the nonsymplectic interval.** In some sense $\varepsilon(\Lambda)$ measures how much Λ deviates from the trivial loop $\Lambda_t = \Lambda_0$ for all t.

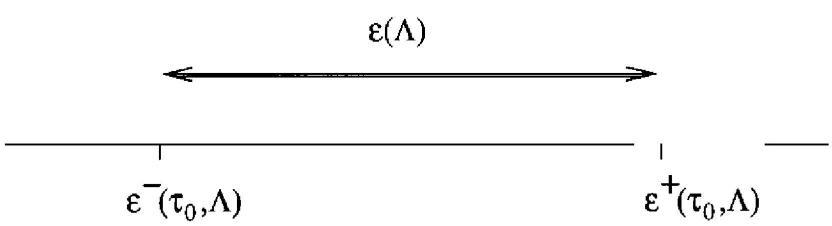

Figure 4.2: The non-symplectic interval

We will first state and prove a technical but elementary lemma and then state a theorem which relates the width of the non-symplectic interval to the invariants from the previous section.

Lemma 4.3.2 *Given a 2-form τ and two vector fields X and Y then*

$$\iota(X)\iota(Y)\tau^{n+1} = (n+1)\Big(\tau(X, Y)\tau^{n} \quad n\iota(X)\tau \wedge \iota(Y)\tau \wedge \tau^{n-1}\Big).$$

In particular,

$$n dF \wedge dG \wedge \omega^{n-1} = \{F, G\}\omega^{n}.$$

Proof: First of all we have that

$$\iota(X)\tau^{n+1} = (n+1)\big(\iota(X)\tau\big) \wedge \tau^{n}$$

and in general we have that given two forms α and β we get

$$\iota(X)\big(\alpha \wedge \beta\big) = \big(\iota(X)\alpha\big) \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \big(\iota(X)\beta\big)$$

The proof will now go by induction. Clearly the Statement it true for $n = 0$, for $n = 1$ we get

$$\begin{aligned} \iota(X)\iota(Y)\tau^2 &= \iota(X)\big(2(\iota(Y)\tau) \wedge \tau\big) \\ &= 2(\iota(X)\iota(Y)\tau) \wedge \tau - 2\iota(Y)\tau\ AL(X)?. \end{aligned}$$

Now assume the Statement holds for $n - 1$, then we get

$$\begin{aligned} \iota(X)\iota(Y)\tau^{n+1} &= \iota(Y)\big((\iota(X)\tau) \wedge \tau^n + 5 \wedge \iota(X)\tau^n\big) \\ &= (\iota(Y)\iota(X)\tau) \wedge 5" - (\iota(X)\tau) \wedge \iota(Y)\tau^n \\ &\quad + (\iota(Y)\tau) \wedge \iota(X)\tau^n + \tau \wedge \iota(Y)\iota(X)\tau^n \\ &= \tau(X, Y)\tau^n - n(\iota(X)\tau) \wedge \iota(Y)\tau \wedge \tau^{n-1} \\ &\quad + n(\iota(Y)\tau) \wedge \iota(X)\tau \wedge \tau^{n-1} \\ &\quad + \tau \wedge \Big(n\tau(X, Y)\tau^{n-1} \\ &\qquad - n(n-1)\iota(X)\tau \wedge \iota(Y)\tau \wedge \tau^{n-2}\Big) \\ &= (n+1)\tau(X, Y)\tau^n - n(n+1)\iota(X)\tau \wedge \iota(Y)\tau \wedge \tau^{n-1} \end{aligned}$$

The second Statement follows from applying this formula to ω and observing that $\omega^{n+1} = 0$. □

Theorem 4.3.3 ***For every exact Lagrangian loop Λ C D x M***

$$\varepsilon(\Lambda) \leq \chi(\Lambda).$$

Proof: Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop and $F, G : D \times M \to \mathbb{R}$ be smooth functions such that the functions $H_t : M \to \mathbb{R}$ defmed by (4.4) satisfy $dH_t|_{\Lambda_t} = \partial_t\Lambda_t$ for every t. For every smooth function $c : D \to \mathbb{R}$ let $\tau_c \in$ T(A) be given by (4.3). In particular, τ_0 is given by (4.3) with $c = 0$. We shall prove that

$$\varepsilon^+(\tau_0, \Lambda) \leq \int_D \max_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy, \tag{4.16}$$

$$\varepsilon^-(\tau_0, \Lambda) \geq \int_D \min_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy. \tag{4.17}$$

From Lemma 4.3.2 we get that

$$\begin{aligned}
{\tau_c}^{n+1} &= (n+1)(\partial_x G - \partial_y F + c)dx \wedge dy \wedge \omega^n \\
&\quad + n(n+1)dF \wedge dx \wedge dG \wedge dy \wedge \omega^{n-1} \\
&= (n+1)(\partial_x G - \partial_y F - \{F, G\} + c)dx \wedge dy \wedge \omega^n \\
&= (n+1)(c - \Omega_{\tau_0})dx \wedge dy \wedge \omega^n.
\end{aligned} \tag{4.18}$$

This shows that τ_c is nondegenerate if and only if $c(x, y) \neq \Omega_{\tau_0}(x, y, z)$ for all $(x + iy, z) \in D \times M$. Fix a number s satisfying

$$s > \int_D \max_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy.$$

Choose a smooth function $c : D \to \mathbb{R}$ such that

$$c(x, y) > \max_{z \in M} \Omega_{\tau_0}(x, y, z)$$

for all $x + iy \in D$ and

$$\int_D c\, dxdy = s.$$

Then τ_c is nondegenerate and represents the class $[\tau_c] = [\tau_0] + s\sigma$. This proves (4.16) and (4.17) follows from a similar argument. It follows from (4.16) and (4.17) that

$$\begin{aligned}
\varepsilon(\Lambda) &= \varepsilon^+(\tau_0, \Lambda) - \varepsilon^-(\tau_0, \Lambda) \\
&\leq \int_D \left(\max_{z \in M} \Omega_{\tau_0}(x, y, z) - \min_{z \in M} \Omega_{\tau_0}(x, y, z) \right) dxdy \\
&= \|\Omega_{\tau_0}\|.
\end{aligned}$$

Since the curvature of τ_0 is equal to the curvature of τ_c for every c it follows that $\varepsilon(\Lambda) \leq \|\Omega_\tau\|$ for every $\tau \in \mathcal{T}(\Lambda)$ and hence $\varepsilon(\Lambda) \leq \chi(\Lambda)$. This proves the theorem. □

Combining the above with Theorem 4.2.4 we get

Corollary 4.3.4

$$\varepsilon(\Lambda) \leq \chi(\Lambda) = \nu(\Lambda)$$

Remark 4.3.5 Let us denote

$$T(A) := \left\{[\tau] \in H^2(D \times M, \Lambda; \mathbb{R}) \mid \tau \in \mathcal{T}(\Lambda)\right\}. \tag{4.19}$$

Lemma 4.3.1 shows that this set is a l-dimensional affine subspace of $H^2(D \times M, \Lambda;$ JR). Denote

$$\text{T*(A)} := \{[\tau] \quad \tau \in \mathcal{T}^{\pm}(\Lambda)\}$$

These sets are open and connected. Openness follows from the definition of $\mathcal{T}^{\pm}(\Lambda)$. To prove connectedness, let $\tau_i \in \mathcal{T}^+(\Lambda)$ be given by (4.3) with F, G, c replaced by F_i, G_i, c_i for $i = 0,$ 1. By (4.18), $c_i > \Omega_{\tau_i}$. Assume without loss of generality that $s(\tau_1, \tau_0) \geq 0$. Then the path

$$[0, 1] \to T^+(\Lambda) : t \mapsto [\tau_0] + ts(\tau_1, \tau_0)\sigma$$

connects $[\tau_0]$ with $[\tau_1]$. This shows that the sets $T^{\pm}(\Lambda)$ are connected. The complement T(A) \ (T-(A) **U** $T^+(\Lambda)$) is compact and connected. It can be expressed in the form

$$T(\Lambda) \setminus (T^-(\Lambda) \cup \text{T+(A)}) = \{[\tau_0] + s\sigma \mid \varepsilon^-(\tau_0, \Lambda) \leq s \leq \varepsilon^+(\tau_0, \Lambda)\}$$

for every $\tau_0 \in$ 7(A). We do not know whether this complement is always nonempty or, equivalently, whether $\varepsilon(\Lambda)$ is always nonnegative.

Chapter 5

Loops on the 2-torus

In this chapter we will construct a 2-dimensional example for which we can explicitly compute $\nu(\Lambda)$. We will also see that in this case the difference between exact Lagrangian isotopy and Lagrangian isotopy is a genuine one. This chapter contains two sections. In the first section we will construct the example. The second section is devoted to an auxiliary lemma that is needed in the proof of the first section.

5.1 Hofer length versus area

Consider the 2-torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the Standard symplectic form

$$\omega = dx \wedge dy$$

and let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the projection. Let $[x, y] = \pi(x, y) \in \mathbb{T}^2$ denote the equivalence class of all points $(x, y) + \mathbb{Z}^2$. Let

$$B_r = \{(s, t) \in \mathbb{R}^2 \quad s^2 + t^2 \leq r^2\}$$

be a disc of radius r and suppose that $S \subset \mathbb{T}^2$ is the image of an embedding $B_1 \to \mathbb{T}^2$. Define

$$\Lambda_t := \Lambda_t(S) := \{[x, y + t] \mid [x, y] \in \partial S\} \tag{5.1}$$

(see Figure 5.1).

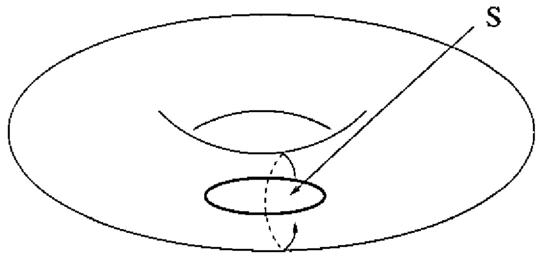

Figure 5.1: A Lagrangian loop on the 2-torus

Theorem 5.1.1 *Let S c $\mathbb{T}^2$ be a closed embedded disc and $t \mapsto$ A, be the exact Lagrangian loop defined by (5.1). Then*

$$\nu(\Lambda) = \text{area}(S).$$

Proof: We prove that $\nu(\Lambda) \leq \text{area}(S)$. To see this, choose smooth functions x, $y : \mathbb{R} \to \mathbb{R}$ with θ as coordinate on $\mathbb{R}$ such that

$$x(\theta + 1) = x(\theta), \qquad y(\theta + 1) = y(\theta),$$

and the map $\iota_t : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2$ defined by

$$\iota_t(\theta) := [x(\theta), y(\theta) + t]$$

is an embedding with $\iota_t(\mathbb{R}/\mathbb{Z}) = \Lambda_t$. Then

$$\alpha_t := \omega(\partial_t \iota_t, d\iota_t \cdot) = -\dot{x} d\theta \in \Omega^1(\mathbb{R}/\mathbb{Z}).$$

Hence $\alpha_t = dh$, where $h_t = -x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. Hence

$$\|\partial_t \Lambda_t\| = \|dh_t\| = \max x - \min x$$

and this implies

$$\ell(\Lambda) = \max x - \min x.$$

By Proposition 5.2.1 in the next section, two loops $t \mapsto$ A,(S) and $t \mapsto \Lambda_t$ (S'), associated to two embedded discs S, S' $\subset \mathbb{T}^2$ via (5.1). are Hamiltonian isotopic if and only if S and S' have the same area. Now for every

$\delta > 0$ there exists an embedded disc S' (as illustrated in Figure 5.2, which represents the fundamental domain of the torus) such that

$$\operatorname{area}(S) = \operatorname{area}(S'), \qquad \max x' - \min x' < \operatorname{area}(S) + \delta,$$

where $x', y' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ are chosen such that the map

$$\iota'(\theta) = [x'(\theta), y'(\theta)]$$

defines an embedding $\mathbb{R}/\mathbb{Z} \to \mathbb{T}^2$ whose image is $\partial S'$. Hence the length of the loop $t \mapsto \Lambda_t(S')$ is bounded above by $\operatorname{area}(S) + \delta$. Thus we have proved that

$$\nu(\Lambda) \leq \operatorname{area}(S).$$

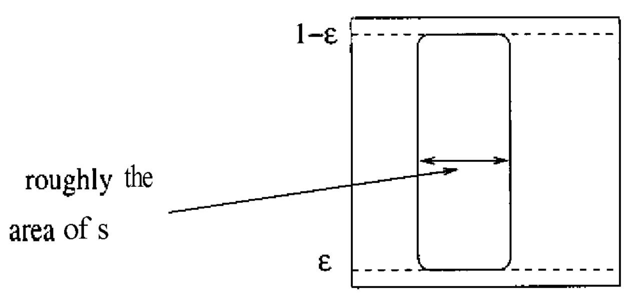

Figure 5.2: Minimizing the length

To show the reverse inequality let $t \mapsto \Lambda'_t$ be an exact Lagrangian loop that is Hamiltonian isotopic to Λ. Then

$$\Lambda'_0 = \partial S',$$

where $S' \subset \mathbb{T}^2$ is a smoothly embedded closed disc of the same area as S. In view of Lemma 3.0.3 we can choose a Hamiltonian isotopy $\psi_t : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\psi_t(\Lambda'_0) = \Lambda'_t.$$

We shall prove that

$$\operatorname{area}(S) \leq \ell\left(\{\psi_t\}_{0 \leq t \leq 1}\right). \tag{5.2}$$

To see this, choose an embedded closed discs $\tilde{S} \subset \mathbb{R}^2$ such that $\pi(\tilde{S}) = S'$ and let $\tilde{\psi}_t : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of ψ_t. Since A' is Hamiltonian isotopic to Λ we have $\tilde{\psi}_{t+1}(\tilde{S}) = \tilde{\psi}_t(\tilde{S}) + (0, 1)$ and hence

$$\tilde{\psi}_1(\tilde{S}) \cap \tilde{S} = \emptyset.$$

Let $\tilde{H}_t : \mathbb{R}^2 \to \mathbb{R}$ be the Hamiltonian functions that generate $\tilde{\psi}_t$ and have mean value zero over the fundamental domain $[0, 1]^2$. Choose $R > 1$ such that $\tilde{\psi}_t(\tilde{S}) \subset B_R$ for every $t \in [0, 1]$ and let $\beta : \mathbb{R}^2 \to [0, 1]$ be a compactly supported cutoff function such that $\beta|_{B_R} \equiv 1$. Then the functions

$$\hat{H}_t := \beta \tilde{H}_t$$

generate a compactly supported Hamiltonian isotopy $\hat{\psi}_t$ of $\mathbb{R}^2$ that satisfies

$$\hat{\psi}_1(\tilde{S}) \cap \tilde{S} = \emptyset.$$

Now it follows from the *energy-capacity inequality* in Hofer [H] that the *displacement energy* of $\tilde{S}$ is bounded below by the area. Hence

$$\text{area}(\tilde{S}) \leq d(\text{id}, \hat{\psi}_1) \leq \ell(\{\hat{\psi}_t\}_{0 \leq t \leq 1}) = \ell(\{\psi_t\}_{0 \leq t \leq 1}).$$

Since

$$\text{area}(\tilde{S}) = \text{area}(S') = \text{area}(S),$$

this proves (5.2). It follows from (5.2) and (3.8) that

$$\text{area}(S) \leq \ell(\Lambda')$$

for every exact Lagrangian loop A' that is Hamiltonian isotopic to A. Hence $\text{area}(S) \leq \nu(\Lambda)$. □

Theorem 5.1.1 shows that the invariant $\nu(\Lambda)$ is not necessarily invariant under Lagrangian isotopy, but only under exact Lagrangian isotopy. As an example consider two embedded circles S and S', enclosing discs of different area, as in Figure 5.3. It is quickly seen that there exists a family of Lagrangian submanifolds S_s, $0 \leq s \leq 1$ with So = S and St = S' (shrinking the circle). However, by Proposition 5.2.1 in the next section, the two loops $t \mapsto \Lambda_t(S)$ and $t \mapsto \Lambda_t(S')$, generated via (5.1) are not Hamiltonian isotopic since the areas the discs enclosed by S and S' are not equal, although they can clearly be connected by a path of Lagrangian loops, use $\Lambda_{s,t} = \Lambda_t(S_s)$.

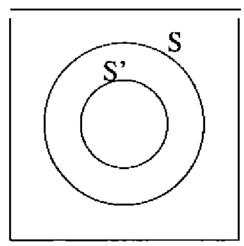

Figure 5.3: Two embedded circles

The techniques of proof are specitic to the 2-dimensional case. To establish lower bounds for our invariants in higher dimensions we shall use existence theorems for pseudoholomorphic discs.

5.2 Symplectic isotopy on Riemann surfaces

The following results are known. However, we could not find proofs in the literatme and present them here for the sake of completeness.

Proposition 52.1 *Let Σ be a compact connected oriented Riemann surface with area form ω and S, $S' \subset \Sigma$ be two closed embedded discs with the same area. Then there exists a Hamiltonian symplectomorphism $\psi : \Sigma \to \Sigma$ such that $\psi(S) = S'$.*

The proof relies on the following three lemmata. The first asserts that, in dimension 2, a symplectomorphism is smoothly isotopic to the identity if and only if it is symplectically isotopic to the identity. For the 2-torus this follows from the characterization of Hamiltonian symplectomotphisms in Conley-Zehnder [CZ, Theorem 6]. In general the proof is a parametrized Version of Moser isotopy. The work of Seidel [S1] shows that in higher dimensions it is no longer true that a symplectomorphism smoothly isotopic to the identity is if and only if it symplectically isotopic to the identity is.

Lemma *5.2.2 Let Σ be a compact oriented Riemann surface with area form ω and $\psi : \Sigma \to \Sigma$ be a symplectomorphism. Then ψ is smoothly* *isotopic to the identity if and only if it is symplectically isotopic to the identity.*

Proof: Let $[0, 1] \to \mathrm{Diff}(\Sigma) : t \mapsto \psi_t$ be a smooth isotopy such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. Define

$$\omega_t := \psi_{t*}\omega, \qquad \omega_{s,t} := s\omega + (1 - s)\omega_t$$

for $0 \le s, t \le 1$. Then $\omega_{s,0} = \omega_{s,1} = \omega_{1,t} = \omega$ and $\omega_{0,t} = \omega_t$ for all s and t. Fix a Riemannian metric on Σ with volume form ω and let $\alpha_t \in \Omega^1(\Sigma)$ bedefinedby

$$d\alpha_t = \omega_t - \omega, \qquad \alpha_t \in \operatorname{im} d^*.$$

Choose $X_{s,t} \in \mathrm{Vect}(\Sigma)$ such that $\iota(X_{s,t})\omega_{s,t} = \alpha_t$ and define a family of diffeomorphisms $\psi_{s,t} \in \mathrm{Diff}(\Sigma)$ by

$$\partial_s\psi_{s,t} = X_{s,t} \circ \psi_{s,t}, \qquad \psi_{0,t} = \psi_t.$$

Then

$$\begin{aligned}\partial_s(\psi_{s,t}{}^*\omega_{s,t}) &= \psi_{s,t}{}^*(\partial_s\omega_{s,t} + \mathcal{L}_{X_{s,t}}\omega_{s,t}) \\ &= \psi_{s,t}{}^*(\omega - \omega_t + d\alpha_t) \\ &= 0\end{aligned}$$

and since $\psi_{0,t}{}^*\omega_{0,t} = \omega$ we get that $\psi_{s,t}{}^*\omega_{s,t} = \omega$ for all s and t. Moreover, $\psi_{s,0} = \mathrm{id}$ and $\psi_{s,1} = \psi$ for all s. Hence $t \mapsto \psi_{1,t}$ is the required symplectic isotopy from id to ψ. □

Note that the above proof relies on the fact that we are in two dimensions. Upon taking the Lie derivative we use that the exterior derivative of a 2-form always vanishes, which is clearly no longer true in higher dimensions.

Lemma 5.2.3 *Let Σ be a compact oriented Riemann surface, $S \subset \Sigma$ be an embedded closed disc, und $\omega_0, \omega_1 \in \Omega^2(\Sigma)$ be two area forms such that*

$$\int_\Sigma (\omega_1 - \omega_0) = \int_S (\omega_1 - \omega_0) = 0.$$

Then there exists a smooth isotopy $\psi_t : \Sigma \to \Sigma$ such that

$$\psi_0 = \mathrm{id}, \qquad \psi_1{}^*\omega_1 = \omega_0, \qquad \psi_t(S) = S$$

for every $t \in [0, 1]$.

Proof: The result follows again from Moser isotopy. We prove that there exists a 1-form $\alpha \in \Omega^1(\Sigma)$ such that

$$d\alpha + \omega_1 - \omega_0 = 0, \qquad \alpha|_{T\partial S} = 0. \tag{5.3}$$

To see this, choose any 1-form $\beta \in \Omega^1(\Sigma)$ such that $d\beta + \omega_1 - \omega_0 = 0$. Such a 1-form exists since ω_0 and ω_1 are area forms with the same area hence they are cohomologous. Then the integral of β over ∂S vanishes and so $\beta|_{\partial S}$ is exact. Hence there exists a smooth function $f : \Sigma \to \mathbb{R}$ such that $(\beta - df)|_{T\partial S} = 0$ and the 1-form $\alpha := \beta - df$ satisfies (5.3). Now let $\omega_t := t\omega_1 + (1 - t)\omega_0$ and define $X_t \in \mathrm{Vect}(\Sigma)$ by $\iota(X_t)\omega_t = \alpha$. Note that since α vanishes on $T\partial S$ we can deduce that X_t must be tangent to ∂S for all t. Define now $\psi_t \in \mathrm{Diff}(\Sigma)$ by

$$\partial_t\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

Then ψ_t preserves ∂S. As before we have $\partial_t({\psi_t}^*\omega_t) = 0$ and hence ${\psi_t}^*\omega_t = \omega_0$ for every t. This proves the lemma. □

Lemma 5.2.4 *Let Σ be a compact connected Riemann surface and $S, S' \subset \Sigma$ be two embedded discs. Then there exists a diffeomorphism $f : \Sigma \to \Sigma$ such that f is isotopic to the identity and $f(S) = S'$.*

Proof: Choose orientation preserving embeddings $\phi, \phi' : B_1 \to \Sigma$ such that $\phi(B_1) = S$ and $\phi'(B_1) = S'$. We prove the result in four steps, see Figure 5.4 as an illustration.

Step 1: *There exists a diffeomorphism $g : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies $g \circ \phi(0) = \phi'(0)$.*

Choose a path $\gamma : [0, 1] \to \Sigma$ such that $\gamma(0) = \phi(0)$ and $\gamma(1) = \phi'(0)$. Next choose a smooth family of vector fields $X_t \in \mathrm{Vect}(\Sigma)$ such that $X_t(\gamma(t)) = \dot\gamma(t)$ for every t. Then the diffeomorphisms $g_t : \Sigma \to \Sigma$, defined by $\partial_t g_t = X_t \circ g_t$ and $g_0 = \mathrm{id}$, satisfy $g_t(\gamma(0)) = \gamma(t)$ for every t. Hence g_1 satisfies the requirements of Step 1.

Step 2: *ϕ can be chosen such that $d(g \circ \phi)(0) = d\phi'(0)$.*

Define $\Psi \in \mathbb{R}^{2\times 2}$ by

$$d(g \circ \phi)(0)\Psi = d\phi'(0)$$

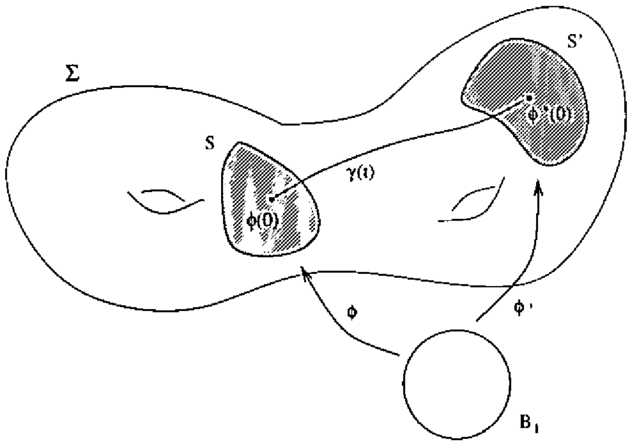

Figure 5.4: Embedded discs in a Riemann surface

Then det $\Psi > 0$ and hence there exists a path

$$[0, 1] \to GL(2) : t \mapsto \Psi(t)$$

such that W(0) = 11 and $\Psi(1) = \Psi$. Choose a family of vector fields X, : $B_1 \to \mathbb{R}^2$ that vanish near the boundary and satisfy

$$X_t(0) = 0, \qquad dX_t(0) = \dot{\Psi}(t)\Psi(t)^{-1}.$$

Here $dX_t(0) : T_0B_1 \to T_0\mathbb{R}^2$. Let $\psi_t : B_1 \to B_1$ be the isotopy generated by X,. Then

$$\psi_t(0) = 0, \qquad d\psi_t(0) = \Psi(t)$$

for every t. To see this observe that we have

$$\begin{aligned} \frac{d}{dt} d\psi_t(0) &= d\frac{d}{dt}\psi_t(0) \\ &= d(X_t \circ \psi_t)(0) \\ &= dX_t(\psi_t(0))d\psi_t(0) \\ &= \dot{\Psi}(t)\Psi(t)^{-1}d\psi_t(0) \end{aligned}$$

and integrate with respect to t. Now replace ϕ by $\phi \circ \psi_1$.

Step *3: ϕ can be chosen such that $g \circ \phi(z) = \phi'(z)$ for $|z|$ sufficiently small.*

By Step 2, we may assume that $d(g \circ \phi)(0) = d\phi'(0)$. Choose $\delta > 0$ such that $\phi'(B_\delta) \subset g(S)$ and consider the function

$$h := \phi^{-1} \circ g^{-1} \circ \phi' : B_\delta \to B_1.$$

This function is an embedding and satisfies $dh(0) = 1\!\!1$. Choose a smooth cutoff function $\beta : [0, 1] \to [0, 1]$ such that $\beta(r) = 1$ for $r \leq 1/3$ and $\beta(r) = 0$ for $r \geq 2/3$. For $0 < \varepsilon < \delta$ define $h_\varepsilon : B_1 \to B_1$ by

$$h_\varepsilon(z) := \beta(|z|/\varepsilon)h(z) + (1 - \beta(|z|/\varepsilon))z.$$

Then h_ε is a diffeomorphism for $\varepsilon > 0$ sufficiently small and

$$g \circ \phi \circ h_\varepsilon(z) = \phi'(z)$$

for $|z| < \varepsilon/3$. Hence the embedding $\phi \circ h_\varepsilon$ satisfies the requirements of Step 3 for $\varepsilon > 0$ sufficiently small.

Step 4: We prove the lemma.

By Step 3, there exist embeddings $\phi, \phi' : B_1 \to \Sigma$, a constant $\varepsilon > 0$, and a diffeomorphism $g : \Sigma \to \Sigma$ such that g is isotopic to the identity and

$$|z| < \varepsilon \qquad \Longrightarrow \qquad g \circ \phi(z) = \phi'(z).$$

Choose $\delta > 0$ such that ϕ and ϕ' extend to embeddings of $B_{1+\delta}$ into Σ. Choose a smooth function $\rho : [0, 1 + \delta] \to [0, 1 + \delta]$ such that $\dot\rho(r) > 0$ for every r and

$$\rho(r) = \begin{cases} r, & \text{for } r \leq \varepsilon/2, \\ 1, & \text{for } r = \varepsilon, \\ r, & \text{for } r \geq 1 + \delta/2. \end{cases}$$

Let $f : \Sigma \to \Sigma$ be given by

$$f(\phi(z)) := \phi(\rho(|z|)z/|z|)$$

for $z \in B_{1+\delta}$ and by $f = \mathrm{id}$ in $\Sigma \setminus \phi(B_{1+\delta})$. Then f is isotopic to the identity and $f \circ \phi(B_\varepsilon) = S$. Similarly, there exists a diffeomorphism $f' : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies $f' \circ \phi'(B_\varepsilon) = S'$. The diffeomotphism $f' \circ g \circ f^{-1}$ is isotopic to the identity and maps S to S'. This proves the lemma. □

Proof **of Proposition 52.1:** By Lemma 5.2.4, there exists a diffeomorphism $f : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies f(S) = S'. Since S and S' have the same area, we obtain

$$\int_{\Sigma}(f^*\omega - \omega) = \int_{S}(f^*\omega - \omega) = 0.$$

By Lemma 5.2.3, there exists a diffeomorphism $\psi : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies

$$\psi^* f^*\omega = \omega, \qquad \psi(S) = s.$$

Hence $\phi := f \circ \psi$ is isotopic to the identity and

$$\phi^*\omega = \omega, \qquad \phi(S) = S'.$$

By Lemma 5.2.2, ϕ is symplectically isotopic to the identity. Let $t \mapsto \phi_t$ be a symplectic isotopy such that $\phi_0 = \text{id}$ and $\phi_1 = \phi$. Then the embedded discs $S_t := \phi_t(S)$ all have the same area and $S_0 =$ S, $S_1 =$ S'. Hence $t \mapsto \partial S_t$ is an exact Lagrangian path (this follows by combining the fact that the area is preserved and the definition of exact). By Lemma 3.0.4, there exists a Hamiltonian isotopy $t \mapsto \psi_t$ of Σ such that $\psi_t(\partial S_0) = \partial S_t$ for all t. Hence ψ_1 (S) = S' and this proves the proposition. □

Chapter 6

Relative Gromov invariants

Throughout we assume that our symplectic manifold (M, ω) is compact and $\dim M = 2n$. The relative Gromov invariants of an exact Lagrangian loop $\Lambda \subset D \times M$ are defined in terms of holomorphic sections of the bundle $D \times M \to D$ with boundary values in Λ, where $\Lambda = \{(e^{2\pi it}, \Lambda_t) \subset D \times M\}$. Let us denote by $\mathrm{Map}_\Lambda(D, M)$ the space of smooth functions $u : D \to M$ that satisfy $u(e^{2\pi it}) \in \Lambda_t$ for every $t \in \mathbb{R}$. The **Maslov class** is a function

$$\mu_\Lambda : \mathrm{Map}_\Lambda(D, M) \to \mathbb{Z}$$

defined as follows. Given $u \in \mathrm{Map}_\Lambda(D, M)$ choose a trivialization of the tangent bundle u^*TM. Then the tangent spaces $T_{u(e^{2\pi it})}\Lambda_t$ define a loop of Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$ and $\mu_\Lambda(u)$ is defined as the Maslov index of this loop (cf. [RS1] or see Appendix B). This integer is independent of the choice of the trivialization used to define it, and it depends only on the homology class of u in $H_2(D \times M, \Lambda; \mathbb{Z})$. We shall assume throughout that the pair (M, Λ_0) is **monotone,** i.e. there exists a $\lambda > 0$ such that, for every smooth map $v \in \mathrm{Map}_{\Lambda_0}(D, M)$,

$$\int_D v^*\omega = \lambda \mu_{\Lambda_0}(v).$$

Here μ_{Λ_0} denotes the Maslov class corresponding to the constant loop $t \mapsto \Lambda_0$. The **minimal Maslov number** of the pair (M, Λ_0) is defined by

$$N := \inf \left\{\mu_{\Lambda_0}(v) \mid v : (D, \partial D) \to (M, \Lambda_0), \ \mu_{\Lambda_0}(v) > 0\right\},$$

hence $N > 0$. In this chapter we shall define relative Gromov invariants for every tuple $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ with $0 \le t_1 < \cdots < t_k < 1$ and every class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ that satisfies $n \pm \mu_\Lambda(A) \le N - 2$. The invariants are homology classes

$$\mathrm{Gr}^{\pm}_{A,t}(\Lambda) \in H_{n \pm \mu_\Lambda(A)}(\Lambda_t; \mathbb{Z}_2),$$

where $\Lambda_t := \Lambda_{t_1} \times \cdots \times \Lambda_{t_k}$. These homology classes arise from certain moduli spaces $\mathcal{M}_A(\tau, \pm J)$ of (anti-)holomorphic sections of the bundle $D \times M$ with boundary values in Λ that represent the class A. The points $(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})$ determine an evaluation map

$$\mathrm{ev}_t : \mathcal{M}_A(\tau, \pm J) \to \Lambda_t$$

and $\mathrm{Gr}^{\pm}_{A,t}(\Lambda)$ is defined as the image of the fundamental cycle of the compact manifold $\mathcal{M}_A(\tau, \pm J)$ under the induced homomorphism on homology. We shall work with almost complex structures on $D \times M$ that are compatible with the fibration. Every such structure is determined by a family of almost complex structures on M and a connection 2-form $\tau \in \mathcal{T}(\Lambda)$.

6.1 J-holomorphic discs

Let $\Lambda \subset S^1 \times M$ be an exact Lagrangian loop and $\tau \in \mathcal{T}(\Lambda)$ be a Hamiltonian connection 2-form that preserves Λ. Throughout we shall denote by $\mathcal{J}(M, \omega)$ the space of smooth almost complex structures on TM that are compatible with ω as defined in Chapter 2. Let $D \to \mathcal{J}(M, \omega) : (x, y) \mapsto J_{x,y}$ be a smooth family of such almost complex structures. Associated to the triple (τ, J, Λ) there is a natural boundary value problem for smooth functions $u : D \to M$:

$$\begin{cases} \partial_x u - X_F(u) + J(\partial_y u - X_G(u)) = 0 \\ u(e^{2\pi i t}) \in \Lambda_t, \qquad t \in \mathbb{R}. \end{cases} \tag{6.1}$$

Here τ is the connection 2-form determined by the two functions F and G given by (4.3). We use the following abbreviations $J = J_{x,y}$, $X_F = X_F(x, y, \cdot) \in \mathrm{Vect}(M)$ denotes the family of Hamiltonian vector fields defined by the functions $F = F(x, y, \cdot) : M \to \mathbb{R}$, and similarly for X_G. Following Gromov [Gl] we observe that the solutions of (6.1) can be thought of as pseudo-holomorphic curves in $D \times M$.

Remark 6.1.1 Consider the almost complex structure $\tilde{J}$ on $D \times M$ given by

$$\tilde{J} = \tilde{J}(\tau, J) := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -JX_F + X_G & -X_F - JX_G & J \end{pmatrix}$$

Then $u : D \to M$ is a solution of (6.1) if and only if the function

$$\tilde{u}(x, y) = (x, y, u(x, y)) \tag{6.2}$$

is a $\tilde{J}$-holomorphic curve in $D \times M$, i.e.

$$\partial_x \tilde{u} + \tilde{J} \partial_y \tilde{u} = 0 .$$

If τ is given by (4.3) and we denote a tangent vector

$$\xi\, \partial_x + \eta\, \partial_y + \zeta \in T_{x,y,z}(D \times M)$$

by (ξ, η, ζ) then, for every $\tilde{\zeta} = (\xi, \eta, \zeta) \in T_{x,y,z}(D \times M)$,

$$\begin{aligned}
\tau(\tilde{\zeta}, \tilde{J}\tilde{\zeta}) &= \tau(\xi\, \partial_x + \eta\, \partial_y + \zeta, -\eta\, \partial_x + \xi \partial_y \\
&\qquad + \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&= \omega(\zeta, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad + (\partial_x G - \partial_y F + c)(\xi^2 + \eta^2) \\
&\quad - \eta dF(\zeta) - \xi dF(\xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad + \xi dG(\zeta) - \eta dG(\xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&= \omega(\zeta, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad + (\partial_x G - \partial_y F + c)(\xi^2 + \eta^2) \\
&\quad - \omega(\eta X_F, \zeta) - \omega(\xi X_F, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad + \omega(\xi X_G, \zeta) - \omega(\eta X_G, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&= \omega(\zeta, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad + (\partial_x G - \partial_y F + c)(\xi^2 + \eta^2) \\
&\quad + \omega(\zeta, \eta X_F) - \omega(\xi X_F, \xi(-JX_F + X_G) - \eta(X_F + JX_G) + J\zeta) \\
&\quad - \omega(\zeta, \xi X_G) - \omega(\eta X_G, \xi(-JX_F + X_,) - \eta(X_F + JX_G) + J\zeta)
\end{aligned}$$

$$
\begin{aligned}
&= \omega(\zeta, -\xi J X_F - \eta J X_G + J\zeta) + (\partial_x G - \partial_y F + c)(\xi^2 + \eta^2) \\
&\quad - \omega(\xi X_F, -\xi J X_F - \eta J X_G + J\zeta) - \omega(\xi X_F, \xi X_G - \eta X_F) \\
&\quad - \omega(\eta X_G, -\xi J X_F - \eta J X_G + J\xi) - \omega(\eta X_G, \xi X_G \quad \eta X_F) \\
&= \omega(\zeta, -\xi J X_F - \eta J X_G + J\zeta) + (\partial_x G - \partial_y F + c)(\xi^2 + \eta^2) \\
&\quad - \omega(\xi X_F, -\xi J X_F - \eta J X_G + J\xi) - \xi^2\{F, G\} \\
&\quad - \omega(\eta X_G, -\xi J X_F - \eta J X_G + J\xi) + \eta^2\{G, F\} \\
&= \omega(\zeta - \xi X_F - \eta X_G, -\xi J X_F - \eta J X_G + J\zeta) \\
&\quad + (\xi^2 + \eta^2)((\partial_x G - \partial_y F - \{F, G\} + c) \\
&= \omega(\zeta - \xi X_F - \eta X_G, J(\xi - \xi X_F - \eta X_G)) + (c - \Omega_\tau)(\xi^2 + \eta^2) \\
&= |\zeta - \xi X_F - \eta X_G|^2 + (c - \Omega_\tau)(\xi^2 + \eta^2).
\end{aligned}
$$

Hence $\tilde{J}$ is tamed by τ whenever $c(x, y) - \Omega_\tau(x, y, z) > 0$ for all $(x, y, z) \in D \times M$ which means precisely that $\tau \in \mathcal{T}^+(\Lambda)$ (see (4.18)). On the other hand, consider $\tilde{J}(\tau, -J)$, then one verifies easily

$$\tau(\tilde{\zeta}, \tilde{J}(\tau, -J)\tilde{\zeta}) = -|\zeta - \xi X_F - \eta X_G|^2 + (c - \Omega_\tau)(\xi^2 + \eta^2).$$

The above expression is strictly negative if $c(x, y) - \Omega_\tau(x, y, z) < 0$ for all $(x, y, z) \in D \times M$ So if $\tau \in \mathcal{T}^-(\Lambda)$ then $\tilde{J}(\tau, -J)$ is tamed by $-\tau$.

The energy of a solution u of (6.1) is defined by

$$E(u) := \int_D |\partial_x u - X_F(u)|^2 \, dxdy.$$

The next lemma shows that the solutions of (6.1) that represent a given homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ satisfy a uniform energy bound.

Lemma 6.1.2 *Let $u : D \to M$ be a smoorh solution of (6.1) and denote by $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ rhe homology class represented by the map $\tilde{u} : D \to D \times M$ defined by (6.2). Let $c : D \to \mathbb{R}$ be the function in (4.3) entering the definition oft. Then*

$$E(u) = \langle [\tau], A \rangle + \int_D (\Omega_\tau(x, y, u) - c(x, y))\, dxdy.$$

Proof: We compute

$$
\begin{aligned}
E(u) &= \int_D |\partial_x u - X_F(u)|^2 \, dxdy \\
&= \int_D \omega(\partial_x u - X_F(u), J(\partial_x u - X_F(u))) \, dxdy \\
&= \int_D \omega(\partial_x u - X_F(u), \partial_y u - X_G(u)) \, dxdy \\
&= \int_D \Big(\omega(\partial_x u, \partial_y u) - dF(u)\partial_y u + dG(u)\partial_x u \\
&\qquad + \{F, G\}(u) \Big) \, dxdy \\
&= \int_D \Big(\omega(\partial_x u, \partial_y u) - dF(u)\partial_y u + dG(u)\partial_x u \Big) \, dxdy \\
&\quad + \int_D \Big(\Omega_\tau(x, y, u) - (\partial_y F)(u) + (\partial_x G)(u) \Big) \, dxdy \\
&= \int_D \Big(\tau(\partial_x \tilde{u}, \partial_y \tilde{u}) - c(x, y) \Big) \, dxdy + \int_D \Omega_\tau(x, y, u) \, dxdy \\
&= \int_D \tilde{u}^* \tau \, dxdy + \int_D \Big(\Omega_\tau(x, y, u) - c(x, y) \Big) \, dxdy
\end{aligned}
$$

This proves the lemma. □

Let us denote the moduli space of solutions of (6.1) that represent a given homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ by

$$\mathcal{M}_A(\tau, J) := \{u : D \to M \quad u \text{ satisfies (6.1)}, [\tilde{u}] = A\},$$

where $\tilde{u}$ is defined in (6.2). We shall prove that, for a generic pair (τ, J), this space is a smooth manifold of dimension $n + \mu_\Lambda(A)$. Moreover, if the pair (M, Λ_0) is monotone with minimal Maslov number N and $n + \mu_\Lambda(A) < N$, we shall prove that $\mathcal{M}_A(\tau, J)$ is compact, again for a generic pair (τ, J). The key tool for establishing compactness is the energy bound of Lemma 6.1.2. Under these assumptions the moduli spaces will be used to define Gromov invariants of Λ. The significance of these invariants for exact Lagrangian loops lies in the following Observation.

Lemma *6.1.3 Let Λ be an exact Lagrangian loop and let*

$$A \in H_2(D \times M, \Lambda; \mathbb{Z})$$

be a relative homology class. Suppose that for every $\tau \in \mathcal{T}^+(\Lambda)$ there exists a J such that $\mathcal{M}_A(\tau, J) \neq \emptyset$. Then

$$\varepsilon^+(\tau_0, \Lambda) + \langle [\tau_0], A\rangle \geq 0$$

for every $\tau_0 \in \mathcal{T}(\Lambda)$.

Proof: Let $\tau \in \mathcal{T}^+(\Lambda)$ and $u \in \mathcal{M}_A(\tau, J)$. Let $\tilde{u} : D \to D \times M$ be given by (6.2). Then $\tilde{u}$ is a $\tilde{J}(\tau, J)$-holomorphic curve. By Remark 6.1.1, $\tilde{J}(\tau, J)$ is tamed by τ. Hence

$$0 < \int_D \tilde{u}^*\tau = \langle [\tau], A\rangle = \langle [\tau_0] + s(\tau, \tau_0)\sigma, A\rangle = \langle [\tau_0], A\rangle + s(\tau, \tau_0).$$

The infimum of the numbers on the right is $\langle [\tau_0], A\rangle + \varepsilon^+(\tau_0, \Lambda)$. This proves the lemma. □

A similar estimate for $\varepsilon^-(\tau_0, \Lambda)$ can be obtained by studying anti-holomorphic curves. These are solutions of the equation

$$\begin{cases} \partial_x u - X_F(u) - J(\partial_y u - X_G(u)) = 0 \\ u(e^{2\pi i t}) \in \Lambda_t, \quad t \in \mathbb{R}. \end{cases} \tag{6.3}$$

Let us denote the moduli space of solutions by $\mathcal{M}_A(\tau, -J)$.

Lemma 6.1.4 *Let Λ be an exact Lagrangian loop and let*

$$A \in H_2(D \times M, \Lambda; \mathbb{Z})$$

be a relative homology class. Suppose that for every $\tau \in \mathcal{T}^-(\Lambda)$ there exists a J such that $\mathcal{M}_A(\tau, -J) \neq \emptyset$. Then

$$\varepsilon^-(\tau_0, \Lambda) + \langle [\tau_0], A\rangle \leq 0$$

for every $\tau_0 \in \mathcal{T}(\Lambda)$.

Proof: Let $\tau \in \mathcal{T}^-(\Lambda)$ and $u \in \mathcal{M}_A(\tau, -J)$. Let $\tilde{u} : D \to D \times M$ be given by (6.2). Then $\tilde{u}$ is a $\tilde{J}(\tau, -J)$-holomorphic curve. By Remark 6.1.1, $\tilde{J}(\tau, -J)$ is tamed by $-\tau$. Hence

$$0 > \int_D \tilde{u}^*\tau = \langle[\tau], A\rangle = \langle[\tau_0] + s(\tau, \tau_0)\sigma, A\rangle = \langle[\tau_0], A\rangle + s(\tau, \tau_0).$$

The supremum of the numbers on the right is $\langle[\tau_0], A\rangle + \varepsilon^-(\tau_0, \Lambda)$. This proves the lemma. □

6.2 Fredholm theory

In this section we examine the moduli spaces $\mathcal{M}_A^{\pm}(\tau, J)$ in more detail and show that, for a generic J, these spaces are smooth manifolds of the predicted dimensions $n \pm \mu_\Lambda(A)$. Here $\mathcal{M}_A^+(\tau, J)$ is the moduli space of solutions of (6.1) which represent the class A and $\mathcal{M}_A^-(\tau, J)$ is the moduli space of solutions of (6.3) representing the class A. The arguments are standard (cf. [FHS, MS2]) and we shall only outline the main points and sometimes briefly sketch the proofs. Fix an exact Lagrangian loop $\Lambda \subset S^1 \times M$, a homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$, and a constant $p > 2$. Consider the Banach manifold

$$\mathcal{B} = W^{1,p}_{\Lambda,A}(D, M)$$

of all functions $u : D \to M$ of class $W^{1,p}$ that satisfy the boundary condition $u(e^{2\pi i t}) \in \Lambda_t$ for all t and represent the class A. There is a natural vector bundle $\mathcal{E} \to \mathcal{B}$ with fibres

$$\mathcal{E}_u = L^p(D, u^*TM)$$

and the left hand sides of (6.1) and (6.3) define Fredholm sections $\mathcal{F}^{\pm} : \mathcal{B} \to \mathcal{E}$ given by

$$\mathcal{F}^{\pm}(u) := \mathcal{F}(u; \tau, \pm J) := \partial_x u - X_F(u) \pm J(\partial_y u - X_G(u)).$$

The moduli spaces $\mathcal{M}_A(\tau, \pm J)$ are the zero sets of these sections. The tangent space

$$T_u\mathcal{B} = W^{1,p}_\Lambda(D, u^*TM)$$

consists of all vector fields $\xi \in W^{1,p}(D, u^*TM)$ along u which are of class $W^{1,p}$ and satisfy the boundary condition $\xi(e^{2\pi it}) \in T_{u(e^{2\pi it})}\Lambda_t$. The vertical differential of $\mathcal{F}^{\pm}$ ata zero $u \in \mathcal{M}_{\Lambda}^{\pm}(\tau, J)$ is the linear Operator

$$D_u^{\pm} = D\mathcal{F}^{\pm}(u) : W_{\Lambda}^{1,p}(D, u^*TM) \to L^p(D, u^*TM)$$

given by

$$D_u^{\pm}\xi = \nabla_x\xi - \nabla_\xi X_F(u) \pm J(\nabla_y\xi - \nabla_\xi X_G(u)) \pm (\nabla_\xi J)(\partial_y u - X_G(u)). \tag{6.4}$$

Here V denotes the Levi-Civita connection of the Riemannian metric

$$\langle \cdot, \cdot \rangle_{x,y} = \omega(\cdot, J_{x,y}\cdot)$$

and thus depends on $x + iy \in D$. The expression ∇X_F denotes the covariant derivative of $X_F = X_{F_{x,y}}$ with respect to the Levi-Civita connection at the point $x + iy$. The next theorem follows from the Riemann-Roch theorem for discs (see for example [RS2] for a recent exposition) and the infinite dimensional implicit function theorem (see for example [S, Appendix B]). The proof is standard (see for example [MS2]) and will only be sketched.

Theorem 6.2.1 *For every $u \in W_{\Lambda,A}^{1,p}(D, M)$ the Operators $D_u^{\pm}$ dejined by (6.4) are Fredholm and their indices are*

$$\operatorname{index} D_u^{\pm} = n \pm \mu_\Lambda(u).$$

If $D_u^{\pm}$ is surjective for every $u \in \mathcal{M}_A(\tau, \pm J)$ then $\mathcal{M}_A(\tau, \pm J)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_A(\tau, \pm J) = n \pm \mu_\Lambda(A).$$

Proof: (Sketch) That $D_u^{\pm}$ is an elliptic first order partial differential operators and hence Fredholm follows from general theory. This means that $D_u^{\pm}$ has closed range and finite dimensional kernel and cokemel. The Fredholm index of such an Operator is defined as the dimension of the kernel minus the dimension of the cokemel. The index formula follows from the Riemann-Roch theorem, see [RS2].

In order for $\mathcal{M}_A(\tau, \pm J)$ to be a manifold, it is required that $\mathcal{F}^{\pm}$ is transversal to the zero section. This means that the image of the linearization

$$d\mathcal{F}^{\pm}(u; \tau, \pm J) : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$$

is complementary to the tangent space $T_u\mathcal{B}$ of the zero section. Hence, we must require that the linearized Operator $D_u^{\pm}$, which is the composition of $d\mathcal{F}^{\pm}(u; \tau, \pm J)$ with the projection $\pi_u : T_{(u,0)}\mathcal{E} \to \mathcal{E}_u$, is surjective for every $u \in \mathcal{M}_A(\tau, \pm J)$. Since $D_u^{\pm}$ is also Fredholm, it follows from the infinite dimensional implicit function theorem that $\mathcal{M}_A(\tau, \pm J)$ is a finite dimensional manifold whose tangent space at u is the kernel of $D_u^{\pm}$. □

Fix an exact Lagrangian loop Λ and fix a connection 2-form $\tau \in \mathcal{T}(\Lambda)$. Let us denote by $\mathcal{J}(D; M, \omega)$ the space of all smooth families of smooth almost complex structures $J : D \to \mathcal{J}(M, w)$. We call such a family $J \in \mathcal{J}(D; M, \omega)$ **regular** for (6.1) if D_u^{+} is surjective for every class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every $u \in \mathcal{M}_A(\tau, J)$. Similarly, $J \in \mathcal{J}(D; M, \omega)$ is called **regular** for (6.3) if D_u^{-} is surjective for every $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every $u \in \mathcal{M}_A(\tau, -J)$. We shall denote set of all families of almost complex structures that are regular for (6.1). respectively (6.3), by

$$\mathcal{J}^{\pm}_{\text{reg}}(\tau, \Lambda) \subset \mathcal{J}(D; M, \omega).$$

The proof of the next theorem is a Standard application of the Sard-Smale theorem (cf. [MS2]) and will only be sketched. Strictly speaking the theorem only holds true in the $\mathcal{C}^{\ell}$ category. It follows from Taubes, see Appendix C, **that** we can also apply it to the $\mathcal{C}^{\infty}$ category.

Theorem 6.2.2 *The sets $\mathcal{J}^{\pm}_{\text{reg}}(\tau, \Lambda)$ are of the secondcategory (in the sense of Baire) in $\mathcal{J}(D; M, \omega)$, i.e. they are countable intersections of open and dense subsets of $\mathcal{J}(D; M, \omega)$. In particular, they are dense.*

Proof: (Sketch) Define the universal moduli space by

$$\mathcal{M}^{\pm}(A, \mathcal{T} \times \mathcal{J}) = \{(u; \tau, J) \in \mathcal{B} \times \mathcal{T}(\Lambda) \times \mathcal{J}(D; M, \omega) \mid \mathcal{F}(u; \tau, \pm J) = 0\}.$$

lt follows from the infinite dimensional implicit function theorem that this space is a smooth Banach manifold. [1] Consider the projection

$$\pi^{\pm} : \mathcal{M}^{\pm}(A, \mathcal{T} \times \mathcal{J}) \to \mathcal{J}.$$

One can show, us in [FHS], that $d\pi^{\pm}(u; \tau,$ J) is a Fredholm Operator whose kerne1 is isomorphic to the kernel of $D_u^{\pm}$ and whose image has the same codimension as $D_u^{\pm}$. The Operator $d\pi^{\pm}(u; \tau,$ J) is onto precisely when the Operator $D_u^{\pm}$ is onto. This means that the regular values of $\pi^{\pm}$ are in one-to-one correspondence with the regular $J \in \mathcal{J}(D;$ M, $\omega)$. By the Sard-Smale theorem the set of regular values is of the second category in the sense of Baire. □

Let $\tau_0, \tau_1 \in$ 7(A) and choose regular families of almost complex structures

$$J_0 \in \mathcal{J}^{\pm}_{\text{reg}}(\tau_0, \Lambda), \qquad J_1 \in \mathcal{J}^{\pm}_{\text{reg}}(\tau_1, \Lambda).$$

We saw in Theorem 6.2.1, the spaces $\mathcal{M}_A(\tau_0, \pm J_0)$ and $\mathcal{M}_A(\tau_1, \pm J_1)$ are smooth manifolds of the same dimension. We will now discuss the dependence of these manifolds on the choice of $(\tau,$ J). We will show that these manifolds are cobordant. To construct a cobordism choose a smooth path $[0, 1] \to \mathcal{T}(\Lambda) : \lambda \mapsto \tau_\lambda$ that connects τ_0 to τ_1. Let us denote by

$$\mathcal{J} = \mathcal{J}([0, 1] \times D, J_0, J_1; M, \omega)$$

the space of smooth homotopies $[0, 1] \to \mathcal{J}(D; M, \omega) : \lambda \mapsto J_\lambda$ that connect J_0 to J_1 Given $\{J_\lambda\} \in \mathcal{J}$ denote

$$\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \{(\lambda, u) \mid 0 \le \lambda \le 1,\ u \in \mathcal{M}_A(\tau_\lambda, \pm J_\lambda)\}.$$

In general we cannot find a homotopy such that $J_\lambda \in \mathcal{J}^{\pm}_{\text{reg}}(\tau_\lambda, \Lambda)$ for every λ, that is, $\mathcal{M}_A(\tau_\lambda, \pm J_\lambda)$ may fail to be a manifold of the right dimension for some λ. However, there is always a smooth homotopy such that the space $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is a manifold of the right dimension. Such a homotopy $\{J_\lambda\} \in \mathcal{J}$ is called regular. More explicitly, this means that

[1] Strictly speaking we are dealing with Frechet manifolds here. We should first develop this theory in the $\mathcal{C}^\ell$-category and then use a Taubes' like argument to extend it to the smooth category, but we will omit that here. The arguments can be found in [MS2] or Appendix C.

for every homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every pair $(\lambda, u) \in \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$,

$$\mathrm{im}\, D^{\pm}_{\lambda,u} + \mathbb{R}\xi^{\pm}_{\lambda,u} = L^p(D, u^*TM).$$

Here $D^{\pm}_{\lambda,u}$ is defined by (6.4) with τ and J replaced by τ_λ and J_λ, respectively, and $\xi^{\pm}_{\lambda,u} \in L^p(D, u^*TM)$ is given by

$$\xi^{\pm}_{\lambda,u} := X_{\partial_\lambda F_\lambda}(u) \pm J_\lambda(u) X_{\partial_\lambda G_\lambda}(u) \mp \partial_\lambda J_\lambda(u)(\partial_y u - X_{G_\lambda}(u)).$$

The set of all regular homotopies will be denoted by

$$\mathcal{J}^{\pm}_{\mathrm{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda) \subset \mathcal{J}.$$

The proof of the next theorem is again standard and analogous to the proof of Theorem 6.2.2. It will be omitted.

Theorem 6.2.3 *Let* $[0, 1] \to \mathcal{T}(\Lambda) : \lambda \mapsto \tau_\lambda$ *be a smooth family of connection 2-forms. Suppose that* $J_0 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_0, \Lambda)$ *and* $J_1 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_1, \Lambda)$. *Then the sets* $\mathcal{J}^{\pm}_{\mathrm{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda) \subset \mathcal{J}$ *are of the second category in the sense of Baire. Moreover, if* $\{J_\lambda\} \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda)$ *then* $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ *is a smooth manifold of dimension*

$$\dim \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = n \pm \mu_\Lambda(A) + 1$$

with boundary

$$\partial \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \mathcal{M}_A(\tau_0, \pm J_0) \cup \mathcal{M}_A(\tau_1, \pm J_1).$$

This shows that the manifolds $\mathcal{M}_A(\tau_0, \pm J_0)$ and $\mathcal{M}_A(\tau_1, \pm J_1)$ are cobordant. This will only be of significance once we have established certain compactness properties. This will be the topic of the next section.

6.3 Compactness

Theorem 6.3.1 *Let* $\Lambda \subset S^1 \times M$ *be an exact Lagrangian loop and suppose that the pair* (M, Λ_0) *is monotone. Let* $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ *and denote by* $N \in \mathbb{N}$ *the minimal Maslov number of the pair* (M, Λ_0).

(i) *If*

$$n \pm \mu_\Lambda(A) \leq N - 1$$

then the manifold $\mathcal{M}_A(\tau, \pm J)$ *is compact for every* $\tau \in \mathcal{T}(\Lambda)$ *and every* $J \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau, \Lambda)$.

(ii) *If*

$$n \pm \mu_\Lambda(A) \leq N - 2$$

then $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ *is compact for every smooth path*

$$[0, 1] \to \mathcal{T}(\Lambda) : \lambda \mapsto \tau_\lambda,$$

every $J_0 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_0, \Lambda)$, *every* $J_1 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_1, \Lambda)$, *and every regular homotopy* $\{J_\lambda\} \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda)$.

The proof of Theorem 6.3.1 relies on the following theorem about Gromov compactness for J-holomorphic discs. This result is implicitly contained in Gromov's original paper [Gl] and has been folk knowledge since then. However, the full details of the proof have not so far appeared in the literature. They were recently carried out by Frauenfelder [F] in his Diploma thesis. In his thesis Frauenfelder also discusses the corresponding notion of *stable maps* for pseudoholomorphic discs.

Theorem 6.3.2 (Gromov) *Let* $(\tau^\nu, J^\nu) \in \mathcal{T}(\Lambda) \times \mathcal{J}(D; M, \omega)$ *be a sequence that converges to* $(\tau, J) \in \mathcal{T}(\Lambda) \times \mathcal{J}(D; M, \omega)$ *in the* $\mathcal{C}^\infty$*-topology. Let* $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ *and* $u^\nu \in \mathcal{M}_A(\tau^\nu, \pm J^\nu)$. *If* u^ν *has no* $\mathcal{C}^\infty$*-convergent subsequence then*

(1) *there exist finitely many points* $(x_i, y_i) \in D$ *and maps* $v_i : S^2 \to M$, $i = 1, \ldots, k$,

(2) *there exist finitely many points* $t_j \in \mathbb{R}$ *and maps* $w_j : D \to M$, $j = 1 \ldots, \ell$,

(3) *there exists a map* $u_0 : D \to M$,

such that the v_i *are nonconstant* J_{x_i,y_i}*-(anti)holomorphic spheres for* $i = 1, , k$, *the* w_j *are nonconstant* $J_{e^{2\pi i t_j}}$ *-(anti)holomorphic discs with*

$w_j(\partial D) \subset \Lambda_{t_j}$ for $j = 1, \ldots, \ell$, $u_0 \in \mathcal{M}_{A_0}(\tau, \pm J)$ for some $A_0 \in H_2(D \times M, \Lambda; \mathbb{Z})$, and

$$A = A_0 + \sum_{i=1}^{k}[v_i] + \sum_{j=1}^{\ell}[w_j]. \tag{6.5}$$

Here $[v_i]$ and $[w_j]$ denote the induced relative homology classes in $H_2(D \times M, \Lambda; \mathbb{Z})$ and one of the integers k and ℓ is nonzero.

Remark 6.3.3 **(i)** Let $\widetilde{M}$ be a compact manifold and $\widetilde{L} \subset \widetilde{M}$ be a compact submanifold of half the dimension. Suppose that $\widetilde{\omega}^\nu$ is a sequence of symplectic forms on $\widetilde{M}$ that converges to $\widetilde{\omega}$ in the $\mathcal{C}^\infty$-topology such that $\widetilde{L}$ is a Lagrangian submanifold of $(\widetilde{M}, \widetilde{\omega}^\nu)$ for every ν. Suppose that $\widetilde{J}^\nu$ is a sequence of $\widetilde{\omega}^\nu$-tame almost complex structures on $\widetilde{M}$ that converges in the $\mathcal{C}^\infty$-topology to $\widetilde{J}$. In [F] Frauenfelder proves, in particular, that a sequence of $\widetilde{J}^\nu$-holomorphic discs $\widetilde{u}^\nu : (D, \partial D) \to (\widetilde{M}, \widetilde{L})$ that represent a fixed homology class $\widetilde{A} \in H_2(\widetilde{M}, \widetilde{L}; \mathbb{Z})$ has a subsequence that converges (in a precisely defined sense) to a tree consisting of $\widetilde{J}$-holomorphic spheres in $\widetilde{M}$ and $\widetilde{J}$-holomorphic discs in $\widetilde{M}$ with boundary in $\widetilde{L}$ such that the sum of their homology classes in $H_2(\widetilde{M}, \widetilde{L}; \mathbb{Z})$ is equal to $\widetilde{A}$. The techniques in [F] are an adaptation of those in Hofer-Salamon [HS] for holomorphic spheres to the case of holomorphic discs.

(ii) The moduli space $\mathcal{M}_A(\tau, \pm J)$ does not depend on the function $c : D \to M$ in (4.3). Hence we may assume without loss of generality that the connection forms τ^ν in Theorem 6.3.2 lie in $\mathcal{T}^{\pm}(\Lambda)$. Under this assumption our case fits in this framework. The manifold $\widetilde{M}$ is given by $D \times M$, the submanifold $\widetilde{L}$ by Λ, the symplectic forms $\widetilde{\omega}^\nu$ by $\pm\tau^\nu$, the almost complex structures $\widetilde{J}^\nu$ by $\widetilde{J}(\tau^\nu, \pm J^\nu)$ defined in Remark 6.1.1, and the functions $\widetilde{u}^\nu$ are given by (6.2). They satisfy the requirements of (i).

(iii) Theorem 6.3.2 follows from (i) and (ii) since each bubble in (1) and (2) of Theorem 6.3.2 in the limit curve is contained in a fibre of the (trivial) fibration $D \times M$. To see this, note that each curve v_i appears as the limit of a sequence

$$v_i^\nu(x, y) = u^\nu(x_i^\nu + \varepsilon^\nu x, y_i^\nu + \varepsilon^\nu y),$$

where $x_i^\nu \to x_i$, $y_i^\nu \to y_i$, $\varepsilon^\nu \to 0$, and

$$\lim_{\nu\to\infty} \frac{\varepsilon^\nu}{1-\sqrt{(x_i^\nu)^2+(y_i^\nu)^2}} = 0.$$

One can show that, after passing to a suitable subsequence, the sequence v_i^ν converges to v_i in the $\mathcal{C}^\infty$-topology on the complement of some finite set. The functions v_i^ν satisfy

$$\partial_x v_i^\nu - \varepsilon^\nu X_{F^\nu} + J^\nu(\partial_y v_i^\nu - \varepsilon^\nu X_{G^\nu}) = 0,$$

where the vector fields X_{F^ν}, X_{G^ν}, and the almost complex structure J'' are evaluated at the point $(x_i^\nu + \varepsilon^\nu x, y_i^\nu + \varepsilon^\nu y, v_i^\nu)$. It follows that the limit curve v_i extends to a J_{x_i,y_i}-holomorphic sphere. The holomorphic discs w_j appear as similar limits with $x_j + iy_j = e^{2\pi i t_j}$ and

$$\lim_{\nu\to\infty} \frac{\varepsilon^\nu}{1-\sqrt{(x_j^\nu)^2+(y_j^\nu)^2}} > 0.$$

A similar argument as above, with coordinates on the upper halfplane, then shows that the limit curve w_j is a $J_{e^{2\pi i t_j}}$-holomorphic disc with boundary values in Λ_{t_j}.

(iv) The limit curve in (i) is a stable map consisting of $\tilde{J}$-holomorphic discs and spheres. For closed curves this concept is due to Kontsevich [K]. Some of the components of the stable map may be constant. However, these do not contribute to the homology class and can be neglected for our purposes. If the original sequence $\tilde{u}^\nu$ does not have a $\mathcal{C}^\infty$-convergent subsequence, then the limit curve has more than one nonconstant component. This shows that in Theorem 6.3.2 either k or ℓ is nonzero.

Proof of Theorem 6.3.1: We prove the Statement (ii). Suppose, by contradiction, that $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is not compact and non-empty. Then there exists a sequence

$$(\lambda^\nu, u^\nu) \in \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$$

that has no convergent subsequence. We may assume without loss of generality that λ^ν converges to λ_0. Then, by Theorem 6.3.2, there exist nonconstant $J_{\lambda_0;x_i,y_i}$-(anti)holomorphic spheres $v_i : S^2 \to M$ for $i = 1, \ , k,$

there exist nonconstant $J_{\lambda_0; e^{2\pi i t_j}}$-(anti)holomorphic discs $w_j : (D, \partial D) \to (M, L_{t_j})$ for $j = 1, \ldots, \ell$, and there exists an element $u_0 \in \mathcal{M}_{A_0}(\tau_{\lambda_0}, \pm J_{\lambda_0})$ for some $A_0 \in H_2(D \times M, \Lambda; \mathbb{Z})$ such that (6.5) is satisfied. Since the pair (M, Λ_t) is, by assumption, monotone with minimal Maslov number N for every t we have

$$\pm\mu_\Lambda(v_i) \geq N, \qquad \pm\mu_\Lambda(w_j) \geq N$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, \ell$. Since either k or ℓ is nonzero this implies

$$\begin{aligned} n \pm \mu_\Lambda(A) &= n \pm \mu_\Lambda(A_0) \pm \sum_{i=1}^{k} \mu_\Lambda(v_i) \pm \sum_{j=1}^{\ell} \mu_\Lambda(w_j) \\ &\geq n \pm \mu_\Lambda(A_0) + N. \end{aligned}$$

Since $\{J_\lambda\} \in \mathcal{J}^{\pm}_{\text{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda)$, the moduli space $\mathcal{W}_{A_0}(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is a smooth manifold of dimension

$$\begin{aligned} \dim \mathcal{W}_{A_0}(\{\tau_\lambda\}, \{\pm J_\lambda\}) &= n \pm \mu_\Lambda(A_0) + 1 \\ &\leq n \pm \mu_\Lambda(A) + 1 - N \\ &\leq N - 2 + 1 - N \\ &= -1 < 0. \end{aligned}$$

Hence

$$\mathcal{W}_{A_0}(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \emptyset,$$

in contradiction to the fact that

$$(\lambda_0, u_0) \in \mathcal{W}_{A_0}(\{\tau_\lambda\}, \{\pm J_\lambda\}).$$

Thus we have proved (ii). The proof of (i) is almost word by word the same and will be left to the reader. □

6.4 Gromov invariants

Fix an exact Lagrangian loop $\Lambda \subset S^1 \times M$ and a relative homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$. Throughout we shall assume that the pair (M, Λ_0)

is monotone and

$$n \pm \mu_\Lambda(A) \leq N - 2, \tag{6.6}$$

where $N \in \mathbb{N}$ denotes the minimal Maslov number of the pair (M, Λ_0). Fix a tuple $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ such that $0 \leq t_1 < \cdots < t_k < 1$ and denote

$$\Lambda_{\mathbf{t}} := \Lambda_{t_1} \times \cdots \times \Lambda_{t_k}.$$

For $\tau \in \mathcal{T}(\Lambda)$ and $J \in \mathcal{J}(D; M, \omega)$ we define $\mathrm{ev}_{\mathbf{t}} : \mathcal{M}_A(\tau, \pm J) \to \Lambda_{\mathbf{t}}$ by

$$\mathrm{ev}_{\mathbf{t}}(u) := (u(e^{2\pi i t_1}), \dots, u(e^{2\pi i t_k})).$$

If $J \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau, \Lambda)$ then, by Theorems 6.2.1 and 6.3.1, the moduli space $\mathcal{M}^{\pm}_A(\tau, J)$ is a compact smooth manifold (without boundary) of dimension $n \pm \mu_\Lambda(A)$. It is not necessarily orientable. Let

$$[\mathcal{M}_A(\tau, \pm J)] \in H_{n \pm \mu_\Lambda(A)}(\mathcal{M}_A(\tau, \pm J); \mathbb{Z}_2)$$

denote the fundamental cycle. The Gromov invariants are defined by

$$\mathrm{Gr}^{\pm}_{A,\mathbf{t}}(\Lambda) := \mathrm{ev}_{\mathbf{t}*}[\mathcal{M}^{\pm}_A(\tau, J)] \in H_{n \pm \mu_\Lambda(A)}(\Lambda_{\mathbf{t}}; \mathbb{Z}_2). \tag{6.7}$$

Lemma 6.4.1 *The homology classes $\mathrm{Gr}^{\pm}_{A,\mathbf{t}}(\Lambda) \in H_{n \pm \mu_\Lambda(A)}(\Lambda_{\mathbf{t}}; \mathbb{Z}_2)$ are independent of the choices of the connection 2-form $\tau \in \mathcal{T}(\Lambda)$ and the almost complex structure $J \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau, \Lambda)$ used to define them.*

Proof: Given two connection 2-forms τ_0 and τ_1 and two families of almost complex structures $J_0 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_0, \Lambda)$ and $J_1 \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\tau_1, \Lambda)$. Choose a path $\{\tau_\lambda\}$ connecting τ_0 to τ_1 and choose a corresponding path $\{J_\lambda\} \in \mathcal{J}^{\pm}_{\mathrm{reg}}(\{\tau_\lambda\}, J_0, J_1, \Lambda)$. By theorems 6.2.3 and 6.3.1 (ii), $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is a smooth compact manifold with boundary

$$\partial \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \mathcal{M}_A(\tau_0, \pm J_0) \cup \mathcal{M}_A(\tau_1, \pm J_1).$$

The evaluation map extends in the obvious way to

$$\mathrm{ev}_{\mathbf{t}} : \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) \to \Lambda_{\mathbf{t}}.$$

Since

$$\begin{aligned}
\mathrm{ev}_{\mathbf{t}*}([\mathcal{M}_A(\tau_0, \pm J_0)] - [\mathcal{M}_A(\tau_1, \pm J_1]) \\
&= \mathrm{ev}_{\mathbf{t}*}[\partial \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})] \\
&= \partial(\mathrm{ev}_{\mathbf{t}*}[\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})]) \\
&= 0 \in H_{n \pm \mu_\Lambda(A)}(\Lambda_{\mathbf{t}}; \mathbb{Z}_2)
\end{aligned}$$

we deduce that

$$\mathrm{ev}_{\mathbf{t}*}[\mathcal{M}_A(\tau_0, \pm J_0)] = \mathrm{ev}_{\mathbf{t}*}[\mathcal{M}_A(\tau_1, \pm J_1],$$

which proves the lemma. □

Corollary 6.4.2 *Let* $A^{\pm} \in H_2(D \times M, \Lambda; \mathbb{Z})$ *satisfy (6.6) and suppose rhat*

$$\mathrm{Gr}^{\pm}_{A^{\pm}, \mathbf{t}^{\pm}}(\Lambda) \neq 0$$

for some $\mathbf{t}^{\pm}$. *Then*

$$\varepsilon^{+}(\tau, \Lambda) \geq -\langle [\tau], A^{+} \rangle, \qquad \varepsilon^{-}(\tau, \Lambda) \leq -\langle [\tau], A^{-} \rangle$$

for every $\tau \in \mathcal{T}(\Lambda)$.

Proof: Since $\mathrm{Gr}^{\pm}_{A^{\pm}, \mathbf{t}^{\pm}}(\Lambda) = \mathrm{ev}_{\mathbf{t}*}[\mathcal{M}_A(\tau, \pm J)] \neq 0$ it follows that $\mathcal{M}_A(\tau, \pm J) \neq 0$. Now lemmata 6.1.3 and 6.1.4 state that

$$\varepsilon^{+}(\tau, \Lambda) + \langle [\tau], A^{+} \rangle \geq 0, \qquad \varepsilon^{-}(\tau, \Lambda) + \langle [\tau], A^{-} \rangle \leq 0,$$

which proves the corollary. □

Chapter 7

Complex projective space

In this section we shall use the Gromov invariants to compute the K-area of certain exact Lagrangian loops in CP''. The archetypal example is the half turn of a great circle in the 2-sphere. An explicit computation shows that the Hofer length of this loop is $1/2$. We shall use Corollary 6.4.2 and Theorems 4.2.4 and 4.3.3 to show that this loop minimizes the Hofer length in its Hamiltonian isotopy class.

7.1 Rotations of real projective space

Consider the complex projective space

$$M = \mathbb{C}P^n$$

equipped with symplectic form ω that is induced by the Fubini-Study metric and satisfies the normalization condition

$$\int_{\mathbb{C}P^n} \omega^n = 1.$$

Let $L = \mathbb{R}P^n$ and fix an integer $k \in \{1, , n\}$. As in the introduction, we consider the exact Lagrangian loop

$$\Lambda := \bigcup_{t\in\mathbb{R}} \{e^{2\pi it}\} \times \psi_t(\mathbb{R}P^n), \tag{7.1}$$

where

$$\psi_t([z_0 : \cdots : z_n]) = ([z_0 : e^{\pi i t} z_1 : \cdots : e^{\pi i t} z_k : z_{k+1} : \cdots : z_n]),$$

for some $1 \leq k \leq n$ to be specified later. A tedious computation shows that the Hamiltonian isotopy ψ_t is generated, via (3.3), by the the time independent Hamiltonian function $H_t = H : \mathbb{C}P^n \to \mathbb{R}$ given by

$$H([z_0 : \ldots : z_n]) = \frac{k}{2n+2} - \frac{|z_1|^2 + \cdots + |z_k|^2}{2(|z_0|^2 + \cdots + |z_n|^2)}. \tag{7.2}$$

It can be shown that this function has mean value zero and Hofer norm

$$\|H\| = \max H - \min H = \frac{1}{2}$$

Since H attains its maximum and its minimum on $\Lambda_t = \psi_t(\mathbb{R}P^n)$ it follows that $\ell(\Lambda) = 1/2$.

7.2 The Maslov index

In this section we will show that the example described in the previous section satisfies the assumptions of Theorem 6.3.2 and Corollary 6.4.2

Lemma 7.2.1 *The minimal Maslov number N of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$ is as follows*

$$N = n + 1. \tag{7.3}$$

Proof: For $n = 1$ this is well known (it follows basically from the axioms, see Appendix B). For $n > 1$ consider the homology exact sequence of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$. It has the form

$$0 \to H_2(\mathbb{C}P^n; \mathbb{Z}) \to H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}) \to H_1(\mathbb{R}P^n; \mathbb{Z}) \to 0,$$

because $H_1(\mathbb{C}P^n; \mathbb{Z}) = H_2(\mathbb{R}P^n; \mathbb{Z}) = 0$. Since $H_2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}$ and $H_1(\mathbb{R}P^n; \mathbb{Z}) = \mathbb{Z}_2$ we deduce that $H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z})$ equals either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. Now $\mathbb{R}P^n$ decomposes the line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ into two discs that represent the same homotopy class in $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$. This implies that the homomorphism

$$H_2(\mathbb{C}P^n; \mathbb{Z}) \to H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z})$$

is given by multiplication by two and that $H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}) = \mathbb{Z}$. Let $[\mathbb{C}P^1] \in H_2(\mathbb{C}P^n; \mathbb{Z})$ be the generator. Hence there is an element $A \in H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z})$ such that $2A$ is equal to the image of $[\mathbb{C}P^n]$ under the homomorphism

$$\mathbb{Z} \cong H_2(\mathbb{C}P^n; \mathbb{Z}) \xrightarrow{\times 2} H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}).$$

This implies that A is the generator of $H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}$. Since the Maslov class of $2A \in \pi_2(\mathbb{C}P^n, RP'')$ is equal to (see [RS2])

$$2\langle c_1(T\mathbb{C}P^n), [CP'] \rangle = 2(n+1)$$

we have proved the lemma. □

Lemma 7.2.2 ***Let (M, ω) be a symplectic manifold and $\Lambda \in S^1 \times M$ be an exact Lagrangian loop such that (M, Λ_0) is a monotonepair with minimal Maslov number N. Then***

$$\mu_\Lambda(u_1) \equiv \mu_\Lambda(u_0) \bmod N$$

for all $u_0, u_1 \in \mathrm{Map}_\Lambda(D, M)$.

Proof: If $u_0(e^{2\pi it}) = u_1(e^{2\pi it})$ for every $t \in \mathbb{R}$ then u_0 (with reversed orientation) and u_1 form a sphere and the difference $\mu_\Lambda(u_1) - \mu_\Lambda(u_0)$ is equal to twice the first Chern number of this sphere. Hence the difference of the Maslov numbers is an even multiple of N. This continues to hold whenever $u_0|_{\partial D}$ is homotopic to $u_1|_{\partial D}$ as a section of the bundle $\Lambda \to S^1$.

For any two maps $u_0, u_1 \in \mathrm{Map}_\Lambda$ (D, M) there exists a smooth function $v : (D, \partial D) \to (M, \Lambda_0)$ such that $v(-1) = u_0(1)$ and the connected sum $u_0 \# v$ is homotopic in Λ to u_1 along the boundary. Since, by the definition of the Maslov index in Appendix B, $\mu_\Lambda(u_0 \# v) = \mu_\Lambda(u_0) + \mu_\Lambda(v)$, we find, by what we have just proved

$$\mu_\Lambda(u_1) - \mu_\Lambda(u_0 \# v) = \mu_\Lambda(u_1) - \mu_\Lambda(u_0) - \mu_{\Lambda_0}(v) \in 2N\mathbb{Z}.$$

Since, by the monotonicity assumption, $\mu_{\Lambda_0}(v)$ is an integer multiple of N, the lemma is proved. □

Retuming to the loop $\Lambda \subset S^1 \times$ CP" we observe that the Hamiltonian function H in (7.2) is a Morse-Bott function with critical manifolds

$$C^+ := \{[0 : z_1 : \cdots : z_k : 0 : \cdots : 0] \mid (z_1, \ldots, z_k) \in \mathbb{C}^k \setminus \{0\}\},$$

$$C^- := \{[z_0 : 0 : \cdots : 0 : z_{k+1} : \cdots : z_n] \mid (z_0, z_{k+1}, \ldots, z_n) \in \mathbb{C}^{n-k+1} \setminus \{0\}\}.$$

Note that H attains its minimum, which equals $(k - n - 1)/(2n + 2)$, on C^+ and its maximum, which equals $k/(2n + 2)$, on C^-. Moreover, $C^* \cap \mathbb{R}P^n \subset \Lambda_t$ for every t. Let us denote by

$$A^{\pm} \in H_2(D \times \mathbb{C}P^n, \Lambda; \mathbb{Z})$$

the homology classes represented by the constant functions $D \to \mathbb{C}P^n$ with values in $C^{\pm} \cap \mathbb{R}P^n$. The next lemma shows that Λ has Maslov index $k \in \mathbb{Z}_{n+1}$, that is $k \mod n + 1$, us claimed in the introduction (see (1.2)). It also follows from the next lemma that the homology classes $A^* \in H_2(D \times \mathbb{C}P^n, \Lambda; \mathbb{Z})$ satisfy the condition (6.6) for the definition of the Gromov invariants.

Lemma 7.2.3

$$\mu_\Lambda(A^+) = k - 1 - n, \qquad \mu_\Lambda(A^-) = k. \tag{7.4}$$

Proof: In the case of A^-, consider the constant function

$$u(x, y) \equiv p := [1 : 0 : \cdots : 0]$$

Then a trivialization of the pull back tangent bundle $u^*T\mathbb{C}P^n$ is determined by the coordinate chart $[z_0 : \cdots : z_n] \mapsto (z_1/z_0, \ldots, z_n/z_0)$. In these coordinates the Hamiltonian flow is

$$\zeta \mapsto (e^{\pi i t}\zeta_1, \ldots, e^{\pi i t}\zeta_k, \zeta_{k+1}, \ldots, \zeta_n)$$

Since $T_p\Lambda_0 \cong \mathbb{R}^n \subset \mathbb{C}^n \cong T_p\mathbb{C}P^n$, we see that the Maslov index of the loop $t \mapsto T_p\Lambda_t$ is equal to k, see Theorem B.O. 1. This proves the second equation in (7.4) and the first follows from a similar argument. □

7.3 Computation of the Gromov invariants

Since $N = n + 1$ it follows from Lemma 7.2.3 that the classes $A^{\pm}$ satisfy (6.6), that is

$$n + \mu_\Lambda(A^+) = n + k - 1 - n \le n - 1 = N - 2$$
$$n - \mu_\Lambda(A^-) = n - k \le n - 1 = N - 2$$

and hence the requirements of Theorem 6.3.1. The next theorem shows that the Gromov invariants $\mathrm{Gr}^{\pm}_{A^\pm,0}(\Lambda)$ are nonzero. Here the subscript 0 corresponds to the choice $t = t_1 = 0$ for the evaluation map.

Theorem 7.3.1

$$\mathrm{Gr}^{+}_{A^+,0}(\Lambda) = [\mathbb{R}P^{k-1}] \in H_{k-1}(\mathbb{R}P^n; \mathbb{Z}_2),$$

$$\mathrm{Gr}^{-}_{A^-,0}(\Lambda) = [\mathbb{R}P^{n-k}] \in H_{n-k}(\mathbb{R}P^n; \mathbb{Z}_2).$$

Proof: Let $\tau \in T(A)$ be given by (4.3) with $c = 0$ and

$$F_{x,y} = \frac{-\sin(2\pi t)\rho(r)}{2\pi r} H, \qquad G_{x,y} = \frac{\cos(2\pi t)\rho(r)}{2\pi r} H,$$

where $re^{2\pi i t} = x + iy$ and H is given by (7.2). As in (4.9).

$$\rho : [0, 1] \to [0, 1]$$

is a smooth nondecreasing cutoff function such that $\rho(r) = 0$ for r near 0 and $\rho(r) = 1$ for r near 1. The formula for the curvature introduced in Chapter 4 yields

$$\Omega_\tau(re^{2\pi i t}, z) = -\frac{\dot\rho(r)}{2\pi r} H(z) \tag{7.5}$$

for $z \in \mathbb{C}P^n$ shows that $\Omega_\tau(x, y, z) \ge 0$ for $z \in C^+$ and $\Omega_\tau(x, y, z) \le 0$ for $z \in C^-$. From (7.5) and Lemma 6.1.2 with $c = 0$ and $E(u) = 0$ (since u is constant and takes values in the set of critical values of F and G) we get that

$$\langle[\tau], A^+\rangle = -\int_D \Omega_\tau(x, y, u)\, dx dy = \int_D \frac{\dot\rho(r)}{2\pi r} H(z) r \;\; dr dt$$

and similarly for A^-. Since H attains its minimum on C^+ and its maximum on C^- it is easy to compute the above integrals and we get

$$\langle[\tau], A^+\rangle = \frac{k-1-n}{2n+2}, \qquad \langle[\tau], A^-\rangle = \frac{k}{2n+2}. \tag{7.6}$$

The explicit formulae for F and G show that $C^\pm$ consist entirely of critical points of $F_{x,y}$ and $G_{x,y}$ for all $x+iy \in D$. This shows that the constant functions $u : D \to \mathbb{C}P^n$ with values in $C^+ \cup C^-$ are horizontal for the symplectic connection determined by τ. In explicit terms $\partial_x u = X_F(u)$ and $\partial_y u = X_G(u)$. Hence these constant functions satisfy both equations (6.1) and (6.3) for every $J \in \mathcal{J}(D; \mathbb{C}P^n, \omega)$. The constant functions with values in $(C^+ \cup C^-) \cap \mathbb{R}P^n$ satisfy in addition the boundary condition $u(e^{2\pi it}) \in \Lambda$, for all t. The formula (7.4) shows that the constant solutions with values in $C^+ \cap \mathbb{R}P^n$ and those with values in $C^- \cap \mathbb{R}P^n$ represent different homology classes (since they are distinguished by their Maslov index).

We prove that, for every $J \in \mathcal{J}(D; \mathbb{C}P^n, \omega)$,

$$\mathcal{M}_{A^+}(\tau, J) = \left\{u : D \to C^+ \cap \mathbb{R}P^n \mid du = 0\right\}. \tag{7.7}$$

To see this, let $u \in \mathcal{M}_{A^+}(\tau, J)$. Then, by Lemma 6.1.2 and (7.5),

$$\begin{aligned} 0 &\le E(u) \\ &= \langle[\tau], A^+\rangle + \int_D \Omega_\tau(x, y, u(x, y))\, dxdy \\ &= \frac{k-n-1}{2n+2} - \int_0^1\int_0^1 \dot\rho(r) H(u(re^{2\pi it}))\, drdt \\ &\le \frac{k-n-1}{2n+2} - \int_0^1\int_0^1 \dot\rho(r) \min H\, drdt \\ &= 0. \end{aligned}$$

Hence every $u \in \mathcal{M}_{A^+}(\tau, J)$ satisfies $E(u) = 0$, therefore

$$\dot\rho(r) \ne 0 \qquad \Longrightarrow \qquad H(u(re^{2\pi it})) = \min H.$$

The latter implies that $u(x_0, y_0) \in C^+$ for some point $x_0 + iy_0 \in D$. We would like to show that $u(x, y) = u(x_0, y_0)$ for all $(x, y) \in D$. From

$E(u) = 0$ we conclude that u is a horizontal section of $D \times M$ with respect to τ. Now let $x_1 + iy_1 \in D$, choose a path $[0, 1] \to D : t \mapsto x(t) + iy(t)$ that connects $x_0 + iy_0$ to $x_1 + iy_1$, and define $z : [0, 1] \to M$ by

$$z(t) := u(x(t), y(t)).$$

Then $z(0) \in C^+$ and

$$\dot{z}(t) = \dot{x}(t) X_{F_{x(t),y(t)}}(z(t)) + \dot{y}(t) X_{G_{x(t),y(t)}}(z(t)).$$

Since C^+ consists of critical points of $F_{x,y}$ and $G_{x,y}$ for all $x + iy \in D$ it follows that z(r) $= z(0)$ for all $t \in [0, 1]$. Hence u is constant. The boundary condition shows that this constant lies in $C^+ \cap \mathbb{R}P^n$. This proves (7.7). Hence $\mathcal{M}_{A^+}(\tau, J)$ is diffeomorphic to $\mathbb{R}P^{k-1}$ for every J and, in particular, for every $J \in \mathcal{J}^+_{\text{reg}}(\tau, \Lambda)$. The evaluation map $u \mapsto u(1)$ is obviously an embedding of $\mathcal{M}_{A^+}(\tau, J) \cong \mathbb{R}P^{k-1}$ into WP". A similar assertion holds for $\mathcal{M}_{A^-}$ (t, -J) and this proves the theorem in view of the definition of the Gromov invariants in (6.7). □

7.4 Calculating the invariants ν, χ and ε of projective Lagrangian loops

Identify $\mathbb{C}P^n$ with the quotient S^{2n+1}/S^1. The action of $SU(n + 1)$ descends to the quotient, since for $\lambda \in S^1$ and $w = \lambda z$ for $w, z \in S^{2n+1}$ and for $A \in SU(n + 1)$ we have $Aw = A\lambda z = \lambda Az$ and hence

$$[Aw] = [\lambda Az] = [Az]$$

where [ ] denotes the equivalence class in the quotient. A matrix $A \in SU(n + 1)$ acts as the identity on CP" if $Az = Az$ for some $\lambda \in S^1$ and for all $z \in \mathbb{C}P^n$ this means that $A = \lambda\,\text{id}$. Since $A \in SU(n + 1)$, $1 = \det(A) = \lambda^{n+1}$ and therefore λ is an $n + 1$-th root of unity. So the projective special unitary group can be identified with the quotient

$$PSU(n + 1) = SU(n + 1)/\{e^{\frac{2\pi ik}{n+1}} \mid k = 0, \ldots, n\}.$$

Since $\pi_1(\mathrm{SU}(n+1)) = \pi_0(\mathrm{SU}(n+1)) = 0$ (because $\mathrm{SU}(n+1)$ is simply connected,see the proof of Proposition 2.21 in [MS1]) we deduce that

$$\pi_1(\mathrm{PSU}(n+1)) = \mathbb{Z}_{n+1}.$$

Let $\mathrm{PL}(n+1)$ denote the manifold of projective Lagrangian planes in $\mathbb{C}P^n$, which can be identified with the quotient $\mathrm{PSU}(n+1)/\mathrm{SO}(n+1)$.

Lemma 7.4.1 $\pi_1(\mathrm{PL}(n+1)) = \mathbb{Z}_{n+1}$.

Proof: We have the following long homotopy exact sequence (see for example [Br], Chapter VII)

$$0 = \pi_2(\mathrm{PSU}(n+1) \to \pi_2(\mathrm{PL}(n+1)) \to \pi_1(\mathrm{SO}(n+1)) \xrightarrow{\beta}$$
$$\pi_1(\mathrm{PSU}(n+1)) \to \pi_1(\mathrm{PL}(n+1)) \to \pi_0(\mathrm{SO}(n+1)) = 0,$$

where $\pi_1(\mathrm{SO}(n+1)) = \mathbb{Z}_2$ and $\pi_1(\mathrm{PSU}(n+1)) = \mathbb{Z}_{n+1}$ Here we have also used that π_2 of a compact Lie group is zero (see [BD] Proposition 7.5). For $n+1$ odd the map β can only be the zero map and hence $\pi_1(\mathrm{PSU}(n+1)) \cong \pi_1(\mathrm{PL}(n+1))$, which proves the lemma. If $n+1$ is even, we need to study the map β in more detail. This map is induced by the composition of maps

$$\mathrm{SO}(n+1) \hookrightarrow \mathrm{SU}(n+1) \to \mathrm{PSU}(n+1),$$

which maps id and -id to the same element of $\mathrm{PSU}(n+1)$. Therefore the map β is again the zero map, which proves the lemma also in this case. □

For $k \in \mathbb{Z}$ we denote by $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ the exact Lagrangian loop delined by (1.1) in the introduction, i.e $\Lambda_t^k := \phi_{kt}(\mathbb{R}P^n)$, where

$$\phi_t([z_0 : : z_n]) = [e^{\pi it}z_0 : z_1 : : z_n].$$

If k is divisible by $n+1$ then this loop is contractible (since the Maslov index is defined modulo $n+1$ and since it is a homotopy invariant, divisibility by $n+1$ is equivalent to being contractible). If $k \in (1, , n\}$ and $k \equiv k'$modn $+ 1$ then $\Lambda^{k'}$ is Hamiltonian isotopic to Λ^k. Our main result is the following corollary of theorem 7.3.1

Corollary *7.4.2 If k is not divisible by $n+1$ then*

$$\nu(\Lambda^k) = \chi(\Lambda^k) = \varepsilon(\Lambda^k) = \frac{1}{2}.$$

If k is divisible by $n+1$ then $\nu(\Lambda^k) = \chi(\Lambda^k) = 0$.

Proof: Let $k \in \{1, \ldots, n\}$. Then the loop Λ, given by (7.1) namely

$$\Lambda = \bigcup_{t\in\mathbb{R}} \{e^{2\pi i t}\} \times \psi_t(\mathbb{R}P^n),$$

is Hamiltonian isotopic to Λ^k and hence

$$\varepsilon(\Lambda^k) = \varepsilon(\Lambda), \qquad \chi(\Lambda^k) = \chi(\Lambda), \qquad \nu(\Lambda^k) = \nu(\Lambda).$$

By Theorem 7.3.1, $\mathrm{Gr}^+_{A^+,0}(\Lambda) \neq 0$ and $\mathrm{Gr}^-_{A^-,0}(\Lambda) \neq 0$. Hence, by Corollary 6.4.2 and (7.6),

$$\varepsilon^+(\tau, \Lambda) \geq -\langle[\tau], A^+\rangle = \frac{n+1-k}{2n+2},$$
$$\varepsilon^-(\tau, \Lambda) \leq -\langle[\tau], A^-\rangle = -\frac{k}{2n+2}.$$

Here $\tau \in \mathcal{T}(\Lambda)$ denotes the connection 2-form introduced in the proof of Theorem 7.3.1. Hence

$$\varepsilon(\Lambda) = \varepsilon^+(\tau, \Lambda) - \varepsilon^-(\tau, \Lambda) \geq \frac{1}{2}$$

Recalling from section 7.1 that $\nu(\Lambda) \leq \ell(\Lambda) = 1/2$ the result now follows from the inequality

$$\varepsilon(\Lambda) \leq \chi(\Lambda) = \nu(\Lambda)$$

proved in Corollary 4.3.4. □

Remark 7.4.3 Our invariants do not distinguish between Λ^j and Λ^k unless one of the numbers is divisible by $n+1$ and the other is not. However, if

$$\gcd(j, n+1) \neq \gcd(k, n+1)$$

then the iterated loops Λ^{mj} and Λ^{mk} have different invariants for some m. To see this stippose, without loss of generality, that $\gcd(j, n+1) < \gcd(k, n+1)$ and denote

$$m := \frac{n+1}{\gcd(k, n+1)} < \frac{n+1}{\gcd(j, n+1)}$$

Then mk is divisible by $n + 1$ whereas mj is not. By Corollary 7.4.2,

$$\nu(\Lambda^{mj}) \neq \nu(\Lambda^{mk}).$$

In the case of Hamiltonian loops the analogue of the line T(A) has a natural basepoint and in that case there are separate invariants $\varepsilon^+(P)$ and $\varepsilon^-(P)$ that contain finer information than their difference.

Remark 7.4.4 We conjecture that the constant loop $\Lambda^0 = S^1 \times \mathbb{R}P^n$ satisfies $\varepsilon(\Lambda^0) = 0$. This does not follow from the techniques of this paper. The homology class $A^0 \in H^2(D \times \mathbb{C}P^n, S^1 \times \mathbb{R}P^n; \mathbb{Z})$, represented by the constant maps $D \to \mathbb{R}P^n$, satisfies $\mu_{\Lambda^0}(A^0) = 0$. Hence A^0 does not satisfy our condition (6.6) for the definition of the Gromov invariants, although the arguments of Theorem 7.3.1 carry over to the constant loop Λ^0 with $A^+ = A^- = A^0$. It should be possible to circumvent the problem of compactness in the sense of Gromov described above by using the invariants introduced in Cieliebak–Gaio–Salamon [CGS]. We expect that these techniques apply to the constant loop Λ^0 in CP^n.

Remark 7.4.5 Let (M, ω) be a symplectic $2n$-manifold and L be a closed n-manifold with $H^1(L; \mathbb{R}) = 0$. In [W] Weinstein considers the space of all pairs (Λ, ρ) where $\Lambda \subset M$ is a Lagrangian submanifold diffeomorphic to L and ρ is a volume form on Λ (or a smooth measure in the nonorientable case). He interprets this space as the cotangent bundle of $\mathcal{L} = \mathcal{L}(M, \omega, L)$ and examines the symplectic action functional on the loop space of $T^*\mathcal{L}$. In [D] Donaldson interprets this cotangent bundle as a symplectic quotient of the space of all embeddings $\iota : L \to M$ with vanishing cohomology class $\iota^*[\omega]$ by the group of volume preserving diffeomorphisms of L (with respect to a given smooth measure). The group action is Hamiltonian and the zero set of the moment map is the space of Lagrangian embeddings of L into (M, ω). It would be interesting to examine analogues of the invariants studied in the present paper for loops in $T^*\mathcal{L}$ and relate these to the work of Weinstein and Donaldson.

Chapter 8

A non-trivial Legendrian 'knot'

In this chapter we present an example of a non-trivial Legendrian 2-sphere in the 1-jet bundle of $\mathbb{R}^2$. A certain familiarity with contact geometry is assumed and we claim no completeness here. For definitions and proofs of theorems that we use, we give the necessary references.

8.1 Contact geometry

See e.g. [MS1] for a more rigourous introduction in contact geometry. A **contact structure** on a manifold M of dimension 2n + 1 is a field of hy-Perplanes ξ c TM which is as far as possible from being integrable. The complete non-integrability of ξ can be expressed by the inequality

$$\alpha \wedge (\mathrm{da})" \neq 0$$

where ξ is locally described by $\xi = \ker \alpha$ for a local I-form α. For simplicity, we will assume that ξ is transversally orientable so that it can be globally described as the kernel of some 1 -form α. The model example of a contact manifold is the 1-jet bundle $\mathcal{J}^1(N) = T^*$ N $\times \mathbb{R}$ of an n-manifold N with the 1-form α locally given by $\alpha = dz - \lambda$ described in section 8.2.

A diffeomorphism $\psi : M \to M$ is called a **contactomorphism** if it preserves ξ so that

$$\psi^*\alpha = e^h\alpha,$$

for some function $h : M \to \mathbb{R}$. **A contact isotopy** is a smooth 1-parameter family ψ_t of contactomorphisms such that ψ_0 = id. For non-compact manifolds we will always assume that the isotopy has compact support. An integral submanifold L of ξ of dimension n is called **a Legendrian submanifold** of (M, ξ) i.e. TL c ξ. We call two Legendrian submanifolds L_0 and L_1 **Legendrian isotopic** if there exists a Legendrian isotopy ψ_t (this means that $\psi_t(L)$ is Legendrian for $0 \leq t \leq 1$) such that $\psi_0(L_0) = L_0$ and $\psi_1(L_0) = L_1$. The question when two Legendrian submanifolds are Legendrian isotopic is one of great interest and has at the moment of writing only been answered in a few special cases, see for example [EF].

Remark 8.1.1 The Legendrian isotopy extension theorem, see for a proof e.g. [Tr], states that for a closed Legendrian submanifold L of (M, ξ) and an isotopy $\iota_t : L \to M$ with $\iota_0(L) = L$ and $\iota_t(L)$ Legendrian for all $t \in [0, 1]$ there exists a contact isotopy ψ_t : (M, ξ) $\to$ (M, ξ) such that $\psi_t|_L = \iota_t$. This justifies the name Legendrian isotopic instead of for example contact isotopic.

8.2 Legendrian knots in $\mathcal{J}^1(\mathbb{R}^n)$

Consider the 1-jet space $\mathcal{J}^1(\mathbb{R}^n) \cong T^*\mathbb{R}^n \times \mathbb{R}$ of real valued functions on $\mathbb{R}^n$ with coordinates $(x, y, z) := (x_1, \ldots, x_n, y_1, \ldots, y_n, z)$. This is a contact manifold with contact form

$$\alpha = dz - \lambda,$$

where $\lambda = \lambda_{\mathrm{can}} = \sum_{i=1}^n y_i dx_i$. Observe that the zero section

$$\{(x, y, z) \in \mathcal{J}^1(\mathbb{R}^n) \mid y = z = 0\}$$

is a Legendrian submanifold and so is the 1-jet

$$j^1 f := \{(x, df(x), f(x)) \in \mathcal{J}^1(\mathbb{R}^n)\}$$

of a function $f : \mathbb{R}^n \to \mathbb{R}$. A **Legendrian knot** is an embedding $\iota : \mathbb{R}^n \to \mathcal{J}^1(\mathbb{R}^n)$ such that $\iota(\mathbb{R}^n)$ is Legendrian and such that $\iota(\mathbb{R}^n)$ coincides with the zero section outside a compact set. We also say that $\iota(\mathbb{R}^n)$ is flat at infinity. In this chapter we study Legendrian knots. Let

$$\pi : \mathcal{J}^1(\mathbb{R}^n) \to \mathcal{J}^0(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}$$

denote the obvious projection sending (x, y, z) to (x, z). We call the image n(L) of a Legendrian submanifold L its **wave** front. Note that the wave front determines the Legendrian submanifold uniquely. So it suffices to study the wave fronts of Legendrian submanifolds to determine whether the corresponding Legendrian submanifolds are Legendrian isotopic to each other or not.

If $n = 1$ there are the so called Bennequin number and the Maslov number. In case of the topological unknot, theses can be used to determine whether two oriented Legendrian knots are isotopic or not, see [EF]. These are calculated by studying the wave front. The Bennequin number $\text{tb}(L)$ of a Legendrian knot L is equal to the number of positive crossings minus the number of negative crossings minus half the number of cusps (a crossing is positive when the two rays exiting the crossing are on the same side of the vertical and negative otherwise). The Maslov number $\mu(L)$ is equal to half the number of cusps passed downward minus half the numbers of cusps passed upward, see eg. [FT]. Consider the following wave front, see Figure 8.1.

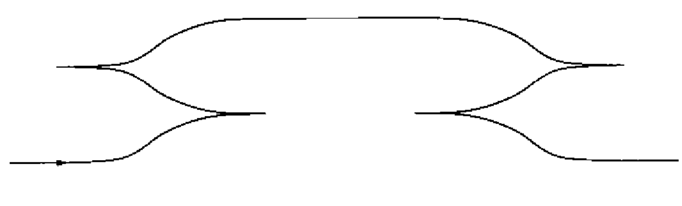

Figure 8.1: A non-trivial and a trivial Legendrian knot

This knot can easily be seen to be smoothly isotopic to the zero section. Consider the lift to $\mathcal{J}^1$ (W) see Figure 8.2.

Now project the knot onto the xy-plane. We then get an ordinary knot diagram, see Figure 8.3. We know that two knots are isotopic (sometimes also called equivalent) if we can deform one knot diagram into the other through Reidemeister moves, see e.g. [L]. It is easily seen that this is possible here.

On the other hand, this knot is not Legendrian isotopic to the zero section, which can be seen by computing the above mentioned invariants. The wavefront of this knot has no crossings, two upward cusps and two down-

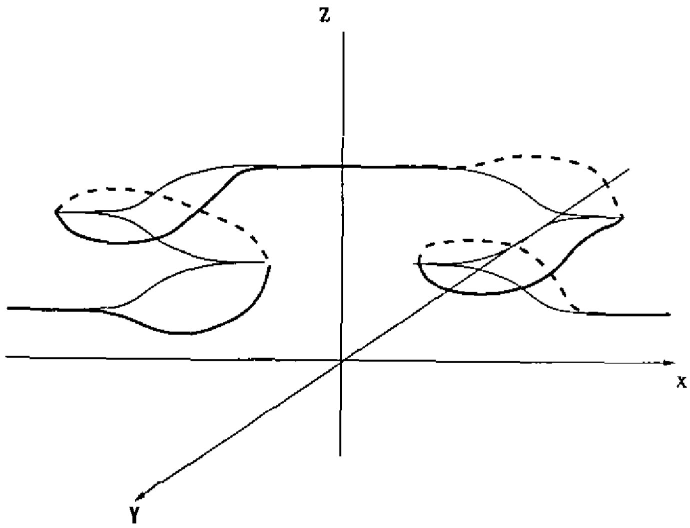

Figure 8.2: Lifting the knot

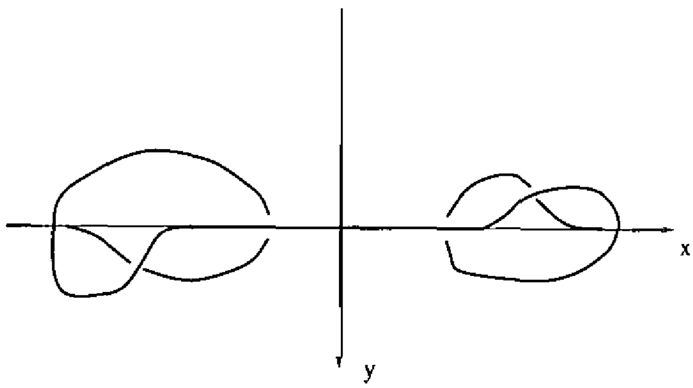

Figure 8.3: Projecting the knot on the xy-plane

ward cusps and therefore the Bennequin number is -2 and the Maslov number is 0. For the zero section these numbers are both zero and since these numbers are in this case Legendrian invariants, we deduce that these two knots are not Legendrian isotopic.

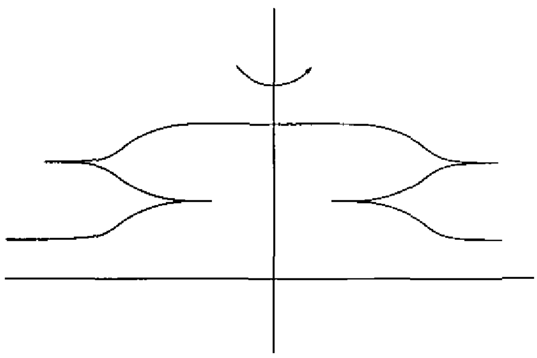

Figure 8.4: A non-trivial Legendrian 2-sphere

Consider now the following wave front, see Figure 8.4, of a 2-dimensional Legendrian knot in $\mathcal{J}^1(\mathbb{R}^2)$, which is a generalisation of the above described one dimensional knot, obtained by rotating the wave front around the z-axis. We quote the following theorem from [RS] where we have applied the remark about codimension 3 and the exercise about manifolds, which are not closed.

Theorem 8.2.1 *Suppose $f_0, f_1 : M^m \to int\ N^n$ are homotopic embeddings, which are jixed outside a compact m-manifold $M_0 \subset int\ M$, and suppose $n - m \geq 3$. Then $f_0(M)$ and $f_1(M)$ are ambient isotopic by an isotopy supported in a compact set in int N.*

This theorem applies by setting $M = \mathbb{R}^2$ which is not closed and by setting N $= \mathcal{J}^1(\mathbb{R}^2)$. We know that the above embedding of $\mathbb{R}^2$ into $\mathcal{J}^1(\mathbb{R}^2)$ is homotopic to the standard one (i.e. the l-jet of the zero function). This follows from the Hopf's degree theorem for spheres. Computing the degree outside the compact subset we see that the number of inverse image points is one and hence the embedding is homotopic to the Standard one. Theorem 8.2.1 implies that the embedding is also isotopic through an ambient isotopy to the standard embedding.

However, in higher dimensions there are so far no Legendrian knot invariants, like the Bennequin number and the Maslov number, to determine whether they are also Legendrian isotopic. So we need to use other tools to determine whether or not this is Legendrian isotopic to the trivial knot. In the next section we will prove the following theorem.

Theorem *8.2.2 The above wave front is not Legendrian isotopic to the zero section.*

8.3 Generating functions

We observed earlier that the 1-jet of a function determines a Legendrian submanifold of $\mathcal{J}^1(\mathbb{R}^n)$. It is, however, not true that every Legendrian submanifold can be described as the I-jet of a function. In this section we will extend the idea of I-jets and introduce generating functions. We will see how they can be used to say something about Legendrian submanifolds.

Consider the trivial vector bundle $E := \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ for some $k \in \mathbb{N}$. Let S : $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $(x, q) \mapsto S(x, q)$ be a function that equals zero for $|x| \gg 0$ and whose fiber derivative is transverse to zero, that is, the Jacobian of $\frac{\partial S}{\partial q}(x, q)$ has maximal rank whenever $\frac{\partial S}{\partial q}(x, q) = 0$. Observe that

$$\Sigma_S := \{(x, q) \in E; \frac{\partial S}{\partial q}(x, q) = 0)$$

is a submanifold of E. Define the map $\iota : \Sigma_S \to \mathcal{J}^1(\mathbb{R}^n)$ by

$$\iota(x, q) := (x, \frac{\partial S}{\partial x}(x, q), S(x, q))$$

then $\iota(\Sigma_S)$ is an (immersed) Legendrian submanifold. If for $|q| \gg 0$, $S(x, q) = Q(q)$, where Q is a non degenerate quadratic form, S is called a **generating function quadratic at infinity** abbreviated by **g.f.q.i.**. Given a Legendrian submanifold L then S is said to be a g.f.q.i. for L if $\iota(\Sigma_S) = L$, ι is an embedding and dim Σ_S = n.

Theorem 8.2.2 is proved by combining the following two theorems, which we will only state (for proofs see the references).

Theorem 8.3.1 (Bhupal [B], Theret [Th]) *Let L be a Legendrian submanifold of $\mathcal{J}^1(\mathbb{R}^n)$. If L is isotopic to the zero section then L has a g.f.q.i..*

This theorem has first been proved in the Lagrangian case by Viterbo [V] and Sikorav [Si]. Theorem 8.3.1 may also be attributed to Chekanov, although he has not published it.

Theorem 8.3.2 (Chaperon [Ch],Joukovskaia [Jou1], [Jou2]) *Let L be a Legendrian submanifold of $\mathfrak{J}^1(\mathbb{R}^n)$. If L possesses a g.f.q.i. then its wave front $\pi(L)$ has a Lipschitz continuous section.*

Proof of Theorem 8.2.2: Figure 8.4 shows that the wave front does not possess a Lipschitz continuous section (it does not even have a continuous section). By Theorem 8.3.2 L does not have a g.f.q.i. and hence, by Theorem 83.1, L is not Legendrian isotopic to the zero section. This proves the theorem. □

Chapter 9

Travelling wave solutions of a fourth order semi-linear diffusion equation

And now .... for something completely different. This chapter has appeared as the joint paper [AH].

9.1 Introduction

In this chapter we are interested in travelling wave solutions, that is solutions of the form $u = u(x, t) = u(x - ct)$ for some c, of the fourth order equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + f(u), \qquad f(u) = (u - a)(1 - u^2), \qquad (9.1)$$

where $-1 < a \leq 0$ and $\gamma > 0$. We are looking for solutions that connect the two stable states $u(x, t) = \pm 1$ of the ordinary differential equation $u' = (u - a)(1 - u^2)$. This equation has many applications in e.g. population genetics and pattern formation, for references see [PT1].

When $\gamma = 0$ a travelling wave solution is given by

$$u(x, t) = \tanh\left(\frac{x - a\sqrt{2}t}{\sqrt{2}}\right), \tag{9.2}$$

with wave speed ca $= a\sqrt{2}$, which is negative if $a < 0$. The wave profile is independent of a and for $a = 0$ this travelling wave solution is a stationary solution of (9.1) with $\gamma = 0$.

Equation (9.1) with $\gamma = 0$ with a slightly more general non-linearity has been studied extensively in [AW1], [AW2], [FM] amongst many others. It has been shown that for this type of non-linearity there exists a unique (except for symmetry and translation) travelling wave solution, i.e. a solution of the form $u(x, t) = u(\xi)$ where $\xi = x - ct$ for some c, connecting the stable states $u = \pm 1$.

If $\gamma > 0$ and a $= 0$ equation (9.1)) is called the Extended Fisher-Kolmogorov equation (EFK), which is a fourth order extension of the classical Fisher-Kolmogorov equation (FK). In a series of papers [PT1], [PT2], [PT3], [PT4], [PTV] existence results on stationary solutions and their properties have been proved. In these papers one studies, in view of the symmetry of the non-linearity, odd solutions and hence the conditions $u(0) = 0$, $u''(0) = 0$ and $u(\infty) = 1$ are imposed. One distinguishes two different cases $\gamma \leq 1/8$ and $\gamma > 1/8$ where the behaviour of solutions is different.

In both cases an energy identity can be used to reduce the order of the equation. If $\gamma \leq 1/8$ the order can be reduced further by assuming monotonicity of the solutions and the remaining problem is of second order. In [PT1] this is used to show that there is a unique solution for $\gamma \leq 1/8$. If $\gamma > 1/8$ monotonicity is lost, This can be seen by linearising the equation at $u = \pm 1$. The eigenvalues are now complex so that any solution converging to $u = \pm 1$ must be oscillatory. In [PT4] a shooting method is used to prove the existence of families of different kinks. In [PTV] the variational structure of the stationary equation is used to prove existence of odd equilibrium solutions connecting the stable states u = fl.

In this chapter we look for travelling wave solutions of equation (9.1). The resulting travelling wave equation neither has a conserved energy nor a variational structure and also the symmetry is lost. Thus the methods of [PT1] and [PTV] cannot be applied directly here. For small γ however, equation (9.1) can be seen as aperturbation of (9.1) with $\gamma = 0$ and it is this view that is taken in this chapter. With the methods of geometric Singular perturbation theory as developed in [Fe] and [J] we prove the following

Theorem 9.1.1 *For $\gamma > 0$ sufficiently small there exists a $c = c(\gamma)$ for which there is a travelling wave solution of (9.1) connecting the steady states $u = \pm 1$. The rate of change of the wave speed with respect to γ is given by*

$$\frac{dc}{d\gamma}_{\gamma=0} = -1/5\sqrt{2}a(2a^2 - 3).$$

The chapter is divided as follows. In section 2 we describe how geometric perturbation theory is used to construct a locally invariant manifold M_γ for the travelling wave equation when γ is small and positive. In section 3 we use this manifold to obtain a travelling wave solution. In the last section we compute the rate at which the wave speed changes when the fourth order term is added.

9.2 Geometric Singular Perturbation Theory

Our approach in this section is similar to that in [GJ] where existence of a travelling wave solution is proved for a sixth order equation, but our calculations are more explicit. After substituting $u = u(\xi)$, where $\xi = x - ct$, and setting $\gamma = \varepsilon^2$ where $\varepsilon > 0$ in (9.1) we obtain the following boundary value problem:

$$(P_\varepsilon)\begin{cases} -\varepsilon^2 u'''' + u'' + cu' + (u - a)(1 - u^2) = 0 \quad \text{on} \quad \mathbb{R} \\ \lim_{\xi\to-\infty} u(\xi) = -1, \quad \lim_{\xi\to\infty} \lim_{\xi\to\infty} u(\xi) = 1, \end{cases}$$

where primes mean differentiation with respect to ξ.

We can write the differential equation in problem (P_ε) as a first order system

$$(S_\varepsilon)\begin{cases} u' = v \\ v' = w \\ \varepsilon w' = z \\ \varepsilon z' = w + cu + (u - a)(1 - u^2), \end{cases}$$

and setting $\xi = \varepsilon\eta$ we obtain

$$(F_\varepsilon)\begin{cases} \dot{u} = \varepsilon v \\ \dot{v} = \varepsilon w \\ \dot{w} = z, \\ \dot{z} = w + cv + (u - a)(1 - u^2), \end{cases}$$

where dots denote differentiation with respect to η. Note that (S_ε) is singular at $\varepsilon = 0$ because (So) is not a well-defined dynamical system in $\mathbb{R}^4$. Having set $\xi = \varepsilon\eta$ we overcome this problem. The time scale given by ξ is said tobe slow whereas that for η is fast, hence the corresponding systems are called the slow system (S_ε) and the fast system (F_ε). The latter is well-defined for all ε including $\varepsilon = 0$. For $\varepsilon \neq 0$, (F_ε) and (S_ε) are equivalent and the critical points are (- 1 , 0, 0, 0), (a, 0, 0, 0) and (1, 0, 0, 0).

If $\varepsilon = 0$ we define Mn to be the two dimensional manifold of critical points of (F_0):

$$M_0 := \{(u, v, w, z) \in \mathbb{R}^4 | \quad z = 0, w = -cv - (u - a)(1 - u^2)\}.$$

Geometric perturbation theory uses both the above Systems: (F_ε) provides us with an invariant manifold M_ε close to M_0 and we study the flow of (S_ε) restricted to this manifold. The main theorem that we use is the invariant manifold theorem due to Fenichel and we use the Version formulated by Jones [J]. In our context this theorem yields the following:

Theorem 9.2.1 (Fenichel) *If M_0 is a normally hyperbolic manifold, then for all $R > 0$, for all open intervals I with $c_0 \in I$ and for all $k \in \mathbb{N}$ there exists an $\varepsilon_0 > 0$ depending on R, I and k such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a manifold M_ε, given by*

$$M_\varepsilon = \{(u, v, w, z) \in \mathbb{R}^4 | \, w = \phi(u, v, c, \varepsilon), z = \psi(u, v, c, \varepsilon), \\ (u, v) \in B_R(0), c \in I\}$$

with ϕ and ψ in $C^k(\overline{B_R(0)} \times \overline{I} \times [0, \varepsilon_0])$, which is locally invariant under the flow of (F_ε).

In order to apply this theorem we must ensure that the hypothesis on M_0 is satisfied. The radius R that we choose must be so large that Mo $\cap$ $B_R(0)$ contains the connection from -1 to 1 at $\varepsilon = 0$. We also fix $k \geq 2$.

For M_0 to be normally hyperbolic we must check that the eigenvalues μ associated to the eigenvectors of the linearised problem for any point in M_0 at $\eta = 0$, which are transversal to the tangent space, have non-zero real part. Note that c can either be seen as a Parameter in which case M_0 is a two-dimensional manifold, parametrised by u and v, or as an extra variable in which case we need to add the equation $c' = 0$ and M_0 becomes a three-dimensional manifold.

The linearisation of (F_0) at the point $(u, v, w, z) \in M_0$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ f'(u) & c & 1 & 0 \end{pmatrix}$$

Its set of eigenvalues is always $(0, 0, -1, 1\}$. Only the two zero eigenvalues have eigenvectors tangent to Ma. Thus M_0 is normally hyperbolic. If we view c as a variable instead of as a Parameter, Mo remains normally hyperbolic.

Since ϕ and ψ are C^k functions in u, v, c and ε we can write down their Taylor series in ε, i.e.

$$\phi(u, v, c, \varepsilon) = \sum_{i=0}^{k} \phi_i(u, v, c)\varepsilon^i + \Phi(u, v, c, \varepsilon)\varepsilon^k, \tag{9.3}$$

$$\psi(u, v, c, \varepsilon) = \sum_{i=0}^{k} \psi_i(u, v, c)\varepsilon^i + \Psi(u, v, c, \varepsilon)\varepsilon^k, \tag{9.4}$$

where Φ and Ψ are continuous in $\varepsilon = 0$ with $\Phi(u, v, c, 0) = 0$ and $\Psi(u, v, c, 0) = 0$. In the remainder of this section we compute the coefficients of ϕ and ψ explicitly. Clearly we have

$$\phi_0(u, v, c) = -cv - (u - a)(1 - u^2) \quad \text{and} \quad \psi_0(u, v, c) = 0$$

Since M_ε is locally invariant, the fast vector field

$$(\varepsilon v, \eta w, z, w + cv + f(u))$$

is perpendicular to the two normals

$$\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}, -1, 0\right) \text{ and } \left(\frac{\partial\psi}{\partial u}, \frac{\partial\psi}{\partial v}, 0, -1\right)$$

of M_ε. Taking the inner product of the fast vector field with each of these normals we obtain the following two coupled non-linear partial differential equations

$$\psi(u, v, c, \varepsilon) = \varepsilon\left(v\frac{\partial\phi(u, v, c, \varepsilon)}{\partial u} + \phi(u, v, c, \varepsilon)\frac{\partial\phi(u, v, c, \varepsilon)}{\partial v}\right), \quad (9.5)$$

$$\begin{aligned}\phi(u, v, c, \varepsilon)+cv + f(u) =& \\ &\varepsilon\left(v\frac{\partial\psi(u, v, c, \varepsilon)}{\partial u} + \phi(u, v, c, \varepsilon)\frac{\partial\psi(u, v, c, \varepsilon)}{\partial v}\right),\end{aligned} \quad (9.6)$$

for ϕ and ψ. We now successively compute the coefficients in (9.3) and (9.4) from (9.5) and (9.6). We already know the zero-th order coefficients for ϕ and ψ and substituting the zero-th Orderterm of ϕ into (9.5) gives the first order term for ψ. Substituting the zero-th order term of ψ, which equals zero, into (9.6) we see that the first order term of ϕ vanishes. Thus

$$\begin{aligned}&\phi_1(u, v, c) = 0 \quad \text{and}\\ &\psi_1(u, v, c) = v(1 + c^2 - 3u^2 + 2au) + c(u - a)(1 - u^2).\end{aligned}$$

Similarly we find second order terms

$$\begin{aligned}\phi_2(u, v, c) =& [-v^2(-6u + 2a) + 2v\{-2u(u - a) + 1 - u^2\}c - c^3v \\ &+ (1 - u^2)(u - a)\{-2u(u - a) + 1 - u^2\} \\ &- (1 - u^2)(u - a)c^2]\end{aligned} \quad (9.7)$$

$$\psi_2(u, v, c) = 0. \quad (9.8)$$

Continuing in this way we can solve for all the coefficients whereby we remark that all the odd coefficients of ϕ and all the even ones of ψ are zero.

9.3 The flow on M_ε : construction of the travelling wave

In this section we prove the existence of a travelling wave solution for (9.1) for sufficiently small γ by showing that the heteroclinic orbit corresponding to (9.2) as a solution of the second order problem (Po), is a transversal intersection of the unstable and stable manifolds of respectively $u = -1$ and $u = 1$. We consider the slow equations restricted to the invariant manifold M_ε in Theorem 9.2.1. The resulting reduced slow system is well defined for $\varepsilon = 0$:

$$(S'_\varepsilon)\begin{cases} u' = v \\ v' = w = \phi(u, v, c, \varepsilon), \end{cases}$$

$$(S'_0)\begin{cases} u' = v \\ v' = -cv - f(u). \end{cases}$$

The latter are the phase plane equations for the second order travelling wave equation in (Pu). This system has three non-degenerate critical points (-1, 0), $(a, 0)$ and $(1, 0)$ and thus it follows from the implicit function theorem that for ε small there are still three critical points, which depend in a C^k fashion on c and ε. Since these three points must correspond to the three critical points (- 1 , 0, 0, 0), (a, 0, 0, 0) and (1 , 0, 0, 0) of the full system (S_ε), they are independent of ε and c.

For $\varepsilon = 0$ and $c = c_0 = a\sqrt{2}$ the travelling wave solution (9.2) corresponds to a saddle connection in the phase plane of (S'_0) connecting the saddle points (- 1, 0) and (1, 0). This connection is given by

$$v = \frac{1}{\sqrt{2}}(1 - u^2).$$

By the stable manifold theorem we can for small ε still parametrize the unstable manifold of (- 1, 0) and the stable manifold of (1, 0) in the phase plane of (S'_ε) locally as C^k functions of u. Denoting these functions by $h_0(u, c, \varepsilon)$ and $h_1(u, c, \varepsilon)$, we have $h_0(-1, c, \varepsilon) = 0$ and $h_1(1, c, \varepsilon) = 0$. Using smooth dependence on initial data we can continue h_0 and h_1 to $u = 0$ if ε is small. We want to show that, for possibly even smaller ε, there exists a unique $c = c(\varepsilon)$ such that $h_0(0, c(\varepsilon), \varepsilon) = h_1(0, c(\varepsilon), \varepsilon)$. Thus we introduce

$$G(c,\varepsilon) := h_0(0,c,\varepsilon) - h_1(0,c,\varepsilon).$$

The existence of $c(\varepsilon)$ will follow from the implicit function theorem if we prove that $\frac{\partial G}{\partial c}(c_0, 0) \neq 0$.

For $v \neq 0$ we can rewrite (S'_ε) as

$$\frac{dv}{du} = \frac{\phi(u,v,c,\varepsilon)}{v} \tag{9.9}$$

When we first differentiate (9.9) with respect to c and set $c = c_0$ and $\varepsilon = 0$, we get

$$\frac{dw}{du}(u) = -1 + \frac{f(u)}{h_0^2(u,c_0,0)}w(u) \quad \text{for } w(u) = \frac{\partial h_0}{\partial c}(u, c_0, 0). \tag{9.10}$$

Since $h_0(-1, c, \varepsilon) = 0$ we have $w(-1) = 0$. Note that all the higher order terms of ϕ have disappeared because we have set $\varepsilon = 0$. Similarly, we get for $\tilde{w}(u) = \frac{\partial h_1}{\partial c}(u, c_0, 0)$, that

$$\frac{d\tilde{w}}{du}(u) = -1 + \frac{f(u)}{h_1^2(u,c_0,0)}\tilde{w}(u) \tag{9.11}$$

and $\tilde{w}(1) = 0$. Since $h_0(u, c_0, 0) = h_1(u, c_0, 0) = \frac{1}{\sqrt{2}}(1-u^2)$, (9.10) and (9.11) both read

$$\frac{dw}{du} = -1 + \frac{2(u-a)}{1-u^2}w, \tag{9.12}$$

whence, in view of $w(-1) = \tilde{w}(1) = 0$,

$$w(0) = -\int_{-1}^{0} (1-s)^{1-a}(1+s)^{1+a}\,ds$$

and

$$\tilde{w}(0) = \int_0^1 (1-s)^{1-a}(1+s)^{1+a}\,ds.$$

Thus

$$\frac{\partial G}{\partial c}(c_0, 0) = w(0) - \tilde{w}(0) = -\int_{-1}^{1}(1 \quad s)^{1-a}(1+s)^{1+a}ds =: I_1 < 0. \tag{9.13}$$

Hence by the implicit function theorem there exists a neighbourhood U of 0 such that we can find a C^k map $c : U \to \mathbb{R}$ such that $G(c(\varepsilon), \varepsilon) = 0$ for all $\varepsilon \in U$. So we have found a connecting orbit of (S'_ε) from $(-1, 0)$ to $(1, 0)$ for ε sufficiently close to 0 and thus we have proved existence of a travelling wave equation of equation (9.1) for sufficiently small $\gamma > 0$.

9.4 Rate of change of the wave speed

The implicit function theorem also gives us the dependence of the wave speed on ε or rather on $\gamma = \varepsilon^2$, because as we saw earlier, the only non-vanishing terms in ϕ, see (9.5) and (9.6) and thereafter, are the even powers of ε. We next compute $\frac{\partial G}{\partial \gamma}$.

As before we start with (9.9) and differentiating with respect to γ followed by setting $\gamma = 0$ and $c = c_0$ we get, in view of (9.7) and (9.8),

$$\frac{dz}{du}(u) = -2\sqrt{2}u(3u^2 - 2) + \frac{2(u-a)}{1-u^2}z(u) \quad \text{for} \quad z(u) = \frac{\partial h_0}{\partial \gamma}(u, c_0, 0). \tag{9.14}$$

Similar calculations as before show that for z and $\tilde{z}(u) = \frac{\partial h_1}{\partial \gamma}(u, c_0, 0)$

$$\begin{aligned}\frac{\partial G}{\partial \gamma}(c_0, 0) &= z(0) - \tilde{z}(0)\\ &= -2\sqrt{2}\int_{-1}^{1} s(3s^2-2)(1-s)^{1-a}(1+s)^{1+a}ds =: I_2,\end{aligned} \tag{9.15}$$

whence, with (9.13), using either the calculus of residues or gamma functions,

$$\left.\frac{dc}{d\gamma}\right|_{\gamma=0} = -\frac{\partial G}{\partial \gamma}\left(\frac{\partial G}{\partial c}\right)^{-1} = -\frac{I_2}{I_1} = -\frac{1}{5}\sqrt{2}a(2a^2-3).$$

Note that $\frac{dc}{d\gamma}$ is negative, so by adding a fourth order perturbation to the second order travelling wave equation the wave speed, which is negative for our choice of a, decreases. In other words the absolute value of the wave speed increases under the perturbation of (9.1) with $\gamma = 0$ with the fourth order term in (9.1).

In the special case when $a = 0$ the wave speed $c_0 = 0$ and the travelling wave is an odd stationary Solution. It is shown in [PT1] that this 'kink' Solution satisfies

$$\frac{1}{\sqrt{2}(1+4\gamma)^{\frac{1}{4}}} < u'(0, y) < \frac{1}{\sqrt{2}} \text{ for } 0 < \gamma < \frac{1}{8}. \tag{9.16}$$

Here $u'(0,\gamma)$ is the derivative of u with respect to ξ. From our calculations we have

$$\begin{aligned}\frac{\partial u'}{\partial \gamma}(0, 0) &= \frac{\partial h_0}{\partial \gamma}(0, 0, 0) = z(0) \\ &= -\int_{-1}^{0} 2\sqrt{2}s(3s^2 - 2)(1 - s)(1 + s)ds = -\frac{1}{\sqrt{2}},\end{aligned}$$

which is consistent with (9.16) and shows that the lower bound is sharp.

Appendix A

Symplectic fibrations

In this appendix we will briefly discuss some facts about symplectic connections and their curvature. The manifolds under consideration are closed i.e. compact and without boundary. All that is written down here can be found in [MS1] and [GLS] and we have added it for the sake of completeness only. Most lemmata will be stated without proof, but we will give precise references for the proofs. We will start by recalling some definitions.

A.1 Symplectic connections

A smooth map $\pi : P \to B$ between smooth manifolds is said to be a **locally trivial fibration** with fibre M (also a smooth manifold) if there is an open cover $\{U_\alpha\}$ of B and a collection of diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times M$ such that

$$\mathrm{pr} \circ \phi_\alpha = \pi$$

where $\pi : \pi^{-1}(U_\alpha) \to U_\alpha$ and $\mathrm{pr} : U_\alpha \times M \to U_\alpha$. We always assume that B is 2-dimensional although this is not necessary. The maps ϕ_α are called **local trivialisations**. Denote by $M_b := \pi^{-1}(b)$ the fibre over $b \in B$ and by $\phi_\alpha(b) : M_b \to M$ the restriction of ϕ_α to M_b followed by the projection onto M. The maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(M)$ delined by

$$\phi_{\alpha\beta}(b) := \phi_\beta(b) \circ \phi_\alpha(b)^{-1}$$

for $b \in U_\alpha \cap U_\beta$ are called the **transition functions.** A fibration is said to have **structure group** $G \subset \mathrm{Diff}(M)$ if the transition functions all take values in G.

We are interested in the case that the tibre is a compact symplectic manifold (M, ω) and the structure group is $G \subset \mathrm{Symp}(M, \omega)$ i.e.

$$\phi_{\alpha\beta}(b) \in \mathrm{Symp}(M, \omega)$$

for all α, β and all $b \in U_\alpha \cap U_\beta$. In this case we call $\pi : P \to B$ a **symplectic fibration.** Each tibre M_b carries a natural symplectic structure $\omega_b \in \Omega^2(M_b)$ defined by

$$\omega_b = \phi_\alpha^*(b)\omega$$

for $b \in U_\alpha$. Since

$$\phi_\beta^*(b)\omega = (\phi_{\alpha\beta}(b) \circ \phi_\alpha(b))^*\omega = \phi_\alpha^*(b) \circ \phi_{\alpha\beta}^*(b)\omega = \phi_\alpha^*(b)\omega$$

it follows that ω_b is well defined.

Given a symplectic fibration $\pi : P \to B$ Define $\mathrm{Vert}_x := \ker d\pi(x)$ for $x \in P$ to be the vertical tangent space to the fibre. A connection is a field of horizontal subspaces $\mathrm{Hor}_x \subset T_x P$ such that

$$T_x P = \mathrm{Hor}_x \oplus \mathrm{Vert}_x \text{ for all } x \in P. \tag{A.1}$$

Now every path $\gamma : [0, 1] \to B$ determines a diffeomorphism

$$\Phi_\gamma : M_{\gamma(0)} \to M_{\gamma(1)}$$

assigning to $x_0 \in M_{\gamma(0)}$ the endpoint $x_1 \in M_{\gamma(1)}$ of the unique horizontal lift of the path γ. This diffeomorphism is called the **holonomy** of the path γ. We call a connection a **symplectic connection** if the holonomy along every path preserves the symplectic structures in the fibres. Let τ be a 2-form in $\Omega^2(P)$ which restricts to ω on the fibre that is

$$\omega_b = \iota_b^*\tau,$$

for every $b \in B$. Here $\iota_b : M_b \to P$ denotes the inclusion of the tibre. Then τ defines a natural connection in the following way

$$\mathrm{Hor}_x = \{\xi \in T_x P \mid \tau(\xi, \eta) = 0 \text{ for all } \eta \in \mathrm{Vert}_x\}. \tag{A.2}$$

Conversely every connection is given by such a 2-form, see [MS1] page 211. The following lemma tells us when a connection is symplectic.

Lemma A.l.l *Assume that $\tau \in \Omega^2(P)$ restricts on the fibre to the symplectic form ω. Then the connection dejined by τ is symplectic if and only if τ is vertically closed. This means thatfor all $x \in P$ andfor all vertical tangent vectors $\eta_1, \eta_2 \in Vert_x$, the 1-form $d\tau(\eta_1, \eta_2, \cdot) \in Q'(P)$ satisjies*

$$d\tau(\eta_1, \eta_2, \cdot) = 0.$$

Proof: See [MS1] Lemma 6.18. □

We call a 2-form that restricts to ω on the fibre and which is vertically closed a **connection 2-form.** One can prove the following.

Theorem A.1.2 *Every symplectic fibration admits a symplectic connection.*

Proof: See [GLS] Theorem 1.2.5. □

It is, however, not true that every symplectic connection can be represented by a closed 2-form.

Lemma A.1.3 *Let $\pi : P \to B$ be a symplectic fibration and let Γ be a symplectic connection on P. Then the following Statements are equivalent*

(i) *There exists a closed connection 2-form $\tau \in \Omega^2(P)$ generating the connecting Γ via (A.2).*

(ii) *The holonomy of Γ around any contractible loop in B is Hamiltonian.*

Proof: See [MS 1] Theorem 6.2.1. □

Note that the closed 2-form generating the connection Γ is not unique. If τ_1, $\tau_2 \in \Omega^2(P)$ are two closed connection 2-forms generating the same connection then they differ by $\pi^*\sigma$ where σ is a closed form in $\Omega^2(B)$ (see [MS 1] Theorem 6.2.1 for a proof).

A connection is called a **Hamiltonian connection** if the holonomy around every loop in the base is a Hamiltonian symplectomorphism. A fibration $\pi : P \to B$ is called a **Hamiltonian fibration** if it admits a Hamiltonian connection. Note that Lemma A. 1.3 tells us that if the base is contractible then the existence of a closed connection 2-form implies that the fibration is Hamiltonian. The following theorem characterises Hamiltonian fibrations.

Theorem A.1.4 *Let $\pi : P \to B$ be a symplectic fibration whose restriction to every loop in B admits a symplectic trivialisation. (This means that for every loop $\gamma \subset B$ there exists a diffeomorphism $h : \pi^{-1}(\gamma) \to \gamma \times M$ such that $h(b)^*\omega = \omega_b$). Then the following Statements are equivalent*

(i) *$\pi : P \to B$ is a Hamiltonian fibration.*

(ii) *The structure group reduces to* $\mathrm{Ham}(M, \omega)$.

(iii) *There exists a closed connection 2-form $\tau \in \Omega^2(P)$.*

(iv) *There exists a cohomology class $a \in H^2(P; \mathbb{R})$ which restricts to the class of the symplectic structure on the fibre.*

Proof: See [MS 1] Theorem 6.36. □

A.2 Symplectic curvature

Given a symplectic connection Γ and a connection 2-form τ generating this connection. We will explain here what we mean by the curvature of Γ. From the splitting (A.1) we get a map

$$\Lambda^2 \, \mathrm{Hor} \to \mathrm{Vert}$$

which measures the extent to which the horizontal bundle fails to be integrable. Given $b \in B$. Let T_bB be the tangent space to the base, which can be identified with horizontal space Hor, at the point $p \in P$ with $\pi(p) = b$ and let M_b be the fibre above b in P. We obtain a map

$$\Lambda^2 T_bB \to \mathrm{Vect}(M_b); (v, W) \mapsto [v^\sharp, w^\sharp]^{\mathrm{vert}}.$$

Here $v^\sharp$ and $w^\sharp$ denote the horizontal lifts to P of the vector fields v and w on B and $[v^\sharp, w^\sharp]^{\mathrm{vert}}$ is the vertical component of the commutator $[v^\sharp, w^\sharp]$, see Figure (A.1).

This map is by definition the curvature of the connection and is denoted by Ω_τ. It vanishes if and only if the horizontal distribution is integrable. Since the connection is symplectic, the image of this map is contained in the Lie algebra of symplectic vector fields on M_b. The following lemma is known as the curvature identity.

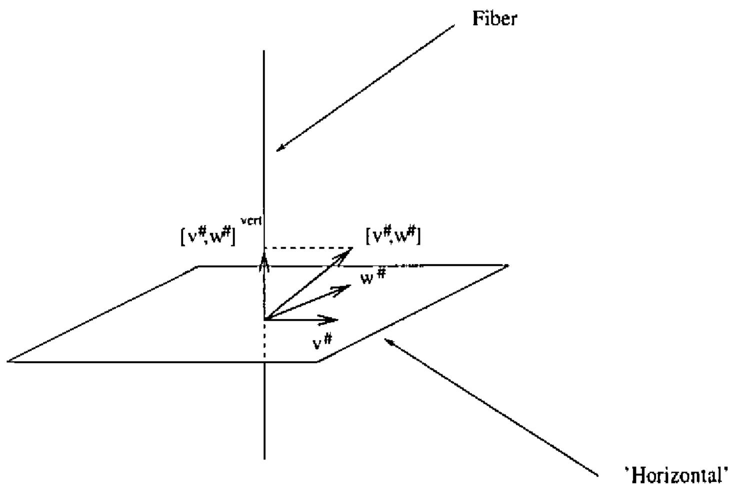

Figure A.1: Symplectic curvature

Lemma A.2.1 (Curvature identity) *Given a connection 2-form τ and two vector fields $v_1, v_2 : B \to TB$ then the curvature satisfies*

$$\iota(\Omega_\tau(v_1, v_2))\tau := \iota([v_1^\sharp, v_2^\sharp]^{vert})\tau \overset{fibre}{=} \iota(v_1^\sharp)\iota(v_2^\sharp)d\tau + d\big(\iota(v_1^\sharp)\iota(v_2^\sharp)\tau\big),$$

where we write $\overset{fibre}{=}$ to mean that the restrictions of the two sides to any fibre agree.

Proof: See [MS1] Lemma 6.28. □

It shows in particular that if the connection is Hamiltonian, i.e. the connection 2-form τ is closed then the curvature is a globally Hamiltonian vector field its Hamiltonian function being the restriction of $\big(\iota(v_1^\sharp)\iota(v_2^\sharp)\tau\big)$ to the fibre M_b.

A.3 Coupling form and weak coupling

Guillemin, Lerman and Sternberg [GLS] address the following question: When does a symplectic connection Γ admit a closed connection form? They show that if the fibre is a compact, connected, simply connected symplectic manifold then there exists a unique closed connection 2-form τ_Γ on

P with the property that $\pi_* \tau_\Gamma^{n+1} = 0$. This form τ_Γ is called the **coupling form** associated to the symplectic connection Γ. Here n is half the dimension of the fibre and π_* is the Gysin map or 'fibre integration' map. This means that for any $2n+2$-form τ on P, $\pi_* \tau$ is the 2-form on B given by

$$\iota(v_1 \wedge v_2)\pi_*\tau(b) = \int_{M_b} \iota(v_1^\sharp \wedge v_2^\sharp)\tau .$$

where $v_1^\sharp$ and $v_2^\sharp$ denote the horizontal lifts of v_1 and v_2. Any other closed connection 2-form is of the form

$$\tau_\Gamma + \pi^*\sigma,$$

where σ is a closed 2-form on B (for a proof see [GLS] Theorem 1.4.3). Note that although the coupling form is closed, it is not necessarily symplectic (although it restricts to a symplectic form on the fibre, it may degenerate in horizontal directions). A first attempt to make it symplectic would be to add a horizontal correction term. This might however not work since the coupling form itself might already have a nonzero horizontal component (even adding a symplectic horizontal correction term might not work). One way to make the coupling form symplectic is via **weak coupling:** Rescale the symplectic form on the fibre by a fixed (but small) positive constant. The effect of this rescaling on the coupling form is to multiply it by this constant as well. In [GLS] the following is shown.

The weak coupling construction: For $\varepsilon > 0$ sufficiently small there exists a smooth family of closed 2-forms ω_t on P with $t \in [0, \varepsilon)$ such that

- $\omega_0 = \pi^*\sigma$
- $[\omega_t] = t\,[\omega_\Gamma] + \pi^*[\sigma] \in H^2(P, \mathbb{R})$
- $\iota_b^*\omega_t = t\,\omega_b$
- ω_t is symplectic for all $t > 0$.

Appendix B

The Maslov index

In this appendix we recall how the Maslov index is defined for loops of Lagrangian subspaces of $\mathbb{C}^n$ and how this definition is used to define a general Maslov index for loops of Lagrangian submanifolds. We closely follow [MS1] or see [RS1].

Consider $\mathbb{C}^n$ with the standard symplectic form ω_0. Denote by A(n) the set of Lagrangian planes. The fundamental group of A(n) is isomorphic to the integers (see e.g. [MS1] Lemma 2.29) and an explicit isomorphism is given by the Maslov index homomorphism, $\mu : \pi_1(A(n)) \to \mathbb{Z}$, which is defined by the following theorem.

Theorem B.O.l *There exists a unique jiinctor μ, called the Maslov index homomorphism, which assigns an integer $\mu(L_t)$ to every loop of Lagrangian subspaces $\mathbb{R}/\mathbb{Z} \to A(n)$ sending t to L_t and which satisfies the following axioms:*

- **(homotopy)** *Two loops in* A(n) *are homotopic if and only if they have the same Maslov index.*

- *(***product***) For any two loops $L_t \in A(n)$ and $\Psi_t \in Sp(n)$ (the group of linear symplectomorphisms) we have that*

$$\mu(\Psi_t L_t) = \mu(L_t) + 2\mu(\Psi_t).$$

In particular, a constant loop $L_t = L_0$ has Maslov index 0.

- **(direct sum)** *If $n = n' + n''$ and we identify $\Lambda(n') \oplus \Lambda(n'')$ in the obvious way with a submanifold of $\Lambda(n)$ then*

$$\mu(L'_t \oplus L''_t) = \mu(L'_t) + \mu(L''_t).$$

- **(normalisation)** *The loop $\mathbb{R}/\mathbb{Z} \to \Lambda(1)$ defined by*

$$L_t = e^{\pi i t}\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$$

has Maslov index 1.

Proof: See [MS1] Theorem 2.33. □

Given a symplectic manifold (M, ω) and a Lagrangian submanifold L. We can now define the Maslov index as a homomorphism from $\pi_2(M, L)$ to $\mathbb{Z}$ as follows. Given a class $A \in \pi_2(M, L)$, choose a representative

$$u : (D, \partial D) \to (M, L),$$

Choose a trivialisation of the tangent bundle $u^*TM \simeq D \times \mathbb{C}^n$. This defines us a map $\gamma_u : S^1 \to \Lambda(n)$; $e^{2\pi i t} \mapsto T_{u(e^{2\pi i t})}L$ where we have identified ∂D with S^1 We define thc Maslov index of the class A by

$$\mu(A) = \mu(\gamma_u).$$

One can verify that this index is independent of the choice of representative u. This Maslov index is invariant under symplectic isotopies of M.

Appendix C

Taubes' argument

In the chapter 6 we have skipped over the fact that if the almost complex structure is smooth then the manifolds in question are not Banach but only Frechet manifolds (modelled on complete metric spaces) in which case we cannot apply an infinite dimensional Version of the Sard Smale theorem. To overcome this problem Taubes (see [MS2]) showed that it suffices to prove the theorem for $\mathcal{C}^\ell$ almost complex structures since one can deduce the theorem for smooth structures from this. In this appendix we will outline Taubes' argument.

C.1 From $\mathcal{C}^\ell$ to $\mathcal{C}^\infty$

Define $\mathcal{J}^\ell(M, \omega)$ to be the space of almost complex structures of class $\mathcal{C}^\ell$ on TM that are compatible with ω. Let $\mathcal{J}^\ell(D; M, \omega)$ be the class of all families of almost complex structures $J : D \to \mathcal{J}^\ell(M, \omega)$ of class C^ℓ. For the sake of simplicity we will denote by D_u either D_u^+ or D_u^- in equation (6.4). Here u is either a solution of (6.1) or a solution of (6.3). For brevity we will call u an J-curve. An almost complex structure $J \in \mathcal{J}^\ell(D; M, \omega)$ is called regular if D_u is surjective for all relative homology classes A and for every J-curve u. We denote this set by $\mathcal{J}^\ell_{\text{reg}}(\tau, \Lambda)$. Theorem 6.2.2, stating that the sets $\mathcal{J}^\pm_{\text{reg}}(\tau, \Lambda)$ are of the second category (in the sense of Baire) in $\mathcal{J}(D; M$, w), truely holds for this case and it shows that $\mathcal{J}^\ell_{\text{reg}}(\tau, \Lambda)$ is dense in $\mathcal{J}^\ell(D; M, \omega)$ with respect to the $\mathcal{C}^\ell$-topology. Define for $K \in$ IV the set

$$\mathcal{J}_{\text{reg},K}(\tau,\Lambda) \subset \mathcal{J}(D;M,\omega)$$

of all smooth almost complex structures $J \in \mathcal{J}(D;M,\omega)$ such that the Operator D_u is onto for every J-curve u that satisfies

$$\|du\|_{L^\infty} \le K.$$

Note that for every K, $\mathcal{J}_{\text{reg}}(\tau,\Lambda)$ c $\mathcal{J}_{\text{reg},K}(\tau,\Lambda)$ and that

$$\mathcal{J}_{\text{reg}}(\tau,\Lambda) = \bigcap_{K\in\mathbb{N}} \mathcal{J}_{\text{reg},K}(\tau,\Lambda).$$

So it suffices to prove that each $\mathcal{J}_{\text{reg},K}(\tau,\Lambda)$ is open and dense in $\mathcal{J}(D;M,\omega)$ with respect to the $\mathcal{C}^\infty$-topology in order to deduce that $\mathcal{J}_{\text{reg}}(\tau,\Lambda)$ is dense in $\mathcal{J}(D;M,\omega)$ with respect to the $\mathcal{C}^\infty$-topology.

We will first show that $\mathcal{J}_{\text{reg},K}(\tau,\Lambda)$ is open or equivalently that $\mathcal{J}(D;M,\omega)\setminus\mathcal{J}_{\text{reg},K}(\tau,\Lambda)$ is closed. Given a sequence

$$J_\nu \in \mathcal{J}(D;M,\omega)\setminus\mathcal{J}_{\text{reg},K}(\tau,\Lambda)$$

which converges to $J \in \mathcal{J}(D;M,\omega)$ in the $\mathcal{C}^\infty$-topology. Then there exists a sequence of J_ν-curves u_ν which satisfy $\|du_\nu\|_{L^\infty} \le K$ such that the operator D_{u_ν} is not surjective. It follows from elliptic bootstrapping (see [MS2] Appendix B) that u_ν has a subsequence u'_ν which converges uniformly with all derivatives to a smooth J-curve u. This limit curve clearly still satisfies $\|du\|_{L^\infty} \le K$ We will now show that D_u is not surjective. Assume that D_u is surjective. For every $\delta > 0$ there exists ν such that D_{u_ν} is not surjective and for which we have

$$\|D_u - D_{u_\nu}\| \le \delta.$$

Hence for all ξ we have

$$|D_u\xi - D_{u_\nu}\xi| \le \delta\,|\xi|.$$

Since D_{u_ν} is not surjective we can find a v with $|v| = 1$ such that

$$\text{dist}\,(v, \text{im}\, D_{u_\nu}) = 1$$

Since D_u is surjective we can find ξ with $D_u\xi = v$ and $|\xi| \leq \frac{1}{\|D_u\|}$. So we get

$$1 = \operatorname{dist}(v, \operatorname{im} D_{u_\nu}) = \operatorname{dist}(D_u\xi, \operatorname{im} D_{u_\nu}) \leq |D_u\xi - D_{u_\nu}\xi| \leq \delta\,|\xi| \leq \frac{\delta}{\|D_u\|}.$$

Choose $\delta = \frac{1}{2}\|D_u\|$ then we obtain a contradiction, which shows that D_u cannot be surjective. This proves that $J \notin \mathcal{J}_{\mathrm{reg},K}(\tau, \Lambda)$ and therefore $\mathcal{J}_{\mathrm{reg},K}(\tau, \Lambda)$ is open.

Denote the set of all $J \in \mathcal{J}^\ell(D; M, \omega)$ such that the Operator D_u is onto for every J-curve u of class $\mathcal{C}^\ell$ that satisfies $\|du\|_{L^\infty} \leq K$ by $\mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \Lambda)$. We claim that

$$\mathcal{J}_{\mathrm{reg},K}(\tau, \Lambda) = \mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \Lambda) \cap \mathcal{J}(D; M, \omega).$$

It is clear that $\mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \Lambda) \cap \mathcal{J}(D; M, \omega) \subset \mathcal{J}_{\mathrm{reg},K}(\tau, \Lambda)$. To see this note that if $J \in \mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \Lambda) \cap \mathcal{J}(D; M, \omega)$ then the Operator D_u has tobe onto for a group of $\mathcal{C}^\ell$ curves, whereas for J to lie in $\mathcal{J}_{\mathrm{reg},K}(\tau, \Lambda)$ it only has to be onto for a group of $\mathcal{C}^\infty$ curves (a smaller group). Hence the set itself is smaller. The other inclusion follows from elliptic regularity (we have stated the proposition here in a general form) and remark 3.2.3 in [MS2].

Proposition C.1.1 (Elliptic regularity) *Assume $J \in \mathcal{J}^\ell$ is an almost complex structure of class $\mathcal{C}^\ell$ with $\ell \geq 1$. If $u : D \to M$ is a J-holomorphic curve of class $W^{1,p}$ with $p > 2$ then u is of class $\mathcal{C}^\ell$.*

Proof: See [MS2], Theorem B.4.1. □

A similar argument as above shows that

$$\mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \Lambda) \subset \mathcal{J}^\ell(D; M, \omega) \text{ is open w.r.t. the } \mathcal{C}^\ell\text{-topology.}$$

We already saw that

$$\mathcal{J}^\ell_{\mathrm{reg}}(\tau, \omega) \subset \mathcal{J}^\ell(D; M, \omega) \text{ is dense w.r.t. the } \mathcal{C}^\ell\text{-topology.}$$

Since $\mathcal{J}^\ell_{\mathrm{reg}}(\tau, \omega) \subset \mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \omega)$ this implies that

$$\mathcal{J}^\ell_{\mathrm{reg},K}(\tau, \omega) \subset \mathcal{J}^\ell(D; M, \omega) \text{ is dense w.r.t. the } \mathcal{C}^\ell\text{-topology.}$$

We will show that this implies that $\mathcal{J}_{\text{reg},K}(\tau, \omega)$ is dense in $\mathcal{J}(D; M, \omega)$ with respect to the $\mathcal{C}^\ell$-topology.

Let $J \in \mathcal{J}(D; M, \omega)$ **c** $\mathcal{J}^\ell(D; M, \omega)$. Since $\mathcal{J}^\ell_{\text{reg},K}(\tau, \omega)$ is dense in $\mathcal{J}^\ell(D; M, \omega)$ we can $\mathcal{C}^\ell$-approximate J by an almost complex structure $J' \in \mathcal{J}^\ell_{\text{reg},K}(\tau, \omega)$. Now $\mathcal{J}^\ell_{\text{reg},K}(\tau, \omega)$ is open and dense in $\mathcal{J}^\ell(D; M, \omega)$ and $\mathcal{J}(D; M, \omega)$ is dense in $\mathcal{J}^\ell(D; M, \omega)$, so from Lemma C.1.2 it follows that

$$\mathcal{J}^\ell_{\text{reg},K}(\tau, \omega) \cap \mathcal{J}(D; M, \omega) \subset \mathcal{J}^\ell(D; M, \omega)$$

is dense w.r.t. the $\mathcal{C}^\ell$-topology.

So we can approximate J' by J” $\in \mathcal{J}^\ell_{\text{reg},K}(\tau, \omega)$ fl $\mathcal{J}(D; M, \omega)$ in the $\mathcal{C}^\ell$-topology. This shows that

$$\mathcal{J}_{\text{reg},K}(\tau, \omega) \subset \mathcal{J}(D; M, \omega)$$

is dense w.r.t. the $\mathcal{C}^\ell$-topology.

The above argument holds for any ℓ, so given a $J \in \mathcal{J}(D; M, \omega)$ choose a sequence $J_\nu \in \mathcal{J}_{\text{reg},K}(\tau, \omega)$ such that

$$\|J - J_\nu\|_{\mathcal{C}^\nu} \leq 2^{-\nu}.$$

Thus J_ν converges to J in the $\mathcal{C}^\infty$-topology. This proves that the set $\mathcal{J}_{\text{reg},K}(\tau, \omega)$ is of second category and since $\mathcal{J}_{\text{reg}}(\tau, \omega)$ is a countable intersection of open and dense sets it must be dense itself.

Lemma C.1.2 *Let X be a topological space. Let A* **c** *X be open and dense and let B* **c** *X be dense. Then $A \cap B$ is dense in X.*

Proof: Given $p \in$ X and an open neighbourhood V_p of p. Since A is dense in X, there exists an $a \in A$ such that $a \in V_p$. Since A is open we can find an open neighbourhood V_a of a that lies entirely in $V_p \cap A$. Since B is dense in X we can find ab $\in V_a$ and hence $b \in$ *V,,*. Therefore $b \in B \cap A$. □

Bibliography

[AH] M. Akveld & J. Hulshof, Travelling wave solutions of a fourth order semi-linear diffusion equation, *Appl. Math. Letters.* **11(3)** (1998), 115–120.

[AS] M. Akveld & D. Salamon, Loops of Lagrangian submanifolds and pseudoholomorphic discs, accepted by *GAFA*, May 2000.

[AW1] D.C. Aronson & H.F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation,* J.A. Goldstein, ed., Partial Differential Equations and Related Topics, Lecture Notes in Math. 446, Springer-Verlag, Berlin, 1975.

[AW2] D.G. Aronson & H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math. 30* (1978), 33-76.

[B] M. Bhupal, A generalisation of the Morse inequalities, to appear in *Australian Mathematical Society (Series A) (2000).*

[BP] M. Bialy & L. Polterovich, Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms, *Duke. Math. J. 76* (1994), 273-292.

[Br] G.E. Bredon, *Topology und geometry*, Springer-Verlag, 1997.

[BD] T. Bröcker & T. tom Dieck, *Representation of compact Lie groups*, Springer-Verlag, 1985.

[BT] R. Bott & W.T. Tu, *Differential forms in algebraic topology*, Springer-Verlag, 1995.

[Ch] M. Chaperon, On generating families, in *The Floer memorial,* Birkhäuser, 1995.

[Cl Y. Chekanov, Invariant Finsler metrics on the space of Lagrangian embeddings, *Math.Zeitschrift* **234(3)** (2000), 605–619.

[CGS] K. Cieliebak, R. Gaio, & D.A. Salamon, J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions, Preprint, ETH-Zürich, September 1999.

[CZ] C.C. Conley & E. Zehnder, The Birkhoff Lewis fixed point theorem and a conjecture of V.I. Arnold, *Invent.Math. 73* (1983), 33–49.

[D] SK. Donaldson, Moment maps and diffeomorphisms, Sir Michael Atiyah: a great mathematician of the twentieth century. *Asian J. Math. 3* (1999), no. 1, 1–15.

[EF] Y. Eliashberg & M. Fraser, Classification of topologically trivial Legendrian knots, *CRM Proc.Lecture Notes* **15** (1998). 17-5 1.

[F] U. Frauenfelder, Gromov *convergence of pseudoholomorphic discs,* Diplomarbeit, ETH-Zürich, February 2000.

[Fe] N. Fenichel, Persistence and smoothness of invariant manifolds for *flows, Indiana Univ. Math. Journal* **21** (1971), 193–226.

[FHS] A. Floer, H. Hofer, & D.A. Salamon, Transversality in elliptic Morse theory for the symplectic action, *Duke Math. J. 80* (1995), 25 1–292.

[FM] P.C. Fife & J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling wave solutions, *Arch. Rational Mech. und Anal. 65* (1977), 335-362.

[FT] D. Fuchs & S. Tabachnikov, Invariants of Legendrian and transverse knotsa in the standard contact space, *Topology 36* (1997) , 1025-1053.

[Gl] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math. 82* (1985), 307–347.

[G2] M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, in *Functional Analysis on the Eve of the 21st Century,* Birkhäuser, 1996.

[GJ] R. Gardner & C.R.K.T. Jones, Travelling waves of a perturbed diffusion equation arising in a phase field model, *Indiana Univ. Math.* Journal 38 (1989), 1197–1222.

[GLS] V. Guillemin, E. Lerman & S. Sternberg *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, 1996.

[Ha] A. Haefliger, Plongements différentiables de variétés dans variétés, *Comment.Math.Helv. 36* (1961), 47-82.

[H] H. Hofer, On the topological properties of symplectic maps, *Proc. Royal Soc. Edinburgh* **115A** (1990), 25-38.

[HS] H. Hofer & D.A. Salamon, Gromov compactness and stable maps, in preparation.

[J] C.K.R.T. Jones, *Geometric Singular Perturbation Theory* in Dynamical Systems, eds. R. Johnson, Springer-Verlag, Berlin Heidelberg, 1995.

[Jou1] T. Joukovskaia, Metamorphoses of the Chaperon-Sikorav weak solutions of Hamilton-Jacobi equations, Topology *3, J. Math. Sci. 82* (1996), no. 5, 3737–3746.

[Jou2] T. Joukovskaia,Singularities of a minimax function (Russian) *Uspekhi Mat. Nauk* 51 (1996), no. 2(308), 159–160.

[K] M. Kontsevich, Enumeration of rational curves via torus actions, in "The moduli space of curves (Texel Island, 1994)", 335-368, *Progr. Math,* 129, Birkhauser, 1995.

[L] W.B.R. Lickorish, *An introduction to knot theory,* Graduate Texts in Mathematics, Springer, 1997.

[LMP] F. Lalonde, D. McDuff, & L. Polterovich, Topological rigidity of Hamiltonian loops and quantum homology, *Znvent. Math.* 135 (1999), no. 2, 369–385.

[MS1] D. McDuff & D.A. Salamon, *Introduction to symplectic topology*, Oxford mathematical Monographs, Oxford University Press, 1998.

[MS2] D. McDuff & D.A. Salamon, *J-holomorphic curves and quarzturn cohomology*, University Lecture Series, Vol. 6, American Mathematical Society, Providence, RI, 1994.

[Mi] D. Milinković, Geodesics on the space of Lagrangian submanifolds in cotangent bundles, Preprint, UCI, March 2000.

[M] J. Milnor, *Topology from the Differential Viewpoint*, University of Virginia Press, 1965.

[Pl] L. Polterovich, Gromov's K-area and symplectic rigidity, *Geom. Funct. Anal. 6* (1996), 726–739.

[P2] L. Polterovich, Symplectic aspects of the first eigenvalue, *J. Reine Angew. Math. 502* (1998), 1–17.

[P3] L. Polterovich, Precise measurements in symplectic topology, in *"Proceedings of the 2nd European Congress of Mathematics"*, Vol. 2, Birkhäuser, 1998, pp, 159–166.

[P4] L. Polterovich, Geometry on the group of Hamiltonian diffeomorphisms, *Doc. Math. J. DMV*, Extra Volume ICM 1998, Vol. 11, 401-410.

[P5] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, to appear in Birkhäuser Verlag.

[PT1] L.A. Peletier & W.C. Troy, Spatial patterns described by the Extended Fisher-Kolmogorov (EFK) Equation: Kinks, *Diff. and Int. Equ., 8* (1995). 1279–1304.

[PT2] L.A. Peletier & W.C. Troy, Spatial patterns described by the Extended Fisher-Kolmogorov (EFK) Equation: Periodic solutions, to appear in SIAM Journal of Math. Anal.

[PT3] L.A. Peletier & W.C. Troy, Chaotic spatial patterns described by the Extended Fisher-Kolmogorov equation, *Journal of Diff. Equ.* **129** (1996), 458-508.

[PT4] L.A. Peletier & W.C. Troy, A topological shooting method and the existence of kinks of the Extended Fisher-Kolmogorov equation, *Topological Methods in Nonlinear Analysis* 6 (1995), 331-355.

[PTV] L.A. Peletier, W.C. Troy & R.C.A.M.van der Vorst, Stationary solutions of a fourth order non-linear diffusion equation, *Diff. Equ.* **31** (1995) (Russian), 327-338.

[RS1] J. Robbin & D.A. Salamon, The Maslov index for paths, *Topology* 32 (1993), 827-844.

[RS2] J. Robbin & D.A. Salamon, Immersed curves in the Riemann sphere, in preparation.

[RS] C.P. Rourke & B.J. Sanderson, *Introduction to peicewise-linear topolgy,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1972.

[S] D.A. Salamon, *Spin Geometry and Seiberg-Witten invariants*, to appear in Birkhäuser Verlag.

[S1] P. Seidel, *Floer homology and the symplectic isotopy Problem,* PhD thesis, Oxford, 1997.

[S2] P. Seidel, π_1 of symplectic automorphism groups and invertibles in quantum homology rings, Geom. *Funct. Anal.* 7 (1997), 1046–1095.

[S3] P. Seidel, On the group of symplectic automorphisms of $\mathbb{C}P^m \times \mathbb{C}P^n$, *Amer. Math. Soc. Transl. Ser.* 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[S4] P. Seidel, On Hamiltonian loops, unpublished notes.

[Si] J.-C. Sikorav, Problèmes d'intersection et de points fixes en géométrie hamiltonienne, *Comment.Math.Helvetici 62* (1987), 62-73.

[Th] D. Théret, *Utilisation des fonctions génératrices en géométrie symplectique globale,* PhD thesis, Université Denis Diderot (Paris 7), 1995.

[T] R. Thom, Quelques propriétés globales des variétés différentiables, *Comment.Math.Helvetici* 28 (1954), 17-86.

[Tr] L. Traynor, Legendrian circular helix links, *Math.Proc.Cambridge Philos.Soc.*, 122 1997, 301–314.

[V] C. Viterbo, Symplectic topology as the geometry of generating functions, *Math.Ann.* 292 (1992). 685–710.

[W] A. Weinstein, Connections of Berry and Hannay type for moving Lagrangian submanifolds, *Advances in Mathematics* 82 (1990), 133-159.

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