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Triconnected Planar Graphs of Maximum Degree Five are Subhamiltonian*

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Abstract

We show that every triconnected planar graph of maximum degree five is subhamiltonian planar. A graph is subhamiltonian planar if it is a subgraph of a Hamiltonian planar graph or, equivalently, if it admits a 2-page book embedding. In fact, our result is stronger because we only require vertices of a separating triangle to have degree at most five, all other vertices may have arbitrary degree. This degree bound is tight: We describe a family of triconnected planar graphs that are not subhamiltonian planar and where every vertex of a separating triangle has degree at most six. Our results improve earlier work by Heath and by Bauernöppel and, independently, Bekos, Gronemann, and Raftopoulou, who showed that planar graphs of maximum degree three and four, respectively, are subhamiltonian planar. The proof is constructive and yields a quadratic time algorithm to obtain a subhamiltonian plane cycle for a given graph.

As one of our main tools, which might be of independent interest, we devise an algorithm that, in a given 3-connected plane graph satisfying the above degree bounds, collapses each maximal separating triangle into a single edge such that the resulting graph is biconnected, contains no separating triangle, and no separation pair whose vertices are adjacent.

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1 Introduction

The structure of spanning a.k.a. Hamiltonian paths and cycles in graphs has been a fruitful subject of intense research over many decades, both from a combinatorial and from an algorithmic point of view. For general graphs, sufficient conditions for the existence of a Hamiltonian cycle typically involve rather strong assumptions on the degree, such as in

* Due to space constraints, some proofs in this extended abstract are only sketched or omitted entirely.

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Dirac's Theorem [9] (minimum degree $\geq n/2$), Ore's Theorem [16] (degree sum of every nonadjacent vertex pair $\geq n$), or Asratian and Khachatrian's Theorem [1] ($\deg(u)+\deg(w) = |\mathrm{N}(u)+\mathrm{N}(v)+\mathrm{N}(w)|$, for every induced path uvw). Planar graphs provide a lot more structure so that by a famous theorem of Tutte, 4-connectivity suffices to guarantee the existence of a Hamiltonian cycle [19], which can be computed in linear time [7]. In contrast, deciding Hamiltonicity is NP-complete for 3-connected cubic planar graphs [11]. Finally, *maximal* planar graphs of degree at most six are Hamiltonian [10]. We observe that both vertex degree and connectivity are crucial parameters concerning Hamiltonicity.

Hamiltonian cycles are also of interest in the context of graph embeddings. Specifically, in a *book embedding*, all vertices are embedded on a line called *spine*, and every edge is embedded in a halfplane, called *page*, bounded by the spine. No two edges (on the same page) cross. If k pages are used, then the corresponding embedding is a k-page book embedding. Note that a k-page book embedding with $k \leq 2$ is plane. Bernhart and Kainen [4] characterized those graphs that can be embedded on k pages, for $k \leq 2$. For $k = 2$ these are the subhamiltonian planar[1] graphs, that is, subgraphs of Hamiltonian planar graphs, cf. Figure 1a. Hence not all planar graphs can be embedded on two pages; in fact, it is NP-complete to decide whether a given planar graph is subhamiltonian [20], even if all vertices have degree at most seven [2]. However, no planar graph is too far away from being subhamiltonian: Subdividing at most $n/2$ of the up to $3n-6$ edges of a planar graph on n vertices yields a subhamiltonian planar graph [6].

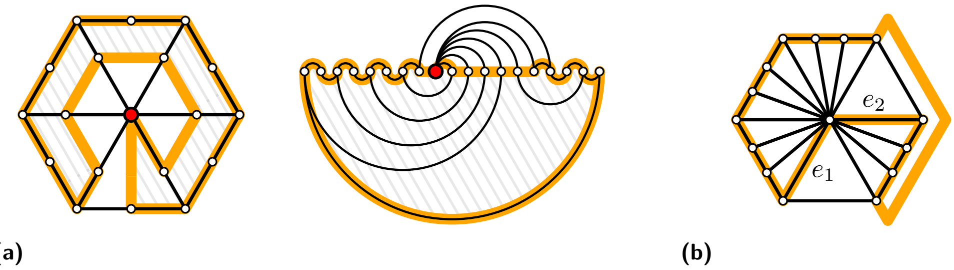


Figure 1 (a) A nonhamiltonian graph with a subhamiltonian cycle and a corresponding two-page book embedding. (b) A subhamiltonian cycle using e_1 and e_2 in a wheel with at least four spokes.

Despite a plethora of results concerned with Hamiltonian cycles in planar graphs and book embeddings, several fundamental questions are still open. Let us give just two prominent examples to illustrate this point. For once, there is Barnette's Conjecture: "Every 3-connected cubic bipartite planar graph is Hamiltonian." And then there is the question if every planar graph can be embedded on three pages. Yannakakis showed, improving a series of earlier results, that four pages are sufficient for every planar graph [22]. However, a corresponding lower bound is still elusive, in spite of initial claims [21].

In this paper, we investigate what upper bound on the vertex degree guarantees that a planar graph is subhamiltonian planar. Heath [14] showed that maximum degree three suffices. Later, Bauernöppel [2] and, independently, Bekos et al. [3] showed that every planar graph of degree at most four is subhamiltonian planar. On the negative side, Bauernöppel [2] described planar graphs of vertex degree at most seven that are not subhamiltonian (these graphs are biconnected but not 3-connected). In a preprint, Guan and Yang [12] show that planar graphs of maximum degree five can be embedded on three pages. Our goal is to further relax the degree bound so as to (ultimately) determine what is the largest k so that every

[1] We prefer the term "subhamiltonian planar" over just "subhamiltonian" because the former is more descriptive as a shortcut for "subgraph of a Hamiltonian planar graph".

planar graph of degree at most k is subhamiltonian planar. Specifically, we are interested in the case where the given graph is 3-connected and hence the combinatorial embedding is unique. Along these lines, we make a natural next step by considering the case of maximum degree five. In fact, we prove a much stronger statement where the degree restriction applies to vertices of separating 3-cycles only.

▶ **Theorem 1.** *Let G be a 3-connected simple planar graph on n vertices where every vertex of a separating 3-cycle has degree at most five. Then G is subhamiltonian planar. Moreover, a subhamiltonian plane cycle for G can be computed in $O(n^2)$ time.*

▶ **Corollary 2.** *Every 3-connected simple planar graph with maximum vertex degree five can be embedded on two pages, and such an embedding can be computed in quadratic time.*

We also show that the degree bound in Theorem 1 is tight.

▶ **Theorem 3.** *There exists a family of 3-connected simple planar graphs that are not subhamiltonian planar and where every vertex of a separating 3-cycle has degree at most six.*

Organization. We begin by introducing some terminology in Section 2. In Section 3, we study three special cases for which Theorem 1 is easily proven. In Section 4, we proceed with a high-level overview of the proof of Theorem 1, which is then developed in some more detail in Sections 5–7. We conclude with an example to illustrate Theorem 3 in Section 8.

2 Notation

All graphs in this paper are undirected. We denote by $\mathrm{V}(G)$ the vertex set and by $\mathrm{E}(G)$ the edge set of a graph G. For a set of edges $E \subseteq \mathrm{E}(G)$ we use $\mathrm{V}(E)$ to denote the set of vertices that are incident to at least one edge in E. For a face f of G we denote by ∂f the closed walk in G that traverses the vertices and edges on the boundary of f in anticlockwise direction. If G is biconnected, then ∂f is a cycle, for every face f of G. A *Hamiltonian* cycle for a graph is a simple cycle through all vertices and a graph is *Hamiltonian* if it contains a Hamiltonian cycle. An *augmentation* of a graph $G = (V, E)$ is a supergraph $A = (V, E')$ with $E' \supseteq E$. If G is a plane graph, then a *plane augmentation* H of G is an augmentation that respects the embedding of G, that is, if Γ denotes the embedding of G and Γ' denotes the embedding of H, then $\Gamma'|_G = \Gamma$. A *subhamiltonian cycle* for a graph G is a Hamiltonian cycle in some augmentation of G.

We distinguish between separating 3-cycles as a notion for both abstract and embedded graphs and separating triangles in embedded graphs. A *separating* 3*-cycle* is a 3-cycle whose removal disconnects the graph. A *separating triangle* is a 3-cycle C of an embedded graph G such that both the interior and the exterior region bounded by C contain some vertex of G.

For a cycle C in a plane graph G denote by G_C^+ the plane subgraph of G that contains all vertices and edges on C and exterior to C. Similarly denote by G_C^- the plane subgraph of G that contains all vertices and edges on C and interior to C. The *inside* of C refers to the interior of the bounded region enclosed by C. So a vertex of G *inside of* C is a vertex of $G_C^- \setminus C$. Analogously a vertex *outside of* C is a vertex of $G_C^+ \setminus C$. A separating triangle T is called *trivial* if $G_T^- \simeq K_4$ and *nontrivial*, otherwise.

Our algorithm uses a decomposition of the graph into its triconnected components, which can be efficiently maintained via the SPQR-tree data structure [13, 15]. We use the terminology by Gutwenger and Mutzel [13].

3 Three simple cases

It suffices to prove Theorem 1 for an arbitrary plane embedding of the given graph, which we assume to be represented as a doubly-connected edge list (DCEL) [8]. In fact, by 3-connectivity, the combinatorial plane embedding is unique up to the choice of the outer face, and there is no difference between separating 3-cycles and separating triangles. So let G be an embedded 3-connected simple planar graph (with a fixed outer face) where every vertex incident to a separating triangle has degree at most five. By combining known results, it is easy to deal with the case that G has no separating triangle:

▶ **Theorem 4.** *If an embedded simple planar graph $\mathcal{G} = (V, E)$ does not contain any separating triangle, then for any two distinct edges $e_1, e_2 \in E$ there is a plane augmentation $\mathcal{H}$ of $\mathcal{G}$ that contains a Hamiltonian cycle C using e_1 and e_2. Moreover, the cycle C can be computed in $O(n^2)$ time, where n is the number of vertices in $\mathcal{G}$. If both e_1 and e_2 are on the outer face of $\mathcal{G}$, then C can be computed in linear time.*

Proof Sketch. Augment $\mathcal{G}$ using an algorithm by Biedl, Kant, and Kaufmann [5] to obtain, if possible, a 4-connected planar graph, which contains the desired Hamiltonian cycle due to a theorem of Sanders [17, Corollary 2]. If such an augmentation is impossible, then the graph can be augmented to a wheel, in which the desired cycle is easily found; see Figure 1b.

Algorithmically, the bottleneck is Sanders' theorem. While the original formulation is purely existential, a recently developed algorithmic version by Schmid and Schmidt [18] allows to compute such a cycle in quadratic time. For the special case where both e_1 and e_2 are on the outer face, we can use the linear time algorithm by Chiba and Nishizeki [7]. ◀

In order to be able to argue inductively, we prove a stronger statement than necessary, namely a version of Theorem 1 where, similar to the statement in Theorem 4, two edges of the desired subhamiltonian cycle may be prescribed:

▶ **Theorem 5.** *Let $\mathcal{G} = (V, E)$ be an embedded 3-connected simple planar graph on n vertices with outer face $T_\circ$, where every vertex that is incident to a separating triangle has degree at most five. Further, let $F \subset E$ be a set of at most two edges so that*

(P1) *all edges in F are edges of $T_\circ$;*
(P2) *if $T_\circ$ is not a triangle, then $F = \emptyset$;*
(P3) *if $F \neq \emptyset$, then no vertex of $T_\circ$ is incident to a separating triangle of $\mathcal{G}$; and*
(P4) *if $F \neq \emptyset$, then either at least one vertex of $T_\circ$ has degree three in $\mathcal{G}$, or all vertices of $T_\circ$ have degree four in $\mathcal{G}$.*

Then there is a plane augmentation of $\mathcal{G}$ that contains a Hamiltonian cycle C that uses all edges from F. Moreover, the cycle C can be computed in $O(n^2)$ time.

The proof, sketched in Sections 5–7, is carried out by induction on the number of vertices. *In this context*, one can easily show that:

▶ **Lemma 6.** *Suppose that the statement of Theorem 5 holds for all graphs with at most $n-1$ vertices, where $n \geq 6$. Then it also holds for every graph on n vertices that contains at least one nontrivial separating triangle.*

Proof Sketch. Let T be a nontrivial separating triangle in $\mathcal{G}$. Replace the interior of T with a single vertex d and inductively find a subhamiltonian cycle C in the resulting smaller graph. Then, inductively obtain a subhamiltonian cycle C' for $\mathcal{G}_T^-$. For this second inductive call, we make use of the ability to prescribe edges, depending on how C visits d, in order to ensure that the cycles C and C' may be glued together to obtain a subhamiltonian cycle for $\mathcal{G}$. ◀

The degree restriction basically enforces that the separating triangles of G are pairwise vertex-disjoint. However, there are some exceptional configurations where two separating triangles share an edge. Our next goal is to classify these configurations precisely.

A *double kite* is a subgraph $U \simeq K_4$ of an embedded graph $\mathcal{G}$ so that exactly two of the four triangles of U are separating in $\mathcal{G}$. The two separating triangles are said to *define* the double kite. Note that $\mathcal{G}$ may contain multiple double kites, see Figure 2a. We refer $\mathcal{G}$ itself as a *trivial double kite* if it is 3-connected, contains a double kite, and has precisely 6 vertices.

▶ **Lemma 7.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let T_1, T_2 be two distinct separating triangles of $\mathcal{G}$ such that every vertex incident to T_1 has degree at most five. Then if T_1 and T_2 share a vertex, the triangles T_1 and T_2 define a double kite.*

▶ **Observation 8.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let T_1, T_2 be two distinct trivial separating triangles of $\mathcal{G}$ such that all vertices incident to T_1 or T_2 have degree at most five. Then if T_1 and T_2 share a vertex, the graph $\mathcal{G}$ is a trivial double kite.*

For a trivial double kite G, the statement of Theorem 5 is easily verified. Hence, from now on we may assume that the separating triangles of G are trivial and pairwise vertex-disjoint.

4 Proof overview

To prove Theorem 5 we proceed in three steps.

In the first step (Section 5), we destroy all separating triangles using edge collapses. An edge $e = ab$ of a separating triangle $T = abc$ is *collapsed* by contracting (the three edges of) the triangle that is spanned by the endpoints of e and the single vertex d inside T into a new vertex x and merging the two edges ac and bc to a new edge cx; see Figure 2b. Observe that an edge collapse can be performed in constant time in a DCEL.

(a) **(b)**

Figure 2 (a) A graph with three double kites. (b) Collapsing the edge $e = ab$.

A collapse operation may create multiple edges if a and b have a common neighbor $z \notin \{c, d\}$. However, we will assert that an edge ab is collapsed only if the triangle abz is not separating, i.e., one of its sides is empty. In particular, if T is not part of a double kite, we may collapse any of its edges by Lemma 7. Our assertion ensures that whenever an edge collapse creates multiple edges, we may merge them into a singleton edge without losing information about the embedding. Hence, we may assume that the graph resulting from an edge collapse is always simple. Moreover, note that the collapse operation does not increase the degree of any vertex. In particular, the degree of c decreases by two. The degree of the new vertex x is at most five. Hence, collapsing an edge in G results in a graph that satisfies the degree constraints, unless the collapse creates a new separating triangle.

We describe a procedure to find a set $K \subset E$ of edges such that the graph G' obtained by simultaneously collapsing all edges of K does not contain any separating triangle.

In the second step (Section 6), we augment G' by *stellating* every face, that is, for each nontriangular face f of G' we insert a new vertex v_f into f and we add an edge between v_f and each vertex on the boundary of f. By choosing the set K suitably, we ensure that the graph G'' resulting from these stellations does not contain any separating triangle. Thus, using Theorem 4 we obtain a subhamiltonian cycle C'' for G''.

Finally, in the third step (Section 7), we iteratively revert the edge collapses. Due to the stellated faces, we have some control over the possible ways that C'' visits the vertex pairs in G'' that result from collapsing a triangle of G. We will show that a series of local rerouting steps suffices to transform C'' into a subhamiltonian cycle for G.

5 Collapsing edges

Recall that G is an embedded 3-connected simple planar graph in which every separating triangle is trivial and all vertices incident to a separating triangle have degree at most five. Moreover, G is not a trivial double kite. Let S denote the set of separating triangles of G, which are pairwise vertex-disjoint by Observation 8. We want to find a set $K \subset E$ of edges so that (1) every edge in K is incident to a triangle in S; (2) every triangle in S is incident to exactly one edge from K; and (3) the graph obtained by simultaneously collapsing the edges of K does not contain any separating triangle.

Obviously, there exists a set $\hat{K}$ of edges that satisfies (1) and (2). Suppose that (3) does not hold for $\hat{K}$, that is, the graph $\hat{G}$ obtained from G by collapsing the edges in $\hat{K}$ contains a separating triangle T. By (2) we know that T is not a triangle in G, but T corresponds to a separating k-cycle in G, for $k \geq 4$. By (1), (2), and since the triangles in S are pairwise vertex-disjoint, at most every other edge of a cycle in G is in $\hat{K}$ and, therefore, we have $k \leq 6$. In other words, every separating triangle in $\hat{G}$ corresponds to a separating k-cycle in G where $k \in \{4, 5, 6\}$ and exactly $k-3$ edges are in $\hat{K}$. Moreover, for any such separating k-cycle in G, both the interior and the exterior must contain at least one vertex that is not the interior vertex of a triangle from S because by (2) every interior vertex of a triangle from S disappears in some collapse.

Inspired by these observations, we call a cycle C of an embedded graph *hyperseparating* if both the interior and the exterior contain at least one vertex that is not the interior vertex of a trivial separating triangle. We define an *inhibitor* to be a hyperseparating k-cycle I, where $k \in \{4, 5, 6\}$, for which at least $k-3$ edges are incident to a trivial separating triangle, for an illustration refer to Figure 3d. We refer to these at least $k-3$ edges as *constrained*. An edge e of an embedded graph is *constrained* if there exists an inhibitor I so that e is a constrained edge of I and *unconstrained*, otherwise. An inhibitor of length $k \in \{4, 5, 6\}$ is also called a *k-inhibitor*.

Note that a set $\hat{K}$ of edges that satisfies (1) and (2) also satisfies (3) if for every inhibitor of length k in G, no more than $k-4$ of its constrained edges are in $\hat{K}$. Next we study the:

5.1 Structure of inhibitors

First, let us observe that 4-inhibitors of G are chordless.

▶ **Observation 9.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let $T = abc$ be a trivial separating triangle of $\mathcal{G}$ with inner vertex d such that all vertices incident to T have degree at most five. Further, let $abxy$ be a 4-inhibitor constraining ab. Then $\{x, y\} \cap \{c, d\} = \emptyset$.*

▶ **Lemma 10.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let $T = abc$ be a trivial separating triangle of $\mathcal{G}$ with inner vertex d such that all vertices incident to T have degree at most five. Suppose that ab is constrained by a 4-inhibitor $I_{ab} = abxy$. Then I_{ab} is chordless, unless ab is incident to two separating triangles that define a double kite.*

Proof. Assume without loss of generality that by is a chord of I_{ab}. By Observation 9, $x, y \notin \{c, d\}$ and, so, the degree of b is saturated; see Figure 3a. We claim that by is on the side of I_{ab} that contains c and d. To see this, assume the contrary. Since I_{ab} is separating, there is some vertex z on the side of I_{ab} that does not contain c, d. The chord by splits this side of I_{ab} into two triangles, one of which contains z, see Figure 3b and 3c. However, by 3-connectivity, this implies that b has a neighbor in this triangle, in contradiction to the degree bound of b. So the claim holds, that is, by is indeed on the side of I_{ab} that contains c, d. Then aby is a triangle that separates x from c, d, see Figure 3d. By Lemma 7, the triangles abc and aby define a double kite. ◀

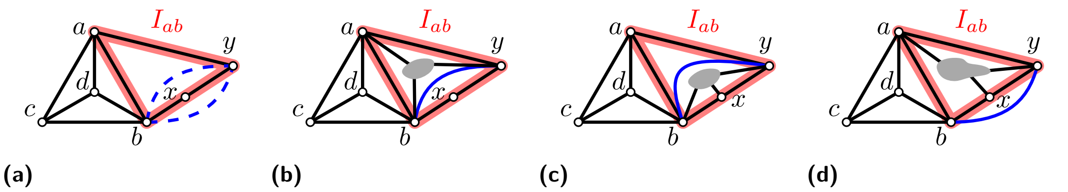


Figure 3 A 4-inhibitor $I_{ab} = abxy$ that constrains the edge ab of the separating triangle abc.

A necessary requirement for the existence of a set K of edges to collapse is that our graph does not contain a separating triangle for which every edge is constrained by a 4-inhibitor. We show that this requirement is met, except for some specific special cases.

▶ **Lemma 11.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let $\mathcal{I}$ be a set of 4-inhibitors of $\mathcal{G}$. Further, let T be a trivial separating triangle of $\mathcal{G}$ such that all vertices incident to T have degree at most five. Finally, assume that $\mathcal{G}$ contains no separating triangle that together with T defines a double kite.*

Then either (1) T is incident to at least one edge that is not constrained by a 4-inhibitor of $\mathcal{I}$; or (2) $\mathcal{G}$ contains a subgraph $\mathcal{G}'$ that is isomorphic to one of the graphs $\mathcal{G}_1, \mathcal{G}_2$ depicted in Figure 4 such that each thick (colored) edge of $\mathcal{G}'$ is incident to some 4-inhibitor of $\mathcal{I}$.

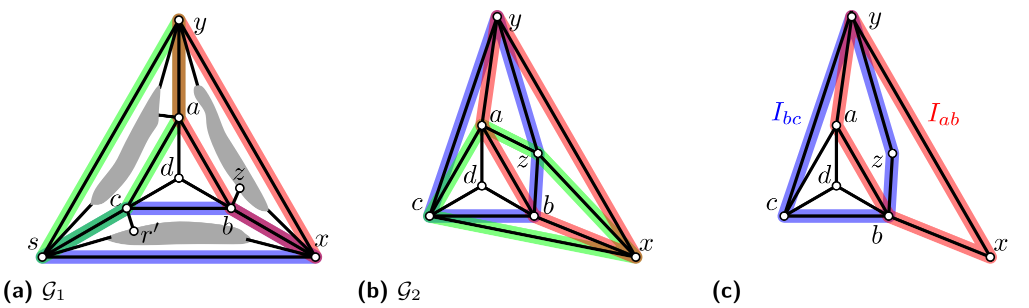


Figure 4 If all three edges of a separating triangle in the graph $\mathcal{G}$ are constrained by a 4-inhibitor, there is a subgraph $\mathcal{G}'$ isomorphic to $\mathcal{G}_1$ or $\mathcal{G}_2$. The gray parts in (a) represent arbitrary subgraphs.

Proof Sketch. Assume that each edge of $T = abc$ is constrained by a 4-inhibitor of $\mathcal{I}$, and that $\mathcal{G}$ contains no subgraph $\mathcal{G}'$ as in the claim. We denote the inhibitors of $\mathcal{I}$ constraining ab, bc, and ac by $I_{ab} = abxy$, $I_{bc} = bcst$, and $I_{ac} = acvw$, respectively. By Lemma 10, we have $c, d \notin I_{ab}$, $a, d \notin I_{bc}$, and $b, d \notin I_{ac}$, where d denotes the inner vertex of T. By 3-connectivity and since I_{ab} is separating, we may assume w.l.o.g. that b has a neighbor z which is located on the side of I_{ab} that does not contain c; see Figure 4c. Due to the degree bound of b, we have $t \in \{x, z\}$.

First, assume that $t = z$. By planarity, s belongs to I_{ab}; and by Lemma 10 applied to I_{bc} we have $s = y$ and, thus, $I_{bc} = bcyz$. We study the third inhibitor $I_{ac} = acvw$. We have $v \neq y$, as otherwise ay would be a chord of I_{ac}, contradicting Lemma 10. The triangle acy is nonseparating by Lemma 7 and the assumption that T does not define a double kite. Thus, v is located on the side of I_{bc} that does not contain a and, hence, by planarity, v or w has to belong to I_{bc}. Since $c \in I_{ac} \cap I_{bc}$, Lemma 10 implies that the only other vertex of I_{ac} in this intersection is z. As z and c are located on distinct sides of I_{ab}, the vertex v must belong to I_{ab} and, thus, $v = x$ and $I_{ac} = acxz$. Altogether, this establishes that $\mathcal{G}$ contains a subgraph $\mathcal{G}' \simeq \mathcal{G}_2$ as in the statement, which yields the desired contradiction.

It remains to consider the case that $t = x$. Using similar arguments as in the previous paragraph, we obtain a contradiction to the assumption that $\mathcal{G}$ does not contain a subgraph $\mathcal{G}'' \simeq G_1$ as in the statement. ◀

Similiar to the proof of Lemma 6, we can deal with the special case that there is a separating triangle in S such that each of its edges is constrained using the inductive framework of the proof of Theorem 5. So assume from now on that G does not contain a subgraph isomorphic to $\mathcal{G}_1$ or $\mathcal{G}_2$ as in Lemma 11. Our plan is to use Lemma 11 to identify an unconstrained edge, which is then collapsed. This procedure is then iterated until no separating triangle is left. The edges collapsed during this process form the desired set K.

Due to the degree bound, it is easy to determine in quadratic time the set of *ineligible* edges, that is, edges of separating triangles that are constrained by 4-inhibitors. Whenever an edge collapses, we check all separating triangles in the constant size neighborhood in order to update the set of ineligible edges in constant time. These edges are never considered for inclusion into the set $\mathcal{K}$ of edges to collapse.

An edge collapse is 3-*connectivity preserving* if the graph resulting from the collapse is 3-connected. As long as there is an eligible edge whose collapse preserves 3-connectivity, we collapse it. In order to test whether this is the case for an edge ab of a separating triangle $T = abc$, it suffices to determine and check all pairs of distinct faces f_a and f_b of $\mathcal{G}_T^+$ incident to a and b, respectively, and a vertex $v \neq c$ such that $v \in \partial f_a \cap \partial f_b$. This test can be performed in linear time, given that there is only a constant number of choices for f_a and f_b. Observe that negative test results are robust under 3-connectivity preserving collapses; whereas, in general, positive results are not. Hence, it suffices to consider every eligible edge at most once, and so the time spent on these tests is quadratic overall.

However, in general an edge collapse may reduce the connectivity of the graph. In this case, we plan to recurse on the triconnected components of the graph. To make such a recursion work in the context of our overall proof strategy, we must take special care concerning the vertices of separation pairs. Specifically, as we will discuss in the following section, we should never create a separation pair whose vertices are adjacent.

5.2 Avoiding adjacent separation pairs

Recall that we plan to stellate each face of the graph G' that is obtained by simultaneously collapsing all edges in K, and that we need to ensure that the resulting graph G'' does not contain any separating triangle. Consider a face f of G' and assume that its stellation creates a separating triangle $s = abv_f$ where v_f is the new vertex inserted into f. Note that the vertices a and b are incident to f. Therefore, the edge $ab \in \mathrm{E}(G')$ is a chord of ∂f and, moreover, a, b is a separation pair of G'.

In order to avoid this situation, it suffices to choose the set K subject to the following additional constraint: (4) the graph obtained by simultaneously collapsing the edges of K does not create an *adjacent* separation pair, i.e., a separation pair a, b where a and b are adjacent. Inspired by this observation, we devise a strengthened version (Lemma 13) of Lemma 11. For its proof, we require the following observation:

▶ **Observation 12.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let $T = abc$ be a trivial separating triangle of $\mathcal{G}$ with inner vertex d. Further, assume that collapsing ab results in a graph $\mathcal{G}'$ that is 2-connected but not 3-connected; and let p, q be a separation pair of $\mathcal{G}'$. Then we may assume that p is the new vertex that is created by the collapse of ab and that $q \notin \{c, d\}$.*

▶ **Lemma 13.** *Let $\mathcal{G}$ be an embedded 3-connected simple planar graph and let $\mathcal{I}$ be a set of 4-inhibitors of $\mathcal{G}$. Further, let $T = abc$ be a trivial separating triangle of $\mathcal{G}$ with inner vertex d such that all vertices incident to T have degree at most five. Finally, assume that $\mathcal{G}$ contains no separating triangle that together with T defines a double kite.*

Then either (1) T is incident to at least one edge e such that (i) e is not constrained by a 4-inhibitor of $\mathcal{I}$; and (ii) collapsing e does not create an adjacent separation pair; or (2) $\mathcal{G}$ contains a subgraph $\mathcal{G}'$ that isomorphic to one of the graphs $\mathcal{G}_1, \mathcal{G}_2$ depicted in Figure 4 such that each thick (colored) edge of $\mathcal{G}'$ is incident to some 4-inhibitor of $\mathcal{I}$.

5.3 Choosing the set of edges to collapse

We are now prepared to discuss how the desired edge set K is obtained.

▶ **Theorem 14.** *Let $\mathcal{H}$ be an embedded 3-connected simple planar graph on n vertices where every vertex incident to a separating triangle has degree at most five and where every separating triangle is trivial. Further, let $\mathcal{S}$ denote the set of separating triangles in $\mathcal{H}$. Assume that $\mathcal{H}$ is not a trivial double kite and that $\mathcal{H}$ contains no subgraph isomorphic to $\mathcal{G}_1$ or $\mathcal{G}_2$. Then we can compute in $O(n^2)$ time a set $\mathcal{K} \subset \mathrm{E}(\mathcal{H})$ of edges so that:*

(I1) *every edge in $\mathcal{K}$ is incident to a triangle in $\mathcal{S}$;*
(I2) *every triangle in $\mathcal{S}$ is incident to exactly one edge from $\mathcal{K}$;*
(I3) *$\mathcal{H}$ contains no k-inhibitor I such that $\mathcal{K}$ contains $k-3$ or more edges of I; and*
(I4) *the graph $\mathcal{H}''$ obtained by simultaneously collapsing the edges of $\mathcal{K}$ is biconnected and does not contain an adjacent separation pair.*

Proof Sketch. The proof is by induction on the number $|\mathcal{S}|$ of separating triangles. Let e be an edge of $\mathcal{H}$ that satisfies Property (1) of Lemma 13. Let $\mathcal{H}'$ be the embedded simple planar graph obtained by collapsing e in $\mathcal{H}$. We show that $\mathcal{H}'$ is not a trivial double kite or contains a subgraph isomorphic to $\mathcal{G}_2$. We also show that e can always be chosen such that $\mathcal{H}'$ contains no $\mathcal{G}_1$. If $\mathcal{H}'$ is 3-connected, we inductively obtain an edge set $\mathcal{K}'$ that satisfies the Properties (I1)–(I4) and, hence, $\mathcal{K} = \mathcal{K}' \cup \{e\}$ satisfies these properties for $\mathcal{H}$.

It remains to deal with the case that for *every* edge e that satisfies Property (1) of Lemma 13, collapsing e results in a graph $\mathcal{H}'$ that is biconnected, but not 3-connected. In this case, we recurse on the triconnected components of $\mathcal{H}'$. In particular, from the fact that $\mathcal{H}'$ contains no adjacent separation pair, we may derive that each separating triangle of $\mathcal{H}'$ appears, with all its real edges, in some rigid triconnected component R of $\mathcal{H}'$, which is a simple 3-connected planar graph. The graph R may contain separating triangles that do not belong to $\mathcal{S}$, namely the separating triangles that are incident to virtual edges. Keeping in mind that virtual edges correspond to separation pairs of $\mathcal{H}'$, and that $\mathcal{H}'$ contains no adjacent separation pairs, virtual edges of R may be thought of as paths (in $\mathcal{H}'$) of length at least two. As a consequence, we may ignore such *virtual* separating triangles. Similarly, regarding Property (I3), when picking the next edge to collapse in R, we do not need to worry about 4-inhibitors that have virtual edges. Formally, this is realized by proving a generalized version of the theorem, in which each edge of $\mathcal{H}$ is labeled as either real or virtual and in which the Properties (I1)–(I4) are relaxed accordingly.

The relaxed versions of Property (I1) and (I2) are easy to achieve. Regarding Property (I4), the main challenge is to ensure that the separation pairs already present in $\mathcal{H}'$ do not become adjacent when performing further edge collapses. Let p, q be a separation pair of $\mathcal{H}'$. By Observation 12, we may assume that p is the vertex created by collapsing e. Since the *real* separating triangles of $\mathcal{H}'$ are pairwise vertex-disjoint, we conclude that in order for p and q to become adjacent, there must exist a path psq in $\mathcal{H}'$, where sq is incident to a triangle T of $\mathcal{S}$. Let R be the rigid triconnected component of $\mathcal{H}'$ that contains T. We compute an edge set $\mathcal{K}_R$ for R that satisfies Properties (I1)–(I4). We show that if sq is constrained by a 4-inhibitor in $\mathcal{H}$, we may simply obtain $\mathcal{K}_R$ by induction as the Properties (I1)–(I4) of $\mathcal{K}_R$ already suffice to guarantee that p, q do not become adjacent. If sq is not constrained by a 4-inhibitor, there are choices of $\mathcal{K}_R$ for which p, q become adjacent even though Properties (I1)–(I4) hold for $\mathcal{K}_R$. We show that this is only possible in a very constrained special case, where

- either R has constant size (in this case we select $\mathcal{K}_R$ explicitly, rather than inductively);
- or we can apply a strategy ϱ to replace e with a new edge $\varrho(e)$, so that ϱ is acyclic, that is, $\varrho^i(e) \neq e$, for all $i \in \mathbb{N}$.

Finally, we show that Property (I4) implies Property (I3). To maintain the triconnected components, we use a decremental data structure by Holm et al. [15, Theorem 11] that allows to dynamically maintain an SPQR-tree under a sequence of edge contractions or deletions in $O(n \log^2 n)$ total time. Note that an edge collapse can be implemented using edge contractions and deletions. The runtime of our algorithm is dominated by the replacement strategy ϱ. We show that a single replacement step can be performed in constant time. After at most a linear number of replacements, we obtain an edge that can safely be collapsed. Hence, as the number of collapses is linear, the overall complexity is quadratic. ◀

6 Stellation

Let $K \subset \mathrm{E}(G)$ be a set of edges to collapse as described in Theorem 14, and let G' denote the graph that results from simultaneously collapsing the edges from K in G. Then by Property (I3) of Theorem 14 the graph G' does not contain any separating triangle. Let G'' denote the graph that results from stellating all faces in G', that is, for every nontriangular face f of G' we insert a new vertex v_f into f and we add an edge between v_f and each vertex on ∂f. As discussed in Section 5.2, the following lemma is an easy consequence of Property (I4).

▶ **Lemma 15.** *The graph G'' does not contain any separating triangle.*

Therefore, we can apply Theorem 4 to G'' to obtain a Hamiltonian cycle C'' for G''. It remains to address the case that one or two edges of the outer face $T_\circ$ of G are prescribed.

▶ **Observation 16.** *If any edge of $T_\circ$ is prescribed, then $T_\circ$ is also the outer face of G''.*

Proof. By Property (P3) we know that $T_\circ$ is also the outer face of G'. By Property (P2) we know that $T_\circ$ a triangle and, therefore, it is not subdivided when going from G' to G''. ◀

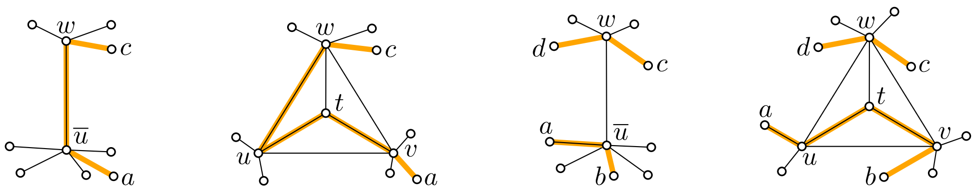
By Observation 16 we can pass any possibly prescribed edge of $T_\circ$ to Theorem 4 so that the obtained Hamiltonian cycle C'' for G'' passes through the(se) prescribed edge(s).

7 Reconstructing Collapses

As a final step to prove Theorem 1, it remains to undo the edge collapses, that is, to go back from the modified graph G'' to the original graph G. Our algorithm maintains a subhamiltonian plane cycle in the current graph, starting with a subhamiltonian plane cycle in G''. Then the separating triangles of G are processed in a certain order, which is incrementally computed as part of the algorithm. At every step of the algorithm we handle one separating triangle and include its vertices into the current working cycle. For some steps we may choose this separating triangle freely among the remaining ones, while in other steps that choice might be dictated by the previous step.

During the whole reconstruction process, we modify the current cycle in specific ways only. In particular, we only modify edges of the cycle that are incident to a separating triangle of G (including vertices that result from the collapse of an edge in K). By Observation 16 and (P3) this assertion suffices to ensure that prescribed edges on the outer face of G (if any) are part of the cycle that is constructed. Our algorithm proceeds in up to three phases.

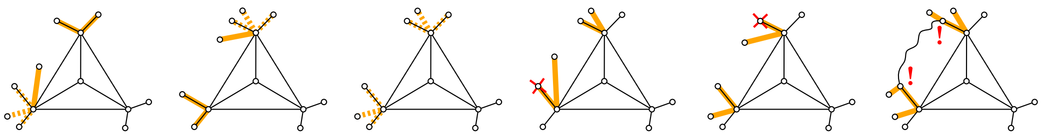
Phase 1. First, we reconstruct *some* collapsed triangles, while maintaining a subhamiltonian plane cycle in the current graph. We consider only triangles that are visited by the current working cycle in a specific way so that after reconstructing them the cycle can easily be extended to visit all vertices of and in the interior of those triangles; see Figure 5 for examples, where the solid orange segments depict edges of the current cycle.



(a) before **(b)** after **(c)** before **(d)** after

Figure 5 Examples of two easy reconstructions in Phase 1 (where uv was collapsed to $\overline{u}$).

At the end of the first phase, we move on to the graph G, that is, we reconstruct all remaining triangles at once. The previously Hamiltonian cycle then becomes a nonspanning cycle in a plane augmentation of G that visits all vertices of G but a pair of vertices from each of the *remaining* triangles (that have not been reconstructed during the first phase already). We can classify these remaining triangles – up to symmetry – into five different types according to how they interact with the current cycle. This classification is illustrated in Figure 6(a)–(e), where the dotted orange segments depict three different options for an edge of the current cycle, and the red crosses mark vertices that are not part of a remaining separating triangle.



(a) **(b)** **(c)** **(d)** **(e)** **(f)**

Figure 6 The six types of remaining triangles during Phase 2 of the reconstruction.

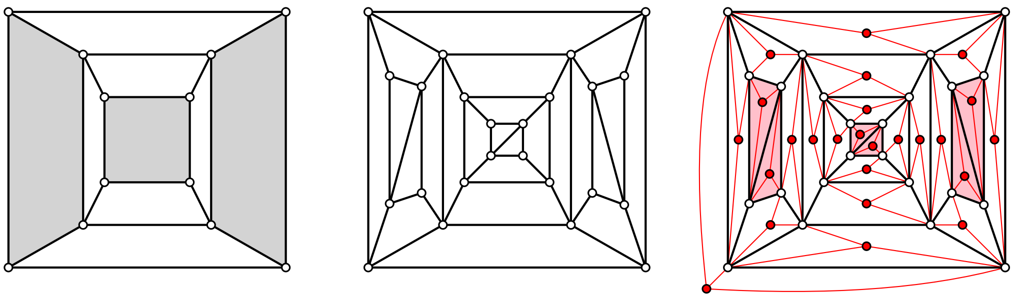
Phase 2 and 3. During the second phase we then maintain this classification: although a triangle may change its type, it always remains one of these five types. As in the first phase, we process the triangles one by one. Processing a triangle amounts to extending the current cycle to visit the two missing vertices. At the end of Phase 2 we either have a subhamiltonian plane cycle for G (and are done), or we are in a situation where *all* remaining triangles to be handled are of a very specific type with respect to the current cycle, which is illustrated in Figure 6f; the vertices labeled with a red exclamation mark are also part of a remaining separating triangle. Note that this type is a specialization of the more general type depicted in Figure 6c. The remaining triangles – if any – are then processed during a third phase, at the end of which we obtain the desired subhamiltonian plane cycle for G.

Remarks. The complexity of the reconstruction algorithm is linear overall. To illustrate the challenges for this reconstruction process and why it is important to control the interaction of the remaining triangles with the cycle under construction, consider the examples shown in Figure 7. If the cycle was to visit a remaining triangle in one of the ways depicted, then there is no way to locally modify the cycle to visit the missing vertices. Note that due to the stellation, these two configurations are avoided in the beginning of Phase 1.

8 Conclusions

The graph G_3 whose construction is depicted in Figure 8 is a member of the infinity family described in Theorem 3, which shows that our degree restriction in Theorem 1 is necessary in general. The most prominent open question is whether all planar graphs of degree ≤ 6 are subhamiltonian.

Figure 7 Two types of remaining triangles to avoid: There is no easy way to extend the cycle.



(a) $C_4 \square P_3$ **(b)** F_3 **(c)** G_3

Figure 8 The construction of G_3, a 3-connected planar graph that is not subhamiltonian and where every vertex of a separating triangle has degree at most six. We start from the Cartesian product $C_4 \square P_3$, where we pick three pairwise nonadjacent faces (shaded in (a)). Then we plant a rectangular prism on each picked face, obtaining the frame F_3 (b). Finally, to obtain G_3 we add a new vertex in every face of F_3 and connect it to three vertices on the boundary (c). The separating triangles of G_3 are shaded red; their vertices have degree six. The red vertices form an independent set, and no edge between any two red vertices can be added while maintaining planarity. As there are 25 red vertices and 24 black vertices, no plane augmentation of G_3 is Hamiltonian.

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