Manifestly supersymmetric RG flows

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Abstract: Renormalisation group (RG) equations in two-dimensional $\mathcal{N} = 1$ supersymmetric field theories with boundary are studied. It is explained how a manifestly $\mathcal{N} = 1$ supersymmetric scheme can be chosen, and within this scheme the RG equations are determined to next-to-leading order. We also use these results to revisit the question of how brane obstructions and lines of marginal stability appear from a world-sheet perspective.

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Contents

1 Introduction

The behaviour of supersymmetric D-branes under deformations of the closed string background has a surprisingly rich and interesting structure. For example, even if the closed string remains supersymmetric under the deformation, the same may not be true in the presence of a D-brane. If this is the case one says that the deformation is 'obstructed' by the D-brane. Another phenomenon that has attracted a lot of attention recently [1–3] concerns lines of marginal stability: a supersymmetric brane may decay into a superposition of (supersymmetric) D-branes as the string background is modified (in a supersymmetric fashion).

These phenomena were recently studied from the point of view of the world-sheet [4]. From this perspective obstructions are associated with bulk perturbations for which the (adjusted) boundary condition does not preserve the $\mathcal{N} = 2$ super(conformal) symmetry any longer (see also [5, 6]). Indeed, the supervariation of these perturbations vanishes only up to total (bosonic) derivatives, which in the presence of a D-brane generically lead to non-vanishing boundary contributions [7, 8]. The latter can sometimes be cancelled by adding new boundary terms to the sigma model action, but this is not necessarily possible and the D-brane may therefore be obstructed. However, it is always possible to add a suitable boundary term so that an $\mathcal{N} = 1$ subalgebra is preserved. This is important from the point of view of string theory since the $\mathcal{N} = 1$ algebra describes a gauge symmetry in this context (and hence must be preserved for consistency). Furthermore, lines of marginal stability appear when this boundary deformation (that is added to preserve the $\mathcal{N} = 1$ supersymmetry) becomes relevant.

Given that it is always possible to preserve an $\mathcal{N} = 1$ supersymmetry it should be possible to formulate the combined bulk and boundary deformation problem in a *manifestly* $\mathcal{N} = 1$ *supersymmetric* fashion. In fact, as we shall explain in this paper, the boundary correction term has a natural interpretation in terms of a superspace description of the problem. This observation can be used to formulate a renormalisation group scheme in which the $\mathcal{N} = 1$ supersymmetry is (manifestly) preserved. Given what we said above, this is a very natural scheme for superstring calculations.

Within this scheme we then analyse the coupled renormalisation group equations, thereby combining the superspace approach of e.g. [4] with methods of perturbed conformal field theory (see in particular [5, 9]). Among other things we identify the precise coupling constant which controls the bulk induced boundary RG source term of [5], and we explain how the change in conformal dimension of a boundary field can be calculated [9] in this context. We also apply these techniques to the case of (cc) perturbations of B-type branes. In our supersymmetric scheme the first order bulk induced boundary RG source term always vanishes for marginal boundary fields. However, the (cc) bulk perturbation may change the conformal dimension of a marginal boundary field, and thus induce an instability. Finally, we compare these findings with results that had been obtained previously using matrix factorisation techniques [10] (see also [11–16] for related recent work). In particular, we show (at least in an example) that the boundary field that becomes relevant is precisely the one predicted from the analysis of [10].

For the case of a (ca) perturbation of a B-type brane, on the other hand, one does not expect any obstructions, and thus generically the full $\mathcal{N} = 2$ supersymmetry should be preserved. This suggests that the (ca) deformation problem should have a manifestly $\mathcal{N} = 2$ supersymmetric formulation, and this again turns out to be true. However, there *exist* lines of marginal stability in this context. They are associated to a breakdown of this manifestly $\mathcal{N} = 2$ supersymmetric scheme.

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Supersymmetric D-branes have been studied extensively, using sigma model techniques, in the past, see for example [17–23]. More recently, a manifestly $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric formulation for D-branes has also been given and interpreted in terms of generalised geometry [24–27]. Here we study how the manifestly $\mathcal{N}=1$ supersymmetric formulation can be maintained under supersymmetric bulk deformations.

The paper is organised as follows. In section 2 we explain how to formulate the deformation problem in a manifestly $\mathcal{N}=1$ supersymmetric fashion. In section 3 we determine the RG equations in the associated scheme. These results are then applied to (cc) perturbations of B-type branes in section 4.1. In section 4.2, we explain how a manifestly $\mathcal{N}=2$ supersymmetric description is available for (ca) perturbations of B-type branes, and section 5 contains our conclusions. There are a number of appendices in which some of the more technical material has been collected.

2 Manifestly supersymmetric theories with boundary

In this section we want to discuss manifestly $\mathcal{N}=1$ supersymmetric field theories on two-dimensional manifolds with boundaries. After introducing some basic notation we will explain how one can write the perturbation in superspace in a manifestly supersymmetric manner.

2.1 Superfields and OPEs

Let us begin by discussing some important aspects of two-dimensional supersymmetric field theories. We will work with the standard $\mathcal{N}=(1,1)$ superspace (see e.g. [28])

$$\mathbb{R}^{(2|1,1)} = \{z, \bar{z}, \theta, \bar{\theta}\}\,, \tag{2.1}$$

with a boundary along the line $z=\bar{z}$ (for more details see also appendix A.1). A scalar superfield is of the form

$$\Phi(z,\bar{z},\theta,\bar{\theta}) = \phi(z,\bar{z}) + \theta\chi(z,\bar{z}) + \bar{\theta}\bar{\chi}(z,\bar{z}) + \theta\bar{\theta}F(z,\bar{z})\;. \tag{2.2}$$

We denote by h and $\bar{h}$ the (left- and right-moving) conformal dimension of the lowest component of Φ, and by $\Delta = h+\bar{h}$ the total conformal dimension. For the following it is important to characterise the behaviour of two such superfields as they approach each other. Introducing labels $I, J, \ldots$ to distinguish them, the bulk operator product expansion (OPE) takes the form

$$\begin{aligned}
&\Phi_I(z_1,\bar{z}_1,\theta_1,\bar{\theta}_1)\,\Phi_J(z_2,\bar{z}_2,\theta_2,\bar{\theta}_2) \qquad\qquad (2.3)\\
&= \sum_K |z_{12}|^{\Delta_K-\Delta_I-\Delta_J} C^{(1)}_{IJK}\Big[\Phi_K+\cdots\Big] + \sum_L |z_{12}|^{\Delta_L-\Delta_I-\Delta_J-1} C^{(2)}_{IJL}\Big[\theta_{12}\bar{\theta}_{12}\Phi_L+\cdots\Big]\\
&\quad + \sum_\alpha |z_{12}|^{\Delta_\alpha-\Delta_I-\Delta_J-\frac{1}{2}} C^{(3)}_{IJ\alpha}\Big[\theta_{12}\Xi_\alpha+\cdots\Big] + \sum_\beta |z_{12}|^{\Delta_\beta-\Delta_I-\Delta_J-\frac{1}{2}} C^{(4)}_{IJ\beta}\Big[\bar{\theta}_{12}\Xi_\beta+\cdots\Big]\,,
\end{aligned}$$

where z_{12} and θ_{12} are defined by

$$z_{12} = \frac{1}{2}(z_1-z_2) - \theta_1\theta_2\,, \qquad\qquad \theta_{12} = \frac{1}{2}(\theta_1-\theta_2)\,, \tag{2.4}$$

and the fields appearing on the right hand side are all evaluated at

$$\hat{z}_{12} = \frac{1}{2}(z_1 + z_2) \ , \qquad\qquad \hat{\theta}_{12} = \frac{1}{2}(\theta_1 + \theta_2) \ . \tag{2.5}$$

The ellipses refer to terms that are either of higher order in $|z_{12}|$ or involve superdescendants of Φ_I and Ξ_α. Using nomenclature from conformal field theory, the first line in (2.3) describes the even-even and odd-odd fusion rules, respectively, while the second line gives the contribution of the even-odd and odd-even fusion rules. The superfields Ξ_α that appear there are fermionic, but cannot be written in terms of superdescendants of Φ_I. They will not play a role in the following.

In the presence of a boundary we also need to describe the behaviour as the bulk field approaches the boundary. To this end it is convenient to rewrite the superfield (2.2) as

$$\Phi(x, y, \theta^+, \theta^-) = \phi(x, y) + \theta^+\chi^+(x, y) + \theta^-\chi^-(x, y) + \theta^+\theta^- G(x, y) \, , \tag{2.6}$$

where we have introduced real coordinates via $z = x + iy$ and $\theta^\pm = (\theta \pm \bar{\theta})$, as well as

$$\chi^+ = \frac{1}{2}(\chi + \bar{\chi}) \, , \qquad \chi^- = \frac{1}{2}(\chi - \bar{\chi}) \, , \qquad G = -\frac{1}{2}F \ . \tag{2.7}$$

It is now convenient to expand (2.6) in powers of the variable $\tilde{y} = y - \theta\bar{\theta}$, which becomes small in the vicinity of the boundary. The most generic expression which can be written down is of the following form

$$\begin{aligned} &\Phi_I(x, y, \theta^+, \theta^-) \\ &\quad = \sum_i B^{(1)}_{Ii} (2\tilde{y})^{h_i - \Delta_I} \Big[\Pi_i(x, \theta^+) + \cdots \Big] + \sum_a B^{(2)}_{Ia} (2\tilde{y})^{h_a - \Delta_I + \frac{1}{2}} \Big[\mathcal{D}^+ \Psi_a(x, \theta^+) + \cdots \Big] \\ &\qquad + \theta^- \Bigg(\sum_j B^{(3)}_{Ij} (2\tilde{y})^{h_j - \Delta_I} \Big[\mathcal{D}^+ \Pi_j(x, \theta^+) + \cdots \Big] \\ &\qquad\qquad + \sum_b B^{(4)}_{Ib} (2\tilde{y})^{h_b - \Delta_I - \frac{1}{2}} \Big[\Psi_b(x, \theta^+) + \cdots \Big] \Bigg) \ , \end{aligned} \tag{2.8}$$

where $\mathcal{D}^+$ is the spinor derivative defined in appendix A.1 and $B^{(1,3)}_{Ii}$ and $B^{(2,4)}_{Ia}$ are some expansion coefficients. Π_i and Ψ_a are the most generic boundary superfields which can be written using just a single Grassmann variable. They have an expansion as

$$\Pi_i(x, \theta^+) = \pi_i(x) + \theta^+ \chi_i(x) \, , \tag{2.9}$$

$$\Psi_a(x, \theta^+) = \psi_a(x) + \theta^+ \rho_a(x) \, , \tag{2.10}$$

where we note that Π_i is bosonic, while Ψ_a is fermionic. Finally, h_i and h_a are the conformal dimensions of π_i and ψ_a respectively. From a superspace point of view, (2.8) corresponds to a decomposition of the $\mathcal{N} = (1, 1)$ superfield in terms of the $\mathcal{N} = 1$ superfields (2.9) and (2.10) (and their (super)derivatives). In the language of conformal field theory, the terms in the first line correspond to the even fusion rules with respect to θ^- — the two terms are even and odd with respect to θ^+, respectively — while the terms in the second and third line are in the odd fusion channel with respect to θ^-. The coefficients can, for example, be determined by analysing the fusion rules, using the techniques of appendix B.2 in [4].

We will also need the OPEs of the boundary superfields with one another, in particular

$$\begin{aligned}\Psi_a(x_1,\theta_1^+)\Psi_b(x_2,\theta_2^+) = &\sum_i D^{(1)}_{abi}|x_{12}|^{h_i-h_a-h_b}\left[1+(h_i-h_a-h_b)\theta_{12}^+\hat{\mathcal{D}}_{12}^+\right]\Pi_i(\hat{x}_{12},\hat{\theta}_{12}^+)\\ &+\sum_c D^{(2)}_{abc}|x_{12}|^{h_c-h_a-h_b-\frac{1}{2}}\left[|x_{12}|\hat{\mathcal{D}}_{12}^+ + \theta_{12}^+\right]\Psi_c(\hat{x}_{12},\hat{\theta}_{12}^+)+\cdots\,,\end{aligned} \tag{2.11}$$

where we have introduced the variables

$$x_{12}=\frac{1}{2}(x_1-x_2)+\theta_1^+\theta_2^+\,,\quad \hat{x}_{12}=\frac{1}{2}(x_1+x_2)\,,\quad \theta_{12}^+=\frac{1}{2}(\theta_1^+-\theta_2^+)\,,\quad \hat{\theta}_{12}^+=\frac{1}{2}(\theta_1^++\theta_2^+)\,, \tag{2.12}$$

along with their corresponding spinor derivative $\hat{\mathcal{D}}_{12}^+$. From a (boundary) CFT point of view the two lines of (2.11) again represent the even and odd fusion channel, respectively. For later convenience we have also explicitly displayed the first super-descendants, which correspond to the terms proportional to θ_{12}^+, along with the appropriate numerical factors.

2.2 Superactions

With these preparations we can now explain how to formulate manifestly supersymmetric perturbations of an $\mathcal{N}=1$ superconformal field theory in the presence of a boundary.

2.2.1 Bulk deformations

As is well known, we can write a supersymmetric bulk deformation as an integral over the standard superspace (2.1)

$$S_{\text{bulk}}=\lambda\int d^2z\int d\theta\int d\bar{\theta}\;\Phi(z,\bar{z},\theta,\bar{\theta})\,, \tag{2.13}$$

where λ is the coupling constant corresponding to Φ. In general this deformation is however only supersymmetric as long as we consider the theory on manifolds without a boundary. Indeed, as explained in [29–31], the integral in (2.13) is not invariant under generic coordinate transformations if the supermanifold over which the integral is taken has a boundary [32–36].[1] This is just a reformulation of the fact that the deformation (2.13) breaks supersymmetry in the presence of a boundary, as was for example already discovered in [7].

There are several possibilities for how to generalise the Berezin integration to supermanifolds with boundaries. For example, one can formulate the integral as a generalised contour integral [29, 37], or treat the Berezin integral as a differential operator and introduce a special type of differential form to allow integration over arbitrary supermanifolds [38]. Here we shall follow a different approach [30], and define the integral as the integral over the full superspace, restricted to a certain domain. Let $u(x,y,\theta,\bar{\theta})=0$ be

[1] For a simple example of this claim see appendix B.

the defining equation of the boundary, with $u(x,y,\theta,\bar\theta) < 0$ corresponding to the interior. (In particular, the carrier, i.e. the 'bosonic' piece of the superboundary is described by the equation $u(x,y,0,0) = 0$.) The invariant integral measure of the supermanifold with boundary is then

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int d^2\theta \; \vartheta\left(-u(x,y,\theta,\bar\theta)\right) \; . \tag{2.14}$$

Here $\vartheta(x)$ is the analytic continuation of the characteristic function, which is defined via its Grassmann expansion. For a boundary function

$$u(x,y,\theta,\bar\theta) = u_0(x,y) + \theta\bar\theta\, u_1(x,y) \, , \tag{2.15}$$

we have the definition

$$\vartheta\left(-u(x,y,\theta,\bar\theta)\right) := \Theta(-u_0(x,y)) - \theta\bar\theta\, u_1(x,y)\, \delta(u_0(x,y)) \, , \tag{2.16}$$

where Θ and δ on the right hand side are the usual Heaviside step-function and its first derivative (the Dirac delta-function), respectively.

For the case of the upper half-plane we take the boundary function to be

$$u(x,y,\theta,\bar\theta) = -\tilde y = -y + \theta\bar\theta \; . \tag{2.17}$$

The choice $u_1(x,y) = 1$ is motivated by the requirement that the boundary of the supermanifold should have codimension $(1|1)$ as (A.9) demands; for example, had we taken $u_1(x,y) = 0$, the superspace integral would still be invariant, but the boundary would only have co-dimension $(1|0)$. The invariant bulk deformation (2.13) then becomes

$$\begin{aligned} S^{\rm inv}_{\rm bulk} &= \lambda \int d^2 z \int d\theta \int d\bar\theta \, \vartheta\big(y - \theta\bar\theta\big)\, \Phi(z,\bar z,\theta,\bar\theta) \\ &= \lambda \left[\int_{y>0} d^2 z \int d\theta \int d\bar\theta \, \Phi(z,\bar z,\theta,\bar\theta) + \int_{-\infty}^{\infty} dx \int d\theta \int d\bar\theta \, \theta\bar\theta\, \Phi(x,y=0,\theta,\bar\theta) \right] , \end{aligned} \tag{2.18}$$

where we have used (2.16). This result is actually familiar from a field theory point of view, see [7, 8]. Indeed, if we consider the supersymmetry-variation of (2.13) with respect to (A.2), we obtain

$$\delta_{\mathcal N=1} S_{\rm bulk} = -\, 2\lambda \int_{y>0} d^2 z \int d\theta \int d\bar\theta \, \left(\epsilon\theta\partial_z - \bar\epsilon\bar\theta\partial_{\bar z}\right) \Phi(z,\bar z,\theta,\bar\theta) \; . \tag{2.19}$$

Using the basis (A.7) and (A.8), as well as the boundary conditions (A.9), integration by parts then leads to

$$\delta_{\mathcal N=1} S_{\rm bulk} = \frac{\lambda}{2} \int_{-\infty}^{\infty} dx \int d\theta^+ \epsilon_- \Phi(x,y=0,\theta^+,\theta^-=0) \, . \tag{2.20}$$

In general this term is non-vanishing, thus showing that (2.13) by itself is not supersymmetric [7]. In order to restore supersymmetry one therefore has to add to (2.13) a pure boundary term whose supervariation precisely cancels (2.20). This is exactly what the second term in (2.18) achieves.

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2.2.2 Boundary deformations

For the discussion of the coupled bulk-boundary RG equations we shall also need pure boundary perturbations. The (1|1) superspace on the boundary does not have any boundary points itself, and thus the usual superspace measure will be appropriate. The manifestly supersymmetric boundary deformation is thus of the form

$$S_{\rm bdy} = \mu \int dx \int d\theta^{+}\, \Psi(x,\theta^{+})\,, \tag{2.21}$$

where Ψ is a fermionic superfield, and μ is the corresponding coupling constant. Since we have a single fermionic integration in this case, the resulting expression is bosonic, as must be the case for a term that can be added to the action. Because of this reason we cannot write down a similar term involving the bosonic boundary superfield Π. The only bosonic term we could write down would be of the form

$$S'_{\rm bdy} = \gamma \int dx \int d\theta^{+}\theta^{+}\, \Pi(x,\theta^{+})\,, \tag{2.22}$$

but this is not supersymmetric.

3 Renormalisation group equations

Now we can turn to the analysis of the renormalisation group (RG) equations. We shall only consider the manifestly supersymmetric deformation terms from above. In particular, we want to show that the RG equations close among the coupling constants λ_I and μ_a, corresponding to the manifestly supersymmetric deformations (2.18) and (2.21), respectively. We will work to leading order in the bulk couplings, but to next-to-leading order in the boundary couplings.

3.1 The supersymmetric scheme

In the following we shall work in a Wilsonian scheme, which is also sometimes referred to as the 'OPE-scheme' since the coefficients of the leading order terms of the RG-equation are proportional to OPE coefficients [9]. To derive the RG equations we consider the expansion of the free energy $e^{\Delta S}$ with

$$\Delta S = \sum_I \lambda_I \int d^2z \int d^2\theta\, \vartheta\left(y-\theta\bar\theta\right)\Phi_I(z,\bar z,\theta,\bar\theta) + \sum_a \mu_a \int dx \int d\theta^{+}\, \Psi_a(x,\theta^{+}) \tag{3.1}$$

in powers of the coupling constants (λ_I,μ_a). Obviously these integrals are in general divergent, and we need to regularise them, for example by introducing a cut-off ℓ. Since the free energy is a physical quantity it should not depend on the value of this cut-off. This then requires, as we shall see, that the coupling constants λ_I and μ_a are functions of ℓ. However, in order to be able to re-absorb changes in ℓ into a redefinition of the manifestly supersymmetric terms parametrised by (λ_I,μ_a), we need to choose our regulator prescription carefully. In general, the divergencies arise when either (i) two bulk fields come close together; (ii) two boundary fields come close together; or (iii) a bulk field comes close to the boundary. For case (i) and (ii) our regulator will simply cut off the integrals so that two bulk fields or two boundary fields do not come closer than ℓ. As regards the third divergence, the naive prescription would be to prevent the bulk field from getting closer than $\frac{\ell}{2}$ to the boundary. However, this prescription would not preserve supersymmetry since we would run into the same problems as above. We shall therefore use the same idea as there and implement the cut-off by multiplying the bulk integral by $\vartheta\left(\tilde{y}-\frac{\ell}{2}\right)$.

As usual we shall make our coupling constants dimensionless by multiplying them by a suitable power of ℓ, i.e. by writing the perturbation as

$$\Delta S = \sum_I \lambda_I \ell^{\Delta_I - 1} \int d^2 z \int d^2\theta \, \vartheta\left(y - \theta\bar{\theta}\right) \vartheta\left(y - \theta\bar{\theta} - \frac{\ell}{2}\right) \Phi_I(z,\bar{z},\theta,\bar{\theta}) \\ + \sum_a \mu_a \ell^{h_a - \frac{1}{2}} \int dx \int d\theta^+ \, \Psi_a(x,\theta^+) \ . \tag{3.2}$$

If we change the cut-off ℓ by $\ell \mapsto \ell(1+\delta t)$, then from this explicit dependence of the integrals on ℓ we get the usual leading RG terms

$$\dot{\lambda}_I = (1-\Delta_I)\lambda_I + \cdots \ , \qquad \dot{\mu}_a = (\tfrac{1}{2} - h_a)\mu_a + \cdots \ . \tag{3.3}$$

In addition, we also have a contribution from the *implicit* dependence on ℓ via the cut-off prescription. Let us first consider the contribution from the first order bulk deformation. Since the dependence on ℓ comes from the contribution where the bulk field is close to the boundary, we may use the bulk-boundary OPE (2.8) to write

$$\begin{aligned} A_I &= \int d^2 z \int d^2\theta \, \vartheta\left(y - \frac{\ell}{2} - \theta\bar{\theta}\right) \langle \Phi_I(z,\bar{z},\theta,\bar{\theta}) \cdots \rangle \\ &= \int d^2 z \int d^2\theta \, \vartheta\left(y - \frac{\ell}{2} + \frac{1}{2}\theta^+\theta^-\right) \Bigg[\sum_i B^{(1)}_{Ii} (2\tilde{y})^{h_i - \Delta_I} \langle \Pi_i(x,\theta^+) \cdots \rangle \\ &\quad + \sum_a B^{(2)}_{Ia} (2\tilde{y})^{h_a - \Delta_I + \frac{1}{2}} \langle (\mathcal{D}^+\Psi_a)(x,\theta^+) \cdots \rangle \\ &\quad + \theta^- \sum_a B^{(4)}_{Ia} (2\tilde{y})^{h_a - \Delta_I - \frac{1}{2}} \langle \Psi_a(x,\theta^+) \cdots \rangle + \cdots \Bigg] \ . \end{aligned} \tag{3.4}$$

Here we have dropped the $\mathcal{D}^+\Pi_j$ term since it leads, after θ^+ integration, to a total derivative (with respect to x), which we can ignore. The final ellipses describe terms appearing at higher order in y. Next we perform the θ^- integration to obtain

$$\begin{aligned} A_I = &\int d^2 z \int d\theta^+ \theta^+ \sum_i B^{(1)}_{Ii} 2^{h_i - \Delta_I} \left[\Theta\left(y - \frac{\ell}{2}\right)(h_i - \Delta_I) + \delta\left(y - \frac{\ell}{2}\right) y \right] \\ &\qquad\qquad \times y^{h_i - \Delta_I - 1} \langle \Pi_i(x,\theta^+) \cdots \rangle \\ &+ \int d^2 z \int d\theta^+ \theta^+ \sum_a B^{(2)}_{Ia} 2^{h_a - \Delta_I + \frac{1}{2}} \left[\Theta\left(y - \frac{\ell}{2}\right)(h_a - \Delta_I + \frac{1}{2}) + \delta\left(y - \frac{\ell}{2}\right) y \right] \\ &\qquad\qquad \times y^{h_a - \Delta_I - \frac{1}{2}} \langle (\mathcal{D}^+\Psi_a)(x,\theta^+) \cdots \rangle \\ &+ \frac{1}{2} \int d^2 z \int d\theta^+ \Theta\left(y - \frac{\ell}{2}\right) \sum_a B^{(4)}_{Ia} (2y)^{h_a - \Delta_I - \frac{1}{2}} \langle \Psi_a(x,\theta^+) \cdots \rangle + \cdots \ . \end{aligned} \tag{3.5}$$

In a final step, we can perform the y-integration. In the first term, for $h_i \neq \Delta_I$ both contributions from the square bracket cancel each other on the boundary (i.e. for $y \to \ell/2$), leaving just an IR-divergence for $y \to \infty$ which we shall, as usual, ignore. For $h_i = \Delta_I$ on the other hand, the first term of the square bracket is absent, while the second one gives an ℓ-independent contribution which therefore does not contribute to the RG equations. A similar argument also applies to the second term. Finally, in the last line we can simply perform the y-integration directly. The ℓ-dependence of the resulting contribution can then be absorbed into redefining μ_a; more specifically, we obtain in this manner the correction to the second equation of (3.3)

$$\dot{\mu}_a = \left(\frac{1}{2} - h_a\right) \mu_a + \frac{1}{2} \sum_I B^{(4)}_{Ia} \lambda_I + \cdots \ . \tag{3.6}$$

This is the supersymmetric analogue of the bulk-induced source term of [5]. Note that only a source term for the supersymmetric boundary perturbation corresponding to Ψ_a is switched on, but not for the supersymmetry breaking perturbation involving Π_i.

3.2 Higher order contributions

In the following we want to study the quadratic terms in the RG equations for the boundary coupling constant μ_a. These arise from the implicit ℓ-dependence of two types of correlators that we shall discuss in turn.

3.2.1 The boundary two-point function

The implicit ℓ-dependence of the boundary two-point function leads, by the usual computation (see for example [39]), to a further correction of (3.6)

$$\dot{\mu}_a = (\tfrac{1}{2} - h_a)\mu_a + \frac{1}{2} \sum_I B^{(4)}_{Ia} \lambda_I + \sum_{b,c} D^{(2)}_{abc}\, \mu_b \mu_c + \cdots \ . \tag{3.7}$$

For the consistency of our manifestly supersymmetric scheme, it is important that only a correction term corresponding to μ_a is switched on. This is not obvious since, on the face of it, we also get a contribution from the first line of (2.11), giving rise to a source term for the non-supersymmetric coupling corresponding to (2.22). The resulting term is of the form

$$\dot{\gamma}_i \sim \mathcal{D}_{iab}\, \mu_a \mu_b = \frac{1}{2} \big(\mathcal{D}_{iab} + \mathcal{D}_{iba}\big)\, \mu_a \mu_b \, , \tag{3.8}$$

where γ_i is the coupling constant corresponding to Π_i in (2.22), and

$$\begin{aligned} \mathcal{D}_{iab} = \lim_{\ell \to 0} \ell^{h_a + h_b - h_i + 1} \frac{\partial}{\partial \ell} \int dx \int d\theta_1^+ \int d\theta_2^+ \int d\theta_3^+ \theta_3^+ \Theta(|x| - \ell) \\ \times \langle \Psi_a(x, \theta_1^+) \Psi_b(0, \theta_2^+) \Pi_i^*(\infty, \theta_3^+) \rangle \, , \end{aligned} \tag{3.9}$$

with Π_i^* the conjugate field to Π_i. Using (2.11) and performing the θ_1^+- and θ_2^+-integrals, it is straight-forward to check that this correlator is a total derivative in x. Another way to see this is to use methods of conformal field theory. After performing the θ^+ integrals the integrand of $\mathcal{D}_{iab}$ becomes

$$\begin{aligned}\mathcal{I}_{iab} &= \langle \pi_i | (G_{-1/2}\psi_a)(x)\,(G_{-1/2}\psi_b)(0)\rangle \\ &= \langle \pi_i | \Delta_{x,0}(G_{-1/2})\Big[(G_{-1/2}\psi_a)(x)\,\psi_b(0)\Big]\rangle - \langle \pi_i | (G_{-1/2}G_{-1/2}\psi_a)(x)\,\psi_b(0)\rangle \\ &= -\langle \pi_i | (L_{-1}\psi_a)(x)\,\psi_b(0)\rangle = -\frac{d}{dx}\langle \pi_i | \psi_a(x)\,\psi_b(0)\rangle\,, \end{aligned} \tag{3.10}$$

where we have used the same notation as in [4]. The term proportional to $\Delta_{x,0}(G_{-1/2})$ vanishes since π_i is a highest weight state (but we do not need to assume that either ψ_a or ψ_b are highest weight). In the final line we have used that L_{-1} is the derivative operator, thus implying that the integrand is indeed a total derivative. The integral $\mathcal{D}_{iab}$ therefore only gets contributions from $\pm\infty$, as well as from $x = \pm\ell$. The former are IR effects which we can ignore. On the other hand, the contributions from $x = \pm\ell$ cancel between $\mathcal{D}_{iab}$ and $\mathcal{D}_{iba}$, and thus the contribution (3.8) to the RG equation actually vanishes. Thus, at least to this order, no supersymmetry-breaking term is induced.

3.2.2 The bulk boundary correlator

The other interesting contribution comes from the implicit ℓ-dependence of the correlator involving one bulk and one boundary field. In this case, the ℓ-dependence appears only in the $\vartheta\left(\tilde{y} - \frac{\ell}{2}\right)$ term of the bulk integral. In fact, following the same arguments as in (3.5), the only contribution (except for total derivatives, see (2.8)) comes again from the final line of (3.5). Obviously, we have to be careful in evaluating the precise coefficient since it now involves the correlation function with the insertion of an additional boundary field, see [9]. In particular, we need to worry about the divergence as the boundary field that is switched on by the bulk field approaches the boundary field in the correlator. As in the discussion in section 3.2.1 this will contribute to the RG equation for $\dot{\mu}_a$. On the face of it, it will also give rise to a source term for $\dot{\gamma}_i$. However, by a similar reasoning as in (3.10) and (2.8), it is clear that the corresponding integrand is a total derivative. Thus the only interesting contribution (apart from IR effects which we ignore) comes from the term where the bulk induced boundary field is evaluated on either side of the boundary field. However, these two terms cancel since the boundary correlator is local, i.e. independent of the order of the fields. (This is a consequence of the fact that one of the two boundary fields comes from a local bulk field; we also assume that ψ_a does not change the boundary condition, as is usually the case for moduli.) Thus again, there is no source term for the supersymmetry-breaking coupling (2.22), and hence the scheme closes (at least to this order) on the supersymmetry-preserving fields.

The complete RG-equations to this order are then of the form

$$\begin{aligned}\dot{\lambda}_I &= (1-\Delta_I)\lambda_I + \cdots\,, \\ \dot{\mu}_a &= \left(\frac{1}{2} - h_a\right)\mu_a + \frac{1}{2}\sum_I B^{(4)}_{Ia}\lambda_I + \sum_{b,c} D^{(2)}_{abc}\,\mu_b\mu_c + \sum_{I,b}\mathcal{E}_{Iab}\,\lambda_I\mu_b + \cdots\,,\end{aligned} \tag{3.11}$$

where $\mathcal{E}_{Iab}$ is given by the integral (for a similar computation in the purely bosonic case see [9])

$$\begin{aligned}\mathcal{E}_{Iab} = & -\lim_{\ell\to 0} \ell^{\Delta_I+h_b-h_a} \int d^2z \int d^2\theta \int d\theta_1^+ \int d\theta_2^+ \frac{\partial}{\partial\ell}\vartheta\left(y-\frac{\ell}{2}-\theta\bar\theta\right) \\ & \times \langle \Phi_I(z,\bar z,\theta,\bar\theta)\Psi_b(0,\theta_1^+)\Psi_a^*(\infty,\theta_2^+)\rangle \\ & -\frac{1}{2}\lim_{\ell\to 0}\sum_c \ell^{h_b+h_c-h_a+\frac{1}{2}} B^{(4)}_{Ic}\int dx\int d\theta_1^+\int d\theta_2^+\int d\theta_3^+ \langle\Psi_c(x,\theta_1^+)\Psi_b(0,\theta_2^+)\Psi_a^*(\infty,\theta_3^+)\rangle.\end{aligned} \tag{3.12}$$

Here the last line stems from lower order counter-terms and simply subtracts the poles of the first term that would lead to divergencies after integration. To evaluate the first line one uses

$$\frac{\partial}{\partial\ell}\vartheta\left(y-\frac{\ell}{2}-\theta\bar\theta\right) = -\frac{1}{2}\delta\left(y-\frac{l}{2}\right)+\frac{1}{2}\theta\bar\theta\,\delta'\left(y-\frac{l}{2}\right)\ . \tag{3.13}$$

4 Applications to the $\mathcal{N}=2$ case

Next we want to apply these general methods to study the behaviour of D-branes in string theory. As we have mentioned before, in the context of string theory it is important to preserve the $\mathcal{N}=1$ supersymmetry since it is a gauge symmetry. Our manifestly $\mathcal{N}=1$ supersymmetric scheme is therefore the appropriate language for this problem. In particular, we can use it to re-visit the RG analysis of [4] and study how the results of that paper relate to the matrix factorisation analysis of [10].

4.1 Obstructions and RG flows from (cc) perturbations

In the following we shall study B-type boundary conditions under (cc) and (ca) bulk perturbations; because of mirror symmetry this then also covers the case of A-type branes. As was explained for example in [4], the perturbation of a B-type brane by a (ca) deformation is never obstructed, while obstructions can arise in the (cc) case. We shall therefore concentrate on the (cc) case in the following and come back to the (ca) case below (see subsection 4.2). Using our manifestly supersymmetric scheme, a (cc) perturbation takes the form

$$S^{\text{chiral}}_{\text{bulk}} = \sum_I \lambda_I \int d^2z \int d\theta^{(+)} \int d\theta^{(-)} \vartheta\left(y-\theta^{(+)}\theta^{(-)}\right)\Phi_I^{(cc)}(z,\bar z,\theta^{(+)},\theta^{(-)})\ . \tag{4.1}$$

Here we have used similar conventions as in [19, 20] (see also appendix A.2), and $\Phi^{(cc)}$ is a chiral superfield characterised by the following analyticity properties

$$\bar{\mathcal{D}}_{(+)}\Phi_I^{(cc)}(z,\bar z,\theta^{(+)},\theta^{(-)}) = \bar{\mathcal{D}}_{(-)}\Phi_I^{(cc)}(z,\bar z,\theta^{(+)},\theta^{(-)}) = 0\,, \tag{4.2}$$

with the spinor derivatives $\bar{\mathcal{D}}_{(\pm)}$ given in (A.15).

The deformation (4.1) is manifestly $\mathcal{N}=1$ supersymmetric since the correction term that was introduced by hand in [4] (see eq. (2.13) of that paper) is now automatically included. We are therefore in the framework of the previous section, and thus the RG

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boundary fields	h	q
$\pi = (2,-2,0)^{\otimes 5}$	1	2
$\psi_1 = \{(2,-2,0)^{\otimes 4} \otimes (3,3,0)\}$	11/10	1
$\psi_2 = \{(2,-2,0)^{\otimes 3} \otimes (3,3,2) \otimes (3,3,0)\}$	17/10	1
$\psi_3 = \{(2,-2,0)^{\otimes 2} \otimes (3,3,2)^{\otimes 2} \otimes (3,3,0)\}$	23/10	1
$\psi_4 = \{(2,-2,0) \otimes (3,3,2)^{\otimes 3} \otimes (3,3,0)\}$	29/10	1
$\psi_5 = \{(3,3,2)^{\otimes 4} \otimes (3,3,0)\}$	7/2	1

Table 1. Boundary fields switched on by the (cc) bulk modulus ϕ of (4.3).

equations (3.11) apply. To lowest order, the qualitative behaviour of the RG flow depends then simply on whether $B^{(4)}_{Ia}$ is non-zero for a marginal field Ψ_a.

Actually, it is clear on general grounds that $B^{(4)}_{Ia} \neq 0$ only for irrelevant Ψ_a. To see this we observe that the coefficient $B^{(4)}_{Ia}$ describes the bulk-boundary OPE of the bulk field Φ_I with a G-descendant of the boundary field ψ_a. In the (cc) case, the bulk field Φ has U(1)-charges $q = \bar{q} = 1$, and thus the U(1)-charge of the boundary field in question must at least be $q = 1$. But then its conformal dimension satisfies $h \geq \frac{1}{2}$, and the case $h = \frac{1}{2}$ is excluded since the $G^{+}_{-1/2}$ descendant is then a null-vector. Thus no RG flow is directly switched on, as was already observed in [4].

On the other hand, the matrix factorisation analysis of [10] suggests that the (cc) bulk deformation triggers an RG flow on the boundary. As we have just seen, to leading order no RG flow is switched on. However, higher order terms may also lead to an RG flow. In particular, the $\mathcal{E}\lambda\mu$ term describes the change of conformal dimension of the boundary field corresponding to μ as a consequence of the bulk deformation [9]. If this term is positive for a marginal boundary field Ψ_a, the field Ψ_a becomes relevant and thus triggers an instability of the boundary condition.

In order to see whether this does indeed happen, let us study an explicit example. We consider the quintic at the Gepner point (for our notation and some useful relations see appendix C) with the tensor product boundary condition corresponding to $L_i = 1$ (see [40, 41]). To be specific, let us analyse the (cc) perturbation corresponding to the bulk field

$$\phi = (1,-1,0)^{\otimes 5} \otimes \overline{(1,-1,0)^{\otimes 5}}\,, \tag{4.3}$$

where we use the same conventions as in [42]. As the bulk field is brought to the boundary it can switch on the boundary fields given in table 1 — in the conventions of section 2, these are the lowest components of the superfields appearing in (2.8). Here curly brackets denote all possible permutations of the five factors, and we have used that

$$(1,-1,0) \otimes (1,-1,0) = (3,3,2) \oplus (2,-2,0)\ . \tag{4.4}$$

Note that $(2,-2,0)$ is a primary field, while $(3,3,2)$ is a G-descendant of $(3,3,0)$. The field π is bosonic, and is in fact precisely the boundary field that is present in the manifestly $\mathcal{N} = 1$ supersymmetric formulation, see (2.18). The other fields ψ_a, $a = 1,\ldots,5$, are the lowest components of fermionic superfields Ψ_a. As explained above on general grounds, they have indeed $h > 1/2$ and thus lead to irrelevant perturbations.

The matrix factorisation analysis of [10] suggests that the boundary modulus corresponding to π should be switched on by the bulk perturbation. The corresponding modulus field is obtained by applying a full unit of spectral flow $\mathcal{S}$ to the field π

$$\hat{\psi} = \mathcal{S}\pi = (1,1,0)^{\otimes 5} \; . \tag{4.5}$$

The field $\hat{\psi}$ has indeed $h = \frac{1}{2}$ and $q = -1$, and it is the lowest component of a fermionic boundary superfield $\widehat{\Psi}$. Based on the matrix factorisation analysis we would therefore expect that this field becomes tachyonic as a consequence of the bulk perturbation. To see whether this is the case we need to study the correlator

$$\mathcal{E} \sim \langle (G^-_{-1/2}\bar{G}^-_{-1/2}\phi)\,(G^+_{-1/2}\hat{\psi})\,(G_{-1/2}\psi_b)^*\rangle \,, \tag{4.6}$$

where ψ_b is a marginal boundary field. Actually, as explained just before (3.5), only the channel where the bulk field switches on a G-descendant of one of the ψ_a boundary fields contributes to $\mathcal{E}$. Thus the relevant correlator is

$$\mathcal{E} \sim \langle (G^-_{-1/2}\psi_a)\,(G^+_{-1/2}\hat{\psi})\,(G_{-1/2}\psi_b)^*\rangle \; . \tag{4.7}$$

Using (C.5) one can indeed show that the correlator is only non-zero if $\psi_b = \hat{\psi}^*$, leading to the RG equation

$$\dot{\mu}^* = \mathcal{E}\,\lambda\,\mu \,, \tag{4.8}$$

where μ^* is the coupling constant for $\hat{\Psi}^*$, while λ corresponds to the bulk deformation (4.3). Since the bulk perturbation must be real, it must also involve Φ^*, and this leads to the RG term

$$\dot{\mu} = \mathcal{E}\,\lambda\,\mu^* \,, \tag{4.9}$$

where μ is the coupling constant for $\hat{\Psi}$. Taking these two equations together it is then clear that the conformal dimension of the fields $\hat{\psi}_\pm = \hat{\psi} \pm \hat{\psi}^*$ is shifted by $\pm\mathcal{E}\lambda$. Irrespective of the sign of $\mathcal{E}$, one of the two fields therefore becomes relevant and thus triggers an instability. At least qualitatively, the corresponding flow should be the flow predicted in [10] from the matrix factorisation point of view. A detailed comparison is, however, difficult because it is not clear how to identify the RG scheme from the matrix factorisation analysis.

We have also checked this conclusion for other perturbations of other tensor product branes, and the situation is always exactly as above. On the other hand, the analysis is different for the permutation brane case of [10] since at the permutation point the effective superpotential has a zero of higher order. In terms of the above RG analysis this translates to the statement that the $\mathcal{E}$ coefficient is zero, and that only a higher order correlator, involving a larger number of boundary moduli, is non-zero. This therefore agrees again nicely with the expectations from [10].

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4.2 Manifestly $\mathcal{N}=2$ description of (ca) perturbations

For the case of a (ca) deformation, not only the $\mathcal{N}=1$ supersymmetry can always be preserved (by adding suitable boundary terms to the action), but also the $\mathcal{N}=2$ symmetry [4]. This suggests that the (ca) case should allow for a manifestly $\mathcal{N}=2$ formulation. Using similar conventions as in [19, 20] (see also appendix A.2) the (ca) deformation can indeed be written as

$$S_{\rm inv}^{\rm twist} = \sum_I \lambda_I \int d^2z \int d\theta^{(+)} \int d\bar{\theta}^{(-)} \vartheta\left(y - \theta^{(+)}\bar{\theta}^{(-)}\right) \Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) \, . \tag{4.10}$$

Here $\Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)})$ is a twisted chiral superfield which satisfies the following analyticity properties

$$\bar{\mathcal{D}}_{(+)}\Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) = \mathcal{D}_{(-)}\Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) = 0 \, . \tag{4.11}$$

Here $\bar{\mathcal{D}}_{(+)}$ and $\mathcal{D}_{(-)}$ are two of the $\mathcal{N}=2$ spinor derivatives which are defined in (A.15).

The deformation (4.10) preserves the full $\mathcal{N}=2$ supersymmetry since the supervariation (A.17) leads to

$$\begin{aligned} &\delta_{\mathcal{N}=2} S_{\rm inv}^{\rm twist} \qquad (4.12)\\ &= 2i \sum_I \lambda_I \int d^2z \int d\theta^{(+)} \int d\bar{\theta}^{(-)} \left[\epsilon_{(+)}\bar{\theta}^{(-)}\partial_- - \bar{\epsilon}_{(-)}\theta^{(+)}\partial_+\right] \Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) \\ &\quad + 2i \sum_I \lambda_I \int d^2z \int d\theta^{(+)} \int d\bar{\theta}^{(-)} \delta(y) \left[\epsilon_{(+)}\bar{\theta}^{(-)} + \theta^{(+)}\bar{\epsilon}_{(-)}\right] \Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) \, , \end{aligned}$$

which vanishes upon partial integration.

One may wonder how lines of marginal stability appear in this formulation. Expanding out the ϑ function in (4.10), we can write $S_{\rm inv}^{\rm twist}$ as

$$S_{\rm inv}^{\rm twist} = \sum_I \lambda_I \int d^2z \int d\theta^{(+)} \int d\bar{\theta}^{(-)} \left[\Theta(y) - \theta^{(+)}\bar{\theta}^{(-)}\delta(y)\right] \Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) \, .$$

The second term in the bracket describes a boundary term that needs to be switched on in order to preserve the $\mathcal{N}=2$ supersymmetry. This boundary term can be written as

$$-\sum_{i,I} B_{Ii}\lambda_I \int d^2z \int d\theta^{(+)} \int d\bar{\theta}^{(-)}\theta^{(+)}\bar{\theta}^{(-)}\delta(y)(2y)^{h_i-\Delta_I}\Pi_i^{(ca)}(x,\theta^{(+)},\bar{\theta}^{(-)}) + \cdots \, , \tag{4.13}$$

where we have used the bulk boundary OPE which in the $\mathcal{N}=2$ context takes the form

$$\Phi_I^{(ca)}(z,\bar{z},\theta^{(+)},\bar{\theta}^{(-)}) = \sum_i B_{Ii}(2\tilde{y})^{h_i-\Delta_I}\Pi_i^{(ca)}(x,\theta^{(+)},\bar{\theta}^{(-)}) \, . \tag{4.14}$$

Here $\Pi^{(ca)}$ is a bosonic boundary multiplet, which depends on both $\theta^{(+)}$ and $\bar{\theta}^{(-)}$. It is clear from (4.13) that the additional boundary term is only well-defined for $h_i \geq \Delta_I$ but is divergent otherwise. In particular, if we consider a deformation by a bulk modulus

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($\Delta_I = 1$) the boundary correction is only well-defined if $h_i \geq 1$, i.e. if the boundary fields switched on by Φ are marginal or irrelevant.

Starting from a point in moduli space where all $h_i \geq 1$, we reach a line of marginal stability as one of them becomes relevant [4]. At this point the manifestly supersymmetric scheme ceases to be well-defined and becomes rather formal. Thus the above manifestly $\mathcal{N} = 2$ supersymmetric description is not in conflict with the existence of lines of marginal stability.

5 Conclusions

In this paper we have shown that there exists a manifestly $\mathcal{N} = 1$ supersymmetric RG scheme (at least up to next-to-leading order) for the coupled problem of $\mathcal{N} = 1$ preserving bulk and boundary perturbations. Since the $\mathcal{N} = 1$ superconformal symmetry is a gauge symmetry of superstring theory, this is the appropriate scheme in this context. We have applied our results to the study of B-type branes under (cc) deformations. In particular, we have shown that the bulk-induced source for relevant and marginal boundary perturbations always vanishes in this case, even if the brane is obstructed. The obstruction manifests itself rather in that a boundary modulus becomes tachyonic, thus triggering an RG-flow in the corresponding direction in moduli space. We have also seen that our results agree, at least qualitatively, with the predictions of [10]. A quantitative comparison is problematic since it is not clear how to identify the RG scheme from the matrix factorisation analysis.

For (ca) perturbations of B-type branes, on the other hand, no obstructions are believed to appear (see for example [4]). This is reflected in the fact that a manifestly $\mathcal{N} = 2$ supersymmetric RG scheme exists in this case (see section 4.2). Lines of marginal stability manifest themselves from this point of view as a breakdown of this scheme. It would be interesting to study this more explicitly in examples, and see whether this perspective can shed any light on the wall-crossing formulae of $\mathcal{N} = 2$ theories, see for example [2, 3].

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A Superspace conventions

In this appendix we outline our conventions for the $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 2)$ superspace which we will use throughout this work.

A.1 $\mathcal{N} = (1, 1)$ conventions

Let us begin by describing the standard $\mathcal{N} = (1, 1)$ superspace, which is spanned by the coordinates of (2.1). We will first discuss our notation for the bulk, and then introduce a boundary along the line $z = \bar{z}$.

A.1.1 Bulk superspace

The supercharges corresponding to the coordinates given in (2.1) read

$$\mathcal{Q} = \frac{\partial}{\partial\theta} - \theta\partial_z\,, \qquad \text{and} \qquad \bar{\mathcal{Q}} = \frac{\partial}{\partial\bar\theta} - \bar\theta\partial_{\bar z}\,, \tag{A.1}$$

which are combined with constant spinors ϵ and $\bar\epsilon$ to give the following supervariation

$$\delta_{\mathcal{N}=1} = \epsilon\mathcal{Q} - \bar\epsilon\bar{\mathcal{Q}}\ . \tag{A.2}$$

For later convenience we also introduce the covariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\partial_z\,, \qquad \text{and} \qquad \bar{\mathcal{D}} = \frac{\partial}{\partial\bar\theta} + \bar\theta\partial_{\bar z}\,, \tag{A.3}$$

which satisfy the important relation

$$\mathcal{D}^2 = \partial_z\,, \qquad \text{and} \qquad \bar{\mathcal{D}}^2 = \partial_{\bar z}\ . \tag{A.4}$$

Moreover, in view of dealing with superconformal field theories, we mention that the conformal dimensions of the bosonic and fermionic variables are

$$h_z = h_{\bar z} = -1\,, \qquad \text{and} \qquad h_\theta = h_{\bar\theta} = -\frac{1}{2}\,, \tag{A.5}$$

which, in particular, implies that the conformal dimension of the integral measure of the superspace (2.1) is

$$h_{\int dz \int d\bar z \int d\theta \int d\bar\theta} = -1\ . \tag{A.6}$$

A.1.2 Boundary superspace

Next we introduce a boundary along the line $z = \bar z$. For most of the computations in the main body of this work it is much more convenient to switch to a real basis for the bosonic coordinates. More precisely we introduce

$$z = x + iy\,, \qquad \text{and} \qquad \theta = \frac{1}{2}(\theta^+ + \theta^-)\,, \qquad \text{and} \qquad \epsilon = \frac{1}{2}(\epsilon_+ + \epsilon_-)\,, \tag{A.7}$$

$$\bar z = x - iy\,, \qquad \text{and} \qquad \bar\theta = \frac{1}{2}(\theta^+ - \theta^-)\,, \qquad \text{and} \qquad \bar\epsilon = \frac{1}{2}(\epsilon_+ - \epsilon_-)\ . \tag{A.8}$$

In this basis the boundary is given by the line $y = 0$. At this locus only the sum of the bulk supercharges (A.1) will remain unbroken[2], which entails for the Grassmann variables the following trivial boundary condition

$$\theta = \bar\theta\big|_{y=0}\,, \quad \text{i.e.} \quad \theta^- = 0\big|_{y=0}\,, \qquad \text{and} \qquad \epsilon = -\bar\epsilon\big|_{y=0}\,, \quad \text{i.e.} \quad \epsilon_+ = 0\big|_{y=0}\ . \tag{A.9}$$

Thus we can view the boundary as a one-dimensional superspace spanned by the variables

$$\mathbb{R}^{(1|1)} = \{x, \theta^+\}\ . \tag{A.10}$$

For completeness, we also introduce the corresponding supercharge and spinor derivative

$$\mathcal{Q}^+ = \frac{\partial}{\partial\theta^+} - \theta^+\frac{\partial}{\partial x}\,, \qquad \text{and} \qquad \mathcal{D}^+ = \frac{\partial}{\partial\theta^+} + \theta^+\frac{\partial}{\partial x}\ . \tag{A.11}$$

[2] Strictly speaking the most generic boundary condition would be $\mathcal{Q} - e^{2\pi i\eta}\,\bar{\mathcal{Q}}\big|_{y=0} = 0$ for an arbitrary phase η. Since this phase, however, will not play any role in our computations we simply choose $\eta = 0$.

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A.2 $\mathcal{N} = (2,2)$ conventions

Similar to the $\mathcal{N} = (1,1)$ superspace of the previous section we will now also outline our notations for the $\mathcal{N} = (2,2)$ standard superspace, which will be relevant for section 4.2. We begin again with the bulk superspace, and then introduce a boundary at $z = \bar{z}$.

A.2.1 Bulk superspace

We shall use standard $\mathcal{N} = (2,2)$ superspace, which we will split in two light-cone sectors as

$$\mathbb{R}^{(2|2,2)} = \mathbb{R}_L^{(1|2)} \times \mathbb{R}_R^{(1|2)} = \{x_+, \theta^{(+)}, \bar{\theta}^{(+)}\} \times \{x_-, \theta^{(-)}, \bar{\theta}^{(-)}\} \ . \tag{A.12}$$

Translations in the Grassmann directions (i.e. $\mathcal{N} = 2$ supersymmetry transformations) are generated by the supercharges

$$\mathcal{Q}_{(\pm)} = \frac{\partial}{\partial\theta^{(\pm)}} + i\bar{\theta}^{(\pm)}\partial_\pm \,, \qquad \bar{\mathcal{Q}}_{(\pm)} = -\frac{\partial}{\partial\bar{\theta}^{(\pm)}} - i\theta^{(\pm)}\partial_\pm \,, \tag{A.13}$$

where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. The only non-vanishing anti-commutators of these generators are

$$\{\mathcal{Q}_{(\pm)}, \bar{\mathcal{Q}}_{(\pm)}\} = -2i\partial_\pm \ . \tag{A.14}$$

For completeness, we also introduce the corresponding spinor derivatives, which take the form

$$\mathcal{D}_{(\pm)} = \frac{\partial}{\partial\theta^{(\pm)}} - i\bar{\theta}^{(\pm)}\partial_\pm \,, \qquad \bar{\mathcal{D}}_{(\pm)} = -\frac{\partial}{\partial\bar{\theta}^{(\pm)}} + i\theta^{(\pm)}\partial_\pm \,, \tag{A.15}$$

and which anti-commute with all the $\mathcal{Q}_{(\pm)}$ and $\bar{\mathcal{Q}}_{(\pm)}$, and have only the following non-trivial anti-commutator relations

$$\{\mathcal{D}_{(\pm)}, \bar{\mathcal{D}}_{(\pm)}\} = 2i\partial_\pm \ . \tag{A.16}$$

Introducing the constant spinors $\epsilon_{(\pm)}$ and $\bar{\epsilon}_{(\pm)}$ we can parametrise the supervariation as

$$\delta_{\mathcal{N}=2} = \epsilon_{(+)}\mathcal{Q}_{(-)} - \epsilon_{(-)}\mathcal{Q}_{(+)} - \bar{\epsilon}_{(+)}\bar{\mathcal{Q}}_{(-)} + \bar{\epsilon}_{(-)}\bar{\mathcal{Q}}_{(+)} \,. \tag{A.17}$$

A.2.2 Boundary superspace

We will now introduce a boundary in the bosonic coordinates. In the basis of (A.12) we choose the line $x_+ = x_-$. Just as in the $\mathcal{N} = (1,1)$ case, only half of the four supercharges (A.13) are preserved along this line. In fact, ignoring irrelevant phase factors, there are two distinct boundary conditions

$$\text{A-type:} \qquad \epsilon \equiv \epsilon_{(+)} = \bar{\epsilon}_{(-)} \,, \qquad \text{and} \qquad \bar{\epsilon} = \bar{\epsilon}_{(+)} = \epsilon_{(-)} \,, \tag{A.18}$$

$$\text{B-type:} \qquad \epsilon \equiv \epsilon_{(+)} = -\epsilon_{(-)} \,, \qquad \text{and} \qquad \bar{\epsilon} = \bar{\epsilon}_{(+)} = -\bar{\epsilon}_{(-)} \ . \tag{A.19}$$

Throughout this work we will just consider B-type boundary conditions; this is not a restriction since we may use mirror symmetry to obtain the corresponding statements for A-type boundary conditions.

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B Generic superspace coordinate transformations

In this appendix we illustrate, with a simple example, the fact that the usual Berezin integration fails to be invariant in the presence of a boundary of the carrier manifold. Let us consider an arbitrary function $F(x,y,\theta,\bar{\theta})$ which lives on the superspace (2.1) and which has the following Grassmann expansion

$$F(x,y,\theta,\bar{\theta}) = F_{(0,0)}(x,y) + \theta F_{(1,0)}(x,y) + \bar{\theta} F_{(0,1)}(x,y) + \theta\bar{\theta} F_{(1,1)}(x,y) \ . \tag{B.1}$$

Let us consider an integral of this function over the supermanifold (2.1), where we have a boundary at the line $y = 0$. A typical integral using the naive integral prescription for compact supermanifolds is for example given by

$$\mathcal{I}^{\text{Ber}} = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \int d\bar{\theta} \int d\theta \, F(x,y,\theta,\bar{\theta}) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \, F_{(1,1)}(x,y) \ . \tag{B.2}$$

Now suppose we make the coordinate transformation

$$y \mapsto \tilde{y} = y - \theta\bar{\theta} \tag{B.3}$$

before the Grassmann integration. By the naive Berezin rules we would find

$$\begin{aligned} \mathcal{I}^{\text{Ber}} &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\tilde{y} \int d\bar{\theta} \int d\theta \, F(x,\tilde{y}+\theta\bar{\theta},\theta,\bar{\theta}) \\ &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\tilde{y} \left(F_{(1,1)}(x,\tilde{y}) + \partial_{\tilde{y}} F_{(0,0)}(x,\tilde{y}) \right) \\ &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\tilde{y} \, F_{(1,1)}(x,\tilde{y}) - \int_{-\infty}^{\infty} dx F_{(0,0)}(x,0) \, , \end{aligned} \tag{B.4}$$

which differs from (B.2) by an additional integral over the boundary of the carrier (i.e. an integral along the line $y = 0$). The reason for this discrepancy is that the Berezin transformation rules have only instructed us to transform the integrand, but they fail to also adapt the boundaries of the bosonic y integration.

Let us therefore consider a manifestly invariant integral. In order to make contact with section 2.2 we choose an integral with a boundary function $u(x,y,\theta,\bar{\theta}) = -y + \theta\bar{\theta}$

$$\begin{aligned} \mathcal{I}^{\text{inv}} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int d\bar{\theta} \int d\theta \, \vartheta(y - \theta\bar{\theta}) F(x,y,\theta,\bar{\theta}) \\ &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \, F_{(1,1)}(x,y) - \int_{-\infty}^{\infty} dx \, F_{(0,0)}(x,0) \, , \end{aligned} \tag{B.5}$$

where we have used the expansion (2.16). Now let us again study this integral after the coordinate transformation (B.3)

$$\mathcal{I}^{\text{inv}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tilde{y} \int d\bar{\theta} \int d\theta \, \Theta(\tilde{y}) \, F(x,\tilde{y}+\theta\bar{\theta},\theta,\bar{\theta}) \ . \tag{B.6}$$

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As we can see, there is no need to change the integration range of the $\tilde{y}$ variable, since it is anyway unbounded. Calculating the Grassmann integrals in (B.6) we then obtain

$$\begin{aligned}\mathcal{I}^{\text{inv}} &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tilde{y} \int d\bar{\theta} \int d\theta\, \Theta(\tilde{y}) \left(F(x,\tilde{y},\theta,\bar{\theta}) + \theta\bar{\theta}\, \partial_{\tilde{y}} F(x,\tilde{y},\theta,\bar{\theta})\right) \\ &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\tilde{y}\, F_{(1,1)}(x,\tilde{y}) - \int_{-\infty}^{\infty} dx\, F_{(0,0)}(x,0)\,, \end{aligned} \tag{B.7}$$

which indeed agrees with (B.5). With this modified prescription the superspace integral is thus invariant under generic coordinate transformations.

C Gepner point description

In view of the particular example studied in section 4.1 we compile here some basic notations and useful relations. At the Gepner point, the quintic Calabi-Yau can be described as a $\mathbb{Z}_5$ orbifold of a five-fold product of $\mathcal{N}=2$ minimal models, each with $k=3$. The central charge of a single minimal model at level k is given by

$$c = \frac{3k}{k+2}\,, \tag{C.1}$$

and thus the total charge of five copies with $k=3$ gives $c_{\text{tot}}=9$, as is appropriate for a Calabi-Yau manifold.

The representations $\mathcal{H}_{(l,m,s)}$ of a single minimal model are labelled by triples of integers (l,m,s), where $l=0,1,\ldots,k$, while m and s are defined modulo $2k+4$ and 4, respectively. All three labels have to sum up to an even integer

$$l+m+s = 0 \bmod 2\,, \tag{C.2}$$

and we have the *field identification*

$$(l,m,s) \sim (k-l, m+k+2, s+2)\;. \tag{C.3}$$

States with s even belong to the Neveu-Schwarz (NS)-sector, while states with s odd live in the Ramond (R)-sector. If $|m-s|\leq l$ and $s\in\{-1,0,1,2\}$ the conformal weight and U(1) charge of the ground state is given by

$$h(l,m,s) = \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8}\,, \qquad \text{and} \qquad q(l,m,s) = \frac{s}{2} - \frac{m}{k+2}\;. \tag{C.4}$$

Finally, the fusion rules are simply described by

$$(l_1,m_1,s_1)\otimes(l_2,m_2,s_2) = \sum_{l=|l_1-l_2|}^{\min(l_1+l_2,2k-l_1-l_2)} (l, m_1+m_2, s_1+s_2)\,, \tag{C.5}$$

where the sum over l is over every second l.

The full state-space at the Gepner point is then spanned by

$$\bigotimes_{i=1}^{5} \mathcal{H}_{(l_i,m_i+n,s_i)} \otimes \bar{\mathcal{H}}_{(l_i,m_i-n,\bar{s}_i)}\,, \tag{C.6}$$

where $\mathcal{H}_{(l_i,m_i,s_i)}$ denotes the (l_i, m_i, s_i) representation in the i-th minimal model, and $n = 0, 1, \ldots, 4$ describes the twist sectors of the $\mathbb{Z}_5$ orbifold. s_i and $\bar{s}_i$ are either all odd (R-sector) or all even (NS-sector). Finally, the (cc) fields take the general form

$$\Phi^{(cc)}_{l_1,l_2,l_3,l_4,l_5} = \prod_{i=1}^{5} (l_i, -l_i, 0) \otimes \overline{(l_i, -l_i, 0)}\,, \tag{C.7}$$

where $l_i \leq k_i = 3$. It follows from (C.4) that these states indeed have $h = \frac{q}{2}$ and $\bar{h} = \frac{\bar{q}}{2}$.

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