Diss. ETH No. 14710

Ballistic Random Motions in Random Media

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY
ZURICH

for the degree of
Doctor of Mathematics

presented by
LIAN SHEN
Dipl. Phys. ETH-Zürich
born Oct. 13, 1971
citizen of P. R. China

accepted on the recommendation of
Prof. Dr. A.-S. Sznitman, examiner
Prof. Dr. E. Bolthausen, co-examiner

2002

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Acknowledgements

First of all, I would like to thank my supervisor Prof. A.-S. Sznitman for guiding me into this wonderful field, his constant support and his wise suggestions during the years.

I would also like to thank Prof. E. Bolthausen for being the co-reference of this thesis, and for his support during the sabbatical of Prof. Sznitman.

Next, I would like to thank my former colleague and friend Martin Zerner for his advice and help. I thank Michael Loulakis for the joyful year we share in ETH-Zürich; Ms. Freweini Tewelde for her practical help in D-MATH. Definitively, my colleagues and friends in D-MATH at ETH-Zürich deserve my thanks for the enjoyable time we spent together: Martin Baumann, Elisabeth Bourqui, Thierry Delmotte, Stefano Demarta, Barbara Gentz, Laurent Goergen, Anderea Höing, Marco Jost, Alex Juri, Michael Kupper, Franz Merkl, Stéfphane Moine, Marianne Pfister, Tom Schmitz, Uwe Schmock, Michael Studer, Mario Wüthrich, and many more, whose names I might forget to include here. The wonderful memory will stay for ever.

For my friends outside mathematics (Dongliang Chang, Chuang Cen, Jianming Chen, Matthias Herrmannstorfer, Harald Krueger, Changqing Li, Olivier Sarbach, Rosemarie Wettstein and Hai Zhang) I want to say: "thank you for making my life in Switzerland more attractive".

Finally, I want to thank my parents and fiancee for their love, understanding and moral support. This thesis is dedicated to them.

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Abstract

We consider two different types of random motions in random media (RMRM), which are Markov processes when the random medium is fixed. We study their asymptotic properties, esp. the strong law of large numbers and a functional central limit theorem.

The first type of RMRM is the discrete random walks in random bond environment on $\mathbb{Z}^d$; i.e. the random environment is realized through i.i.d. random variables on the nearest neighbor bonds of $\mathbb{Z}^d$, and the random walks are Markov chains on $\mathbb{Z}^d$ under fixed environment. We show that for any anisotropy strength the strong law of large numbers and functional central limit theorem hold.

The second type of RMRM is the continuous diffusion in random environment on $\mathbb{R}^d$, which is the distribution of the solution for some stochastic differential equations. In this case, the random environment is incorporated into the drift term and the diffusion matrix. We provide a sufficient condition, under which the strong law of large numbers and a functional central limit theorem hold. We also apply these results to an explicit class of gradient-type anisotropic diffusion in random environment, and show that the sufficient condition is fulfilled for this class of examples.

For both models, we apply a strategy of introducing certain regeneration times, which was developed by Sznitman and Zerner in their investigation of random walks in i.i.d. random environment. These regeneration times provide us a Markovian structure in the first model, and a renewal structure in the continuous diffusion model, which are the key tools for the investigation in this thesis.

Zusammenfassung

Wir betrachten zwei verschiedene Modelle für zufällige Bewegungen in zufälligen Umgebungen. Für eine feste Umgebung ist die Bewegung ein Markov Prozess. Wir studieren die asymptotischen Eigenschaften von diesen Prozessen, insbesondere beweisen wir das starke Gesetz der grossen Zahlen und ein funktionalen zentraler Grenzwertsatz.

Das erste Modell ist die diskrete Irrfahrt in zufälliger Kanten-Umgebung auf $\mathbb{Z}^d$, d.h. die zufällige Umgebung wird beschrieben durch unabhängige identisch verteilte Zufallsvariablen auf den Kanten in $\mathbb{Z}^d$. Wir zeigen das starke Gesetz der grossen Zahlen und einen funtionalen zentraler Grenzwertsatz für dieses Modell.

Das zweite Modell ist die stetige Diffusion in zufälliger Umgebung auf $\mathbb{R}^d$, welche Lösung von gewissen stochastischen Differentialgleichungen ist. In diesem Fall, ist die zufällige Umgebung durch den Driftterm und die Diffusionsmatrix repräsentiert. Wir geben eine hinreichende Bedingung, so dass das starkes Gesetz der grossen Zahlen und ein funktionaler zentraler Grenzwertsatz gelten. Wir wenden diese Resultate auf eine explizite Klasse von anisotropischen Diffusionen in zufälliger Umgebung an. Wir zeigen, dass die hinreichende Bedingung erfüllt ist.

Für beide Modelle führen wir Erneuerungszeiten ein. Diese Strategie wurde durch Sznitman und Zerner in ihrer Untersuchung von Irrfahrten in einer zufällign i.i.d. Umgebung eingeführt. Diese Erneuerungszeiten liefern eine Markov Struktur in dem ersten Modell und eine Erneuerungsstruktur in dem stetigen Modell.

Introduction

It has always been very difficult for me to explain to my family what I am working on and why this should be interesting for other people. I still owe them an answer. Here is it. Although it is only an attempt, I still hope they would find in the past four years I have done something useful and interesting; this might justify the waiting for me to pursue this degree. I also hope that my fiancee may have now more understanding for my decision of staying in Zurich to finish this thesis, instead of joining her.

Motivation

In the nature, many phenomena can be described through random motions, like diffusion of gas particles and conduction of heat or electrons in *homogenous* material. Classical models of random motions, e.g. random walks and Brownian motion, are used successfully describing these transportation phenomena in homogenous media.

As we know, there can never be absolutely pure material in the nature. So, random walks or Brownian motion can only be considered as the first level approximation. To provide more precise description of naturally occurred events, people come to the idea of simulating the impurity of the material by *random media*. This is where *random motions in random media* comes from.

To take account of the random media in our investigation, we must average over both the random motion *and* the random media. Thus, we confront with the interaction between these two randomness. The interplay between these two makes the investigation much more complex.

In the models of this thesis the random media do not depend on time, and the random motion under a fixed random medium is a Markov process, i.e. given the present location the rest of the past is irrelevant for predicting the next move. For example, consider a random walker, his next move only depends on his current location, not on where he has been before. In our models, the influence of the random media is expressed through the jump probability in each move or transition density.

As the title says, in this thesis we study the influence of random media on the random motion, esp. we are interested in the asymptotic properties of the random motion.

The first models for random motions in random media can be dated back to Fatt [12], studying random conductivity in disordered media physics; Chernov [5] and Temkin [47], in their investigation of DNA replication. Further applications gain more and more importance in biology, chemistry and physics. Esp., in the past thirty years, random motions in random media have been an active research field within probability and statistical physics, see the review books Bolthausen-Sznitman [3], Havlin-Bunde [16] and Hughes [17]. Many interesting effects have been discovered and new questions been posed, we are getting more and more insight in this fields. But there are still many important questions unanswered. I hope this thesis might contribute to the understanding of the RMRM. This might be the second motivation for me.

Let me also point out, despite of the fact that the title of this thesis contains the word "ballistic", this thesis does *not* have any relation to weapons or guns, although we have been asked repeatedly if we were working for the Swiss army. It is just a nickname of certain asymptotic property that we will explain shortly.

Models and Results

In this thesis, we are studying two different types of random motions in random media. The first one is called the "random walks in random bond environment". The other model is the "diffusions in random environment".

First, let us describe the general frame of our work in words, before going into more details. In both types, the random medium is realized

on some abstract probability space $(\Omega, \mathscr{A}, \mathbb{P})$. As we mentioned above, the random motion under a given random environment $\omega \in \Omega$ is then a Markov process, $(X_t)_{t \geq 0}$. The law of this Markov process starting in x is sometimes called the "quenched law", we denote it by P_x^ω. Our goal is to study the asymptotic behavior of the random motion averaged over the randomness of the motion *and* the environment, i.e. w.r.t. the product measure $\mathsf{P}_0 := \mathbb{P} \times \mathsf{P}_0^\omega$ (it is called the annealed measure).

Let us now explain what the word *ballistic* means. If X_t/t converges P_0-almost surely to a deterministic non-vanishing velocity $v \in \mathbb{R}^d$, as $t \to \infty$, we say that the random motion in random media is ballistic.

The key technique we use in this thesis is embedding certain regeneration times into the random motion, which we will describe in more detail later when we describe the models separately. It was introduced by Sznitman and Zerner in [46] in their investigation of random walks in i.i.d. random environment on $\mathbb{Z}^d$.

Previous to this technique, most of the progress made in the investigation of the random motions in random media is related to a method called "process of the environment viewed from the particle". Basically, one looks at the process of the random environment carried by the random walker, and observes that this process is also a Markov process, then one tries to apply ergodic theories to this process, see Kipnis-Varadhan [21], Kozlov [22], Molchanov [28], Olla [31], Papanicolaou-Varadhan [33]. One shortcoming of this method is that in order to apply it successfully, one need to find an invariant measure for this environment process, which is absolutely continuous w.r.t. the static distribution $\mathbb{P}$. And this is not always very easy and obvious.

Random Walks in Random Bond Environment

In this model, the random environment is given through i.i.d. random variables $\big(\omega(b)\big)_{b \in \mathbb{B}^d}$, where $\mathbb{B}^d$ denotes the set of the nearest neighbor bonds on $\mathbb{Z}^d$, see also Figure 0.1. We also assume that $\omega(b)$, $b \in \mathbb{B}^d$, take values in some compact subset $\mathbb{I} \subset (0, \infty)$ and have common distribution μ. More formally, a random environment $\omega = \big(\omega(b)\big)_{b \in \mathbb{B}^d}$ is an element of the product space $\Omega = \mathbb{I}^{\mathbb{B}^d}$ endowed with the canonical product measure $\mathbb{P} = \mu^{\mathbb{B}^d}$ and the canonical product Borel-σ-algebra $\mathscr{A} = \big(\mathscr{B}(\mathbb{I})\big)^{\mathbb{B}^d}$.
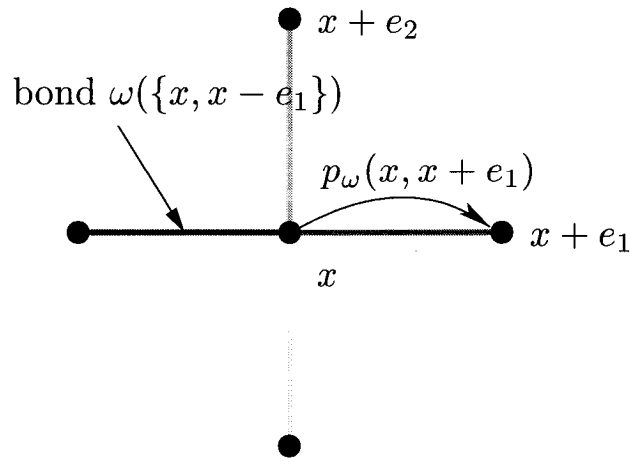
Figure 0.1: *Bond Model*

Given a realization ω of the environment, the random motion in ω is a Markov chain with transition kernel $p_\omega(x, x + e)$, that means

$$(0.1) \qquad \begin{cases} \mathsf{P}_x^\omega[X_{n+1} = X_n + e | X_0, \cdots, X_n] \overset{\mathsf{P}_x^\omega\text{-a.s.}}{=} p_\omega(X_n, X_n + e), \\ \mathsf{P}_x^\omega[X_0 = x] = 1. \end{cases}$$

with $\sum_{|e|=1} p_\omega(x, x + e) = 1$, ($e$ denotes unit vectors in $\mathbb{Z}^d$), see also Figure 0.1.

The typical example we have in mind is the following

$$(0.2) \qquad p_\omega(x, x + e) = \frac{\omega(\{x, x + e\})\, e^{\lambda \ell . e}}{\sum_{|e'|=1} \omega(\{x, x + e'\})\, e^{\lambda \ell . e'}},$$

where $\lambda > 0$ is some given constant (it is an anisotropy strength). We will provide more general setting than (0.2).

Let us mention that for the type of models we consider here, the questions of the existence of an effective, non-vanishing velocity was asked by Lebowitz and Rost, see [27], in their investigation of the Einstein relation.

We address this question in one of our main results for this model:

$$(0.3) \qquad \mathsf{P}_0\text{-a.s.} \quad \frac{X_n}{n} \to v, \quad \text{as } n \to \infty,$$

where v is a deterministic *non-vanishing* velocity, see Theorem 1.5.1. That means the random walk is *ballistic*. Moreover, we show in Theorem

1.5.3 that the process $B^n_\cdot$,

$$B^n_\cdot := \frac{X_{[\cdot n]} - [\cdot n]v}{\sqrt{n}},$$

($[t]$ denotes the integer part of $t \geq 0$) converges in law under the annealed measure P_0 to a d-dimensional Brownian motion with non-degenerate covariance matrix given in (1.5.16), as $n \to \infty$.

One special aspect of our work is that our results hold for *arbitrarily small* anisotropy strength λ (for instance, in (0.2) as soon as $\lambda > 0$, the walk is ballistic). We do not need any Kalikow-like condition as for the i.i.d. random walks in random environment, see Kalikow [20], Sznitman-Zerner [46].

A degenerated case of this example is discussed in the physics literature. It corresponds to the anisotropic random walk on the infinite percolation cluster, cf. page 136 – 146 in Bunde-Havlin, [16]. In this case the random variable $\omega(b)$ only takes the values 0 or 1.

The strategy employed to derive these two theorems is to construct an embedded Markov chain structure under the annealed measure P_0, which has a "small state space", cf. Corollary 1.3.6. The times τ_k, $k \geq 1$, defined in (1.3.12) and (1.3.26), play a central role here. In essence τ_k is the k-th time, when the random walker comes to a new maximum in the direction ℓ and then never comes back below this level. The true definition is in fact more sophisticated, cf. Remark 1.3.2. The random variables consisting of $\tau_{k+1} - \tau_k$, $X_{\tau_{k+1}} - X_{\tau_k}$ and the value of some bonds connected to X_{τ_k}, $k \geq 1$, build a Markov chain, as shown in Corollary 1.3.6. In Theorem 1.3.8 the ergodicity of this Markov chain is shown. In Section 1.5 we show the strong law of large numbers and central limit theorem for the model under consideration. The limit velocity and covariance matrix are expressed through τ_1 and X_{τ_1}, see Theorem 1.5.1, 1.5.3. Let us mention that the above strategy is in the same spirit as the renewal structure attached to certain regeneration times τ_k for i.i.d. random walks in random environment model in Sznitman-Zerner [46]. But unlike what happens for the i.i.d. random walks in random environment model, the times τ_k in our model do not yield a renewal structure, but rather lead to a Markov structure with a small state space, see Theorem 1.3.3 and Corollary 1.3.6. This comes from the fact that the transition kernel $p_\omega(x, x+e)$ depends on all bonds connected to x, therefore the jump probabilities $p_\omega(x, x+e)$ and $p_\omega(x+e, x+e+e')$

are not independent under $\mathbb{P}$ (they depend both on the value of the bond $\omega(\{x, x + e\})$.

Diffusions in Random Environment

This is a continuous time model of random motions in random media.

To describe the random environment, we consider some probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with a group of measure preserving transformations, $(t_x)_{x \in \mathbb{R}^d}$, acting ergodically on Ω, we refer the beginning of Section 2.2 for more precise description of the space Ω. Let $b(\cdot) : \Omega \to \mathbb{R}^d$ and $\sigma(\cdot) : \Omega \to \mathbb{R}^{d \times d}$ now be two bounded measurable functions, and we write $b(x, \omega) = b(t_x(\omega))$ and $\sigma(x, \omega) = \sigma(t_x(\omega))$. We assume that $b(\cdot, \omega)$ and $\sigma(\cdot, \omega)$ are Lipschitz continuous for all $\omega \in \Omega$. We also assume that $\sigma\sigma^t(x, \omega)$ is uniformly elliptic, i.e. for some $\nu > 0$, $\frac{1}{\nu}|y|^2 \le |\sigma^t(x, \omega) y|^2 \le \nu |y|^2$ holds for all $\omega \in \Omega$, $x, y \in \mathbb{R}^d$.

Further, we assume an independence condition, which we call R-separation. To explain this, let us denote with $\mathscr{H}_F$ the σ-algebra generated by $b(x, \omega)$, $\sigma(x, \omega)$, for $x \in F$. We assume that there exists a constant $R > 0$ such that for all Borel subsets F, F' in $\mathbb{R}^d$ such that if the Euclidean distance between F and F' is bigger than R, $\mathscr{H}_F$ and $\mathscr{H}_{F'}$ are $\mathbb{P}$-independent.
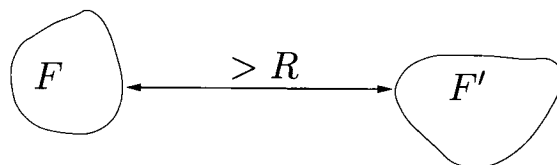
Figure 0.2: *R-separation*

Let us mention two examples of such random vectors $b(x, \omega)$ and random matrices $\sigma(x, \omega)$ respectively. The convolution of a Poissonian point process with a Lipschitz continuous vector-valued, or matrix-valued, function supported in a ball of radius $R/2$ yields after truncation a possible example, cf. Sznitman [42], page 185. Another possible example is to use the Gaussian field, described in [1], section 1.6 and 2.3. After convolution and truncation, we get another example. (The formula (2.3.4) on page 28 in Adler [1] need be changed to $X(x) = \int g(x - \lambda) \, dZ(\lambda)$,

where $g(\lambda)$ is some vector- or matrix-valued Lipschitz continuous function, compactly supported in a ball of radius $R/2$.)

The diffusion in the random environment ω is the law P_x^ω (it is called the quenched law) of the solution of the stochastic differential equation

$$(0.4) \qquad \begin{cases} dX_t(\omega) = b(X_t, \omega)\, dt + \sigma(X_t, \omega)\, dB_t\,, \\[2mm] X_0 = x\,, \qquad x \in \mathbb{R}^d,\ \omega \in \Omega\,, \end{cases}$$

with d-dimensional Brownian motion $(B_t)_{t \geq 0}$ and $b(x, \omega)$, $\sigma(x, \omega)$ described above.

Again, we are interested in the asymptotic behavior of the diffusion under the annealed measure $\mathsf{P}_x \overset{\text{def}}{=} \mathbb{P} \times \mathsf{P}_x^\omega$.

The aim of Chapter 2 is to provide a sufficient condition, see (2.3.1-i), under which the strong law of large numbers holds, that is:

$$(0.5) \qquad \mathsf{P}_0\text{-a.s.} \quad \frac{X_t}{t} \to v, \quad \text{as } t \to \infty,$$

where v is a deterministic and *non-vanishing* velocity, cf. Theorem 2.3.2. Further, we show in Theorem 2.3.3 that the slightly stronger condition (2.3.1-ii) guarantees a functional central limit theorem, namely as s tends to infinity, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued process

$$B^s_\cdot = \frac{X_{s\cdot} - sv\cdot}{\sqrt{s}},$$

converges under the annealed measure P_0 in law to a non-degenerate d-dimensional Brownian motion with covariance matrix given in (2.3.12).

The derivation of this sufficient condition (2.3.1) is based on the strategy of constructing some regeneration times τ_k, $k \geq 1$, similar to those defined in Sznitman-Zerner [46], and providing a renewal structure, cf. Theorem 2.2.5. The sufficient condition is then expressed in terms of the transience of the diffusion $X_\cdot$ in some direction ℓ and the finiteness of the first (or the second) moment of τ_1 conditioned on no-backtracking, cf. (2.3.1). There are several ways to construct these regeneration times τ_k. In the spirit of [6], [48], we introduce additional Bernoulli variables. In essence, the first regeneration time τ_1 is the first integer time, at which the diffusion process reaches a local maximum in a given direction $\ell \in S^{d-1}$, the auxiliary Bernoulli variable takes value 1, and from then

on the process never backtracks. The regeneration times τ_k, $k \geq 2$, are then obtained by iteration of this procedure. For the true definition, we refer to (2.2.12) – (2.2.17), (2.2.22). In our construction we take special advantage of the diffusion structure to couple the Bernoulli variables with the diffusion process, cf. Theorem 2.2.5, the resulting renewal structure gives us a good control over the trajectory of the diffusion, see Remark 2.2.6, and we also have a convenient Markov structure, cf. Corollary 2.2.2. This provides a key tool for studying asymptotic behavior of the diffusion in a random environment.

As an illustration of our results, we study a class of gradient-type diffusion processes, for which $\sigma = \mathbb{1}$ and $b(x,\omega) = \nabla V(x,\omega)$, where $V(\cdot,\omega)$ has uniformly bounded and Lipschitz continuous derivatives, further we assume there exist a unit vector $\ell \in \mathbb{R}^d$, $A, B > 0$ and $\lambda > 0$ such that

$$(0.6) \qquad Ae^{2\lambda\ell\cdot x} \leq e^{2V(x,\omega)} \leq Be^{2\lambda\ell\cdot x}, \text{ for all } x \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

In the case where $\lambda = 0$, the diffusive behavior of the process has been extensively investigated, cf. [7], [31], [32], however we do not know of any result when $\lambda > 0$. We show in Section 2.4 that when $\lambda > 0$, (no matter how small λ is) the sufficient condition (2.3.1) is fulfilled (in fact, we prove the much stronger exponential estimates under $\hat{\mathsf{P}}_x^\omega$, cf. Theorem 2.4.9 and Corollary 2.4.10, which can also be used to deduce certain large deviation controls, cf. [43], [44]). As a result, the above mentioned law of large numbers and functional central limit theorem hold, see Theorem 2.4.11. The class under consideration includes especially the case where $b(x,\omega) = \nabla\tilde{V}(\omega,x) + \lambda\ell$, for some bounded $\tilde{V} \in C^1(\mathbb{R}^d, \mathbb{R})$, with bounded and Lipschitz continuous derivatives. Let us mention that this situation is a generalization of the discrete bond model studied in Chapter 1.

Organization of the Thesis

The main part of this thesis consists two separate articles. Chapter 1 corresponds to the first article, which is to appear in the *Annals of Applied Probability*. It deals with the discrete bond model. In a setting more general than (0.2) we show the ballistic behavior and the central limit theorem.

Chapter 2 is an article submitted for publication. In this chapter, we investigate the continuous time diffusions in random environment. We

provide a sufficient condition for the ballistic behavior and central limit theorem. We also apply our results to the gradient-type models described around (0.6).

Chapter 1

Asymptotic Properties of Certain Anisotropic Walks in Random Media

ABSTRACT

We discuss a class of anisotropic random walks in a random media on $\mathbb{Z}^d$, $d \geq 1$, which have reversible transition kernels when the environment is fixed. The aim is to derive a strong law of large numbers and a functional central limit theorem for this class of models. The technique of the environment viewed from the particle does not seem to apply well in this setting. Our approach is based on the technique of introducing certain times similar to the regeneration times in the work concerning random walks in i.i.d. random environment by Sznitman-Zerner [46]. With the help of these times we are able to construct an ergodic Markov structure.

1.1   Introduction

There are many works investigating random motions in random media. The point of view of the "environment viewed from the particle" has played an important role in the progresses made so far, cf. De

Masi-Ferrari-Goldstein-Wick [7], Kozlov [22], Olla [31] and Papanico-laou-Varadhan [33], and also the lectures [3] of Sznitman. Lawler showed in [26] the central limit theorem for driftless random walks in random environments by using this technique. This technique has mostly been successful when one can find an explicit invariant measure of the Markov chain of the environment viewed from the particle, which is absolutely continuous with respect to the static distribution of the environment, especially when this invariant measure is reversible.

In this article we want to study a class of anisotropic random walks in random media, which are reversible Markov chains when the environment is fixed, but for which the chain of the environment viewed from the particle has no obvious invariant measure absolutely continuous to the static measure. Paradoxically, we are able to apply a strategy, which has been used in the investigation of a genuinely non-reversible model: the i.i.d. random walks in random environment, cf. Sznitman [43] and Sznitman-Zerner [46]. The principal aim of the present work is to derive a strong law of large numbers with non-vanishing limiting velocity and a functional central limit theorem for the anisotropic random motion in random environment under consideration. Incidentally, let us mention that for the type of models we consider here, the question of the existence of an effective, non-vanishing velocity was asked by Lebowitz and Rost, see [27], in their investigation of the Einstein relation.

Let us describe our model in details. First we denote with $\mathbb{B}^d$ the set of nearest neighbor bonds on $\mathbb{Z}^d$. The random environment is given through i.i.d. non-negative random variables $\omega(b) \in \mathbb{I} \subset (0, \infty)$, $b \in \mathbb{B}^d$, with common distribution μ. Here $\mathbb{I}$ denotes a compact interval of $(0, \infty)$. A random environment $\omega = \big(\omega(b)\big)_{b \in \mathbb{B}^d}$ is an element of the product space $\Omega := \mathbb{I}^{\mathbb{B}^d}$ endowed with the canonical product measure $\mathbb{P} = \mu^{\otimes \mathbb{B}^d}$ and the canonical product σ-algebra $\mathscr{A} = (\mathscr{B}(\mathbb{I}))^{\mathbb{B}^d}$, where $\mathscr{B}(\mathbb{I})$ denotes the σ-algebra of Borel subsets of $\mathbb{I}$.

In our model we have a nearest neighbor jump transition kernel $p_\omega(x, x+e)$, i.e. $\sum_{|e'|=1} p_\omega(x, x+e') = 1$, where e' denotes unit vectors in $\mathbb{Z}^d$ and $|\cdot|$ the L^1-norm in $\mathbb{R}^d$. Further we assume that the kernel fulfills the ellipticity condition:

(1.1.1) $p_\omega(x, x+e) \geq \kappa > 0$, for unit vectors $e \in \mathbb{Z}^d, x \in \mathbb{Z}^d, \omega \in \Omega$,

and it is reversible, i.e. there exists a positive measure $(m_\omega(x))_{x \in \mathbb{Z}^d}$

such that

$$(1.1.2) \qquad m_\omega(x)\,p_\omega(x, x+e) = m_\omega(x+e)\,p_\omega(x+e, x)\,,$$

for all $\omega \in \Omega$, $x \in \mathbb{Z}^d$, $|e| = 1$. We also assume that $p_\omega(x, x+e)$ has the form:

$$(1.1.3) \qquad p_\omega(x, x+e) = f\left(\left(\omega(\{x, x+e'\})\right)_{|e'|=1}, e\right)\,,$$

for all $x \in \mathbb{Z}^d$ and unit vectors e. This means that the transition kernel $p_\omega(x, x+e)$ depends only on the value of ω for bonds connected to x, in the same way for all $x \in \mathbb{Z}^d$. This is a translation invariance assumption on the jump mechanism.

In addition, we assume there exists a nearest neighbor random walk on $\mathbb{Z}^d$ with jumps distributed according to the law $(q(e))_{|e|=1, e \in \mathbb{Z}^d}$, $q(e) \neq 0$, for all $|e| = 1$, such that with

$$(1.1.4) \qquad \begin{aligned} \lambda &:= \frac{1}{2}\left\|\sum_e (e \log q(e))\right\| > 0\,, \\ \ell &:= \frac{1}{2\lambda}\sum_e e \log q(e) \in S^{d-1}\,, \end{aligned}$$

($\|\cdot\|$ denoting the L^2-norm in $\mathbb{R}^d$) there exist constants $0 < A < B$, such that

$$(1.1.5) \qquad Ae^{2\lambda\ell.x} \leq m_\omega(x) \leq Be^{2\lambda\ell.x}\,, \quad \text{for all } \omega \in \Omega, x \in \mathbb{Z}^d\,,$$

where $x.y$ always denotes the standard scalar product of $x, y \in \mathbb{R}^d$ throughout this article.

For instance, if we choose for given $\lambda > 0$ and $\ell \in S^{d-1}$

$$(1.1.6) \qquad p_\omega(x, x+e) = \frac{\omega(\{x, x+e\})\,e^{\lambda\ell.e}}{\sum_{|e'|=1} \omega(\{x, x+e'\})\,e^{\lambda\ell.e'}}\,,$$

then the conditions (1.1.1), (1.1.2), (1.1.3) and (1.1.5) hold for suitable choices of κ, A and B, provided $q(e) = \frac{e^{\lambda\ell.e}}{\sum_{e'} e^{\lambda\ell.e'}}$ and $m_\omega(x) = e^{2\lambda\ell.x}\sum_e \omega(\{x, x+e\})\,e^{\lambda\ell.e}/\sum_{e'} e^{\lambda\ell.e'}$, (the last denominator is simply a matter of normalization).

Actually, (1.1.6) is a special case of a transition probability with the form:

$$(1.1.7) \qquad p_\omega(x, x+e) = \frac{\omega(\{x, x+e\})\,q(e)}{\sum_{|e'|=1} \omega(\{x, x+e'\})\,q(e')}\,,$$

and (1.1.7) fulfills all the conditions (1.1.1) – (1.1.5) for suitable choices of κ, A, B, the reversible measure for (1.1.7) being now:

$$m_\omega(x) = e^{2\lambda\ell.x} \sum_e \omega(\{x, x+e\})\, q(e)\,,$$

with λ and ℓ from (1.1.4).

With these assumptions over p_ω, the random walk in the random environment ω is the Markov chain $(X_n)_{n \geq 0}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$, with state space $\mathbb{Z}^d$ and "quenched law" P_x^ω, for $x \in \mathbb{Z}^d$:

$$(1.1.8) \qquad \begin{cases} \mathsf{P}_x^\omega[X_{n+1} = X_n + e | X_0, \cdots, X_n] \\ \qquad\qquad\qquad\quad \overset{\mathsf{P}_x^\omega\text{-a.s.}}{=} p_\omega(X_n, X_n + e)\,, \\ \mathsf{P}_x^\omega[X_0 = x] = 1\,, \end{cases}$$

where e denotes unit vectors in $\mathbb{Z}^d$. The "annealed law" P_x is then defined as the semi-direct product on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$:

$$(1.1.9) \qquad\qquad \mathsf{P}_x := \mathbb{P} \times \mathsf{P}_x^\omega\,, \text{ with } x \in \mathbb{Z}^d\,.$$

A degenerate case of the above model is discussed in the physics literature. It corresponds to the anisotropic random walk on the infinite percolation cluster, cf. page 136 – 146, in Bunde-Havlin, [16]. In this case the random variable $\omega(b)$ only takes the values 0 or 1. Although random walks an the infinite cluster have been discussed in the isotropic case, cf. [7], we know of no mathematical reference in the anisotropic situation.

The main goal of this article is to show in Theorem 1.5.1 that

$$\frac{X_n}{n}$$ converges P_0-a.s. to a deterministic non-degenerate velocity v,

Further we prove in Theorem 1.5.3 that the process $B^n_.$,

$$(1.1.10) \qquad\qquad B_t^n = \frac{X_{[tn]} - [tn]v}{\sqrt{n}}, t \geq 0\,,$$

with $[t]$ denoting the integer part of $t \geq 0$, converges in law under the annealed measure P_0 to a d-dimensional Brownian motion with non-degenerate covariance matrix, as $n \to \infty$.

One special aspect of our work is that our results hold for arbitrarily small anisotropy strength λ. We do not need any Kalikow-like condition as for the i.i.d. random walks in random environment, see Kalikow [20], Sznitman-Zerner [46] and Sznitman [43].

The strategy employed to derive these two theorems is to construct an embedded Markov chain structure under the annealed measure P_0, which has a "small state space", cf. Corollary 1.3.6. The times τ_k, $k \geq 1$, defined in (1.3.12) and (1.3.26), play a central role here. In essence τ_k is the k-th time, when the random walker comes to a new maximum in the direction ℓ and then never comes back below this level. The true definition is in fact more sophisticated, cf. Remark 1.3.2. The random variables consisting of $\tau_{k+1} - \tau_k$, $X_{\tau_{k+1}} - X_{\tau_k}$ and the value of some bonds connected to X_{τ_k}, $k \geq 1$, build a Markov chain, as shown in Corollary 1.3.6. In Theorem 1.3.8 the ergodicity of this Markov chain is shown. Let us mention that the above strategy is in the same spirit as the renewal structure attached to certain regeneration times τ_k for i.i.d. random walks in random environment model in Sznitman-Zerner [46] and Sznitman [43]. But unlike what happens for the i.i.d. random walks in random environment model, the times τ_k in our model do not yield a renewal structure, but rather lead to a Markov structure with a small state space, see Theorem 1.3.3 and Corollary 1.3.6. This comes from the fact that the transition kernel $p_\omega(x, x + e)$ depends on all bonds connected to x, therefore the jump probabilities $p_\omega(x, x + e)$ and $p_\omega(x + e, x + e + e')$ are not independent under $\mathbb{P}$.

Let us explain the organization of this article.

In Section 1.2 we make full use of the ellipticity condition (1.1.1) and the reversibility assumption (1.1.2) – (1.1.5) on $(X_n)_{n \geq 0}$ under the quenched law P_x^ω to derive a key estimate in Theorem 1.2.2. In particular with the help of this estimate we prove that the random walk has a strict positive probability of never coming below its initial level, cf. Corollary 1.2.3, and at the end of Section 1.2 we show that P_x^ω-a.s. $(X_n)_{n \geq 0}$ tends to $+\infty$ in the direction ℓ.

In Section 1.3 the times τ_k, $k \geq 1$, are introduced, cf. (1.3.12) and (1.3.26), and the embedded Markov chain $(Y_n)_{n \geq 0}$ under the annealed measure P_0 is constructed in Corollary 1.3.6. Its ergodicity is then discussed in Theorem 1.3.8.

In Section 1.4 we use the key estimate of Theorem 1.2.2 to derive the integrability properties of X_{τ_1} and τ_1. Our main result is presented in

Corollary 1.4.4.

In Section 1.5, with the help of the embedded Markov chain $(Y_n)_{n \geq 0}$ constructed in Section 1.3 and the integrability property of τ_1 proved in Corollary 1.4.4, a strong law of large numbers for $(X_n)_{n \geq 0}$ under the annealed measure P_0 is proved in Theorem 1.5.1. Further we are able to prove a functional central limit theorem for the process $B^n_{\cdot}$ in Theorem 1.5.3.

Let me finally thank my advisor Prof. A.-S. Sznitman for guiding me to this area and his advices during the completion of this work. I would also like to thank my former colleague Martin Zerner for his friendly help and discussion.

1.2 Notations, Reversible Structure and a Key Estimate

In this section we use the ellipticity condition (1.1.1) and the specific reversibility assumption (1.1.2) – (1.1.5) on the quenched Markov chain (1.1.8) to show that the random walk has a positive probability of no-backtracking, cf. Corollary 1.2.3, and derive transience in direction ℓ, cf. Corollary 1.2.4. We first provide a uniform lower bound for the generalized principal Dirichlet eigenvalue in Theorem 1.2.1, which will be useful to prove our key estimate in Theorem 1.2.2.

Before doing so we introduce some further notations needed throughout this article.

In this article, c and c_j, $j \in \mathbb{N}$ always stand for positive constants, which depend only on the quantities $(\kappa, d, A, B, q(\cdot))$, which are introduced in (1.1.1) – (1.1.5).

We denote by $(\theta_n)_{n \geq 0}$ the canonical shift on $(\mathbb{Z}^d)^{\mathbb{N}}$, and by $\mathscr{F}_n$, $n \geq 0$, the canonical filtration of $(X_n)_{n \geq 0}$, i.e. $\mathscr{F}_n = \sigma\{X_0, \cdots, X_n\}$, for $n \geq 0$.

The *exit time* T_U for $U \subset \mathbb{Z}^d$ is given by:

$$(1.2.1) \qquad\qquad T_U = \inf\{n \geq 0 : X_n \notin U\},$$

and for $u \in \mathbb{R}$ we introduce

(1.2.2)
$$\begin{cases} T_u = \inf\{n \geq 0 : \ell.(X_n - X_0) \geq u\}\,, \\ \tilde{T}_u = \inf\{n \geq 0 : \ell.(X_n - X_0) < u\}\,. \end{cases}$$

Further we shall also need the *first backtracking time* defined through

(1.2.3)
$$D = \inf\{n \geq 0 : \ell.X_n < \ell.X_0\}\,.$$

1.2.1   Principal Dirichlet Eigenvalue

Keeping in mind the reversible structure stated in (1.1.2) – (1.1.5), we introduce for each $\omega \in \Omega$ the scalar product on the space of functions $f : \mathbb{Z}^d \to \mathbb{R}$ and its associated norm:

(1.2.4)
$$(f, g)_{m_\omega} := \sum_{x \in \mathbb{Z}^d} m_\omega(x) f(x) g(x)\,, \quad \|f\|_{m_\omega} := \sqrt{(f, f)_{m_\omega}}\,,$$

for $f, g : \mathbb{Z}^d \to \mathbb{R}$. For $\omega \in \Omega$, $U \subset \mathbb{Z}^d$ non-empty, we introduce $\Lambda_\omega(U)$:

(1.2.5) $\Lambda_\omega(U)$

$$:= \inf\left\{ \frac{\mathcal{E}_{m_\omega}(f, f)}{\sum_x m_\omega(x) f(x)^2} : f \neq 0, f|_{U^c} = 0, f \in L^2(m_\omega) \right\}\,,$$

with the Dirichlet form

$$\mathcal{E}_{m_\omega}(f, g)$$
$$= \frac{1}{2} \sum_{x,y} m_\omega(x) p_\omega(x, y) \big(f(x) - f(y)\big)\big(g(x) - g(y)\big)\,, \quad f, g \in L^2(m_\omega)\,,$$

where for $x, y \in \mathbb{Z}^d$ we use the following convention:

$$p_\omega(x, y) := \begin{cases} p_\omega(x, x + e), & \text{for } y = x + e\,, \text{ with } |e| = 1\,, \\ 0, & \text{otherwise;} \end{cases}$$

and by $f|_{U^c}$ we mean the restriction of f to the complement U^c of $U \subset \mathbb{Z}^d$.

With a slight abuse of language, we refer to $\Lambda_\omega(U)$ as the principal Dirichlet eigenvalue attached to U, it is in fact the bottom of the spectrum of the bounded self-adjoint operator $1 - \mathsf{P}_U^\omega$ on $L^2(m_\omega)$, where

$P_{U,\omega}$ is defined through

$$(1.2.6) \qquad \begin{cases} P_{U,\omega} := P_{U,\omega}^1, & \text{provided} \\ (P_{U,\omega}^n f)(x) := E_x^\omega[f(X_n), T_U > n], n \in \mathbb{N}, f : \mathbb{Z}^d \to \mathbb{R}. \end{cases}$$

The next theorem provides a uniform lower bound for $\Lambda_\omega(U)$:

Theorem 1.2.1

$$(1.2.7) \qquad \inf_{U,\omega \in \Omega} \Lambda_\omega(U) = \varepsilon > 0,$$

where U varies over the collection of non-empty subsets of $\mathbb{Z}^d$.

Consequently

$$(1.2.8) \qquad \left\| P_{U,\omega}^n \right\|_{L^2(m_\omega)} \le e^{-n\gamma}, \quad \text{with } \gamma = \log \frac{1}{1-\varepsilon},$$

for all $U \subset \mathbb{Z}^d$ and all $\omega \in \Omega$.

Proof: We begin with the proof of (1.2.7). The ellipticity condition (1.1.1) and assumption (1.1.5) imply that for $x, y \in \mathbb{Z}^d$

$$m_\omega(x)\, p_\omega(x,y) \ge A\,\kappa\,\tilde{m}(x) q(x,y),$$

with

$$\tilde{m}(x) = e^{2\lambda \ell . x} \text{ and } q(x,y) = \begin{cases} q(e), & \text{for } y = x + e, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore $\Lambda_\omega(U) \ge \frac{A\kappa}{B}\,\tilde{\Lambda}(U)$, with

$$(1.2.9) \quad \tilde{\Lambda}(U) := \inf \left\{ \frac{\sum_{x,y} \tilde{m}(x)\, q(x,y)\big(f(x) - f(y)\big)^2}{2 \sum_x \tilde{m}(x)\, f^2(x)} : \right.$$

$$\left. f \ne 0, f|_{U^c} = 0, f \in L^2(\tilde{m}) \right\}.$$

So we only need to provide a positive lower bound in the context of the deterministic random walk with jump probability $\big(q(e)\big)_{|e|=1}$. Further,

because $\Lambda_\omega(\mathbb{Z}^d) = \inf_{U \neq \emptyset} \Lambda_\omega(U)$ and for $f \in L^2(m_\omega)$ we have

$$\frac{\mathcal{E}_{m_\omega}(f,f)}{(f,f)} = \lim_{U \uparrow \mathbb{Z}^d} \frac{\mathcal{E}_{m_\omega}(f1_U, f1_U)}{(f1_U, f1_U)}$$

we see that $\Lambda_\omega(\mathbb{Z}^d) = \inf_{U \neq \emptyset, \text{ finite}} \Lambda_\omega(U)$, hence we can assume without loss of generality that $\sup\{|\ell.z| : z \in U\} < \infty$.

Let us denote the canonical law of this random walk starting in x by Q_x and its expectation value by $\mathsf{E}_x^{\mathsf{Q}}$. Because

$$2\lambda\ell.\left(\sum_{|e|=1} eq(e)\right) = \sum_{j=1}^{d}\left(q(e_j) - q(-e_j)\right)\left(\log q(e_j) - \log q(-e_j)\right) > 0,$$

(recall λ and ℓ are given in (1.1.4)) we can find $0 < c < 1$ and $\delta > 0$ small enough such that

$$(1.2.10) \qquad \mathsf{E}_x^{\mathsf{Q}}[e^{-\delta\ell.(X_1 - X_0)}] \leq c < 1.$$

Defining $\eta = -\log c > 0$, we observe that $\exp\{-\delta\ell.X_n + \eta n\}$ is a Q_x-supermartingale. The stopping theorem implies that

$$(1.2.11) \qquad \mathsf{E}_x^{\mathsf{Q}}\left[\exp\{-\delta\ell.(X_{T_U} - x) + \eta T_U\right] \leq 1, \text{ for all } x \in U.$$

Let $L := \sup\{|\ell.(z - x)| : z \in U\} < \infty$, and since $-\delta\ell.(X_{T_U} - x) \geq -\delta(L + 1)$ we find:

$$\sup_{x \in U} \mathsf{E}_x^{\mathsf{Q}}\left[\exp\{-\delta(L+1) + \eta T_U\}\right] \leq 1,$$

which implies

$$(1.2.12) \qquad \sup_{x \in U} \mathsf{E}_x^{\mathsf{Q}}[e^{\eta T_U}] \leq e^{\delta\rho}, \text{ with } \rho = \sup_{x \in U}\{\ell.x\} - \inf_{x \in U}\{\ell.x\} + 1.$$

Notice also

$$\tilde{\Lambda}(U) = 1 - \sup\left\{\frac{(f, \mathsf{Q}_U f)_{\tilde{m}}}{(f,f)_{\tilde{m}}} : f \neq 0, f|_{U^c} = 0, f \in L^2(\tilde{m})\right\},$$

with the sub-Markov kernel Q_U defined through

$$(1.2.13) \qquad \begin{cases} \mathsf{Q}_U := \mathsf{Q}_U^1, \qquad \text{provided} \\ (\mathsf{Q}_U^n f)(x) = \mathsf{E}_x^{\mathsf{Q}}[f(X_n), T_U > n], \quad n \in \mathbb{N}, f : \mathbb{Z}^d \to \mathbb{R}. \end{cases}$$

We observe also $Q_U^n = (Q_U)^n$ and Q_U is a bounded self-adjoint operator on $L^2(\tilde{m})$ with respect to the canonical scalar product $(\cdot, \cdot)_{\tilde{m}}$ attached to $\tilde{m}$.

It now suffices to show that $\|Q_U\|_{L^2(\tilde{m})} \le e^{-\eta/2}$ to prove (1.2.7). To show this we observe:

$$\|Q_U^n f\|_{L^2(\tilde{m})}^2 = \sum_{x \in U} \tilde{m}(x) \, (Q_U^n f)^2(x)$$

$$\overset{\text{Jensen}}{\le} \left(1, Q_U^n f^2\right)_{\tilde{m}} = \left(Q_U^n 1, f^2\right)_{\tilde{m}}$$

$$(1.2.14) \qquad \overset{(1.2.13)}{=} \sum_y \tilde{m}(y) \, Q_y[T_U > n] \, f^2(y) \le e^{-\eta n} e^{\delta \rho} \|f\|_{L^2(\tilde{m})}^2 \,,$$

where the Chebychev inequality $Q_y[T_U > n] \le \mathsf{E}_y^Q[\exp\{\eta T_U - \eta n\}] \overset{(1.2.12)}{\le} e^{-\eta n} e^{\delta \rho}$ is used in the last step. Taking the n-th root, it follows from Theorem VI.6 on page 192 in Reed-Simon, cf. [34], that $\|Q_U\|_{L^2(\tilde{m})} \le e^{-\eta/2}$, and hence (1.2.7) follows. (1.2.8) is an immediate consequence of the fact that $\Lambda_\omega(U) = 1 - \|P_{U,\omega}\|_{L^2(m_\omega)}$ and $P_{U,\omega}^n = (P_{U,\omega})^n$. $\qquad \square$

1.2.2   Key Estimate

Thanks to Theorem 1.2.1 we can prove the key estimate of this section:

Theorem 1.2.2
There exist constants $c_1 > 0$ and $c_2 > 0$ such that for $m \in \mathbb{N}$

$$(1.2.15) \qquad \sup_{x \in \mathbb{Z}^d, \omega \in \Omega} \mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} < T_{2^m}\right] \le c_1 \, e^{-c_2 2^m} \,.$$

Proof: Let $U \subset \mathbb{Z}^d$ be finite, then (1.2.6) and (1.2.8) imply that for all $\omega \in \Omega$, $x \in U$,

$$m_\omega(x) \, \mathsf{P}_x^\omega[T_U > n] = \left(1_{\{x\}}, P_{U,\omega}^n 1_U\right)_{L^2(m_\omega)}$$

$$\le \|1_{\{x\}}\|_{L^2(m_\omega)} \cdot \|1_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n}$$

$$(1.2.16) \qquad = \sqrt{m_\omega(x)} \cdot \|1_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n}$$

Using the assumption (1.1.5), $\mathsf{P}_x^\omega[T_U > n]$ can be estimated from above by

$$\mathsf{P}_x^\omega[T_U > n] \le \|\mathbb{1}_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n} / \sqrt{m_\omega(x)}$$

(1.2.17)

$$\le \frac{1}{\sqrt{A}} \, e^{-\lambda \ell . x} \cdot \|\mathbb{1}_U\|_{L^2(m_\omega)} \cdot e^{-\gamma n} .$$

Let U now be a box centered at x with width L in the ℓ direction and size L^2 in the directions normal to ℓ, that is with a rotation R of space $\mathbb{R}^d$ such that $R(e_1) = \ell$:

(1.2.18) $$U := \Big\{ z \in \mathbb{Z}^d : \big|(z-x).\ell\big| < \frac{L}{2},$$

$$\sup_{j \ge 2} \big|R(e_j).(z-x)\big| < \frac{L^2}{2} \Big\}.$$

With $r_{\max} := \sup\{\ell.z : z \in U\} < \infty$, we see from (1.1.5) that for $L \ge 1$

$$\|\mathbb{1}_U\|_{L^2(m_\omega)} \le c_3 L^d e^{\lambda r_{\max}} .$$

Thereafter for

(1.2.19) $$n \ge \frac{\lambda L}{\gamma} ,$$ (recall that γ is defined in (1.2.8))

it follows from (1.2.17) that

$$\mathsf{P}_x^\omega[T_U > n] \le \frac{c_3}{\sqrt{A}} e^{-\lambda \ell . x} L^d e^{\lambda r_{\max}} e^{-\gamma n} \le \frac{c_3}{\sqrt{A}} L^d e^{-\frac{\lambda}{2}L}$$

(1.2.20) $$\le c_4 e^{-\frac{\lambda}{4}L} .$$

The boundary of U is defined through

(1.2.21) $$\partial U = \{z \notin U : \exists y \in U, |z-y| = 1\} ,$$

with $|\cdot|$ denoting the L^1-norm on $\mathbb{R}^d$. Now we divide it into $\partial U = \partial_+ U \cup \partial_- U \cup \partial_0 U$, with

(1.2.22) $$\begin{cases} \partial_+ U := \{z \in \partial U : \ell.(z-x) \ge \frac{L}{2}\} , \\ \partial_- U := \{z \in \partial U : \ell.(z-x) \le -\frac{L}{2}\} , \\ \partial_0 U := \partial U \backslash (\partial_+ U \cup \partial_- U) , \end{cases}$$

and set $L = 2^{m+1}$ in the above definition of U, we observe that

(1.2.23)
$$\mathsf{P}^\omega_x \left[\tilde{T}_{-2^m} < T_{2^m} \right]$$
$$\leq \mathsf{P}^\omega_x \left[T_U > \tfrac{\lambda L}{\gamma} \right] + \mathsf{P}^\omega_x \left[T_U \leq \tfrac{\lambda L}{\gamma}, X_{T_U} \notin \partial_+ U \right].$$

Using (1.2.20) the first term on the right hand side of (1.2.23) can be estimated by

(1.2.24)
$$\mathsf{P}^\omega_x \left[T_U > \tfrac{\lambda L}{\gamma} \right] \leq c_4 e^{-\frac{\lambda}{4} L}.$$

To estimate the second term, we use Carne's inequality [1] for reversible Markov chains, cf. Theorem 1 in Carne [4]:

(1.2.25) $\mathsf{P}^\omega_x[X_k = y]$
$$\leq 2\sqrt{\frac{m_\omega(y)}{m_\omega(x)}} \cdot \exp\left\{ -\frac{|x-y|^2}{2k} \right\}, \quad \text{for all } x, y \in \mathbb{Z}^d, \omega \in \Omega,$$

with $|\cdot|$ denoting the L^1-norm on $\mathbb{R}^d$.

Because $|x-y|^2 \geq \|x-y\|^2$, the second term can now be estimated through

$$\mathsf{P}^\omega_x \left[T_U \leq \tfrac{\lambda L}{\gamma}, X_{T_U} \notin \partial_+ U \right] \leq \sum_{k \leq \frac{\lambda L}{\gamma}} \sum_{\substack{y \in \\ \partial_0 U \cup \partial_- U}} \mathsf{P}^\omega_x[X_k = y]$$

(1.2.26)
$$\leq \frac{2\lambda L}{\gamma} \left[\sum_{y \in \partial_0 U} \sqrt{\frac{m_\omega(y)}{m_\omega(x)}} \, e^{-\frac{\gamma \|x-y\|^2}{2\lambda L}} + \sum_{y \in \partial_- U} \sqrt{\frac{m_\omega(y)}{m_\omega(x)}} \, e^{-\frac{\gamma \|x-y\|^2}{2\lambda L}} \right]$$

By using (1.1.5) again the first sum on the right hand side of (1.2.26) can be estimated by

(1.2.27)
$$\sum_{y \in \partial_0 U} \sqrt{\frac{m_\omega(y)}{m_\omega(x)}} e^{-\frac{\gamma \|x-y\|^2}{2\lambda L}} \leq c_5 L^{2d-3} \sqrt{\frac{B}{A}} \, e^{\lambda L} e^{-\frac{L^4 \gamma}{8\lambda L}}$$
$$\leq c_6 \, e^{-c_7 L^3},$$

---

[1] There is a small typo in the paper: x and y are interchanged on the right hand side of the inequality.

and the second sum by

$$\sum_{y \in \partial_- U} \sqrt{\frac{m_\omega(y)}{m_\omega(x)}} e^{-\frac{\gamma \|x-y\|^2}{2\lambda L}} \leq \sum_{y \in \partial_- U} \sqrt{\frac{B}{A}} e^{-c_8 \lambda L} e^{-c_9 L}$$

$$\leq c_{10} L^{2(d-1)} e^{-(c_9 + \lambda c_8)L}$$

$$(1.2.28) \qquad\qquad\qquad\qquad \leq c_{11} e^{-c_{12} L}.$$

Putting the above inequalities together:

$$(1.2.29) \quad \mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} < T_{2^m} \right] \leq c_1 e^{-c_2 2^m}, \quad \text{for } x \in \mathbb{Z}^d, \omega \in \Omega, m \geq 0.$$

$\square$

1.2.3 Transience

The next corollary of Theorem 1.2.2 will be useful in Section 1.3 and Section 1.4.

Corollary 1.2.3
There exists $c_{13} > 0$ such that for all $x \in \mathbb{Z}^d$ and $\omega \in \Omega$

$$(1.2.30) \qquad\qquad \mathsf{P}_x^\omega[D = \infty] \geq c_{13} > 0,$$

where D is the first backtracking time defined in (1.2.3).

Proof: With the notation $U_x^m := \{z \in \mathbb{Z}^d : |\ell.(z - x)| < 2^m\}$, the ellipticity condition (1.1.1) and the strong Markov property imply that $\mathsf{P}_y^\omega[T_{U_x^m} = \infty] = 0$ for all $y \in U_x^m$, $\omega \in \Omega$. Therefore (1.2.15) implies

$$(1.2.31) \qquad\qquad \inf_{x,\omega} \mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^m} \right] \geq 1 - c_1 e^{-c_2 2^m}.$$

Let $m := \inf\{k \geq 1 : 1 > c_1 e^{-c_2 2^k}\}$, we claim for any $n \geq m+1$, $x \in \mathbb{Z}^d$, $\omega \in \Omega$:

$$(1.2.32) \qquad\qquad \mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^n - 2^m} \right] \geq \prod_{k=m}^{n-1} (1 - c_1 e^{-c_2 2^k}).$$

We show this by induction. The case $n = m + 1$ is immediate from (1.2.31). The step $n \to n + 1$ follows easily by the strong Markov property and (1.2.31):

$$\mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^{n+1}-2^m} \right]$$

$$\geq \mathsf{E}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^n-2^m}, \mathsf{P}_{X_{T_{2^n-2^m}}}^\omega [\tilde{T}_{-2^n} > T_{2^n}] \right]$$

$$\geq \mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^n-2^m} \right] (1 - c_1 e^{-c_2 2^n}).$$

From (1.2.32) it is clear that for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$:

$$\mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^n-2^m} \right] \geq \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0,$$

and hence

$$\mathsf{P}_x^\omega \left[\tilde{T}_{-2^m} > T_{2^k-2^m}, \text{ for all } k > m \right] \geq \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0.$$

Therefore by using ellipticity condition (1.1.1) and the strong Markov property again we find that P_x^ω-a.s.

$$\mathsf{P}_x^\omega [D = \infty]$$

$$\geq \kappa^{c2^m} \mathsf{E}_x^\omega \left[\mathsf{P}_{X_{T_{2^m}}}^\omega [\tilde{T}_{-2^m} > T_{2^k-2^m}, \text{ for all } k > m] \right]$$

$$\geq \kappa^{c2^m} \prod_{k \geq m} (1 - c_1 e^{-c_2 2^k}) > 0, \text{ for all } x \in \mathbb{Z}^d, \omega \in \Omega.$$

This finishes the proof.   □

As an application of the above corollary we prove the transience of X_n in the direction ℓ under the quenched law P_x^ω:

Corollary 1.2.4
The random walk is transient and $\mathsf{P}_x^\omega [\lim_n \ell.X_n = \infty] = 1$, *for all* $x \in \mathbb{Z}^d$, $\omega \in \Omega$.

Proof: At first we show

$$(1.2.33) \qquad \mathsf{P}_x^\omega \left[\inf_n \ell.X_n = -\infty \right] = 0, \text{ for all } x \in \mathbb{Z}^d, \omega \in \Omega.$$

Indeed with

$$D_1 := D \text{ and } D_{m+1} := D \circ \theta_{D_m} + D_m, m \geq 1,$$

we find

$$\sup_{x \in \mathbb{Z}^d} \mathsf{P}_x^\omega [\inf_n \ell.X_n = -\infty] \leq \sup_{x \in \mathbb{Z}^d} \mathsf{P}_x^\omega [D_m < \infty, \forall m]$$

$$\leq \sup_{x \in \mathbb{Z}^d} \mathsf{E}_x^\omega \left[D_1 < \infty, \mathsf{P}_{X_{D_1}}^\omega [D_m < \infty, \forall m] \right]$$

$$\leq \sup_{x \in \mathbb{Z}^d} \mathsf{P}_x^\omega [D_1 < \infty] \times \sup_{y \in \mathbb{Z}^d} \mathsf{P}_y^\omega [D_m < \infty, \forall m]$$

$$\leq (1 - c_{13}) \cdot \sup_{y \in \mathbb{Z}^d} \mathsf{P}_y^\omega [D_m < \infty, \forall m]$$

where we used (1.2.30) in the last step. Because $1 - c_{13} < 1$, it follows that $\sup_x \mathsf{P}_x^\omega [D_m < \infty, \forall m] = 0$, and hence (1.2.33).

Now we claim that for $h > 0$ and $u \in \mathbb{R}$:

(1.2.34) P_x^ω-a.s., $\{\ell.(X_n - x) < u \text{ i.o. }\} \subset \{\ell.(X_n - x) < u - h \text{ i.o. }\}$.

To verify this, we observe that from the ellipticity condition (1.1.1) there exist a large enough integer $N > 0$ and $c > 0$, such that

(1.2.35) $\mathsf{P}_x^\omega [\tilde{T}_{-h} \leq N] \geq c$, for all $\omega \in \Omega, x \in \mathbb{Z}^d$.

Then we define a sequence of auxiliary stopping-times $(\tilde{V}_k)_{k \geq 0}$:

- $\tilde{V}_0 := 0$, $\tilde{V}_1 := \inf\{n \geq 0 : \ell.(X_n - x) < u\}$,

- $\tilde{V}_{k+1} := \tilde{V}_1 \circ \theta_{\tilde{V}_k + N} + \tilde{V}_k + N \leq \infty$, for $k \geq 1$,

and let $G_k = \{\tilde{V}_k < \infty\}$, $1_{H_k} = 1_{\{\tilde{T}_{-h} \leq N\}} \circ \theta_{\tilde{V}_k}$. We observe that $G_k \in \mathscr{F}_{\tilde{V}_k}$ and $H_k \in \mathscr{F}_{\tilde{V}_{k+1}}$. Using the strong Markov property and (1.2.35) we find

(1.2.36) $\mathsf{P}_x^\omega [H_k | \mathscr{F}_{\tilde{V}_k}] \geq c 1_{G_k}$, for all $x \in \mathbb{Z}^d, \omega \in \Omega, k \geq 1$.

Therefore it follows from Borel-Cantelli's second lemma, cf. page 240 in Durrett, [9], that

(1.2.37) P_x^ω-a.s., $\sum_{k \geq 1} 1_{H_k} = \infty$ on $\left\{ \sum_{k \geq 1} 1_{G_k} = \infty \right\}$,

which implies (1.2.34).

An immediate consequence of (1.2.34) is: for $u' \in \mathbb{R}$, P_x^ω-a.s.

$$(1.2.38) \qquad \begin{aligned} \{\ell.X_n < u' \text{ f.o. }\} &\subset \bigcap_{h \in \mathbb{N}} \{\ell.X_n < u' + h \text{ f.o. }\} \\ &= \{\lim \ell.X_n = \infty\}. \end{aligned}$$

Due to (1.2.33) we have $\mathsf{P}_x^\omega[\inf \ell.X_n > -\infty] = 1$, and since $\{\inf \ell.X_n > -\infty\} \subset \bigcup_{u' \in \mathbb{Z}} \{\ell.X_n < u' \text{ f.o. }\}$, it follows from (1.2.38) that

$$\mathsf{P}_x^\omega[\lim_n \ell.X_n = \infty] = 1.$$

$\square$

1.3   Embedded Markov Chain and Ergodicity

In this section we will define the regeneration times τ_k, $k \geq 1$, introduce the resulting Markov chain under the annealed measure P_0, and then show that this Markov chain has an invariant probability measure, with which the chain is ergodic.

1.3.1   The first no-backtracking time τ_1

At first let us introduce some further notations.

With $t_x : \Omega \to \Omega$, $x \in \mathbb{Z}^d$, we denote the spatial shift operator:

$$(1.3.1) \qquad (t_x\omega)(\{y, z\}) := \omega(\{y + x, z + x\}), \quad \text{with } \{y, z\} \in \mathbb{B}^d.$$

Let us also denote by $\mathscr{E}$ the set of unit vectors in $\mathbb{Z}^d$, which maximize $\{\ell.e\}$ and fix one such vector from $\mathscr{E}$, call it $\tilde{e}$:

$$(1.3.2) \qquad \begin{cases} \mathscr{E} := \{e \in \mathbb{Z}^d : |e| = 1, \\ \qquad\qquad \ell.e = \max\{\ell.e' : e' \in \mathbb{Z}^d, |e'| = 1\}\}, \\ \tilde{e} \in \mathscr{E} \text{ fixed.} \end{cases}$$

With the help of this $\tilde{e}$ we are able to introduce the set of *maximizing bonds* containing the point $x - \tilde{e}$:

$$(1.3.3) \qquad \mathcal{B}^x := \left\{ b \in \mathbb{B}^d : b = \{x - \tilde{e}, x - \tilde{e} + e\}, e \in \mathscr{E} \right\}.$$

and separate $\mathbb{B}^d$ into two subsets, $\mathcal{R}^x$ and $\mathcal{L}^x$ ($\mathcal{R}$ and $\mathcal{L}$ respectively stand for "right" and "left" of the point $x \in \mathbb{Z}^d$):

$$(1.3.4) \qquad \begin{cases} \mathcal{R}^x := \left\{ \{y, z\} \in \mathbb{B}^d : \max(\ell.z, \ell.y) \geq \ell.x \right\}, \\ \mathcal{L}^x := (\mathbb{B}^d \backslash \mathcal{R}^x) \bigcup \mathcal{B}^x, \end{cases}$$

so that

$$(1.3.5) \qquad \mathcal{R}^x \cap \mathcal{L}^x = \mathcal{B}^x.$$

We depict $\mathcal{L}^x$ and $\mathcal{R}^x$ for $d = 2$ in Figure 1.3.1, where solid lines are bonds in $\mathcal{L}^x$, dashed lines are bonds in $\mathcal{R}^x$ and the two thick lines are bonds in $\mathcal{B}^x$.
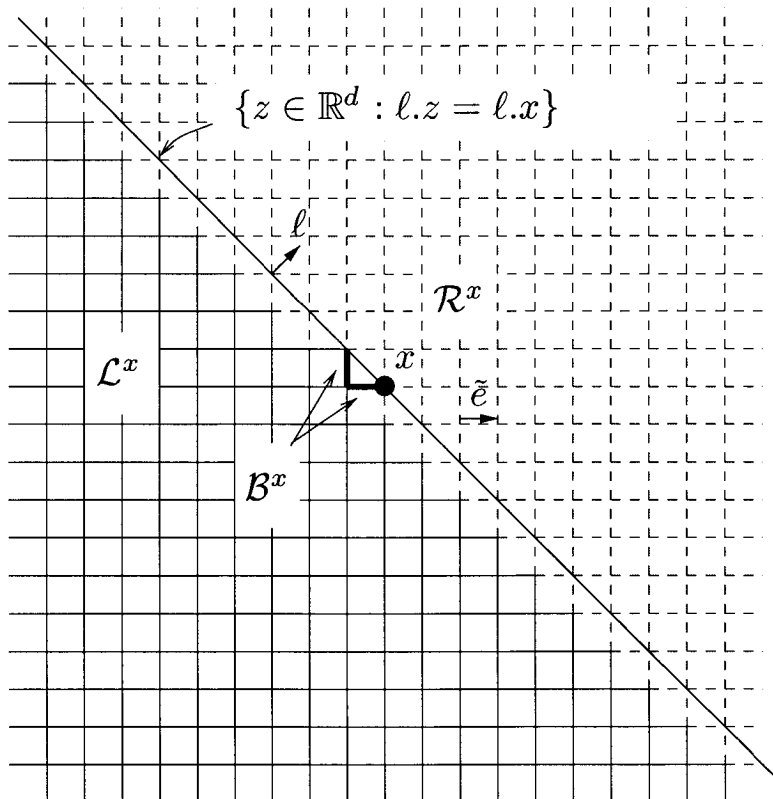

Figure 1.3.1: $\mathcal{L}^x$ *and* $\mathcal{R}^x$

Further we introduce two sequences of $(\mathscr{F}_n)_{n\geq 0}$ stopping times S_k, $k \geq 0$ and R_k, $k \geq 1$, and a sequence of successive maxima in the direction $\ell \in \mathbb{R}^d$, M_k, $k \geq 0$ (we recall the definition of D in (1.2.3)):

$$(1.3.6) \quad \begin{cases} S_0 := 0, \quad M_0 := \ell.X_0\,, \\ S_1 := \inf\left\{n \geq 2 : X_n - X_{n-1} = \tilde{e}; X_{n-1} - X_{n-2} = \tilde{e};\right. \\ \qquad\qquad \left. \ell.X_m \leq \ell.X_{n-2}\,, \forall m \leq n-2\,.\right\}\,, \\ R_1 := D \circ \theta_{S_1} + S_1\,, \\ M_1 := \sup\left\{\ell.X_m : 0 \leq m \leq R_1\right\}\,, \end{cases}$$

and inductively for $k \geq 1$:

$$(1.3.7) \quad \begin{cases} S_{k+1} := \inf\left\{n \geq R_k : X_n - X_{n-1} = \tilde{e}; X_{n-1} - X_{n-2} = \tilde{e};\right. \\ \qquad\qquad \left. \ell.X_m \leq \ell.X_{n-2}, \forall m \leq n-2\,.\right\}\,, \\ R_{k+1} := D \circ \theta_{S_{k+1}} + S_{k+1}\,, \\ M_{k+1} := \sup\left\{\ell.X_m : 0 \leq m \leq R_{k+1}\right\}\,. \end{cases}$$

Clearly we have $0 = S_0 < S_1 \leq R_1 \leq S_2 \leq \cdots \leq \infty$, and the inequalities are strict if the left member is finite.

Now let us introduce

$$(1.3.8) \qquad K := \inf\{k \geq 1 : S_k < \infty, R_k = \infty\}\,.$$

Before defining τ_1 as S_K, we first prove the finiteness of K:

Lemma 1.3.1

$$(1.3.9) \qquad \mathsf{P}_x^\omega[K < \infty] = 1\,, \text{ for all } x \in \mathbb{Z}^d\,, \omega \in \Omega\,.$$

Proof: At first we show $\mathsf{P}_x^\omega[S_1 < \infty] = 1$, for all $x \in \mathbb{Z}^d, \omega \in \Omega$. To this end we introduce a sequence of auxiliary $(\mathscr{F}_n)_{n\geq 0}$ stopping times $\tilde{S}_k$, $k \geq 0$:

- $\tilde{S}_0 = 0$,

- $\tilde{S}_{k+1} = \inf\{n \geq \tilde{S}_k + 2 : \ell.X_m \leq \ell.X_n, \forall m \leq n\}$,

in words, $\tilde{S}_{k+1}$ is the first time, at least 2 steps later than $\tilde{S}_k$, when the walk reaches a new maximum.

Because from Corollary 1.2.4 we have P_x^ω-a.s. $\ell.X_n \xrightarrow{n\to\infty} \infty$, it follows that P_x^ω-a.s. $\tilde{S}_k < \infty$ and $\tilde{S}_k \xrightarrow{k\to\infty} \infty$, for all $x \in \mathbb{Z}^d, \omega \in \Omega$.

We prove now by induction that there exists a constant $c \in (0,1)$ such that

$$(1.3.10) \qquad \mathsf{P}_x^\omega[S_1 > \tilde{S}_k] \leq c^k, \quad \text{for all } k, x \in \mathbb{Z}^d, \omega \in \Omega,$$

which implies by Borel-Cantelli-Lemma immediately that

$$(1.3.11) \qquad \mathsf{P}_x^\omega[S_1 = \infty] = 0, \quad \text{for all } x \in \mathbb{Z}^d, \omega \in \Omega.$$

For $k = 0$, (1.3.10) is immediate. Assume then (1.3.10) up to k. Because of ellipticity (1.1.1) there exists a $c > 0$ such that $\sup_{y,\omega} \mathsf{P}_y^\omega[(X_1 - X_0, X_2 - X_1) \neq (\tilde{e}, \tilde{e})\}] \leq c < 1$. Using the strong Markov property we get:

$$\mathsf{P}_x^\omega[S_1 > \tilde{S}_{k+1}]$$
$$\leq \mathsf{E}_x^\omega[S_1 > \tilde{S}_k; (X_{\tilde{S}_k+1} - X_{\tilde{S}_k}, X_{\tilde{S}_k+2} - X_{\tilde{S}_k+1}) \neq (\tilde{e}, \tilde{e})]$$
$$= \mathsf{E}_x^\omega[S_1 > \tilde{S}_k, \mathsf{P}_{X_{\tilde{S}_k}}^\omega[(X_1 - X_0, X_2 - X_1) \neq (\tilde{e}, \tilde{e})]]$$
$$\leq c\,\mathsf{P}_x^\omega[S_1 > \tilde{S}_k] \leq c^{k+1}.$$

The claim (1.3.10) follows.

Now we return to the proof of finiteness of K: by (1.2.30), we know that $\sup_{y,\omega} \mathsf{P}_y^\omega[D < \infty] \leq 1 - c_{13} < 1$, therefore for $k \geq 1$

$$\mathsf{P}_x^\omega[R_k < \infty] = \mathsf{E}_x^\omega[S_k < \infty, \mathsf{P}_{X_{S_k}}^\omega[D < \infty]]$$
$$\leq (1 - c_{13})\,\mathsf{P}_x^\omega[S_k < \infty]$$
$$\leq (1 - c_{13})\,\mathsf{P}_x^\omega[R_{k-1} < \infty],$$

with the convention $R_0 = 0$. By induction it is $\mathsf{P}_x^\omega[R_k < \infty] \leq (1 - c_{13})^k$, for all $x \in \mathbb{Z}^d, \omega \in \Omega$, from which we deduce that P_x^ω-a.s. $\sum_{k\geq 1} 1_{\{R_k < \infty\}} < \infty$, for all $x \in \mathbb{Z}^d, \omega \in \Omega$. It is only possible when

$$\mathsf{P}_x^\omega[K < \infty] = 1, \quad \text{for all } x \in \mathbb{Z}^d, \omega \in \Omega.$$

$\square$

Now we are ready to define

$$(1.3.12) \qquad\qquad \tau_1 := S_K \,,$$

and certainly we have

$$(1.3.13) \qquad \mathsf{P}_x^\omega[\tau_1 < \infty] = 1 \,, \text{ for all } x \in \mathbb{Z}^d, \omega \in \Omega \,.$$

Let us give the meaning of τ_1: The random variable τ_1, when finite, is on the one hand the first time n, at which $\ell.X_{n-2}$ reaches a maximum and the next two steps have increment $\tilde{e} \in \mathscr{E}$: i.e. $\ell.X_{\tau_1-2} \geq \ell.X_m$ for all $m \leq \tau_1 - 2$, and $X_{\tau_1} - X_{\tau_1-1} = \tilde{e}$, $X_{\tau_1-1} - X_{\tau_1-2} = \tilde{e}$; on the other hand it is a time such that after τ_1, $\ell.X_n$ never becomes smaller than $\ell.X_{\tau_1}$.

Remark 1.3.2

In the definition of S_k, $k \geq 1$, we chose quite artificially that the random walk $(X_n)_{n\geq 0}$ has increments $\tilde{e}$ in the previous two steps before S_k. Indeed, we can also choose any number of steps larger than two, and this will not affect our later discussion, as the proof of Theorem 1.4.3 shows.

Loosely speaking, we want to reduce the common dependency of the bonds involved before and after time τ_1 to only finitely many bonds, namely to $\{b \in \mathcal{B}^{X_{\tau_1}}\}$ (recall (1.3.3) for the definition of $\mathcal{B}^x$). To achieve this we need that the walker performs at least two steps in the direction $\tilde{e} \in \mathscr{E}$ just before time τ_1. This reduction of dependency is essential to the proof of Theorem 1.3.3.

Before going to the key result of this section, let us introduce some further notations used in the remainder of this article.

Recall the definition of $\mathscr{E}$, $\tilde{e}$ in (1.3.2) and that $\mathbb{I} \subset \mathbb{R}_+$ is the compact interval given above (1.1.1). We introduce for each $x \in \mathbb{Z}^d$

$$(1.3.14) \qquad a_x := \big(\omega(\{x - \tilde{e}, x - \tilde{e} + e\})\big)_{e \in \mathscr{E}} = \big(\omega(b)\big)_{b \in \mathcal{B}^x} \in \mathbb{I}^{\mathscr{E}} \,,$$

and for $a \in \mathbb{I}^{\mathscr{E}}$

$$(1.3.15) \qquad \mathbb{P}_x^a := \delta_a\Big(\big(\omega(\{x - \tilde{e}, x - \tilde{e} + e\})\big)_{e \in \mathscr{E}}\Big) \bigotimes \int_{b \in (\mathbb{B}^d \setminus \mathcal{B}^x)} \otimes d\mu(\omega(b))$$

as well as for the annealed measure

$$(1.3.16) \qquad\qquad \mathsf{P}_x^a = \mathbb{P}_x^a \times \mathsf{P}_x^\omega \, .$$

We also need the σ-algebra $\mathscr{G}_1$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$, describing the history of path and environment involved before τ_1:

$$(1.3.17) \qquad \mathscr{G}_1 := \sigma \left\{ \tau_1, (X_{\tau_1 \wedge m})_{m \geq 0}; \{\omega(b) : b \in \mathcal{L}^{X_{\tau_1}}\} \right\} \, ,$$

i.e. $\mathscr{G}_1$ is generated by the sets

$$(1.3.18) \qquad\qquad \{\tau_1 = m\} \cap \{X_{\tau_1} = x\} \cap A \, ,$$

with $m \geq 0$, $x \in \mathbb{Z}^d$, $A \in \sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_m$, and

$$(1.3.19) \qquad\qquad \{\tau_1 = \infty\} \cap A \, , \text{ with } A \in \mathscr{A} \otimes \mathscr{F}_\infty \, .$$

(Recall $\mathscr{A}$ is defined above (1.1.1).)

The key step in the study of the embedded Markov chain structure mentioned in Section 1.1 is now

Theorem 1.3.3
Let f, g, h be bounded and respectively $\sigma\{X_n : n \geq 0\}$-, $\sigma\{\omega(b) : b \in \mathcal{R}^0\}$- and $\mathscr{G}_1$-measurable functions, then for $a \in \mathbb{I}^{\mathscr{E}}$:

$$(1.3.20) \quad \mathsf{E}_0^a \left[f(X_{\tau_1 + \cdot} - X_{\tau_1}) \, g \circ t_{X_{\tau_1}} h \right] = \mathsf{E}_0^a \left[h \, \mathsf{E}_0^{a_{X_{\tau_1}}} [fg | D = \infty] \right] \, ,$$

where t_x is the spatial shift operator introduced in (1.3.1).

Proof: The left hand side of (1.3.20) is

$$\mathsf{E}_0^a [f(X_{\tau_1 + \cdot} - X_{\tau_1}) \, g \circ t_{X_{\tau_1}} \, h]$$

$$= \sum_{k \geq 1, x \in \mathbb{Z}^d} \mathsf{E}_0^a \big[f(X_{\tau_1 + \cdot} - X_{\tau_1}) \, g \circ t_{X_{\tau_1}} \, h,$$

$$(1.3.21) \qquad\qquad\qquad S_k < \infty, R_k = \infty, X_{S_k} = x \big]$$

$$= \sum_{k,x} \mathbb{E}_0^a \big[\mathsf{E}_0^\omega [f(X_{S_k + \cdot} - x) \, h,$$

$$S_k < \infty, R_k = \infty, X_{S_k} = x] \, g \circ t_x \big] \, .$$

Observe that on the event $\{\tau_1 = S_k\} \cap \{X_{\tau_1} = x\}$, there exists a bounded $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_{S_k}$-measurable variable $h_{k,x}$, which coincides with h. Indeed, from the definition of $\mathscr{G}_1$ in (1.3.18), by applying the monotone class theorem, cf. page 280 in Durrett, [9], on any set $\{\tau_1 = m\} \cap \{X_{\tau_1} = x\}$ there exists $\tilde{h}_{m,x}$ which is bounded $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_m$-measurable and coincides with h. Now we can define

$$h_{k,x} := \sum_{m \geq 0} \tilde{h}_{m,x} 1_{\{S_k = m\}},$$

so that $h_{k,x}$ is $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_{S_k}$-measurable, and coincides with h on $\{\tau_1 = S_k\} \cap \{X_{\tau_1} = x\}$.

As a result, the rightmost hand side of (1.3.21) equals

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega[f(X_{S_k+\cdot} - x)h_{k,x}, S_k < \infty, \right.$$

$$\left. D \circ \theta_{S_k} = \infty, X_{S_k} = x] \, g \circ t_x \right],$$

applying the strong Markov property at the stopping time S_k yields

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x}] \, \mathsf{E}_x^\omega[f(X_\cdot - x), D = \infty] \, g \circ t_x \right].$$

Because by definition of S_k in (1.3.7), $X_{S_k-1} - X_{S_k-2} = X_{S_k} - X_{S_k-1} = \tilde{e}$ and $\ell.X_m \leq \ell.X_{S_k-2}$ for all $m \leq S_k - 2$, and also because $\ell.e \leq \ell.\tilde{e}$ for all unit vectors $e \in \mathbb{Z}^d$, it follows that $\{X_m, X_m + e\} \in \mathcal{L}^{X_{S_k}}$, for all $m \leq S_k - 1$. Therefore $\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x}]$ is $\sigma\{\omega(b) : b \in \mathcal{L}^x\}$-measurable. On the other hand, due to the restriction $D = \infty$, $\mathsf{E}_x^\omega[f(X_\cdot - x), D = \infty] \cdot g \circ t_x$ is $\sigma\{\omega(b) : b \in \mathcal{R}^x\}$-measurable. Because $\mathcal{L}^x \cap \mathcal{R}^x \neq \emptyset$, these two random variables are not $\mathbb{P}$-independent. Fortunately, by our definition of S_k, we observe the dependence of $\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x}]$ and $\mathsf{E}_x^\omega[f(X_\cdot - x), D = \infty] \cdot g \circ t_x$ is concentrated on $\{\omega(b) : b \in \mathcal{B}^x\}$. (Here we see that it is necessary in the definition of S_k to have the random walk $(X_n)_{n \geq 0}$ going at least two steps in the direction $\tilde{e} \in \mathscr{E}$ before time S_k, otherwise $\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x}]$ is not $\sigma\{\omega(b) : b \in \mathcal{L}^x\}$-measurable.) Using this fact and Fubini's theo-

rem, the last expression equals:

$$\sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega \left[S_k < \infty, X_{S_k} = x, h_{k,x} \right] \times \right.$$

$$\left. \mathbb{E}_x^{a_x} \left[\mathsf{E}_x^\omega [f(X. - x), D = \infty] \, g \circ t_x \right] \right]$$

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega \left[S_k < \infty, X_{S_k} = x, h_{k,x} \right] \mathsf{E}_x^{a_x} \left[f(X. - x) \, g \circ t_x, D = \infty \right] \right],$$

using then the translation invariance of $\mathbb{P}$ measure we have $\mathsf{E}_x^{a_x} \left[f(X. - x) \, g \circ t_x, D = \infty \right] = \mathsf{E}_0^{a_x} \left[f(X. - 0) \, g \circ t_0, D = \infty \right]$, therefore the rightmost side of last expression equals now:

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega \left[S_k < \infty, X_{S_k} = x, h_{k,x} \right] \mathsf{E}_0^{a_x} \left[fg, D = \infty \right] \right]$$

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega \left[S_k < \infty, X_{S_k} = x, h_{k,x} \right] \mathsf{P}_0^{a_x} [D = \infty] \, \mathsf{E}_0^{a_x} [fg | D = \infty] \right].$$

This means

$$\mathsf{E}_0^a \left[f(X_{\tau_1+}. - X_{\tau_1}) \, g \circ t_{X_{\tau_1}} h \right]$$

$$(1.3.22) \qquad = \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega [S_k < \infty, X_{S_k} = x, h_{k,x}] \times \right.$$

$$\left. \mathsf{P}_0^{a_x} [D = \infty] \, \mathsf{E}_0^{a_x} [fg | D = \infty] \right].$$

By taking specially $f = g = 1$, we get from the above equation

$$(1.3.23) \qquad \mathsf{E}_0^a[h] = \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega [S_k < \infty, X_{S_k} = x, h_{k,x}] \, \mathsf{P}_0^{a_x} [D = \infty] \right].$$

Define now $\varphi(a) := \mathsf{E}_0^a[fg | D = \infty]$, and note that $\varphi(a_x)$ is $\sigma\{\omega(b) : b \in \mathcal{B}^x\}$-measurable, hence $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_{S_k}$-measurable, and thereafter $h_{k,x} \, \varphi(a_x)$ is $\sigma\{\omega(b) : b \in \mathcal{L}^x\} \otimes \mathscr{F}_{S_k}$-measurable and coincides with the $\mathscr{G}_1$-measurable function $h\varphi(a_{X_{\tau_1}})$ on $\{\tau_1 = S_k\} \cap \{X_{\tau_1} = x\}$.

Substitute h through $h\varphi(a_{X_{\tau_1}})$ in (1.3.23), we find

$$\mathsf{E}_0^a[h\varphi(a_{X_{\tau_1}})]$$

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x} \cdot \varphi(a_x)] \; \mathsf{P}_0^{a_x}[D = \infty] \right]$$

$$= \sum_{k,x} \mathbb{E}_0^a \left[\mathsf{E}_0^\omega[S_k < \infty, X_{S_k} = x, h_{k,x}] \; \mathsf{P}_0^{a_x}[D = \infty] \; \mathsf{E}_0^{a_x}[fg|D = \infty] \right].$$

Comparing this with (1.3.22) yields our claim (1.3.20). $\square$

Remark 1.3.4
Define

$$(1.3.24) \qquad \psi(X_{\cdot}, \omega) := \left(X_{\tau_1 + \cdot} - X_{\tau_1}; t_{X_{\tau_1}} \omega \right) \in (\mathbb{Z}^d)^{\mathbb{N}} \times \Omega,$$

(1.3.20) can also be expressed as

$$(1.3.25) \qquad \mathsf{E}_0^a[(fg) \circ \psi \, h] = \mathsf{E}_0^a \left[h \mathsf{E}_0^{a_{X_{\tau_1}}}[fg|D = \infty] \right].$$

1.3.2   The k-th no-backtracking time τ_k and the Markov Structure

Because $\{D = \infty\} = \{D \geq \tau_1\} \in \mathscr{G}_1$, we can define on $\{\tau_1 < \infty\}$ a non-decreasing sequence of random variables inductively, by viewing τ_k, $k \geq 1$, as a function of $X_{\cdot}$:

$$(1.3.26) \qquad \tau_{k+1}(X_{\cdot}) := \tau_1(X_{\cdot}) + \tau_k \left(X_{\tau_1 + \cdot} - X_{\tau_1} \right), \text{ for } k \geq 1,$$

and set by convention $\tau_{k+1} = \infty$ on $\{\tau_k = \infty\}$. Because of (1.3.13) and Theorem 1.3.3 we observe that P_0-a.s. $\tau_k < \infty$, for all $k \geq 1$. One could ask why we do not use the equivalent formula $\tau_{k+1} = \tau_k(X_{\cdot}) + \tau_1(X_{\tau_k + \cdot} - X_{\tau_k})$ as definition for τ_{k+1}, the reason will be clear in the proof of Theorem 1.3.5 below.

With τ_{k+1}, $k \geq 1$, introduced, we are now ready to introduce σ-algebra $\mathscr{G}_{k+1}$ for $k \geq 1$:

$$(1.3.27) \quad \mathscr{G}_{k+1} :=$$

$$\sigma \left\{ \tau_1, \cdots, \tau_k, \tau_{k+1}; \left(X_{\tau_{k+1} \wedge m} \right)_{m \geq 0}; \omega(b), b \in \mathcal{L}^{X_{\tau_{k+1}}} \right\},$$

describing the history of the path and environment involved before time τ_{k+1}.

With $\bar{\mathscr{G}}_k := \sigma\{\tau_1, \cdots, \tau_k; \ (X_{\tau_k \wedge m})_{m \geq 0}; \ \omega(b), b \in \mathcal{R}^0 \cap \mathcal{L}^{X_{\tau_k}}\}$, which is clearly included in $\mathscr{G}_k$, we also have

$$(1.3.28) \qquad \mathscr{G}_{k+1} = \sigma\left\{\mathscr{G}_1 \cup \psi^{-1}(\bar{\mathscr{G}}_k)\right\},$$

with ψ introduced in (1.3.24).

The main result showing the embedded Markov chain structure comes in the next theorem, displaying the conditional distribution of the joint random variables $\left((X_{\tau_k+n} - X_{\tau_k})_{n \geq 0}; (\tau_{k+n} - \tau_k)_{n \geq 0}; t_{X_{\tau_k}} \omega(b), b \in \mathcal{R}^{X_{\tau_k}}\right)$ given $\mathscr{G}_k$, $k \geq 1$:

Theorem 1.3.5
Let f, g, h_k be bounded and respectively $\sigma\{X_n : n \geq 0\}$-, $\sigma\{\omega(b) : b \in \mathcal{R}^0\}$- and $\mathscr{G}_k$-measurable functions with $k \geq 1$. Then for $a \in \mathbb{I}^{\mathscr{E}}$:

$$(1.3.29) \quad \mathsf{E}_0^a[f(X_{\tau_k+\cdot} - X_{\tau_k})\, g \circ t_{X_{\tau_k}}\, h_k] = \mathsf{E}_0^a\Big[h_k\, \mathsf{E}_0^{a^{X_{\tau_k}}}[fg | D = \infty]\Big].$$

Proof: We prove (1.3.29) by induction. The case $k = 1$ is just Theorem 1.3.3. For the step k to $k+1$, we observe that in view of (1.3.28) it is sufficient to show (1.3.29) for $h_{k+1} = h_1\, h_k \circ \psi$, while h_1 and h_k are bounded and respectively $\mathscr{G}_1$ and $\bar{\mathscr{G}}_k \subset \mathscr{G}_k$ measurable. For such an h, the right hand side of (1.3.29) equals

$$\mathsf{E}_0^a[f(X_{\tau_{k+1}+\cdot} - X_{\tau_{k+1}})\, g \circ t_{X_{\tau_{k+1}}}\, h_1 h_k \circ \psi]$$
$$= \mathsf{E}_0^a[f(X_{\tau_k+\cdot} - X_{\tau_k}) \circ \psi\, (g \circ t_{X_{\tau_k}} \circ \psi)\, (h_k \circ \psi)\, h_1],$$

applying now (1.3.25), the right hand side of last expression equals

$$= \mathsf{E}_0^a\Big[h_1\, \mathsf{E}_0^{a^{X_{\tau_1}}}[f(X_{\tau_k+\cdot} - X_{\tau_k})\, g \circ t_{X_{\tau_k}}\, h_k | D = \infty]\Big]$$
$$= \mathsf{E}_0^a\Big[h_1\, \mathsf{E}_0^{a^{X_{\tau_1}}}[f(X_{\tau_k+\cdot} - X_{\tau_k})\, g \circ t_{X_{\tau_k}}\, h_k, D = \infty] / \mathsf{P}_0^{a^{X_{\tau_1}}}[D = \infty]\Big],$$

and because $h_k\, 1_{\{D=\infty\}}$ is $\mathscr{G}_k$-measurable, we can use the induction

assumption, and find

$$= \mathsf{E}_0^a \left[h_1 \, \mathsf{E}_0^{a_{X_{\tau_1}}} \left[\mathsf{E}_0^{a_{X_{\tau_k}}} \left[fg \middle| D = \infty \right] h_k, D = \infty \right] \middle/ \mathsf{P}_0^{a_{X_{\tau_1}}} \left[D = \infty \right] \right]$$

$$= \mathsf{E}_0^a \left[h_1 \, \mathsf{E}_0^{a_{X_{\tau_1}}} \left[\mathsf{E}_0^{a_{X_{\tau_k}}} \left[fg \middle| D = \infty \right] h_k \middle| D = \infty \right] \right]$$

$$= \mathsf{E}_0^a \left[h_1 \, \mathsf{E}_0^{a_{X_{\tau_k}} \circ \psi} \left[fg \middle| D = \infty \right] h_k \circ \psi \right]$$

$$= \mathsf{E}_0^a \left[h_1 h_k \circ \psi \, \mathsf{E}_0^{a_{X_{\tau_{k+1}}}} \left[fg \middle| D = \infty \right] \right]$$

$$= \mathsf{E}_0^a \left[h_{k+1} \, \mathsf{E}_0^{a_{X_{\tau_{k+1}}}} \left[fg \middle| D = \infty \right] \right] ,$$

where we applied (1.3.25) backwards in the third line, and this finishes the proof. $\square$

As an immediate consequence we get

Corollary 1.3.6
Let

$$(1.3.30) \qquad\qquad \Gamma := \mathbb{N} \times \mathbb{Z}^d \times \mathbb{I}^{\mathscr{E}}$$

with its canonical product σ-algebra and let $y_k = (j^k, z^k, a^k) \in \Gamma$, $k \geq 0$. For $a \in \mathbb{I}^{\mathscr{E}}$ and $G \subset \Gamma$ measurable let also

$$(1.3.31) \qquad\qquad \tilde{R}(a; G) := \mathsf{P}_0^a[(\tau_1, X_{\tau_1}, a_{X_{\tau_1}}) \in G | D = \infty] .$$

Then under P_0 the Γ-valued random variables (with convention $\tau_0 = 0$)

$$(1.3.32) \quad Y_k := (J_k, Z_k, A_k) := (\tau_{k+1} - \tau_k, X_{\tau_{k+1}} - X_{\tau_k}, a_{X_{\tau_{k+1}}}) , k \geq 0 ,$$

define a Markov chain on the state space Γ, which has transition kernel

$$(1.3.33) \qquad \mathsf{P}[Y_{k+1} \in G | Y_0 = y_0, \cdots, Y_k = y_k] = \tilde{R}(a^k; G) ,$$

and initial distribution

$$(1.3.34) \qquad \tilde{\Lambda}(G) := \mathsf{P}_0[(\tau_1, X_{\tau_1}, a_{X_{\tau_1}}) \in G] .$$

Similarly, on the state space $\mathbb{I}^{\mathscr{E}}$ the random variables

$$(1.3.35) \qquad\qquad A_k = a_{X_{\tau_{k+1}}} , \ k \geq 0 ,$$

also define a Markov chain under P_0. *With* $a \in \mathbb{I}^{\mathscr{E}}$ *and* $B \subset \mathbb{I}^{\mathscr{E}}$ *measurable, its transition kernel is*

$$(1.3.36) \qquad R(a; B) := \mathsf{P}_0^a \big[a_{X_{\tau_1}} \in B | D = \infty \big] = \sum_{\substack{j \in \mathbb{N} \\ z \in \mathbb{Z}^d}} \tilde{R}\big(a; (j, z, B)\big) \, ,$$

and the initial distribution is

$$(1.3.37) \qquad \Lambda(B) := \mathsf{P}_0 \big[a_{X_{\tau_1}} \in B \big] = \sum_{\substack{j \in \mathbb{N} \\ z \in \mathbb{Z}^d}} \tilde{\Lambda}\big((j, z, B)\big) \, .$$

1.3.3   Doeblin Condition, Invariant Measure and Ergodicity

In this part we will show that the transition kernel $\tilde{R}(a; \cdot)$ has an invariant distribution and it is ergodic. At first we need

Lemma 1.3.7
There exists a unique probability measure ν *on* $\mathbb{I}^{\mathscr{E}}$ *and two constants* $c > 0$, $c_{15} > 0$ *such that for* $m \geq 0$:

$$(1.3.38) \qquad \sup_{a \in \mathbb{I}^{\mathscr{E}}} \| R^m(a; \cdot) - \nu(\cdot) \|_{\mathsf{var}} \leq c \, e^{-c_{15} m} \, ,$$

where $\| \cdot \|_{\mathsf{var}}$ *denotes the variational norm on the space of measure on* $\mathbb{I}^{\mathscr{E}}$.

Further, this probability measure ν *is invariant with respect to the transition kernel* R, *i.e.* $\nu R = \nu$, *and the Markov chain* $(A_k)_{k \geq 0}$, *defined in (1.3.35), with transition kernel* R *and initial distribution* ν *on the state space* $\mathbb{I}^{\mathscr{E}}$ *is ergodic.*

Moreover, the initial distribution $\Lambda(\cdot)$ *given in (1.3.37) is absolutely continuous with respect to* $\nu(\cdot)$.

Proof: At first we show that the kernel $R(a; \cdot)$ satisfies the Doeblin condition, cf. page 178 in Revuz, [35]:

$$(1.3.39) \qquad R(a; B) \geq \kappa^2 c_{13} \big(\otimes_{\mathscr{E}} \mu \big)(B) \, , \text{ for all measurable } B \subset \mathbb{I}^{\mathscr{E}} \, ,$$

where we recall that μ is the distribution of $\omega(b)$ on $\mathbb{I}$. Indeed the ellipticity condition (1.1.1) implies:

$$
\begin{aligned}
R(a; B) \quad &= \quad \mathsf{P}_0^a \big[a_{X_{\tau_1}} \in B \big| D = \infty \big] \\
&= \quad \mathbb{E}_0^a \big[\mathsf{P}_0^\omega \big[a_{X_{\tau_1}}, D = \infty \big] \big] / \mathsf{P}_0^a [D = \infty] \\
&\geq \quad \mathbb{E}_0^a \big[\mathsf{P}_0^\omega [X_1 = \tilde{e}, X_2 = 2\tilde{e}, D \circ \theta_2 = \infty], a_{X_2} \in B \big] \\
&= \quad \mathbb{E}_0^a \big[\mathsf{P}_0^\omega [X_1 = \tilde{e}, X_2 = 2\tilde{e}] \, \mathsf{P}_{2\tilde{e}}^\omega [D = \infty], a_{2\tilde{e}} \in B \big] \\
&\geq \quad \kappa^2 \, \mathbb{E}_0^a \big[\mathsf{P}_{2\tilde{e}}^\omega [D = \infty], a_{2\tilde{e}} \in B \big] \\
\overset{(1.2.30)}{\geq} \quad &\kappa^2 \, c_{13} \mathbb{P}_0^a [a_{2\tilde{e}} \in B] = \kappa^2 c_{13} \big(\otimes_{\mathscr{E}} \mu \big)(B) .
\end{aligned}
$$

Applying Theorem 6.15 in Nummelin, [29], the Doeblin condition immediately implies that there exists an invariant measure ν and (1.3.38) holds. (Doeblin condition implies that the kernel is small and aperiodic in the terminology of [29], cf. page 15, 20 and 21 of [29].) The uniqueness is a trivial consequence of (1.3.38).

In view of (1.3.38) the ergodicity of $(A_n)_{n \geq 0}$ follows from Proposition 2.4 in Chapter 6 of Revuz [35].

To prove that the initial distribution $\Lambda(\cdot)$ is absolutely continuous with respect to the invariant measure $\nu(\cdot)$, we observe that Doeblin condition (1.3.39) also implies

$$
\begin{aligned}
\nu(B) &= \int \nu(da) R(a; B) \geq \kappa^2 c_{13} \int \nu(da) \big(\otimes_{\mathscr{E}} \mu \big)(B) \\
&= \kappa^2 c_{13} \big(\otimes_{\mathscr{E}} \mu \big)(B) .
\end{aligned}
$$

Therefore $\nu(B) = 0$ implies $\big(\otimes_{\mathscr{E}} \mu \big)(B) = 0$, and hence

$$
\Lambda(B) \leq \sum_{z \in \mathbb{Z}^d} \mathsf{P}_0 [a_z \in B] = \sum_{z \in \mathbb{Z}^d} \big(\otimes_{\mathscr{E}} \mu \big)(B) = 0 ,
$$

i.e. Λ is absolutely continuous with respect to ν, and this finishes the proof. $\qquad \square$

With this lemma we can now prove

Theorem 1.3.8 (Ergodicity)
$\tilde{\nu} := \nu \tilde{R}$ is the unique invariant distribution for the transition kernel $\tilde{R}$,

for which the relation,

$$(1.3.40) \qquad \sup_{a \in \mathbb{I}^{\mathscr{E}}} \left\| \tilde{R}^m(a; \cdot) - \tilde{\nu}(\cdot) \right\|_{\text{var}} \leq c_{14}\, e^{-c_{15}m}, \quad m \geq 0,$$

holds for some $c_{14} > 0$. With initial distribution equal $\tilde{\nu}$, the Markov chain $(Y_k)_{k \geq 0}$ defined in (1.3.32) is ergodic. Moreover, the law of the Markov chain $(Y_{k+1})_{k \geq 0}$ under P_0 is absolutely continuous with respect to the law of the chain with initial distribution $\tilde{\nu}$.

Proof: We observe that for any bounded and measurable function f on $\mathbb{I}^{\mathscr{E}}$ we have $\tilde{R}f = Rf$ and thereafter $\tilde{\nu}\tilde{R} = \nu\tilde{R}\tilde{R} = \nu R\tilde{R} = \nu\tilde{R} = \tilde{\nu}$. This means that $\tilde{\nu}$ is an invariant probability measure with respect to $\tilde{R}$ on Γ. From $R\tilde{R} = \tilde{R}^2$ and (1.3.38) it follows that $\|\tilde{R}^{m+1}(a; \cdot) - \tilde{\nu}(\cdot)\|_{\text{var}} \leq c\, e^{-c_{15}m}$ for $m \geq 0$, and hence (1.3.40) with some constant $c_{14} > 0$. Applying again Proposition 2.4 in Chapter 6 of Revuz [35], the ergodicity of $(Y_k)_{k \geq 0}$ with initial distribution $\tilde{\nu}$ follows.

From Corollary 1.3.6 we know that, the initial distribution of $(Y_{k+1})_{k \geq 0}$ under P_0 is $\tilde{\Lambda}\tilde{R}$. From Lemma 1.3.7, Λ is absolutely continuous with respect to ν, therefore the absolute continuity of the law $(Y_{k+1})_{k \geq 0}$ under P_0 with respect to the law with initial distribution $\tilde{\nu}$ follows immediately from the obvious relations $\tilde{\Lambda}\tilde{R} = \Lambda\tilde{R}$ and $\tilde{\nu}\tilde{R} = \nu\tilde{R}$. $\qquad \square$

1.4 Integrability Properties of $\ell.X_{\tau_1}$ and τ_1

As a last step of preparation towards the strong law of large numbers and the functional central limit theorem mentioned in Section 1.1, we will show in this section that for $c > 0$ small enough, $\sup_{x,\omega} \mathsf{E}_x^{\omega}\left[e^{c\tau_1}\right] < \infty$. The proof will be divided in several auxiliary lemmas.

Lemma 1.4.1
There exists $c_{16} > 0$ such that for all $\omega \in \Omega$, $x \in \mathbb{Z}^d$

$$(1.4.1) \qquad \mathsf{E}_x^{\omega}\left[\exp\{c_{16}\ell.(X_{S_1} - X_0)\}\right] \leq 1 + \frac{c_{13}}{4},$$

with c_{13} given in (1.2.30).

Proof: At first we define a sequence of auxiliary $(\mathscr{F}_n)_{n \geq 0}$ stopping times: (recall the definition of $\tilde{e}$ in (1.3.2))

- $N_0 := 0; \quad N_1 := \inf\{m \geq 0 : \ell.(X_m - X_0) \geq 2\ell.\tilde{e}\}$;

- $N_{k+1} := N_k + N_1 \circ \theta_{N_k}$, for $k \geq 1$.

Observe that for all $k \geq 1$:

$$2\ell.\tilde{e} \leq \ell.(X_{N_k} - X_{N_{k-1}}) \leq 3\ell.\tilde{e} \quad \text{and} \quad N_k - N_{k-1} \geq 2.$$

Therefore we have $\ell.(X_{S_1} - X_0) \leq 3k(\ell.\tilde{e})$ on $\{N_{k-1} < S_1 \leq N_k\}$, and hence

$$\mathsf{E}_x^\omega\left[e^{c\ell.(X_{S_1} - X_0)}\right] = \sum_{k \geq 1} \mathsf{E}_x^\omega\left[e^{c\ell.(X_{S_1} - X_0)}, N_{k-1} < S_1 \leq N_k\right]$$

$$(1.4.2) \qquad\qquad \leq \sum_{k \geq 1} e^{3ck\ell.\tilde{e}} \mathsf{P}_x^\omega[N_{k-1} < S_1 \leq N_k] .$$

Because for all $y \in \mathbb{Z}^d$, $\omega \in \Omega$:

$$\mathsf{P}_y^\omega[N_{k+1} < S_1]$$
$$\leq \mathsf{P}_y^\omega[N_k < S_1, (X_{N_k+1} - X_{N_k}, X_{N_k+2} - X_{N_k+1}) \neq (\tilde{e}, \tilde{e})]$$
$$\leq (1 - \kappa^2)\mathsf{P}_y^\omega[N_k < S_1] ,$$

where we used the ellipticity condition (1.1.1) in the last step, the right-most hand side of (1.4.2) can be estimated further by

$$\sum_{k \geq 1} e^{3ck\ell.\tilde{e}} \mathsf{P}_x^\omega[N_{k-1} < S_1 \leq N_k] \leq \sum_{k \geq 1} e^{3ck\ell.\tilde{e}} \mathsf{P}_x^\omega[N_{k-1} < S_1]$$

$$(1.4.3) \qquad\qquad \leq \sum_{k \geq 1} e^{3ck\ell.\tilde{e}}(1 - \kappa^2)^{k-1} < \infty ,$$

provided c is small enough.

Take now $c_0 > 0$ and $m_0 \in \mathbb{N}$ such that $\sum_{k > m_0} e^{3c_0 k\ell.\tilde{e}}(1 - \kappa^2)^{k-1} < \frac{c_{13}}{8}$, (1.4.2) and (1.4.3) imply that for all $c < c_0$:

$$\mathsf{E}_x^\omega\left[e^{c\ell.(X_{S_1} - X_0)}\right] \leq \sum_{m \leq m_0} e^{3cm_0\ell.\tilde{e}} \mathsf{P}_x^\omega[N_{m-1} < S_1 \leq N_m] + \frac{c_{13}}{8}$$

$$\leq e^{3cm_0\ell.\tilde{e}} \mathsf{P}_x^\omega[S_1 \leq N_{m_0}] + \frac{c_{13}}{8} .$$

Thereafter there exists $c_{16} \in (0, c_0)$ small enough such that $e^{3c_{16}m_0\ell.\tilde{e}} < 1 + \frac{c_{13}}{8}$ and that finishes our proof. $\square$

Let us introduce the random variable

$$(1.4.4) \qquad M := \sup\{\ell.(X_n - X_0) : 0 \le n \le D\},$$

which is the maximal displacement in the direction ℓ before backtracking. It will turn out that M is a key variable later in studying integrability properties of $\ell.X_{\tau_1}$. Because for all $a \in \mathbb{I}^{\mathscr{E}}$, $\mathsf{P}_0^a[D = \infty] > 0$, we cannot expect $M < \infty$ P_0^a-a.s.. Nevertheless we claim:

Lemma 1.4.2
There exists some $c_{17} > 0$ small enough such that

$$(1.4.5) \qquad \left(1 + \tfrac{c_{13}}{4}\right) \cdot \left\{ \sup_{\substack{x \in \mathbb{Z}^d \\ \omega \in \Omega}} \mathsf{E}_x^\omega\left[e^{c_{17}M}, D < \infty\right]\right\} \le 1 - \frac{c_{13}}{2}.$$

Proof: At first we show that

$$(1.4.6) \quad \mathsf{P}_x^\omega\left[2^m \le M < 2^{m+1}, D < \infty\right]$$

$$\le c_{18}\, e^{-c_{19}2^{m+1}}, \quad \text{for all } x \in \mathbb{Z}^d, \omega \in \Omega.$$

Recall the definition (1.2.18) for the box U centered in x with width L in the direction ℓ and size L^2 in the direction normal to ℓ, also recall (1.2.22) for its boundary $\partial U = \partial_+ U \cup \partial_- U \cup \partial_0 U$ and set $L = 2^{m+1}$, we observe that

$$\mathsf{P}_x^\omega\left[2^m \le M < 2^{m+1}, D < \infty\right]$$

$$\le \mathsf{P}_x^\omega\left[T_U > \tfrac{\lambda L}{\gamma}\right] + \mathsf{P}_x^\omega\left[T_U \le \tfrac{\lambda L}{\gamma}, X_{T_U} \notin \partial_+ U\right]$$

$$+ \mathsf{P}_x^\omega\left[T_U \le \tfrac{\lambda L}{\gamma}, X_{T_U} \in \partial_+ U, \mathsf{P}_{X_{T_U}}^\omega[\tilde{T}_{-2^m} < T_{2^m}]\right].$$

By (1.2.24) – (1.2.29) the first two terms together are $\le c_1\, e^{-c_2 2^m}$. By (1.2.15), third term is less or equal to $c_1 e^{-c_2 2^m}$. Putting them together the claim (1.4.6) follows.

With (1.4.6) in mind we show in the second step that $\sup_{x,\omega} \mathsf{E}_x^\omega[e^{cM}, D < \infty] \le 1 - \frac{3c_{13}}{4}$, provided $c > 0$ small enough. This can be seen by

the obvious estimate

$$\mathsf{E}_x^\omega[e^{cM}, D < \infty]$$

$$\leq e^{2c}\mathsf{P}_x^\omega[0 \leq M < 1, D < \infty]$$

$$+ \sum_{m \geq 0} \mathsf{P}_x^\omega[2^m \leq M < 2^{m+1}, D < \infty] \, e^{c2^{m+1}}$$

$$\leq \mathsf{P}_x^\omega[D < \infty]e^{c2^{m_0+1}} + \sum_{m > m_0} \mathsf{P}_x^\omega[2^m \leq M < 2^{m+1}, D < \infty]e^{c2^{m+1}}$$

$$\leq \mathsf{P}_x^\omega[D < \infty]e^{c2^{m_0+1}} + \sum_{m > m_0} c_{18} \, e^{(c-c_{19})2^{m+1}} \, .$$

Now, let $c_0 = \frac{c_{19}}{2}$ and $m_0 \in \mathbb{N}$ be chosen such that $\sum_{m > m_0} c_{18} \exp\{(c_0 - c_{19})2^{m+1}\} \leq \frac{c_{13}}{8}$, the rightmost hand side above is

$$\leq (1 - c_{13})e^{c2^{m_0+1}} + \frac{c_{13}}{8} \leq 1 - \frac{3c_{13}}{4} \, ,$$

with $0 < c < c_0$ small enough. Our claim follows immediately. $\quad\square$

With the help of these two lemmas we can now provide the integrability of $\mathsf{E}_x^\omega[e^{c\ell.X_{\tau_1}}]$:

Theorem 1.4.3
There exists $c_{20} > 0$ small enough such that

$$(1.4.7) \qquad \sup_{\substack{x \in \mathbb{Z}^d \\ \omega \in \Omega}} \mathsf{E}_x^\omega[\exp\{c_{20}\ell.(X_{\tau_1} - X_0)\}] < \infty \, .$$

Proof: Since

$$(1.4.8) \quad \begin{aligned} &\mathsf{E}_x^\omega\left[e^{c\ell.(X_{\tau_1} - X_0)}\right] \\ &= \sum_{k \geq 1} \mathsf{E}_x^\omega\left[e^{c\ell.(X_{S_k} - X_0)}, S_k < \infty, D \circ \theta_{S_k} = \infty\right] \\ &\leq \sum_{k \geq 1} \mathsf{E}_x^\omega\left[e^{c\ell.(X_{S_k} - X_0)}, S_k < \infty\right] \, , \end{aligned}$$

in view of (1.4.1) it suffices to show that $\sup_{x,\omega} \sum_{k \geq 2} \mathsf{E}_x^\omega[\exp\{c\ell.(X_{S_k} - X_0)\}, S_k < \infty] < \infty$.

To this end we define another sequence of auxiliary $(\mathscr{F}_n)_{n \geq 0}$ stopping times (recall the definition of M_k in (1.3.7)):

$$(1.4.9) \qquad V_k := \inf\{n \geq R_k : \ell.X_n \geq M_k\} \, , \text{ for } k \geq 1 \, ,$$

i.e. V_k is the first time after R_k such that the random walker $(X_n)_{n \geq 0}$ reaches a maximum in the direction ℓ again.

It is clear that $R_k \leq V_k \leq S_{k+1}$, and the inequalities are strict if $S_{k+1} < \infty$.

We observe that for $k \geq 2$:

$$\ell.(X_{S_k} - X_0) = \ell.X_{S_k} - \ell.X_{V_{k-1}} + \ell.(X_{V_{k-1}} - X_0)$$
$$\leq \ell.(X_{S_1} - X_0) \circ \theta_{V_{k-1}} + \ell.(X_{V_{k-1}} - X_0),$$

whence

$$\mathsf{E}^\omega_x \left[e^{c\ell.(X_{S_k} - X_0)}, S_k < \infty \right]$$

$$\leq \mathsf{E}^\omega_x \left[e^{c\ell.(X_{S_k} - X_0)}, V_{k-1} < \infty \right]$$

$$\leq \mathsf{E}^\omega_x \left[e^{c\ell.(X_{V_{k-1}} - X_0)}, V_{k-1} < \infty, \mathsf{E}^\omega_{X_{V_{k-1}}} \left[e^{c\ell.(X_{S_1} - X_0)} \right] \right]$$

$$(1.4.10) \quad \overset{(1.4.1)}{\leq} \mathsf{E}^\omega_x \left[e^{c\ell.(X_{V_{k-1}} - X_0)} \left(1 + \tfrac{c_{13}}{4} \right), V_{k-1} < \infty \right].$$

Further we observe that

$$\ell.(X_{V_{k-1}} - X_0) = \ell.(X_{V_{k-1}} - X_{S_{k-1}}) + \ell.(X_{S_{k-1}} - X_0)$$
$$\leq M_{k-1} + 1 - \ell.X_{S_{k-1}} + \ell.(X_{S_{k-1}} - X_0)$$
$$= M \circ \theta_{S_{k-1}} + 1 + \ell.(X_{S_{k-1}} - X_0).$$

Therefore with the strong Markov property the rightmost hand side of (1.4.10) can be further estimated by

$$\leq e^c \mathsf{E}^\omega_x \left[\exp\{c(M \circ \theta_{S_{k-1}} + \ell.(X_{S_{k-1}} - X_0))\} \left(1 + \tfrac{c_{13}}{4} \right), R_{k-1} < \infty \right]$$

$$= e^c \mathsf{E}^\omega_x \left[e^{c\ell.(X_{S_{k-1}} - X_0)}, S_{k-1} < \infty, \left(1 + \tfrac{c_{13}}{4} \right) \mathsf{E}^\omega_{X_{S_{k-1}}} \left[e^{cM}, D < \infty \right] \right],$$

and this is by (1.4.5) and induction

$$\leq e^c \left(1 - \tfrac{c_{13}}{2} \right) \mathsf{E}^\omega_x \left[e^{c\ell.(X_{S_{k-1}} - X_0)}, S_{k-1} < \infty \right]$$

$$\leq \left(e^c \left(1 - \tfrac{c_{13}}{2} \right) \right)^k,$$

provided $0 < c \leq c_{17}$.

Therefore we can find $c_{20} \in (0, c_{17})$ small enough such that $e^{c_{20}}(1 - \frac{c_{13}}{2}) < 1$. Therefore

$$\sum_{k \geq 2} \mathsf{E}_x^\omega \left[e^{c_{20}\ell.(X_{S_k} - X_0)}, S_k < \infty \right] \leq \sum_{k \geq 2} \left(e^{c_{20}}(1 - \tfrac{c_{13}}{2}) \right)^k < \infty.$$

And with (1.4.8) this finishes our proof.   □

As a corollary we obtain an estimate on the tail of τ_1 and its integrability properties:

Corollary 1.4.4
There exists $c_{21} > 0$ and $c_{22} > 0$ such that for $u \in \mathbb{N}$

$$(1.4.11) \qquad\qquad \sup_{\substack{x \in \mathbb{Z}^d \\ \omega \in \Omega}} \mathsf{P}_x^\omega[\tau_1 > u] \leq c_{21}\, e^{-c_{22}u},$$

and consequently

$$(1.4.12) \qquad\qquad \sup_{\substack{x \in \mathbb{Z}^d \\ \omega \in \Omega}} \mathsf{E}_x^\omega \left[e^{c_{23}\tau_1} \right] \leq c_{24} < \infty,$$

for some $c_{23} > 0$ and $c_{24} > 0$.

Proof: Recall $\gamma = \log \frac{1}{1-\varepsilon}$ from Theorem 1.2.1 and choose $u \in \mathbb{N}$, $u \geq \frac{2\lambda}{\gamma}$. We denote with U the box defined in (1.2.18), with center x, width $\frac{\gamma}{2\lambda}u$ in the direction ℓ and size $(\frac{\gamma}{2\lambda}u)^2$ in the direction normal to ℓ.

By Chebychev inequality and with c_{20} from (1.4.7) we observe

$$\mathsf{P}_x^\omega[\tau_1 > u]$$
$$\leq \mathsf{P}_x^\omega \left[\tau_1 > u, \ell.(X_{\tau_1} - X_0) \leq \tfrac{\gamma}{4\lambda}u \right] + \mathsf{P}_x^\omega \left[\ell.(X_{\tau_1} - X_0) > \tfrac{\gamma}{4\lambda}u \right]$$
$$\leq \mathsf{P}_x^\omega \left[\tau_1 > u, \ell.(X_{\tau_1} - X_0) \leq \tfrac{\gamma}{4\lambda}u \right] + e^{-c_{20}\frac{\gamma}{4\lambda}u}\, \mathsf{E}_x^\omega[e^{c_{20}\ell.(X_{\tau_1} - X_0)}]$$
$$\leq \mathsf{P}_x^\omega \left[\tau_1 > u, \ell.(X_{\tau_1} - X_0) \leq \tfrac{\gamma}{4\lambda}u \right] + c_{25}\, e^{-c_{26}u},$$

further we have

$$\mathsf{P}_x^\omega \left[\tau_1 > u, \ell.(X_{\tau_1} - X_0) \leq \tfrac{\gamma}{4\lambda}u \right] \leq \mathsf{P}_x^\omega \left[T_{\frac{\gamma}{4\lambda}u} > u \right]$$
$$\leq \mathsf{P}_x^\omega \left[T_{\frac{\gamma}{4\lambda}u} > T_U \right] + \mathsf{P}_x^\omega \left[T_U = T_{\frac{\gamma}{4\lambda}u} > u \right]$$
$$\leq \mathsf{P}_x^\omega \left[T_U > \tfrac{u}{2} \right] + \mathsf{P}_x^\omega \left[T_U \leq \tfrac{u}{2}, X_{T_U} \notin \partial_+ U \right]$$
$$+ \mathsf{P}_x^\omega \left[T_U = T_{\frac{\gamma}{4\lambda}u} > u \right].$$

Using the same argument as in (1.2.23) — (1.2.29) the first two terms on the right hand side together can be estimated uniformly: for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$ and for all $u \in \mathbb{N}$,

$$(1.4.13) \qquad \mathsf{P}_x^\omega \big[T_U > \tfrac{u}{2} \big] + \mathsf{P}_x^\omega \big[T_U \leq \tfrac{u}{2}, X_{T_U} \notin \partial_+ U \big] \leq c_{27}\, e^{-c_{28} u} \,,$$

and by (1.2.20) the last term can also be estimated uniformly: for all $x \in \mathbb{Z}^d$, $\omega \in \Omega$ and $u \in \mathbb{N}$, $u \geq \frac{2\lambda}{\gamma}$,

$$(1.4.14) \qquad \mathsf{P}_x^\omega [T_U > u] \leq c_4\, e^{-\frac{\gamma}{8} u} \,,$$

because in our construction of U, $u \geq \frac{\lambda}{\gamma} \frac{\gamma}{2\lambda} u = \frac{u}{2}$, the condition (1.2.19) is fulfilled.

Altogether we get that for all $u \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $\omega \in \Omega$:

$$(1.4.15) \qquad \mathsf{P}_x^\omega [\tau_1 > u] \leq c_{21}\, e^{-c_{22} u} \,,$$

our claim (1.4.11) follows immediately.

And finally, (1.4.12) is an easy consequence of (1.4.11). $\qquad\square$

1.5 Law of Large Numbers and Central Limit Theorem

In this section we will provide the main results of this article: at first a strong law of large numbers, moreover we are able to prove a functional central limit theorem. Some parts of the proofs presented in this section are similar to the proofs of Theorem 2.3 on page 1864 of Sznitman-Zerner, [46] and Theorem 4.1 on pages 130 – 131 of Sznitman, [43].

Theorem 1.5.1 (Strong Law of Large Numbers)
Under the assumption (1.1.1) – (1.1.5) we have

$$(1.5.1) \qquad \mathsf{P}_0\text{-}a.s., \quad \frac{X_n}{n} \xrightarrow{n \to \infty} v = \frac{\mathsf{E}^\Pi [X_{\tau_1}]}{\mathsf{E}^\Pi [\tau_1]} \,, \quad and\ \ell.v > 0 \,,$$

where

$$(1.5.2) \qquad \begin{aligned} \Pi[\cdot] &:= \int \nu(da)\, \mathsf{P}_0^a[\cdot\,|D = \infty]\ and \\ \mathsf{E}^\Pi[\cdot] &:= \int \nu(da)\, \mathsf{E}_0^a[\cdot\,|D = \infty] \,, \end{aligned}$$

(We recall that ν is the unique invariant distribution on $\mathbb{I}^{\mathscr{E}}$ given in Lemma 1.3.7.).

Proof: Let $Y_k = (J_k, Z_k, A_k) = (\tau_{k+1} - \tau_k, X_{\tau_{k+1}} - X_{\tau_k}, a_{X_{\tau_{k+1}}})$, $k \geq 0$, be the random variables on Γ defined in (1.3.32). We know from Theorem 1.3.8 that the Markov chain $(Y_k)_{k\geq 0}$ with initial distribution $\tilde{\nu}$ is stationary and ergodic, further the law of $(Y_{k+1})_{k\geq 0}$ under P_0 is absolutely continuous with respect to the law with initial distribution $\tilde{\nu}$. Therefore from the Birkhoff's ergodic theorem, cf. page 341 in Durrett, [9], it follows that for any $f \in L^1(\Gamma, \tilde{\nu})$, P_0-a.s.:

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) \xrightarrow{n\to\infty} \int d\tilde{\nu}\, f\,.$$

Applying this formula to $f(y) = j$ and $f(y) = z$ for $y = (j, z, a) \in \Gamma$, we find that P_0-a.s.:

$$(1.5.3)\qquad \begin{aligned} \frac{\tau_n - \tau_1}{n-1} &\to \int d\tilde{\nu} J_1 = \int \nu(da)\, \mathsf{E}_0^a[\tau_1 | D = \infty] = \mathsf{E}^\Pi[\tau_1] < \infty\,, \\ \frac{X_{\tau_n} - X_{\tau_1}}{n-1} &\to \int d\tilde{\nu} Z_1 = \int \nu(da)\, \mathsf{E}_0^a[X_{\tau_1} | D = \infty] = \mathsf{E}^\Pi[X_{\tau_1}]\,, \end{aligned}$$

as $n \to \infty$, where the finiteness follows from (1.4.12). We also observe that $\ell.v > 0$, because P_0-a.s. $\ell.X_{\tau_1} > 0$ by definition (1.3.6), (1.3.7) and (1.3.12), and $\mathsf{E}^\Pi[|X_{\tau_1}|] \leq \mathsf{E}^\Pi[\tau_1] < \infty$.

From (1.3.13) we observe that P_0-a.s. $\frac{\tau_1}{n-1} \to 0$, as $n \to \infty$. Therefore (1.5.3) implies that

$$(1.5.3^*)\qquad \begin{aligned} \frac{1}{n}\tau_n &\xrightarrow{n\to\infty} \int d\tilde{\nu} J_1 = \int \nu(da)\, \mathsf{E}_0^a[\tau_1 | D = \infty] = \mathsf{E}^\Pi[\tau_1]\,, \\ \frac{1}{n}X_{\tau_n} &\xrightarrow{n\to\infty} \int d\tilde{\nu} Z_1 = \int \nu(da)\, \mathsf{E}_0^a[X_{\tau_1} | D = \infty] = \mathsf{E}^\Pi[X_{\tau_1}]\,. \end{aligned}$$

Now let us define a non-decreasing sequence k_n, $n \geq 0$, which tends to $+\infty$ P_0-a.s., such that

$$(1.5.4)\qquad \tau_{k_n} \leq n < \tau_{k_n+1}\,, \text{ (with the convention } \tau_0 = 0).$$

Dividing the above inequality by k_n and using $(1.5.3^*)$, we find that P_0-a.s.:

$$(1.5.5)\qquad \frac{k_n}{n} \xrightarrow{n\to\infty} \frac{1}{\mathsf{E}^\Pi[\tau_1]}\,.$$

Further we observe that:

$$(1.5.6) \qquad \frac{X_n}{n} = \frac{X_{\tau_{k_n}}}{n} + \frac{X_n - X_{\tau_{k_n}}}{n},$$

then in view of $(1.5.3^*)$ and $(1.5.5)$, we obtain that P_0-a.s.:

$$(1.5.7) \qquad \frac{X_{\tau_{k_n}}}{n} = \frac{X_{\tau_{k_n}}}{k_n} \cdot \frac{k_n}{n} \xrightarrow{n \to \infty} \frac{\mathsf{E}^\Pi[X_{\tau_1}]}{\mathsf{E}^\Pi[\tau_1]},$$

and by $(1.5.5)$ again, that P_0-a.s.:

$$\frac{|X_n - X_{\tau_{k_n}}|}{n} \le \frac{\tau_{k_n+1} - \tau_{k_n}}{n} = \frac{\tau_{k_n+1}}{k_n+1} \cdot \frac{k_n+1}{n} - \frac{\tau_{k_n}}{k_n} \cdot \frac{k_n}{n} \xrightarrow{n \to \infty} 0.$$

Combining this with $(1.5.6)$ and $(1.5.7)$, we have proved that P_0-a.s. $\frac{X_n}{n} \xrightarrow{n \to \infty} v$, with v given in $(1.5.1)$. $\qquad \square$

We are now able to derive a functional central limit theorem for the process

$$(1.5.8) \qquad B_t^n = \frac{1}{\sqrt{n}} \big(X_{[tn]} - [tn]v \big), \quad t \ge 0,$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}_+$.

We denote by $D_{\mathbb{R}^d}[0, \infty)$ the set of $\mathbb{R}^d$-valued functions on $[0, \infty)$, which are right-continuous and possess left limits (also called càdlàg functions). We endow this set with the Skorohod topology, cf. page 117 in Ethier-Kurtz, [11], and its Borel-σ-algebra, so that B^n defines a $D_{\mathbb{R}^d}[0, \infty)$-valued random variable.

To simplify notations let us temporarily denote the law of the Markov chain $(Y_m)_{m \ge 0}$ with invariant distribution $\tilde{\nu}$ by $\mathsf{P}_{\tilde{\nu}}[\cdot]$ and its expectation value by $\mathsf{E}_{\tilde{\nu}}[\cdot]$. Further we use x^T to denote the transposed vector of $x \in \mathbb{R}^d$.

At first

Lemma 1.5.2
Let $f(y) := z - jv$ for $y = (j, z, a) \in \Gamma$ and v from Theorem 1.5.1. Then

$$(1.5.9) \qquad \sup_{a \in \mathbb{I}^\mathscr{E}} \big(\tilde{R}|f| \big)(a) < \infty,$$

where we recall that $|\cdot|$ denotes the L^1-norm on $\mathbb{R}^d$. Further the $\mathbb{R}^d$-valued random variables

$$(1.5.10) \qquad F(a) := \sum_{m=1}^{\infty} (\tilde{R}^m f)(a)\,, \quad G_n := \sum_{m=1}^{n} f(Y_m)\,,$$

$$W_n := G_n + F(A_n)\,, \quad n \geq 1\,,$$

(with notations from (1.3.32) and (1.3.35)) are well defined, and under P_0, $(W_n)_{n \geq 1}$ is a $(\mathscr{H}_n)_{n \geq 1}$-martingale w.r.t. $\mathscr{H}_n := \sigma\{Y_1, \cdots, Y_n\}$. We use the convention $W_0 := 0$ and $\mathscr{H}_0$ equals to the trivial σ-algebra.

Finally, on the space $D_{\mathbb{R}^d}[0, \infty)$ the partial sum $\frac{1}{\sqrt{n}} G_{[n\cdot]}$ converges under P_0 in law to a d-dimensional Brownian motion with covariance matrix $\mathbf{K}$:

$(1.5.11)$

$$\mathbf{K} = \mathsf{E}_{\tilde{\nu}}\left[(W_2 - W_1)(W_2 - W_1)^T\right]$$

$$= \mathsf{E}^{\Pi}\left[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^T\right]$$

$$+ \sum_{m=1}^{\infty} \mathsf{E}^{\Pi}\left[(X_{\tau_1} - \tau_1 v)\left(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v\right)^T\right]$$

$$+ \sum_{m=1}^{\infty} \mathsf{E}^{\Pi}\left[\left(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v\right)\left(X_{\tau_1} - \tau_1 v\right)^T\right]\,,$$

where the last two terms converge in all matrix norms. (We recall the definition of E^{Π} in (1.5.2).)

Proof: (1.5.9) follows immediately from (1.2.30) and (1.4.12), because:

$$\sup_{a \in \mathbb{I}^{\mathscr{E}}} (\tilde{R}|f|)(a) \leq \sup_a \mathsf{E}_0^a[|X_{\tau_1}| + |v|\tau_1|D = \infty]$$

$$\leq (1 + |v|) \sup_a \mathsf{E}_0^a[\tau_1|D = \infty] < \infty\,.$$

With this, we can now show that

$$(1.5.12) \qquad \sup_{a \in \mathbb{I}^{\mathscr{E}}} |F(a)| < c_{29} < \infty\,.$$

Indeed, Theorem 1.5.1 implies that $\tilde{\nu}\tilde{R}f = \tilde{\nu}f = 0$ and hence for $a \in \mathbb{I}^{\mathscr{E}}$, $m \geq 1$,

$$
\begin{aligned}
\left|(\tilde{R}^m f)(a)\right| &\leq \left|(\tilde{R}^{m-1} \circ (\tilde{R}f))(a) - \tilde{\nu}\tilde{R}f\right| + \left|\tilde{\nu}\tilde{R}f\right| \\
&\leq \left\|\tilde{R}^{m-1}(a; \cdot) - \tilde{\nu}(\cdot)\right\|_{\mathrm{var}} \cdot \left\|\tilde{R}f\right\|_{L^\infty} \\
&\leq c_{14}\, e^{-c_{15}(m-1)} \left\|\tilde{R}f\right\|_{L^\infty},
\end{aligned}
$$
(1.5.13)

where (1.3.40) is used in the last step, and this with (1.5.9) proves (1.5.12).

To show that $(W_n)_{n \geq 1}$ is a $(\mathscr{H}_n)_{n \geq 1}$-martingale, we observe from Corollary 1.3.6 that for $n \geq 1$:

$$
\begin{aligned}
\mathsf{E}_0[W_{n+1} - W_n | \mathscr{H}_n] &= \mathsf{E}_0[f(Y_{n+1}) + F(A_{n+1}) - F(A_n) | \mathscr{H}_n] \\
&= (\tilde{R}f)(A_n) + (\tilde{R}F)(A_n) - F(A_n) = 0.
\end{aligned}
$$

Now we show that under P_0:

$$
\frac{1}{\sqrt{n}} W_{[n \cdot]} \xrightarrow{n \to \infty} B(\cdot) \quad \text{in law on } D_{\mathbb{R}^d}[0, \infty),
$$
(1.5.14)

where $B(\cdot)$ is a $\mathbb{R}^d$-valued Brownian motion with covariance matrix $\mathbf{K}$ given by the first line of (1.5.11). With (1.5.14) proved, we can replace $W_{[n \cdot]}$ by $G_{[n \cdot]}$ in (1.5.14), because of (1.5.12).

To show (1.5.14), we observe at first that

$$
\mathsf{E}_0\left[\left(\frac{1}{\sqrt{n}} \sup_{1 \leq k \leq [nT]} |W_k - W_{k-1}|\right)^4\right] \leq \frac{1}{n^2} \sum_{1 \leq k \leq [nT]} \mathsf{E}_0\left[|W_k - W_{k-1}|^4\right]
$$

$$
\leq \frac{1}{n^2} \mathsf{E}_0[(|X_{\tau_1} - v\tau_1| + c_{29})^4]
$$

$$
+ \frac{[nT] - 1}{n^2} \sup_{a \in \mathbb{I}^{\mathscr{E}}} \mathsf{E}_0^a[(|X_{\tau_1} - v\tau_1| + 2c_{29})^4 | D = \infty]
$$

$$
\xrightarrow{n \to \infty} 0, \quad \text{by (1.4.12) and (1.2.30)},
$$

where we used (1.5.10), (1.5.12) and Corollary 1.3.6 in the second line.

At second, by the Birkhoff's ergodic theorem, cf. page 341 in Durrett, [9], we get from Theorem 1.3.8 that $\mathsf{P}_{\tilde{\nu}}$-a.s., and hence P_0-a.s.:

$$
\sum_{k=1}^{[nt]} \frac{1}{n}(W_{k+1} - W_k)(W_{k+1} - W_k)^T \xrightarrow{n \to \infty} t\mathsf{E}_{\tilde{\nu}}[(W_2 - W_1)(W_2 - W_1)^T].
$$

and the same limit holds true for a sum from $k = 0$ to $[nt]$.

Thereafter, (1.5.14) follows immediately from the martingale central limit theorem, cf. Theorem 1.4 (a), Remark 1.5 on page 339 – 340 in Ethier-Kurtz, [11].

It remains to show the second equality in (1.5.11). We show at first that the last two terms in (1.5.11) are well defined, i.e. the series converges in any matrix norm. Let $\| \cdot \|$ be an arbitrary matrix norm, then with the notations of (1.3.32) we have for $m \geq 1$:

$$\left\| \mathsf{E}^{\Pi}[(X_{\tau_1} - \tau_1 v)(X_{\tau_{m+1}} - X_{\tau_m} - (\tau_{m+1} - \tau_m)v)^T] \right\|$$
$$(1.5.15) \quad = \left\| \mathsf{E}_{\tilde{\nu}}[(Z_0 - J_0 v)(Z_m - J_m v)^T] \right\| = \left\| \mathsf{E}_{\tilde{\nu}}[f(Y_0)\,(\tilde{R}^m f)(A_0)^T] \right\|$$
$$\leq c' \sup_a (\tilde{R}^m |f|)(a) \cdot \mathsf{E}_{\tilde{\nu}}[|Z_0 - J_0 v|]$$

where $c' > 0$ is a dimension dependent constant. Thereafter it follows now from (1.5.13) that the rightmost hand side above is

$$\leq c' c_{14} e^{-c_{15}(m-1)} \left\| \tilde{R}f \right\|_{L^\infty} \cdot \mathsf{E}_{\tilde{\nu}}[|Z_0 - J_0 v|]$$
$$\leq c_{30}\, e^{-c_{15}m} \,.$$

Consequently, the right hand side of (1.5.11) converges in any matrix norm.

To verify the second equality, we put in the definition of W_m, $m = 1, 2$:

$$\begin{aligned}
\mathbf{K} &= \mathsf{E}_{\tilde{\nu}}[(W_2 - W_1)(W_2 - W_1)^T] \\
&= \mathsf{E}_{\tilde{\nu}}\left[\{f(Y_2) + F(A_2) - F(A_1)\}\{f(Y_2) + F(A_2) - F(A_1)\}^T\right] \\
&= \mathsf{E}_{\tilde{\nu}}\left[f(Y_2)f(Y_2)^T\right] + \mathsf{E}_{\tilde{\nu}}\left[f(Y_2)F(A_2)^T\right] + \mathsf{E}_{\tilde{\nu}}\left[F(A_2)f(Y_2)^T\right] \\
&\quad + \mathsf{E}_{\tilde{\nu}}\left[F(A_2)F(A_2)^T\right] - \mathsf{E}_{\tilde{\nu}}\left[F(A_1)\big(f(Y_2) + F(A_2)\big)^T\right] \\
&\quad + \mathsf{E}_{\tilde{\nu}}\left[F(A_1)F(A_1)^T\right] - \mathsf{E}_{\tilde{\nu}}\left[(f(Y_2) + F(A_2))F(A_1)^T\right].
\end{aligned}$$

Using the fact that $\tilde{\nu}$ is the invariant distribution of the kernel $\tilde{R}$, and applying the Markov property, we see that the second and third line on the right hand side of the above equation vanish.

Now put in the definition of F from (1.5.10), the second equality of (1.5.11) follows from (1.5.15) and Corollary 1.3.6. This finishes our proof. $\square$

Thanks to this lemma, we can now prove

Theorem 1.5.3 (Functional Central Limit Theorem)

Under assumption (1.1.1) – (1.1.5), the $D_{\mathbb{R}^d}[0,\infty)$-valued random variable B^n defined in (1.5.8) converges under P_0 in law to a d-dimensional Brownian motion with a non-degenerate covariance matrix

$$
(1.5.16) \qquad \frac{\mathbf{K}}{\mathsf{E}^\Pi[\tau_1]} \, ,
$$

with $\mathbf{K}$ given in (1.5.11) and $\mathsf{E}^\Pi[\cdot]$ defined in (1.5.2).

Proof: Let k_n, $n \geq 0$ be the sequence introduced in (1.5.4). Then (1.5.5) and Dini's theorem, cf. page 129 in Dieudonné, [8], imply that P_0-a.s.:

$$
(1.5.17) \qquad \text{for all } T > 0, \ \sup_{0 \leq t \leq T} \left| \frac{k_{[tn]}}{n} - \frac{t}{\mathsf{E}^\Pi[\tau_1]} \right| \xrightarrow{n \to \infty} 0 \, .
$$

Further, for the random variables B_t^n and G_n respectively defined in (1.5.8) and (1.5.10), we observe that P_0-a.s. for any $T > 0$:

$$
\sup_{0 \leq t \leq T} \left| B_t^n - \frac{G_{k_{[tn]}}}{\sqrt{n}} \right| \leq (1 + |v|) \sup_{0 \leq k \leq k_{[nT]}} \frac{\tau_{k+1} - \tau_k}{\sqrt{n}} \, ,
$$

and

$$
(1.5.18) \qquad \sup_{0 \leq k \leq k_{[nT]}} \frac{\tau_{k+1} - \tau_k}{\sqrt{n}} \xrightarrow{n \to \infty} 0 \, , \text{ in } \mathsf{P}_0\text{-probability.}
$$

To see (1.5.18), we observe that thanks to Corollary 1.3.6, and since $k_n \leq n$, for $u > 0$:

$$
\mathsf{P}_0 \left[\sup_{0 \leq k \leq k_{[nT]}} \frac{\tau_{k+1} - \tau_k}{\sqrt{n}} > u \right]
$$

$$
\leq \mathsf{P}_0[\tau_1 > \sqrt{n}u] + nT \sup_{a \in \mathbb{I}^{\mathscr{E}}} \mathsf{P}_0^a[\tau_1 > \sqrt{n}u | D = \infty] \xrightarrow{n \to \infty} 0 \, ,
$$

where we used (1.4.11) and (1.2.30) in the last step.

Therefore, the Skorohod-distance of B^n and $\frac{G_{k_{[n \cdot]}}}{\sqrt{n}}$, cf. page 117 in Ethier-Kurtz, [11], tends to 0 in P_0-probability, as $n \to \infty$.

From this fact, (1.5.17) and Lemma 1.5.2 we obtain that, under P_0, B^n converges in law to a d-dimensional Brownian motion with covariance matrix $\frac{\mathbf{K}}{\mathsf{E}^\Pi[\tau_1]}$.

What remains to prove is the non-degeneracy of $\mathbf{K}$.

If $w^T \mathbf{K} w = 0$ for some $w \in \mathbb{R}^d$, it follows from the first line of (1.5.11) that
$$\mathrm{P}_{\tilde{\nu}}[w.f(Y_2) = w.F(A_1) - w.F(A_2)] = 1\,,$$
and since from (1.5.12) we know that F is bounded, we can find some constant $c_{31} > 0$ such that

(1.5.19) $$\mathrm{P}_{\tilde{\nu}}[w.f(Y_2) \in (-c_{31}, c_{31})] = 1\,.$$

Because $\tilde{\nu}$ is the invariant distribution of $\tilde{R}$ we obtain (recall the definition of Π in (1.5.2))

(1.5.20)
$$\begin{aligned}
1 &= \mathrm{P}_{\tilde{\nu}}[w.f(Y_1) \in (-c_{31}, c_{31})] \\
&= \Pi[(v.w)\tau_1 \in (X_{\tau_1}.w - c_{31}, X_{\tau_1}.w + c_{31})]\,.
\end{aligned}$$

Let now $r > 2\sqrt{d}$ and $H = \{z \in \mathbb{Z}^d : \ell.z < r + 2\ell.\tilde{e}\}$. Then for all $x \in \partial H$ we can construct a path in H such that $X_0 = 0$, $X_{S_1} = x$. To see this, we first notice that with the argument on page 102 in Sznitman, [43], the set $\{z \in \mathbb{Z}^d : 0 \le \ell.z < r\}$ is connected. Therefore there is a path connecting 0 and $x - 2\tilde{e}$, which remains in $\{z \in \mathbb{Z}^d : 0 \le \ell.z < r\}$ except for the last point. By inserting a loop at each step of this path, which goes back to the previous point and then returns to the current position, we can make sure that X_{S_1} does not occur within $\{z \in \mathbb{Z}^d : 0 \le \ell.z < r\}$. Now let the modified path go two steps in the direction $\tilde{e}$ after it reaches $x - 2\tilde{e}$, we get a path $(X_n)_{n \ge 0}$ with $X_0 = 0$ and $X_{S_1} = x$.

This and (1.2.30) together imply that for each $x \in \partial H$ there exists $n \in \mathbb{N}$ such that for all $a \in \mathbb{I}^{\mathscr{E}}$:

$$\mathrm{P}_0^a[X_{\tau_1} = x, \tau_1 = S_1 = n, D = \infty] > 0\,.$$

Using a nearest neighbor loop of length $2k$, $k \in \mathbb{N}$, inserted at the first jump step, we get from the ellipticity condition (1.1.1) that for all $k \in \mathbb{N}$ and $a \in \mathbb{I}^{\mathscr{E}}$:

(1.5.21) $$\mathrm{P}_0^a[X_{\tau_1} = x, \tau_1 = S_1 = n + 2k, D = \infty] > 0\,.$$

On the other hand it follows from (1.5.20) and (1.5.21) that for $x \in \partial H$, there exists $n \in \mathbb{N}$ such that:

$$(2k + n)(v.w) \in (x.w - c_{31}, x.w + c_{31})\,, \quad \text{for all } k \in \mathbb{N}\,.$$

This is only possible when

$$(1.5.22) \qquad\qquad v.w = 0\,.$$

Taking now limits points in ∂H, we observe from (1.5.20) that

$$(1.5.23) \qquad\qquad w.y = 0, \text{ for all } y \perp \ell\,,$$

hence w is co-linear to ℓ. But since $v.\ell > 0$, (1.5.22) implies that $w = 0$, which completes our proof. $\qquad\qquad \square$

Chapter 2

On Ballistic Diffusions in Random Environment

ABSTRACT

In this article we investigate diffusions in random environment. We provide a sufficient condition for a strong law of large numbers with non-vanishing limiting velocity and a functional central limit theorem. In the course of this work we introduce certain regeneration times and obtain a renewal structure. As an illustration, we apply our results to a class of anisotropic gradient-type diffusions in random environment, where the technique of the environment viewed from the particle does not apply well.

2.1 Introduction

Random motions in random media has been a very active research area over the last twenty years, both in the discrete and continuous settings. The method of the "environment viewed from the particle" has played an important role, see for instance [21], [22], [28], [31], [33]. In the continuous setting, there has been a special emphasis on the gradient-type or the incompressible drift situations, and most of the progress has occurred when there is an explicit invariant measure for the process of the environment viewed from the particle, which is absolutely continu-

ous with respect to the static distribution of the random medium, see
[7], [23], [24], [25], [30], [31], [33] . However, the general setting is still
poorly understood. On the other hand, progress has been made recently
in the discrete setting, see [3], [6], [44], [45], [46], [48]. One appeal of
the continuous theory is that, unlike in the discrete setting (cf. [3]),
imposing independence assumptions on the environment at the level of
bonds or sites, is not relevant anymore. Related to this feature, some
arguments of the discrete theory are not applicable to the continuous
setting.

The present article investigates diffusions in random environment in the
continuous setting, in situations where a priori no invariant measure of
the the process of the environment viewed from the particle is known to
exist. We provide a sufficient condition, under which the process sat-
isfies a strong law of large numbers with non-vanishing velocity, which
can further be refined by a central limit theorem. In particular, under
this condition, the diffusion in random environment exhibits a ballistic
behavior. We use a strategy which has been successful in the discrete
setting. We construct certain regeneration times which provide a re-
newal structure, see [46]. As an application of our results, we show the
ballistic behavior of a concrete class of diffusion processes in random
environment, which is a natural generalization of some discrete models
mentioned in [27], which were studied in [38].

We now describe the setting in more details. We denote with $(\Omega, \mathscr{A}, \mathbb{P})$
a probability space and with $G = \left\{ t_x : x \in \mathbb{R}^d \right\}$ a group of measure
preserving transformations, acting ergodically on Ω, for details see the
beginning of Section 2.2. We consider bounded measurable functions
$b(\cdot) : \Omega \to \mathbb{R}^d$ and $\sigma(\cdot) : \Omega \to \mathbb{R}^{d \times d}$, as well as two constants $\bar{b}$, $\bar{\sigma} > 0$
such that

$$(2.1.1) \qquad |b(\omega)| \leq \bar{b} < \infty, \quad |\sigma(\omega)| \leq \bar{\sigma} < \infty,$$

where $| \cdot |$ denotes Euclidean norm both for vectors and $d \times d$-matrices.
We write

$$(2.1.2) \qquad b(x, \omega) = b(t_x(\omega)), \quad \sigma(x, \omega) = \sigma(t_x(\omega)).$$

We assume that $b(\cdot, \omega)$ and $\sigma(\cdot, \omega)$ are Lipschitz continuous, i.e. there
exists a constant $K > 0$ such that for all $\omega \in \Omega$, $x, y \in \mathbb{R}^d$,
$$(2.1.3)$$
$$|b(x, \omega) - b(y, \omega)| \leq K |x - y| \quad \text{and} \quad |\sigma(x, \omega) - \sigma(y, \omega)| \leq K |x - y|.$$

Further, we assume that $\sigma\sigma^t(x,\omega)$ is uniformly elliptic, that means, there is a constant $\nu > 0$ such that for all $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$(2.1.4) \qquad \frac{1}{\nu}|y|^2 \leq |\sigma^t(x,\omega)\,y|^2 \leq \nu|y|^2\,,$$

where σ^t stands for the transposed matrix of σ. For a Borel subset $F \subset \mathbb{R}^d$, we define the σ-algebra generated by $b(x,\omega)$, $\sigma(x,\omega)$, for $x \in F$:

$$(2.1.5) \qquad \mathscr{H}_F \stackrel{\text{def}}{=} \sigma\big\{b(x,\omega),\sigma(x,\omega) : x \in F\big\}\,,$$

and assume an independence condition, which we call R-separation. Namely, there exists an $R > 0$, such that for all Borel subsets F, F' in $\mathbb{R}^d$ with $d(F,F') \stackrel{\text{def}}{=} \inf\{|x - x'| : x \in F, x' \in F'\} > R$,

$$(2.1.6) \qquad \mathscr{H}_F \text{ and } \mathscr{H}_{F'} \text{ are } \mathbb{P}\text{-independent.}$$

Let us mention two examples of such random vectors $b(x,\omega)$ and random matrices $\sigma(x,\omega)$ respectively. The convolution of a Poissonian point process with a Lipschitz continuous vector-valued, or matrix-valued, function supported in a ball of radius $R/2$ yields after truncation a possible example, cf. [42], page 185. Another possible example is to use the Gaussian field, described in [1], section 1.6 and 2.3. After convolution and truncation, we get another example. (The formula (2.3.4) on page 28 in [1] need be changed to $X(x) = \int g(x - \lambda)\,dZ(\lambda)$, where $g(\lambda)$ is some vector- or matrix-valued Lipschitz continuous function, compactly supported in a ball of radius $R/2$.)

We denote by $\big(C(\mathbb{R}_+,\mathbb{R}^d),\mathscr{F},\mathsf{W}\big)$ the canonical Wiener space, and with $(B_t)_{t\geq 0}$ the d-dimensional canonical Brownian motion, (which is independent from $(\Omega,\mathscr{A},\mathbb{P})$). The diffusion process in the random environment ω is the law P_x^ω (which is sometimes called the quenched law) on $\big(C(\mathbb{R}_+,\mathbb{R}^d),\mathscr{F}\big)$ of the solution of the stochastic differential equation:

$$(2.1.7) \qquad \begin{cases} dX_t(\omega) = b(X_t,\omega)\,dt + \sigma(X_t,\omega)\,dB_t\,, \\ X_0 = x\,, \qquad x \in \mathbb{R}^d,\ \omega \in \Omega\,. \end{cases}$$

The aim of this article is to study the asymptotic properties of X. under the "annealed law":

$$(2.1.8) \qquad \mathsf{P}_x \stackrel{\text{def}}{=} \mathbb{P} \times \mathsf{P}_x^\omega\,.$$

We provide a sufficient condition, see (2.3.1-i), under which the strong law of large numbers holds, that is:

$$\mathsf{P}_0\text{-a.s.} \qquad \frac{X_t}{t} \to v, \quad \text{as } t \to \infty,$$

where v is a deterministic and *non-vanishing* velocity, (cf. Theorem 2.3.2). Further, we show that the stronger condition (2.3.1-ii) guarantees a functional central limit theorem, namely as s tends to infinity, the $C(\mathbb{R}_+, \mathbb{R}^d)$-valued process

$$B_{\cdot}^s \overset{\text{def}}{=} \frac{1}{\sqrt{s}}(X_{s\cdot} - sv\cdot),$$

converges in law, under the annealed measure P_0, to a non-degenerate d-dimensional Brownian motion with covariance matrix $\mathbf{K}$, (cf. Theorem 2.3.3).

The derivation of this sufficient condition (2.3.1) is based on the strategy of constructing some regeneration times τ_k, $k \geq 1$, similar to those defined in [46], and providing a renewal structure, cf. Theorem 2.2.5. The sufficient condition is then expressed in terms of the transience of the diffusion $X_\cdot$ in some direction ℓ and the finiteness of the first (or the second) moment of τ_1 conditioned on no-backtracking, cf. (2.3.1). There are several ways to construct these regeneration times τ_k. In the spirit of [6], [48], we introduce additional Bernoulli variables. In essence, the first regeneration time τ_1 is the first integer time, at which the diffusion process reaches a local maximum in a given direction $\ell \in S^{d-1}$, the auxiliary Bernoulli variable takes value 1, and from then on the process never backtracks. The regeneration times τ_k, $k \geq 2$, are then obtained by iteration of this procedure. For the true definition, we refer to (2.2.12) – (2.2.17), (2.2.22). In our construction we take special advantage of the diffusion structure to couple the Bernoulli variables with the diffusion process, the resulting renewal structure, cf. Theorem 2.2.5, gives us a good control over the trajectory of the diffusion, see Remark 2.2.6, and we also have a convenient Markov structure, cf. Corollary 2.2.2. This provides a key tool for studying asymptotic behavior of the diffusion in a random environment. Further applications of this renewal structure and Theorems 2.3.2, 2.3.3 will follow.

As an illustration of our results, we study a class of reversible diffusion processes, for which $\sigma = \mathbb{1}$ and $b(x, \omega) = \nabla V(x, \omega)$, where $V(\cdot, \omega)$ has uniformly bounded and Lipschitz continuous derivatives, and there exist

a unit vector $\ell \in \mathbb{R}^d$, $A, B > 0$ and $\lambda > 0$ such that

$$(2.1.9) \qquad Ae^{2\lambda\ell\cdot x} \leq e^{2V(x,\omega)} \leq Be^{2\lambda\ell\cdot x}, \text{ for all } x \in \mathbb{R}^d \text{ and } \omega \in \Omega.$$

In the case where $\lambda = 0$, the diffusive behavior of the process has been extensively investigated, cf. [7], [31], [32], however we do not know of any result when $\lambda > 0$. We show in this article that when $\lambda > 0$, (no matter how small λ is) the sufficient condition (2.3.1) is fulfilled (in fact, we prove the much stronger exponential estimates under $\hat{P}_x^\omega$, cf. Theorem 2.4.9 and Corollary 2.4.10, which can also be used to deduce certain large deviation controls, cf. [43], [44]). As a result, the above mentioned law of large numbers and functional central limit theorem hold, see Theorem 2.4.11. The class under consideration includes the case where $b(x,\omega) = \nabla\tilde{V}(\omega, x) + \lambda\ell$, for some bounded $\tilde{V} \in C^1(\mathbb{R}^d, \mathbb{R})$, with bounded and Lipschitz continuous derivatives. Let us mention that this situation is closely related to some of the models studied by Lebowitz and Rost in [27], where the existence of an effective limiting velocity is mentioned as an open question.

Let us finally describe how this article is organized.

In Section 2.2, we enlarge the probability space with coupled Bernoulli random variables, cf. Theorem 2.2.1. We then define the regeneration times $(\tau_k)_{k\geq 1}$, cf. (2.2.12) – (2.2.17), and we provide the crucial renewal structure in Theorem 2.2.5.

In Section 2.3, the sufficient condition is expressed in terms of the transience of the diffusion in the direction ℓ and the (square) integrability of τ_1 conditioned on no-backtracking, cf. (2.3.1). With the help of the renewal structure constructed in Section 2.2, we are able to show the ballistic behavior of $(X_t)_{t\geq 0}$ in Theorem 2.3.2, and a functional central limit theorem in Theorem 2.3.3.

In Section 2.4, we will apply the results from the previous sections to the specific class of models described in (2.1.9). An important role is played by estimates on the exit distribution and exit time of the diffusion processes from a large cylinder with axis parallel to ℓ, cf. Proposition 2.4.2 and 2.4.3. The main integrability properties of X_{τ_1} and τ_1 are derived in Theorem 2.4.9 and Corollary 2.4.10, and our main result is stated in Theorem 2.4.11.

Finally, in the appendix, we collect some results about continuous local martingales and linear parabolic partial differential equations of second order, which are used throughout this article.

2.2   The Renewal Structure

In this section we will enlarge the probability space $\left(C(\mathbb{R}_+, \mathbb{R}^d), \mathscr{F}, \mathsf{P}_x^\omega\right)$ to $\left(C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^\mathbb{N}, \mathscr{F} \otimes \mathscr{S}, \hat{\mathsf{P}}_x^\omega\right)$, by adding some suitable auxiliary i.i.d. Bernoulli random variables, see (2.2.6) and Theorem 2.2.1.

On the enlarged space $\left(\Omega \times C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^\mathbb{N}, \mathscr{A} \otimes \mathscr{F} \otimes \mathscr{S}, \hat{\mathsf{P}}_x\right)$, see (2.2.11), we will define the regeneration times τ_k, $k \geq 1$, and discover the resulting renewal structure under the new annealed measure $\hat{\mathsf{P}}_0$, see Theorem 2.2.4 and Theorem 2.2.5.

For the random environment $(\Omega, \mathscr{A}, \mathbb{P})$, we assume that for all $x, y \in \mathbb{R}^d$, t_x is a mapping on Ω with $t_0 = 1$ and $t_{x+y} = t_x \circ t_y$; the mapping $(x, \omega) \mapsto t_x(\omega)$ is $(\mathscr{B} \otimes \mathscr{A}, \mathscr{A})$-measurable, with $\mathscr{B}$ denoting the the Borel σ-algebra on $\mathbb{R}^d$; t_x preserves the $\mathbb{P}$-measure; and for $A \in \mathscr{A}$ such that $t_x(A) = A$ for all x, then $\mathbb{P}[A] \in \{0,1\}$. We recall that under these assumptions $\left\{t_x : x \in \mathbb{R}^d\right\}$ is a group of strongly continuous unitary operators on $L^2(\Omega, \mathscr{A}, \mathbb{P})$, cf. page 223 in [19].

2.2.1   The Coupling Construction

We first need to introduce further notations. Let $\ell \in \mathbb{R}^d$ be a given unit vector, and let

$$(2.2.1) \qquad U^x \overset{\text{def}}{=} B_{6R}(x + 5R\ell), \quad B^x \overset{\text{def}}{=} B_R(x + 9R\ell),$$

be the two subsets shown in Figure 2.2.1.

We also introduce for open set G in $\mathbb{R}^d$ and $u \in \mathbb{R}$ the $(\mathscr{F}_t)_{t \geq 0}$-stopping times: $((\mathscr{F}_t)_{t \geq 0}$ denotes the canonical right continuous filtration on $\left(C(\mathbb{R}_+, \mathbb{R}^d), \mathscr{F}\right))$

$$(2.2.2) \qquad \begin{cases} T_G \overset{\text{def}}{=} \inf\{t \geq 0 : X_t \notin G\}, \\[2mm] T_u \overset{\text{def}}{=} \inf\{t \geq 0 : \ell \cdot (X_t - X_0) \geq u\}, \\[2mm] \tilde{T}_u \overset{\text{def}}{=} \inf\{t \geq 0 : \ell \cdot (X_t - X_0) \leq u\}, \end{cases}$$
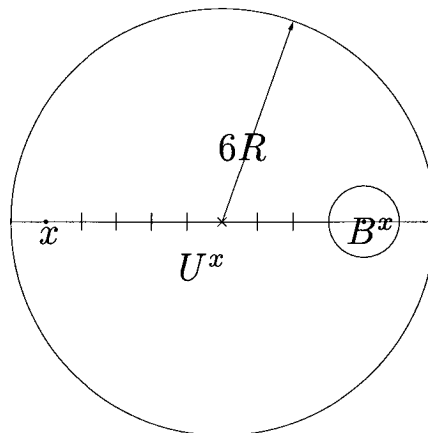
Figure 2.2.1: *Sets U^x and B^x*

and the maximal relative displacement to X_0 the process $(\ell \cdot X_s)_{s \geq 0}$ has reached within time t,

$$(2.2.3) \qquad M(t) \overset{\text{def}}{=} \sup \left\{ \ell \cdot (X_s - X_0) : 0 \leq s \leq t \right\}.$$

We denote by $p_\omega(s, x, y)$ the transition density under P_x^ω, which is a continuous function of $s > 0$, $x, y \in \mathbb{R}^d$ such that $\mathsf{P}_x^\omega[X_s \in G] = \int_G dy \, p_\omega(s, x, y)$, for all open set $G \subset \mathbb{R}^d$, cf. [13], page 139 – 141. We also introduce the sub-transition density $p_{\omega, U^x}(s, x, y)$, which is a continuous function in $s > 0$, $x \in \mathbb{R}^d$ and $y \in U^x$, fulfilling:

$$(2.2.4) \qquad \mathsf{P}_x^\omega\left[X_s \in G, T_{U^x} > s\right] = \int_G dy \, p_{\omega, U^x}(s, x, y),$$

for all open set $G \subset U^x$.

Under our assumptions on the drift term $b(\cdot, \omega)$ and the diffusion matrix $\sigma\sigma^t(\cdot, \omega)$, there exists a constant $\varepsilon(\nu, d, \bar{b}, \bar{\sigma}, R, K) \in (0, 1)$ such that for all $\omega \in \Omega$,

$$(2.2.5) \qquad p_{\omega, U^x}(1, x, y) \geq \frac{2\varepsilon}{|B_R|} > 0, \text{ for all } x \in \mathbb{R}^d \text{ and } y \in B^x,$$

where $|B_R|$ denotes the volume of B_R. We refer to Corollary 2.5.5 in the Appendix for the proof of (2.2.5).

With the help of (2.2.5), we are going to use a coupling construction enlarging our probability space $(C(\mathbb{R}_+, \mathbb{R}^d), \mathscr{F}, \mathsf{P}_x^\omega)$ to include some auxiliary i.i.d. Bernoulli random variables $(\lambda_m)_{m \in \mathbb{N}}$.

Before providing this coupling construction, let us give some other notations. We denote by λ_j the canonical coordinates on $\{0,1\}^{\mathbb{N}}$ (the variables λ_j will turn out to be i.i.d. Bernoulli random variables with success probability ε). Further, let $\mathscr{S}_m \stackrel{\text{def}}{=} \sigma\{\lambda_0, \cdots, \lambda_m\}$, $m \in \mathbb{N}$, denote the canonical filtration on $\{0,1\}^{\mathbb{N}}$ generated by $(\lambda_m)_{m \in \mathbb{N}}$ and $\mathscr{S} \stackrel{\text{def}}{=} \sigma\{\bigcup_m \mathscr{S}_m\}$ be the canonical σ-algebra. To simplify notation let us write for $t \geq 0$:

$$(2.2.6) \qquad \mathscr{Z}_t \stackrel{\text{def}}{=} \mathscr{F}_t \otimes \mathscr{S}_{\lceil t \rceil}, \quad \mathscr{Z} \stackrel{\text{def}}{=} \mathscr{F} \otimes \mathscr{S} = \sigma\left\{\bigcup_{m \in \mathbb{N}} \mathscr{Z}_m\right\},$$

with $\lceil t \rceil \stackrel{\text{def}}{=} \{n \in \mathbb{N} : t \leq n\}$. We also introduce the shift operators $\{\theta_m : m \in \mathbb{N}\}$, with

$$\theta_m : \left(C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^{\mathbb{N}}, \mathscr{Z}\right) \to \left(C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^{\mathbb{N}}, \mathscr{Z}\right),$$

such that

$$(2.2.7) \qquad\qquad \theta_m (X_\cdot, \lambda_\cdot) = (X_{m+\cdot}, \lambda_{m+\cdot}),$$

for $X_\cdot \in C(\mathbb{R}_+, \mathbb{R}^d)$ and $\lambda_\cdot \in \{0,1\}^{\mathbb{N}}$.

Now we can state the coupling construction.

Theorem 2.2.1 (Coupling Construction)
For every $\omega \in \Omega$ and $x \in \mathbb{R}^d$ there exists a probability measure $\hat{\mathsf{P}}_x^\omega$ on $\left(C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^{\mathbb{N}}, \mathscr{Z}\right)$ depending measurably on ω and x, such that

1. *Under $\hat{\mathsf{P}}_x^\omega$, $(X_t)_{t \geq 0}$ is P_x^ω-distributed, and the λ_m, $m \geq 0$, are i.i.d. Bernoulli variables with success probability ε (recall (2.2.5)).*

2. *Under $\hat{\mathsf{P}}_x^\omega$, λ_m ($m \geq 1$) is independent of $\mathscr{F}_m \otimes \mathscr{S}_{m-1}$, and conditioned on $\mathscr{Z}_m$, $X_\cdot \circ \theta_m$ has the same law as $X_\cdot$ under $\hat{\mathsf{P}}_{X_m, \lambda_m}^\omega$, where for $\lambda = 0, 1$, $\hat{\mathsf{P}}_{x,\lambda}^\omega$ denotes the law $\hat{\mathsf{P}}_x^\omega[\,\cdot\,|\lambda_0 = \lambda]$.*

3. *$\hat{\mathsf{P}}_{x,1}^\omega$ almost surely, $X_s \in U^x$ for $s \in [0,1]$ (recall (2.2.1)).*

4. *Under $\hat{\mathsf{P}}_{x,1}^\omega$, X_1 is uniformly distributed on B^x (recall (2.2.1)).*

Proof: Given a probability kernel $\hat{\mathsf{P}}_{x,\lambda}^\omega[X_\cdot \in O]$, for $O \in \mathscr{F}_1$, $x \in \mathbb{R}^d$, $\lambda \in \{0,1\}$ and $\omega \in \Omega$, there will be a unique probability kernel $\hat{\mathsf{P}}_x^\omega$ on $\mathscr{Z}$, for $x \in \mathbb{R}^d$, $\omega \in \Omega$, such that under $\hat{\mathsf{P}}_x^\omega$:

- λ_m is a Bernoulli random variable with success probability ε, independent of $\mathscr{F}_m \otimes \mathscr{S}_{m-1}$, when $m \geq 1$;

- For $O \in \mathscr{F}_1$, the conditional expectation $\hat{\mathsf{P}}^\omega_x[\theta_m^{-1}(X. \in O)|\mathscr{Z}_m]$ equals $\hat{\mathsf{P}}^\omega_{X_m,\lambda_m}[O]\ \hat{\mathsf{P}}^\omega_x$-a.s..

Here is how we define $\hat{\mathsf{P}}^\omega_{x,\lambda}[X. \in O]$ for $O \in \mathscr{F}_1$, $x \in \mathbb{R}^d$, $\omega \in \Omega$ and $\lambda \in \{0,1\}$, namely we set

$$(2.2.8) \qquad \hat{\mathsf{P}}^\omega_{x,\lambda_0=1}[X. \in O] = \frac{1}{|B_R|}\int_{B^x} dy\ \mathsf{P}^{\omega,1}_{x,y}[O|T_{U^x} > 1],$$

and

$$(2.2.9) \quad \hat{\mathsf{P}}^\omega_{x,\lambda_0=0}[X. \in O]$$

$$= \frac{1}{1-\varepsilon}\left\{\mathsf{P}^\omega_x[O] - \frac{\varepsilon}{|B_R|}\int_{B^x} dy\ \mathsf{P}^{\omega,1}_{x,y}[O|T_{U^x} > 1]\right\},$$

where $\mathsf{P}^{\omega,1}_{x,y}$ is the bridge measure from x to y in time 1 under P^ω_x; i.e. $\mathsf{P}^{\omega,1}_{x,y}$ is the unique probability measure on $\left(C([0,1],\mathbb{R}^d),\mathscr{F}_1\right)$ such that for all $O_s \in \mathscr{F}_s, s < 1$:

$$\mathsf{P}^{\omega,1}_{x,y}[O_s] = \frac{1}{p_\omega(1,x,y)}\ \mathsf{E}^\omega_x[O_s, p_\omega(1-s,X_s,y)].$$

The proof of the existence of this bridge measure can be found in [42], page $137 - 139$. Although the proof in [42] is for the Brownian bridge, it can still be used for the proof of $\mathsf{P}^{\omega,1}_{x,y}$ with little modification. The only change one need to do is in the proof of (A.8) on page 138, namely one need to use the inequality $1/p_\omega(t-s,X_s,y) \geq \varphi(t-s)^{\frac{d}{2}}\exp\left\{\frac{\mu(X_s-y)^2}{2(t-s)}\right\}$, $\mu > 0, \varphi > 0$, which can be found in [13], page 141.

Observe that $p_{\omega,U^x}(1,x,y) = p_\omega(1,x,y)\ \mathsf{P}^{\omega,1}_{x,y}[T_{U^x} > 1]$ and $\mathsf{P}^\omega_x[X. \in O, T_{U^x} > 1, X_1 \in B'] = \int_{B'} p_{\omega,U^x}(1,x,y)\cdot\mathsf{P}^{\omega,1}_{x,y}[X. \in O|T_{U^x} > 1]\ dy$, so in view of (2.2.5), $\hat{\mathsf{P}}^\omega_{x,\lambda}$ is well defined. It is then straightforward to see that the resulting $\hat{\mathsf{P}}^\omega_x$ fulfills 1, 2, 3, 4. $\qquad\square$

As a consequence, we have

Corollary 2.2.2 (Markov Property)
Under $\hat{\mathsf{P}}^\omega_x$, the joint process $(X_m,\lambda_m)_{m\in\mathbb{N}}$ is a time homogeneous Markov chain, with respect to the filtration $(\mathscr{Z}_m = \mathscr{F}_m \otimes \mathscr{S}_m)_{m\in\mathbb{N}}$, and in fact

$\hat{\mathsf{P}}_x^\omega$-a.s.

$$(2.2.10) \qquad \hat{\mathsf{P}}_x^\omega[(X_.,\lambda_.) \circ \theta_m \in \star | \mathscr{Z}_m] = \hat{\mathsf{P}}_{X_m,\lambda_m}^\omega[(X_.,\lambda_.) \in \star].$$

Finally, let us introduce the new annealed measure on $\left(\Omega \times C(\mathbb{R}_+,\mathbb{R}^d) \times \{0,1\}^{\mathbb{N}}, \mathscr{A} \otimes \mathscr{Z}\right)$, see also (2.1.8):

$$(2.2.11) \qquad \hat{\mathsf{P}}_x \overset{\text{def}}{=} \mathbb{P} \times \hat{\mathsf{P}}_x^\omega \quad \text{and} \quad \hat{\mathsf{E}}_x \overset{\text{def}}{=} \mathbb{E} \times \hat{\mathsf{E}}_x^\omega,$$

and observe that by property 1 in Theorem 2.2.1, $(X_t)_{t\geq 0}$ has same distribution under $\hat{\mathsf{P}}_x$ and P_x.

2.2.2   The Regeneration Times τ_k

In this part, we will define the regeneration times τ_k, $k \in \mathbb{N}$, and discover the resulting renewal structure.

To define the first regeneration time τ_1, we need to introduce a sequence of integer-valued $(\mathscr{Z}_t)_{t\geq 0}$-stopping times N_k, for which the condition $\lambda_{N_k} = 1$ holds, and at these times the process $(\ell \cdot X_s)_{s\geq 0}$ reaches essentially a local maxima (within a small variation). Then τ_1 is the first $N_k + 1$, $k \geq 1$, such that the process $(\ell \cdot X_t)_{t\geq 0}$ never goes below $\ell \cdot X_{N_k+1} - R$ after $N_k + 1$.

To define N_k, we introduce the integer-valued $(\mathscr{F}_t)_{t\geq 0}$-stopping times $(\tilde{N}_k)_{k\geq 1}$, which are essentially the times when $(\ell \cdot X_s)_{s\geq 0}$ reaches local maxima (also within a small variation). Then, we choose $\tilde{N}_1$ to be the first N_k with $\lambda_{N_k} = 1$.

Here is how we precisely define them: First, we introduce for $a > 0$ the $(\mathscr{F}_t)_{t\geq 0}$-stopping times $V_k(a)$, $k \geq 0$: V_0 is the first time $\left(\ell \cdot (X_s - X_0)\right)_{s\geq 0}$ reaches a, and V_{k+1} is the first time $(\ell \cdot X_s)_{s\geq 0}$ reaches R above the local maximum it reached till $\lceil V_k \rceil$, that is (recall $M(a)$ in (2.2.3) and T_u in (2.2.2)),

$$(2.2.12) \qquad \begin{aligned} V_0(a) &\overset{\text{def}}{=} T_a \,; \quad V_1(a) \overset{\text{def}}{=} T_{M(\lceil V_0(a)\rceil)+R} \,; \\ V_{k+1}(a) &\overset{\text{def}}{=} T_{M(\lceil V_k(a)\rceil)+R} \,. \end{aligned}$$

Then, we define $\tilde{N}_1(a)$ to be the first $\lceil V_k \rceil$, $k \geq 0$, such that $|\ell \cdot (X_s - X_{V_k})| \leq \frac{R}{2}$ for all $s \in [V_k, \lceil V_k \rceil]$; and $\tilde{N}_{k+1}(a)$ to be $\tilde{N}_1(3R)$ shifted after

$\tilde{N}_k(a)$ (it is *not* $\tilde{N}_1(a)$ after $\tilde{N}_k(a)$, the reason for this comes from our definition of N_{k+1} later in (2.2.15)):

$$(2.2.13) \quad \begin{cases} \tilde{N}_1(a) \overset{\text{def}}{=} \inf \Big\{ \lceil V_k(a) \rceil : k \geq 0, \\ \qquad\qquad \sup_{s \in [V_k, \lceil V_k \rceil]} |\ell \cdot (X_s - X_{V_k})| \leq \tfrac{R}{2} \Big\}, \\ \tilde{N}_{k+1}(a) \overset{\text{def}}{=} \tilde{N}_1(3R) \circ \theta_{\tilde{N}_k(a)} + \tilde{N}_k(a), \quad k \geq 1, \\ N_1(a) \overset{\text{def}}{=} \inf \Big\{ \tilde{N}_k(a) : k \geq 1, \lambda_{\tilde{N}_k(a)} = 1 \Big\}; \end{cases}$$

(by convention we set $\tilde{N}_{k+1} = \infty$ if $\tilde{N}_k = \infty$). We illustrate in Figure 2.2.2 the situation, where $\tilde{N}_2(a)$ is $\lceil V_0(3R) \rceil$ after $\tilde{N}_1(a)$.
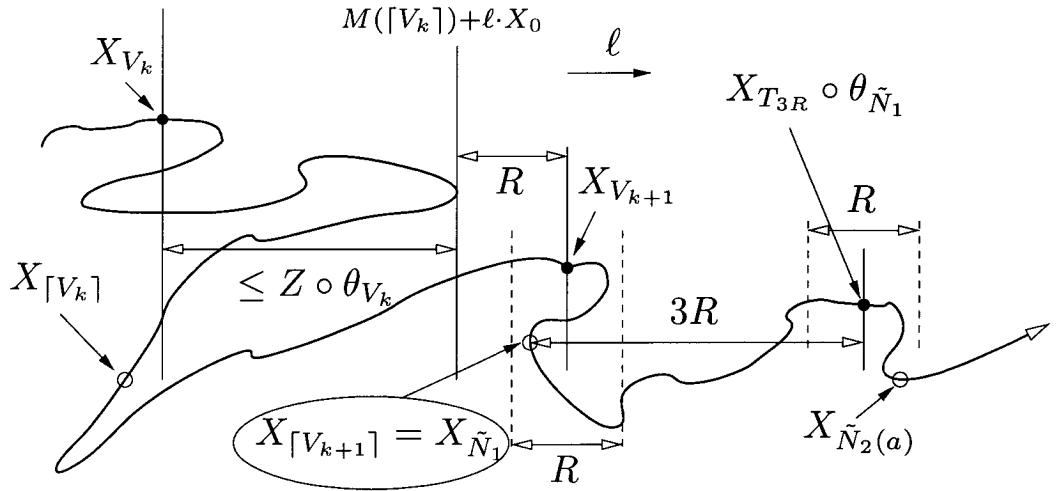

Figure 2.2.2: $V_k(a)$ *and* $\tilde{N}_m(a)$

Observe that $\tilde{N}_k$, $k \geq 1$, are integer-valued, bigger or equal to 1, and P_x^ω-a.s. $\sup_{s \leq \tilde{N}_k} \ell \cdot (X_s - X_{\tilde{N}_k}) \leq R$, i.e. within a variation of R, $\ell \cdot X_{\tilde{N}_k}$ reaches a local maximum. Now we can define the $(\mathcal{Z}_t)_{t \geq 0}$-stopping times (recall (2.2.2)):

$$(2.2.14) \quad \begin{cases} S_1 \overset{\text{def}}{=} N_1(3R) + 1; \quad J_1 \overset{\text{def}}{=} S_1 + \tilde{T}_{-R} \circ \theta_{S_1}; \\ R_1 \overset{\text{def}}{=} \lceil J_1 \rceil = S_1 + D \circ \theta_{S_1}; \end{cases}$$

with $D \overset{\text{def}}{=} \lceil \tilde{T}_{-R} \rceil$.

Now we shall define the integer-valued $(\mathcal{Z}_t)_{t \geq 0}$-stopping time N_{k+1}, $k \geq 1$, which is bigger than R_k such that $\lambda_{N_{k+1}} = 1$, and the process

$(\ell \cdot X_s)_{s\geq 0}$ does not go above $\ell \cdot X_{N_{k+1}} + R$ until time N_{k+1}. More precisely:

$$(2.2.15) \qquad N_{k+1} \overset{\text{def}}{=} R_k + N_1(a_k) \circ \theta_{R_k} \quad \text{with}$$
$$a_k \overset{\text{def}}{=} M(R_k) - \ell \cdot (X_{R_k} - X_0) + R \,,$$

(the shift θ_{R_k} is not applied to a_k in the above definition, cf. Figure 2.2.3).
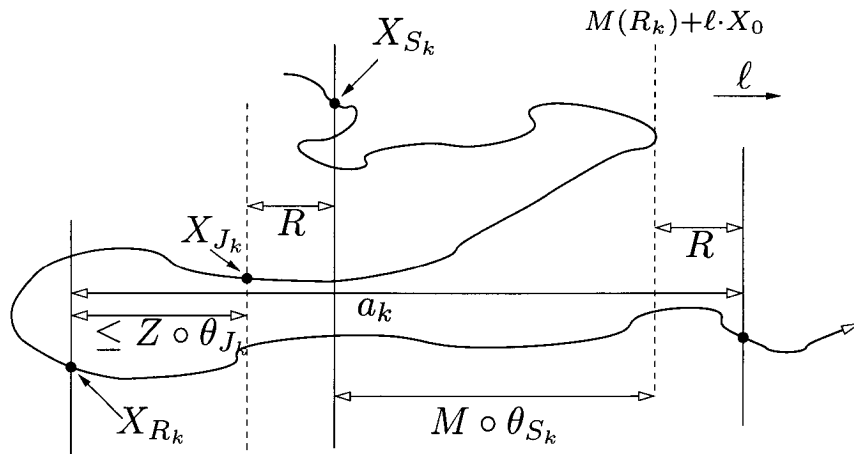

Figure 2.2.3:

The quantity a_k in (2.2.15) is used to make sure that N_{k+1} is an integer bigger than R_k, such that $\sup_{s\leq N_{k+1}} \ell \cdot X_s \leq \ell \cdot X_{N_{k+1}} + R$ (here is why we defined the stopping times $\big(V_k(a)\big)_{k\geq 0}$ for a general a).

As in (2.2.14), we define the $(\mathscr{Z}_t)_{t\geq 0}$-stopping times:

$$(2.2.16) \qquad \begin{cases} S_{k+1} \overset{\text{def}}{=} N_{k+1} + 1; \quad J_{k+1} \overset{\text{def}}{=} S_{k+1} + \tilde{T}_{-R} \circ \theta_{S_{k+1}} \,; \\[2mm] R_{k+1} \overset{\text{def}}{=} \lceil J_{k+1} \rceil = S_{k+1} + D \circ \theta_{S_{k+1}} \,. \end{cases}$$

Observe that for all $k \in \mathbb{N}$, the $(\mathscr{Z}_t)_{t\geq 0}$-stopping times N_k, S_k and R_k are integer-valued, possibly equal to infinity. Of course we have $1 \leq N_1 \leq S_1 \leq J_1 \leq R_1 \leq N_2 \leq S_2 \leq J_2 \leq R_2 \cdots \leq \infty$.

With the help of these stopping times, the first *regeneration time* is defined, as in [46], by

$$(2.2.17) \qquad \tau_1 \overset{\text{def}}{=} \inf\{S_k : S_k < \infty, R_k = \infty\} \leq \infty \,.$$

Again, τ_1 is integer-valued, and $\tau_1 \geq 2$, because $N_1 \geq 1$.

With this definition, we see that on the event $\{\tau_1 < \infty\}$, $\hat{\mathsf{P}}_x$-a.s., $\ell \cdot X_s \leq \ell \cdot X_{\tau_1 - 1} + R \leq \ell \cdot X_{\tau_1} - 7R$, for $s \leq \tau_1 - 1$, see also Theorem 2.2.1 and Figure 2.2.1, i.e. $(X_s)_{s \leq \tau_1 - 1}$ remains in the half space $\mathcal{L}(\ell \cdot X_{\tau_1} - 7R)$, with $\mathcal{L}(a) \overset{\text{def}}{=} \{z \in \mathbb{R}^d : z \cdot \ell \leq a\}$ for $a \in \mathbb{R}$. On the other hand, because the process $(\ell \cdot X_t)_{t \geq 0}$ never goes below $\ell \cdot X_{\tau_1} - R$ after τ_1, i.e. $(X_t)_{t \geq \tau_1}$ belongs to the half space $\mathcal{R}(\ell \cdot X_{\tau_1} - R)$, where for $a \in \mathbb{R}$, $\mathcal{R}(a) \overset{\text{def}}{=} \{z \in \mathbb{R}^d : z \cdot \ell \geq a\}$. This will turn out to be an important issue in the proof of Theorem 2.2.4.

We will see in Proposition 2.2.7 below that the $\hat{\mathsf{P}}_0$ almost sure finiteness of τ_1 is equivalent to P_0-a.s., $\lim_{t \to \infty} \ell \cdot X_t = \infty$. For the time being we begin with

Lemma 2.2.3
Suppose that $\hat{\mathsf{P}}_0$-a.s. $\tau_1 < \infty$, then $\mathsf{P}_0[D = \infty] > 0$.

Proof: We prove this by contradiction. If $\mathsf{P}_0[D = \infty] = 0$, it follows from the stationarity of $\mathbb{P}$-measure that $\int dx\, \mathsf{P}_x[D = \infty] = 0$. Thereafter, by Fubini's theorem, there exists a $\mathbb{P}$-null-set $\Upsilon \subset \Omega$, such that for all $\omega \notin \Upsilon$, outside a Lebesgue-null-set $\mathcal{N}(\omega) \subset \mathbb{R}^d$, $\mathsf{P}_x^\omega[D = \infty] = \mathsf{P}_x^\omega[\tilde{T}_{-R} = \infty] = 0$ holds.

Because by our assumptions (2.1.1), (2.1.3) and (2.1.4), the transition density $p_\omega(t, x, y)$ exists for all $\omega \in \Omega$ and $t > 0$, it follows from the Markov property of $(X_t)_{t \geq 0}$ under P_x^ω that for $\omega \notin \Upsilon$ and for all $x \in \mathbb{R}^d$, $\mathsf{P}_x^\omega[\cap_{\substack{q \in \mathbb{Q} \\ q > 0}} \tilde{T}_{-R} \circ \theta_q < \infty] = 1$.

Therefore, for ω outside the $\mathbb{P}$-null-set Υ and all $x \in \mathbb{R}^d$, $\mathsf{P}_x^\omega[\tilde{T}_{-\frac{R}{2}} < \infty] = 1$, which implies by the strong Markov property that P_x^ω-a.s. $\liminf_t X_t \cdot \ell = -\infty$. This contradicts the assumption $\hat{\mathsf{P}}_0[\tau_1 < \infty] = 1$. $\square$

Let us define on the space $\left(\Omega \times C(\mathbb{R}_+, \mathbb{R}^d) \times \{0, 1\}^{\mathbb{N}}, \mathscr{A} \otimes \mathscr{X}\right)$ the σ-algebra $\mathscr{G}$, which is generated by the sets of the form:

$$(2.2.18) \qquad \{\tau_1 = m\} \cap O_{m-1} \cap \{X_{m-1} \cdot \ell > a\} \cap \{X_m \in G\} \cap F_a,$$

$(m \geq 2,\ a \in \mathbb{R})$ with $O_{m-1} \in \mathscr{X}_{m-1}$, $G \subset \mathbb{R}^d$ open, and $F_a \in \mathscr{H}_{\mathcal{L}(a+R)}$ (recall $\mathscr{H}$ in (2.1.5) and $\mathcal{L}$ below (2.2.17)). The situation is shown in Figure 2.2.4.
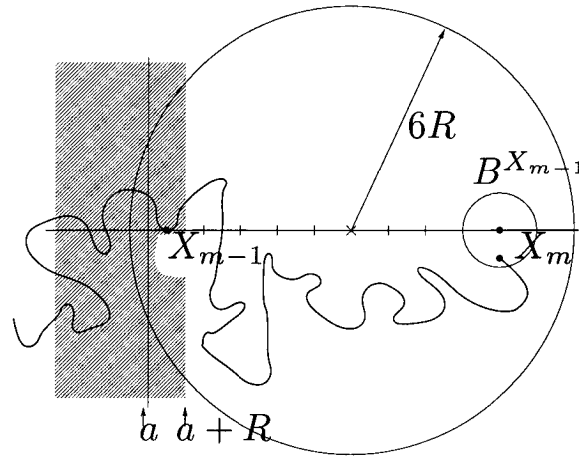
Figure 2.2.4:

Essentially, the σ-algebra $\mathscr{G}$ describes the history of the Bernoulli variables $\lambda.$, the path of the process $(X_t)_{t\geq 0}$, and the random environment ω possibly contributing before time $\tau_1 - 1$.

The key step in the study of the renewal structure mentioned in the introduction is now:

Theorem 2.2.4

Assume that $\hat{\mathsf{P}}_0$-*a.s.* $\tau_1 < \infty$. *Let* $x \in \mathbb{R}^d$, *and* f, g, h *be bounded functions, which are respectively* $\mathscr{X}$- *(recall (2.2.6))*, $\mathscr{H}_{\mathcal{R}(-R)}$- *(recall $\mathscr{H}$ in (2.1.5) and $\mathcal{R}$ below (2.2.17))*, *and* $\mathscr{G}$-*measurable. Then*

$$(2.2.19) \qquad \begin{aligned} &\hat{\mathsf{E}}_x \left[f(X_{\tau_1+\cdot} - X_{\tau_1}, \lambda_{\tau_1+\cdot})\, g \circ t_{X_{\tau_1}}\, h \right] \\ =& \hat{\mathsf{E}}_0 \left[f(X_{\cdot}, \lambda_{\cdot})\, g | D = \infty \right] \cdot \hat{\mathsf{E}}_x[h] \,, \end{aligned}$$

where t_y, $y \in \mathbb{R}^d$, *is the shift operator from the beginning of Section 2.2.*

Proof: By Lemma 2.2.3, we know that $\mathsf{P}_0[D = \infty] = \hat{\mathsf{P}}_0[D = \infty] > 0$, and the right hand side of (2.2.19) is well-defined.

Since the σ-algebra $\mathscr{G}$ is generated by sets of the form in (2.2.18), which form a π-system, it is sufficient to prove (2.2.19) for $h = 1_{\{\tau_1 = m\}} \cdot 1_{F_a} \cdot 1_{O_{m-1}} \cdot 1_{X_m \in G} \cdot 1_{\{X_{m-1} \cdot \ell > a\}}$, with $O_{m-1} \in \mathscr{X}_{m-1}$, $G \subset \mathbb{R}^d$ open, and $F_a \in \mathscr{H}_{\mathcal{L}(a+R)}$.

For this special h, the left hand side of (2.2.19) is now:

$$\hat{\mathsf{E}}_x\left[f(X_{\tau_1+\cdot} - X_{\tau_1}, \lambda_{\tau_1+\cdot})\, g \circ t_{X_{\tau_1}}\, h\right]$$

$$=\hat{\mathsf{E}}_x\left[f(X_{m+\cdot} - X_m, \lambda_{m+\cdot})\, g \circ t_{X_m};\right.$$

$$\left.\tau_1 = m, O_{m-1}, X_{m-1} \cdot \ell > a, F_a, X_m \in G\right].$$

Observe that $\{\tau_1 = m\} \cap O_{m-1} = \tilde{O}_{m-1} \cap \{D \circ \theta_m = \infty\} \cap \{\lambda_{m-1} = 1\}$, for some $\tilde{O}_{m-1} \in \mathscr{X}_{m-1} \cap \{X_{m-1} \cdot \ell + R \geq X_t \cdot \ell, \forall t \leq m-1\}$, therefore the last expression is now:

$$(2.2.20) \quad = \mathbb{E}\Bigg\{\hat{\mathsf{E}}_x^\omega\Bigg[\hat{\mathsf{E}}_x^\omega\left[f(X_{m+\cdot} - X_m, \lambda_{m+\cdot})\, g \circ t_{X_m};\right.$$

$$\left.X_m \in G, D \circ \theta_m = \infty \Big| \mathscr{X}_{m-1}\right];$$

$$\left.F_a, \tilde{O}_{m-1}, X_{m-1} \cdot \ell > a, \lambda_{m-1} = 1\right]\Bigg\}.$$

By the Markov property, cf. (2.2.10), we observe that P_x^ω-a.s. on the event $\{\lambda_{m-1} = 1\}$,

$$\hat{\mathsf{E}}_x^\omega\left[f(X_{m+\cdot} - X_m, \lambda_{m+\cdot})\, g \circ t_{X_m}; X_m \in G, D \circ \theta_m = \infty \Big| \mathscr{X}_{m-1}\right]$$

$$=\hat{\mathsf{E}}_{X_{m-1},1}^\omega\left[f(X_{1+\cdot} - X_1, \lambda_{1+\cdot})\, g \circ t_{X_1}; X_1 \in G, D \circ \theta_1 = \infty\right]$$

$$=\hat{\mathsf{E}}_{X_{m-1},1}^\omega\left[\hat{\mathsf{E}}_{X_1,\lambda_1}^\omega[f(X_\cdot - X_0, \lambda_\cdot)\, g \circ t_{X_0}; D = \infty],\ X_1 \in G\right].$$

Note that, by Theorem 2.2.1, λ_1 is independent of X_1 under the measure $\hat{\mathsf{P}}_{y,1}^\omega$, for all $y \in \mathbb{R}^d$; and using property 4 of Theorem 2.2.1, the last expression is:

$$=\frac{1}{|B_R|}\int_{B^{X_{m-1}}\cap G} dy\,\hat{\mathsf{E}}_y^\omega\left[f(X_\cdot - y, \lambda_\cdot)\, g \circ t_y, D = \infty\right].$$

Plugging this formula into (2.2.20) and using Fubini's theorem, the left hand side of (2.2.19) now equals

$$\frac{1}{|B_R|}\int dy\,\mathbb{E}\Bigg\{\hat{\mathsf{E}}_x^\omega\left[\hat{\mathsf{E}}_y^\omega\left[f(X_\cdot - y, \lambda_\cdot)\, g \circ t_y, D = \infty\right];\right.$$

$$\left.F_a, \tilde{O}_{m-1}, X_{m-1} \cdot \ell > a, \lambda_{m-1} = 1, \{y \in B^{X_{m-1}} \cap G\}\right]\Bigg\}.$$

Set $V \stackrel{\text{def}}{=} \{F_a, \tilde{O}_{m-1}, X_{m-1} \cdot \ell > a, \lambda_{m-1} = 1, y \in B^{X_{m-1}} \cap G\}$, the last expression equals

$$(2.2.21) \qquad \frac{1}{|B_R|} \int dy \, \mathbb{E}\left\{ \hat{\mathsf{P}}_x^\omega[V] \cdot \hat{\mathsf{E}}_y^\omega[f(X. - y, \lambda.), D = \infty] \cdot g \circ t_y \right\}.$$

Observe that $1_{\{y \in B^{X_{m-1}}\}}$ is zero for $y \cdot \ell - 8R \le X_{m-1} \cdot \ell$, see also Figure 2.2.4. Therefore, in the above integral we only need to consider y such that $a < y \cdot \ell - 8R$, and thus $F_a \in \mathcal{H}_{\mathcal{L}(y \cdot \ell - 7R)}$. Also observe that for the $\tilde{O}_{m-1}$ introduced above (2.2.20), we have $\tilde{O}_{m-1} \subset \{X_{m-1} \cdot \ell + R \ge X_t \cdot \ell, \forall t \le m - 1\}$. Therefore, we see that $\hat{\mathsf{P}}_x^\omega[V]$ is $\mathcal{H}_{\mathcal{L}(y \cdot \ell - 7R)}$-measurable.

On the other hand, since g is $\mathcal{H}_{\mathcal{R}(-R)}$-measurable and due to the restriction $D = \infty$, we observe that $\hat{\mathsf{E}}_y^\omega[f(X. - y, \lambda.), D = \infty] \cdot g \circ t_y$ is $\mathcal{H}_{\mathcal{R}(y \cdot \ell - R)}$-measurable.

As a result of R-separation, cf. (2.1.6), we see that $\hat{\mathsf{P}}_x^\omega[V]$ and $\hat{\mathsf{E}}_y^\omega[f(X. - y, \lambda.), D = \infty] \cdot g \circ t_y$ are independent under the $\mathbb{P}$-measure. Using this observation, (2.2.21) equals

$$\int dy \, \hat{\mathsf{E}}_x\left[\tfrac{1_V}{|B_R|}\right] \cdot \hat{\mathsf{E}}_y[f(X. - y, \lambda.) \, g \circ t_y, D = \infty]$$

$$= \left(\int dy \, \hat{\mathsf{E}}_x\left[\tfrac{1_V}{|B_R|}\right]\right) \cdot \hat{\mathsf{E}}_0[f(X., \lambda.) \, g, D = \infty],$$

where we used the stationarity of the $\mathbb{P}$-measure in the last step. By taking $f = g = 1$, we get from the above calculation that $\hat{\mathsf{E}}_x[h] = \hat{\mathsf{P}}_0[D = \infty] \cdot \int dy \, \hat{\mathsf{E}}_x\left[\tfrac{1_V}{|B_R|}\right]$, therefore the left hand side of (2.2.19) is now

$$\hat{\mathsf{E}}_0\left[f(X., \lambda.) \, g, D = \infty\right] \cdot \frac{\hat{\mathsf{E}}_x[h]}{\hat{\mathsf{P}}_0[D = \infty]}$$

$$= \hat{\mathsf{E}}_0\left[f(X., \lambda.) \, g | D = \infty\right] \cdot \hat{\mathsf{E}}_x[h].$$

This finishes the proof. $\qquad \square$

We now define inductively on the event $\{\tau_1 < \infty\}$ a non-decreasing sequence of random variables, by viewing τ_k, $k \ge 1$, as a function of $(X., \lambda.)$:

$$(2.2.22) \qquad \begin{aligned} &\tau_{k+1}\big((X., \lambda.)\big) \\ &\stackrel{\text{def}}{=} \tau_1\big((X., \lambda.)\big) + \tau_k\big((X_{\tau_1 + \cdot} - X_{\tau_1}, \lambda_{\tau_1 + \cdot})\big), \quad k \ge 1, \end{aligned}$$

and set by convention $\tau_{k+1} = \infty$ on $\{\tau_k = \infty\}$. We observe that for each k, τ_k is either infinite or a positive integer. Of course, $\tau_{k+1} = \tau_k\big((X_.,\lambda_.)\big) + \tau_1\big((X_{\tau_k+.} - X_{\tau_k}, \lambda_{\tau_k+.})\big)$, but we prefer the definition (2.2.22) in view of the proof of the renewal structure promised in the introduction: (in the next theorem, we set $\tau_0 = 0$)

Theorem 2.2.5 (Renewal Structure)
Assume that $\hat{\mathsf{P}}_0$-a.s., $\tau_1 < \infty$. Then under the measure $\hat{\mathsf{P}}_0$, the random variables $Z_k \overset{def}{=} \big(X_{(\tau_k+.)\wedge(\tau_{k+1}-1)} - X_{\tau_k}; X_{\tau_{k+1}} - X_{\tau_k}; \tau_{k+1} - \tau_k\big)$, $k \geq 0$, are independent. Furthermore, Z_k, $k \geq 1$, under $\hat{\mathsf{P}}_0$, have the distribution of $Z_0 = \big(X_{.\wedge(\tau_1-1)} - X_0; X_{\tau_1} - X_0; \tau_1\big)$ under $\hat{\mathsf{P}}_0[\,\cdot\,|D = \infty]$.

Proof: Let us define on the space $\big(\Omega \times C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^{\mathbb{N}}, \mathscr{A} \otimes \mathscr{X}\big)$ the σ-algebra $\mathscr{G}_{n+1}$, which is generated by $(Z_k)_{0 \leq k \leq n}$. It suffices to show that for h bounded and $\mathscr{G}_{n+1}$-measurable, $n \geq 0$,

$$(2.2.23) \qquad \hat{\mathsf{E}}_0[h, Z_{n+1} \in *] = \hat{\mathsf{E}}_0[h] \cdot \hat{\mathsf{P}}_0[Z_0 \in *|D = \infty].$$

We prove this by induction. The case $n = 0$ follows from Theorem 2.2.4, because $\mathscr{G}_1 \subset \mathscr{G}$, with $\mathscr{G}$ defined in (2.2.18). For the step $n \to n+1$, we observe that because $\mathscr{G}_{n+1}$ is generated by $\mathscr{G}_1$ and $\theta_{\tau_1}^{-1}(\mathscr{G}_n)$, without loss of generality we can assume that $h = h_1 \cdot h_n \circ \theta_{\tau_1}$, with $h_n \in \mathscr{G}_n$ and $h_1 \in \mathscr{G}_1$. It follows from Theorem 2.2.4 that the left hand side of (2.2.23) equals

$$\hat{\mathsf{E}}_0\big[(h_n 1_{\{Z_n \in *\}}) \circ \theta_{\tau_1} \cdot h_1\big]$$

$$= \hat{\mathsf{E}}_0\big[h_n 1_{\{Z_n \in *\}}\,;\, D = \infty\big] \cdot \frac{\hat{\mathsf{E}}_0[h_1]}{\mathsf{P}_0[D = \infty]}.$$

Observe that $\{D = \infty\} = \{\tilde{T}_{-R} = \infty\} = \{\tilde{T}_{-R} \geq \tau_1\} = \{D \geq \tau_1\}$ (the equalities hold $\hat{\mathsf{P}}_0$-a.s). Indeed, we only need to show the last equality: from the definition of D, it is obvious that $\{\tilde{T}_{-R} \geq \tau_1\} \subset \{D \geq \tau_1\}$; to the opposite direction, we see that $D \geq \tau_1$ implies $\tilde{T}_{-R} > \tau_1 - 1$, and in addition because $(X_{N_j} - X_0) \cdot \ell \geq 3R$ for all $j \geq 1$, cf. (2.2.14), and $\tilde{T}_{-R} \circ \theta_{\tau_1} = \infty$, $\tilde{T}_{-R} = \infty$ follows. Then, we observe that up-to a $\hat{\mathsf{P}}_0$-null-set, $\{D \geq \tau_1\}$ lies in $\mathscr{G}_1$, (indeed, $\hat{\mathsf{P}}_0$-a.s. $\{D \geq \tau_1 = m\} = \{D > m - 1\} \cap \{\tau_1 = m\}$, thus by (2.2.18), the claim follows), therefore $h_n \cdot 1_{\{D=\infty\}} \in \mathscr{G}_n$. Hence, it follows by the induction assumption that

the right hand side of the previous expression equals

$$= \hat{P}_0 \big[Z_0 \in * | D = \infty \big] \cdot \hat{E}_0 \big[h_n; D = \infty \big] \cdot \frac{\hat{E}_0[h_1]}{P_0[D = \infty]}$$

$$= \hat{P}_0 \big[Z_0 \in * | D = \infty \big] \cdot \hat{E}_0 \big[h_1 h_n \circ \theta_{\tau_1} \big] \,.$$

This finishes the proof. □

Remark 2.2.6
In the above theorem, the renewal structure is proved for trajectory between times τ_k and $\tau_{k+1} - 1$, unlike in [46]. Nevertheless, we have very good control over the trajectory between times τ_k and τ_{k+1}, because by our construction $\lambda_{\tau_{k+1}-1} = 1$, hence, $\hat{P}_0$-a.s. $X_s \in B^{X_{\tau_{k+1}-1}}$, for all $s \in [\tau_{k+1} - 1, \tau_{k+1}]$. I.e. the path between $\tau_{k+1} - 1$ and τ_{k+1} remains in a ball of radius $6R$, see also Figure 2.2.4.

Proposition 2.2.7
$\hat{P}_0$-a.s. $\tau_1 < \infty$ if and only if P_0-a.s. $\lim_{t \to \infty} X_t \cdot \ell = \infty$.

Proof: If $\hat{P}_0$-a.s. $\tau_1 < \infty$, then it follows from Theorem 2.2.5 that $\hat{P}_0$-a.s. $\tau_m < \infty$, for all $m \geq 1$, and by definition of τ_m that $\hat{P}_0$-a.s. $\lim_{m \to \infty} X_{\tau_m} \cdot \ell = \infty$. Therefore, $\lim_{t \to \infty} X_t \cdot \ell = \infty$.

To show the opposite direction, we first claim that $\hat{P}_0$-a.s. $N_1 < \infty$, and hence $S_1 < \infty$. Let us define

$$(2.2.24) \qquad Z \stackrel{\text{def}}{=} \sup_{s \leq 1} |X_s - X_0| \quad \text{and} \quad A \stackrel{\text{def}}{=} \big\{ Z > \tfrac{R}{2} \big\},$$

and observe that because of the assumption (2.1.1) and (2.1.4) it follows from the Support Theorem of Stroock-Varadhan, cf. [2], page 25, that there exists a constant $c_0(K, \bar{b}, \bar{\sigma}, \nu, R, d) > 0$ such that for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$:

$$(2.2.25) \qquad P_x^\omega[A^c] \geq c_0 > 0 \,.$$

Since $\lim_{t \to \infty} X_t \cdot \ell = \infty$, P_0-a.s., we see that there exists a $\mathbb{P}$-null-set $\Upsilon \subset \Omega$ such that for all $\omega \notin \Upsilon$, P_0^ω-a.s. $V_k(3R) < \infty$ for all $k \in \mathbb{N}$, cf. (2.2.12) for the definition of V_k. Let us define

$$(2.2.26) \qquad A_k \stackrel{\text{def}}{=} \Big\{ \sup_{s \in [V_k, \lceil V_k \rceil]} |\ell \cdot (X_s - X_{V_k})| > \tfrac{R}{2} \Big\}, \quad k \geq 0 \,,$$

then it follows from induction and the strong Markov property that for $n \in \mathbb{N}$ and $\omega \notin \Upsilon$, $\mathsf{P}_0^\omega \big[\bigcap_{0 \leq k \leq n} A_k \big] \leq (1 - c_0)^n$. As a result, for all $\omega \notin \Upsilon$, $\mathsf{P}_0^\omega \big[\tilde{N}_1(3R) = \infty \big] \leq \mathsf{P}_0^\omega \big[\bigcap_{k \geq 0} A_k \big] = 0$. By the stationarity of $\mathbb{P}$-measure, we see that P_x-a.s. $\tilde{N}_1 < \infty$, for all $x \in \mathbb{R}^d$. Therefore, $\int dx\, \mathsf{P}_x[\tilde{N}_1 = \infty] = 0$, so it follows from Fubini's theorem that there is a $\mathbb{P}$-null-set $\Psi \subset \Omega$, such that for all $\omega \notin \Psi$, outside a Lebesgue-null-set $\mathcal{N}(\omega) \subset \mathbb{R}^d$, $\mathsf{P}_x^\omega[\tilde{N}_1 = \infty] = 0$. Using the positivity of $p_\omega(n, y, z)$, with a somewhat similar argument as in the last two paragraphs of the proof of Lemma 2.2.3, we see by induction that $\mathsf{P}_0[\tilde{N}_m = \infty] = 0$, for $m \geq 1$.

Clearly, for arbitrary $n \geq 1$, $\hat{\mathsf{P}}_0[N_1(3R) = \infty] \leq \hat{\mathsf{P}}_0 \big[\lambda_{\tilde{N}_m(3R)} = 0, \forall m \leq n \big] \leq (1 - \varepsilon)^n$ holds. As a result, $\hat{\mathsf{P}}_0$-a.s. $N_1 < \infty$.

We now can prove that $\hat{\mathsf{P}}_0$-a.s. $\tau_1 < \infty$. To show this we note that by similar computations as in the proof of Theorem 2.2.4, (see (2.2.20), (2.2.21)):

$$\hat{\mathsf{P}}_0[R_k < \infty] = \mathbb{E} \left[\hat{\mathsf{P}}_0^\omega[N_k < \infty, D \circ \theta_{N_k+1} < \infty] \right]$$

$$= \sum_{m \geq 2} \mathbb{E} \left[\hat{\mathsf{P}}_0^\omega[N_k = m - 1, D \circ \theta_m < \infty] \right]$$

$$= \sum_{m \geq 2} \mathbb{E} \left[\hat{\mathsf{P}}_0^\omega \left[N_k = m - 1, \hat{\mathsf{P}}_{X_{m-1}, 1}^\omega \big[\hat{\mathsf{P}}_{X_1, \lambda_1}^\omega [D < \infty] \big] \right] \right]$$

$$= \sum_{m \geq 2} \frac{1}{|B_R|} \int dy\, \mathbb{E} \left[\hat{\mathsf{P}}_0^\omega \big[\Gamma, \lambda_{m-1} = 1, \, y \in B^{X_{m-1}} \big] \cdot \hat{\mathsf{P}}_y^\omega[D < \infty] \right],$$

for some $\Gamma \in \mathscr{F}_{m-1} \otimes \mathscr{S}_{m-2}$ such that $\{N_k = m-1\} = \Gamma \cap \{\lambda_{m-1} = 1\}$. We observe that $\Gamma \subset \{X_{m-1} \cdot \ell + R \geq X_t \cdot \ell, \forall t \leq m-1\}$, hence as in the proof of Theorem 2.2.4, $\hat{\mathsf{P}}_0^\omega \big[\Gamma, y \in B^{X_{m-1}}, \lambda_{m-1} = 1 \big]$ and $\hat{\mathsf{P}}_y^\omega[D < \infty]$ are $\mathbb{P}$-independent, therefore the last expression equals

$$= \sum_{m \geq 2} \frac{1}{|B_R|} \int dy\, \hat{\mathsf{P}}_0 \big[\Gamma, \lambda_{m-1} = 1, \, y \in B^{X_{m-1}} \big] \cdot \mathsf{P}_0[D < \infty]$$

$$= \hat{\mathsf{P}}_0[S_k < \infty] \cdot \mathsf{P}_0[D < \infty] \leq \hat{\mathsf{P}}_0[R_{k-1} < \infty] \cdot \mathsf{P}_0[D < \infty],$$

(it is not hard to see that the last inequality above is indeed an equality) so by induction we obtain that

$$(2.2.27) \qquad \qquad \hat{\mathsf{P}}_0[R_k < \infty] \leq \mathsf{P}_0[D < \infty]^k.$$

On the other hand, as in the proof of Lemma 2.2.3, P_0-a.s. $\lim_t X_t \cdot \ell = \infty$ implies $\mathsf{P}_0[D = \infty] > 0$. Therefore, from (2.2.27) and $\hat{\mathsf{P}}_0$-a.s. $S_1 < \infty$, $S_{k+1} < \infty$ on $\{R_k < \infty\}$ we see that $\hat{\mathsf{P}}_0$-a.s. $\inf\{k \geq 1 : S_k < \infty, R_k = \infty\} < \infty$, which proves $\hat{\mathsf{P}}_0$-a.s. $\tau_1 < \infty$. $\square$

2.3   Law of Large Numbers and Central Limit Theorem

In this section we will provide a sufficient condition to derive a strong law of large numbers and a functional central limit theorem. Some parts of the proofs presented in this section are similar to the proofs of Theorem 2.3 on page 1864 in [46], and of Theorem 4.1 on page 130 – 131 in [43]. We will also use some classical results about local martingales, which are presented in the Appendix on page 90.

We begin with

Lemma 2.3.1
Under (2.3.1-i), (2.3.2-i) holds:

(2.3.1-i) P_0-*a.s.* $\displaystyle\lim_{t\to\infty} \ell \cdot X_t = \infty$ *and* $\hat{\mathsf{E}}_0\big[\tau_1 \big| D = \infty\big] < \infty$,

(2.3.2-i) $\hat{\mathsf{E}}_0\big[\, |X_{\tau_1}| \,\big| D = \infty\big] < \infty$.

Analogously, under (2.3.1-ii), (2.3.2-ii) holds:

(2.3.1-ii) P_0-*a.s.* $\displaystyle\lim_{t\to\infty} \ell \cdot X_t = \infty$ *and* $\hat{\mathsf{E}}_0\big[\tau_1^2 \big| D = \infty\big] < \infty$

(2.3.2-ii) $\hat{\mathsf{E}}_0\big[\, |X_{\tau_1}|^2 \big| D = \infty\big] < \infty$.

Proof: First, we prove the implication (2.3.1-ii) $\Rightarrow$ (2.3.2-ii). From Lemma 2.2.3 we see that $\mathsf{P}_0[D = \infty] > 0$, and hence $\hat{\mathsf{E}}_x\big[\tau_1^2 \big| D = \infty\big]$ is well-defined. Further, because τ_1 only takes integer value bigger or equal to 2, we can write

(2.3.3) $\displaystyle\hat{\mathsf{E}}_0\big[\, |X_{\tau_1}|^2 \big| D = \infty\big] = \sum_{n=2}^{\infty} \hat{\mathsf{E}}_0\big[\, |X_n|^2, \tau_1 = n \big| D = \infty\big]$.

Observe that P_0^ω-a.s. (and therefore $\hat{\mathsf{P}}_0^\omega$-a.s.)

$$|X_n|^2 = \left| \int_0^n b(X_s, \omega) \, ds + \int_0^n \sigma(X_s, \omega) \, dW_s \right|^2$$
$$\leq 2\bar{b}^2 n^2 + 2 \left| Y_n(\omega) \right|^2,$$

where W_s is some suitable $\mathscr{F}_t$ Brownian motion, $\bar{b}$ appears in (2.1.1), and

$$(2.3.4) \qquad Y_t(\omega) := \int_0^t \sigma(X_s, \omega) \, dW_s$$

is an $(\mathscr{F}_t)_{t \geq 0}$ local martingale under P_0^ω. Thus, the right hand side of (2.3.3) is

$$\leq 2\bar{b}^2 \, \hat{\mathsf{E}}_0[\tau_1^2 | D = \infty] + 2 \sum_{n=2}^{\infty} \hat{\mathsf{E}}_0 \left[|Y_n|^2, \tau_1 = n \middle| D = \infty \right].$$

By Hölder's inequality with $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, each term in the summation of the last display can be estimated by

$$\hat{\mathsf{E}}_0 \left[|Y_n|^2, \tau_1 = n \middle| D = \infty \right]$$
$$\leq \hat{\mathsf{E}}_0 \left[|Y_n|^{2p} \middle| D = \infty \right]^{\frac{1}{p}} \cdot \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}}$$
$$\leq \frac{1}{\hat{\mathsf{P}}_0[D = \infty]^{\frac{1}{p}}} \hat{\mathsf{E}}_0 \left[|Y_n|^{2p} \middle| D = \infty \right]^{\frac{1}{p}} \cdot \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}},$$

From the assumption (2.1.4), we see that $\langle Y^i(\omega) \rangle_t \leq \nu t$ for all $\omega \in \Omega$, $i = 1, \cdots, d$, so we can apply (2.5.1) in the appendix and obtain that the rightmost side of the above expression is smaller than

$$(2.3.5) \qquad \frac{c(p, d, \nu)}{\hat{\mathsf{P}}_0[D = \infty]^{\frac{1}{p}}} \cdot n \, \hat{\mathsf{P}}_0 \left[\tau_1 = n \middle| D = \infty \right]^{\frac{1}{q}}.$$

Coming back to (2.3.3), we see that in order to show $\hat{\mathsf{E}}_0[|X_{\tau_1}|^2 | D = \infty] < \infty$, it suffices to prove $\sum_{n=2}^{\infty} n \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}} < \infty$, for some $q > 1$.

To this end, observe that by assumption (2.3.1-ii), $\hat{\mathsf{E}}_0[\tau_1^2 | D = \infty] =$

$\sum_{n=2}^{\infty} n^2 \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty] < \infty$, and hence with Hölder's inequality:

$$
\sum_n n \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}}
$$

(2.3.6)
$$
= \sum_n n^{1-2/q} \, n^{2/q} \, \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}}
$$

$$
\leq \left(\sum_n n^{(1-\frac{2}{q})p} \right)^{\frac{1}{p}} \cdot \left(\sum_n n^2 \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty] \right)^{\frac{1}{q}} < \infty,
$$

provided q close to 1, i.e. p close to ∞.

For the implication (2.3.1-i) $\Rightarrow$ (2.3.2-i), we proceed similarly as above. Instead of (2.3.6), we use

$$
\sum_n \sqrt{n} \, \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty]^{\frac{1}{q}}
$$

$$
\leq \left(\sum_n n^{(\frac{1}{2} - \frac{1}{q})p} \right)^{\frac{1}{p}} \cdot \left(\sum_n n \, \hat{\mathsf{P}}_0[\tau_1 = n | D = \infty] \right)^{\frac{1}{q}} < \infty,
$$

for q close to 1. This completes the proof. $\square$

Now we are ready to prove the strong law of large numbers:

Theorem 2.3.2 (Strong Law of Large Numbers)
Assume (2.3.1-i), then

(2.3.7)
$$
\mathsf{P}_0\text{-}a.s. \quad \frac{X_t}{t} \xrightarrow{t \to \infty} v \stackrel{def}{=} \frac{\hat{\mathsf{E}}_0[X_{\tau_1} | D = \infty]}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]},
$$

and $\ell \cdot v > 0$.

Proof: Because $X_.$ has same distribution under $\hat{\mathsf{P}}_0$ and P_0, it is sufficient to show that $\hat{\mathsf{P}}_0$-a.s. $\frac{X_t}{t} \xrightarrow{t \to \infty} v$.

Further, from our construction of S_k and τ_1, see (2.2.14), (2.2.16) and (2.2.17), it is clear that $\hat{\mathsf{P}}_0$-a.s. $X_{\tau_1} \cdot \ell > 0$, thus $\ell \cdot v > 0$ is immediate.

By Theorem 2.2.5, the strong law of large numbers applied on the i.i.d. random variables $(\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n})$, $n \geq 1$, shows that $\hat{\mathsf{P}}_0$-a.s.

(2.3.8)
$$
\frac{X_{\tau_n}}{n} \xrightarrow{n \to \infty} \hat{\mathsf{E}}_0[X_{\tau_1} | D = \infty], \qquad \frac{\tau_n}{n} \xrightarrow{n \to \infty} \hat{\mathsf{E}}_0[\tau_1 | D = \infty].
$$

For each $t > 0$, we define a non-decreasing integer-valued function $k(t)$, which tends to infinity $\hat{\mathsf{P}}_0$-a.s., such that

(2.3.9) $\tau_{k(t)} \leq t < \tau_{k(t)+1}$, (with the convention $\tau_0 = 0$) .

Dividing the above inequality by $k(t)$ and using (2.3.8), we find that $\hat{\mathsf{P}}_0$-a.s.

(2.3.10) $$\frac{k(t)}{t} \xrightarrow{t \to \infty} \frac{1}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]} .$$

Further, we observe that, because of

$$\frac{X_t}{t} = \frac{X_{\tau_{k(t)}}}{t} + \frac{X_t - X_{\tau_{k(t)}}}{t} ,$$

and in view of (2.3.8) and (2.3.10), $\hat{\mathsf{P}}_0$-a.s.

$$\frac{X_{\tau_{k(t)}}}{t} \xrightarrow{t \to \infty} \frac{\hat{\mathsf{E}}_0[X_{\tau_1} | D = \infty]}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]} ,$$

we can show (2.3.7) by proving

$$\hat{\mathsf{P}}_0\text{-a.s.} \quad \frac{X_t - X_{\tau_{k(t)}}}{t} \xrightarrow{t \to \infty} 0 .$$

To prove this, we observe that since $X.$ is the solution of the stochastic differential equation (2.1.7), we have $\hat{\mathsf{P}}_0$-a.s.

$$\frac{1}{t} \left| X_t - X_{\tau_{k(t)}} \right| \leq \bar{b} \, \frac{|t - \tau_{k(t)}|}{t} + \frac{2}{t} \sup_{s \leq t} |Y_s| ,$$

with the $(\mathscr{F}_t)_{t \geq 0}$ local martingale $Y_t(\omega)$ defined in (2.3.4). In view of (2.3.9) and (2.3.10), the first term in the last expression tends to zero $\hat{\mathsf{P}}_0$-a.s.. Applying (2.5.2), the second term P_0^ω-a.s. tends to zero, as t tends to infinity. □

We are now able to state and prove the promised functional central limit theorem:

Theorem 2.3.3 (Functional Central Limit Theorem)
Let us assume (2.3.1-ii). Define for each $s > 0$ the process $B^s : \big(\Omega \times C(\mathbb{R}_+, \mathbb{R}^d), \mathscr{A} \otimes \mathscr{F}\big) \to \big(C(\mathbb{R}_+, \mathbb{R}^d), \mathscr{F}\big)$, with

(2.3.11) $$B_t^s = \frac{X_{st} - stv}{\sqrt{s}} , t \geq 0 .$$

Then, under the P_0*-measure, the* $C(\mathbb{R}_+, \mathbb{R}^d)$*-valued random variable* $B^s_{\cdot}$
converges in law, as $s \to \infty$*, to a d-dimensional Brownian motion* $B_{\cdot}$*,*
which has the non-degenerated covariance matrix

$$(2.3.12) \qquad \mathbf{K} \overset{def}{=} \frac{\hat{\mathsf{E}}_0[(X_{\tau_1} - v\tau_1)(X_{\tau_1} - v\tau_1)^t | D = \infty]}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]} .$$

Before proving this theorem, let us recall some classical facts about
weak convergence on $C(\mathbb{R}_+, \mathbb{R}^d)$, which will be used throughout the
proof. (For a detailed treatment, we refer to Chapter 3 in [11], and
Section 3.1 in [39])

On the space $C(\mathbb{R}_+, \mathbb{R}^d)$ we define the metric

$$(2.3.13) \quad \rho(Y_{\cdot}; Z_{\cdot})$$

$$\overset{def}{=} \sum_{m=1}^{\infty} \frac{1}{2^m} \sup_{0 \le s \le m} (|Y_s - Z_s| \wedge 1) \le 1, \quad Y_{\cdot}, Z_{\cdot} \in C(\mathbb{R}_+, \mathbb{R}^d),$$

which induces the topology of uniform convergence on compact intervals
of $\mathbb{R}_+$. If on some probability space, say $(\Xi, \mathcal{A}, \mathsf{P})$, $Y^n_{\cdot}$ and $Z^n_{\cdot}$ are
sequences of continuous $\mathbb{R}^d$-valued stochastic processes, and the laws of
the processes $Y^n_{\cdot}$ converges weakly to some probability measure Q on
$C(\mathbb{R}_+, \mathbb{R}^d)$, further if $\rho(Y^n_{\cdot}, Z^n_{\cdot})$ converges in probability P to 0, then
the sequence of laws of the processes $Z^n_{\cdot}$ converges weakly to Q.

Proof of Theorem 2.3.3: It suffices to prove that $B^s_{\cdot} \xrightarrow{s \to \infty} B_{\cdot}$ in law
under $\hat{\mathsf{P}}_0$, because $X_{\cdot}$ has the same distribution under $\hat{\mathsf{P}}_0$ and P_0. The
proof is divided in 5 steps. In Step 1 – 3, we prove that for integer-
valued s, $B^s_{\cdot} \xrightarrow{s \to \infty} B_{\cdot}$ in law under $\hat{\mathsf{P}}_0$. In Step 4, we generalize this to
non-integer s. And in the last step, Step 5, the non-degeneracy of the
covariance matrix $\mathbf{K}$ is proved.

Step 1: Define

$$Z_j \overset{def}{=} (X_{\tau_{j+1}} - X_{\tau_j}) - v(\tau_{j+1} - \tau_j), \quad j \ge 1,$$

$$S_n \overset{def}{=} \sum_{j=1}^{n} Z_j = X_{\tau_{n+1}} - X_{\tau_1} - v(\tau_{n+1} - \tau_1),$$

and let S_t be the linear interpolation of S_n, with the convention $S_0 = 0$.

In view of Theorem 2.2.5 and the definition of v in (2.3.7), the random variables $Z_j, j \geq 1$, are i.i.d., centered under $\hat{P}_0$, and, thanks to our assumption (2.3.1-ii) and Lemma 2.3.1, square integrable.

The Wiener & Donsker's Invariance Principle, cf. page 172 in [39], implies that under the $\hat{P}_0$-measure

$$(2.3.14) \qquad \frac{1}{\sqrt{n}} S_{n\cdot} \xrightarrow{n \to \infty} \tilde{B}_\cdot \quad \text{in law},$$

where $\tilde{B}_\cdot$ is a d-dimensional Brownian motion with covariance matrix $\tilde{A} = \hat{E}_0[\tau_1 | D = \infty] \cdot \mathbf{K}$. (The theorem stated in [39] is for the case with covariance matrix equals $\mathbb{1}$. To get our result, we observe that, as we will show below in Step 5, the matrix $\tilde{A}$ is positive definite, hence $\tilde{A}^{-\frac{1}{2}}(\frac{1}{\sqrt{n}} S_{n\cdot})$ converges under $\hat{P}_0$ in law to a Brownian motion with covariance matrix $\hat{E}_0[(\tilde{A}^{-\frac{1}{2}} Z_1)(\tilde{A}^{-\frac{1}{2}} Z_1)^t] = \mathbb{1}$. Thereafter, (2.3.14) follows.)

Step 2: For each $n \in \mathbb{N}$, define a non-decreasing sequence $j(n) \in \mathbb{N}$ (with the convention $j(0) = 0$), which tends to infinity $\hat{P}_0$-a.s., such that

$$(2.3.15) \qquad \tau_{j(n)} \leq n < \tau_{j(n)+1},$$

and let $j(t)$ be its linear interpolation.

The goal of this step is to show that under $\hat{P}_0$

$$(2.3.16) \qquad \frac{1}{\sqrt{n}} S_{(j(n\cdot)-1)_+} \xrightarrow{n \to \infty} B_\cdot \quad \text{in law},$$

where $B_\cdot$ is a d-dimensional Brownian motion with the covariance matrix $\mathbf{K}$.

As a result of (2.3.14), we have $\frac{1}{\sqrt{n}} S_{\frac{n\cdot}{\hat{E}_0[\tau_1|D=\infty]}} \xrightarrow{n \to \infty} B_\cdot$ in law under $\hat{P}_0$, so in view of the comments after Theorem 2.3.3, it suffices to show

$$(2.3.17) \qquad \hat{E}_0\left[\rho\left(\frac{1}{\sqrt{n}} S_{(j(n\cdot)-1)_+}; \frac{1}{\sqrt{n}} S_{\frac{n\cdot}{\hat{E}_0[\tau_1|D=\infty]}} \right) \right] \xrightarrow{n \to \infty} 0.$$

To prove this, we pick $\delta > 0$ arbitrarily small, and choose $T \in \mathbb{N}$ large such that $\sum_{m>T} \frac{1}{2^m} \leq \delta$. Because $\frac{1}{\sqrt{n}} S_{n\cdot} \xrightarrow{n \to \infty} \tilde{B}_\cdot$ in law under $\hat{P}_0$, the laws of $\frac{1}{\sqrt{n}} S_{n\cdot}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$ are tight, so there is a compact set

$K_\delta \subset C(\mathbb{R}_+, \mathbb{R}^d)$, for the topology of uniform convergence on compact intervals, such that $\sup_n \hat{\mathsf{P}}_0 \left[\frac{1}{\sqrt{n}} S_{n\cdot} \notin K_\delta \right] \leq \delta$, and by the Arzela-Ascoli Theorem, cf. page 369 in [37], there exists some $\eta(\delta) > 0$ such that

$$(2.3.18) \qquad \sup_n \hat{\mathsf{P}}_0 \left[\sup_{\substack{|t - t'| \leq \eta, \\ t, t' \leq T}} \frac{1}{\sqrt{n}} |S_{nt} - S_{nt'}| \geq \delta \right] \leq \delta.$$

On the other hand, we observe that $|j(t) - j([t])| \leq 1$, $t \in \mathbb{R}_+$, ($[t]$ denotes the integer part of t), and

$$(2.3.19) \qquad j(m) \leq m, \text{ for all } m \in \mathbb{N}.$$

From (2.3.9), we also see that $j(n) = k(n)$ for all $n \in \mathbb{N}$, hence (2.3.10) implies $\hat{\mathsf{P}}_0$-a.s. $\frac{j(n)}{n} \xrightarrow{n \to \infty} \frac{1}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]}$. Applying Dini's second lemma, we obtain that

$$\hat{\mathsf{P}}_0\text{-a.s., for all } U > 0, \quad \sup_{0 \leq t \leq U} \left| \frac{(j(tn) - 1)_+}{n} - \frac{t}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]} \right| \xrightarrow{n \to \infty} 0.$$

Hence, for n large enough we get

$$\hat{\mathsf{P}}_0 \left[\sup_{0 \leq t \leq T} \left| \frac{(j(tn) - 1)_+}{n} - \frac{t}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]} \right| \geq \eta(\delta) \right] \leq \delta.$$

Coming back to (2.3.18), we obtain

$$\hat{\mathsf{E}}_0 \left[\sup_{0 \leq t \leq T} \frac{1}{\sqrt{n}} \left| S_{(j(tn) - 1)_+} - S_{\frac{tn}{\hat{\mathsf{E}}_0[\tau_1 | D = \infty]}} \right| \wedge 1 \right] \leq 3\delta,$$

for sufficiently large n. The claim (2.3.17), and hence (2.3.16) follow.

Step 3: We show in this step that under $\hat{\mathsf{P}}_0$

$$(2.3.20) \qquad \frac{1}{\sqrt{n}} B^n_\cdot \xrightarrow{n \to \infty} B_\cdot \text{ in law}.$$

As stated in the comments after Theorem 2.3.3, it suffices to show that

$$(2.3.21) \qquad \hat{\mathsf{E}}_0 \left[\rho \left(B^n_\cdot ; \frac{1}{\sqrt{n}} S_{(j(n\cdot) - 1)_+} \right) \right] \xrightarrow{n \to \infty} 0.$$

To this end, choose $T > 0$. Then we have

$$\sup_{t \leq T} \left| B_t^n - \frac{1}{\sqrt{n}} S_{(j(tn)-1)_+} \right|$$

$$(2.3.22) \qquad \leq \sup_{t \leq T} \frac{1}{\sqrt{n}} \left| S_{(j(tn)-1)_+} - S_{(j([tn])-1)_+} \right|$$

$$+ \sup_{t \leq T} \left| B_t^n - \frac{1}{\sqrt{n}} S_{(j([tn])-1)_+} \right|.$$

Observe that the first term on the right hand side of (2.3.22) is bounded from above by

$$(2.3.23) \quad \frac{|v|}{\sqrt{n}} \sup_{0 \leq m \leq j([Tn])} \left(\tau_{m+1} - \tau_m \right) + \sup_{0 \leq m \leq j([Tn])} \frac{1}{\sqrt{n}} \left| X_{\tau_{m+1}} - X_{\tau_m} \right|,$$

which, as we will see, converges to 0 in $\hat{\mathsf{P}}_0$-probability. Indeed, in view of Theorem 2.2.5 and (2.3.19), for any $u > 0$:

$$\hat{\mathsf{P}}_0 \left[\frac{1}{\sqrt{n}} \sup_{0 \leq m \leq j([Tn])} \left(\tau_{m+1} - \tau_m \right) > u \right]$$

$$\leq \hat{\mathsf{P}}_0 \left[\tau_1 > \sqrt{n}u \right] + [nT] \, \hat{\mathsf{P}}_0 \left[\tau_1 > \sqrt{n}u \mid D = \infty \right]$$

$$\leq \hat{\mathsf{P}}_0 \left[\tau_1 > \sqrt{n}u \right] + \frac{nT}{nu^2} \, \hat{\mathsf{E}}_0 \left[\tau_1^2, \tau_1 > \sqrt{n}u \mid D = \infty \right] \xrightarrow{n \to \infty} 0,$$

by assumption (2.3.1-ii). Similar result holds for the second term in (2.3.23), by (2.3.2-ii) we have:

$$\hat{\mathsf{P}}_0 \left[\frac{1}{\sqrt{n}} \sup_{0 \leq m \leq j([Tn])} \left| X_{\tau_{m+1}} - X_{\tau_m} \right| > u \right]$$

$$\leq \hat{\mathsf{P}}_0 [|X_{\tau_1}| > \sqrt{n}u] + \frac{nT}{nu^2} \, \hat{\mathsf{E}}_0 [|X_{\tau_1}|^2, |X_{\tau_1}| > \sqrt{n}u \mid D = \infty] \xrightarrow{n \to \infty} 0.$$

Let us now consider the second term on the right hand side of (2.3.22). We claim that it also converges in $\hat{\mathsf{P}}_0$-probability to 0. To show this, we start with the easy fact that $\hat{\mathsf{P}}_0$-a.s., the second term on r.h.s. of (2.3.22) is smaller than

$$\sup_{t \leq T} \frac{1}{\sqrt{n}} \left\{ \int_{\tau_{j([nt])}}^{nt} \left(|v| + |b(X_s, \omega)| \right) ds + \int_0^{\tau_1} \left(|v| + |b(X_s, \omega)| \right) ds \right\}$$

$$(2.3.24) \qquad \qquad + \sup_{t \leq T} \frac{1}{\sqrt{n}} \left\{ \left| Y_{nt} - Y_{\tau_{j([nt])}} \right| + \left| Y_{\tau_1} \right| \right\},$$

with Y_t defined in (2.3.4). The first term in (2.3.24) is bounded from above by

$$\frac{\bar{b} + |v|}{\sqrt{n}} \sup_{t \leq T} (nt - \tau_{j([nt])} + \tau_1)$$

$$\leq \frac{2(\bar{b} + |v|)}{\sqrt{n}} \left(\sup_{0 \leq m \leq j([nT])} \left(\tau_{m+1} - \tau_m \right) \right),$$

which converges to 0 in $\hat{\mathsf{P}}_0$-probability, as shown above.

The last term in (2.3.24) is smaller than

$$\frac{1}{\sqrt{n}} |Y_{\tau_1}| + \frac{1}{\sqrt{n}} \sup_{t \leq T} |Y_{nt} - Y_{\tau_{j([nt])}}|,$$

which, we claim, converges also to 0 in $\hat{\mathsf{P}}_0$-probability. Indeed, for all $u > 0$:

$$\hat{\mathsf{P}}_0 \left[\sup_{t \leq T} |Y_{nt} - Y_{\tau_{j([nt])}}| > \sqrt{n}u \right]$$

$$\leq \hat{\mathsf{P}}_0 \left[\sup_{t \leq T} |Y_{nt} - Y_{\tau_{j([nt])}}| > \sqrt{n}u; \sup_{0 \leq m \leq j([nT])} \left(\tau_{m+1} - \tau_m \right) \leq \sqrt{n} \right]$$

(2.3.25)

$$+ \hat{\mathsf{P}}_0 \left[\sup_{0 \leq m \leq j([nT])} |\tau_{m+1} - \tau_m| > \sqrt{n} \right].$$

We know already from above that the second term on the r.h.s. of (2.3.25) converges to 0, as $n \to \infty$. For the first term on the r.h.s. in (2.3.25), we observe that it can be further estimated from above by

(2.3.26)
$$\hat{\mathsf{P}}_0 \left[\sup_{m \leq [nT]} \sup_{0 \leq s \leq \sqrt{n}} |Y_{m+s} - Y_m| > \sqrt{n}u \right]$$

$$\leq \sum_{m=0}^{[nT]} \hat{\mathsf{P}}_0 \left[\sup_{0 \leq s \leq \sqrt{n}} |Y_{m+s} - Y_m| > \sqrt{n}u \right].$$

Applying the Bernstein's inequality, cf. page 153 – 154 in [36], we obtain that for any $m \in \mathbb{N}$:

$$\mathsf{P}_0^\omega \left[\sup_{s \leq \sqrt{n}} |Y_{m+s} - Y_m| > \sqrt{n}u \right] \leq 2d\, e^{-\frac{u^2 \sqrt{n}}{2\nu}},$$

thus the right hand side of (2.3.26) tends to 0. This completes the proof of (2.3.20).

Step 4: In this step we study $B_{\cdot}^{s}$ for $s \in \mathbb{R}_+$ tending to infinity, and extends (2.3.20).

The proof is very similar the one given in Step 2. We consider $s_n \to \infty$. For $\delta > 0$ arbitrarily small, we define $T \in \mathbb{N}$ such that $\sum_{m>T} \frac{1}{2^m} \leq \delta$.

From (2.3.21) we know that under $\hat{P}_0$, with $B_{\cdot}$ as in (2.3.16),

$$(2.3.27) \qquad \frac{X_{[s_n]\cdot} - v[s_n]\cdot}{\sqrt{[s_n]}}, \quad \text{and hence} \quad \frac{X_{[s_n]\cdot} - v[s_n]\cdot}{\sqrt{s_n}},$$

converges in law to $B_{\cdot}$, as $n \to \infty$.

Therefore, the laws of $\frac{1}{\sqrt{s_n}}(X_{[s_n]\cdot} - v[s_n]\cdot)$ are tight, and for any $T > 0$ and $\delta > 0$, one can find $\eta(\delta) > 0$ such that:

$$\sup_n \hat{P}_0 \left[\sup_{\substack{|t-t'| \leq \eta, \\ t,t' \leq T}} \frac{(X_{[s_n]t} - v[s_n]t) - (X_{[s_n]t'} - v[s_n]t')}{\sqrt{s_n}} \geq \delta \right] \leq \delta.$$

Since $\sup_{t \leq T} \left| t - \frac{s_n}{[s_n]}t \right| \xrightarrow{n \to \infty} 0$, we obtain that for large n

$$\hat{P}_0 \left[\sup_{0 \leq t \leq T} \frac{1}{\sqrt{s_n}} \left| (X_{[s_n]t} - v[s_n]t) - (X_{s_n t} - v s_n t) \right| \geq \delta \right] \leq \delta,$$

and from (2.3.27) we deduce our claim.

Step 5: In this final step we will prove the non-degeneracy of the co-variance matrix $\mathbf{K}$. First, we let $H \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d : 14R < z \cdot \ell < 15R\}$ be a strip in $\mathbb{R}^d$. We claim that for any $n \geq 3$ and $x \in H$,

$$(2.3.28) \qquad \begin{aligned} &\hat{P}_0[X_{\tau_1} \in B_R(x); \ n = S_1 = \tau_1; D = \infty] \\ =&\hat{P}_0[X_n \in B_R(x); \ n = S_1 < D] \cdot P_0[D = \infty] > 0. \end{aligned}$$

To show this, we prove in the first step that for any $x \in H$

$$(2.3.29) \qquad \hat{P}_0[X_n \in B_R(x); \ n = S_1 < D] > 0.$$

To see this, we observe that for all $\omega \in \Omega$, $x \in H$, with $\tilde{B} \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d : |B^z \cap B_R(x)| \geq |B_R|/2\}$ (recall (2.2.1)), we have (see (2.2.13), (2.2.14)

and Theorem 2.2.1):

$$\hat{\mathsf{P}}_0^\omega \left[X_n \in B_R(x); \ n = S_1 < D \right]$$

(2.3.30)
$$\geq \frac{1}{2} \, \hat{\mathsf{P}}_0^\omega \left[X_{n-1} \in \tilde{B}; \ N_1 = n - 1; \ \tilde{T}_{-R} > n - 1 \right]$$

$$\geq \frac{\varepsilon}{2} \, \mathsf{P}_0^\omega \left[X_{n-1} \in \tilde{B}; \ \tilde{N}_1(3R) = \lceil V_1(3R) \rceil = n - 1; \right.$$

$$\left. \tilde{T}_{-R} > n - 1 \right].$$

Because the path in the next figure belongs to the event on the right hand side of (2.3.30), with the Support Theorem of Stroock-Varadhan, cf. page 25 in [2], the right hand side in (2.3.30) is positive, for all $\omega \in \Omega$. This proves (2.3.29).
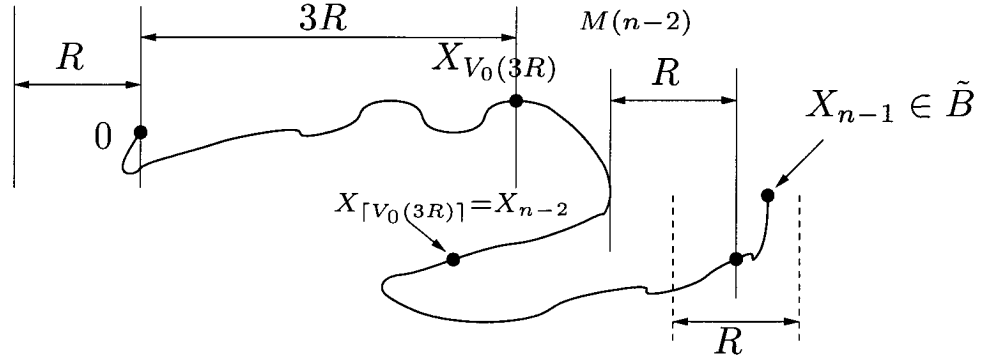

Figure 2.3.1:

To finish the proof of (2.3.28), we only need to prove the first equality in (2.3.28). To do this, we proceed as in the proof of Theorem 2.2.4:

$$\hat{\mathsf{P}}_0[X_{\tau_1} \in B_R(x); \ n = S_1 = \tau_1; \ D = \infty]$$

$$= \hat{\mathsf{P}}_0[X_n \in B_R(x); \ n = S_1 < D; \ D \circ \theta_n = \infty]$$

$$= \mathbb{E}\left\{ \hat{\mathsf{P}}_0^\omega \left[X_{n-1} \in \hat{B}; \ \lambda_{n-1} = 1; \ \Gamma; \ X_1 \circ \theta_{n-1} \in B_R(x); \ D \circ \theta_n = \infty \right] \right\},$$

with $\hat{B} \overset{\text{def}}{=} \left\{ z \in \mathbb{R}^d : B^z \cap B_R(x) \neq \emptyset \right\}$ and some $\Gamma \in \mathscr{F}_{n-1} \otimes \mathscr{S}_{n-2}$. By the Markov property, cf. Corollary 2.2.2, and similar calculations as in the proof of Theorem 2.2.4, see page 60, the last expression equals

$$= \frac{1}{|B_R|} \int dy \, \mathbb{E}\left\{ \hat{\mathsf{P}}_0^\omega[V] \cdot \hat{\mathsf{P}}_y^\omega[D = \infty] \right\}$$

$$= \frac{1}{|B_R|} \left(\int dy \, \hat{\mathsf{P}}_0[V] \right) \cdot \mathsf{P}_0[D = \infty],$$

with $V \overset{\text{def}}{=} \left\{ X_{n-1} \in \hat{B}; \; \Gamma; \; \lambda_{n-1} = 1; \; y \in B^{X_{n-1}} \cap B_R(x) \right\}$, where, as in the proof of Theorem 2.2.4, we have used that $\hat{\mathsf{P}}_0^\omega[V]$ and $\mathsf{P}_y^\omega[D = \infty]$ are $\mathbb{P}$-independent, and the $\mathbb{P}$-measure is translation invariant. On the other hand, we observe that by the identical calculation $\hat{\mathsf{P}}_0[X_n \in B_R(x); \; n = S_1 < D] = \int dy \; \frac{1}{|B_R|} \hat{\mathsf{E}}_0[V]$ holds, the first equality in (2.3.28) follows immediately.

With the help of (2.3.28) we can now prove the non-degeneracy of the covariance matrix $\mathbf{K}$. Clearly, for any $w \in \mathbb{R}^d$, $w^t \mathbf{K} w \geq 0$, i.e. $\mathbf{K}$ is positive semi-definite. We prove the non-degeneracy by contradiction. If $w^t \mathbf{K} w = 0$ for some unit vector $w \in \mathbb{R}^d$, then $\hat{\mathsf{P}}_0[w \cdot (X_{\tau_1} - \tau_1 v) = 0 | D = \infty] = 1$.

Combine this with (2.3.28), we obtain that for any given $x \in H$, and for all $n \geq 3$: $\hat{\mathsf{P}}_0\left[w \cdot x - R \leq n \, (w \cdot v) \leq w \cdot x + R; \; \tau_1 = n \big| D = \infty \right] > 0$, which implies $w \cdot v = 0$. Coming back to the above inequality, we see that $|w \cdot x| \leq R$ for $x \in H$, by taking limits of points in H, we obtain that $w \cdot z = 0$, for all z such that $z \cdot \ell = 0$. Since $v \cdot \ell > 0$, it follows that $w = 0$. This, combined with $\hat{\mathsf{E}}_0[\tau_1 | D = \infty] < \infty$, proves the non-degeneracy of the matrix $\mathbf{K}$, and hence finish the proof of Theorem 2.3.3. $\qquad\square$

2.4 Application to an Anisotropic gradient-type Diffusion

In this section we will apply the results from the previous sections to a class of anisotropic diffusion processes in a random medium, which is reversible when the environment is fixed. The class under consideration is a specialization of (2.1.7) with $\sigma = \mathbb{1}$ and $b(x, \omega) = \nabla V(x, \omega)$, where for each $\omega \in \Omega$, $V(\cdot, \omega) \in C^1(\mathbb{R}^d, \mathbb{R})$ has bounded and Lipschitz-continuous derivatives; in addition we assume that for some $\ell \in S^{d-1}$, $A, B > 0$ and $\lambda > 0$,

$$(2.4.1) \qquad A e^{2\lambda \ell \cdot x} \leq e^{2V(x,\omega)} \leq B e^{2\lambda \ell \cdot x}, \quad \text{for } x \in \mathbb{R}^d, \omega \in \Omega.$$

We will prove the existence of an effective, non-vanishing velocity, and a functional central limit theorem in Theorem 2.4.11.

Let us mention that in this section $c, \tilde{c}, \hat{c}$ and C always denote some

positive constants, which do not depend on $x \in \mathbb{R}^d$ and $\omega \in \Omega$. They need not to be the same in each occurrence.

2.4.1 Key Estimates

We will now derive estimates on the exit distribution and exit time of the diffusion process from a large cylinder with axis parallel to ℓ, cf. Proposition 2.4.1 and 2.4.2. We will then derive the transience of the process in direction ℓ, cf. Corollary 2.4.6.

Let us introduce

$$(2.4.2) \quad m_\omega(dx) \stackrel{\text{def}}{=} \exp\left\{2V(x,\omega)\right\} dx\,, \quad m(dx) \stackrel{\text{def}}{=} \exp\left\{2\lambda\ell \cdot x\right\} dx\,,$$

and the corresponding scalar product $(\,\cdot\,;\,\cdot\,)_{m_\omega}$ on $L^2(m_\omega)$, respectively $(\,\cdot\,;\,\cdot\,)_m$ on $L^2(m)$. Observe that due to (2.4.1), the norms $\|\cdot\|_{L^2(m_\omega)}$ and $\|\cdot\|_{L^2(m)}$ are equivalent, hence $L^2(m_\omega) = L^2(m)$ for all $\omega \in \Omega$.

Further, let us denote by $(P_\omega^t)_{t \geq 0}$ the semi-group corresponding to the solution of this stochastic differential equation, that is, $(P_\omega^t f)(x) = \mathsf{E}_x^\omega[f(X_t)]$ for f bounded and Borel-measurable. Observe that for each $\omega \in \Omega$ the differential operator

$$L_\omega = \frac{1}{2}\Delta + \nabla V(x,\omega) \cdot \nabla$$

is the generator of the semi-group $(P_\omega^t)_{t \geq 0}$, cf. page 251 in [10]. One can easily check that $(f; L_\omega g)_{m_\omega} = (g; L_\omega f)_{m_\omega}$ for $f, g \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. From (2.3) in [15] we observe that $m_\omega(dx)$ is the reversible measure to P_ω^t, i.e. $(f; P_\omega^t g)_{m_\omega} = (g; P_\omega^t f)_{m_\omega}$, for $f, g \in L^1(m_\omega)$ and bounded (the operator L_ω has the form of (3.4) in [15], therefore the assumption for (2.3) in [15] is fulfilled).

Let us now introduce the Dirichlet form $\mathscr{E}_{m_\omega}$ corresponding to the operator L_ω, or the semi-group P_ω^t,

$$(2.4.3) \qquad \mathscr{E}_{m_\omega}(f, g) \stackrel{\text{def}}{=} \lim_{t \downarrow 0} \frac{1}{t}\left((1 - P_\omega^t)f; g\right)_{m_\omega},$$

with its definition domain

$$\mathscr{D}_{m_\omega} \stackrel{\text{def}}{=} \left\{ f \in L^2(m_\omega) : \lim_{t \downarrow 0} \frac{1}{t}\left((1 - P_\omega^t)f; f\right)_{m_\omega} < \infty \right\}.$$

It follows from Remark (2.12) and the proof of Theorem (2.3) in [15] that $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ is a core of $\mathscr{E}_{m_\omega}$. Further, from (2.4.1), we have

$$(2.4.4) \quad \begin{cases} \mathscr{D}_{m_\omega} = \mathscr{D} = \left\{ f \in L^2(m) : \frac{\partial}{\partial x_i} f \in L^2(m), i = 1, \cdots, d \right\}, \\[2mm] \mathscr{E}_{m_\omega}(f, g) = \frac{1}{2} \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} f; \frac{\partial}{\partial x_i} g \right)_{m_\omega}, \qquad f, g \in \mathscr{D}, \\[2mm] A\,\mathscr{E}_m(f, f) \leq \mathscr{E}_{m_\omega}(f, f) \leq B\,\mathscr{E}_m(f, f), \qquad f \in \mathscr{D}, \end{cases}$$

with $\mathscr{E}_m(f, g) = \frac{1}{2} \sum_{i=1}^d \left(\frac{\partial}{\partial x_i} f; \frac{\partial}{\partial x_i} g \right)_m$.

For each $\omega \in \Omega$ and open subset U of $\mathbb{R}^d$, we introduce the bottom of the Dirichlet spectrum of operator $-L_\omega$ in U:

$$(2.4.5) \qquad \Lambda_\omega(U) = \inf \left\{ \frac{\mathscr{E}_{m_\omega}(f, f)}{(f; f)_{m_\omega}} : f \in C_c^\infty(U), f \neq 0 \right\} \geq 0.$$

Proposition 2.4.1

$$(2.4.6) \qquad \inf_{U, \omega \in \Omega} \Lambda_\omega(U) > 0,$$

where U varies over the collection of non-empty open subsets of $\mathbb{R}^d$. The bounded self-adjoint operator $P_{\omega,U}^t$ on $L^2(m_\omega)$, which is defined by

$$\left(P_{\omega,U}^t f \right)(x) \overset{def}{=} \mathsf{E}_x^\omega \left[f(X_t), T_U > t \right], \text{ for } t > 0 \text{ and } f \in L^2(m_\omega), \text{ satisfies}$$

$$(2.4.7) \qquad \sup_{\omega, U} \left\| P_{\omega,U}^t \right\|_{m_\omega} \leq \exp \left\{ -\frac{\gamma t}{\lambda} \right\}, \quad t > 0,$$

for some $\gamma > 0$, with $\| \cdot \|_{m_\omega}$ denoting the operator norm in $L^2(m_\omega)$.

Proof: Observe that because of (2.4.1) the inequality $\frac{1}{B}(f; f)_{m_\omega} \leq (f; f)_m$ holds for all $f \in L^2(m) = L^2(m_\omega)$; and similarly $\frac{1}{A}\mathscr{E}_{m_\omega}(f, f) \geq \mathscr{E}_m(f, f)$ holds for all $f \in C_c^\infty(U)$. Therefore, for U open subset of $\mathbb{R}^d$, $\Lambda_\omega(U) \geq \frac{A}{B}\Lambda(U)$, for all $\omega \in \Omega$, where $\Lambda(U)$ is defined, analogously to $\Lambda_\omega(U)$ in (2.4.5), with $\mathscr{E}_m$ instead of $\mathscr{E}_{m_\omega}$.

It thus suffices to find a lower bound for $\inf_U \Lambda(U)$. Further, because $\Lambda(\mathbb{R}^d) = \inf_{U \neq \emptyset} \Lambda(U)$ and (2.4.5) also holds for $\Lambda(U)$, we can assume that U is open and bounded.

Observe that the measure $m(dx) = e^{2\lambda \ell \cdot x}\, dx$ is (up to a multiplication factor) the reversible measure for Brownian motion with constant drift $\lambda \ell$, and $\mathscr{E}_m$ is just the corresponding Dirichlet form. Let us denote the canonical law of this diffusion process starting in x by Q_x and its expectation value by E_x^Q. Then $\exp\{-\delta \ell \cdot X_t + \alpha t\}$ is a Q_x-martingale, provided $\alpha = \delta \lambda - \frac{\delta^2}{2}$. Choosing $\delta > 0$ small enough, we can make $\alpha > 0$. The stopping theorem implies that for any bounded open set $U \subset \mathbb{R}^d$ containing x, $E_x^Q\big[\exp\{-\delta \ell \cdot (X_{T_U} - x) + \alpha T_U\}\big] = 1$. With $\rho \overset{\mathrm{def}}{=} \sup\{|\ell \cdot (z - z')| : z, z' \in U\}$, we have $-\delta \ell \cdot (X_{T_U} - x) \geq -\delta\rho$, hence $\sup_{x \in U} E_x^Q[\exp\{\alpha T_U\}] \leq e^{\delta\rho}$.

Now, let us introduce the bounded self-adjoint operator Q_U^t on $L^2(m)$, which is defined by $(Q_U^t f)(x) \overset{\mathrm{def}}{=} E_x^Q[f(X_t), T_U > t]$, with $t > 0$ and $f \in L^2(m)$. We claim that for all $t > 0$:

$$\sup_{\substack{U \text{ open} \\ U \neq \emptyset}} \big\|Q_U^t\big\|_m \leq e^{-\alpha t/2},$$

with $\|\cdot\|_m$ denoting the operator norm in $L^2(m)$. To show this, we observe that for $f \in L^2(m)$:

$$\|Q_U^t f\|_{L^2(m)}^2 = \int_U m(dx)\,(Q_U^t f)^2(x) \overset{\text{Jensen}}{\leq} \big(1_U; Q_U^t f^2\big)_m$$

$$= \big(Q_U^t 1_U; f^2\big)_m = \int m(dy)\, Q_y[T_U > t] f^2(y) \leq e^{-\alpha t} e^{\delta\rho} \|f\|_{L^2(m)}^2,$$

where Chebychev's inequality $Q_y[T_U > t] \leq E_y^Q\big[e^{\alpha(T_U - t)}\big] \leq e^{-\alpha t + \delta\rho}$ is used in the last step. Hence, $\|Q_U^{nt}\|_m^2 \leq e^{-\alpha nt + \delta\rho}$, $n \in \mathbb{N}$. Taking the n-th root, it follows from Theorem VI.6 on page 192 in [34], that $\|Q_U^t\|_m \leq e^{-\alpha t/2}$, and our claim follows. This implies that $\Lambda(U) \geq \frac{\alpha}{2} > 0$, and (2.4.6) follows. Finally, (2.4.7) is just an easy consequence of (2.4.6), cf. Theorem 4.4.2 in [14]. $\qquad\square$

Let $U(L)$ now be a cylinder centered at x with height $4L$ in the direction ℓ and radius $4L^2 > 0$ in the directions normal to ℓ, that is,

$$(2.4.8) \quad U(L) \overset{\mathrm{def}}{=} \Big\{z \in \mathbb{R}^d : |(z - x) \cdot \ell| < 2L;$$

$$|(z - x) \cdot e| < 4L^2, \forall e \perp \ell, |e| = 1\Big\}.$$

Proposition 2.4.2
There exist two constants $c_1 > 0$ and $\tilde{c}_1 > 0$ such that for all $L > 0$

$$(2.4.9) \qquad \sup_{x,\omega} \mathsf{P}_x^\omega \left[T_{U(L)} \geq \tfrac{4}{\gamma} L \right] \leq \tilde{c}_1 e^{-c_1 L} .$$

Proof: Observe that for $t \geq 1$,

$$\mathsf{P}_x^\omega \left[T_{U(L)} > t \right]$$
$$\leq \mathsf{P}_x^\omega \left[X_1 \in B_L(x), T_{U(L)} \circ \theta_1 > t - 1 \right] + \mathsf{P}_x^\omega \left[X_1 \notin B_L(x) \right] .$$

By (2.5.5), there exist constants $\tilde{c} > 0$ and $c > 0$ such that the second term on the right-hand side above is smaller than $\tilde{c} e^{-cL^2}$ for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$, hence it suffices to study the first term in the above expression.

By the Markov property, the first term above is

$$\mathsf{P}_x^\omega \left[X_1 \in B_L(x), T_{U(L)} \circ \theta_1 > t - 1 \right]$$
$$= \mathsf{E}_x^\omega \left[X_1 \in B_L(x), \mathsf{P}_{X_1}^\omega [T_U > t - 1] \right]$$
$$= \left(1_{B_L(x)}(\cdot) \, p_\omega(1, x, \cdot) \, e^{-2V(\cdot, \omega)}; (P_{\omega,U}^{t-1} 1_U)(\cdot) \right)_{m_\omega}$$
$$\leq \left\| 1_{B_L(x)}(\cdot) \, p_\omega(1, x, \cdot) \, e^{-2V(\cdot, \omega)} \right\|_{L^2(m_\omega)} \times \left\| P_{\omega,U}^{t-1} \right\|_{m_\omega} \times \left\| 1_U \right\|_{L^2(m_\omega)} .$$

Because there exists a constant $c > 0$ such that $p_\omega(1, x, y) \leq c$ for all $\omega \in \Omega$, $x \in \mathbb{R}^d$ and $y \in B_L(x)$, cf. (2.5.9), we obtain for the first term on the rightmost side in the above expression that

$$\left\| 1_{B_L(x)} \, p_\omega(1, x, \cdot) e^{-2V} \right\|_{m_\omega}^2 \leq c^2 \int dy \, 1_{B_L(x)}(y) e^{-2V(y,\omega)}$$
$$\leq \tilde{c} \, L^d \, e^{-2\lambda \ell \cdot x} e^{2\lambda L} ,$$

for some $\tilde{c} > 0$, where we used (2.4.1) in the last step. Similarly, we can estimate $\| 1_U \|_{m_\omega}$ by:

$$\| 1_U \|_{m_\omega}^2 = \int dy \, 1_U(y) \, e^{2V(y,\omega)} \leq B \int dy \, 1_U(y) \, e^{2\lambda \ell \cdot y}$$
$$\leq c \, L^{2d-1} \, e^{2\lambda \ell \cdot x} e^{4\lambda L} .$$

Putting them with (2.4.7) together, we obtain for $t \geq \tfrac{4L}{\gamma} \vee 1$ that

$$\mathsf{P}_x^\omega \left[T_U > t \right] \leq \tilde{c} \, L^{c(d)} \, e^{3\lambda L} e^{-\gamma(t-1)} \leq \tilde{c} e^{-cL} ,$$

for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$. Therefore, we can find $\tilde{c}_1 > 0$ and $c_1 > 0$ such that $\mathsf{P}_x^\omega \left[T_{U(L)} \geq \frac{4}{\gamma} L \right] \leq \tilde{c}_1 e^{-c_1 L}$. $\square$

Let us divide the boundary of $U(L)$, cf. (2.4.8), into $\partial U(L) = \partial_+ U(L) \cup \partial_- U(L) \cup \partial_0 U(L)$, with

$$(2.4.10) \quad \begin{cases} \partial_+ U(L) \overset{\text{def}}{=} \{z \in \partial U(L) : \ell \cdot (z - x) \geq 2L\}, \\ \partial_- U(L) \overset{\text{def}}{=} \{z \in \partial U(L) : \ell \cdot (z - x) \leq -2L\}, \\ \partial_0 U(L) \overset{\text{def}}{=} \partial U(L) \backslash \left(\partial_+ U(L) \cup \partial_- U(L) \right). \end{cases}$$

The following estimate will play an important role:

Proposition 2.4.3
There exist two constants $c_2 > 0$ and $\tilde{c}_2 > 0$ such that for all $L > 0$:

$$(2.4.11) \quad \sup_{x,\omega} \mathsf{P}_x^\omega \left[T_{U(L)} < \tfrac{4L}{\gamma} ; X_{T_{U(L)}} \notin \partial_+ U \right] \leq \tilde{c}_2 e^{-c_2 L}.$$

Proof: Without loss of generality let us assume $L > \frac{\gamma}{4}$. Observe that, with $I_n \overset{\text{def}}{=} [n, n+1)$, $n \geq 0$, we have

$$\mathsf{P}_x^\omega \left[T_U < \tfrac{4L}{\gamma} ; X_{T_U} \notin \partial_+ U \right]$$

$$\leq \mathsf{P}_x^\omega \left[T_U \in I_0 \right] + \sum_{n=1}^{\lceil \frac{4L}{\gamma} \rceil - 1} \mathsf{P}_x^\omega \left[T_U \in I_n, X_{T_U} \notin \partial_+ U \right].$$

Also observe that in the above expression, because of (2.5.5), we have for the the first term on the right hand side

$$\mathsf{P}_x^\omega \left[T_U \in I_0 \right] \leq \mathsf{P}_x^\omega \left[\sup_{s \leq 1} |X_s - X_0| > 2L \right] \leq \tilde{c} e^{-cL^2}, \quad x \in \mathbb{R}^d, \omega \in \Omega.$$

For the terms in the sum, we notice that for $n \geq 1$:

$$\mathsf{P}_x^\omega \left[T_U \in I_n, X_{T_U} \notin \partial_+ U \right]$$

$$\leq \mathsf{P}_x^\omega \left[X_1 \in B_{\frac{L}{2}}(x), T_U \in I_n, X_{T_U} \notin \partial_+ U \right] + \mathsf{P}_x^\omega \left[X_1 \notin B_{\frac{L}{2}}(x) \right],$$

and $P_x^\omega\left[X_1 \notin B_{\frac{L}{2}}(x)\right] \leq P_x^\omega\left[\sup_{s\leq 1}|X_s - X_0| > \frac{L}{2}\right] \overset{(2.5.5)}{\leq} \tilde{c}e^{-cL^2}$.
Hence, we only need to prove that

$$\sum_{1\leq n<(4L/\gamma)} P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), T_U \in I_n, X_{T_U} \notin \partial_+ U\right] \leq \tilde{c}e^{-cL}.$$

To this end, we notice that

$$P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), T_U \in I_n, X_{T_U} \notin \partial_+ U\right]$$

$$\leq P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), T_U \in I_n, X_n \in U_0 \cup U_-\right]$$

$$+ P_x^\omega\left[T_U \in I_n, \sup_{s\leq 1}|X_s - X_0| \circ \theta_n > \frac{L}{2}\right],$$

with $U_0(L) \overset{\text{def}}{=} \left\{z \in \mathbb{R}^d : \exists y \in \partial_0 U(L), |y - z| < \frac{L}{2}\right\}$ and $U_-(L) \overset{\text{def}}{=} \left\{z \in \mathbb{R}^d : \exists y \in \partial_- U(L), |y - z| < \frac{L}{2}\right\}$. We see with (2.5.5) that the expression above is

$$\leq P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), T_U \in I_n, X_n \in U_0 \cup U_-\right] + \tilde{c}e^{-cL^2}.$$

Thus, it suffices to show that $\sum_{1\leq n<(4L/\gamma)} P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), X_n \in U_0 \cup U_-\right] \leq \tilde{c}e^{-cL}$. To prove this, we observe that with for $U_j = U_0$ or $U_j = U_-$, it follows from the Markov property and $p_\omega(1, x, y) \leq c$, cf. (2.5.9), that

$$P_x^\omega\left[X_1 \in B_{\frac{L}{2}}(x), X_n \in U_j\right]$$

$$(2.4.12) \qquad = \int_{B_{\frac{L}{2}}(x)} dz\, p_\omega(1, x, z)\left(P_\omega^{n-1}1_{U_j}\right)(z)$$

$$\leq c\, e^{-2\lambda\ell\cdot x}e^{\lambda L}\left(P_\omega^{n-1}1_{U_j}; 1_{B_{\frac{L}{2}}(x)}\right)_m.$$

By Theorem 1.8 of [41] on page 290, there exists a constant $C > 0$ such that for all $\omega \in \Omega$, and any open sets $U, B \subset \mathbb{R}^d$:

$$(2.4.13) \quad \left(P_\omega^{n-1}1_U; 1_B(x)\right)_m \leq \sqrt{m(B)}\sqrt{m(U)}\exp\left\{-\frac{\rho(B,U)^2}{4C(n-1)}\right\},$$

where $\rho(\cdot, \cdot)$ is a pseudo metric on $\mathbb{R}^d$, which is defined for open subsets F and F' in $\mathbb{R}^d$ through

$$\rho(F, F') = \sup\left\{\psi(F, F') : \psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}),\ d\Gamma(\psi, \psi) \leq dm\right\},$$

with $\psi(F, F') \stackrel{\text{def}}{=} \inf\{|\psi(x) - \psi(y)| : x \in F, y \in F'\}$, cf. page 290 in [41], and see page 277 in [41] for the definition of $\Gamma(\cdot, \cdot)$. For our $\mathscr{E}_m$, one can easily compute that $d\Gamma(\psi, \psi) = e^{2\lambda\ell \cdot x}|\nabla\psi|^2\, dx$ for $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. Thereafter, we obtain that $\rho(F, F') \geq \inf\{|x - y| : x \in F, y \in F'\}$. (See also the second example on page 278 in [41]). Actually, the Dirichlet form $\mathscr{E}_m$ plays the role of $\mathscr{E}$, and $\mathscr{E}_{m_\omega}$ the role of $\mathscr{E}_t$ in [41]. They are symmetric and strongly local, hence with (2.4.4) the condition (UP) on page 279, and the assumption for $\mathscr{E}$ on page 277 in [41] is fulfilled.

Through simple computation, we get that for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$

$$m\big(B_{\frac{L}{2}}(x)\big) \leq ce^{2\lambda\ell \cdot x}e^{\lambda L}L^d\,,$$

$$m\big(U_0(L)\big) \leq \tilde{c}e^{2\lambda\ell \cdot x}e^{5\lambda L}L^{2d-2}\,,$$

$$m\big(U_-(L)\big) \leq \tilde{c}e^{2\lambda\ell \cdot x}e^{-3\lambda L}L^{2d-1}\,.$$

Hence, for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$ we obtain from (2.4.13) that

$$\mathsf{P}_x^\omega\Big[X_1 \in B_{\frac{L}{2}}(x), X_n \in U_-\Big] \leq \hat{c}L^{k(d)}\exp\Big\{-\tfrac{\gamma L^2}{16CL}\Big\} \leq \tilde{c}\,e^{-cL}\,,$$

because $\rho\big(B_{L/2}(x), U_-(L)\big) \geq L$ and $n < 4L/\gamma$. Similarly, because $\rho\big(B_{L/2}(x), U_0(L)\big) \geq 4L^2 - L$, we obtain for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$ that

$$\mathsf{P}_x^\omega\Big[X_1 \in B_{\frac{L}{2}}(x), X_n \in U_0\Big] \leq \hat{c}L^{\tilde{n}(d)}e^{4\lambda L}\exp\Big\{-cL^3\Big\} \leq \tilde{c}\,e^{-cL}\,.$$

Collecting the above results, we see that (2.4.11) is proved.   □

With the help of the previous two propositions, we obtain:

Corollary 2.4.4
There exist two constants $c_3 > 0$ and $\tilde{c}_3 > 0$ such that for $m \in \mathbb{N}$,

$$(2.4.14)\qquad \sup_{x,\omega} \mathsf{P}_x^\omega\Big[\tilde{T}_{-2^mR} < T_{2^mR}\Big] \leq \tilde{c}_3\exp\Big\{-c_32^mR\Big\}\,,$$

where $R > 0$ is the constant from R-separation above (2.1.6).

Proof: Let $4L = 2^{m+1}R$ in the definition of $U(L)$ in (2.4.8), and observe that

$$\mathsf{P}_x^\omega\Big[\tilde{T}_{-2^mR} < T_{2^mR}\Big] \leq \mathsf{P}_x^\omega\Big[T_U \geq \tfrac{4L}{\gamma}\Big] + \mathsf{P}_x^\omega\Big[T_U < \tfrac{4L}{\gamma}, X_{T_U} \notin \partial_+U\Big]\,.$$

Our claim follows immediately from the previous two propositions. □

The next two corollaries will be useful when checking the assumptions of Theorem 2.3.2 and 2.3.3.

Corollary 2.4.5
There exists a constant $c_4 > 0$ such that

$$(2.4.15) \qquad \inf_{x,\omega} \mathsf{P}_x^\omega[D = \infty] \geq c_4 > 0,$$

where D is the first backtracking time defined below (2.2.14).

Corollary 2.4.6
The process $(X_t)_{t \geq 0}$ is transient and $\mathsf{P}_x^\omega\big[\lim_{t \to \infty} \ell \cdot X_t = \infty\big] = 1$ for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$. Hence by Proposition 2.2.7, $\hat{\mathsf{P}}_x$-a.s. $\tau_1 < \infty$.

The proof of these two corollaries is just a slight variation on the proof of Corollary 2.3 and 2.4 in [38], where we apply the Support Theorem of Stroock-Varadhan, cf. page 25 in [2], instead of ellipticity directly.

2.4.2 Integrability Properties

In this part we use the results from the previous part to prove that $\sup_{x,\omega} \hat{\mathsf{E}}_x^\omega[e^{c\tau_1}] < \infty$ for some $c > 0$, and derive the main result of this section. The proof is divided into several propositions.

First, let us introduce the random variable

$$(2.4.16) \qquad M \stackrel{\text{def}}{=} \sup\left\{\ell \cdot (X_t - X_0) : 0 \leq t \leq \tilde{T}_{-R}\right\},$$

i.e. M is the maximal relative displacement of $X_\cdot$ in the direction ℓ before it goes R below its origin. It will turn out that M is an important variable in studying the integrability properties of $\ell \cdot X_{\tau_1}$. Because $\inf_{x,\omega} \mathsf{P}_x^\omega[\tilde{T}_{-R} = \infty] \geq c_4 > 0$, cf. (2.4.15), we cannot expect $M < \infty$ P_x^ω-a.s.. Nevertheless, we have the next proposition.

Proposition 2.4.7
There exists a constant $c_7 > 0$ small enough such that

$$(2.4.17) \qquad \sup_{x,\omega} \mathsf{E}_x^\omega\big[e^{c_7 M}, \tilde{T}_{-R} < \infty\big] \leq 1 - \frac{c_4}{2},$$

where c_4 is the constant defined in (2.4.15).

Proof: With the help of (2.4.14),the proof of this proposition is a slight variation of the proof of Lemma 4.2 in [38], ($\tilde{T}_{-R}$ plays the role of the variable D in (4.5) of [38]).   □

Now we shall prove the integrability of $\exp\{c\ell \cdot X_{\tau_1}\}$ under the extended quenched measure $\hat{\mathsf{P}}_x^\omega$. We recall the $(\mathscr{Z}_t)_{t \geq 0}$-stopping times $\big(V_k(a)\big)_{k \geq 0}$, $\big(\tilde{N}_k(a)\big)_{k \geq 0}$ and $N_1(a)$ defined in (2.2.12), (2.2.13), and the events $(A_k)_{k \geq 0}$ introduced in (2.2.26).

As we will see in the proof of Theorem 2.4.9, $\exp\{c\ell \cdot (X_{N_1(a)} - X_0) - ca\}$ will play a key role in studying the integrability of $\exp\{c\ell \cdot (X_{\tau_1} - X_0)\}$ under $\hat{\mathsf{P}}_x^\omega$. Let us start with:

Proposition 2.4.8
For each $\tilde{c}_5 > 0$ there is a $c_5 > 0$, such that:

$$(2.4.18) \qquad \sup_{\substack{x,\omega \\ a>0}} \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c_5\big(\ell \cdot (X_{N_1(a)} - X_0) - a\big)\right\}\right] \leq 1 + \tilde{c}_5.$$

Proof: First, we claim that for each $\tilde{c}_6 > 0$, there exists a $c > 0$, which tends to 0 as $\tilde{c}_6$ tends to 0, such that

$$(2.4.19) \qquad \sup_{\substack{x,\omega \\ a>0}} \mathsf{E}_x^\omega \left[\exp\left\{ c\big(\ell \cdot (X_{\tilde{N}_1(a)} - X_0) - a\big)\right\}\right] \leq 1 + \tilde{c}_6.$$

To see this, we observe that because for any x and ω, P_x^ω-a.s. $\lim_t \ell \cdot X_t = +\infty$, cf. Corollary 2.4.6, hence $V_k(a) < \infty$, $k \geq 0$, we can show with the same proof as the one given in the proof of Proposition 2.2.7 (instead of $3R$ we simply use a) that for all x, ω and for any $a > 0$, P_x^ω-a.s. $\tilde{N}_1(a) < \infty$. Notice, (we drop the "a" from all $V_k(a)$ and $\tilde{N}_1(a)$)

$$\mathsf{E}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{\tilde{N}_1(a)} - X_0)\right\}\right]$$

$$= \mathsf{E}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{\lceil V_0 \rceil} - X_0)\right\}, \tilde{N}_1 = \lceil V_0 \rceil \right]$$

$$+ \sum_{k \geq 1} \mathsf{E}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{\lceil V_k \rceil} - X_0)\right\}, \tilde{N}_1 = \lceil V_k \rceil \right].$$

Further, we notice that the first term on the right hand side is smaller than $\exp\left\{c(a + \frac{R}{2})\right\}$, since $\ell \cdot (X_{V_0} - X_0) = a$ and $\ell \cdot (X_{\lceil V_0 \rceil} - X_{V_0}) \leq R/2$ on the event $\left\{\lceil V_0 \rceil = \tilde{N}_1\right\}$. We also observe that for $k \geq 1$, $\ell \cdot (X_{\lceil V_k \rceil} - X_{V_k}) \leq \frac{R}{2}$ on the event $\left\{\tilde{N}_1 = \lceil V_k \rceil\right\}$; and $\ell \cdot (X_{V_k} - X_{V_{k-1}}) \leq$

$R + Z \circ \theta_{V_{k-1}}$, with Z defined in (2.2.24). So, it follows from the strong Markov property that for $k \geq 1$,

$$
\mathsf{E}_x^\omega \left[e^{c\ell \cdot (X_{\lceil V_k \rceil} - X_0)}, \tilde{N}_1 = \lceil V_k \rceil \right]
$$

$$
\leq e^{\frac{cR}{2}} \, \mathsf{E}_x^\omega \left[\exp \{ c\ell \cdot (X_{V_k} - X_0) \} ; A_0, \cdots, A_{k-1} \right]
$$

$$
\leq e^{\frac{cR}{2}} \, \mathsf{E}_x^\omega \left[e^{c \left(\ell \cdot (X_{V_{k-1}} - X_0) + Z \circ \theta_{V_{k-1}} + R \right)} ; A_0, A_1, \cdots, A_{k-1} \right]
$$

$$
\leq e^{\frac{cR}{2}} \, \mathsf{E}_x^\omega \left[e^{c\ell \cdot (X_{V_{k-1}} - X_0)} ; A_0, \cdots, A_{k-2} ; \mathsf{E}_{X_{V_{k-1}}}^\omega \left[e^{c(R+Z)} ; A \right] \right] ,
$$

$(A_0, \cdots, A_{k-2}$ are omitted when $k = 1$). It follows from (2.5.7) that for $c > 0$ small enough $\sup_{x,\omega} \mathsf{E}_x^\omega [e^{c(Z+R)} ; A] \leq 1 - \frac{c_0}{4}$, where the constant $c_0 > 0$ is defined in (2.2.25). Therefore, by induction we observe that the last expression is smaller than

$$
e^{\frac{cR}{2}} \left(1 - \tfrac{c_0}{4} \right)^k \mathsf{E}_x^\omega \left[\exp \{ c\ell \cdot (X_{V_0} - X_0) \} \right] = e^{c(a + \frac{R}{2})} \left(1 - \tfrac{c_0}{4} \right)^k .
$$

Hence, for $c > 0$ small enough we obtain that

$$
\sup_{\substack{x,\omega \\ a > 0}} \mathsf{E}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_1(a)} - X_0) - ca \right\} \right]
$$

$$
\leq e^{\frac{cR}{2}} \sum_{k \geq 0} (1 - \tfrac{c_0}{4})^k =: C < \infty .
$$

To get (2.4.19), we observe that by Chebychev's inequality, for $\hat{c} \in (0, c)$,

$$
(2.4.20) \quad \sup_{\substack{x,\omega \\ a}} \mathsf{E}_x^\omega \left[\exp \left\{ \hat{c}\ell \cdot (X_{\tilde{N}_1(a)} - X_0) - \hat{c}a \right\} \right]
$$

$$
\leq 1 + \hat{c}C \int_0^\infty dz \, e^{\hat{c}z} e^{-cz} \leq 1 + \tilde{c}_6 ,
$$

provided $\hat{c}$ is small enough. This proves (2.4.19).

Now, observe that it follows from the definition of $N_1(a)$ in (2.2.13) and

the strong Markov property, cf. Corollary 2.2.2, that

$$\hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{N_1(a)} - X_0) \right\} \right]$$

$$= \sum_{k \geq 1} \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{\tilde{N}_k(a)} - X_0)}; N_1(a) = \tilde{N}_k(a) \right]$$

$$= \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_1(a)} - X_0) \right\}; \lambda_{\tilde{N}_1(a)} = 1 \right]$$

$$+ \sum_{k \geq 1} \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_k(a)} - X_0) \right\}; \lambda_{\tilde{N}_1(a)} = \cdots = \lambda_{\tilde{N}_k(a)} = 0 \right. ;$$

$$\hat{\mathsf{E}}_{X_{\tilde{N}_k(a)},0}^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_1(3R)} - X_0) \right\}; \lambda_{\tilde{N}_1(3R)} = 1 \right] \Bigg] .$$

Similar to (2.4.19), we can find $c_6 > 0$ such that $\hat{\mathsf{E}}_{x,0}^\omega \left[\exp\{c_6\ell \cdot (X_{\tilde{N}_1(a)} - X_0) - c_6 a\} \right] < 1 + \tilde{c}_6$. Further, we observe that under the measure $\hat{\mathsf{P}}_{x,\lambda}^\omega$, for any integer-valued $(\mathscr{F}_t)_{t \geq 0}$-stopping time S, λ_S is independent of $\mathscr{F}_S \otimes \mathscr{S}_{S-1}$, see property 2 of Theorem 2.2.1. Therefore, we see that for $c \in (0, c_6)$ the previous expression is smaller than

$$\varepsilon \, \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_1(a)} - X_0) \right\} \right] + \sum_{k \geq 1} \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{\tilde{N}_k(a)} - X_0) \right\}; \right.$$

$$\left. \lambda_{\tilde{N}_1(a)} = \cdots = \lambda_{\tilde{N}_k(a)} = 0 \right] e^{3cR} (1 + \tilde{c}_6) \, \varepsilon \, ,$$

where ε is given in (2.2.5). By induction we obtain that the last expression is

$$\leq \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{\tilde{N}_1(a)} - X_0)} \right] \left\{ \varepsilon + \frac{\varepsilon}{1 - \varepsilon} \sum_{k \geq 1} \left[(1 - \varepsilon) e^{3cR} (1 + \tilde{c}_6) \right]^k \right\}$$

$$\leq e^{ca} C < \infty \, ,$$

for some $C > 0$ independent of a, provided $\tilde{c}_6 > 0$ and $c > 0$ are small enough. That is, $\sup_{x,\omega,a} \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c\ell \cdot (X_{N_1(a)} - X_0) - ca \right\} \right] \leq C < \infty$. Our claim follows by a similar computation as in (2.4.20). □

Theorem 2.4.9
There exists a constant $c_8 > 0$ such that

$$(2.4.21) \qquad \sup_{x,\omega} \hat{\mathsf{E}}_x^\omega \left[\exp \left\{ c_8 \ell \cdot (X_{\tau_1} - X_0) \right\} \right] < \infty .$$

Proof: Observe that

$$\hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{\tau_1} - X_0) \right\} \right]$$

$$(2.4.22) \qquad = \sum_{k \geq 1} \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{S_k} - X_0)}, S_k < \infty, D \circ \theta_{S_k} = \infty \right]$$

$$\leq \sum_{k \geq 1} \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{S_k} - X_0)}, S_k < \infty \right] \overset{\text{def}}{=} \sum_{k \geq 1} h_k \, ,$$

and because for any x and ω, $\ell \cdot (X_{S_1} - X_{N_1(3R)}) \leq 10R$, $\hat{\mathsf{P}}_x^\omega$-a.s. (cf. Theorem 2.2.1), Proposition 2.4.8 implies that $h_1 < \infty$. So it suffices to show that $\sum_{k \geq 1} h_{k+1} < \infty$. To show this, we observe that (cf. (2.2.15))

$$\ell \cdot (X_{S_{k+1}} - X_0) \leq 10R + \ell \cdot (X_{R_k} - X_0) + \ell \cdot (X_{N_1(a_k)} - X_0) \circ \theta_{R_k} \, ,$$

with $a_k = M(R_k) - \ell \cdot (X_{R_k} - X_0) + R \in \mathscr{X}_{R_k}$, see also Figure 2.2.3. We recall that the shift θ_{R_k} is *not* applied to a_k. Therefore, by the strong Markov property, cf. Corollary 2.2.2, we have:

$$\hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{S_{k+1}} - X_0) \right\}, S_{k+1} < \infty \right]$$

$$(2.4.23) \qquad \leq e^{10cR} \, \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{R_k} - X_0)}, R_k < \infty; \hat{\mathsf{E}}_{X_{R_k}}^\omega \left[e^{c\ell \cdot (X_{N_1(a_k)} - X_0)} \right] \right]$$

$$\leq e^{10cR} \, \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{R_k} - X_0) \right\}, R_k < \infty; (1 + \tilde{c}_5) e^{ca_k} \right] ,$$

where we applied Proposition 2.4.8 in the last step, provided $c \in (0, c_5)$.

From Figure 2.2.3 we also observe that with M from (2.4.16) and Z from (2.2.24), the following inequalities hold:

$$a_k \leq Z \circ \theta_{J_k} + M \circ \theta_{S_k} + 2R \, ,$$

$$\ell \cdot (X_{R_k} - X_0) \leq \ell \cdot (X_{J_k} - X_0) + Z \circ \theta_{J_k} \, .$$

We put them into the rightmost side of (2.4.23), apply the strong Markov property at time S_k, cf. Corollary 2.2.2, then apply the strong Markov property for the process $(X_t)_{t \geq 0}$ at time J_k. We obtain

$$\hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{S_{k+1}} - X_0) \right\}; S_{k+1} < \infty \right]$$

$$\leq e^{12cR} \, \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{J_k} - X_0) + cM \circ \theta_{S_k}}; J_k < \infty, (1 + \tilde{c}_5) \mathsf{E}_{X_{J_k}}^\omega \left[e^{2cZ} \right] \right]$$

$$\leq e^{12cR} \, \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{J_k} - X_0) + cM \circ \theta_{S_k} \right\}; J_k < \infty, (1 + \tilde{c}_5)^2 \right] ,$$

provided $\sup_{x,\omega} \mathsf{E}_x^\omega[e^{2cZ}] \leq 1 + \tilde{c}_5$, for $c > 0$ small enough (cf. (2.5.6)). Hence, by the strong Markov property again, the above expression is

$$\leq e^{11cR} (1 + \tilde{c}_5)^2 \, \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{S_k} - X_0) \right\}, S_k < \infty; \right.$$

$$\left. \underbrace{\mathsf{E}_{X_{S_k}}^\omega \left[e^{cM}, \tilde{T}_{-R} < \infty \right]}_{\leq 1 - \frac{c_4}{2} \text{ by } (2.4.17)} \right]$$

$$\leq (1 - \alpha) \, \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{S_k} - X_0) \right\}, S_k < \infty \right],$$

for some $\alpha > 0$, provided $\tilde{c}_5 > 0$ and $c \in (0, c_5)$ are small enough such that $e^{11cR} (1 + \tilde{c}_5)^2 (1 - \frac{c_4}{2}) < 1 - \alpha$. By induction the last expression is:

$$\leq (1 - \alpha)^k \, \hat{\mathsf{E}}_x^\omega \left[\exp\left\{ c\ell \cdot (X_{S_1} - X_0) \right\}, S_1 < \infty \right].$$

Coming back to (2.4.22), we obtain

$$\sup_{x,\omega} \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{\tau_1} - X_0)} \right]$$

$$\leq \sup_{x,\omega} \hat{\mathsf{E}}_x^\omega \left[e^{c\ell \cdot (X_{S_1} - X_0)}, S_1 < \infty \right] \cdot \sum_{k \geq 0} (1 - \alpha)^k < \infty.$$

$$\square$$

As a corollary, we obtain an exponential estimate on the tail of τ_1. Let us point out that such an estimate together with Theorem 2.4.9 and the renewal structure of Theorem 2.2.5 can be used to derive large deviation controls, see [43], [44].

Corollary 2.4.10
There exist constants $c_9 > 0$ and $\tilde{c}_9 > 0$ such that for $u \in \mathbb{N}$

$$(2.4.24) \qquad \sup_{x,\omega} \hat{\mathsf{P}}_x^\omega \left[\tau_1 > u \right] \leq \tilde{c}_9 \, \exp\{ -c_9 u \}.$$

Proof: Observe that for $u \geq 6R/\gamma$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$:

$$\hat{\mathsf{P}}_x^\omega \left[\tau_1 > u \right] \leq \hat{\mathsf{P}}_x^\omega \left[\tau_1 > u, \ell \cdot (X_{\tau_1} - X_0) \leq \frac{\gamma}{2} u - 3R \right]$$

$$+ \hat{\mathsf{P}}_x^\omega \left[\ell \cdot (X_{\tau_1} - X_0) > \frac{\gamma}{2} u - 3R \right].$$

By Chebychev's inequality and Theorem 2.4.9, the last term on the right hand side is smaller than $\tilde{c}e^{-cu}$, for some $\tilde{c} > 0$ and $c \in (0, c_8)$. Hence it suffices to study the first term on the right hand side of the above expression. Let U now be the cylinder defined in (2.4.8), which is centered in x, has height $4L = \gamma u$ in the direction ℓ and radius $4L^2$ in the directions normal to ℓ. With the observation that P_x^ω-a.s. $\sup_{s \leq \tau_1} \ell \cdot (X_s - X_{\tau_1}) < 3R$, cf. Figure 2.2.4, we see that for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$:

$$\hat{\mathsf{P}}_x^\omega \left[\tau_1 > u, \ell \cdot (X_{\tau_1} - X_0) \leq \tfrac{\gamma}{2}u - 3R\right] \leq \mathsf{P}_x^\omega \left[T_{\frac{\gamma}{2}u} > u\right]$$

$$\leq \mathsf{P}_x^\omega \left[T_U < T_{\frac{\gamma}{2}u}\right] + \mathsf{P}_x^\omega \left[T_U = T_{\frac{\gamma}{2}u} > u\right]$$

$$\leq \mathsf{P}_x^\omega \left[T_U \geq u\right] + \mathsf{P}_x^\omega \left[T_U < u, X_{T_U} \notin \partial_+ U\right] + \mathsf{P}_x^\omega \left[T_U = T_{\frac{\gamma}{2}u} > u\right].$$

Observe that by Proposition 2.4.2 the first and the third term in the above expression are smaller than $\tilde{c}e^{-cu}$ for suitable $\tilde{c} > 0$ and $c > 0$, and by Proposition 2.4.3 the second term is also smaller than $\tilde{c}e^{-cu}$. This finishes our proof. $\qquad\square$

We come now to the main result of this section:

Theorem 2.4.11
Let $(X_t)_{t \geq 0}$ be the (unique strong) solution to the stochastic differential equation $dX_t = dW_t + \nabla V(X_t, \omega)\, dt$ and $X_0 = x$, where for each $\omega \in \Omega$, $V(\cdot, \omega) \in C^1(\mathbb{R}^d, \mathbb{R})$ has bounded and Lipschitz-continuous derivatives, and $Ae^{2\lambda \ell \cdot x} \leq V(x, \omega) \leq Be^{2\lambda \ell \cdot x}$ holds for some $\ell \in S^{d-1}$, $A, B > 0$ and $\lambda > 0$. Then

$$\mathsf{P}_0\text{-}a.s. \quad \frac{X_t}{t} \xrightarrow{\ t \to \infty\ } v,$$

with a deterministic $v \in \mathbb{R}^d$, which is given in (2.3.7), and $\ell \cdot v > 0$; further the processes $\left(\frac{X_{st} - vst}{\sqrt{s}}\right)_{t \geq 0}$ converge in law under P_0, as $s \to \infty$, to a non-degenerate d-dimensional Brownian motion with covariance matrix $\mathbf{K}$ given in (2.3.12).

Proof: It follows from (2.4.15) and Corollary 2.4.10 that the condition (2.3.1) is fulfilled. Our claims follow from Theorem 2.3.2 and Theorem 2.3.3. $\qquad\square$

2.5   Appendix

2.5.1   Some Facts about Local Martingales

Lemma 2.5.1
On some probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathsf{P})$, *let* $(Y_t)_{t \geq 0}$ *be a continuous local martingale satisfying* $Y_0 = 0$ *and* $\langle Y \rangle_t \leq \nu t$ *for* $t \geq 0$. *Then for* $p > 1$ *there is a constant* $c(p, \nu) > 0$ *such that*

$$(2.5.1) \qquad \mathsf{E}\Big[\sup_{s \leq t} |Y_s|^p\Big] \leq c(p, \nu)\, t^{\frac{p}{2}} \,,$$

and

$$(2.5.2) \qquad \mathsf{P}\text{-}a.s. \quad \frac{1}{t} \sup_{s \leq t} |Y_s| \xrightarrow{t \to \infty} 0 \,.$$

Proof: The Bernstein's inequality, cf. page 153 – 154 in [36], shows that

$$(2.5.3) \qquad \mathsf{P}\Big[\sup_{s \leq t} |Y_s| \geq a\Big] \leq 2\, \exp\Big\{-\frac{a^2}{2\nu t}\Big\} \,,$$

hence

$$\mathsf{E}\Big[\sup_{s \leq t} |Y_s|^p\Big] \leq p \int_0^\infty y^{p-1} \exp\Big\{-\frac{y^2}{2\nu t}\Big\}\, dy =: c(p, \nu) t^{\frac{p}{2}} \,.$$

For (2.5.2), it suffices to prove that P-a.s. $\frac{1}{n} \sup_{s \leq n} |Y_s| \xrightarrow{n \to \infty} 0$. To see this, we observe that from (2.5.3) it follows that for $a > 0$

$$(2.5.4) \qquad \sum_{n \geq 1} \mathsf{P}\Big[\frac{1}{n} \sup_{s \leq n} |Y_s| \geq a\Big] \leq 2 \sum_{n \geq 1} \exp\Big\{-\frac{a^2 n}{2\nu}\Big\} < \infty \,,$$

and the claim follows from Borel-Cantelli's lemma.   □

From this lemma we easily get the next two corollaries.

Corollary 2.5.2
Let $(X_t)_{t \geq 0}$ *be the solution of the stochastic differential equation (2.1.7), whose coefficients satisfy (2.1.1), (2.1.3) and (2.1.4). Then there exist two constants* $c > 0$ *and* $\tilde{c} > 0$ *depending only on* $(d, \nu, \bar{b})$ *such that for all* $x \in \mathbb{R}^d$, $\omega \in \Omega$ *and* $L > 0$,

$$(2.5.5) \qquad \sup_{x, \omega} \mathsf{P}_x^\omega\Big[\sup_{s \leq 1} |X_s - X_0| \geq L\Big] \leq \tilde{c}\, e^{-cL^2} \,.$$

Proof: Observe that for all $x \in \mathbb{R}^d$ and for all $\omega \in \Omega$, P_x^ω-a.s. $X_t - X_0 = \int_0^t b(X_s, \omega) \, ds + Y_t(\omega)$, with the P_x^ω-local martingale $Y_t(\omega) := \int_0^t \sigma(X_s, \omega) \, dW_s$. Further we observe by our assumption (2.1.4) that $\langle Y^j(\omega) \rangle_t \leq \nu t$ for all $j = 1, \ldots, d$ and $\omega \in \Omega$. Therefore with our assumption $|b| \leq \bar{b}$, it follows immediately from the Bernstein's inequality (2.5.3) that

$$\mathsf{P}_x^\omega \Big[\sup_{s \leq 1} |X_s - X_0| \geq L \Big] \leq \mathsf{P}_x^\omega \Big[\sup_{s \leq 1} |Y_s(\omega)| \geq (L - \bar{b}) \Big] \leq \tilde{c} e^{-cL^2} \, .$$

$\square$

Corollary 2.5.3

Let $Z(\omega) := \sup_{s \leq 1} |X_s - X_0|$, then for all $\alpha > 0$ there exists a constant $\delta(\alpha, d, \nu, \bar{b}) > 0$ such that

$$(2.5.6) \qquad \sup_{x, \omega} \mathsf{E}_x^\omega \big[e^{\delta Z} \big] \leq 1 + \alpha \, .$$

Further, let $A \in \mathscr{F}_1$ be an event such that $\sup_{x, \omega} \mathsf{P}_x^\omega[A] \leq 1 - 2\beta$ for some $\beta > 0$, then there exists a constant $\delta(\beta, d, \nu, \bar{b}) > 0$ such that

$$(2.5.7) \qquad \sup_{x, \omega} \mathsf{E}_x^\omega \big[e^{\delta Z} ; A \big] \leq 1 - \beta \, .$$

Proof: Because $Z(\omega) \leq \sup_{s \leq 1} |Y_s(\omega)| + \bar{b}$, we get for $0 < \delta < 1$ that

$$\mathsf{E}_x^\omega \big[e^{\delta Z} \big] \leq e^{\delta \bar{b}} \, \mathsf{E}_x^\omega \Big[\exp\{\delta \sup_{s \leq 1} |Y_s|\} \Big]$$

$$= e^{\delta \bar{b}} \Big(1 + \delta \int_0^\infty da \, e^{\delta a} \underbrace{\mathsf{P}_x^\omega \Big[\sup_{s \leq 1} |Y_s| \geq a \Big]}_{\leq 2d \exp\{-a^2/(2d\nu)\}} \Big) \leq e^{\delta \bar{b}} \big(1 + \delta \, c(\bar{b}, \nu, d) \big) \, ,$$

for some $c(\bar{b}, \nu, d) > 0$ and this proves (2.5.6). To prove (2.5.7) we observe by Hölder's inequality that for $p, q > 0$ such that $1/p + 1/q = 1$:

$$\mathsf{E}_x^\omega \big[e^{\delta Z} ; A \big] \leq \mathsf{E}_x^\omega \big[e^{\delta p Z} \big]^{\frac{1}{p}} \, \mathsf{P}_x^\omega[A]^{\frac{1}{q}} \leq (1 + \alpha)^{\frac{1}{p}} (1 - 2\beta)^{\frac{1}{q}} \leq 1 - \beta \, ,$$

by choosing δ small and p large enough. $\square$

2.5.2   Some Results about Parabolic PDE

In this part we will collect some results about parabolic partial differential equations, which we use throughout this article. For detailed treatment we refer to the article by Il'in, Kalashnikov and Oleinik, [18], section 4.

Proposition 2.5.4
We consider the linear parabolic equation of second order $\frac{\partial u}{\partial t} = Lu$, where

$$(2.5.8) \qquad L = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j},$$

with the coefficients a_{ij} and b_k satisfying for all $x, y \in \mathbb{R}^d$

$$|a_{ij}(x) - a_{ij}(y)| + |b_k(x) - b_k(y)| \le C|x - y|^{\delta},$$

$$|a_{ij}(x)| + |b_k(x)| \le K, \quad a_{ij}(x) = a_{ji}(x),$$

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \frac{1}{\nu}\sum_{j=1}^{d}\xi_j^2, \quad \xi \in \mathbb{R}^d,$$

for some $C > 0$, $K > 0$, $\nu > 0$ and $\delta > 0$. Then there exists a unique fundamental solution $Z(t, x, y)$ of $\frac{\partial u}{\partial t} = Lu$, such that for $t \le 1$

$$(2.5.9) \qquad |Z(t, x, y)| \le \frac{M}{t^{d/2}} \exp\left\{ -\frac{\mu|x-y|^2}{t} \right\},$$

for some constants $M(\nu, C, K, d, \delta) > 0$ and $\mu(\nu, C, K, d, \delta) > 0$. Further, there exist two constants $a(\nu, C, K, d, \delta) > 0$ and $\tilde{M}(\nu, C, K, d, \delta) > 0$ such that for $|x - y|^2 < at$ and $t \in (0, 1]$

$$(2.5.10) \qquad Z(t, x, y) \ge \frac{\tilde{M}}{t^{d/2}}.$$

The claims (2.5.9) and (2.5.10) are just the statement (4.16) and (4.75) in [18]. The authors of [18] did not state on which the constants M, μ, a and $\tilde{M}$ really depend on, but by working through their computation, cf. page 63–82, one can see that these constants only depend on (ν, C, K, d, δ).

As a consequence of the previous proposition we get the next corollary.

Corollary 2.5.5

Let U^x and B^x be the open set defined in (2.2.1). Under the assumption (2.1.1), (2.1.3) and (2.1.4), there exist two constants $\tilde{M}(\nu, d, \bar{b}, \bar{\sigma}, K) > 0$ and $a(\nu, d, \bar{b}, \bar{\sigma}, K) > 0$ (recall the constants $\nu, d, \bar{b}, \bar{\sigma}$ and K are defined in Section 2.1), such that for all $\omega \in \Omega$, $1 \geq t > 0$ and $|x - y|^2 \leq at$, the transition density $p_\omega(t, x, y)$ satisfies

$$(2.5.11) \qquad p_\omega(t, x, y) \geq \frac{\tilde{M}}{t^{d/2}},$$

and there exists a constant $\varepsilon(\nu, d, \bar{b}, \bar{\sigma}, R, K) > 0$ such that the subtransition density $p_{\omega, U^x}(1, x, y)$ (recall (2.2.4)) satisfies

$$(2.5.12) \qquad p_{\omega, U^x}(1, x, y) \geq \frac{2\varepsilon}{|B_R|},$$

for all $y \in B^x$.

Proof: With $a_{ij} = \left(\sigma \sigma^t\right)_{ij}$ we see from (2.1.1), (2.1.3) and (2.1.4) that the assumptions of Proposition 2.5.4 are fulfilled. Hence, (2.5.11) follows immediately from Proposition 2.5.4.

To prove (2.5.12), first we observe that because of (2.5.10) there is $t_0 \in (0, 1]$ such that $\sqrt{at_0} \leq \frac{R}{4}$ and for all $t \leq t_0$,

$$\frac{\tilde{M}}{t^{d/2}} \geq \frac{2M}{t_0^{d/2}} \exp\{-\tfrac{\mu R^2}{16t_0}\}$$

holds, in addition the function

$$t \mapsto \frac{M}{t^{d/2}} \exp\left\{ - \tfrac{\mu R^2}{16t} \right\}$$

is monotone increasing on $\{t : t \leq t_0\}$. Now let $G = B_{\frac{R}{2}}(x)$ and $y \in B_{\sqrt{at_0}}(x)$, we observe that on the event $\{T_G < t \leq t_0\}$, the inequality $p_\omega(t - T_G, X_{T_G}, y) \leq \frac{M}{t^{d/2}} \exp\left\{ - \tfrac{\mu R^2}{16t} \right\}$ follows from the monotonicity mentioned above. Hence, by Duhamel's formula, cf. page 331 in [40]:

$$p_{\omega, G}(t, x, y)$$
$$= p_\omega(t, x, y) - \mathsf{E}_x^\omega\left[T_G < t, p_\omega(t - T_G, X_{T_G}, y)\right], \text{ for all } x, y \in G,$$

there is $\tilde{\varepsilon}(\nu, d, \bar{b}, \bar{\sigma}, R, K) > 0$ so that $p_{\omega, G}(t, x, y) \geq \tilde{\varepsilon} > 0$, for $t \leq t_0$ and $|x - y| \leq \sqrt{at}$.

By iteration, it is straightforward to see that $\inf_{\omega, y \in B^x} p_{\omega, U^x}(1, x, y) > 0$. $\qquad \square$

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Curriculum Vitae

Name: Lian Shen
Birth Date: October 13, 1971
Birth Place: Shanghai, China
Citizenship: P.R. China

Education

Sep. 1978 — Jul. 1984:   Primary school in Beijing.

Sep. 1984 — Jul. 1990:   Middle school in Beijing.

Nov. 1991 — May. 1992:   German course, Fribourg, Switzerland.

Oct. 1992 — Oct. 1997:   Diplom study in theoretical physics at ETH-Zürich. Thesis: Positive Energy Theorem in Einstein's General Relativity Theory.

Sep. 1998 — Jul. 2002:   Teaching assistent and Ph.D. student under supervison of Prof. A.-S. Sznitman, Department of Mathematics, Swiss Federal Institute of Technology, (ETH-Zürich), Switzerland.