

# Buy-and-Hold Property for Fully Incomplete Markets when Super-replicating Markovian Claims

**Working Paper**

**Author(s):**

Neufeld, Ariel

**Publication date:**

2017-07-04

**Permanent link:**

<https://doi.org/10.3929/ethz-b-000219155>

**Rights / license:**

In Copyright - Non-Commercial Use Permitted

**Originally published in:**

arXiv

# Buy-and-Hold Property for Fully Incomplete Markets when Super-replicating Markovian Claims

Ariel Neufeld\*

July 6, 2017

## Abstract

We show that when the price process  $S$  represents a fully incomplete market, the optimal super-replication of any Markovian claim  $g(S_T)$  with  $g(\cdot)$  being nonnegative and lower semicontinuous is of buy-and-hold type. As both (unbounded) stochastic volatility models and rough volatility models are examples of fully incomplete markets, one can interpret the buy-and-hold property when super-replicating Markovian claims as a natural phenomenon in incomplete markets.

*Keywords* Super-replication; Fully incomplete markets; Stochastic volatility

*AMS 2010 Subject Classification* 91G10; 91G20

## 1 Introduction

Fully incomplete markets were introduced in [5]. Roughly speaking, a financial market is fully incomplete if for any volatility process  $\alpha$  one can find an equivalent local martingale measure  $\mathbb{Q}$  under which  $\alpha$  is close to the volatility process  $\nu$  of the price process  $S$ . It turns out that it is a natural appearance for incomplete markets to be fully incomplete. Indeed, it was shown in [5] that stochastic volatility models (with unbounded volatility) like the Heston model [11], the Hull–White model [14] and the Scott model [15], as well as rough volatility models like the one in [9] where the log-volatility is a fractional Ornstein-Uhlenbeck process are all examples of fully incomplete markets.

The key property of fully incomplete markets, which is the main result in [5], is the following. When a financial agent is allowed to invest in the risky asset  $S$  and statically in up to finite many liquid options, the classical super-replication price (defined with respect to the given initial law  $\mathbb{P}$  of the price process) of a (possibly path-dependent) European option  $G(S)$  with  $G : C[0, T] \rightarrow \mathbb{R}_+$  being uniformly continuous and bounded coincides with the robust super-replication price, where the super-hedging property must hold for any path. This follows from the result proven in [5] that for fully incomplete markets, the set of all equivalent local martingale measures are weakly dense in the set of all local martingale

---

\*Department of Mathematics, ETH Zurich, [ariel.neufeld@math.ethz.ch](mailto:ariel.neufeld@math.ethz.ch). Financial support by the Swiss National Foundation grant SNF 200020\_172815 is gratefully acknowledged.

measures defined on the continuous path space. For more papers related to robust pricing, in particular to duality results, we refer to [1, 2, 3, 6, 7, 10, 12, 13] to name but a few.

The goal of this (short) paper is to further investigate the super-replication property in fully incomplete markets in the special case where one is only allowed to trade in the risky asset (i.e. no liquid options available), and the contingent claim is of Markovian type, i.e. of the form  $g(S_T)$  where  $S_T$  denotes the value of the stock at maturity. The main result of this paper states that in that case, even for unbounded and non-continuous payoff functions  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , the classical super-replication price of  $g(S_T)$  coincides with the (even more robust) buy-and-hold super-replication price where one can only buy (or sell) stocks at the initial time and keeps his position till maturity to super-replicate the sold option. To prove our result, we apply techniques which were developed in [5].

Our paper is motivated by the result in [4] stating that for stochastic volatility models with unbounded volatility, the classical super-replication price of a Markovian claim  $g(S_T)$  coincides with the buy-and-hold super-replication price, even for unbounded and non-continuous payoff functions  $g$ . A similar result was obtained in [8] for European Call options only, but in more general stochastic volatility models including e.g. the Heston model, which is not covered in [4] as there strong regularity conditions were imposed on the coefficients of the SDE for the price process which is not fulfilled by square root models.

Summing up, our contribution is twofold. First, our result enlarges the class of price processes from stochastic volatility models to the richer class of fully incomplete markets for which the buy-and-hold property holds when super-replicating  $g(S_T)$ . Second, for stochastic volatility models, our main result generalizes [4, 8] in the sense that we recover the buy-and-hold property when super-replicating  $g(S_T)$  for (even more) general stochastic volatility models than in [8] for unbounded non-continuous payoff functions  $g$  as in [4].

The remainder of this paper is organized as follows. In Section 2, we introduce the setup and state the main theorem of this paper. Then, we provide the proof of the main result in Section 3.

## 2 Setup and Main Result

Let  $T$  be a finite time horizon and  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a filtered probability space, where  $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$  satisfies the usual conditions. Consider a financial market which consists of one constant bank account  $B_t \equiv 1$  for all  $t \in [0, T]$  and one risky asset with price process

$$dS_t = S_t \nu_t dW_t, \quad S_0 \equiv s_0 > 0, \quad (2.1)$$

where  $\nu = \{\nu_t\}_{t=0}^T$  is an  $\mathbb{F}$ -progressively measurable process with given initial point  $\nu_0 > 0$  satisfying  $\int_0^T \nu_s^2 ds < \infty$   $\mathbb{P}$ -a.s., and  $W = \{W_t\}_{t=0}^T$  denotes a  $\mathbb{P}$ - $\mathbb{F}$ -Brownian motion.

The definition for the financial market (2.1) to be fully incomplete was introduced in [5, Definition 2.1]. Let  $\mathcal{C}(\nu_0)$  be the set of all continuous, strictly positive stochastic processes  $\alpha \equiv \{\alpha_t\}_{t=0}^T$  which are adapted with respect to the filtration  $\mathbb{F}^W$  generated by  $W$  completed by the null sets, and satisfy both that  $\alpha_0 = \nu_0$ , and that  $\alpha$  and  $\frac{1}{\alpha}$  are uniformly bounded.

**Definition 2.1.** The financial market introduced in (2.1) is called fully incomplete if for any  $\epsilon > 0$  and any process  $\alpha \in \mathcal{C}(\nu_0)$  there exists a probability measure  $\mathbb{Q} \ll \mathbb{P}$  such that  $\{W_t\}_{t=0}^T$  is a  $\mathbb{Q}$ - $\mathbb{F}$ -Brownian motion and

$$\mathbb{Q}(\|\alpha - \nu\|_\infty > \epsilon) < \epsilon, \quad (2.2)$$

where  $\|u - v\|_\infty := \sup_{0 \leq t \leq T} |u_t - v_t|$  denotes the uniform distance between  $u$  and  $v$ .

Observe that due to the structure of the financial market in (2.1), by taking convex combinations of the form  $\lambda\mathbb{P} + (1 - \lambda)\mathbb{Q}$ , we see that it is equivalent in Definition 2.1 to require  $\mathbb{Q} \approx \mathbb{P}$  instead of  $\mathbb{Q} \ll \mathbb{P}$ .

It turns out that being fully incomplete is a natural phenomenon for incomplete markets. More precisely, the following proven in [5] holds true.

**Proposition 2.2.** [5, Proposition 2.3]:

I. Consider the following stochastic volatility model for the volatility process  $\nu$  in (2.1):

$$d\nu_t = a(t, \nu_t) dt + b(t, \nu_t) d\widehat{W}_t + c(t, \nu_t) dW_t, \quad \nu_0 > 0,$$

where  $\{\widehat{W}_t\}_{t=0}^T$  is a  $\mathbb{P}$ - $\mathbb{F}$ -Brownian motion which is independent of  $W$ . Assume that the SDE in (2.1) has a unique strong solution and the solution is strictly positive. If the functions  $a, b, c : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$  are continuous and for any  $t \in [0, T]$ ,  $x > 0$  we have  $b(t, x) > 0$ , then the corresponding financial market defined in (2.1) is fully incomplete.

II. Let here  $\mathbb{F}$  be the usual augmentation of the filtration generated by  $W$  and  $\nu$ . Assume a decomposition  $\nu_t = \nu_t^{(1)} \nu_t^{(2)}$  where  $\nu^{(1)}$  is adapted to the filtration generated by  $W$ , and  $\nu^{(2)}$  is independent of  $W$ . Moreover, assume that  $\nu^{(1)}, \nu^{(2)}$  are strictly positive and continuous processes. If  $\ln \nu^{(2)}$  has a conditional full support (CFS) property (for definition of (CFS), see e.g. [5, Chapter 2]), then the market given by (2.1) is fully incomplete.

As a consequence, popular stochastic volatility models like the Heston model, Hull–White model and the Scott model, as well as rough volatility models like the one in [9] are examples of fully incomplete markets.

To recall the main property of fully incomplete markets, let  $\mathfrak{A}^\mathbb{P}$  denote the set of all  $\mathbb{F}$ -progressively measurable processes  $\{\gamma_t\}_{t=0}^T$  with  $\int_0^T \gamma_t^2 \nu_t^2 S_t^2 dt < \infty$   $\mathbb{P}$ -a.s. such that the stochastic integral  $\int \gamma dS$  is uniformly bounded from below. The robust setup is defined as follows. Let  $\{\mathbb{S}_t\}_{t=0}^T$  be the canonical process on the space  $C[0, T]$ , i.e.  $\mathbb{S}_t(\omega) = \omega(t)$ ,  $\omega \in C[0, T]$ , and  $\mathbb{F}_t^\mathbb{S} = \sigma\{\mathbb{S}_u : u \leq t\}$ ,  $t \in [0, T]$  denotes the canonical filtration. The set  $\mathfrak{A}$  consists of processes  $\{\gamma_t\}_{t=0}^T$  which are  $\mathbb{F}_t^\mathbb{S}$ -adapted and of bounded variation with left-continuous paths such that the process  $\int \gamma d\mathbb{S}$  is uniformly bounded from below, where here  $\int \gamma d\mathbb{S}$  is defined by

$$\int_0^t \gamma_u d\mathbb{S}_u := \gamma_t \mathbb{S}_t - \gamma_0 \mathbb{S}_0 - \int_0^t \mathbb{S}_s d\gamma_s, \quad t \in [0, T],$$

using the standard Lebesgue-Stieltjes integral for the last integral. Finally, define  $\mathfrak{S}$  to be the set of all paths in  $C[0, T]$  which are strictly positive and starts in  $S_0$ . Then the main theorem in [5] states the following.

**Theorem 2.3.** [5, Theorem 3.1]: Assume that the financial market defined in (2.1) is fully incomplete. For  $i := 1, \dots, N$ , let  $h_i(S)$  with  $h_i : C[0, T] \rightarrow \mathbb{R}$  bounded, uniformly continuous be a static position with price  $\mathcal{P}_i$ . Consider a (path-dependent) European option with payoff  $G(S)$ , where  $G : C[0, T] \rightarrow \mathbb{R}$  is a bounded and uniformly continuous function. Then the classical super-replication price

$$V_{h_1, \dots, h_N}^{\mathbb{P}}(G) := \inf \left\{ x + \sum_{i=1}^N c_i \mathcal{P}_i \mid \exists (\gamma_t) \in \mathfrak{A}^{\mathbb{P}} \text{ s.t. } x + \sum_{i=1}^N c_i h_i(S) + \int_0^T \gamma_t dS_t \geq G(S) \text{ } \mathbb{P}\text{-a.s.} \right\}$$

coincides with the robust price

$$V_{h_1, \dots, h_N}(G) := \inf \left\{ x + \sum_{i=1}^N c_i \mathcal{P}_i \mid \exists (\gamma_t) \in \mathfrak{A} \text{ s.t. } x + \sum_{i=1}^N c_i h_i(\mathbb{S}) + \int_0^T \gamma_t d\mathbb{S}_t \geq G(\mathbb{S}) \forall \mathbb{S} \in \mathfrak{G} \right\},$$

i.e. we have  $V_{h_1, \dots, h_N}^{\mathbb{P}}(G) = V_{h_1, \dots, h_N}(G)$ .

In this paper, we analyze the super-replication property when the financial market defined in (2.1) is fully incomplete in the special case where the option is of Markovian type, i.e. the option is of the form  $g(S_T)$  for some function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , and there are no liquid options to trade with. In this case, the classical super-replication price of  $g(S_T)$  is given by

$$V_0^{\mathbb{P}}(g) := \inf \left\{ x \in \mathbb{R} \mid \exists (\gamma_t) \in \mathfrak{A}^{\mathbb{P}} \text{ s.t. } x + \int_0^T \gamma_t dS_t \geq g(S_T) \text{ } \mathbb{P}\text{-a.s.} \right\}.$$

The robust price  $V_0(g)$  is defined analogously. Another (even more robust) super-replication price is the one where only buy-and-hold strategies are allowed. Its formal definition is

$$V_0^{B\&H}(g) := \inf \left\{ x \in \mathbb{R} \mid \exists \Delta \in \mathbb{R} \text{ s.t. } x + \Delta(S_T(\omega) - S_0) \geq g(S_T(\omega)) \forall \omega \in \Omega \right\}.$$

Clearly, for any option  $G$  in any financial market

$$V_0^{\mathbb{P}}(G) \leq V_0(G) \leq V_0^{B\&H}(G),$$

and by [5] we know that in fully incomplete markets, a priori,  $V_0^{\mathbb{P}}(G) = V_0(G) \leq V_0^{B\&H}(G)$  holds true for options  $G$  which are bounded and uniformly continuous.

The goal of this paper is to show that for Markovian claims, the above inequalities are in fact true equalities, even for unbounded and non-continuous payoff functions. Moreover, the price and the optimal (buy-and-hold) strategy can be calculated explicitly.

**Theorem 2.4.** Assume that the financial market defined in (2.1) is fully incomplete. Then for any nonnegative payoff function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  being lower semicontinuous, the super-replication price of the corresponding Markovian claim  $g(S_T)$  satisfies

$$V_0^{\mathbb{P}}(g) = V_0^{B\&H}(g) = \widehat{g}(S_0), \tag{2.3}$$

where  $\widehat{g}$  denotes the concave envelope of  $g$ . Moreover, an optimal (buy-and-hold) strategy exists and is explicitly defined by

$$\gamma \equiv \partial_+ \widehat{g}(S_0). \tag{2.4}$$

**Remark 2.5.** By the cash-invariance property of both  $V_0^{\mathbb{P}}$  and  $V_0^{B\&H}$ , the condition that  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is nonnegative could be relaxed by the requirement to be bounded from below.

### 3 Proof of Theorem 2.4

We start the proof by first recalling the well-known (trivial) inequalities in (2.3), namely:

**Lemma 3.1.** *For any Markovian payoff  $g(S_T)$ , we have*

$$V_0^{\mathbb{P}}(g) \leq V_0^{B\&H}(g) \leq \widehat{g}(S_0),$$

and for  $\gamma \equiv \partial_+ \widehat{g}(S_0)$ , the pair  $(\widehat{g}(S_0), \gamma)$  is a buy-and-hold super-replicating strategy, i.e.

$$\widehat{g}(S_0) + \partial_+ \widehat{g}(S_0) (S_T(\omega) - S_0) \geq g(S_T(\omega)) \quad \forall \omega \in \Omega.$$

*Proof.* Clearly,  $V_0^{\mathbb{P}}(g) \leq V_0^{B\&H}(g)$ . Moreover, by the definition of  $\widehat{g}$  being the smallest concave function bigger than  $g$ , we obtain for any  $\omega \in \Omega$  that

$$\widehat{g}(S_0) + \partial_+ \widehat{g}(S_0) (S_T(\omega) - S_0) \geq \widehat{g}(S_T(\omega)) \geq g(S_T(\omega)).$$

□

Next, we show that it is sufficient to prove the results in Theorem 2.4 for bounded, nonnegative payoff function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which are Lipschitz continuous. Beforehand, let us quickly introduce the following notion, which we will use frequently in the rest of this paper. For any  $x > 0$  and any (sufficiently integrable) progressively measurable process  $\alpha = \{\alpha_t\}_{t=0}^T$  we denote the corresponding stochastic exponential with respect to  $W$  by

$$\mathbf{S}_t^{\alpha, x} := x e^{\int_0^t \alpha_v dW_v - \frac{1}{2} \int_0^t \alpha_v^2 dv}, \quad t \in [0, T].$$

**Lemma 3.2.** *If (2.3) and (2.4) in Theorem 2.4 hold true for any bounded, nonnegative payoff function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which is Lipschitz continuous, then it also hold true for any nonnegative payoff function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  being lower semicontinuous.*

*Proof.* Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be any nonnegative, lower semicontinuous function. Due to Lemma 3.1, it remains to show that  $V_0^{\mathbb{P}}(g) \geq \widehat{g}(S_0)$ . Define the sequence of functions

$$\widetilde{g}_n(x) := \inf_{y \geq 0} \{g(y) + n|x - y|\}, \quad x \geq 0;$$

$$g_n(x) := \min \{\widetilde{g}_n(x), n\}, \quad x \geq 0.$$

We see that for each  $n$ , the function  $g_n$  is bounded, nonnegative and Lipschitz continuous (with Lipschitz constant  $n$ ). Moreover, the sequence  $(g_n)$  converges non-decreasingly to  $g$ . Let  $\mathcal{T}$  denote all  $\mathbb{F}$ -stopping times. Using [4, Lemma 5.4] and the monotone convergence theorem, we see that

$$\widehat{g}(S_0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_\tau^{1, S_0})] = \sup_{n \in \mathbb{N}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[g_n(\mathbf{S}_\tau^{1, S_0})] = \sup_{n \in \mathbb{N}} \widehat{g}_n(S_0) \quad (3.1)$$

By the assumption that (2.3) and (2.4) holds true for bounded, nonnegative payoff functions which are Lipschitz continuous and as the super-replication price is monotone in the claim

$$\sup_{n \in \mathbb{N}} \widehat{g}_n(S_0) = \sup_{n \in \mathbb{N}} V_0^{\mathbb{P}}(g_n) \leq V_0^{\mathbb{P}}(g) \quad (3.2)$$

Thus, by (3.1)–(3.2) we obtained  $\widehat{g}(S_0) \leq V_0^{\mathbb{P}}(g)$  as desired. □

Due to Lemma 3.2, it suffices to prove Theorem 2.4 for bounded, Lipschitz continuous payoff functions  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . To do so, we first start with a Lemma regarding an upper bound for the concave envelope.

**Lemma 3.3.** *Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be bounded, nonnegative and Lipschitz continuous. Then*

$$\widehat{g}(s) \leq \sup_{\alpha \in \mathcal{C}(\nu_0)} \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_T^{\alpha, s})], \quad \forall s > 0.$$

*Proof.* Introduce the set  $\mathcal{A}$  of all nonnegative progressively measurable processes with respect to the filtration  $\mathbb{F}^W$  generated by the Brownian motion  $W$  satisfying  $\int_0^T \alpha_t^2 dt < \infty$   $\mathbb{P}$ -a.s. and for which there exists a constant  $C > 0$  (which may depend on  $\alpha$ ), such that  $\frac{1}{C} \leq \mathbf{S}^{\alpha, 1} \leq C$ . By the same argument as in [5, Lemma 7.1], the function  $\mathfrak{G} : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$\mathfrak{G}(s) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_T^{\alpha, s})]$$

is concave and satisfies  $g \leq \mathfrak{G}$ . By the minimality property of the concave envelope of  $g$ , this means that also

$$\widehat{g}(s) \leq \mathfrak{G}(s), \quad \forall s > 0.$$

Then, by the same approximation argument as in [5, Lemma 7.2], where in this step we use the Lipschitz property of  $g$ , we obtain that

$$\mathfrak{G}(s) \leq \sup_{\alpha \in \mathcal{C}(\nu_0)} \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_T^{\alpha, s})], \quad \forall s > 0,$$

which implies the result.  $\square$

Now we are able to finish the proof of Theorem 2.4.

*Proof of Theorem 2.4.* We follow a similar argument as the one in [5, Theorem 4.2]. Let  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be bounded, nonnegative and Lipschitz continuous. By Lemma 3.1 and Lemma 3.2, it remains to show that  $V_0^{\mathbb{P}}(g) \geq \widehat{g}(S_0)$ . Thus, we fix any  $x > V_0^{\mathbb{P}}(g)$  and need to show that  $x \geq \widehat{g}(S_0)$ . By definition of  $x$ , there exists an  $\mathbb{F}$ -predictable,  $S$ -integrable process  $\{\gamma_t\}_{t=0}^T$  such that

$$x + \int_0^T \gamma_t dS_t \geq g(S_T) \quad \mathbb{P}\text{-a.s.}$$

Fix any  $\varepsilon > 0$ . By Lemma 3.3, there exists  $\alpha \in \mathcal{C}(\nu_0)$  such that

$$\widehat{g}(S_0) - \varepsilon \leq \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_T^{\alpha, S_0})]. \quad (3.3)$$

Next, choose any  $\delta > 0$  small enough (i.e.  $\delta \ll \varepsilon$ ). As the financial market defined in (2.1) is fully incomplete, there exists  $\mathbb{Q} \ll \mathbb{P}$  such that  $W$  is a  $\mathbb{Q}$ - $\mathbb{F}$ -Brownian motion and

$$\mathbb{Q}(\|\alpha - \nu\|_{\infty} \geq \delta) < \delta. \quad (3.4)$$

As  $(x, \gamma)$  is a super-replicating strategy, the supermartingale property of the gain process yields

$$\mathbb{E}^{\mathbb{Q}}[g(S_T)] \leq \mathbb{E}^{\mathbb{Q}}\left[x + \int_0^T \gamma_t dS_t\right] \leq x. \quad (3.5)$$

Now, define the stopping time

$$\tau := \inf \{t \geq 0 \mid |\alpha_t - \nu_t| \geq \delta\} \wedge T.$$

Then, by definition of  $\tau$

$$\begin{aligned} \frac{1}{2} \int_0^\tau |\alpha_t^2 - \nu_t^2| dt &= \frac{1}{2} \int_0^\tau |\alpha_t^2 - (\alpha_t - (\alpha_t - \nu_t))^2| dt \\ &= \frac{1}{2} \int_0^\tau |\alpha_t^2 - (\alpha_t^2 - 2\alpha_t(\alpha_t - \nu_t) + (\alpha_t - \nu_t)^2)| dt \\ &= \frac{1}{2} \int_0^\tau |2\alpha_t(\alpha_t - \nu_t) - (\alpha_t - \nu_t)^2| dt \\ &\leq \frac{1}{2} T \delta (2\|\alpha\|_\infty + \delta), \end{aligned}$$

as well as due to the Itô isometry

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^\tau (\alpha_t - \nu_t) dW_t\right)^2\right] \leq \delta^2 T.$$

Thus, Chebyshev's inequality implies that

$$\begin{aligned} &\mathbb{Q}\left(\frac{1}{2} \int_0^\tau |\alpha_t^2 - \nu_t^2| dt + \left|\int_0^\tau (\alpha_t - \nu_t) dW_t\right| \geq 2\sqrt{\delta}\right) \\ &\leq \frac{\frac{1}{2} \mathbb{E}^{\mathbb{Q}}\left[\int_0^\tau |\alpha_t^2 - \nu_t^2| dt\right]}{\sqrt{\delta}} + \frac{\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^\tau (\alpha_t - \nu_t) dW_t\right)^2\right]}{\delta} \\ &\leq \frac{\frac{1}{2} T \delta (2\|\alpha\|_\infty + \delta)}{\sqrt{\delta}} + \frac{\delta^2 T}{\delta} \\ &\leq c(\sqrt{\delta} + \delta). \end{aligned} \quad (3.6)$$

for some constant  $c$  which may depend on  $\varepsilon$  (but not on  $\delta$ ). Now, since by definition, the price process  $S$  is defined as  $S \equiv \mathbf{S}^{\nu, S_0}$ , we deduce from (3.6) that for sufficiently small  $\delta$

$$\mathbb{Q}\left(|\ln S_\tau - \ln \mathbf{S}_\tau^{\alpha, S_0}| > \varepsilon\right) \leq \mathbb{Q}\left(\frac{1}{2} \int_0^\tau |\alpha_t^2 - \nu_t^2| dt + \left|\int_0^\tau (\alpha_t - \nu_t) dW_t\right| > \varepsilon\right) \leq c(\sqrt{\delta} + \delta). \quad (3.7)$$

Next, define the event

$$U_\varepsilon := \{\tau < T\} \cup \{|\ln S_\tau - \ln \mathbf{S}_\tau^{\alpha, S_0}| > \varepsilon\}$$



It is elementary to check that as  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is Lipschitz continuous with respect to some Lipschitz-constant  $K > 0$ , we have for any  $x > 0, y > 0$  with  $|\ln(x) - \ln(y)| \leq \epsilon$ , that  $g(x) \geq g(y) - Ky(e^\epsilon - 1)$ . Therefore, using (3.5) and the definition of (the complement of)  $U_\epsilon$  yields

$$\begin{aligned} x &\geq \mathbb{E}^{\mathbb{Q}}[g(S_T)] \geq \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{U_\epsilon^c} g(S_T)] \\ &\geq \mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})] - K(e^\epsilon - 1) \mathbb{E}^{\mathbb{Q}}[\mathbf{S}_T^{\alpha, S_0}] - \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{U_\epsilon} g(\mathbf{S}_T^{\alpha, S_0})] \\ &\geq \mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})] - K(e^\epsilon - 1)S_0 - \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{U_\epsilon} g(\mathbf{S}_T^{\alpha, S_0})]. \end{aligned} \quad (3.8)$$

Now, before letting  $\epsilon$  go to zero, we first need to analyze  $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{U_\epsilon} g(\mathbf{S}_T^{\alpha, S_0})]$ . To that end, see that by the Lipschitz property of  $g \geq 0$

$$\mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})^2] \leq \mathbb{E}^{\mathbb{Q}}[(g(0) + K\mathbf{S}_T^{\alpha, S_0})^2] \leq 2g(0)^2 + 2K^2 \mathbb{E}^{\mathbb{Q}}[(\mathbf{S}_T^{\alpha, S_0})^2].$$

Moreover, as  $\alpha \in \mathcal{C}(\nu_0)$  is uniformly bounded, we see that for some constant  $C > 0$ ,

$$\mathbb{E}^{\mathbb{Q}}[(\mathbf{S}_T^{\alpha, S_0})^2] \leq C\mathbb{E}^{\mathbb{Q}}[\mathbf{S}_T^{2\alpha, S_0}] < \infty.$$

Hence we conclude that  $\mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})^2] < \infty$ . Furthermore, observe that

$$\{\tau < T\} \subseteq \{\|\alpha - \nu\|_\infty \geq \delta\}.$$

By the Cauchy-Schwarz inequality, (3.4) and (3.7), this implies that for sufficiently small  $\delta$

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{U_\epsilon} g(\mathbf{S}_T^{\alpha, S_0})] \leq \mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})^2]^{\frac{1}{2}} \mathbb{Q}[U_\epsilon]^{\frac{1}{2}} \leq C(\delta + \sqrt{\delta})^{\frac{1}{2}} < \epsilon.$$

Therefore, we deduce from (3.8) that

$$x \geq \mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})] - K(e^\epsilon - 1)S_0 - \epsilon. \quad (3.9)$$

Recall that under both measures  $\mathbb{P}$  and  $\mathbb{Q}$ , the process  $W$  is an  $\mathbb{F}$ -Brownian motion, and that the process  $\alpha$  is progressively measurable with respect to the smaller filtration  $\mathbb{F}^W$  generated by  $W$ . In particular,  $\alpha = \varphi(W)$  for some progressively measurable map  $C[0, T] \rightarrow C[0, T]$ . Therefore, the law of  $\mathbf{S}_T^{\alpha, S_0}$  under  $\mathbb{P}$  and  $\mathbb{Q}$  are the same. This implies together with (3.3) that

$$\begin{aligned} x &\geq \mathbb{E}^{\mathbb{Q}}[g(\mathbf{S}_T^{\alpha, S_0})] - K(e^\epsilon - 1)S_0 - \epsilon \\ &= \mathbb{E}^{\mathbb{P}}[g(\mathbf{S}_T^{\alpha, S_0})] - K(e^\epsilon - 1)S_0 - \epsilon \\ &\geq \widehat{g}(S_0) - K(e^\epsilon - 1) - 2\epsilon. \end{aligned}$$

As  $\epsilon > 0$  was arbitrarily chosen, we can now let  $\epsilon$  go to zero to obtain desired inequality

$$x \geq \widehat{g}(S_0).$$

□

## References

- [1] B. Acciaio, M. Beiglböck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *Math. Finance*, 26(2):233–251, 2016.
- [2] D. Bartl, M. Kupper, D. Prömel, and L. Tangpi. Duality for pathwise superhedging in continuous time. *arXiv:1705.02933*, 2017.
- [3] M. Burzoni, M. Frittelli, and M. Maggis. Model-free superhedging duality. *To appear in Ann. Appl. Probab.* *arXiv:1506.06608*, 2015.
- [4] J. Cvitanić, H. Pham, and N. Touzi. Super-replication in stochastic volatility models under portfolio constraints. *J. Appl. Probab.*, 36(2):523–545, 1999.
- [5] Y. Dolinsky and A. Neufeld. Super-replication in fully incomplete markets. *Math. Finance*, *forthcoming*, 2016.
- [6] Y. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields*, 160(1–2):391–427, 2014.
- [7] Y. Dolinsky and H. M. Soner. Martingale optimal transport in the Skorokhod space. *Stochastic Processes and their Applications*, 125:3657–4020, 2015.
- [8] R. Frey and C.A. Sin. Bounds on european option prices under stochastic volatility. *Math. Finance*, 9:97–116, 1999.
- [9] J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *preprint; arxiv: 1410.3394*, 2014.
- [10] G. Guo, X. Tan, and N. Touzi. Tightness and duality of martingale transport on the skorokhod space. *To appear in Stochastic Process. Appl.* *arXiv:1507.01125*, 2015.
- [11] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.
- [12] D. Hobson. Robust hedging of the lookback option. *Finance Stoch.*, 2(4):329–347, 1998.
- [13] Z. Hou and J. Obłój. On robust pricing-hedging duality in continuous time. *preprint. arXiv:1503.02822*, 2015.
- [14] J. Hull and A. White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42(2):281–300, 1987.
- [15] L.O. Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative Analysis*, 22(4):419–438, 1987.