A correspondence of good G-sets under partial geometric quotients

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Abstract For a complex variety $\widehat{X}$ with an action of a reductive group $\widehat{G}$ and a geometric quotient $\pi : \widehat{X} \to X$ by a closed normal subgroup $H \subset \widehat{G}$, we show that open sets of X admitting good quotients by $G = \widehat{G}/H$ correspond bijectively to open sets in $\widehat{X}$ with good $\widehat{G}$-quotients. We use this to compute GIT-chambers and their associated quotients for the diagonal action of PGL_2 on $(\mathbb{P}^1)^n$ in certain subcones of the PGL_2-effective cone via a torus action on affine space. This allows us to represent these quotients as toric varieties with fans determined by convex geometry.

Keywords Geometric invariant theory · Good quotients · Toric varieties · Variation of GIT

Mathematics Subject Classification 14L24 · 14L30

1 Introduction

Let G be a reductive group acting on a variety X, then an important question in Geometric Invariant theory is to classify the open G-invariant subsets U of X having a good quotient under the action of G. Define

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$$\mathcal{U}_{(X,G)} = \left\{ \begin{array}{l} U \subset X \text{ nonempty, open } G\text{-invariant such that a good} \\ \text{quotient } U \to U /\!\!/ G \text{ exists (in schemes over } \mathbb{C}) \end{array} \right\}, \qquad (1)$$

$$\mathcal{U}_{(X,G)}^{\mathrm{pr}} = \left\{ \begin{array}{l} U \subset X \text{ nonempty, open } G\text{-invariant such that a good} \\ \text{quotient } U \to U /\!\!/ G \text{ exists with } U /\!\!/ G \text{ a projective variety} \end{array} \right\}, \qquad (2)$$

$$\mathcal{U}_{(X,G)}^{\mathrm{pr,g}} = \left\{ \begin{array}{l} U \subset X \text{ nonempty, open } G\text{-invariant such that an affine, geometric} \\ \text{quotient } U \to U /\!\!/ G \text{ exists and such that } U /\!\!/ G \text{ is a projective variety} \end{array} \right\}. \qquad (3)$$

In this note we describe a geometric situation, in which these collections of open sets for two pairs (X, G), $(\widehat{X}, \widehat{G})$ can be identified.

Definition 1 Let G, H be reductive, linear algebraic groups and let G act on a variety X. A good (resp. geometric) H-lift of (X, G) is the data of

– a reductive algebraic group $\widehat{G}$ containing H as a closed normal subgroup together with an identification $\widehat{G}/H = G$,
– a variety $\widehat{X}$ with an action of $\widehat{G}$,
– a morphism $\pi : \widehat{X} \to X$, which is
  • $\widehat{G}$-equivariant with respect to the action of $\widehat{G}$ on X induced by the action of G and the morphism $\widehat{G} \to \widehat{G}/H = G$,
  • a good (resp. geometric) quotient for the induced action of H on $\widehat{X}$.

Then we prove the following

Theorem 1 *Let $\pi : \widehat{X} \to X$ be a good H-lift of (X, G) for X a variety with an action of the reductive group G. Then the map $U \mapsto \pi^{-1}(U)$ induces an injection $\mathcal{U}_{(X,G)} \to \mathcal{U}_{(\widehat{X},\widehat{G})}$. Moreover, for $U \in \mathcal{U}_{(X,G)}$ the map π induces a natural isomorphism $\pi^{-1}(U) /\!\!/ \widehat{G} \cong U /\!\!/ G$. Thus we also get an injection $\mathcal{U}_{(X,G)}^{pr} \to \mathcal{U}_{(\widehat{X},\widehat{G})}^{pr}$.*

If π is a geometric H-lift, the correspondence above is a bijection and it induces a bijection $\mathcal{U}_{(X,G)}^{pr,g} \to \mathcal{U}_{(\widehat{X},\widehat{G})}^{pr,g}$.

It has already been observed by various authors (see Białynicki-Birula 2002, Theorem 6.1.5 for an overview) that for a geometric H-lift $\pi : \widehat{X} \to X$ a G-invariant open subset $U \subset X$ has a good/geometric quotient by G iff $\pi^{-1}(U)$ has a good/geometric quotient by $\widehat{G}$. Below we give a self-contained argument that also includes the case of good H-lifts.

The structure of the paper is as follows: as a motivation for the definition of $\mathcal{U}_{(X,G)}^{\mathrm{pr}}$, we show in Sect. 2 how its elements relate to G-linearized line bundles on X for X a smooth variety. This makes a connection to the classical approach of Mumford et al. (1994), obtaining (semi)stable sets from linearized line bundles. We give two special situations where we can identify the elements of $\mathcal{U}_{(X,G)}^{\mathrm{pr,g}}$ with chambers of the G-effective cone in $\mathrm{NS}^G(X)$:

– for X a projective homogeneous variety, G reductive,
– for $X = \mathbb{A}^n$ and $G = T$ a torus acting linearly.

In Sect. 3 we give the proof of Theorem 1 and describe situations where H-lifts appear naturally. Moreover we show that H-lifts where the action of H on $\widehat{X}$ is free give an

identification $\mathrm{Pic}^G(X) \cong \mathrm{Pic}^{\widehat{G}}(\widehat{X})$ compatible with forming semistable sets. Finally in Sect. 5 we give explicit examples, showing firstly how to compute parts of the VGIT-decomposition of the G-effective cone for the componentwise action of $G = \mathrm{PGL}_2$ on $(\mathbb{P}^1)^n$ using toric quotients. We also show how the classical Gelfand–MacPherson correspondence fits into our framework.

Conventions

In the paper we are going to work over the complex numbers. For us, a *good quotient* of the action of an algebraic group G on a scheme X is a morphism $p : X \to Y$ to a scheme Y satisfying

1. p is surjective, affine and H-invariant,
2. $p_*(\mathcal{O}_X^G) = \mathcal{O}_Y$, where $\mathcal{O}_X^G$ is the sheaf of G-invariant functions on X,
3. for Z_1, $Z_2 \subset X$ closed, disjoint G-invariant subsets, their images $p(Z_1)$, $p(Z_2)$ are also closed and disjoint.

On the other hand, for p to be a *geometric quotient* we require the properties above, except that it is affine, but additionally we want the fibres of geometric points under p to be orbits of G. This is the definition of Mumford et al. (1994).

2 Motivation: projective quotients from linearized line-bundles

Let X be a smooth, irreducible variety with an action of a connected reductive group G. We want to study good quotients of open sets $U \subset X$ by G which are projective varieties.

Lemma 1 *Let X be a smooth, irreducible variety with an action of a connected reductive group G. Then the open sets U in $\mathcal{U}_{(X,G)}^{pr}$ are all of the form $U = X^{ss}(L)$ for a G-linearized line bundle L on X.*

Proof By Białynicki-Birula (2002, Theorem 6.1.5), all open G-invariant sets $U \subset X$ with a good quotient $U /\!\!/ G$ that is a quasi-projective variety are saturated subsets of some $X^{ss}(L)$ for L a G-linearized line bundle. Let $\pi : X^{ss}(L) \to X^{ss}(L) /\!\!/ G$ be the corresponding good quotient. Then we have $U /\!\!/ G \subset X^{ss}(L) /\!\!/ G$ contained as an open subset. If $U /\!\!/ G$ is in addition projective, this inclusion is an isomorphisms. But as U is a saturated open subset, we have $U = \pi^{-1}(\pi(U)) = \pi^{-1}(U /\!\!/ G) = X^{ss}(L)$ as claimed. □

Remark 1 We can generalize the setting above to X being a normal variety if we work with G-linearized Weil divisors instead of line bundles, as described in Hausen (2004). However, as our applications work with smooth X, we stay in the more classical setting of line bundles.

An advantage of the condition "$U /\!\!/ G$ is projective" in comparision with "U is maximal with respect to saturated inclusion in $\mathcal{U}_{(X,G)}$" is that this can be verified intrinsically only from the action of G on U (without reference to the ambient variety

X). Thus the definition of $\mathcal{U}^{\text{pr}}$ is compatible with the restriction to open G-invariant subsets in the following sense.

Corollary 1 *Let X be a variety with an action of a reductive group G. Let $U_0 \subset X$ be an open G-invariant subset. Then we have $\mathcal{U}^{pr}_{(U_0, G)} = \mathcal{U}^{pr}_{(X, G)} \cap \{U : U \subset U_0\}$ (similarly for $\mathcal{U}^{pr,g}$).*

In the following two sections, we are going to present situations where the dependence of $X^{ss}(L)$ of L has been studied before and where the classes of G-linearized line bundles are partitioned into cones on which $X^{ss}(L)$ is constant.

2.1 Actions of reductive groups on smooth projective varieties

Let X be an irreducible, smooth projective variety acted upon by a connected, reductive linear algebraic group G. In this situation, Dolgachev and Hu defined in (1998) the G-ample cone $C^G(X) \subset \text{NS}^G(X)_{\mathbb{R}}$ inside the Neron-Severi group of G-linearized line bundles. It is spanned by the homology classes of G-linearized ample line bundles L such that $X^{ss}(L) \neq \emptyset$. In Dolgachev and Hu (1998, Theorem 3.3.2), they show that if this cone has nonempty interior, it contains open chambers such that two elements $L, L' \in C^G(X)$ are in the same chamber σ iff we have

$$X^{ss}(L) = X^s(L) = X^s(L') = X^{ss}(L') =: X^{ss}(\sigma).$$

Furthermore, as X is projective, for any L in a chamber as above, we have that $X^{ss}(L) /\!\!/ G$ is projective. This shows that the set of chambers of $C^G(X)$ injects into $\mathcal{U}^{\text{pr,g}}_{(X, G)}$ by sending a chamber σ to $X^{ss}(\sigma)$. Note however, that this inclusion can be strict: in Białynicki-Birula and Święcicka (1989) the authors give an example of a smooth projective variety X with an action of a torus T together with an open set $U \subset X$, which has a projective geometric quotient $U /\!\!/ T$ but is not of the form $X^s(L)$ for L ample and G-linearized. For a treatment of the behaviour of $X^{ss}(L)$ when L is outside the ample cone see Berchtold and Hausen (2006).

However, for certain special varieties X the correspondence between chambers of $C^G(X)$ and elements of $\mathcal{U}^{\text{pr,g}}_{(X, G)}$ is bijective.

Proposition 1 *Let X be an irreducible, smooth projective variety acted upon by a connected, reductive linear algebraic group G. Assume that every effective divisor is semiample (i.e. some positive power is base-point free) and that $C^G(X)$ has nonempty interior with all walls having positive codimension. Then the chambers of $C^G(X)$ are in bijection with $\mathcal{U}^{pr,g}_{(X, G)}$ via $\sigma \mapsto X^{ss}(\sigma)$.*

Proof Let $U \in \mathcal{U}^{\text{pr,g}}_{(X, G)}$ then we need to show that U is of the form $U = X^s(L') = X^{ss}(L')$ for some ample G-linearized line bundle L'. By Lemma 1 a priori we only know that $U = X^{ss}(L)$ for some G-linearized (not necessarily ample) L. As $U \to U /\!\!/ G$ is a geometric quotient, all orbits in U are fibres of this map and hence closed, so $U = X^s(L) = X^{ss}(L)$. As U is nonempty, the bundle L must have at least one section, so its associated divisor is effective and thus semiample by assumption.

We want to show that for m sufficiently large and a suitable L_0 in the interior of $C^G(X)$ the line bundle $L' = L^{\otimes m} \otimes L_0$ satisfies $U \subset X^s(L') = X^{ss}(L')$. Then as in the proof of Lemma 1 we see that this inclusion is already the identity. But as such L' are ample and G-effective, this finishes the proof. In the following, we can use the Hilbert-Mumford criterion to determine the (semi)stable points of L'.

For this recall from Dolgachev and Hu (1998, Section 1.1) the construction of the function $M^\bullet(x) : \mathrm{Pic}^G(X)_{\mathbb{R}} \to \mathbb{R}$ for $x \in X$. To define it let $\lambda : \mathbb{C}^* \to G$ be a 1-parameter subgroup of G then, as X is proper, the map $\mathbb{C}^* \to X, t \mapsto \lambda(t).x$ has a limit z over $t = 0$. The point z is fixed by λ and for $L \in \mathrm{Pic}^G(X)$, $\mathbb{C}^*$ acts on the fibre L_z of L over z with weight $r =: \mu(x, \lambda)$. Let T be a maximal torus in G and let $\| \ \|$ be a Weyl-invariant norm on the group of 1-parameter subgroups of T tensor $\mathbb{R}$. Then for any 1-parameter subgroup λ of G define $\|\lambda\|$ to be the norm of a suitable conjugate of λ contained in T. We set

$$M^L(x) = \sup_{\lambda \ \text{1-PSG of } G} \frac{\mu^L(x, \lambda)}{\|\lambda\|}.$$

By Dolgachev and Hu (1998, Lemma 3.2.5) the function $M^\bullet(x)$ factors through $\mathrm{NS}^G(X)_{\mathbb{R}}$ and satisfies

$$M^{L_1+L_2}(x) \leq M^{L_1}(x) + M^{L_2}(x), \ M^{mL}(x) = m M^L(x)$$

for $L_1, L_2 \in \mathrm{Pic}^G(X)$, $m > 0$. For L' ample G-linearized, x is semistable (properly stable) with respect to L' iff $M^{L'}(x) \leq 0$ ($M^{L'}(x) < 0$).

For our given L, we first show that $M^L(x) < 0$ for all $x \in X^s(L)$. Observe that as L is semiample, by (Ressayre 2010, Corollary 1) there exists a 1-parameter subgroup λ of G with $M^L(x) = \mu^L(x, \lambda)/\|\lambda\|$. Thus it suffices to show $\mu^L(x, \lambda) < 0$ for all 1-parameter subgroups λ. As x is stable, there exists an invariant section s of some tensor power of L with $x \in X_s$ and $Gx \subset X_s$ closed. We claim that then $z = \lim_{t \to 0} \lambda(t)x$ is not contained in X_s. Indeed assume otherwise, then $z \in \overline{Gx} \cap X_s = Gx$, so $z = gx$ for some $g \in G$. However then the stabilizer G_{gx} contains all of $\lambda(\mathbb{C}^*)$, so it is not finite. But then also the stabilizer of x is not finite and we obtain a contradiction, as our assumptions imply that all stable points are properly stable.[1] We conclude that $s(z) = 0$ and by Ressayre (2010, Proposition 1) this implies $\mu(x, \lambda) < 0$.

Let $L_0 \in C^G(X)$ such that for all $0 < r \ll 1$ we have that $L + rL_0$ is contained in a chamber of $C^G(X)$. As there are only finitely many walls (Dolgachev and Hu 1998, Theorem 3.3.3), which are all of positive codimension, such L_0 exist. For $m \gg 0$ an integer, we have that $L' = L^{\otimes m} \otimes L_0$ is ample and for a fixed $x \in X^s(L)$ we know

$$M^{L'}(x) \leq m M^L(x) + M^{L_0}(x).$$

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[1] In Dolgachev and Hu (1998) finiteness of stabilizers was part of the definition of a stable point.

As $M^L(x) < 0$ we can choose m sufficiently big such that $M^{L'}(x) < 0$ and hence $x \in X^s(L')$. The subsets

$$Y_m = X^s(L) \backslash X^s(L^{\otimes m} \otimes L_0)$$

form a descending chain of closed subsets of $X^s(L)$ and for all $x \in X^s(L)$ there exists m with $x \notin Y_m$. By Noetherian induction we can thus choose m_0 such that $U = X^s(L) \subset X^s(L^{\otimes m} \otimes L_0)$ for all $m \geq m_0$. But by the choice of L_0 we have that $L' = L^{\otimes m} \otimes L_0$ is contained in a chamber of $C^G(X)$ for m sufficiently large.   □

The condition that every effective divisor D is semiample is for instance satisfied for homogeneous projective varieties $X = G/P$. Indeed, in this case

$$G \mapsto \mathrm{Pic}(X), g \mapsto \mathcal{O}(g.D),$$

where $g.D$ is the translate of D by g, is a family of line bundles over G. By Popov (1974, Proposition 7), $\mathrm{Pic}(X)$ is discrete and hence the map above is constant and equal to $\mathcal{O}(D)$. But as the G-translates of $X\backslash D$ cover X, this shows that $\mathcal{O}(D)$ is base-point free.

Note that spaces of the form G/P are examples of Mori dream spaces, since in particular they are spherical varieties. Mori dream spaces have been introduced in Hu and Keel (2000) and their Geometric Invariant theory admits a description via their Cox ring (see Arzhantsev and Hausen 2009).

2.2 Toric quotients of affine space

In this section, we explain how for linear actions $(\mathbb{C}^*)^n \curvearrowright \mathbb{C}^r$ we can compute open sets in $\mathcal{U}_{(\mathbb{C}^r, (\mathbb{C}^*)^n)}$ via elementary and algorithmically accessible operations involving fans and polyhedra. We closely follow Cox et al. (2011, Section 14) in notation and presentation.

Let an algebraic torus $G = (\mathbb{C}^*)^n$ act faithfully, linearly on the affine space $X = \mathbb{C}^r$. By a suitable change of coordinates, we may assume that G acts by diagonal matrices. For $\mathbf{t} = (t_1, \ldots, t_n) \in G$ and $\beta = (\beta^1, \ldots, \beta^n) \in \mathbb{Z}^n$ write

$$\mathbf{t}^\beta = t_1^{\beta^1} t_2^{\beta^2} \cdots t_n^{\beta^n}.$$

Then after coordinate change, the action of $\mathbf{t} \in G$ on $x \in \mathbb{C}^r$ is given by

$$\mathbf{t}.x = \mathrm{diag}(\mathbf{t}^{\beta_1}, \ldots, \mathbf{t}^{\beta_r})x$$

for integer vectors $\beta_1, \ldots, \beta_r \in \mathbb{Z}^n$. Note that via the identification of $\mathbb{Z}^n$ with the character group $\widehat{G}$ of G, the β_i are simply the restrictions of the characters $\mathbf{t} \mapsto t_i$ of $(\mathbb{C}^*)^r$ along the map $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^r \subset \mathrm{GL}(\mathbb{C}^r)$ specifying the action. Let

$$\gamma : \mathbb{Z}^r \cong \widehat{(\mathbb{C}^*)^r} \to \widehat{G} \cong \mathbb{Z}^n$$

be this restriction map (such that $\gamma(e_i) = \beta_i$). The assumption that the action is faithful implies that γ is surjective (Cox et al. 2011, Lemma 14.2.1). Let $\delta : M \to \mathbb{Z}^r$ be the kernel of γ. Setting $N = \mathrm{Hom}(M, \mathbb{Z})$, the map δ is given by

$$\delta(m) = (\langle m, v_1 \rangle, \ldots, \langle m, v_r \rangle)$$

for some $v_1, \ldots, v_r \in N$. Below we will see that the vectors $\beta_1, \ldots, \beta_r$ control the linearizations and GIT-chambers for quotients of $\mathbb{C}^r$ by G and these quotients are toric varieties of fans in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with rays spanned by some of the vectors $v_1, \ldots, v_r$.

For this note that, as all line bundles on $\mathbb{C}^r$ are trivial, the G-linearized line bundles $L = \mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$ are specified by characters $\chi \in \widehat{G}$ via

$$\mathbf{t}.(x, y) = (\mathbf{t}.x, \chi(\mathbf{t})y), \quad \text{with } \mathbf{t} \in G, x \in \mathbb{C}^r, y \in \mathbb{C}.$$

Denote by $(\mathbb{C}^r)_{\chi}^{ss}$, $(\mathbb{C}^r)_{\chi}^{s}$ the (semi)stable points with respect to these linearizations and by $\mathbb{C}^r /\!\!/_{\chi} G$ the categorical quotient of $(\mathbb{C}^r)_{\chi}^{ss}$ by G. Let $\widehat{G}_{\mathbb{R}} = \widehat{G} \otimes_{\mathbb{Z}} \mathbb{R}$ and let $C_{\beta} \subset \widehat{G}_{\mathbb{R}}$ be the cone spanned by $\beta_1 \otimes 1, \ldots, \beta_r \otimes 1$. Note that as γ was surjective, we have $\dim C_{\beta} = \dim \widehat{G}_{\mathbb{R}}$. Then we have the following results:

1. The set $(\mathbb{C}^r)_{\chi}^{ss}$ of semistable points is nonempty iff $\chi \otimes 1 \in C_{\beta}$ (Cox et al. 2011, Proposition 14.3.5).
2. The set $(\mathbb{C}^r)_{\chi}^{s}$ of stable points is nonempty iff $\chi \otimes 1$ is in the interior of C_{β} (Cox et al. 2011, Proposition 14.3.5).
3. The quotient $\mathbb{C}^r /\!\!/_{\chi} G$ is projective for some $\chi \otimes 1 \in C_{\beta}$ iff all β_i are nonzero and C_{β} is strongly convex (i.e. $C_{\beta} \cap (-C_{\beta}) = \{0\}$). In this case all nonempty quotients $\mathbb{C}^r /\!\!/_{\chi} G$ are projective (Cox et al. 2011, Proposition 14.3.10).
4. We have $(\mathbb{C}^r)_{\chi}^{s} = (\mathbb{C}^r)_{\chi}^{ss}$ iff $\chi \otimes 1$ does not lie on a cone $C_{\beta'}$ generated by a subset β' of the β_i with $\dim C_{\beta'} < \dim C_{\beta}$ (Cox et al. 2011, Theorem 14.3.14).

In fact the behaviour of $(\mathbb{C}^r)_{\chi}^{ss}$ as $\chi \otimes 1$ varies in C_{β} is completely determined by the so-called secondary fan Σ_{GKZ} (see Cox et al. 2011, Theorem 14.4.7). This is a rational fan in $\widehat{G}_{\mathbb{R}}$ with support C_{β} such that $(\mathbb{C}^r)_{\chi}^{ss}$ is constant for $\chi \otimes 1$ moving in the relative interior of any of the cones $\sigma \in \Sigma_{\mathrm{GKZ}}$.

Let $\chi \in \widehat{G} \cap C_{\beta}$ be given by $\chi = \sum_{i=1}^{r} a_i \beta_i$, i.e. $\chi = \gamma(\mathbf{a})$. Define the polyhedron

$$P_{\mathbf{a}} = \{m \in \mathbb{M}_{\mathbb{R}} : \langle m, v_i \rangle \geq -a_i, 1 \leq i \leq r\} \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let Σ_{χ} be the normal fan of $P_{\mathbf{a}}$ (see (Cox et al. 2011, Proposition 14.2.10), then it is independent of the choice of $\mathbf{a} \in \gamma^{-1}(\chi)$ and we have that the quotient $\mathbb{C}^r /\!\!/_{\chi} G$ is isomorphic to the toric variety associated to Σ_{χ} (Cox et al. 2011, Theorem 14.2.13).

Moreover, we have an explicit description of the set $(\mathbb{C}^r)_{\chi}^{ss}$ of semistable points. Set

$$I_{\emptyset, \chi} = \{i \in \{1, \ldots, r\} : P_{\mathbf{a}} \cap \{m : \langle m, v_i \rangle = -a_i\} = \emptyset\}.$$

Define the ideal

$$B(\Sigma_\chi, I_{\emptyset,\chi}) = \left(\prod_{i \notin I_{\emptyset,\chi} : v_i \notin \sigma} x_i : \sigma \in \Sigma_\chi \right) \cdot \left(\prod_{i \in I_{\emptyset,\chi}} x_i \right)$$

in $\mathbb{C}[x_1, \ldots, x_r]$. Then $(\mathbb{C}^r)^{ss}_\chi = \mathbb{C}^r \setminus V(B(\Sigma_\chi, I_{\emptyset,\chi}))$ (Cox et al. 2011, Corollary 14.2.22).

In fact, the fan Σ_χ and the set $I_{\emptyset,\chi}$ of indices is also constant on the relative interior of the cones of Σ_{GKZ} and these cones are uniquely indexed by this data and written as $\Gamma_{\Sigma, I_\emptyset}$.

Proposition 2 *The map $\Sigma_{GKZ} \to \mathcal{U}_{(\mathbb{C}^r, G)}$ associating to a cone $\Gamma_{\Sigma, I_\emptyset}$ the set $(\mathbb{C}^r)^{ss}_\chi$ for any $\chi \otimes 1$ in the relative interior of $\Gamma_{\Sigma, I_\emptyset}$ is well-defined and injective.*

If all vectors β_i are nonzero and the cone C_β is strongly convex, the map above is a bijection from Σ_{GKZ} to $\mathcal{U}^{pr}_{(\mathbb{C}^r, G)}$ sending chambers to elements of $\mathcal{U}^{pr,g}_{(\mathbb{C}^r, G)}$.

Conversely, every $U \in \mathcal{U}_{(\mathbb{C}^r, G)}$ is a saturated open set of some $(\mathbb{C}^r)^{ss}_\chi$.

Proof We have already remarked that in the relative interior of σ, the set $(\mathbb{C}^r)^{ss}_\chi$ is constant, so we show injectivity. Assume that $\Gamma_{\Sigma, I_\emptyset}$ and $\Gamma_{\Sigma', I'_\emptyset}$ map to the same set U of semistable points. Then the vanishing ideal I of $\mathbb{C}^r \setminus U$ is the radical of the ideals $B(\Sigma, I_\emptyset)$, $B(\Sigma', I'_\emptyset)$ as defined above. But as these two are ideals generated by square-free monomials, they are already radical [see for instance (Herzog and Hibi 2011, Corollary 1.2.5), so $B(\Sigma, I_\emptyset) = B(\Sigma', I'_\emptyset)$. By Cox et al. (2011, Corollary 14.4.15) this implies $\Gamma_{\Sigma, I_\emptyset} = \Gamma_{\Sigma', I'_\emptyset}$.

The additional conditions on the β_i guarantee that all quotients $(\mathbb{C}^r)^{ss}_\chi \sslash G$ for $\chi \otimes 1 \in C_\beta$ are projective. Assume conversely that we have $U \in \mathcal{U}^{pr,g}_{(\mathbb{C}^r, (\mathbb{C}^*)^n)}$, then by Lemma 1 it is of the form $U = X^{ss}(L)$ for L a G-linearized line bundle corresponding to the character χ of G. As this set is nonempty, we have $\chi \otimes 1 \in C_\beta$ and as the fan Σ_{GKZ} has support C_β, it is contained in the relative interior of one of its cones.

The last statement above is again Lemma 1. $\square$

Remark 2 Let us remark here that the above discussion could be carried out more generally for torus actions on factorial affine varieties, see (Berchtold and Hausen 2006, Section 2). In this case, the GIT fan also has a simple description and explicit computation, see Keicher (2012).

3 Properties of H-lifts

We are now ready to prove Theorem 1. For this, we need the following technical result, which we prove here in lack of a good reference.

Lemma 2 *Let $\pi : Z \to X$ be a surjective morphism of schemes with X reduced. Then π is an epimorphism, i.e. two maps $\varphi_1, \varphi_2 : X \to Y$ to some scheme Y agree iff $\varphi_1 \circ \pi = \varphi_2 \circ \pi$. In particular, good quotients of reduced schemes are epimorphisms.*

Proof For morphisms φ_1, φ_2 as above we have a fibred diagram

$$
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle \delta_Y} \\
X & \xrightarrow{(\varphi_1,\varphi_2)} & Y \times Y
\end{array}
$$

where δ_Y is the diagonal map of Y, which is a locally closed embedding. In particular $W \to X$ is also a locally closed embedding. Assume that $\varphi_1 \circ \pi = \varphi_2 \circ \pi$, then by definition the map π factors through $W \to X$. In particular, $W \to X$ is surjective and hence a closed embedding. But as X is reduced, this means that it is an isomorphism. Then the diagram above shows that $\varphi_1 = \varphi_2$. In particular, if π is a good quotient and Z is reduced, so is X and thus the assumptions above are satisfied.

Proof of Theorem 1 We first note that as π is surjective, the G-invariant subsets U of X inject via π^{-1} into the $\widehat{G}$-invariant subsets $\widehat{U}$ of $\widehat{X}$. If π is a geometric quotient this is a bijection with inverse map given by $\widehat{U} \mapsto \pi(\widehat{U})$. This is well-defined because π sends open H-invariant sets to open sets and it is an inverse to π^{-1} as the fibres of π are orbits.
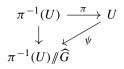
Before we continue, recall the following fact, which is Lemma 5.1 in Ramanathan (1996). Let a reductive algebraic group G' act on schemes Y, Z. If $Y \to Z$ is an affine, G'-equivariant morphism and $Z \to Z /\!\!/ G'$ is a good quotient, then Y also has a good quotient $Y \to Y /\!\!/ G'$ and the induced morphism $Y /\!\!/ G' \to Z /\!\!/ G'$ is affine.

First assume that $U \in \mathcal{U}_{(X,G)}$, so we have a good quotient $U \to U /\!\!/ G = U /\!\!/ \widehat{G}$. The map $\pi|_{\pi^{-1}(U)}$ is $\widehat{G}$-equivariant and affine. Then by the result above, $\pi^{-1}(U)$ has a good quotient $\pi^{-1}(U) /\!\!/ \widehat{G}$, which maps to $U /\!\!/ G$ via a map ψ. We want to show ψ is an isomorphism, so we construct an inverse φ. The H-invariant map $\pi^{-1}(U) \to \pi^{-1}(U) /\!\!/ \widehat{G}$ factors uniquely through a map $\varphi' : U \to \pi^{-1}(U) /\!\!/ \widehat{G}$ as π is a categorical H-quotient and φ' is G-invariant. But as $U \to U /\!\!/ G$ is a quotient for the G-action on U, the map φ' factors uniquely through some map $\varphi : U /\!\!/ G \to \pi^{-1}(U) /\!\!/ \widehat{G}$. We can write the following commutative diagram

$$
\begin{array}{ccccccc}
\pi^{-1}(U) & \xrightarrow{\ \pi\ } & U & \longleftarrow & \pi^{-1}(U) & \longrightarrow & U \\
\downarrow & {\scriptstyle \varphi'}\nearrow & \downarrow & & \downarrow & & \downarrow \\
\pi^{-1}(U)/\!\!/\widehat{G} & \xleftarrow{\varphi} & U/\!\!/G & \xleftarrow{\psi} & \pi^{-1}(U)/\!\!/\widehat{G} & \xleftarrow{\varphi} & U/\!\!/G
\end{array}
$$

Via diagram chase and using that good quotients of reduced schemes are epimorphisms (Lemma 2), we conclude that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both the identity on their domains.

If $U \to U /\!\!/ G$ and π are geometric quotients, then the preimage of some geometric point $p \in U /\!\!/ G$ in U is a G-orbit and thus its preimage in $\pi^{-1}(U)$ is a $\widehat{G}$-orbit, hence $\pi^{-1}(U) \to U /\!\!/ G$ is a geometric quotient.

Now assume that $\pi^{-1}(U)$ has a good quotient map $\pi^{-1}(U) \to \pi^{-1}(U) /\!\!/ \widehat{G}$. For the trivial H-action on the latter space, this is a H-equivariant affine map and clearly the identity on $\pi^{-1}(U) /\!\!/ \widehat{G}$ is a good quotient for the trivial H-action. Thus by the result from Ramanathan (1996), the H-action on $\pi^{-1}(U)$ has a good quotient (which is isomorphic to U, as π is a good H-quotient) and the map $\psi : U \to \pi^{-1}(U) /\!\!/ \widehat{G}$ is affine. To show that it is a good quotient, we consider the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \pi\ } & U \\ \big\downarrow & \swarrow{\scriptstyle\psi} & \\ \pi^{-1}(U)/\!\!/\widehat{G} & & \end{array}$$

and use that π is an epimorphism to show that ψ is surjective, G-invariant and sends disjoint closed G-invariant sets to disjoint closed sets. Given a G-invariant local function f on U, the function $f \circ \pi$ is $\widehat{G}$-invariant, so it factors uniquely through some function g on $\pi^{-1}(U)/\!\!/\widehat{G}$. Again using that π is an epimorphism, we see $f = g \circ \psi$, so indeed f factors through $\pi^{-1}(U)/\!\!/\widehat{G}$. Thus ψ is a good G-quotient.

If $\pi^{-1}(U) \to \pi^{-1}(U)/\!\!/\widehat{G}$ is a geometric quotient, its geometric fibres are orbits of $\widehat{G}$, so the fibres in U are G-orbits and thus ψ is a geometric quotient. $\qquad\square$

Instead of looking at the correspondence of open sets admitting a good quotient induced by H-lifts, we can also directly consider the behaviour of equivariant Picard groups and the corresponding (semi)stable sets. Here we have the following result.

Proposition 3 *Let $\pi : \widehat{X} \to X$ be a good H-lift of (X, G) for X a variety with an action of the reductive group G. Then pullback by π induces an map $\pi^* : Pic^G(X) \to Pic^{\widehat{G}}(\widehat{X})$ and we have*

$$\widehat{X}^{ss}(\pi^* L) = \pi^{-1}(X^{ss}(L))$$

for $L \in Pic^G(X)$. If π is a geometric H-lift and H acts freely on $\widehat{X}$, the map π^* is an isomorphism.

Proof Via the map $\widehat{G} \to G = \widehat{G}/H$ we have a natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^{\widehat{G}}(X)$ by extending a G-action on a line bundle L to a $\widehat{G}$-action. For the $\widehat{G}$-equivariant morphism π we then have a natural pullback map $\mathrm{Pic}^{\widehat{G}}(X) \to \mathrm{Pic}^{\widehat{G}}(\widehat{X})$ and the map π^* above is the composition of these two homomorphisms.

Fix L (the total space of) a G-linearized line bundle on X and let $p : L \to X$ be the corresponding G-equivariant morphism. Then we have a cartesian diagram

$$\begin{array}{ccc} \pi^*(L) & \longrightarrow & L \\ {\scriptstyle\widehat{p}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \widehat{X} & \xrightarrow{\ \pi\ } & X \end{array}$$

where all maps are $\widehat{G}$-equivariant. Now G-invariant global sections of L are G-equivariant sections of p and those correspond bijectively to $\widehat{G}$-equivariant sections of $\widehat{p}$, i.e. global sections of $\pi^*(L)$. Here we use that π is a categorical H-quotient. Thus π^* induces a natural isomorphism $\Gamma(L)^G \cong \Gamma(\pi^*(L))^{\widehat{G}}$. Of course this argument also works after replacing L by $L^{\otimes k}$ for $k \geq 1$.

Now let $x \in X^{ss}(L)$, then there exists a G-invariant section s of some $L^{\otimes k}$ with $x \in X_s = \{x' : s(x') \neq 0\}$ and X_s is affine. But then $\pi^* s$ is a $\widehat{G}$-invariant section of $\pi^* L^{\otimes k}$ and $\widehat{X}_{\pi^* s} = \pi^{-1}(X_s)$ is affine as π is an affine morphism. Hence all elements of $\pi^{-1}(x)$ are $\pi^*(L)$-semistable.

Conversely for $\widehat{x} \in \widehat{X}^{ss}(\pi^*L)$ there exists a $\widehat{G}$-invariant section $\widehat{s}$ of some $\pi^*L^{\otimes k}$ with $\widehat{x} \in \widehat{X}_{\widehat{s}}$ and $\widehat{X}_{\widehat{s}}$ affine. By the argument above, $\widehat{s} = \pi^*s$ for some G-invariant section s of $L^{\otimes k}$ and we only need to show X_s affine. But clearly $\widehat{X}_{\widehat{s}} \to X_s$ is a categorical quotient of the affine variety $\widehat{X}_{\widehat{s}}$ by H and thus X_s is affine by (Mumford et al. 1994, Theorem 1.1). Hence $\pi(\widehat{x})$ is L-semistable.

If the action of H is free on $\widehat{X}$, by Mumford et al. (1994, Proposition 0.9) the map π is a fppf-locally trivial H-torsor. The fact that π^* is an isomorphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}^{\widehat{G}}(\widehat{X})$ then follows from descent along torsors. A concise way to put the proof, using the language of stacks, is the following: the fact that π is a H-torsor implies that there is a canonical isomorphism $X \cong [\widehat{X}/H]$. Taking the quotient stack under the actions of $G = \widehat{G}/H$ on both sides we have

$$[X/G] \cong [[\widehat{X}/H]/(\widehat{G}/H)] \cong [\widehat{X}/\widehat{G}],$$

where in the last isomorphism we use Romagny (2005, Remark 2.4). Taking Picard groups on both sides we see

$$\mathrm{Pic}^G(X) = \mathrm{Pic}([X/G]) \cong \mathrm{Pic}([\widehat{X}/\widehat{G}]) = \mathrm{Pic}^{\widehat{G}}(\widehat{X})$$

and this isomorphism is exactly given by pullback via π. □

In the example presented in Sect. 5, all H-lifts that are used will come from a free H-action on $\widehat{X}$, so we have isomorphisms of Picard groups as above.

4 Applications

In this section we will see several situations, where H-lifts naturally appear and thus allow us to conclude results about the chamber-decompositions of G-effective cones.

4.1 Partial quotients

One possibility to construct H-lifts is basically a reformulation of the definition.

Proposition 4 *Let $\widehat{X}$ be a variety acted upon by a reductive group $\widehat{G}$ and assume a closed, normal subgroup $H \subset \widehat{G}$ acts on $\widehat{X}$ with a good (resp. geometric) quotient $\pi : \widehat{X} \to X$, where X is a variety. Then X carries an induced action of $\widehat{G}/H$ making $(\widehat{X}, \widehat{G})$ a good (resp. geometric) H-lift of $(X, \widehat{G}/H)$.*

Combined with Corollary 1, this tells us the following: assume we are given a $\widehat{G}$-action on $\widehat{X}$ and a closed normal subgroup H of $\widehat{G}$ acting on the open, $\widehat{G}$-invariant set $U_0 \subset \widehat{X}$ with geometric quotient $U_0 /\!\!/ H$. Then the open sets $U \in \mathcal{U}^{\mathrm{pr,g}}_{(\widehat{X},\widehat{G})}$ contained in U_0 are in bijection with $\mathcal{U}^{\mathrm{pr,g}}_{(U_0/\!\!/H,\widehat{G}/H)}$, which is a problem on a smaller-dimensional variety.

4.2 Morphisms to homogeneous spaces

Proposition 5 *Let a reductive group G act on an irreducible variety X and assume we are given a G-equivariant morphism $\varphi : X \to Z$ to a homogeneous G-space Z (i.e. the action of G on Z is transitive). Let $z_0 \in Z$ be a closed point and let $H = G_{z_0}$ be its stabilizer in G, which we assume to be reductive. Consider the variety*

$$\widehat{X} = \{(g, x) : \varphi(gx) = z_0\} \subset G \times X$$

with the action of $G \times H$ given by

$$(g', h).(g, x) = (hg(g')^{-1}, g'x).$$

Then the projection

$$\pi_X : \widehat{X} \to X, (g, x) \mapsto x$$

makes $(\widehat{X}, G \times H)$ a geometric H-lift for (X, G). On the other hand, for $Y = \pi^{-1}(z_0) \subset X$ with the induced action of $H = G_{z_0}$, the map

$$\pi_Y : \widehat{X} \to Y, (g, x) \mapsto gx$$

makes $(\widehat{X}, G \times H)$ a geometric G-lift of (Y, H).

Proof By definition of Y and $\widehat{X}$, we have cartesian diagrams

$$
\begin{array}{ccccc}
\widehat{X} & \xrightarrow{\pi_Y} & Y & \longrightarrow & \{z_0\} \\
\downarrow & & \downarrow & & \downarrow \\
G \times X & \xrightarrow{\sigma} & X & \xrightarrow{\varphi} & Z
\end{array}
\tag{4}
$$

where σ is the action map of G on X. As we are in characteristic zero and as $G \times X$ and X are irreducible, the fibres of the generic point η_Z of Z under φ and $\varphi \circ \sigma$ are geometrically reduced. Hence by Grothendieck (1966, Theorem 9.7.7), the set of closed points in Z whose fibre under φ and $\varphi \circ \sigma$ is geometrically reduced is open and nonempty. But because the G-action on Z is transitive, all these fibres are isomorphic to Y and $\widehat{X}$, respectively. Thus these are varieties over $\mathbb{C}$.

From the formula for the action of $G \times H$ on $\widehat{X}$, it is clear that the maps π_X, π_Y are $G \times H$-equivariant for the induced actions of $G = G \times H/H$ on X and $H = G \times H/G$ on Y.

For the map π_X, observe that we can obtain it using a different cartesian diagram, namely

$$
\begin{array}{ccc}
\widehat{X} & \longrightarrow & G \\
\downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \psi} \\
X & \xrightarrow{\varphi} & Z
\end{array}
$$

where $\psi(g) = g^{-1}z_0$. Clearly ψ is a fpqc-locally trivial H-torsor representing Z as the quotient G/H. But then its base change π_X via φ is still a fpqc-locally trivial H-torsor and thus a geometric quotient.

On the other hand, for π_Y we see from the diagram (4) that it is a base change of the map σ, which clearly is a (trivial) G-torsor (using the automorphism $(g, x) \mapsto (g, g^{-1}x)$ of $G \times X$). Thus it is a (trivial) G-torsor itself and hence a geometric quotient. □

Using Theorem 1 we see that given a G-action on a variety X and a subset Y of X obtained as the fibre of an equivariant map to a G-homogeneous space, we have a bijection between $\mathcal{U}_{(X,G)}$ and $\mathcal{U}_{(Y,G_Y)}$, where G_Y is the subgroup of G leaving Y stable.

Note that in Białynicki-Birula (2002, Definition 15.1) a subvariety $Y \subset X$ such that for all $y \in Y$ we have $H = \{g \in G : gy \in Y\}$ is called a strong H-section of X. If X is normal, (Białynicki-Birula 2002, Lemma 15.2) says that for such a Y the morphism $G \times_H Y \to X$ given by $[(g, y)] \mapsto gy$ is a G-isomorphism. Using this isomorphism, we have a G-equivariant projection $X \cong G \times_H Y \to G/H$ with fibre Y over $[e] \in G/H$, placing us in the situation of Proposition 5. Again it has been noted before that Y has a good/geometric H-quotient iff X has a good/geometric G-quotient (Białynicki-Birula 2002, Corollary 15.3).

5 Examples

5.1 Diagonal action of PGL_2 on $(\mathbb{P}^1)^n$

To illustrate how the techniques above can be used in practice, consider the diagonal action of $G = \mathrm{PGL}_2$ on $X = (\mathbb{P}^1)^n$. We demonstrate how some of the chambers of the G-effective cone can be related to chambers of the cone C_β for a linear action of $(\mathbb{C}^*)^{n-1}$ on $\mathbb{C}^{2n-4}$. Here we can compute the chamber decomposition as well as the resulting quotient varieties using the toric methods we recalled in Sect. 2.2.

The action of G on X has been intensely studied in the past (Kirwan et al. 1987; Kapranov 1993; Mumford et al. 1994; Polito 1995; Hassett 2003). The line bundle $\mathcal{O}(a_1, \ldots, a_n)$ on X carries a (unique) G-linearization iff the sum of the a_i is even, so

$$\mathrm{Pic}^G(X) \cong \left\{ (a_1, \ldots, a_n) \in \mathbb{Z}^n : \sum_{i=1}^n a_i \equiv 0 \bmod 2 \right\} \subset \mathbb{Z}^n$$

and the effective cone is given by

$$(\mathbb{R}_{\geq 0})^n \subset \mathbb{R}^n = \mathrm{Pic}^G(X)_{\mathbb{R}}.$$

We can analyze (semi)stability with respect to a given polarization using the Hilbert-Mumford numerical criterion. For $a = (a_1, \ldots, a_n) \in (\mathbb{Z}_{>0})^n$ with $|a| = \sum_{i=1}^n a_i$ even, a point $p = (p_1, \ldots, p_n) \in X$ is semistable with respect to $\mathcal{O}(a_1, \ldots, a_n)$ iff for all $p \in \mathbb{P}^1$ we have $\sum_{i:p_i=p} a_i \leq |a|/2$. The point p is stable iff all inequalities

above are strict. From this we see that the G-effective ample cone is given by

$$C^G(X) = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n : 0 < a_i \leq \sum_{j \neq i} a_j \right\} \subset \mathbb{R}^n.$$

The criterion above also gives an explicit identification of the VGIT-chamber structure of $C^G(X)$. For $S \subset \{1, \ldots, n\}$ consider the half-space

$$H_S = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n : \sum_{i \in S} a_i \geq \sum_{i \in \{1,\ldots,n\} \setminus S} a_i \right\} \subset \mathbb{R}^n.$$

Then the hyperplanes corresponding to the half-spaces above divide $C^G(X)$ into connected components, which are exactly the chambers of the VGIT-decomposition as in Sect. 2.1.

Though it is easy to determine the various chambers, it is more difficult to compute the quotients associated to them. In Kirwan et al. (1987), some of the quotients are computed for $n = 5, 6, 7, 8$. Using the techniques from the previous sections, we are able to compute these quotients for chambers contained in certain subcones of $C^G(X)$.

For the notation below, recall from Sect. 2.2 that given an action of a torus T on $\mathbb{C}^k$, the set of linearizations of the action is given by the character group $\widehat{T}$. Inside $\widehat{T} \otimes \mathbb{R}$ we have a fan Σ_{GKZ} such that the set of χ-semistable points in $\mathbb{C}^k$ is constant as the linearization χ varies in the relative interior of the cones of Σ_{GKZ}, which are denoted by $\Gamma_{\Sigma, I_\emptyset}$.

Theorem 2 *Let $S \subset \{1, \ldots, n\}$ with $|S| = 2$, then the chambers σ of $C^G(X)$ contained in H_S are in bijective correspondence to the chambers $\Gamma_{\Sigma, I_\emptyset} \in \Sigma_{\text{GKZ}}$ for the action of $T = (\mathbb{C}^*)^{n-1}$ on $\mathbb{C}^{2n-4}$ given by*

$$\begin{aligned}
(t_1, \ldots, &t_{n-2}, s).(x_1, y_1, x_2, y_2, \ldots, x_{n-2}, y_{n-2}) \\
&= (t_1 x_1, s t_1 y_1, t_2 x_2, s t_2 y_2, \ldots, t_{n-2} x_{n-2}, s t_{n-2} y_{n-2}).
\end{aligned} \tag{5}$$

Under this correspondence, the quotient variety associated to σ is the toric variety associated to the fan Σ.

Proof Assume for simplicity of notation $S = \{1, 2\}$ below. As X is smooth and as every effective divisor is semiample on X, by Proposition 1 the chambers of $C^G(X)$ are in bijection with the open sets in $\mathcal{U}_{(X,G)}^{\text{pr,g}}$ by sending σ to X_σ^{ss}. Now for any $a \in C^G(X) \cap \text{int}(H_S)$ and $p \in X$ semistable with respect to a we know $p_1 \neq p_2$ by the Hilbert–Mumford criterion. Thus under the correspondence above, the chambers contained in H_S correspond to the subset

$$\mathcal{U}_{((\mathbb{P}^1)^n \setminus \Delta_{12}, G)}^{\text{pr,g}} \subset \mathcal{U}_{(X,G)}^{\text{pr,g}},$$

where $\Delta_{12} = \{(p_1, \ldots, p_n) : p_1 = p_2\}$. So the projection $\varphi : (\mathbb{P}^1)^n \setminus \Delta_{12} \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta = Z$ to the first two factors is a G-equivariant morphism to the homogeneous

G-space Z. For $z_0 = ([0 : 1], [1 : 0]) \in Z$, the stabilizer G_{z_0} is exactly the diagonal torus

$$H = \mathbb{C}^* = \left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right] : a \in \mathbb{C}^* \right\} \subset \mathrm{PGL}_2.$$

By Proposition 5 we obtain a variety $\widehat{X}$ with an action of $G \times H$ which is a geometric H-lift for (X, G) and a geometric G-lift for $(\varphi^{-1}(z_0), H) = ((\mathbb{P}^1)^{n-2}, \mathbb{C}^*)$. Here the action of $\mathbb{C}^*$ on $(\mathbb{P}^1)^{n-2}$ is given by

$$a.([x_1, y_1], \dots, [x_{n-2}, y_{n-2}]) = ([x_1, ay_1], \dots, [x_{n-2}, ay_{n-2}]).$$

By Theorem 1 the geometric H and G-lifts above give a natural bijection

$$\mathcal{U}^{\mathrm{pr,g}}_{((\mathbb{P}^1)^n \setminus \Delta_{12}, G)} = \mathcal{U}^{\mathrm{pr,g}}_{((\mathbb{P}^1)^{n-2}, \mathbb{C}^*)}.$$

We now approach the pair $((\mathbb{P}^1)^{n-2}, \mathbb{C}^*)$ from a different angle. Of course the space $(\mathbb{P}^1)^{n-2}$ is a geometric quotient of $(\mathbb{C}^2 \setminus \{0\})^{n-2}$ by $(\mathbb{C}^*)^{n-2}$ via the action

$$(t_1, \dots, t_{n-2}).(x_1, y_1, \dots, x_{n-2}, y_{n-2})$$
$$= (t_1 x_1, t_1 y_1, t_2 x_2, t_2 y_2, \dots, t_{n-2} x_{n-2}, t_{n-2} y_{n-2}).$$

The action of $\mathbb{C}^*$ on $(\mathbb{P}^1)^{n-2}$ lifts to a linear action on the prequotient $\mathbb{C}^{2n-4}$, which commutes with the action of $(\mathbb{C}^*)^{n-2}$ above and together they determine the action of $(\mathbb{C}^*)^{n-1}$ given in (5). By Proposition 2 the chambers of the secondary fan Σ_{GKZ} for this toric action correspond to elements of $\mathcal{U}^{\mathrm{pr,g}}_{(\mathbb{C}^{2n-4}, (\mathbb{C}^*)^{n-1})}$ by sending a chamber $\Gamma_{\Sigma, I_\emptyset}$ to $(\mathbb{C}^{2n-4})^{ss}_\chi = (\mathbb{C}^{2n-4})^s_\chi$ for any χ contained in this chamber.

However, for the action above no point $(x_1, y_1, \dots, x_n, y_n)$ with $x_i = y_i = 0$ for some i can be stable (with respect to any character) as it has nonfinite stabilizer. Thus we have

$$\mathcal{U}^{\mathrm{pr,g}}_{(\mathbb{C}^{2n-4}, (\mathbb{C}^*)^{n-1})} = \mathcal{U}^{\mathrm{pr,g}}_{((\mathbb{C}^2 \setminus \{0\})^{n-2}, (\mathbb{C}^*)^{n-1})}.$$
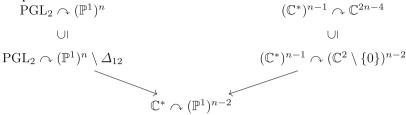
Using Proposition 4 the space $((\mathbb{C}^2 \setminus \{0\})^{n-2}, (\mathbb{C}^*)^{n-1})$ is a $(\mathbb{C}^*)^{n-2}$-lift of $((\mathbb{P}^1)^{n-2}, \mathbb{C}^*)$ as above, so we can identify

$$\mathcal{U}^{\mathrm{pr,g}}_{((\mathbb{P}^1)^{n-2}, \mathbb{C}^*)} = \mathcal{U}^{\mathrm{pr,g}}_{((\mathbb{C}^2 \setminus \{0\})^{n-2}, (\mathbb{C}^*)^{n-1})}.$$

Combining the correspondences above (see also the diagram in Remark 3) we have proved the claim.   □

Remark 3 In the situation of Theorem 2 we can not only relate chambers of the G-effective cones for $((\mathbb{P}^1)^n, \mathrm{PGL}_2)$ and $(\mathbb{C}^{2n-4}, (\mathbb{C}^*)^{n-1})$ abstractly but we can actually find a linear map between the equivariant Picard groups inducing this correspondence. Recall from Proposition 3 that for $\pi : \widehat{X} \to X$ a geometric H-lift with respect to a

free H-action, the pullback by π induces an isomorphism $\pi^* : \mathrm{Pic}^G(X) \to \mathrm{Pic}^{\widehat{G}}(\widehat{X})$ with $\pi^{-1}(X^{ss}(L)) = \widehat{X}^{ss}(\pi^*L)$ for $L \in \mathrm{Pic}^G(X)$. We illustrate again the course of the proof of Theorem 2.

$$\mathrm{PGL}_2 \curvearrowright (\mathbb{P}^1)^n \qquad\qquad (\mathbb{C}^*)^{n-1} \curvearrowright \mathbb{C}^{2n-4}$$

$$\cup| \qquad\qquad\qquad\qquad \cup|$$

$$\mathrm{PGL}_2 \curvearrowright (\mathbb{P}^1)^n \setminus \Delta_{12} \qquad\qquad (\mathbb{C}^*)^{n-1} \curvearrowright (\mathbb{C}^2 \setminus \{0\})^{n-2}$$

$$\mathbb{C}^* \curvearrowright (\mathbb{P}^1)^{n-2}$$

Both arrows at the bottom are (compositions of) geometric H-lifts for free H-actions, so they induce isomorphisms of equivariant Picard groups compatible with forming semistable sets. We have to see how the two inclusions at the top behave in this respect.

The inclusion $(\mathbb{C}^2 \setminus \{0\})^{n-2} \subset \mathbb{C}^{2n-4}$ has complement of codimension 2, so it induces isomorphisms of (equivariant) Picard groups and (invariant) sections of line bundles. Also the complement of the inclusion above consists of points with nonfinite stabilizers. So for every linearization on $\mathbb{C}^{2n-4}$ such that stable and semistable points agree, these sets are anyway contained in $(\mathbb{C}^2 \setminus \{0\})^{n-2}$. Thus on the interior of the chambers in $\mathrm{Pic}^{(\mathbb{C}^*)^{n-1}}(\mathbb{C}^{2n-4})_\mathbb{Q}$ the isomorphism above respects the formation of (semi)stable points. Note that this is not true for all linearizations: for the trivial linearization all of $\mathbb{C}^{2n-4}$ is semistable, but on $(\mathbb{C}^2 \setminus \{0\})^{n-2}$ the trivial linearization has no semistable points (as this variety is not affine).

For the other inclusion $i : (\mathbb{P}^1)^n \setminus \Delta_{12} \hookrightarrow (\mathbb{P}^1)^n$ we have

$$\mathrm{Pic}^G((\mathbb{P}^1)^n \setminus \Delta_{12})_\mathbb{Q} = \mathrm{Pic}^G((\mathbb{P}^1)^n)_\mathbb{Q}/\mathbb{Q}\mathcal{O}(1, 1, 0, \dots, 0)$$

and i^* is the corresponding quotient map. For any G-linearized line bundle L' on $(\mathbb{P}^1)^n \setminus \Delta_{12}$, which is the restriction of a bundle L on $(\mathbb{P}^1)^n$, any invariant section s' of $(L')^{\otimes k}$ extends to a section s of $(L \otimes \mathcal{O}(m, m, 0, \dots, 0))^{\otimes k}$ vanishing on Δ_{12} for $m \gg 0$ (take m greater than the order of the rational section s of $L^{\otimes k}$ along Δ_{12}). Conversely, for $L = \mathcal{O}(a_1, a_2, \dots, a_n)$ on $(\mathbb{P}^1)^n$ with $a_1 + a_2 > a_3 + \dots + a_n$ we consider again the Hilbert–Mumford criterion from above. For $S \subset \{1, \dots, n\}$ and $\Sigma_S(\mathbf{a}) = \Sigma_{s \in S} a_s - \Sigma_{s \notin S} a_s$ we have

- $\Sigma_S(\mathbf{a}) > 0$ for $1, 2 \in S$,
- $\Sigma_S(\mathbf{a}) < 0$ for $1, 2 \notin S$,
- $\Sigma_S(\mathbf{a}) = \Sigma_S(\mathbf{a} + (m, m, 0, \dots, 0))$ for $1 \in S, 2 \notin S$ or $1 \notin S, 2 \in S$.

So we see that twisting L by $\mathcal{O}(m, m, 0, \dots, 0)$ does not change the set of semistable points. In fact this shows that all cones of the VGIT-fan in $\mathrm{Pic}^G((\mathbb{P}^1)^n)_\mathbb{R}$ with relative interior strictly inside the interior of the half-space $H_{\{1,2\}} = \{a : \Sigma_{\{1,2\}} \geq 0\}$ have the cone generated by $\mathcal{O}(1, 1, 0, \dots, 0)$ in their closure and thus as a face. Moreover, the set of semistable points of L is contained in $(\mathbb{P}^1)^n \setminus \Delta_{12}$. All this shows that for any $L \in \mathrm{Pic}^G((\mathbb{P}^1)^n)$ and $L' = i^*L$ its restriction to $(\mathbb{P}^1)^n \setminus \Delta_{12}$, we have

$$((\mathbb{P}^1)^n \setminus \Delta_{12})^{ss}_{L'} = ((\mathbb{P}^1)^n)^{ss}_{L \otimes \mathcal{O}(m,m,0,\dots,0)}$$

for $m \gg 0$. To conclude, inside $\text{Pic}^G((\mathbb{P}^1)^n)_{\mathbb{R}}$ we have the subfan of the VGIT-fan contained in $H_{\{1,2\}}$. Via the map i^* it maps to its quotient fan by the ray $\text{Cone}(\mathcal{O}(1, 1, 0, \ldots, 0))$. Moreover, on the relative interior of the cones in the quotient fan, the set of semistable points is constant and equal to the semistable points on the cone in the preimage containing $\mathcal{O}(1, 1, 0, \ldots, 0)$.

Remark 4 The linear action of $T = (\mathbb{C}^*)^{n-1}$ on $\mathbb{C}^{2n-4}$ that arises above has been studied in the Master thesis of the author. It arises as the canonical representation of the toric variety $\text{Bl}_{n-2}\mathbb{P}^{n-3}$ as a torus quotient of affine space with respect to a symmetric linearization (i.e. the character $(1, \ldots, 1)$ of T). In the thesis a family of chambers of the secondary fan together with their quotients is explicitly identified. The quotients occurring in this family are iterated projective $\mathbb{P}^1$-bundles over some $\text{Bl}_k\mathbb{P}^{n'-3}$ ($k \leq n' - 2$).

5.2 The Gelfand–MacPherson correspondence

The classical Gelfand–MacPherson correspondence (see Gelfand and MacPherson 1982; Kapranov 1993) is a relation between orbit spaces for the natural actions

$$\text{PGL}_k \curvearrowright (\mathbb{P}^{k-1})^n \quad \text{and} \quad (\mathbb{C}^*)^n \curvearrowright G(k, n),$$

where $2 \leq k \leq n$ and $G(k, n)$ is the Grassmannian of k-planes in $\mathbb{C}^n$.

To see how this fits into our framework let $\text{Mat}(k, n)$ be the space of $k \times n$-matrices over $\mathbb{C}$. It carries an action of $\text{GL}_k \times (\mathbb{C}^*)^{n-1}$, where GL_k acts by multiplication from the left and $(\mathbb{C}^*)^{n-1}$ acts by scaling of the columns $2, 3, \ldots, n$ of the matrices. We denote by $\Delta \subset \text{GL}_k$ the diagonal one-parameter subgroup. Let $\text{Mat}'(k, n)$ be the set of matrices such that no column is equal to zero and let $\text{Mat}_0(k, n)$ be the set of matrices of rank k.

Then we have the following diagram

$$\text{GL}_k \times (\mathbb{C}^*)^{n-1} \curvearrowright \text{Mat}'(k, n) \subseteq \text{Mat}(k, n) \supseteq \text{Mat}_0(k, n) \curvearrowleft \text{GL}_k \times (\mathbb{C}^*)^{n-1}$$

$$\Big\downarrow /\!\!/ \Delta \times (\mathbb{C}^*)^{n-1} \qquad\qquad\qquad \Big\downarrow /\!\!/ \text{GL}_k$$

$$\text{PGL}_k \quad \curvearrowright \quad (\mathbb{P}^{k-1})^n \qquad\qquad\qquad G(k, n) \quad \curvearrowleft \quad (\mathbb{C}^*)^{n-1}$$

where the vertical arrows are geometric quotients and in fact geometric H-lifts for $H = \Delta \times (\mathbb{C}^*)^{n-1}, \text{GL}_k$. Moreover, these groups H clearly act freely on $\text{Mat}'(k, n)$, $\text{Mat}_0(k, n)$, respectively. Applying Proposition 3 this means we have isomorphisms

$$\text{Pic}^{\text{PGL}_k}((\mathbb{P}^{k-1})^n) \cong \text{Pic}^{\text{GL}_k \times (\mathbb{C}^*)^{n-1}}(\text{Mat}'(k, n)),$$

$$\text{Pic}^{(\mathbb{C}^*)^{n-1}}(G(k, n)) \cong \text{Pic}^{\text{GL}_k \times (\mathbb{C}^*)^{n-1}}(\text{Mat}_0(k, n)),$$

compatible with the formation of semistable sets.

Finally, since $\text{Mat}'(k, n), \text{Mat}_0(k, n)$ both have complement of codimension at least 2 in $\text{Mat}(k, n)$, both groups on the right are canonically isomorphic to $\text{Pic}^{\text{GL}_k \times (\mathbb{C}^*)^{n-1}}(\text{Mat}_0(k, n))$.

Again we see that if for some $\mathrm{GL}_k \times (\mathbb{C}^*)^{n-1}$-linearized line bundle $\mathcal{L}$ the stable and semistable points of $\mathrm{Mat}(k, n)$ coincide, all such points must have finite stabilizer. But this easily implies $\mathrm{Mat}(k, n)^{ss, \mathcal{L}} \subset \mathrm{Mat}'(k, n) \cap \mathrm{Mat}_0(k, n)$. Thus in particular using Theorem 1 we have

$$(\mathbb{P}^{k-1})^n /\!\!/_\mathcal{L} \mathrm{PGL}_k \cong G(k, n) /\!\!/_\mathcal{L} (\mathbb{C}^*)^{n-1}.$$

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