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# ALMOST SURE CONVERGENCE OF A GALERKIN APPROXIMATION FOR SPDES OF ZAKAI TYPE DRIVEN BY SQUARE INTEGRABLE MARTINGALES

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ABSTRACT. This work describes a Galerkin type method for stochastic partial differential equations of Zakai type driven by an infinite dimensional càdlàg square integrable martingale. Error estimates in the semidiscrete case, where discretization is only done in space, are derived in  $L^p$  and almost sure senses. Simulations confirm the theoretical results.

## 1. INTRODUCTION

The numerical study and simulation of stochastic partial differential equations (SPDEs) has been an active field of research for the last fifteen years. Within the last years the extension of PDEs to SPDEs has become more and more important in applications especially in engineering such as image analysis, surface analysis, filtering [22, 27, 29, 31, 35]. On the other hand side, in finance, people extend finite dimensional systems of stochastic differential equations (SDEs) to infinite dimensional ones [15, 4], i.e. to SPDEs. Explicit solutions to most of the problems do not exist. Therefore it is natural to simulate a discrete version of these SPDEs.

In this paper we study a Galerkin method for the space approximation of the solution of an SPDE of the form

$$(1.1) \quad dX(t) = (A + B)X(t) dt + G(X(t)) dM(t), \quad X(t_0) = X_0,$$

where  $M$  is a càdlàg square integrable martingale with values in a separable Hilbert space  $U$ . Probably the most popular examples of such stochastic processes are Wiener and Lévy processes. The operators  $A$  and  $B$  act on a separable Hilbert space  $H$  and the operator  $G$  is a mapping from  $H$  into the linear operators from  $U$  to  $H$ .

In general, for a numerical treatment of Hilbert space valued stochastic differential equations, approximation has to be done in space and time. There are various approaches possible. So far Galerkin methods are mainly used for PDEs (cf. [36, 17, 34]) but first applications to SPDEs have been done e.g. in [3, 8, 10, 12, 13, 26]. The approximation of mild solutions with colored noise has been done e.g. in [2, 19, 27] and references therein. First approaches to higher order approximation schemes using Taylor expansions were done e.g. in [20] and [21] with additive space-time white noise and with multiplicative colored noise in [5]. Galerkin methods lead to pathwise approximations, also called strong approximations.

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A fully discrete approximation will be done in a forthcoming paper [7] because if the driving noise is not continuous, new problems arise in the proof of almost sure convergence as presented in [5] due to the time regularity of the solution of the SPDE which means that  $X(t) - X(s)$  converges of order  $(t - s)^{1/p}$  in  $L^p$  and this order cannot be improved.

The type of equation studied in this paper appears naturally in the study of Zakai's equation (cf. [37]). Fully discrete approximations of its solution were already studied in [13], while higher order schemes are presented in [28] for a semidiscrete time approximation and in [5] for a semidiscrete space approximation and a fully discrete approximation using a Galerkin method in space and a Crank–Nicolson approach in time. The SPDE of Zakai type, which has been introduced by Zakai for a nonlinear filtering problem, is extended to square integrable martingales and reads then

$$(1.2) \quad du_t(x) = L^*u_t(x) dt + G(u_t(x)) dM_t(x)$$

on a bounded domain  $D \subset \mathbb{R}^d$  with zero Dirichlet boundary conditions on  $\partial D$  and initial condition  $u_0(x) = v(x)$ . The operator  $L^*$  is a second order elliptic differential operator of the form

$$L^*u = \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j a_{ij} u - \sum_{i=1}^d \partial_i f_i u$$

for  $u \in C_c^2(D)$  and it can be split into the operators  $A$  and  $B$  in Equation (1.1). This will be done explicitly in Section 2. Originally the operator  $G$  denotes a pointwise multiplication with a suitable function  $g \in H$ . This is included in the more general assumptions on  $G$  in Equation (1.1) which will be introduced in detail in the next section.

The main result of this paper is that if Equation (1.1) is approximated by the projected SPDE on a finite dimensional subspace of  $H$  with convergence parameter  $h$  and if the corresponding homogeneous deterministic problem converges with order  $O(h^\alpha)$  to the solution of the homogeneous problem, then the approximated SPDE converges with order  $O(h^\alpha)$  in  $L^p$  and almost surely with order  $O(h^{\alpha-\epsilon})$  for any  $\epsilon > 0$  to the mild solution of Equation (1.1). These results are confirmed by simulations of the heat equation driven by Lévy noise.

This work is organized as follows: In Section 2 the framework and the properties of the SPDE and its solution are given. Section 3 introduces the space approximation and its  $L^p$  and almost sure convergence. Examples that meet the assumptions are given. Finally, in Section 4, simulations are provided that give estimates on path and  $L^p$  convergence for  $p = 1, \dots, 5$ .

## 2. FRAMEWORK

Let  $H$  denote the Hilbert space  $L^2(D)$ , where  $D \subset \mathbb{R}^d$  is a bounded domain with piecewise smooth boundary  $\partial D$  and let the subspaces  $H^\alpha$  be the corresponding Sobolev spaces for  $\alpha \in \mathbb{N}$  and  $H_0^\alpha$  those with elements that satisfy zero Dirichlet boundary conditions respectively. For  $\alpha = 0$ , we set  $H^0 = H$  for simplicity of the notation. We are interested in developing a numerical algorithm to estimate the solution of the SPDE

$$(2.1) \quad dX(t) = (A + B)X(t) dt + G(X(t)) dM(t)$$

on the finite time interval  $[0, T]$  with initial condition  $X(0) = X_0$  and zero Dirichlet boundary conditions on  $\partial D$ .  $M$  is a càdlàg square integrable martingale on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  satisfying the “usual conditions” with values in a separable Hilbert space  $(U, (\cdot, \cdot)_U)$ . The space of all càdlàg square integrable martingales taking values in  $U$  with

respect to  $(\mathcal{F}_t)_{t \geq 0}$  is denoted by  $\mathcal{M}^2(U)$ . We restrict ourselves to the following class of martingales

$$\mathcal{C} := \{M \in \mathcal{M}^2(U) : \exists Q \in L_1^+(U) \text{ such that } \forall t \geq s \geq 0, \langle\langle M, M \rangle\rangle_t - \langle\langle M, M \rangle\rangle_s \leq (t-s)Q\},$$

where  $L_1^+(U)$  denotes the space of all linear, nuclear, symmetric, positive-definite operators acting on  $U$ . The operator angle bracket process  $\langle\langle M, M \rangle\rangle_t$  is defined as

$$\langle\langle M, M \rangle\rangle_t = \int_0^t Q_s d\langle M, M \rangle_s,$$

where  $\langle M, M \rangle_t$  denotes the unique angle bracket process from the Doob–Meyer decomposition. The process  $(Q_s, s \geq 0)$  is called the martingale covariance. Examples of such processes are square integrable Lévy martingales, i.e. those Lévy martingales with Lévy measure  $\nu$  that satisfies

$$\int_U \|\varphi\|_U^2 \nu(d\varphi) < +\infty.$$

Since  $Q \in L_1^+(U)$ , there exists an orthonormal basis  $(e_n, n \in \mathbb{N})$  of  $U$  consisting of eigenvectors of  $Q$ . Therefore we have the representation  $Qe_n = \gamma_n e_n$ , where  $\gamma_n \geq 0$  is the eigenvalue corresponding to  $e_n$ . The square root of  $Q$  is defined as

$$Q^{1/2}x := \sum_n (x, e_n)_U \gamma_n^{1/2} e_n, \quad x \in U$$

and  $Q^{-1/2}$  is the pseudo inverse of  $Q^{1/2}$ . Let us denote by  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  the Hilbert space defined by  $\mathcal{H} = Q^{1/2}(U)$  endowed with the inner product  $(x, y)_{\mathcal{H}} = (Q^{-1/2}x, Q^{-1/2}y)_U$  for  $x, y \in \mathcal{H}$ . Let  $L_{HS}(\mathcal{H}, H)$  refer to the space of all Hilbert–Schmidt operators from  $\mathcal{H}$  to  $H$  and  $\|\cdot\|_{L_{HS}(\mathcal{H}, H)}$  denote the corresponding norm.

In what follows we assume a Burkholder–Davis–Gundy type inequality as a generalization of the Itô isometry for square integrable martingales of class  $\mathcal{C}$ .

**Assumptions 2.1.** There exists a positive constant  $C$  depending on  $p \geq 2$  and  $T$  such that

$$(2.2) \quad \mathbb{E}(\|\int_0^t \Psi(s) dM(s)\|_H^p) \leq C \mathbb{E}(\int_0^t \|\Psi(s)\|_{L(U, H)}^p ds)$$

for  $t \in [0, T]$  and a locally bounded predictable process  $\Psi : [0, T] \rightarrow L_{HS}(U, H)$  with

$$\mathbb{E}(\int_0^T \|\Psi(s)\|_{L(U, H)}^p ds) < +\infty.$$

This equation holds e.g. for continuous square integrable martingales because for these processes it holds that (cf. [18])

$$\mathbb{E}(\sup_{0 \leq t \leq T} \|\int_0^t \Psi(s) dM(s)\|_H^p) \leq C \mathbb{E}(\int_0^T \|\Psi(s)\|_{L_{HS}(\mathcal{H}, H)}^p ds)$$

and as  $Q$  is assumed to be trace class we have with Lemma 2.1 in [11]

$$\|\Psi(s)\|_{L_{HS}(\mathcal{H}, H)}^p = \|\Psi(s)Q^{1/2}\|_{L_{HS}(U, H)}^p = \|\Psi(s)\|_{L(U, H)}^p (\text{Tr } Q)^{p/2}.$$

Equation (2.2) is also true for Lévy martingales that satisfy

$$\int_U \|\varphi\|_U^q \nu(d\varphi) < +\infty$$

for  $q \in [2, p]$ , where  $\nu$  denotes the Lévy measure of  $M$  (see Lemma 3.9 in [30]).

For an introduction to Hilbert space valued stochastic differential equations we refer the reader to [33, 14, 11].

The operators  $A$  and  $B$  in Equation (2.1) are derived from  $L^*$  in Equation (1.2). We assume that the functions  $a_{ij}$ , for  $i, j = 1, \dots, d$ , are twice continuously differentiable on  $D$  with continuous extension to the closure  $\bar{D}$ . The operator  $A$  is the unique self-adjoint extension to  $H_0^1$  of the differential operator

$$\sum_{i,j=1}^d \partial_i(a_{ij} \partial_j u), \quad u \in C_c^2(D).$$

$B$  is a first order differential operator given by

$$Bu := \sum_{i=1}^d \partial_i(b_i u), \quad u \in C_c^1(D),$$

for  $f$  continuously differentiable on  $D$  with continuous extension to  $\bar{D}$ , with elements  $b_i$  that are defined as

$$b_i := \frac{1}{2} \sum_{j=1}^d \partial_j a_{ij} - f_i.$$

With the following assumptions, the right hand side of Equation (2.1) is well defined and its solution has certain properties to be shown later. From here on, let  $\alpha \in \mathbb{N}$  be fixed.

**Assumptions 2.2.** The coefficients of  $A$  and  $B$  and the initial condition  $X_0$  satisfy the following conditions:

- (a) For  $i, j = 1, \dots, d$ , the elements  $a_{ij}$  belong to  $C_b^{\alpha+1}(D)$  and  $f_i$  to  $C_b^\alpha(D)$  with continuous extensions to  $\bar{D}$ ,
- (b) there exists  $\delta > 0$  such that for all  $x \in D$  and  $\xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \delta \|\xi\|_{\mathbb{R}^d}^2,$$

- (c)  $X_0$  is  $\mathcal{F}_0$ -measurable and  $\mathbb{E}(\|X_0\|_{H^\alpha}^p) < +\infty$ ,
- (d) there exists  $C > 0$  such that for  $0 \leq \beta \leq \alpha$  and  $\phi \in H^\beta$

$$\|G(\phi)\|_{L(U, H^\beta)} \leq C(1 + \|\phi\|_{H^\beta}),$$

- (e) there exists  $C > 0$  such that for  $0 \leq \beta \leq \alpha$  and  $\phi \in H^\beta$

$$\|G(\phi) - G(\psi)\|_{L(U, H^\beta)} \leq C \|\phi - \psi\|_{H^\beta}.$$

Assumption 2.2(b) implies that the operator  $A$  is dissipative, see e.g. [24]. Then by the Lumer–Phillips theorem, e.g. [16],  $A$  generates a strongly continuous contraction semigroup on  $H$  which we denote by  $S = (S(t), t \geq 0)$ . Furthermore, by Corollary 2 in [23],  $S$  is analytic in the right half-plane. Therefore fractional powers of  $A$  are well defined, cf. [16], and we denote for simplicity  $A_{-\beta} = (-A)^{-\beta}$  and  $A_\beta = A_{-\beta}^{-1}$  for  $\beta > 0$ .

In this context we shall make use of the following lemma — whose statement is also known as *Kato’s conjecture* — and which was proved in [1].

**Lemma 2.3.** *The domain of  $A_{1/2}$  satisfies that its domain  $\mathcal{D}(A_{1/2}) = H_0^1$  and the norm  $\|A_{1/2} \cdot\|_H$  is equivalent to  $\|\cdot\|_{H^1}$ , i.e. there exists  $C > 0$  such that*

$$\|A_{1/2}\phi\|_H \leq C \|\phi\|_{H^1} \quad \text{and} \quad \|\phi\|_{H^1} \leq C \|A_{1/2}\phi\|_H$$

for all  $\phi \in H^1$ .

Assumptions 2.2 also imply by results in Chapter 9 in [33], that Equation (2.1) has a unique mild solution in  $H^\alpha$ , i.e.

$$\sup_{t \in [0, T]} \mathbb{E}(\|X(t)\|_{H^\alpha}^2) < +\infty$$

for all  $T \in (0, +\infty)$ , and the SPDE can be rewritten for all  $t > 0$  in mild form

$$(2.3) \quad X(t) = S(t)X_0 + \int_0^t S(t-s)BX(s) ds + \int_0^t S(t-s)G(X(s)) dM(s).$$

Furthermore,  $X$  has a càdlàg modification by Theorem 9.29 in [33]. From this point on, we denote by  $X$  the càdlàg modification of the solution.

To simplify the notation we introduce the following norm for a mapping  $\Phi$  from  $[0, T] \times \Omega$  into  $H$  with finite  $p$ -th moment for fixed  $p \geq 1$

$$\|\Phi\|_{p, H, t} := \left( \mathbb{E}(\|\Phi(t)\|_H^p) \right)^{1/p}.$$

The next Lemma provides some insight on the space regularity of the mild solution.

**Lemma 2.4.** *Under Assumptions 2.2 the mild solution satisfies  $\sup_{0 \leq t \leq T} \|X\|_{p, H^\alpha, t} < +\infty$  for  $p > 2$ .*

*Proof.* From here on  $C$  denotes varying constants depending on  $p$  and  $T$ .

$$\begin{aligned} \|X\|_{p, H^\alpha, t}^p &= \|S(t)X_0 + \int_0^t S(t-s)BX(s) ds + \int_0^t S(t-s)G(X(s)) dM(s)\|_{p, H^\alpha, t}^p \\ &\leq C(\mathbb{E}\|X_0\|_{H^\alpha}^p + \|\int_0^t S(t-s)BX(s) ds\|_{p, H^\alpha, t}^p \\ &\quad + \|A_{\alpha/2} \int_0^t S(t-s)G(X(s)) dM(s)\|_{p, H, t}^p) \\ &\leq C(\mathbb{E}\|X_0\|_{H^\alpha}^p + \mathbb{E}((\int_0^t \|S(t-s)BX(s)\|_{H^\alpha} ds)^p) \\ &\quad + \mathbb{E}(\int_0^t \|A_{\alpha/2}G(X(s))\|_{L(U, H)}^p ds)) \\ &\leq C(\mathbb{E}\|X_0\|_{H^\alpha}^p + \mathbb{E}((\int_0^t (t-s)^{-1/2}\|X(s)\|_{H^\alpha} ds)^p) \\ &\quad + \mathbb{E}(\int_0^t (1 + \|X(s)\|_{H^\alpha}^p) ds)) \\ &\leq C(1 + \mathbb{E}\|X_0\|_{H^\alpha}^p) + 2 \int_0^t \|X\|_{p, H^\alpha, s}^p ds \leq C(1 + \mathbb{E}\|X_0\|_{H^\alpha}^p) < +\infty, \end{aligned}$$

where we used the boundedness of the contraction semigroup in the first and Equation (2.2) in the second step, Lemma 2.3, Theorem 6.13 in [32], and the definition of the Bochner integral in the third one, and Hölder's and Gronwall's inequality in the fourth.  $\square$

### 3. APPROXIMATION SCHEME AND ORDER OF CONVERGENCE

In this section we derive a semidiscrete approximation scheme for Equation (2.1) and prove the properties of this scheme.

To derive a semidiscrete form of Equation (2.1) we project  $H$  onto a finite subspace  $V_h$  of  $H$ , where suitable spaces are Finite Element spaces. This can for example be done by first discretizing  $D$  by a triangulation defined over a finite number of points. Then let  $(\mathcal{S}_h, h > 0)$  denote a family of Finite Element spaces, consisting of piecewise linear, continuous polynomials with respect to the family of triangulations  $(\mathcal{T}_h, h > 0)$  of  $D$  such that  $\mathcal{S}_h \rightarrow H$  for  $h \rightarrow 0$  and furthermore  $\mathcal{S}_h \subset H_0^1(D)$  for  $h > 0$ . In the general framework let  $(V_h, h > 0)$  be a family of subspaces of  $H$  with orthogonal projection  $P_h$  and norm derived from  $H$ . For  $h \rightarrow 0$  the sequence  $V_h$  is supposed to be dense in  $H$  in the following sense: For all  $\phi \in H$  it holds that

$$\lim_{h \rightarrow 0} \|P_h \phi - \phi\|_H = 0.$$

Furthermore, we assume that the speed of convergence is specified by

$$(3.1) \quad \|(P_h - \mathbb{1})\phi\|_H \leq Ch^\alpha \|\phi\|_{H^\alpha}$$

for  $\phi \in H^\alpha$ . The Finite Element spaces  $(\mathcal{S}_h, h > 0)$  satisfy this inequality for  $\alpha \leq 2$ . Furthermore, Equation (3.1) is satisfied for the space of piecewise polynomials of degree at most  $\alpha - 1$  on a quasi-uniform triangulation (c.f. Theorem 4.28 in [17] and Satz 6.4 in [9]).

The semidiscrete problem that we are interested in is to find  $X_h(t) \in V_h$  such that for  $t \in [0, T]$

$$dX_h(t) = (A_h + B_h)X_h(t) dt + P_h G(X_h(t)) dM(t), \quad X_h(0) = P_h X_0.$$

Here  $A_h := P_h A P_h$ , and  $B_h := P_h B P_h$ . The operator  $S_h(t)$  refers to the discrete analog of  $S(t)$ , formally introduced by  $S_h(t) = e^{-tA_h}$ . The càdlàg semidiscrete mild solution is given by

$$(3.2) \quad X_h(t) = S_h(t)P_h X_0 + \int_0^t S_h(t-s)B_h X_h(s) ds + \int_0^t S_h(t-s)P_h G(X_h(s)) dM(s).$$

By Assumptions 2.2,  $S_h$  is self-adjoint, positive semidefinite on  $H$  and positive definite on  $V_h$ . We assume that for  $\alpha \geq \beta \geq 0$  with  $\phi \in H^\alpha$  and  $t \in [0, T]$ , we have that

$$(3.3) \quad \|(S(t) - S_h(t)P_h)\phi\|_H \leq C h^\alpha t^{-(\alpha-\beta)/2} \|\phi\|_{H^\beta}.$$

This is for example satisfied by the Finite Element spaces  $(\mathcal{S}_h, h > 0)$  as introduced before for  $\alpha = 2$  (see Theorem 3.5 in [36]). In the more general setting of piecewise polynomials of degree at most  $\alpha - 1$ , Theorem 5.7 in [17] as well as Proposition 11.2.2 in [34] imply Equation (3.3). We note that in the proofs of Theorem 3.1 and Theorem 3.2, Equation (3.3) just has to be satisfied for  $\beta = \alpha$  and  $\beta = \alpha - 1$ . If it only holds for  $\beta = \alpha$ , the theorems stay true when the mild solution satisfies  $\sup_{0 \leq t \leq T} \|X\|_{p, H^{\alpha+1}, t} < +\infty$ .

We shall remark here that we do not approximate the noise. If  $U = H$  and  $V_h$  contains a finite subset of the eigenbasis of  $M$ , the noise is automatically finite dimensional (see e.g. [25]). Otherwise this approximation might not be suitable for simulations. In this case it is possible to truncate the series representation of  $M$ . In [6] it is shown for a class of Lévy processes which properties imply that the overall order of convergence is preserved.

The proposed space discretized equation converges uniformly, almost surely with order  $O(h^{\alpha-\epsilon})$  and with order  $O(h^\alpha)$  in  $L^p$  to the mild solution of Equation (2.1), which is stated in the following two theorems.



**Theorem 3.1.**  $X_h$  converges in  $L^p$  to  $X$  of order  $O(h^\alpha)$ , i.e. for all  $p > 0$

$$\sup_{0 \leq t \leq T} \|X_h - X\|_{p,H,t} = O(h^\alpha).$$

*Proof.* We assume first that  $p > 2$  and we have similarly to the proof of almost sure convergence for SDEs driven by continuous martingales in [5]

$$(3.4) \quad \begin{aligned} & \|X - X_h\|_{p,H,t}^p \\ & \leq 3^{p-1} (\|(S - S_h P_h)X_0\|_{p,H,t}^p \\ & + \mathbb{E}(\|\int_0^t S(t-s)BX(s) ds - \int_0^t S_h(t-s)B_h X_h(s) ds\|_H^p) \\ & + \mathbb{E}(\|\int_0^t S(t-s)G(X(s)) dM(s) - \int_0^t S_h(t-s)P_h G(X_h(s)) dM(s)\|_H^p)), \end{aligned}$$

where we applied Hölder's inequality. The first term satisfies for  $\alpha = \beta$  by Equation (3.3)

$$\|(S - S_h P_h)X_0\|_{p,H,t}^p \leq C h^{p\alpha} \mathbb{E}(\|X_0\|_{H^\alpha}^p).$$

The second one is split into

$$\begin{aligned} & \mathbb{E}(\|\int_0^t S(t-s)BX(s) ds - \int_0^t S_h(t-s)B_h X_h(s) ds\|_H^p) \\ & \leq 3^{p-1} (\mathbb{E}(\|\int_0^t (S(t-s) - S_h(t-s)P_h)BX(s) ds\|_H^p) \\ & + \mathbb{E}(\|\int_0^t S_h(t-s)P_h B(\mathbf{1} - P_h)X(s) ds\|_H^p) \\ & + \mathbb{E}(\|\int_0^t S_h(t-s)B_h(X(s) - X_h(s)) ds\|_H^p).) \end{aligned}$$

The first of these expressions is estimated by the properties of the Bochner integral, Equation (3.3) for  $\beta = \alpha - 1$ , Hölder's inequality, and Fubini's theorem

$$\begin{aligned} & \mathbb{E}(\|\int_0^t (S(t-s) - S_h(t-s)P_h)BX(s) ds\|_H^p) \\ & \leq C h^{p\alpha} (\int_0^t (t-s)^{-p/2(p-1)} ds)^{p-1} \|BX\|_{p,H^{\alpha-1},t}^p \leq C h^{p\alpha} \|X\|_{p,H^\alpha,t}^p. \end{aligned}$$

We apply the properties of the Bochner integral again, Theorem 6.13 in [32], Lemma 2.3, and Equation (3.1) to the second expression, which leads similarly to the previous term to

$$\mathbb{E}(\|\int_0^t S_h(t-s)P_h B(\mathbf{1} - P_h)X(s) ds\|_H^p) \leq C h^{p\alpha} \|X\|_{p,H^\alpha,t}^p.$$

Finally, the third term satisfies

$$\mathbb{E}(\|\int_0^t S_h(t-s)B_h(X(s) - X_h(s)) ds\|_H^p) \leq C \mathbb{E}((\int_0^t (t-s)^{-1/2} \|X(s) - X_h(s)\|_H ds)^p)$$

by the properties of the Bochner integral and Theorem 6.13 in [32]. Hölder's inequality for  $p > 2$  leads to

$$\mathbb{E}(\|\int_0^t S_h(t-s)B_h(X(s) - X_h(s)) ds\|_H^p) \leq C \int_0^t \|X - X_h\|_{p,H,s}^p ds.$$

So overall we have for the second term on the right hand side in (3.4)

$$\begin{aligned} \mathbb{E}(\| \int_0^t S(t-s)BX(s) ds - \int_0^t S_h(t-s)B_hX_h(s) ds \|_H^p) \\ \leq C(h^{p\alpha} \|X\|_{p,H^\alpha,t}^p + \int_0^t \|X - X_h\|_{p,H,s}^p ds) \\ \leq C(h^{p\alpha} \sup_{0 \leq s \leq T} \|X\|_{p,H^\alpha,s}^p + \int_0^t \|X - X_h\|_{p,H,s}^p ds). \end{aligned}$$

The third expression on the right hand side of Equation (3.4) is split into the two following terms

$$\begin{aligned} \mathbb{E}(\| \int_0^t S(t-s)G(X(s)) dM(s) - \int_0^t S_h(t-s)P_hG(X_h(s)) dM(s) \|_H^p) \\ \leq 2^{p-1}(\mathbb{E}(\| \int_0^t (S(t-s) - S_h(t-s)P_h)G(X(s)) dM(s) \|_H^p) \\ + \mathbb{E}(\| \int_0^t S_h(t-s)P_h(G(X(s)) - G(X_h(s))) dM(s) \|_H^p)). \end{aligned}$$

The first of these expressions satisfies by Equation (2.2) and the properties of  $G$

$$\begin{aligned} \mathbb{E}(\| \int_0^t (S(t-s) - S_h(t-s)P_h)G(X(s)) dM(s) \|_H^p) &\leq Ch^{p\alpha}(1 + \|X\|_{p,H^\alpha,t}^p) \\ &\leq Ch^{p\alpha}(1 + \sup_{0 \leq s \leq T} \|X\|_{p,H^\alpha,s}^p). \end{aligned}$$

Similarly, Equation (2.2) yields for the other term

$$\begin{aligned} \mathbb{E}(\| \int_0^t S_h(t-s)P_h(G(X(s)) - G(X_h(s))) dM(s) \|_H^p) \\ \leq C \int_0^t \|G(X) - G(X_h)\|_{p,L(U,H),s}^p ds \end{aligned}$$

and the properties of  $G$  imply

$$\|G(X) - G(X_h)\|_{p,L(U,H),s} \leq \|X - X_h\|_{p,H,s}.$$

So overall we have due to the finiteness of  $\|X\|_{p,H^\alpha,t}$  with Gronwall's inequality

$$\|X - X_h\|_{p,H,t}^p \leq C_1 h^{p\alpha} + C_2 \int_0^t \|X - X_h\|_{p,H,s}^p ds \leq Ch^{p\alpha}$$

for constants  $C_1$ ,  $C_2$ , and  $C$  depending on the mild solution,  $T$ , and  $p$  which implies

$$\sup_{0 \leq t \leq T} \|X - X_h\|_{p,H,t} \leq Ch^\alpha.$$

Finally, for  $t \in [0, T]$  and  $p \leq 2$  we have for any  $\tilde{p} > 2$  by Hölder's inequality

$$\|X_h - X\|_{p,H,t} \leq \|X_h - X\|_{\tilde{p},H,t} = O(h^\alpha). \quad \square$$

This theorem implies almost immediately almost sure convergence as stated in the next theorem.

**Theorem 3.2.** *For every  $\epsilon > 0$  and for  $h > 0$  small enough such that  $h$  goes to zero with order  $O(n^{-\delta})$  for  $n \in \mathbb{N}$  and fixed  $\delta > 0$ ,*

$$\sup_{0 \leq t \leq T} \|X(t) - X_h(t)\|_H \leq h^{\alpha-\epsilon} \quad P\text{-a.s.},$$

*i.e. the approximation  $X_h$  introduced in (3.2) converges uniformly, almost surely to  $X$  of order  $O(h^{\alpha-\epsilon})$  for  $h \rightarrow 0$ .*

*Proof.* Let  $\epsilon > 0$ , then Chebyshev's inequality implies with Theorem 3.1 for all  $t \in [0, T]$

$$P(\|X(t) - X_h(t)\|_H \geq h^{\alpha-\epsilon}) \leq h^{-(\alpha-\epsilon)p} \|X - X_h\|_{p,H,t}^p \leq C h^{p\epsilon}.$$

Since  $h = O(n^{-\delta})$ , the corresponding series is convergent for any  $p > (\epsilon\delta)^{-1}$  and therefore by the Borel–Cantelli lemma we get that

$$\|X(t) - X_h(t)\|_H \leq h^{\alpha-\epsilon}, \quad P\text{-a.s.}$$

for all  $t$ . Let  $K = \mathbb{Q} \cap [0, T]$ ,  $q \in K$ , and  $N_q$  a zero set such that for all  $\omega \in N_q^c$ , which denotes the complement of  $N_q$ ,

$$\|X(q, \omega) - X_h(q, \omega)\|_H \leq h^{\alpha-\epsilon},$$

then  $P(\bigcup_{q \in K} N_q) = 0$  and

$$\sup_{q \in K} \|X(q, \omega) - X_h(q, \omega)\|_H \leq h^{\alpha-\epsilon}.$$

We have for  $t \in [0, T)$  that there exists a decreasing series  $(t_n, n \in K)$  that converges to  $t$  and for all  $\omega \in \bigcap_{q \in K} N_q^c$

$$\|X(t_n, \omega) - X_h(t_n, \omega)\|_H \leq h^{\alpha-\epsilon}$$

for all  $n$ . As  $t \mapsto \|X(t) - X_h(t)\|_H$  is càdlàg, the continuity from the right hand side implies that

$$\|X(t, \omega) - X_h(t, \omega)\|_H \leq h^{\alpha-\epsilon}.$$

If  $T$  is not rational, we set  $N = \bigcup_{q \in K} N_q \cup N_T$  which satisfies  $P(N) = 0$  and asymptotically for all  $\omega \in N^c$

$$\sup_{0 \leq t \leq T} \|X(t, \omega) - X_h(t, \omega)\|_H \leq h^{\alpha-\epsilon},$$

which proves the theorem.  $\square$

#### 4. SIMULATIONS

In this section some simulation results are shown. The approximation of the time and the noise are done in such a way that the error of the space approximation dominates. We simulate similarly to [5] the heat equation driven by additive Lévy noise

$$dX(t) = \Delta X(t) dt + dL(t)$$

on the space interval  $[0, \pi]$  and the time interval  $[0, 1]$  with initial condition  $X(0) = \sin(x)$ . The covariance kernel  $C_Q$  of the Lévy process  $L$  is given by

$$C_Q(x, y) = \exp(-10(x - y)^2)$$

and constructed of independent real-valued Lévy processes  $(L_i, i \in \mathbb{N})$ . For every  $i \in \mathbb{N}$  we construct  $L_i = W_i + P_i$ , where  $W_i$  is a Brownian motion and  $P_i$  a compound Poisson process whose jump intensity is 1638.4 and whose jump sizes are given by the product of a Gamma distributed random variable with parameters  $\Gamma(2, 5)$  and a uniformly distributed random

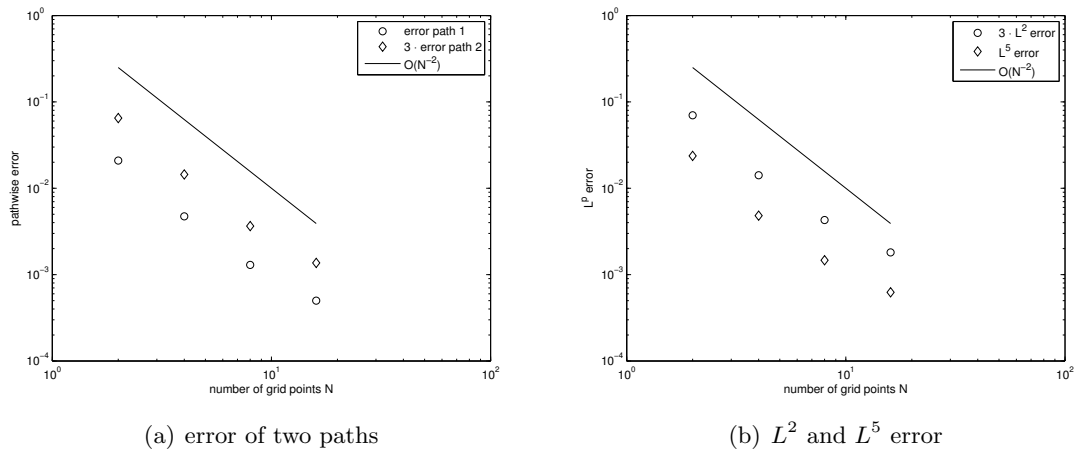


FIGURE 1. Error estimates of the heat equation with additive Lévy noise.

variable on  $\{-1, 1\}$ . The space discretization is done with a Finite Element method and the hat functions, i.e. with the spaces  $(\mathcal{S}_h, h > 0)$  of piecewise linear, continuous polynomials which were introduced in Section 3. Then Theorem 3.1 implies that all moments converge of order 2 and by Theorem 3.2 every path converges asymptotically of order  $2 - \epsilon$  for  $\epsilon > 0$ . Here additionally we use a Crank–Nicolson method for the time stepping and truncate the Karhunen–Loève expansion of the Lévy process according to Lemma 3.1 in [5] to be able to do simulations. We choose the time and noise approximations such that the error of the space approximations dominates the errors.

For the simulation of the error, the solution on a grid with  $2^7$  points in space and  $2^{14}$  points in time was taken as exact solution and compared with the solution on the grids with  $2^n$  points for  $n = 2, \dots, 5$ . In Figure 1(a) the error of two paths of the solution is plotted. As reference, the curve with slope  $N^{-2} = O(h^2)$  is included. The error of path 2, which is denoted with diamonds, is scaled by a factor of 3. Otherwise, the errors of the two paths are almost indistinguishable. Additionally strong errors in  $L^p$  for  $p = 1, \dots, 5$  are estimated by a Monte Carlo method with 1000 paths. The errors are all similar. In Figure 1(b), the scaled error in  $L^2$  and the  $L^5$  error are shown. It attracts attention that the error of approximation with  $2^5$  grid points is not as good as the others. This is due to the chosen reference solution but a simulation on a finer grid would have needed at least  $2^{16}$  time discretization points and increased the computational costs enormously.

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