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Journal Article

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Precision calculation of the recoil–finite-size correction for the hyperfine splitting in muonic and electronic hydrogen

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A B S T R A C T

We present a high-precision calculation of the recoil–finite-size correction to the hyperfine splitting (HFS) in muonic and electronic hydrogen based on nucleon electromagnetic form factors obtained from dispersion theory. This will help guide the upcoming searches of the HFS transition in muonic hydrogen, and will allow a precise determination of the sum of the polarizability and Zemach radius contributions when this transition is found.

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1. Introduction

Laser spectroscopy of muonic hydrogen (μp), an atom formed by a negatively charged muon and a proton, represents an excellent pathway to investigate low-energy properties of the proton. The exquisite sensitivity of the muonic hydrogen energy levels to the proton structure rests on the large muon mass, 207 times larger than the electron mass, that leads to a 10^7 times larger overlap between the atomic wavefunction and the proton compared to regular (electronic) hydrogen, abbreviated as H in what follows.

The measurement of the 2S-2P energy splitting by the CREMA collaboration with 1×10^{-5} relative accuracy [1,2] and its comparison with the corresponding theoretical prediction (we use here the equation of Ref. [3])

$$E_{2P-2S}^{\text{th}} = 206.03470(3) - 5.2275(10)r_p^2 - \Delta E_{2S-2P}^{2\gamma} \quad \text{[meV]} \tag{1}$$

can be used either to extract the proton charge radius r_p with unprecedented accuracy when assuming the two-photon exchange contribution $\Delta E_{2S-2P}^{2\gamma}$ from theory (and measured data from electron-proton scattering), or to extract $\Delta E^{2\gamma}$ when assuming a proton charge radius from H or electron-proton scattering. Using the evaluation of the 2γ-exchange $\Delta E_{2S-2P}^{2\gamma} = -33(2)\mu$ eV by Ref. [4], a proton radius value of $r_p = 0.84099(36)$ fm is obtained from μp [3]. This value is in agreement with the best and most recent determination from electron-nucleon scattering and e^+e^- annihilation data based on dispersion theory, $r_p = 0.840^{+0.003+0.002}_{-0.002-0.002}$ fm [5]. These numbers agree within errors, but clearly the muonic hydrogen result is more precise. Note further that there has been (and still is) some tension with several other determinations from H spectroscopy and electron-proton scattering, that continues to spark lively discussions and triggering more investigations across various fields, such as H spectroscopy or further proton form factor measurements with electron and muon beams. For an update of the present situation we refer to recent review articles, see e.g. [3,6–9].

While the 2S-2P energy splitting is sensitive to electric properties of the proton as the proton charge radius, the hyperfine splitting (HFS) is sensitive also to magnetic properties of the proton as it arises from the interaction between the proton and muon magnetic moments. To leading order, this interaction between magnetic moments yields an energy splitting expressed in terms of the Fermi energy

$$E_{\text{F}} = \frac{8(Z\alpha)^4 m_r^3 (1 + \kappa)}{3mM} = 182.443 \quad \text{[meV]}, \tag{2}$$

where m is the muon mass, M the proton mass, m_r the reduced mass of the μp system, α the fine-structure constant and κ the anomalous magnetic moment of the proton. Radiative, recoil, relativistic and proton structure dependent contributions modify this energy splitting [8,10–13]. For the HFS of the ground state in μp the updated theory takes the form [3]:

$$
\begin{aligned}
E_{\mathrm{HFS}}^{\mathrm{th}}(\mu\mathrm{p}) &= E_F + \Delta E_{\mathrm{QED}} + \Delta E^{2\gamma} \\
&= 182.443 + 1.354(7) + E_F\left(1.01958(13)\Delta_Z^{\mu\mathrm{p}} + 1.01656(4)\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}} + 1.00402\Delta_{\mathrm{pol}}^{\mu\mathrm{p}}\right) \text{ [meV]} .
\end{aligned}
\tag{3}
$$

The second term $\Delta E_{\mathrm{QED}} = 1.354(7)$ meV is the sum of all calculated QED contributions, including minor weak (Z-exchange) and hadronic vacuum polarization contributions. For the HFS, the leading proton structure contribution is given by the two-photon-exchange contribution $\Delta E^{2\gamma}$, which is conventionally divided into a Zemach radius contribution $\Delta_Z^{\mu\mathrm{p}}$, a recoil contribution $\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}}$ and a polarizability contribution $\Delta_{\mathrm{pol}}^{\mu\mathrm{p}}$ [14,15,17–20]. While the sum of these three structure-dependent contributions is unambiguous, the separation between the recoil and polarizability corrections depends upon a protocol [14], here we use the formalism as presented in Ref. [19].

To give an idea of their sizes, these contributions are typically expressed in terms of the Fermi energy E_F, and their value is about $\Delta_Z^{\mu\mathrm{p}} \approx 7500$ ppm, $\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}} \approx 850$ ppm and $\Delta_{\mathrm{pol}}^{\mu\mathrm{p}} \approx 350$ ppm (see e.g. [17]). The small deviations from unity of the numerical coefficients in Eq. (3) arises from radiative corrections. All three coefficients include wavefunction corrections caused by the one-loop electron vacuum polarisation while the coefficient in front of $\Delta_Z^{\mu\mathrm{p}}$ accounts also for the electron-vacuum polarisation insertion in the two-photon exchange diagram [3].

In a dispersive framework [14,15,17–21], all the three contributions forming $\Delta E^{2\gamma}$ can be expressed in terms of phenomenological (measurable) quantities of the proton structure. The Zemach contribution $\Delta_Z^{\mu\mathrm{p}}$ that accounts for the elastic part of the two-photon exchange contribution can be expressed through the electric, $G_E(Q^2)$, and magnetic, $G_M(Q^2)$, Sachs form factors:

$$
\Delta_Z = -2Z\alpha m_r r_Z ,
\tag{4}
$$

where Z is the atomic number and r_Z the Zemach radius defined as [22]

$$
r_Z = -\frac{4}{\pi}\int_0^\infty \frac{\mathrm{d}Q}{Q^2}\left[\frac{G_E(Q^2)G_M(Q^2)}{1+\kappa} - 1\right] .
\tag{5}
$$

For early work on this moment of the charge/magnetization distribution of the proton, see e.g. [22,23], and for the most recent ones, see e.g. [5,24,25]. For a precise definition of the squared momentum transfer Q^2, see Sect. 2.

The so-called recoil contribution, which more precisely is the recoil correction to the Zemach contribution, can also be described solely by form factors. In addition to $G_E(Q^2)$ and $G_M(Q^2)$ in this case also the Dirac $F_1(Q^2)$ and Pauli $F_2(Q^2)$ form factors are used (see the Supplement of Ref. [3] and some earlier references [11,22]):

$$
\begin{aligned}
\Delta_{\mathrm{recoil}} = \frac{Z\alpha}{\pi(1+\kappa)}\int_0^\infty \frac{\mathrm{d}Q}{Q}\Bigg\{ &\frac{G_M(Q^2)}{Q^2}\frac{8mM}{v_l+v}\left(2F_1(Q^2) + \frac{F_1(Q^2)+3F_2(Q^2)}{(v_l+1)(v+1)}\right) \\
&- \frac{8m_r G_M(Q^2)G_E(Q^2)}{Q} - \frac{mF_2^2(Q^2)}{M}\frac{5+4v_l}{(1+v_l)^2}\Bigg\} ,
\end{aligned}
\tag{6}
$$

where $v = \sqrt{1+4M^2/Q^2}$ and $v_l = \sqrt{1+4m^2/Q^2}$. For earlier calculations of this quantity, see e.g. Refs. [14,17]. Recent work on the recoil corrections can be found in Ref. [26]. Differently the polarizability contribution that accounts for the inelastic part of the two-photon exchange contribution can be expressed through integrals over the inelastic structure functions $g_i(x, Q^2)$ and the Pauli form factor $F_2(Q^2)$. The interested reader can find them e.g. in Refs. [14,18]. Note that the polarisability contribution obtained from the dispersive approach [14,19] is derived from the Compton scattering amplitude with finite proton mass so that in this framework no recoil corrections to the polarizability contribution are needed.

A precise evaluation of $\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}}$ is timely given the ongoing experimental efforts carried out by three collaborations that aim at the HFS in μp [27–29] with relative accuracies ranging from 1 to 10 ppm. While comparing with the measured HFS in muonic hydrogen, the theoretical prediction of Eq. (3) can be used to extract the total two-photon exchange contribution $\Delta E^{2\gamma}$, the interpretation of the experimentally obtained $\Delta E^{2\gamma}$ requires a precise knowledge of the recoil contribution. Indeed, in order to extract the polarisability contribution $\Delta_{\mathrm{pol}}^{\mu\mathrm{p}}$ or the Zemach radius r_Z from the measured HFS, the recoil contribution $\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}}$ has to be subtracted from the empirically determined $\Delta E^{2\gamma}$. The purpose of this paper is thus to reduce the uncertainty of $\Delta_{\mathrm{recoil}}^{\mu\mathrm{p}}$, presently on the 5 ppm level [30], to maximize the physics interpretation of the HFS measurements when they will be available.

From the theoretical side, the formalism can be straightforwardly extracted from the muonic case to the H case, by replacing the muon mass by the electron mass and correspondingly the reduced mass of the lepton-proton bound state and the lepton velocity v_l. For completeness and to give a sense of the size of the various corrections we report a summary of the theory in H in a form analogous to Eq. (3). The HFS for the ground state in H from Ref. [3] is

$$
E_{\mathrm{HFS}}^{\mathrm{th}}(\mathrm{H}) = 1418840.082(9) + 1613.024(3) + E_F^{\mathrm{H}}\left(1.01558(13)\Delta_Z^{\mathrm{H}} + 0.99807(13)\Delta_{\mathrm{recoil}}^{\mathrm{H}} + 1.00002\Delta_{\mathrm{pol}}^{\mathrm{H}}\right) \text{ [kHz]}
\tag{7}
$$

where the Fermi energy for hydrogen is $E_F^{\mathrm{H}} = 1418840.082(9)$ kHz.

Evaluating $\Delta_{\mathrm{recoil}}^{\mathrm{H}}$ is interesting for the same reason as in μp, i.e., for dissecting the polarizability and the Zemach radius contributions from the measurement of the HFS in hydrogen. Moreover, an improvement of $\Delta_{\mathrm{recoil}}^{\mathrm{H}}$ can also be used to improve on the prediction of two-photon exchange contribution in μp via the scaling procedure presented in Ref. [13].
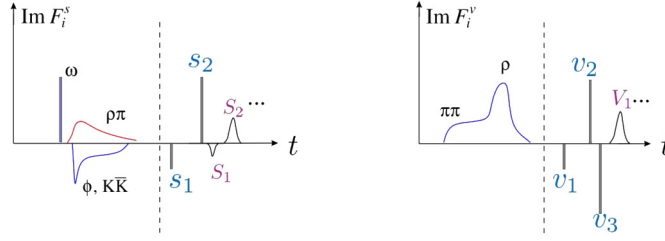
Fig. 1. Cartoon of the nucleon spectral function. Left panel: Isoscalar case. Here, the ω and ϕ mesons are relevant together with the $\pi\rho$ and $K\bar{K}$ continua, while $s_1, s_2, ...$ are narrow and $S_1, S_2, ...$ are broad effective poles. Right panel: Isovector case. Here, the $\pi\pi$ continuum not only generates the ρ but is also visibly enhanced on the left shoulder of the ρ. Further, $v_1, v_2, v_3, ...$ are narrow and $V_1, ...$ are broad effective poles.

This paper is organized in the following way. Sect. 2 contains a brief review of the underlying dispersion-theoretical formalism and recalls the pertinent results from Ref. [5] used here. The results for the recoil correction in muonic as well as electronic hydrogen are displayed and discussed in Sect. 3.

2. Formalism

To set the stage, we briefly define the nucleon electromagnetic form factors. In fact, for the dispersive analysis it is mandatory to consider protons and neutrons together, for details see the review [6]. Only later we will specialize to the proton case (as already done in the introduction). These form factors are given by the matrix element of the electromagnetic current j^μ sandwiched between nucleon states,

$$\langle p'|j_\mu^{\text{em}}|p\rangle = \bar{u}(p')\left[F_1(t)\gamma_\mu + i\frac{F_2(t)}{2M}\sigma_{\mu\nu}q^\nu\right]u(p), \tag{8}$$

with M the nucleon mass (either proton or neutron), $u(p)$ a conventional nucleon spinor and $t = (p' - p)^2$ the four-momentum transfer squared. In the space-like region of relevance here, one often uses the variable $Q^2 = -t > 0$, cf. Eq. (6). The form factors are normalized as

$$F_1^p(0) = 1, \quad F_1^n(0) = 0, \quad F_2^p(0) = \kappa_p, \quad F_2^n(0) = \kappa_n, \tag{9}$$

with $\kappa_p = 1.793$ and $\kappa_n = -1.913$ the anomalous magnetic moment of the proton and the neutron, respectively. Also used are the Sachs form factors, given by

$$G_E(t) = F_1(t) - \tau F_2(t), \quad G_M(t) = F_1(t) + F_2(t), \tag{10}$$

where $\tau = -t/(4M^2)$. The proton charge radius r_p follows as

$$r_p^2 = 6\frac{dG_p^E(t)}{dt}\bigg|_{t=0}. \tag{11}$$

Next, we turn to the dispersive analysis of the nucleon electromagnetic form factors. For a generic form factor $F(t)$, one writes down an unsubtracted dispersion relation of the form:

$$F(t) = \frac{1}{\pi}\int_{t_0}^\infty \frac{\text{Im}\,F(t')}{t' - t - i\epsilon}\,dt', \tag{12}$$

where t_0 is the threshold of the lowest cut of $F(t)$ and the $i\epsilon$ defines the integral for values of t on the cut. In fact, in the isospin basis, $t_0 = 4M_\pi^2$ in the isovector and $t_0 = 9M_\pi^2$ in the isoscalar channel, respectively. The imaginary part $\text{Im}\,F$, the so-called spectral function, encodes the constraints from analyticity and unitarity besides other important physics. These spectral functions are given in terms of continua, narrow vector meson poles as well as broad vector mesons. In the isovector case, the spectral function can be reconstructed up to about ~ 1 GeV^2 from data on pion-nucleon scattering and the pion vector form factor, as most precisely done in Ref. [31]. This in fact not only generates the ρ-meson but also an important enhancement on the left shoulder of the ρ, that is of utmost importance to properly describe the nucleon isovector radii. In the isoscalar spectral function, the ω-meson represents the lowest contribution, that is not affected by uncorrelated three-pion exchange. Further up, in the region of the ϕ-meson, there is a strong competition between $K\bar{K}$ and $\pi\rho$ effects, which to some extent suppresses this part of the spectral function. For momenta above ~ 1 GeV^2, effective narrow poles represent the physics at higher energies. To describe the observed oscillations of the cross sections for $e^+e^- \to p\bar{p}$ and $e^+e^- \to n\bar{n}$ in the timelike region, additional broad poles are required. The spectral functions are further constrained by the normalizations of the form factors given in Eq. (9) as well as the perturbative QCD behaviour, $F_1(t) \sim 1/t^2$ and $F_2(t) \sim 1/t^3$. A cartoon of the spectral functions is given in Fig. 1.

The spectral functions are determined from a fit to the world data set on electron-proton scattering as well as the reactions $e^+e^- \leftrightarrow \bar{p}p, \bar{n}n$, the latter giving the form factors in the timelike region. The fit parameters are the vector meson masses (except for the ω and the ϕ) and the residua as well as the widths for the broad poles. There are two sources of uncertainties that need to be accounted for. First, the statistical error is obtained using a bootstrap procedure and second, the systematic error is calculated from varying the number of vector meson poles so that the total χ^2 does not change by more than 1%. A detailed description of these methods is given in the review [6].
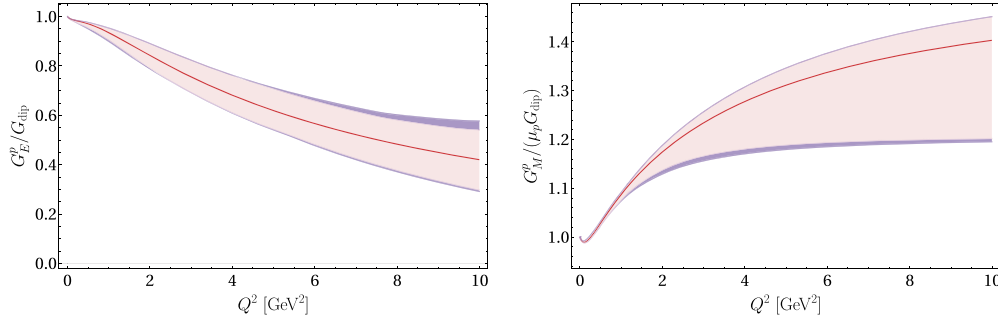
Fig. 2. Electric (left panel) and magnetic (right panel) form factor of the proton from Ref. [5] divided by the canonical dipole form factor are shown by the red lines. The light red band is the statistical uncertainty and the purple band shows the systematic error added in quadrature.

The electric and magnetic form factors of the proton from Ref. [5] normalized to the canonical dipole form, $G_{\text{dip}}(Q^2) = (1 + Q^2/0.71\,\text{GeV}^2)^{-2}$, are shown in Fig. 2 together with their statistical and systematic uncertainties. From these, the proton charge radius and the proton Zemach moment have already been extracted as [5]

$$r_p = 0.840^{+0.003+0.002}_{-0.002-0.002}\,\text{fm}\,, \quad r_z = 1.054^{+0.003+0.000}_{-0.002-0.001}\,\text{fm}\,, \tag{13}$$

where the first error is statistical and the second one is systematic. These values are in good agreement with previous high-precision analyses of the spacelike data alone [6,32] and have comparable errors.

3. Results and discussion

We now turn to the calculation of the recoil correction defined in Eq. (6). Consider first the μp system. We find

$$\Delta^{\mu\text{p}}_{\text{recoil}} = (837.6^{+1.7+2.2}_{-1.0-0.1}) \times 10^{-6} = (837.6^{+2.8}_{-1.0}) \times 10^{-6} = (837.6^{+2.8}_{-1.0})\,\text{ppm}\,, \tag{14}$$

with the first error stemming from the bootstrap and the last one from the variation of the poles (systematic uncertainty). These errors are a few permile, so that this can be considered as a high-precision determination. Compared with the most recent value from Ref. [16], $\Delta^{\mu\text{p}}_{\text{recoil}} = 844(7) \times 10^{-6}$, these numbers agree within errors but our result is more precise.

The analogous value for regular hydrogen is

$$\Delta^{\text{H}}_{\text{recoil}} = (526.9^{+1.1+1.3}_{-0.3-0.2}) \times 10^{-8} = (526.9^{+1.7}_{-0.4}) \times 10^{-8}\,, \tag{15}$$

which is, as expected, two orders of magnitude smaller but with comparable uncertainties as in the μp case. Again, the corresponding number from Ref. [16], $\Delta^{\text{H}}_{\text{recoil}} = 532.8(4.9) \times 10^{-8}$, is about 1% larger but is also a bit less precise. Note that the difference of the two-photon-exchange contribution to the hyperfine splitting between the muonic and electronic hydrogen can be reliably estimated using chiral perturbation theory that is expressed as the difference of the coefficient $c_{4,\text{TPE}}$, that is, $c^{\mu\text{p}}_{4,\text{TPE}} - c^{e\text{p}}_{4,\text{TPE}}$ [13]. Considering only the Zemach (Δ_Z) and recoil (Δ_{recoil}) contributions, we obtain the difference of $c_{4,\text{TPE}}$ is $4.11^{+0.10}_{-0.09}$ that agrees with the chiral perturbation theory value 3.42(62) given by Ref. [13] within errors.

The Zemach radius can be extracted from the HFS measurement using the theory of Eq. (3), with $\Delta^{\mu\text{p}}_{\text{recoil}}$ from this study and assuming $\Delta^{\mu\text{p}}_{\text{pol}}$ from theory. Similarly the polarizability contribution can be extracted using the theory of Eq. (3), with $\Delta^{\mu\text{p}}_{\text{recoil}}$ from this study and taking the Zemach radius from e-p scattering or from H spectroscopy. Pinning down the uncertainty of this recoil–finite-size contribution allows therefore to eliminate the most important higher-order proton-structure dependent contribution that complicates and limits extraction of the leading-order proton-structure effect (Zemach and polarizability contributions) from the μp measurement. The reduced uncertainty of $\Delta^{\mu\text{p}}_{\text{recoil}}$ from this study can become particularly relevant in the scenario that the smaller value of the polarizability contribution predicted by the chiral perturbation theory will be confirmed. Indeed there is presently an interesting tension between the value of $\Delta^{\mu\text{p}}_{\text{pol}}$ predicted in a chiral perturbation theory framework, $\Delta^{\mu\text{p}}_{\text{pol}} = 37(95)$ ppm [3,34,35], and the values obtained from the data-driven approach, e.g. $\Delta^{\mu\text{p}}_{\text{pol}} = 364(89)$ ppm from Ref. [15].

Analogously, the reduced uncertainty of $\Delta^{\text{H}}_{\text{recoil}}$ can be used to improve on the extraction of the polarizability contribution and the Zemach radius from the HFS in H which has been measured with a fractional accuracy of 7×10^{-13} [33]. The relative uncertainty of about 1×10^{-8} of $\Delta^{\text{H}}_{\text{recoil}}$ set also the limit to which theory and experiment can be confronted in H. Testing the hydrogen HFS beyond this relative accuracy requires improving on the proton form factors.

Beside improving the interpretation of the μp HFS measurements when they will be completed, the reduced uncertainty of $\Delta^{\text{H}}_{\text{recoil}}$ can also be used to refine the prediction of two-photon exchange contribution in μp using the scaling procedure presented in Ref. [13]. This serves to narrow down significantly the search range for the HFS transition in μp easing considerably the ongoing experimental efforts.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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