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Sparse Twisted Tensor Frame Discretization of Parametric Transport Operators *

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Abstract

We propose a novel family of frame discretizations for linear, high-dimensional parametric transport operators. Our approach is based on a least squares formulation in the phase space associated with the transport equation and by subsequent Galerkin discretization with a novel, sparse tensor product frame construction in the possibly high-dimensional phase space. The proposed *twisted tensor product frame construction* exploits invariance properties of the parameter space under certain group actions and accounts for propagation of singularities. Specifically, invariance of the parametric transport operator under rotations of the transport direction. We prove convergence rates of the proposed least squares Galerkin discretizations associated with the twisted tensor frames in terms of the number of degrees of freedom. In particular, sparse versions of the twisted tensor frame constructions are proved to break the curse of dimensionality, also for solution classes with low regularity in isotropic Sobolev spaces due to propagating singularities, uniformly with respect to the propagation directions.

AMS Subject Classification: primary 42C40, 65N12 secondary 65N15, 65N30, 42C99

1 Model Transport Problem

Parametric, high-dimensional transport equations such as Vlasov-Poisson, Boltzmann and Radiative Transfer equations appear in numerous models in the physical sciences, but increasingly also in socio-economic models. They are perceived as challenging for efficient deterministic numerical solvers. This is due principally to their hyperbolic nature which mandates efficient numerical treatment of propagating singularities, and to the *high dimensionality* of the phase space on which the parametric transport operator is defined. In addition, the presence of typically *nonlocal* collision operators with possibly nonsmooth collision kernels acting on the high dimensional phase space obstructs efficiency of standard numerical solvers.

In response to the high dimensionality, starting with the work [16] of Nanbu, *randomized discretizations* of particle type have been developed, refined and implemented in the past two decades; without claiming completeness, we mention only [2, 3, 4] and, in particular, the monograph [18] and the references there.

Randomized Boltzmann solution algorithms do not suffer from reduced convergence rates (when measured in terms of the number of degrees of freedom) due to high dimension of the phase space and allow for straightforward implementation, also on massively parallel hardware. Their (dimensionally robust) convergence rate is, however, limited by that of the Monte-Carlo sampling to 1/2. In addition, they do not, generally, offer a strategy towards higher convergence rates by means of a mechanism to deal with propagation of singularities on physical space.

Accordingly, in recent years a number of attempts have been made towards design and development of *efficient, deterministic solvers* for high dimensional parameter dependent transport problems. Among them, we mention spectral-Lagrangean Galerkin discretizations of the Boltzmann equation where the

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velocity space is discretized by Hermite polynomials (e.g. [11] and the references there for more on this approach). These deterministic approaches are efficient as long as the solutions exhibit high smoothness with respect to the parametric variable. Unfortunately, this is not the case for many situations of engineering interest. In addition, the deterministic approaches do not overcome the problem of high dimensionality of the phase space.

In [22], for a model parametric transport problem arising in radiative transport, the use of *sparse tensor product discretizations of angular and physical space* was proposed. This approach was shown to resolve both the above mentioned drawbacks of classical deterministic parametric transport solvers: it allows for local refinement in transport direction space to resolve beam-like solutions optimally, and it allows for isotropic adaptive refinements in physical space in order to afford *isotropic adaptive refinement* of solution singularities. In addition, the hierarchic structure of the multiresolution approximation spaces in physical and transport direction domain allows forming *sparse tensor products* of discretization spaces, thereby overcoming (up to logarithmic factors) the curse of dimensionality. Numerical experiments in [22] confirmed these features of the multiresolution based methods. In [12], it was shown that analogous complexity (albeit without local resolution in direction space) results can be expected by sparse tensorization of spectral discretizations in transport direction space and multiresolutions in physical space, breaking again the curse of dimensionality.

Transport equations, being hyperbolic, exhibit *propagation of singularities*. Approximation of propagating singularities requires *anisotropic mesh refinement concepts* to achieve optimal convergence rates. Anisotropic mesh refinement, however, introduces additional challenges to achieve stability of, say, Petrov-Galerkin discretizations.

The above mentioned deterministic, sparse tensor approaches were based on multiresolution spaces which do not, as a rule, afford anisotropic or directional refinements. Therefore, attention must be paid to discretization concepts which afford, on the one hand, adaptive resolution of propagating singularities and, on the other hand, retain stability during directional adaptation. In [8], an adaptive Petrov-Galerkin discretization of first order transport equations has been proposed which provides stable discretizations of rather general multiresolution approximations of the solution in the physical domain. Extension of this kind of discretization is also feasible for high dimensional, parametric transport problems. In the present paper, we propose a novel, deterministic class of multiresolution discretizations for high-dimensional parametric transport operators. We prove that this class of discretizations resolves both problems mentioned above: reduction of convergence rate due to high dimensional phase spaces is addressed by *sparse tensor products* of hierarchic bases in the physical domain D respectively in the parameter domain $\mathbb{S}^{d-1}$, and resolution of directional, propagating singularities is shown to be feasible with optimal order afforded by the approximation spaces.

To illustrate the key ideas in the simplest setting already of practical interest, see e.g. [17], we develop the discretizations with adaptive, sparse discretization methods of the stationary, monochromatic radiative transfer equation without scattering, i.e of

$$\vec{s} \cdot \nabla u(x, \vec{s}) + \kappa(x, \vec{s}) u(x, \vec{s}) = f(x, \vec{s}) \tag{1}$$

where x ranges in some domain $D \subseteq \mathbb{R}^d$ and the transport direction vector $\vec{s} \in \mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$. "Inflow" boundary conditions are imposed to render the problem well-posed:

$$u(x, \vec{s}) = u_-(x, \vec{s}), \quad x \in \Gamma_-(\vec{s}),$$

where the so-called "inflow boundary" Γ_- with respect to $\vec{n}$ is given by

$$\Gamma_-(\vec{s}) := \{x \in \partial D : \vec{s} \cdot \vec{n}(x) < 0\},$$

with $\vec{n}(x)$ denoting the (outwards oriented) normal field on ∂D. The numerical solution of this equation is challenging for several reasons. First, due to the dimension $2d-1$ of the phase space the curse of dimensionality implies that the number of degrees of freedom in any standard discretization is prohibitively large. Bypassing the curse of dimensionality and the hyperbolic nature of (1) mandates special discretization schemes.

The purpose of this paper is to introduce a family of Galerkin discretizations of the radiative transport problem that is stable and converges at an optimal rate, also in the presence of propagating singularities.

Convergence is understood here in terms of the number of degrees of freedom necessary to compute a solution to a desired accuracy. The extension of the present results to convergence in terms of computational complexity will be addressed in a forthcoming paper.

The main result of the present paper is that Galerkin least squares approximations of transport equations with twisted tensor frames yields the convergence rates as in [22] under realistic smoothness assumptions on the solution. Furthermore our proposed twisted tensor frame discretizations are proved to yield uniformly well-conditioned linear problems.

The outline of this paper is as follows: in Section 2, we precise the problem formulation, introducing in particular notation and assumptions on the physical domain, the transport problem and the boundary conditions. In Section 3, we introduce the concept of twisted tensor products, for phase spaces of the transport problem and for the function spaces on twisted cartesian products of physical domains and transport direction domains. In Section 4, we introduce a Least-Squares type variational formulation of the parametric transport operator which had already been proposed in [15] and which was used also in [22, 12]. We verify the well-posedness of the parametric transport problem in this Least-Squares formulation. Section 5 and 6 contain the core of the new technical material of the present paper. We explain in detail the twisted tensor frame construction for twisted tensor products of function spaces and, moreover, prove stability of Galerkin Least-Squares discretizations of the parametric transport operator in finitely truncated twisted tensor products of multiresolutions in physical and in transport direction space. Importantly, we show that for classes of *directionally smooth solutions* of the parametric transport equations (being piecewise smooth but including, in particular, propagating singularities) optimal convergence rates neither affected by the curse of dimensionality of phase space nor by the low regularity of the solutions in standard tensor products of isotropic Sobolev spaces are achieved. We close with supplementary material required in the main argument of the text in two appendices.

2 Notation and Assumptions

We consider a bounded convex domain $D \subseteq \mathbb{R}^d$ with boundary $\Gamma = \partial D$. Since D is convex, the exterior unit normal vector $\vec{n}(x)$ to D exists for almost every $x \in \partial D$ and is a bounded, measurable vector field with respect to the $d-1$ dimensional surface measure on Γ. In particular, for any transport vector $\vec{s} \in \mathbb{S}^{d-1}$, on Γ the nonnegative function $\omega_-(\vec{s}, x) := (\vec{n}(x) \cdot \vec{s})_+$ is bounded, measurable on Γ with respect to ds.

We denote $L^2(\Gamma)$ the space of all measurable functions on Γ which are square integrable with respect to the surface measure ds over Γ, and by $L^2(\Gamma_-; \omega_-)$ the space of functions on Γ_- which are square integrable with respect to $\omega_- ds$. The prototype of spaces we shall use subsequently is the anisotropic Sobolev space

$$H^{1,0} := H^{1,0}(D) := \left\{ v \in L_2(D) : \frac{d}{dx_1} v \in L_2(D) \right\} . \tag{2}$$

To express the regularity of solutions containing propagating singularities, we embed $H^{1,0}(D)$ in a scale of anisotropic higher order Sobolev spaces in the physical domain D of the following form:

$$H^{s+1,s}(D) := \left\{ v \in L_2(D) : v, \frac{d}{dx_1} v \in H^s(D) \right\} , \; s \geq 0 \tag{3}$$

where $H^s(D)$ denotes the usual (isotropic) Sobolev space on D. From the continuous embedding $H^1((0,1)) \subset C^0([0,1])$ it follows that the trace operator $v \mapsto v|_{\Gamma_-}$ is continuous from $H^{1,0}$ to $L^2(\Gamma_-)$. Accordingly, the space $H^{1,0}_+(D)$ defined by

$$H^{1,0}_+(D) := \left\{ v \in H^{1,0}(D) \; : v|_{\Gamma_-} = 0 \right\} \tag{4}$$

is a closed, linear subspace of $H^{1,0}(D)$. In (4), $\Gamma_- := \{x \in \partial D | \vec{n}(x) \cdot e_1 < 0\}$. In certain cases, we assume that the bounded and convex domain D can be described by two Lipschitz graphs φ_- and φ_+ as $D = \{(x_1, x') : x' \in D', \varphi_-(x') \leq x_1 \leq \varphi_+(x')\}$ with some convex set $D' \subset \mathbb{R}^{d-1}$ and the notation $(x_1, x') = (x_1, x_2, \ldots, x_d)$. Then the inflow boundary is the Lipschitz graph $\Gamma_- = \{(\varphi_-(x'), x') : x' \in D'\}$.

3 Twisted Tensor Products of Function Spaces

Definition 3.1. *For $\vec{s} \in \mathbb{S}^{d-1}$, denote by $R^*_{\vec{s}}$ an orthogonal matrix which takes $e_1 \in \mathbb{R}^d$ to $\vec{s}$, and by $R^*_{\vec{s}}$ its inverse. By $\rho_{\vec{s}}$, we denote the pullback of a function under $R_{\vec{s}}$: for $f \in \mathcal{D}(\mathbb{R}^d)$ and for $\vec{s} \in \mathbb{S}^{d-1}$, $\rho_{\vec{s}} f(x) := f(R^*_{\vec{s}} x)$ and $\rho^*_{\vec{s}} f(x) := f(R_{\vec{s}} x)$. For distributions $f \in \mathcal{D}'(\mathbb{R}^d)$, we define $\rho_{\vec{s}} u$ as usual by duality via smooth testfunctions, i.e.*

$$\forall \varphi \in \mathcal{D}(D): \quad (\rho_{\vec{s}} f, \varphi) = (f, \rho^*_{\vec{s}} \varphi) \ .$$

As in our earlier work [22], we employ variational formulations of (1) in *phase space* contained in $\mathbb{R} \times \mathbb{S}^{d-1}$. A particular role will be played by sets which are invariant under the action of the group $\{R_{\vec{s}} : \vec{s} \in \mathbb{S}^{d-1}\}$: for a bounded Lipschitz domain $D \subset \mathbb{R}^d$, we denote the *twisted product domain* $D \odot \mathbb{S}^{d-1}$ by

$$D \odot \mathbb{S}^{d-1} := \bigcup_{\vec{s} \in \mathbb{S}^{d-1}} R_{\vec{s}} D \times \{\vec{s}\} \ . \tag{5}$$

By construction, twisted product domains $D \odot \mathbb{S}^{d-1} \subseteq \mathbb{R}^d \times \mathbb{S}^{d-1}$ are invariant under the action of $\rho_{\vec{s}}$ for $\vec{s} \in \mathbb{S}^{d-1}$. We define anisotropic function spaces in $D \odot \mathbb{S}^{d-1}$.

Definition 3.2. *For $f \in L_2(D)$ and $g \in L_2(\mathbb{S}^{d-1})$ we define the twisted tensor product $f \odot g$ of the dyad (f, g) by*

$$f \odot g := (\rho_{\vec{s}} f) \otimes g \tag{6}$$

where $\otimes$ denotes the ordinary tensor product on $L^2(\mathbb{S}^{d-1}) \otimes L_2(\mathbb{S}^{d-1}) \simeq L^2(D; L_2(\mathbb{S}^{d-1}))$. By $H^{1,0}(D) \odot L_2(\mathbb{S}^{d-1})$, we denote the norm-closure of all finite linear combinations of twisted dyadic tensor products $f_i \odot g_j$, $(f_i, g_j) \in H^{1,0}(D) \times L_2(\mathbb{S}^{d-1})$ in the norm of the Bochner space $L^2(\mathbb{S}^{d-1}; H^{1,0}(D))$.

For a function f defined on (a twisted tensor product subset of) $\mathbb{R}^d \times \mathbb{S}^{d-1}$ we have $\rho^*_{\vec{s}} f(x, \vec{s}) := f((R^*_{\vec{s}})^* x, \vec{s}) = f(R_{\vec{s}} x, \vec{s})$ and

$$H^{1,0}(D) \odot L_2(\mathbb{S}^{d-1}) = \left\{ f : D \odot \mathbb{S}^{d-1} \to \mathbb{R} : \left\| \|\rho^*_{\vec{s}} f(\cdot, \vec{s})\|_{H^{1,0}(D)} \right\|_{L_2(\mathbb{S}^{d-1})} < \infty \right\} \ .$$

We denote by $H := H(D \odot \mathbb{S}^{d-1}) := H^{1,0}(D) \odot L_2(\mathbb{S}^{d-1})$. This linear space, equipped with the norm

$$\|f\|_H := \left\| \|\rho^*_{\vec{s}} f(\cdot, \vec{s})\|_{H^{1,0}(D)} \right\|_{L_2(\mathbb{S}^{d-1})} \tag{7}$$

is a Banach space. We record some properties of the space H.

Lemma 3.3. *We have*

$$\forall f \in \mathcal{D}'(D \odot \mathbb{S}^{d-1}): \quad \vec{s} \cdot \nabla_x f = \rho_{\vec{s}} \frac{d}{dx_1} (\rho^*_{\vec{s}} f) \ . \tag{8}$$

Proof. For $f \in \mathcal{D}(D \odot \mathbb{S}^{d-1})$, we verify directly that

$$\frac{d}{dx_1} (\rho^*_{\vec{s}} f)(\cdot, \vec{s}) = e_1 \cdot R^*_{\vec{s}} \nabla_x f(R_{\vec{s}} \cdot, \vec{s}) = \vec{s} \cdot \nabla_x f(R_{\vec{s}} \cdot, \vec{s}) = \rho^*_{\vec{s}} (\vec{s} \cdot \nabla_x f)(\cdot, \vec{s}) \ .$$

The proof for $f \in \mathcal{D}'(D \odot \mathbb{S}^{d-1})$ follows by duality. □

Lemma 3.4.

$$H = \left\{ f \in L_2(D \odot \mathbb{S}^{d-1}) : \|\vec{s} \cdot \nabla f\|_{L_2(D \odot \mathbb{S}^{d-1})} < \infty \right\} \ .$$

Proof. For $f \in H$, it holds by the definition of H and by Lemma 3.3

$$\begin{aligned} \infty > \|f\|_H^2 &= \left\| \|\rho^*_{\vec{s}} f(\cdot, \vec{s})\|_{H^{1,0}(D)} \right\|^2_{L_2(\mathbb{S}^{d-1})} \\ &= \int_{\mathbb{S}^{d-1} \times D} \left(|\rho^*_{\vec{s}} f(x, \vec{s})|^2 + |\frac{d}{dx_1} (\rho^*_{\vec{s}} f)(x, \vec{s})|^2 \right) dx d\vec{s} \\ &= \int_{\mathbb{S}^{d-1} \times D} \left(|\rho^*_{\vec{s}} f(x, \vec{s})|^2 + |\rho^*_{\vec{s}} (\vec{s} \cdot \nabla f)(x, \vec{s})|^2 \right) dx d\vec{s} \\ &= \int_{D \odot \mathbb{S}^{d-1}} \left(|f(x, \vec{s})|^2 + |\vec{s} \cdot \nabla f(x, \vec{s})|^2 \right) dx d\vec{s} \end{aligned}$$

whence the assertion. □

By $H_+ := H_+(D \odot \mathbb{S}^{d-1}) \subset H$ we denote the subspace of functions in H which vanish on the inflow boundary of $D \odot \mathbb{S}^{d-1}$, i.e.

$$H_+ := \mathrm{cls}_H \left\{ f \in H \cap C(\overline{D} \odot \mathbb{S}^{d-1}) : \; \forall \vec{s} \in \mathbb{S}^{d-1} \quad \rho^*_{\vec{s}} f(\cdot, \vec{s})|_{\Gamma_-} = 0 \right\} \; .$$

Remark 3.5. *In several applications the transport direction $\vec{s}$ only varies in a subset of $\mathbb{S}^{d-1}$ such as a great circle on $\mathbb{S}^2$, see also [15].*

We are especially interested in the case when the domain D is invariant under the action of $\rho_{\vec{s}}$. For $\vec{s}$ varying in all of $\mathbb{S}^{d-1}$ the only candidates for D are unit balls in $\mathbb{R}^d$, if $\vec{s}$ varies only on a subset of $\mathbb{S}^{d-1}$, more general domains D are possible. Whenever D is invariant we have $D \odot \mathbb{S}^{d-1} = D \times \mathbb{S}^{d-1}$, the usual direct product, and the space H_+ is given by the subspace of $L_2(D \times \mathbb{S}^{d-1})$ defined by the norm $\|f\|_{L_2(D\times\mathbb{S}^{d-1})} + \|\vec{s} \cdot \nabla f\|_{L_2(D\times\mathbb{S}^{d-1})} < \infty$ and the boundary condition $f(\cdot, \vec{s})|_{\Gamma_-(\vec{s})} = 0$. Therefore, in this case, the problem of solving (1) on H_+ coincides with the radiative transport problem considered in previous work, see e.g. [22].

4 Well-Posedness

We are now ready to describe more precisely the radiative transfer equation. We want to solve the equation

$$Au = f \tag{9}$$

where $A : H_+ \to L_2(D \odot \mathbb{S}^{d-1})$ is defined as

$$u(x, \vec{s}) \mapsto \vec{s} \cdot \nabla_x u(x, \vec{s}) + \kappa(x, \vec{s}) u(x, \vec{s}) \; .$$

We also assume that the absorption coefficient κ is defined on $D \odot \mathbb{S}^{d-1}$, nonnegative and sufficiently regular. In order to solve (9) we adopt a least-squares approach and propose to minimize the L_2-residual:

$$u_0 = \mathrm{argmin}_{v \in H_+} \|Av - f\|_{L_2(D\odot\mathbb{S}^{d-1})} \; . \tag{10}$$

By the convexity of the norm $\| \circ \|_{L_2(D\odot\mathbb{S}^{d-1})}$, minimizers exist if the functional $v \mapsto \|Av - f\|^2_{L_2(D\odot\mathbb{S}^{d-1})}$ is lower semicontinuous on H_+. Minimizers of (10) would solve the linear least squares problem: to find

$$u_0 \in H_+ \quad \text{s.t.} \quad (Au_0, Av)_{L_2(D\odot\mathbb{S}^{d-1})} = l(v) := (f, Av)_{L_2(D\odot\mathbb{S}^{d-1})} \text{ for all } v \in H_+. \tag{11}$$

Lower semicontinuity of $v \mapsto \|Av - f\|^2_{L_2(D\odot\mathbb{S}^{d-1})}$ and well-posedness of the least squares problem (11) follow from the following result.

Theorem 4.1. *For $v \in H_+$ we have the norm equivalence*

$$\|Av\|_{L_2(D\odot\mathbb{S}^{d-1})} \sim \|v\|_{H_+}.$$

Proof. The operator A can be represented as

$$Av = \rho_{\vec{s}}(\frac{d}{dx_1}(\rho^*_{\vec{s}} v)) + \rho_{\vec{s}}(\rho^*_{\vec{s}} \kappa \rho^*_{\vec{s}} v).$$

It follows that

$$\begin{aligned}
\|Av\|^2_{L_2(D\odot\mathbb{S}^{d-1})} &= \int_{D\odot\mathbb{S}^{d-1}} |Av(x,\vec{s})|^2 dx d\vec{s} \\
&= \int_{\mathbb{S}^{d-1}} \int_D |\rho^*_{\vec{s}} Av(x,\vec{s})|^2 dx d\vec{s} \\
&= \int_{\mathbb{S}^{d-1}} \int_D |\frac{d}{dx_1}(\rho^*_{\vec{s}} v)(x,\vec{s}) + \rho^*_{\vec{s}} \kappa \rho^*_{\vec{s}} v(x,\vec{s})|^2 dx d\vec{s} \\
&\lesssim \int_{\mathbb{S}^{d-1}} \int_D |\frac{d}{dx_1}(\rho^*_{\vec{s}} v)(x,\vec{s})|^2 + |\rho^*_{\vec{s}} v(x,\vec{s})|^2 dx d\vec{s} \\
&= \|v\|^2_{H_+}.
\end{aligned}$$

This shows one inequality. The converse inequality is shown by an explicit construction of A^{-1} using an explicit representation of solutions by integration along rays and a density argument: we first describe the operator A^{-1} for continuous functions $u \in C(\overline{D} \odot \mathbb{S}^{d-1})$ and then extend to all of $L_2(D \odot \mathbb{S}^{d-1})$ by density. To this end, consider the equation $Av = u$ for $u \in C(\overline{D} \odot \mathbb{S}^{d-1})$ and for $v \in H_+$:

$$Av = u \quad \Leftrightarrow \quad \rho_{\vec{s}}^* A v = \rho_{\vec{s}}^* u \Leftrightarrow \frac{d}{dx_1}(\rho_{\vec{s}}^* v) + \rho_{\vec{s}}^* \kappa \rho_{\vec{s}}^* v = \rho_{\vec{s}}^* u.$$

We can solve this equation explicitly and obtain

$$\rho_{\vec{s}}^* v = \left(\int_{\varphi_-(x')}^{x_1} \exp\left(\int_0^t \rho_{\vec{s}}^* \kappa(r, x', \vec{s}) dr\right) \rho_{\vec{s}}^* u(t, x', \vec{s}) dt \right) \exp\left(- \int_0^{x_1} \rho_{\vec{s}}^* \kappa(r, x', \vec{s}) dr\right), \tag{12}$$

where we use the notation $x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$. Now we estimate

$$\begin{aligned}
\|v\|^2_{L_2(D \odot \mathbb{S}^{d-1})} &= \|\rho_{\vec{s}}^* v\|^2_{L_2(D \times \mathbb{S}^{d-1})} \\
&\lesssim \int_{\mathbb{S}^{d-1}} \int_D \left| \int_{\varphi_-(x')}^{x_1} \exp\left(\int_0^t \rho_{\vec{s}}^* \kappa(r, x', \vec{s}) dr\right) \rho_{\vec{s}}^* u(t, x', \vec{s}) dt \right|^2 dx d\vec{s} \\
&\lesssim \int_{\mathbb{S}^{d-1}} \int_D \int_{\varphi_-(x')}^{x_1} \left| \exp\left(\int_0^t \rho_{\vec{s}}^* \kappa(r, x', \vec{s}) dr\right) \rho_{\vec{s}}^* u(t, x', \vec{s}) \right|^2 dt dx d\vec{s} \\
&\lesssim \int_{\mathbb{S}^{d-1}} \int_D \int_{\varphi_-(x')}^{x_1} \left| \rho_{\vec{s}}^* u(t, x', \vec{s}) \right|^2 dt dx d\vec{s} \\
&\leq \int_{\mathbb{S}^{d-1}} \int_D \int_{\varphi_-(x')}^{\varphi_+(x')} \left| \rho_{\vec{s}}^* u(t, x', \vec{s}) \right|^2 dt dx d\vec{s} \\
&\lesssim \int_{\mathbb{S}^{d-1}} \int_{D'} \int_{\varphi_-(x')}^{\varphi_+(x')} \left| \rho_{\vec{s}}^* u(t, x', \vec{s}) \right|^2 dt dx' d\vec{s} = \|\rho_{\vec{s}}^* u\|^2_{L_2(D \times \mathbb{S}^{d-1})} = \|u\|^2_{L_2(D \odot \mathbb{S}^{d-1})}.
\end{aligned}$$

This shows that

$$\|v\|_{L_2(D \odot \mathbb{S}^{d-1})} \lesssim \|Av\|_{L_2(D \odot \mathbb{S}^{d-1})} \tag{13}$$

for $v \in H_+$, whenever $Av \in C(\overline{D} \odot \mathbb{S}^{d-1})$. This implies that

$$\|v\|_{H_+} \lesssim \|Av\|_{L_2(D \odot \mathbb{S}^{d-1})} + \|v\|_{L_2(D \odot \mathbb{S}^{d-1})} \lesssim \|Av\|_{L_2(D \odot \mathbb{S}^{d-1})},$$

for $v \in H_+$ such that $Av \in C(\overline{D} \odot \mathbb{S}^{d-1})$ (we have used (13) in the last estimation step). By a density argument we get the general inequality

$$\|v\|_{H_+} \lesssim \|Av\|_{L_2(D \odot \mathbb{S}^{d-1})} \quad \text{for all} \quad v \in H_+,$$

completing the proof. □

Corollary 4.2. *For each $f \in L_2(D \odot \mathbb{S}^{d-1})$ there exists a unique weak solution $u_0 \in H_+$ for the radiative transfer equation $Au = f$, satisfying (11).*

Proof. By Theorem 4.1, there exists $c > 0$ such that

$$\forall v \in H_+ : \quad (Av, Av) \geq c\|v\|^2_{H_+} . \tag{14}$$

The unique solvability of (11) follows from the Lax-Milgram Lemma. □

5 Twisted Tensor Frame Construction

5.1 Frames

Since we are interested in discretizations of the radiative transfer equation, we want to represent the solution with respect to a discrete system. We shall consider separately two types of such systems: redundant systems – so called frames – and nonredundant systems – so called Riesz bases [5].

We shall denote infinite vectors and matrices in boldface letters.

Definition 5.1. *Let $\mathcal{H}$ be a Hilbert space. A sequence $\mathcal{F} = (f_n)_{n\in\mathcal{N}}$, $f_n \in \mathcal{H}$, is called a* frame *for $\mathcal{H}$ if, for every $v \in \mathcal{H}$,*

$$\|v\|_{\mathcal{H}}^2 \sim \sum_{n\in\mathcal{N}} |(v, f_n)_{\mathcal{H}}|^2 \,. \tag{15}$$

It can be shown that (15) is equivalent to the fact that the frame operator

$$\mathcal{S} : \begin{cases} \mathcal{H} & \to & \mathcal{H} \\ v & \mapsto & \sum_{n\in\mathcal{N}} (v, f_n)_{\mathcal{H}} f_n \end{cases}$$

is bounded and strictly positive. In that case also the system $\widetilde{\mathcal{F}} := (\tilde{f}_n)_{n\in\mathcal{N}}$, where $\tilde{f}_n := \mathcal{S}^{-1} f_n$, constitutes a frame for $\mathcal{H}$, the so-called canonical dual frame *of $\mathcal{F}$.*

The sequence $\mathcal{F}$ is called a Riesz basis *for $\mathcal{H}$ if it spans $\mathcal{H}$ and we have for every $\mathbf{d} \in \ell^2(\mathcal{N})$*

$$\|\mathbf{d}^\top \mathcal{F}\|_{\mathcal{H}}^2 := \| \sum_{n\in\mathcal{N}} d_n f_n \|_{\mathcal{H}}^2 \sim \|\mathbf{d}\|_{\ell_2(\mathcal{N})}^2 \,. \tag{16}$$

Let $\mathcal{B} \subset \mathcal{H}$ be a Banach space. Then a sequence $\mathcal{F} = (f_n)_{n\in\mathbb{N}}$, $f_n \in \mathcal{B}$, is called Riesz basis *for $\mathcal{B}$ with weight $\mathbf{w}$, if it spans $\mathcal{B}$ and we have, for every $\mathbf{d}$ such that $\mathbf{wd} \in \ell^2(\mathcal{N})$*

$$\|\mathbf{d}^\top \mathcal{F}\|_{\mathcal{B}}^2 \sim \|\mathbf{wd}\|_{\ell_2(\mathcal{N})}^2 := \sum_{n\in\mathcal{N}} (w(n)d(n))^2. \tag{17}$$

The sequence $\mathcal{F}$ is called a Banach frame *for $\mathcal{B}$ with weight $\mathbf{w}$, if it is a frame for $\mathcal{H}$ and*

$$\|\mathbf{d}^\top \mathcal{F}\|_{\mathcal{B}}^2 \le \|\mathbf{wd}\|_{\ell_2(\mathcal{N})}^2, \tag{18}$$

and

$$\|\mathbf{wd}\|_{\ell_2(\mathcal{N})}^2 \le \|v\|_{\mathcal{B}}^2, \ \textit{for } \mathbf{d} = (v, \tilde{\mathcal{F}})_{\mathcal{H}}. \tag{19}$$

In the following we will use greek letters to denote frames or Riesz bases of functions and the corresponding discrete index sets. The next result shows how to construct Riesz bases or (Banach frames) for $H_+(D \odot \mathbb{S}^{d-1})$ from two Riesz bases (or Banach frames) for $H_+^{1,0}(D)$ and $L_2(\mathbb{S}^{d-1})$ by using the twisted tensor product.

Theorem 5.2. *Assume that $\Sigma = (\sigma_\lambda)_{\lambda\in\Lambda}$ constitutes a Banach frame for $\mathcal{B} := H_+^{1,0}(D) \subset \mathcal{H} := L_2(D)$ with weight $\mathbf{w} = (w(\lambda))_{\lambda\in\Lambda}$, and that $\Theta = (\theta_\mu)_{\mu\in M}$ constitutes a frame for $L_2(\mathbb{S}^{d-1})$. Then the system*

$$\Sigma \odot \Theta := (\sigma_\lambda \odot \theta_\mu)_{(\lambda,\mu)\in\Lambda\times M}$$

constitutes a Banach frame for H_+ with weight $(w(\lambda))_{(\lambda,\mu)\in\Lambda\times M}$. If Σ, Θ are Riesz bases, then $\Sigma \odot \Theta$ is a Riesz basis, too, i.e.

$$\|\mathbf{d}^\top (\Sigma \odot \Theta)\|_{H_+}^2 \sim \|\mathbf{wd}\|_{\ell_2(\Lambda\times M)}^2 = \sum_{(\lambda,\mu)\in\Lambda\times M} w(\lambda)^2 d_{\lambda,\mu}^2 \,. \tag{20}$$

Proof. We write

$$\begin{aligned} \|\mathbf{c}^\top (\Sigma \odot \Theta)\|_{H_+}^2 &= \left\| \left\| \rho_{\vec{s}}^* \mathbf{c}^\top (\Sigma \odot \Theta) \right\|_{H_+^{1,0}(D)} \right\|_{L_2(\mathbb{S}^{d-1})}^2 \\ &= \left\| \mathbf{c}^\top (\Sigma \otimes \Theta) \right\|_{H_+^{1,0}(D) \otimes L_2(\mathbb{S}^{d-1})} . \end{aligned}$$

Now the assertion follows from the fact that the tensor product of two frames constitutes a frame for the tensor product Banach space (the same being also true for Riesz bases). ☐

5.2 Construction of Twisted Tensor Frames

5.2.1 Admissible Domains

We now give specific constructions of the frames Σ and Θ. To this end, we introduce a additional assumption on the domain D.

Definition 5.3. *We call D* admissible *if there exist $\delta_1 > \delta_2 > 0$ so that*

$$\delta_2 \geq \varphi_+(x') - \varphi_-(x') \geq \delta_1.$$

In the present section we present a general construction of Banach frames or Riesz bases for $H_+^{1,0}(D)$, provided that D is admissible. We further assume that the reader is familiar with the basic notions of wavelet theory [10]. For an interval $I = [i_l, i_r] \subset \mathbb{R}$ define the Banach spaces

$$H^1_{(0}(I) := \{f \in L_2(I) : f' \in L_2(I) \text{ and } f(i_l) = 0\}$$

and pick a wavelet Banach frame $\Psi = (\psi_\nu)_{\nu \in N}$ of $H^1_{(0}([0,1])$ with weight $\mathbf{w} = (w(\nu))_{\nu \in N}$ or a Riesz basis Ψ of $H^1_{(0}([0,1])$, which means that $\|\mathbf{c}^\top \Psi\|_{H^1_{(0}([0,1])} \sim \|\mathbf{wc}\|_{\ell_2(N)}$, where $w(\nu) := 2^{|\nu|}$, $|\nu|$ denoting the scale parameter of the wavelet index $\nu \in N$. Several constructions of such systems are available; we refer, for example, to [9] and the references there for details. With Ψ in hand, we go on to construct Riesz bases for $H_+^{1,0}(D)$. To this end, we define the dilation operator $D_a : f(\cdot) \mapsto a^{-1/2} f(a^{-1}\cdot)$ and the translation operator $T_y : f(\cdot) \mapsto f(\cdot - y)$. Since D is admissible, it follows that

$$\|f\|^2_{H^1_{(0}([\varphi_-(x'),\varphi_+(x')])} \sim \|D_{(\varphi_+(x')-\varphi_-(x'))^{-1}} T_{-\varphi_-(x')} f\|^2_{H^1_{(0}([0,1])}. \tag{21}$$

Remark 5.4. *The implicit constant in (21) is controlled by the ratio δ_2/δ_1. If we drop the admissibility condition, meaning that $\varphi_+ - \varphi_-$ can get arbitrarily small, this constant degenerates. Later we present a more complicated frame construction for general domains D.*

Now take any frame (or Riesz basis) $\Xi = (\xi_\omega)_{\omega \in \Omega}$ for $L_2(D')$ and define $\Lambda := N \times \Xi$ and

$$\sigma_\lambda := \sigma_{(\nu,\omega)} := T_{\varphi_-(x')} D_{\varphi_+(x')-\varphi_-(x')} \psi_\nu(x_1) \xi_\omega(x'). \tag{22}$$

The so-constructed new system forms a frame or a Riesz basis for $H_+^{1,0}(D)$ as shown in the following lemma:

Lemma 5.5. *The system $\Sigma = (\sigma_\lambda)_{\lambda \in \Lambda}$ is a Banach frame for $H_+^{1,0}(D)$ with weight $(w(\nu))_{(\nu,\omega) \in N \times \Omega}$. If, moreover, Ψ, Ξ are Riesz bases, then Σ is a Riesz basis for $H_+^{1,0}(D)$.*

Proof. We only prove the assertion related to Riesz bases, the frame case can be treated in the same way as in the proof of Theorem 5.2. To show the desired statement we use (21) and estimate as follows:

$$\begin{aligned}
\left\|\mathbf{c}^\top \Sigma\right\|^2_{H_+^{1,0}} &= \int_{D'} \left\|\mathbf{c}^\top \Sigma(\cdot, x')\right\|^2_{H^1_{(0}([\varphi_-(x'),\varphi_+(x')]} dx' \\
&\sim \int_{D'} \left\|\mathbf{c}^\top D_{(\varphi_+(x')-\varphi_-(x'))^{-1}} T_{-\varphi_-(x')} \Sigma(\cdot, x')\right\|^2_{H^1_{(0}([0,1])} dx' \\
&= \left\|\mathbf{c}^\top \Psi \otimes \Xi(\cdot, x')\right\|^2_{H^1_{(0}([0,1]) \otimes L_2(D')}.
\end{aligned}$$

Using again the fact that the tensor product of two Riesz bases is again a Riesz basis for the tensor product space finishes the proof. ☐

5.2.2 General Domains

In general we cannot expect D to be admissible: The most important domain for our purposes, the unit disc is not admissible. If we are not willing to live with the caveat of only computing the solution on a subset of D, we need to construct frames for $H_+^{1,0}(D)$ for general, nonadmissible domains. In this section

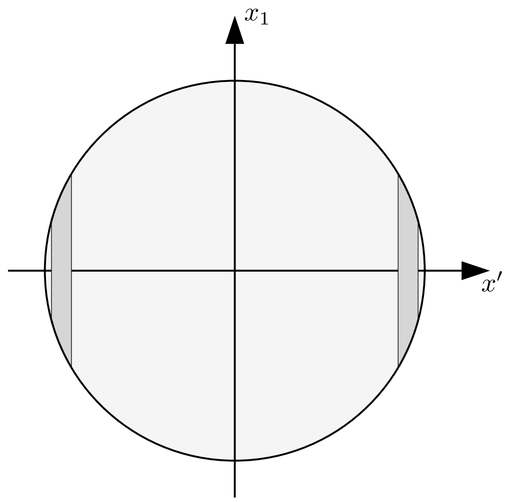

Figure 1: Illustration of the decomposition of the circular domain D into subdomains D_i. The two dark strips denote the subdomain D_2.

we show how to do this by cutting D into admissible parts and building frames using a Partition-Of-Unity. We give the details for

$$D = B^d := \{x \in \mathbb{R}^d : \|x\| < 1\}$$

the unit ball in $\mathbb{R}^d$. In that case we have

$$\varphi_-(x') = -\sqrt{1-(x')^2},\ \varphi_+(x') = -\varphi_-(x') = \sqrt{1-(x')^2}.$$

For $i = 0, 1, 2, \ldots$ decompose the domain $D' = \{x' : (x_1, x') \in B^d\}$ into annuli

$$I_i := \left\{x' \in D' : \varphi_+(x') \in \left[\frac{1}{2}2^{-i}, 2^{-i+1}\right]\right\} \quad \text{and} \quad \hat{I}_i := \left\{x' \in D' : \varphi_+(x') \in \left[\frac{3}{4}2^{-i}, \frac{3}{2}2^{-i}\right]\right\}$$

which both cover D'. Clearly $\hat{I}_i \subset I_i$. On each of these annuli we have

$$\forall x' \in I_i : \quad \frac{1}{2}2^{-i} \leq \varphi_+(x') \leq 2 \cdot 2^{-i}\ . \tag{23}$$

Let $(\gamma_i)_{i\in\mathbb{Z}}$ be a family of smooth functions such that

$$\text{supp } \gamma_i \subset I_i, \tag{24}$$

and

$$\gamma_i \sim 1 \quad \text{on } \hat{I}_i. \tag{25}$$

We also pick an auxiliary partition of unity $(\chi_i)_{i\in\mathbb{N}}$, where χ_i are nonnegative smooth functions with

$$\text{supp } \chi_i \subset \hat{I}_i, \tag{26}$$

and

$$\sum_{i\in\mathbb{N}} \chi_i = 1. \tag{27}$$

The idea is to partition the domain D into admissible overlapping subdomains

$$D_i := D \cap \{(x_1, x') : x' \in I_i\}$$

and to build frames for $H_+^{1,0}(D)$ by aggregating different Riesz bases Σ^i for the spaces

$$H_i := H_+^{1,0}(D_i). \tag{28}$$

In order to construct these bases we pick wavelet Riesz bases $\Psi^i = (\psi^i_{\nu^i})_{\nu^i \in N^i}$ for $H^1_{(0}([-2^{-i}, 2^{-i}])$ and $\Xi^i = (\xi^i_{\omega^i})_{\omega^i \in \Omega^i}$ for $L_2(I_i)$, meaning that

$$\|\mathbf{d}^\top \Psi^i\|_{H^1_{(0}([-2^{-i},2^{-i}])} \sim \|\mathbf{w}^i \mathbf{d}\|_{\ell_2(N^i)}$$

with weights $\mathbf{w}^i = (w^i(\nu^i))_{\nu^i \in N^i}$ and

$$\|\mathbf{d}^\top \Xi^i\|_{L_2(I_i)} \sim \|\mathbf{d}\|_{\ell_2(\Omega^i)},$$

the implicit constants being independent of i. See [1] for such constructions (we provide an explicit, and different construction in the appendix). We put $\Lambda^i := N^i \times \Omega^i$ and

$$\sigma^i_{\lambda^i} := \sigma^i_{(\nu^i,\omega^i)} := D_{2^i\varphi_+(x')}\psi^i_{\nu^i}(x_1)\xi^i_{\omega^i}(x'). \tag{29}$$

Lemma 5.6. *The system $\Sigma^i := \left(\sigma^i_{\lambda^i}\right)_{\lambda^i \in \Lambda^i}$ as defined in (29) constitutes a Riesz basis for H_i: we have*

$$\|\mathbf{d}^\top \Sigma^i\|_{H_i} \sim \|\mathbf{w}^i \mathbf{d}\|_{\ell_2(\Lambda^i)}, \tag{30}$$

where $w^i\left((\nu^i,\omega^i)\right) := w^i(\nu^i)$ and the implicit constant is independent of i.

Proof. The proof proceeds analogous to the proof of Lemma 5.5, noting that by (23) holds $\frac{1}{2} \le 2^i\varphi_+(x') \le 2$ for all $x' \in D_i$. $\square$

Now, using the weight functions γ_i, we patch together the bases Σ^i in order to obtain a Banach frame for $H^{1,0}_+(D)$. Define

$$\Lambda := \bigcup_{i\in\mathbb{Z}} \{i\} \times \Lambda^i,$$

$$\sigma_{(i,\lambda^i)}(x_1, x') := \gamma_i(x')\sigma^i_{\lambda^i}(x_1, x'),$$

and

$$\Sigma := (\sigma_\lambda)_{\lambda\in\Lambda}.$$

Theorem 5.7. *The system Σ constitutes a Banach frame for $H^{1,0}_+(D)$ with weight $\mathbf{w} := (w^i(\nu^i))_{(i,\lambda^i)\in\Lambda}$.*

Proof. We show the equivalent statement

$$\|u\|^2_{H^{1,0}_+(D)} \sim \inf_{\mathbf{d}^\top\Sigma = u} \|\mathbf{w}\mathbf{d}\|^2_{\ell_2(\Lambda)} \tag{31}$$

which is a consequence of Lemma 5.8 and the fact that Σ^i are frames with frame bounds independent of i. $\square$

Lemma 5.8.

$$\|u\|^2_{H^{1,0}_+(D)} \sim \inf_{\substack{u=\sum_{i\in\mathbb{N}} u_i \\ (\gamma_i)^{-1}u_i \in H_i}} \sum_{i\in\mathbb{N}} \|(\gamma_i)^{-1}u_i\|^2_{H_i}.$$

Proof. To obtain the upper estimate we write

$$
\begin{aligned}
\|u\|^2_{H^{1,0}_+(D)} &= \|\sum_i u_i\|^2_{H^{1,0}_+(D)} = \int_D \left|\sum_i u_i\right|^2 + \left|\sum_i \frac{d}{dx_1}u_i\right|^2 dx_1 dx' \\
&= \int_D \sum_{j,k\in\mathbb{N}} \left(u_j u_k + \frac{d}{dx_1}u_j \frac{d}{dx_1}u_k \right) dx_1 dx' \\
&= \int_D \sum_{i\in\mathbb{N}} \left(u_i^2 + u_i u_{i+1} + u_i u_{i-1} + \left(\frac{d}{dx_1}u_i\right)^2 + \frac{d}{dx}u_i \frac{d}{dx_1}u_{i+1} + \frac{d}{dx_1}u_i \frac{d}{dx_1}u_{i-1} \right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \int_{D_i} \left(u_i^2 + \left(\frac{d}{dx_1}u_i\right)^2 \right) dx_1 dx' + 2\int_D \sum_{i\in\mathbb{N}} \left(u_i u_{i+1} + \frac{d}{dx_1}u_i \frac{d}{dx_1}u_{i+1} \right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \|u_i\|^2_{H_i} + 2\int_D \sum_{i\in\mathbb{N}} u_i u_{i+1} + \frac{d}{dx_1}u_i \frac{d}{dx_1}u_{i+1} dx_1 dx' \\
&\leq \sum_{i\in\mathbb{N}} \|u_i\|^2_{H_i} + \int_D \sum_{i\in\mathbb{N}} \left(u_i^2 + \left(\frac{d}{dx_1}u_i\right)^2 + u_{i+1}^2 + \left(\frac{d}{dx_1}u_{i+1}\right)^2 \right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \|u_i\|^2_{H_i} + 2\int_D \sum_{i\in\mathbb{N}} \left(u_i^2 + \left(\frac{d}{dx}u_i\right)^2 \right) dx_1 dx' \\
&= 3\sum_{i\in\mathbb{N}} \|u_i\|^2_{H_i} \lesssim \sum_{i\in\mathbb{N}} \|\gamma_i^{-1} u_i\|^2_{H_i},
\end{aligned}
$$

where we have used the support properties of u_i and the fact that

$$
\|\gamma_i g\|^2_{H_i} = \int_{D_i} |\gamma_i(x')g(x_1,x')|^2 + \left|\gamma_i(x')\frac{d}{dx_1}g(x_1,x')\right|^2 dx_1 dx_2 \leq \|\gamma_i\|^2_\infty \|g\|^2_{H_i}
$$

for general functions g. We still need to show the converse estimate. To this end we write

$$
u = \sum_{i\in\mathbb{N}} \chi_i u \ .
$$

By (25) and (26) it follows that

$$
(\gamma_i)^{-1}\chi_i u \in H_i.
$$

We now claim that

$$
\sum_{i\in\mathbb{N}} \|(\gamma_i)^{-1}\chi_i u\|^2_{H_i} \lesssim \|u\|^2_{H^{1,0}_+(D)}, \tag{32}
$$

which implies the converse inequality. Using the fact that the functions χ_i are nonnegative as well as their support properties we estimate

$$
\begin{aligned}
\|u\|^2_{H^{1,0}_+(D)} &= \int_D \left(\sum_{i\in\mathbb{N}} \chi_i\right)^2 \left(u^2 + \left(\frac{d}{dx_1}u\right)^2\right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \int_D \chi_i^2 \left(u^2 + \left(\frac{d}{dx_1}u\right)^2\right) dx_1 dx' + 2\int_D \sum_{i\in\mathbb{N}} \chi_i\chi_{i+1} \left(u^2 + \left(\frac{d}{dx_1}u\right)^2\right) dx_1 dx' \\
&\geq \sum_{i\in\mathbb{N}} \int_D \chi_i^2 \left(u^2 + \left(\frac{d}{dx_1}u\right)^2\right) dx_1 dx' \\
&= \sum_{i\in\mathbb{Z}} \int_{D_i} \chi_i^2 \left(u^2 + \left(\frac{d}{dx_1}u\right)^2\right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \int_{D_i} \left((\chi_i u)^2 + \left(\frac{d}{dx_1}(\chi_i u)\right)^2\right) dx_1 dx' \\
&= \sum_{i\in\mathbb{N}} \|\chi_i u\|^2_{H_i} \gtrsim \sum_{i\in\mathbb{N}} \|(\gamma_i)^{-1}\chi_i u\|^2_{H_i} .
\end{aligned}
$$

□

Remark 5.9. *Observe that the partition we chose is* not *locally finite: for instance in the case* $d = 2$ *every neighborhood of the two poles* $(0, 1)$, $(0, -1)$ *intersect infinitely many* D_i*'s. Another seemingly pathological property is that the derivatives of the glueing functions* γ_i *grow to infinity as* $i \to \infty$*. However, all this does not matter for our construction since the* γ_i*'s only depend on the variable* x' *in which no derivatives are computed for the* $H^{1,0}$*-norm.*

5.2.3 Frames for H_+

Now that we have constructed an explicit frame for $H^{1,0}_+(D)$ we can pick any L_2-frame Θ for $\mathbb{S}^{d-1}$ and appeal to Theorem 5.2 to construct a twisted tensor frame or even a Riesz basis $\Sigma \odot \Theta$ for H_+:

Theorem 5.10. *Let* Σ *be the Banach frame for* $H^{1,0}_+(D)$ *as constructed above in Sections 5.2.1 and 5.2.2 with weight sequence* $\mathbf{w}$*. Further, let* Θ *be any frame for the Hilbert space* $L_2(\mathbb{S}^{d-1})$*. Then the system* $\Sigma \odot \Theta$ *constitutes a Banach frame for* $H_+(D \odot \mathbb{S}^{d-1})$ *with weight sequence* $(w(\lambda))_{(\lambda,\mu)\in\Lambda\times M}$*. If both* Σ *and* Θ *are Riesz bases then* $\Sigma \odot \Theta$ *is a Riesz basis as well.*

Proof. The statement follows directly from Theorem 5.2. □

Remark 5.11. *There exist many possibilities for the construction of the frame (or Riesz basis)* Θ *for* $L_2(\mathbb{S}^{d-1})$*. We will mainly consider wavelet frames to prove the desired approximation properties below. Useful constructions of wavelets on the sphere can be found e.g. in [20].*

6 Galerkin Discretization

In order to discretize the equation (9) w.r.t. $\Sigma \odot \Theta$ we consider the Gramian matrix

$$\mathbf{A} := \Big(\big(Aw((\lambda,\mu))^{-1}\sigma_\lambda \odot \theta_\mu, Aw((\lambda',\mu'))^{-1}\sigma_{\lambda'} \odot \theta_{\mu'}\big)_{L_2(D\odot\mathbb{S}^{d-1})} \Big)_{(\lambda,\mu),(\lambda',\mu')\in\Lambda\times M}.$$

We have collected all the ingredients to prove our first main result:

Theorem 6.1. *The matrix* $\mathbf{A}$ *defines a bounded operator on* $\ell_2(\Lambda \times M)$ *which is also boundedly invertible on its range. In particular the bi-infinite matrix problem (11) formulated as*

$$\mathbf{Au} = \mathbf{f} := \big((f, Aw((\lambda,\mu))^{-1}\sigma_\lambda \odot \theta_\mu) \big)_{(\lambda,\mu)\in\Lambda\times M}$$

is equivalent and well-conditioned on $\ell_2(\Lambda \times M)$. In particular, if moreover Ξ, Θ, Ψ are Riesz bases, then $\mathbf{A}$ *defines an isomorphism on $\ell^2(\Lambda \times M)$. In this case for every subset $J \subset \Lambda \times M$ the matrices*

$$\mathbf{A}_{J,J} := \Big(\big(A(w(\lambda,\mu))^{-1}\sigma_\lambda \odot \theta_\mu, A(w(\lambda',\mu'))^{-1}\sigma_{\lambda'} \odot \theta_{\mu'} \big)_{L_2(D\odot\mathbb{S}^{d-1})} \Big)_{(\lambda,\mu),(\lambda',\mu')\in J}$$

have a condition number which is bounded independent of J.

Proof. We only show the Riesz basis case since the frame case is covered in Lemma 4.1 of [7]. Due to the fact that $\mathbf{A}$ is a symmetric matrix, it suffices to show that

$$\mathbf{d}^\top \mathbf{A}\mathbf{d} \sim \|\mathbf{d}\|^2_{\ell_2(\Lambda\times M)}. \tag{33}$$

Indeed, this is easy to see since by Theorem 4.1 and the Riesz basis property we have that

$$\begin{aligned} \mathbf{d}^\top \mathbf{A}\mathbf{d} &= \left\| A \left(\sum_{(\lambda,\mu)\in\Lambda\times M} d_{\lambda,\mu}\omega((\lambda,\mu))^{-1}\sigma_\lambda \odot \theta_\mu \right) \right\|^2_{L_2(D\odot\mathbb{S}^{d-1})} \\ &\sim \left\| \sum_{(\lambda,\mu)\in\Lambda\times M} d_{\lambda,\mu}\omega((\lambda,\mu))^{-1}\sigma_\lambda \odot \theta_\mu \right\|^2_{H_+} \\ &\sim \sum_{(\lambda,\mu)\in\Lambda\times M} w((\lambda,\mu))^2 \left(d_{\lambda,\mu} w((\lambda,\mu))^{-1} \right)^2 = \|\mathbf{d}\|^2_{\ell_2(\Lambda\times M)} . \end{aligned}$$

The argument for the more general case of $\mathbf{A}_{J,J}$ follows by running the same argument and only considering $\mathbf{d}$ with supp $\mathbf{d} \subset J$. ☐

In this section we study the approximation rates and the computational complexity of the solution of (9) using the discrete formulation introduced above. We consider a nonadaptive Galerkin scheme incorporating sparse tensor products. In forthcoming work we also study an adaptive method in the spirit of [6, 7]. In our analysis we also need to distinguish whether the domain D is admissible or not. In the first case we can construct Riesz bases which will somewhat simplify things. For the case of general domains we present a slightly more complicated algorithm which is in the spirit of domain decomposition methods.

From now on we shall assume that the frames Ψ, Ξ, Θ considered above are wavelet frames with sufficiently many vanishing moments.

Remark 6.2. *In practice, for a given* non admissible *(due to its invariance under ρ_s) physical domain D with inflow boundary $\Gamma_-(s)$ and source term f one would like to find the solution u_0 of (10) corresponding to this data. Then, it is possible to slightly reduce the domain D in order to obtain a domain $D^{red} \subset D$ which is admissible. Moreover, according to the following result, the solution u_0^{red} of (10) on the domain D^{red} with right-hand side $f^{red} := f|_{D^{red}\odot\mathbb{S}^{d-1}}$ coincides with $u_0|_{D^{red}\odot\mathbb{S}^{d-1}}$, where u_0 is the solution of the original problem on D.*

Proposition 6.3. *Suppose that D is a (in general nonadmissible) domain in $\mathbb{R}^d$ satisfying the assumptions in Section 2. Suppose further that the domain*

$$D^{red} := \left\{ (x_1, x') : x' \in (D^{red})', \varphi_-(x') \le x_1 \le \varphi_+(x') \right\}$$

is admissible for some $(D^{red})' \subset D'$. Let u_0 be the solution of the problem (10) for D and right-hand side $f \in L_2(D \odot \mathbb{S}^{d-1})$. Let u_0^{red} be the solution of the problem (10) for D^{red} and right-hand side $f|_{D^{red}\odot\mathbb{S}^{d-1}}$. Then

$$u_0^{red} = u_0|_{D^{red}\odot\mathbb{S}^{d-1}}.$$

Proof. First note that the inflow boundary of D^{red} is given by

$$\left\{ (\varphi_-(x'), x') : x' \in (D^{red})' \right\}.$$

Therefore the function $u_0|_{D^{red}\odot\mathbb{S}^{d-1}}$ is in $H_+(D^{red}\odot\mathbb{S}^{d-1})$, i.e. it satisfies the necessary inflow-boundary conditions. Furthermore, clearly $u_0|_{D^{red}\odot\mathbb{S}^{d-1}}$ minimizes the problem (10) for D^{red}. By uniqueness of this minimization problem (established in Corollary 4.2) it follows that

$$u_0^{red} = u_0|_{D^{red}\odot\mathbb{S}^{d-1}}.$$

□

Example 6.4. *Consider D the unit ball in $\mathbb{R}^2$. This domain is not admissible. However, if we remove two (arbitrarily small) polar caps we obtain an admissible domain D^{red}. We can then compute the exact solution of the original problem for D restricted to D^{red} with the simple algorithms introduced below for admissible domains.*

6.1 Galerkin Approximation for Admissible Domains

The key in nonadaptive methods is to identify a sequence of nested subspaces that captures an increasing level of detail in the solution. Two such constructions are considered: First the full tensor product case which is defined via twisted tensor products of finite dimensional approximation spaces in space and angle. Due to the problem's phase space having dimension $2d-1$, this approach is not practical, as mentioned previously. For this reason we also study an alternative construction, namely *sparse twisted tensor product approximations*, in Section 6.1.2.

Due to the coercivity (14), the finite-dimensional problems

$$\mathbf{A}_{J,J}\mathbf{u}_J = \mathbf{f}_J, \quad J \subset \Lambda \times M, \tag{34}$$

where $\mathbf{f}_J$ denotes the projection of $\mathbf{f}$ onto the indices in J, admit unique solutions $\mathbf{f}_J$ which are quasioptimal in the sense that

$$\|u_0 - \bar{u}_J\|_H \lesssim \inf_{v\in V_J} \|u_0 - v\|_H, \tag{35}$$

where V_J denotes the linear space spanned by the elements of $\Sigma\odot\Theta$ with index in J, $\bar{u}_J := \mathbf{u}_J^T(2^{-|\nu|}\Sigma\odot\Theta)$, and u_0 the solution of (11), see [22]. Note that by Theorem 6.1 the system (34) is well-conditioned independently of J and possesses a unique solution.

In order to analyze this method we need to address two issues: first, the computational complexity of solving the system (34) up to a desired accuracy needs to be studied. Second, we need to study the approximation properties of the spaces V_J for solutions of (1). We will, in the sequel, focus on the latter issue.

Remark 6.5. *Regarding the first issue we simply remark that it is possible to solve (34) in log-linear time up to a desired accuracy. This is due to the fact that the Gramian matrix $\mathbf{A}$ is s-compressible in the sense of [6] as we show in our forthcoming work [13], and the bounded condition number of all finite sections $\mathbf{A}_{J,J}$, which allows us to give uniform bounds on the number of inexact CG iterations in order to obtain approximate discrete solutions which are consistent to any prescribed power of the meshwidth h, resp. 2^{-j}. In [13] we describe an inexact Richardson iteration procedure to achieve the desired task. This implies that, contrary to [22], we can indeed use the number of degrees of freedom in V_J as measure for the computational complexity required to compute $\mathbf{u}_J$. This will be analyzed in detail in the forthcoming work [13]. We also refer the reader to [6, 19, 14] and the references therein for error and complexity estimates for similar algorithms for classical pseudodifferential operators.*

We next exhibit suitable families V_J of finite dimensional subspaces and study their approximation properties.

Due to the choice of Ψ, Ξ, Θ, the representation systems Σ and Θ possess a natural hierarchical structure. For $\lambda \in \Lambda$ write $|\lambda| := \max(|\nu|, |\omega|)$, where $|\nu| \in N, |\omega| \in \Omega$ denote the scale in Ψ and Ξ, respectively. We also write $|\mu|$ to denote the scale of an index $\mu \in M$.

We start our subspace construction with some definitions: for $j \in \mathbb{N}$ and $I = [0,1]$ define

$$V_j^1 := \mathrm{cls}_{H^1_{(0}(I)}\mathrm{span}\{\psi_\nu : |\nu| \le j\}, \quad V_j^2 := \mathrm{cls}_{L_2(D')}\mathrm{span}\{\xi_\omega : |\omega| \le j\},$$

and

$$V_j^3 := \mathrm{cls}_{L_2(\mathbb{S}^{d-1})}\mathrm{span}\left\{\theta_\mu : |\mu| \le j\right\} .$$

We will assume the validity of the following *Jackson-type inequalities of degree* $s > 0$. These can be easily satisfied, for example when Ψ, Ξ, Θ are wavelet bases with some cancellation properties for their dual basis, see e.g. [10].

$$\inf_{v\in V_j^1} \|u-v\|_{H^1_{(0}(I)} \lesssim 2^{-sj}\|u\|_{H^{1+s}(I)} = 2^{-sj}\sum_{l\le 1+s}\|\partial_x^l u\|_{L_2(I)}, \tag{36}$$

$$\inf_{v\in V_j^2} \|u-v\|_{L_2(D')} \lesssim 2^{-sj}\|u\|_{H^{s}(D')} = 2^{-sj}\sum_{|\mathbf{l}|_1\le s}\|\partial^{\mathbf{l}}_{(x_2,\dots,x_d)} u\|_{L_2(D')}, \tag{37}$$

$$\inf_{v\in V_j^3} \|u-v\|_{L_2(\mathbb{S}^{d-1})} \lesssim 2^{-sj}\|u\|_{H^{s}(\mathbb{S}^{d-1})} = 2^{-sj}\sum_{|\mathbf{l}|_1\le s}\|\partial^{\mathbf{l}}_{\vec{s}} u\|_{L_2(\mathbb{S}^{d-1})}, \tag{38}$$

where we write $|\mathbf{l}|_1 := \sum_{i=1}^{d-1}|l_i|$ for $\mathbf{l} = (l_1,\dots,l_{d-1}) \in \mathbb{N}^{d-1}$. Note that on the sphere $\mathbb{S}^{d-1}$ we define the H^s norm (and the derivatives $\partial^{\mathbf{l}}_{\vec{s}}$) using local charts. Later we will use the operator notation $P_1^j f, P_2^j f, P_3^j f$ for the u attaining the minimum in Equations (36), (37), and (38), respectively.

For the discussion to follow we need to introduce some function spaces.

Definition 6.6. *The smoothness spaces*

$$X^s := \left\{v \in L_2(D\odot\mathbb{S}^{d-1}) : \|v\|_{X^s} < \infty\right\},$$
$$\|v\|_{X^s} := \|\rho_{\vec{s}}^* v\|_{H^{s+1,s}(D)\otimes L_2(\mathbb{S}^{d-1})} + \|\rho_{\vec{s}}^* v\|_{L_2(D)\otimes H^s(\mathbb{S}^{d-1})},$$

and

$$\hat{X}^s := \left\{v \in L_2(D\odot\mathbb{S}^{d-1}) : \|v\|_{\hat{X}^s} < \infty\right\}, \quad \|v\|_{\hat{X}^s} := \|\rho_{\vec{s}}^* v\|_{H^{s+1,s}(D)\otimes H^s(\mathbb{S}^{d-1})}.$$

These spaces will turn out to be natural approximation spaces for twisted tensor frame discretizations. Observe that the norm for $\hat{X}^s$ is slightly stronger than the norm for X^s but much weaker than the spaces $H^{s+1}(D)\otimes H^s(\mathbb{S}^{d-1})$ considered previuously, for example in [22].

6.1.1 Twisted Full Tensor Product Spaces

We start with the most obvious choice for approximation spaces, namely tensor product spaces.

Definition 6.7. *The full tensor product space V_j corresponding to scale j is defined as*

$$V_j := cls_H\, span\left\{\sigma_\lambda \odot \theta_\mu : \max(|\lambda|, |\mu|) \le j\right\}.$$

The next result summarizes the approximation properties of these spaces in H_+.

Theorem 6.8. *Assuming that the Jackson-type inequalities of degree s are valid, we have*

$$\inf_{v\in V_j}\|u-v\|_H^2 \lesssim 2^{-sj}\|u\|_{X^s}.$$

Proof. First we note that

$$\inf_{v\in V_j}\|u-v\|_H = \inf_{\tilde{v}\in V_j^1\otimes V_j^2\otimes V_j^3}\|\rho_{\vec{s}}^* u - \tilde{v}\|_{H^{1,0}(D)\otimes L_2(\mathbb{S}^{d-1})}. \tag{39}$$

Now the result can be deduced from Equations (36), (37), and (38). □

Remark 6.9. *This result is stronger than corresponding results in [22] since the smoothness that is imposed on f is not isotropic but varies with $\vec{s}$. In particular the smoothness in the direction orthogonal to $\vec{s}$ only needs to be 1 (as opposed to 2). Also in the above cited paper boundary conditions have not been incorporated in the theoretical analysis.*

We also need to study the complexity of solving (34) on the spaces V_j. A lower bound is given by the degrees of freedom in the space V_j which is approximately equal to 2^{2d-1}. Therefore, the convergence of the full tensor approximation to the actual solution can at best be

$$\text{error} \gtrsim (\text{number of arithmetic operations})^{-\frac{s}{2d-1}},$$

which becomes worse with d increasing. This phenomenon is commonly called the *curse of dimensionality.* In the next section we present discretization methods which circumvent this curse.

6.1.2 Twisted Sparse Tensor Product Spaces

The problem with Theorem 6.8 is the large number of degrees of freedom in the approximation spaces V_j. With wavelet methods this number is approximately $2^{dj} \cdot 2^{(d-1)j}$ – even for $d = 3$ this is prohibitively large. A common method for high-dimensional problems is the usage of so-called sparse tensor product approximation spaces [23] which are capable of (almost) achieving the same approximation properties under slightly stronger smoothness assumptions but with only about $2^{dj}j$ degrees of freedom, thus giving a substantial reduction in computational cost. This method has first been used in the context of the radiative transport problem in [22] where a different, unstable (in the sense of ill-conditioned linear systems) discretization has been used.

Here we present approximation results with sparse tensor product spaces where the usual tensor product is replaced by the stable twisted tensor product. As it turns out the results obtained are even stronger than those of [22] in terms of the smoothness conditions that need to be imposed on the solution u.

For the sake of keeping the presentation simple, in the analysis to follow below we confine ourselves to the case $D = [0,1]^d$. This is no restriction in generality since, due to the admissibility of D, we can translate the results from the unit cube to D by a simple scaling argument.

Definition 6.10. *The sparse tensor product space $\hat{V}_j$ corresponding to scale j is defined as*

$$\hat{V}_j := cls_H\, span\, \{\sigma_\lambda \odot \theta_\mu : |\lambda| + |\mu| \le j\}\,.$$

The number of elements in $\hat{V}_j$ can asymptotically be bounded by $2^{dj}j$. Now we show that despite the substantial reduction in degrees of freedom, the sparse tensor product spaces provide almost the same approximation rates as the full tensor product spaces, under stronger smoothness assumptions which are "natural" in that they distinguish regularity in the transport direction $\vec{s}$ and transversally to it.

Theorem 6.11. *We have*

$$\inf_{v \in \hat{V}_j} \|u - v\|_H \lesssim j 2^{-sj} \|u\|_{\hat{X}^s}.$$

Proof. The proof proceeds along the same lines as the proof of Theorem 6.8, utilizing known results in approximation with sparse tensor product wavelet spaces [21]. ☐

Due to the special structure of the Riesz basis Σ we can even do better by also considering sparse tensor products for the space $H^{1,0}(D)_+$.

Definition 6.12. *The approximation spaces $\hat{\hat{V}}_j$ corresponding to scale j are given by*

$$\hat{\hat{V}}_j := cls_H\, span\, \{\sigma_\lambda \odot \theta_\mu : (\lambda, \mu) \in J_j\}\,, \quad J_j := \{(\nu, \omega, \mu) : |\nu| + |\omega| + |\mu| \le j\}\,.$$

The spaces $\hat{\hat{V}}_j$ are sparse: their dimension can be bounded asymptotically, as $j \to \infty$, by $2^{(d-1)j}j^2$ which is one order of magnitude smaller than the dimension of $\hat{V}_j$. Nevertheless, as we show next, the spaces $\hat{\hat{V}}_j$ have (up to logarithmic factors) the same approximation properties with only minor additional assumptions on the smoothness of solutions.

Theorem 6.13. *We have the approximation property*

$$\inf_{v \in \hat{\hat{V}}_j} \|u - v\|_H \lesssim j^2 2^{-sj} \|u\|_{H^{s+1}(I) \odot H^s(D') \odot H^s(\mathbb{S}^{d-1})} := j^2 2^{-sj} \|\rho_{\vec{s}}^* u\|_{H^{s+1}(I) \otimes H^s(D') \otimes H^s(\mathbb{S}^{d-1})},$$

where

$$\|f\|^2_{H^{s+1}(I) \otimes H^s(D') \otimes H^s(\mathbb{S}^{d-1})} := \sum_{(l_1, |\mathbf{l}_2|, |\mathbf{l}_3|) \le (s+1, s, s)} \int_{\mathbb{S}^{d-1} \times D} \left| \partial_{x_1}^{l_1} \partial_{(x_2, \ldots, x_d)}^{\mathbf{l}_2} \partial_{\vec{s}}^{\mathbf{l}_3} f(x, \vec{s}) \right|^2 dx d\vec{s}.$$

Proof. Using (39), it suffices to show that

$$\inf_{v \in V_j^1 \hat{\otimes} V_j^2 \hat{\otimes} V_j^3} \|\rho_{\vec{s}}^* u - v\|_{H^{1,0}(D)\otimes L_2(\mathbb{S}^{d-1})} \lesssim 2^{-sj} j^2 \|u\|_{H^{s+1}(I)\otimes H^s(D)\otimes H^s(\mathbb{S}^{d-1})}, \tag{40}$$

where

$$V_j^1 \hat{\otimes} V_j^2 \hat{\otimes} V_j^3 := \operatorname{span}\{\psi_\nu \otimes \xi_\omega \otimes \theta_\mu : |\nu| + |\omega| + |\mu| \le j\} \ .$$

To prove this we write

$$\rho_{\vec{s}}^* u(x, \vec{s}) := \sum_{i_1, i_2, i_3} u_{i_1,i_2,i_3}(x, \vec{s}),$$

where the detail u_{i_1,i_2,i_3} is given by

$$u_{i_1,i_2,i_3} := \left(P_1^{i_1} - P_1^{i_1-1}\right) \otimes \left(P_2^{i_2} - P_2^{i_2-1}\right) \otimes \left(P_3^{i_3} - P_3^{i_3-1}\right) \rho_{\vec{s}}^* u$$

and show that

$$\|\rho_{\vec{s}}^* u - \sum_{i_1+i_2+i_3 \le j} u_{i_1,i_2,i_3}\|_H \lesssim j^2 2^{-sj} \|\rho_{\vec{s}}^* u\|_{H^{s+1}(I)\otimes H^s(D')\otimes H^s(\mathbb{S}^{d-1})}.$$

Indeed, we can write

$$\rho_{\vec{s}}^* u - \sum_{i_1+i_2+i_3 \le j} u_{i_1,i_2,i_3} = \sum_{i_1+i_2+i_3 > j} u_{i_1,i_2,i_3} = I + II,$$

where

$$I := \sum_{i_1+i_2 \le j} \sum_{j-(i_1+i_2)+1}^{\infty} u_{i_1,i_2,i_3} \quad \text{and} \quad II := \sum_{i_1+i_2 > j} \sum_{0}^{\infty} u_{i_1,i_2,i_3} \ .$$

Using (36), (37) and (38) we estimate

$$\begin{aligned}
\|I\|_H &= \left\| \sum_{i_1+i_2 \le j} \left(P_1^{i_1} - P_1^{i_1-1}\right) \otimes \left(P_2^{i_2} - P_2^{i_2-1}\right) \otimes \left(Id - P_3^{j-(i_1+i_2)+1}\right) \rho_{\vec{s}}^* u \right\|_H \\
&\lesssim \sum_{i_1+i_2 \le j} \left\| \left(Id - P_1^{i_1-1}\right) \otimes \left(Id - P_2^{i_2-1}\right) \otimes \left(Id - P_3^{j-(i_1+i_2)+1}\right) \rho_{\vec{s}}^* u \right\|_H \\
&\lesssim j^2 2^{-sj} \|\rho_{\vec{s}}^* u\|_{H^{s+1}(I)\otimes H^s(D')\otimes H^s(\mathbb{S}^{d-1})}.
\end{aligned}$$

We write $II = II_a + II_b$, where

$$II_a := \sum_{i_1 \le j} \sum_{i_2 \ge j+1-i_1} \sum_{i_3 \ge 0} u_{i_1,i_2,i_3}, \quad \text{and} \quad II_b := \sum_{i_1 > j} \sum_{i_2 \ge 0} \sum_{i_3 \ge 0} u_{i_1,i_2,i_3} \ .$$

We estimate II_a:

$$\|II_a\|_H \lesssim \left\| \sum_{i_1 \le j} \left(\left(Id - P_1^{i_1}\right) \otimes \left(Id - P_2^{j-i_1+1}\right) \otimes Id \right) \rho_{\vec{s}}^* u \right\|_H \lesssim j 2^{-sj} \|\rho_{\vec{s}}^* u\|_{H^{s+1}(I)\otimes H^s(D')\otimes H^s(\mathbb{S}^{d-1})}.$$

It remains to consider the term II_b:

$$\|II_b\|_H \lesssim \left\| \left(\left(Id - P_1^j\right) \otimes Id \otimes Id \right) \rho_{\vec{s}}^* u \right\|_H \lesssim 2^{-sj} \|\rho_{\vec{s}}^* u\|_{H^{s+1}(I)\otimes H^s(D')\otimes H^s(\mathbb{S}^{d-1})}.$$

Summing up the estimates for I, II_a, II_b proves the theorem. □

Using the fact that the discrete linear system is uniformly well-conditioned we can deduce the following theorem (observe that we need to rescale the first coordinate so as to be able to apply Theorem 6.13). We denote by $D_a^{x_1}$ the dilation by a factor a in the x_1-coordinate, the same notation is used for the translation operator.

Corollary 6.14. *Assume that the problem (11) admits a solution u_0 with*

$$D^{x_1}_{(\varphi_+(x')-\varphi_-(x'))^{-1}} T^{x_1}_{-\varphi_-(x')} \rho^*_{\vec{s}} u_0 \in H^{s+1}(I) \otimes H^s(D') \otimes H^s(\mathbb{S}^{d-1}) \,.$$

Suppose further that the Jackson-type inequalities (36) – (38) are valid for s. Denote by $\mathbf{u}_j$ the unique solution of the Galerkin equations

$$\mathbf{A}_{J_j,J_j}\mathbf{u}_j = \mathbf{f}_{J_j}.$$

As $j \to \infty$, these approximations converge at the rate

$$\|\mathbf{u} - \mathbf{u}_j\|_2 \lesssim j^2 2^{-sj}.$$

In particular, with $u_j := \mathbf{u}_j^\top(\mathbf{w}^{-1}\Sigma \odot \Theta)$ we have

$$\|u_0 - u_j\|_H \lesssim j^2 2^{-sj},$$

where u_0 is the solution of (11).

Proof. This follows immediately from Theorem 6.13. □

Remark 6.15. *A corollary to this result is that, using sparse twisted tensor product spaces, we are able to compute an approximate solution where the approximation error satisfies*

$$\textit{error} \lesssim (\textit{number of degrees of freedom})^{-\frac{s}{d-1}} \,,$$

if we disregard logarithmic terms. For most applications the relevant dimensions are $d = 2$ or $d = 3$.

Remark 6.16. *By constructing tensor product wavelet bases for $L_2(D')$ and $L_2(\mathbb{S}^{d-1})$ it would be possible to improve this result under further regularity assumptions on (in general unrealistic mixed derivatives in D' and in $\mathbb{S}^{d-1}$ of) the solution, and structural assumptions on the frames Ξ and Θ in D' and $\mathbb{S}^{d-1}$.*

6.2 General Domains

We finally consider the problem of solving the radiative transport problem on general nonadmissible domains using the representation systems derived in Section 5.2.2.

The issue of finding suitable nonadaptive Galerkin schemes on general domains is more subtle, since – contrary to the case where a Riesz basis is at hand – we do not have any information on the spectra of the finite sections $\mathbf{A}_{JJ}$ of the Gramian matrix $\mathbf{A}$. To overcome this problem we now present a domain decomposition method to solve the desired equation on finite dimensional subspaces. In particular we exploit the fact that our frame is of special structure: an aggregation of Riesz bases on different domains.

First we need to introduce some notation. As in Section 5.2.2 we have Riesz bases Ψ^i, Ξ^i of $H^1_{(0}(-2^{-i}, 2^{-i})$ and $L_2(I_i)$, respectively. From them we constructed the Riesz bases Σ^i of the spaces H^i defined in (28). By picking a fixed wavelet basis Θ for $L_2(\mathbb{S}^{d-1})$ we can build Riesz bases $\Sigma^i \odot \Theta$ for $H^i \odot L_2(\mathbb{S}^{d-1})$. The system $\Sigma \odot \Theta$ with $\Sigma := \bigcup_i \gamma_i \Sigma^i$ is a frame for $H_+(D \odot \mathbb{S}^{d-1})$ as shown in Theorem 5.7. We consider the localized problems

$$u_0^i = \operatorname{argmin}_{v \in H_i \odot L_2(\mathbb{S}^{d-1})} \left\| Av - \left(\rho_{\vec{s}} \chi_i \gamma_i^{-1}\right) f \right\|_{L_2(D_i \odot \mathbb{S}^{d-1})}, \; i \in \mathbb{N}. \tag{41}$$

Existence and well-definedness of the solution u_0^i follow from Theorem 4.1. In the method that we now propose we solve the problems (41) separately to a certain accuracy and then glue together the local solutions using the functions γ_i. Define the spaces

$$\hat{V}^i_j := \operatorname{cls}_{H_i} \operatorname{span}\left\{\sigma^i_{\lambda^i} \odot \theta_\mu : (\lambda^i, \mu) \in J^i_j\right\}, \text{ where } J^i_j := \left\{(\nu^i, \omega^i, \mu) : \max(|\nu^i|, |\omega^i|) + |\mu| \le j\right\}.$$

Note that this time the tensor product between the x_1 and x'-variables is not a sparse tensor product. The reason for our choice is that sparse tensoring of these variables would yield cumbersome approximation spaces. With arguments similar to the ones used in the proof of Theorem 6.13 together with Theorem B.1 we can establish that

Theorem 6.17.
$$\inf_{v\in\hat{V}_j^i}\left\|\left(\rho_{\vec{s}}\chi_i(\gamma_i)^{-1}\right)u-v\right\|_{H_i\odot L_2(\mathbb{S}^{d-1})}\lesssim j^2 2^{-sj}\|u\|_{\hat{X}^s}, \tag{42}$$
with a constant independent of i and of j.

Analogous to the case of an admissible domain, a quasioptimal Galerkin approximation u_j^i to (42) can be computed for each index i.

For the numerical solution of the radiative transfer equation (1) we assemble the approximate solution

$$u_{j,k}:=\sum_{i=0}^{k}(\rho_{\vec{s}}\gamma_i)u_j^i \tag{43}$$

for k left unspecified at this point. Note that all computations of the approximate solutions u_j^i can be carried out in parallel. The solution $u_{j,k}$ can be computed by solving a linear system with $k2^{dj}$ degrees of freedom. In order to study the approximation properties of $u_{j,k}$ we further require that the solution u_0 satisfies
$$\|u_0\|_\infty,\|\vec{s}\cdot\nabla u_0\|_\infty<\infty. \tag{44}$$
Since the area of $D_i\odot\mathbb{S}^{d-1}$ is bounded by a constant times 2^{-3i}, assumption (44) implies that

$$\|u_0^i\|_{H^i\odot L_2(\mathbb{S}^{d-1})}=\|\left(\rho_{\vec{s}}\chi_i(\gamma_i)^{-1}\right)u_0\|_{H_i\odot L_2(\mathbb{S}^{d-1})}\lesssim 2^{-3i/2}, \tag{45}$$

with a constant independent of i. Similar to the proof of the upper estimate in Lemma 5.8 we get

$$\begin{aligned}
\|u_0-u_{j,k}\|_H^2 &= \left\|\sum_{i=0}^{k}\left((\rho_{\vec{s}}\chi_i)u_0-(\rho_{\vec{s}}\gamma_i)u_j^i\right)+\sum_{l=k+1}^{\infty}(\rho_{\vec{s}}\chi_l)u_0\right\|_H^2\\
&\lesssim \sum_{i=0}^{k}\left\|(\rho_{\vec{s}}\gamma_i^{-1}\chi_i)u_0-u_j^i\right\|_{H_i\odot L_2(\mathbb{S}^{d-1})}^2+\sum_{l=k+1}^{\infty}\left\|(\rho_{\vec{s}}\gamma_l^{-1}\chi_l)u_0\right\|_{H_i\odot L_2(\mathbb{S}^{d-1})}^2\\
&= \sum_{i=0}^{k}\left\|u_0^i-u_j^i\right\|_{H_i\odot L_2(\mathbb{S}^{d-1})}^2+\sum_{l=k+1}^{\infty}\left\|(\rho_{\vec{s}}\gamma_l^{-1}\chi_l)u_0\right\|_{H_i\odot L_2(\mathbb{S}^{d-1})}^2\\
&\lesssim kj^2 2^{-2sj}+2^{-2k}.
\end{aligned}$$

Therefore, with $k=sj$ we obtain
$$\|u_0-u_{j,k}\|_H\lesssim j^{3/2}2^{-sj}.$$
As $j\to\infty$, the number of degrees of freedom for computing $u_{j,k}$ is $O(j2^{dj})$. We summarize these observations in the following theorem.

Theorem 6.18. *Assume that the problem (11) possesses a solution $u_0\in\hat{X}^s$ and such that (44) holds. Suppose further that the Jackson-type inequalities (36) – (38) are valid for s. Let $u_{j,sj}$ be the defined by (43). Then, as $j\to\infty$, these approximations converge at the rate*

$$\|u_0-u_{j,sj}\|_H\lesssim j^{3/2}2^{-sj}.$$

Therefore, also for general domains we can present a numerical procedure that achieves a rate

$$\text{error}\lesssim(\text{number of degrees of freedom})^{-\frac{s}{d}},$$

(if we disregard logarithmic terms) under the additional assumption that $u_0,\vec{s}\cdot\nabla u_0$ are bounded.

Remark 6.19. *The boundedness condition on u_0 is equivalent to the fact that the RHS f is bounded as can be seen by representing u_0 in terms of the raytracing formula (12). Therefore the additional assumption that we impose can be verified at hand of the given data f.*

7 Conclusion

In this paper we have introduced a novel family of frames well-adapted to the approximation of solutions of parametric linear transport problems. In particular, we proved that discretizing a least squares formulation of the model linear transport equation (1) with respect to these twisted tensor frame systems yields convergence rates in terms of the number of degrees of freedom which are unaffected by propagating singularities, which are free from the curse of dimensionality and which result in linear systems of equations whose condition numbers are uniformly bounded, in terms of the number of degrees of freedom.

We restricted our attention to the study of convergence in terms of the degrees of freedom in the discrete linear systems. We shall show in [13] that our construction allows the same results with convergence understood in terms of the number of arithmetic operations, and remains valid for solution in substantially larger Besov classes. Specifically, [13] will analyze adaptive frame schemes in the spirit of [7].

While in the present work we required the domain D to be symmetric under rotational transforms, we expect that more general convex domains can be treated in a similar way, either by transforming them to the symmetric case or by adjusting the tensor frame construction in an appropriate way.

A Construction of Ψ^i

The aim is to construct Banach frames Ψ^i for $H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)$. Our construction is simple: we just take one fixed Riesz basis $\Psi = (\psi_\nu)_{\nu \in N}$ of $H^1_{(0}([-1,1])$, N being a discrete index set where each $\nu \in N$ consists of a translation parameter and a scale $|\nu|$:

$$\|\mathbf{u}^\top \Psi\|_{H^1_{(0}[-1,1]} \sim \|\mathbf{wu}\|_{\ell_2(N)}, \tag{46}$$

where $\mathbf{w}(\nu) = 2^{|\nu|}$. Then we define $\Psi^i := D_{2^{-|i|}} \Psi = \left(D_{2^{-|i|}} \psi_\nu \right)_{\nu \in N}$.

Theorem A.1. *We have*

$$\|\mathbf{d}^\top \Psi^i\|^2_{H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)} \sim 2^{2|i|} \|\mathbf{wd}\|_{\ell_2(N)} .$$

Proof. By Lemma A.2 we have

$$\|\mathbf{d}^\top \Psi^i\|^2_{H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)} \sim 2^{2|i|} \|\mathbf{d}^\top D_{2^{|i|}} \Psi^i\|^2_{H^1_{(0}([-1,1])} ,$$

the latter expression, by (46) being equivalent to $2^{2|i|}\|\mathbf{wd}\|^2_{\ell_2(N)}$. □

Lemma A.2. *We have the following norm equivalence with the implicit constant independent of i:*

$$\|f\|_{H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)} \sim 2^{|i|} \|D_{2^{|i|}} f\|_{H^1_{(0}([-1,1])}. \tag{47}$$

Proof. Since the dilation operation is an isometry we have

$$\begin{aligned} \|D_{2^{|i|}} f\|_{H^1_{(0}([-1,1])} &= \|D_{2^{|i|}} f\|_{L_2([-1,1])} + \|\frac{d}{dx} D_{2^{|i|}} f\|_{L_2([-1,1])} \\ &= \|D_{2^{|i|}} f\|_{L_2([-1,1])} + 2^{-|i|} \|D_{2^{|i|}} \frac{d}{dx} f\|_{L_2([-1,1])} \\ &= \|f\|_{L_2\left[-2^{-|i|}, 2^{-|i|}\right]} + 2^{-|i|} \|\frac{d}{dx} f\|_{L_2\left[-2^{-|i|}, 2^{-|i|}\right]} . \end{aligned} \tag{48}$$

This immediately implies that

$$\|f\|_{H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)} \leq 2^{|i|} \|D_{2^{|i|}} f\|_{H^1_{(0}[-1,1]}.$$

To show the converse estimate we need to use the fact that $f\left(-2^{-|i|}\right) = 0$. This implies the following Poincaré-Friedrichs-type inequality: for $f \in H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)$ we have

$$\|f\|_{L_2\left[-2^{-|i|}, 2^{-|i|}\right]} \lesssim 2^{-|i|} \|\frac{d}{dx} f\|_{L_2\left[-2^{-|i|}, 2^{-|i|}\right]}. \tag{49}$$

We prove (49): Clearly, $f(x) = \int_{-2^{-|i|}}^{x} f'(t)dt$ and therefore

$$\|f\|_2^2 = \int_{-2^{-|i|}}^{2^{-|i|}} \left| \int_{-2^{-|i|}}^{x} f'(t)dt \right|^2 dx.$$

Jensen's inequality gives that this can be estimated by

$$\int_{-2^{-|i|}}^{2^{-|i|}} \int_{-2^{-|i|}}^{x} |x + 2^{-|i|}| |f'(t)|^2 dt dx \lesssim 2^{-|i|} \int_{-2^{-|i|}}^{2^{-|i|}} \int_{-2^{-|i|}}^{x} |f'(t)|^2 dt dx,$$

which can further be estimated by

$$\lesssim 2^{-|i|} \int_{-2^{-|i|}}^{2^{-|i|}} \int_{-2^{-|i|}}^{2^{-|i|}} |f'(t)|^2 dt dx \lesssim 2^{-2|i|} \|f'\|^2_{L_2\left[-2^{-|i|}, 2^{-|i|}\right]},$$

which is (49). Plugging in (49) into (13) gives that

$$\|f\|_{H^1_{(0}\left(\left[-2^{-|i|}, 2^{-|i|}\right]\right)} \geq 2^{|i|} \|D_{2^{|i|}} f\|_{H^1_{(0}[-1,1]},$$

which completes the proof. □

B Approximation Properties of Σ^i

Since the analysis gets somewhat complicated we restrict ourselves to the case of D being the bivariate unit disc. In this case x' is a real variable and we have $\varphi_+(x') = \left(1 - (x')^2\right)^{1/2}$. Define the points x_i, $i \in \mathbb{N}$ by the condition that

$$\varphi_+(x_i) = 2^{-i}, \quad x_i \geq 0. \tag{50}$$

We use the notation

$$x_i^+ := x_{i+1},\ x_i^- := x_{i-1}, \ \text{ and } x_i^{\pm} := x_{i+1} - x_{i-1}.$$

The annuli I_i are then given by the union of the intervals $[x_i^-, x_i^+] \cup -[x_i^-, x_i^+]$. It is not difficult to see that

$$|x_i^{\pm}| \lesssim 2^{-2i}. \tag{51}$$

Indeed, since

$$|x_i^{\pm}||x_i^+ + x_i^-| = |(x_i^+)^2 - (x_i^-)^2| = |2^{-2(i+1)} - 2^{-2(i-1)}| \lesssim 2^{-2i}$$

and

$$|x_i^+ + x_i^-| \geq \sqrt{\frac{3}{4}},$$

the equation (51) follows. Now we can take a fixed wavelet Riesz basis $\Xi = (\xi_\omega)_{\omega \in \Omega}$ for $L_2([0,1])$ and define

$$\xi_\omega^i := T_{x_i^-} D_{x_i^{\pm}} \xi_\omega.$$

Then $\Xi^i := \left(\xi_\omega^i\right)_{\omega \in \Omega}$ is a Riesz basis for $L_2([x_i^-, x_i^+])$. In an analogous way we can augment this basis to obtain a Riesz basis for $L_2(-[x_i^-, x_i^+])$. In what follows we will only consider the intervals $[x_i^-, x_i^+]$ since the general case is handled in exactly the same fashion. Now, according to Section 5.2.2 we put

$$\sigma_{(i,\lambda)}(x_1, x') := \gamma_i(x') D_{\varphi_+(x')} \psi_\nu(x_1) \xi_\omega^i(x'),\ \lambda = (\nu, \omega).$$

An elementary calculation yields

$$\left\| \left(\frac{d}{dx'}\right)^l \gamma_i(x_i^{\pm} x' + x_i^-) \right\|_\infty, \left\| \left(\frac{d}{dx'}\right)^l 2^{-i} \varphi_+(x_i^{\pm} x' + x_i^-)^{-1} \right\|_\infty, \left\| \left(\frac{d}{dx'}\right)^l 2^{i} \varphi_+(x_i^{\pm} x' + x_i^-) \right\|_\infty \lesssim 1, \tag{52}$$

Define $V_j^i := \operatorname{span}\left\{\sigma^i_{\lambda^i} :\ |\lambda^i| \leq j\right\}$ and $V_j := \operatorname{span}\left\{\psi_\nu \otimes \xi_\omega :\ \max(|\nu|, |\omega|) \leq j\right\}.$

Theorem B.1.
$$\inf_{v\in V_j^i}\|\chi_i(\gamma_i)^{-1}u-v\|_{H_i}\lesssim 2^{-sj}\|u\|_{H^{s+1,s}}.$$

Proof. We estimate as follows:

$$\begin{aligned}
\inf_{v\in V_j^i}\left\|\chi_i(\gamma_i)^{-1}u-v\right\|_{H_i} &\lesssim \inf_{v\in D^{x_1}_{\varphi_+^{-1}}V_j^i} 2^i\left\|D^{x_1}_{\varphi_+^{-1}}\chi_i(\gamma_i)^{-1}u-v\right\|_{H^1_{(0}([-1,1])\otimes L_2(I_i)} \\
&= \inf_{v\in V_j} 2^i\left\|D^{x'}_{(x_i^\pm)^{-1}}T^{x'}_{-x_i^-}D^{x_1}_{\varphi_+^{-1}}\chi_i(\gamma_i)^{-1}u-v\right\|_{H^1_{(0}([-1,1])\otimes L_2([0,1])} \\
&\lesssim 2^{-sj}\left\|2^iD^{x'}_{(x_i^\pm)^{-1}}T^{x'}_{-x_i^-}D^{x_1}_{\varphi_+^{-1}}\chi_i(\gamma_i)^{-1}u\right\|_{H^{s+1,s}([-1,1]\times[0,1])} &(53)\\
&\lesssim 2^{-sj}\left\|\frac{d}{dx_1}2^iD^{x'}_{(x_i^\pm)^{-1}}T^{x'}_{-x_i^-}D^{x_1}_{\varphi_+^{-1}}\chi_i(\gamma_i)^{-1}u\right\|_{H^{s}([-1,1]\times[0,1])} &(54)\\
&= 2^{-sj}\left\|D^{x'}_{(x_i^\pm)^{-1}}T^{x'}_{-x_i^-}D^{x_1}_{\varphi_+^{-1}}\chi_i(\gamma_i)^{-1}2^i\varphi_+\frac{d}{dx_1}u\right\|_{H^{s}([-1,1]\times[0,1])} \\
&\lesssim 2^{-sj}\left\|D^{x'}_{(x_i^\pm)^{-1}}T^{x'}_{-x_i^-}D^{x_1}_{\varphi_+^{-1}}\frac{d}{dx_1}u\right\|_{H^{s}([-1,1]\times[0,1])} &(55)\\
&\le 2^{-sj}\left\|\frac{d}{dx_1}u\right\|_{H^s(D_i)}\le 2^{-sj}\|u\|_{H^{s+1,s}(D_i)}.
\end{aligned}$$

Equation (53) is a classical approximation result for wavelets, (54) follows from the homogeneous boundary condition and the Poincaré-Friedrichs inequality, and (55) follows from (52). □

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