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Geometry and Symmetries in $\mathcal{N} = 4$ super–Yang-Mills theory

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Abstract

In three interdependent parts, the geometry of $\mathcal{N}$-extended superspaces is studied, the symmetries of a novel type of Wilson loop on full $\mathcal{N}=4$ superspace are inspected and an additional symmetry generator residing outside of the hidden Yangian symmetry algebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ of the planar tree-level scattering amplitudes of $\mathcal{N}=4$ super Yang-Mills theory is identified and put into context. In the first part, the geometry of flag manifolds is used to illustrate correspondences between theories on different types of superspaces—most importantly, $\mathcal{N}=4$ SYM on Minkowski superspace and holomorphic Chern-Simons theory on twistor space as well as $\mathcal{N}=3$ SYM on full Minkowski space and complex-real Chern-Simons-theory on $\mathcal{N}=3$ harmonic superspace. The second part develops a treatment of Wilson loops on null polygonal contours in full $\mathcal{N}=4$ superspace and ambitwistor space $\mathbb{A}_{3|4}$ and inspects the symmetries of observables constructed from the one-loop expectation value $\langle\mathcal{W}_n\rangle^{(1)}$. The final part proves the existence of an additional symmetry generator called bonus symmetry $\widehat{\mathfrak{B}}$ of the tree-level amplitudes and leading singularities in planar limit $\mathcal{N}=4$ SYM.

Zusammenfassung

In drei voneinander abhängigen Teilen werden die Geometrie von $\mathcal{N}$-erweiterten Superräumen untersucht, die Konstruktion und Symmetrien von Wilsonschleifen auf lichtartigen polygonalen Integrationskonturen betrachtet und eine neuartige Symmetrie – welche nicht Teil der Yangschen Symmetriealgebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ ist – der Baumdiagramme planarer $\mathcal{N}=4$ Super-Yang-Mills Theorie, wird identifiziert und untersucht. Im ersten Teil wird die Geometrie von Flaggenmanigfaltigkeiten verwendet um Korrespondenzen zwischen Eichtheorien auf verschiedenen Superräumen deutlich zu machen; insbesondere zwischen $\mathcal{N}=4$ SYM über dem vierdimensionalen $\mathcal{N}=4$ Minkowski-Superraum und holomorpher Chern-Simons-Theorie über dem Twistorraum und zwischen $\mathcal{N}=3$ SYM über dem vierdimensionalen $\mathcal{N}=3$ Minkowski-Superraum und komplex-reeller Chern-Simons-Theorie über dem harmonischen $\mathcal{N}=3$ Superraum. Der zweite Teil behandelt Wilsonschleifen auf lichtartigen polygonalen Integrationskonturen auf dem nichtchiralen $\mathcal{N}=4$ Minkowski-Superraum und dem Ambitwistorraum $\mathbb{A}_{3|4}$ und untersucht das Verhalten von Observablen, die aus dem Einschleifenerwartungswert $\langle\mathcal{W}_n\rangle^{(1)}$ gebildet wurden, unter Symmetrieoperationen der Yangschen Symmetriealgebra. Im letzten Teil wird die Existenz eines zusätzlichen Symmetriegenerators der Baumdiagramme und der führenden Singularitäten planarer $\mathcal{N}=4$ SYM-Theorie bewiesen. Dieser zusätzliche Generator trägt den Namen Bonussymmetrie $\widehat{\mathfrak{B}}$.

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First and foremost I thank my parents for their love and their unceasing encouragement.

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A friendly nod goes to the whole of the *HEP group* at ETH Zürich, especially to the members of office HIT K31.7—*Dr. Marius de Leeuw* and *Dr. Johannes Brödel*—and my office mate and "mathematical brother" *Matteo Rosso*, for relaxing and delighting discussions in between bouts of hard science. Thanks also go to *Dr. Song He*, our collaborator on [1].

Part of this work, especially the latter sections of Part 5, has come to be under the IMPRS scholarship while I still worked at the MPI for Gravitational Physics in Potsdam and was enrolled at Humboldt-University in Berlin. I would like to thank both institutions for their hospitality and help. Further cheers go to Till Bargheer, Parikshit Dutta, Philip Fleig, Michael Köhn, Florian Löbbert, Stefan Pfenninger, Bernhard Wurm and everyone else whose names I now cannot remember anymore and who think they should be included in this list.

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Finally, I want to thank *Prof. David Wharam* whose help made me open doors where there had been none.

Thoughts

When I began to work as a researcher, I did not know what research was—how hard it can be. But I learned. I learned that there are things one can do and others one might be able to do, but not yet, and things that one will never do, at least not alone. We learn to become better, to move forward, to do the things we could not do before. Being a researcher is a process of becoming more than what one started with.

This work represents a slice of the research I have done during the four years of my doctoral studies in Potsdam and Zurich. In these four years, times came when I wanted to sit down and give up but I always pushed myself forward and went on. I brooded over problems, I wrote and rend apart, I spotted inadequacies and I strove to overcome them.

Such is the way we walk in our life: Stepping forward, stumbling, falling, failing, and standing up again. Those of us who willingly choose to go through these ordeals shall come out better than they were before—in spirit, in knowledge, in will—or must perish along the way. And perish they will if they don't walk on, or find others that might hold out a hand for them when they fall.

For all those coming after me, this is the hand I hold out: *Fight on, the fruits are sparse, but their nectar is heavenly.*

Contents

1
Introduction

The last 100 years have seen an immense growth in our understanding of the world, and yet, whenever we lift one of the veils that hid the supposed beauty of nature, we tend to be first presented with another veil or find that things have been lurking under it that we wouldn't have wished for to be presented with. This is especially true in physics, which had been declared a closed chapter a number of times already—once by the ancient Greeks, once by our scientific forebears of the late 19$^{\text{th}}$ century. Luckily, our universe is much stranger than we thought, in fact it is much stranger than we can suppose at any given point—if you allow the pun—in time. The two theoretical pillars of modern physics—*quantum theory* and *general relativity*—stand witness to this.

The advent of quantum mechanics took away our firm belief in the clockwork predictability of the universe while the formulation of General Relativity managed to literally take the firmness out of the fabric of time and space. And despite—or because of—their inherent strangeness to human perception, both theories stand strong now after a hundred years of rigorous testing. At the same time, they also present the theoretical researcher with a wealth of mathematical beauty.

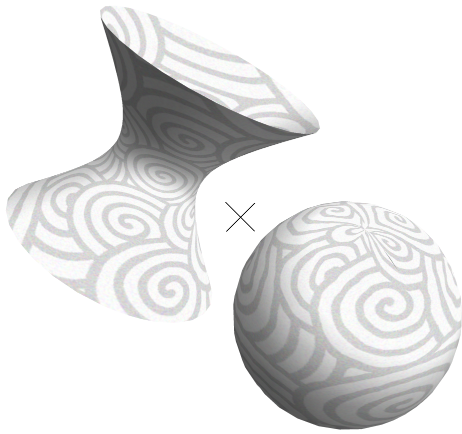
The theory of the small things, quantum theory, proved especially resilient against falsification much to the despair of some physicists and many non-physicists alike. The fact that it does so is fortunate though for all natural sciences, despite the occasional attempt of experimental physicists to shatter our world view by using faulty cables. General relativity on the other hand gave us a much better understanding—compared to Newtonian mechanics—of the workings of the universe. Every new test is only showing how good a theory GR really is, although some predictions like gravitational waves have proven resistant against experimental verification.

Given that we seem to have a very good notion of the natural world with these theories (so good in fact that some researchers are once again declaring our models to be universally true) we were also able to make huge technological progress in the last 100 years. Quantum theory brought us the computer and GR brought us GPS to name a bare minimum for each.

However, for a long time now, the "holy grail" of physics—as some believe it to be—the unification of the big and the small, of quantum field theory and general relativity, evaded even the most dexterous minds. Famously, the quantum theory of Einstein gravity does not exist, we encounter new divergences on every loop level rendering the theory infertile for predictions. The search for a unifying theory led to many failed attempts and culminated in a shift of our point of view from the theoretical concept of particles as point-like structures towards particles as the excitations of extended one-dimensional objects—the so called strings. A musical model of the universe in all its extremes—big and small—was born.

The advent of *string theory* addressed the non-renormalizability of gravity and seemed to bring new hope to this dark vale of tears (of graduate students) by presenting us with a theory that unified gauge degrees of freedom, matter and a theory of quantum gravity. For string theory to work, that is to describe the particle content of the world with bosons and fermions, interactions and matter, amazing steps forward had to be taken: String theory needs supersymmetry to describe fermions, a rather crucial part of a theory of everything. It requires compactification of extra dimensions—the study of the emerging Calabi-Yau-manifolds even started to interest mathematicians. Famously, Maxim Kontsevich first talked about the mathematical background of mirror symmetry in Zurich.

Superstring theory split into *five different descriptions* type I, type IIA, type IIB and the two heterotic string theories with $SO(32)$ and $E_8 \times E_8$ gauge groups respectively. A matter of confusion in itself—after all, theorists thought they were facing one unifying theory not five very different looking ones. Thanks to Edward Witten, we know nowadays that these five theories are very interconnected by dualities and possibly the five shadows of an enigmatic *M-theory* residing in 11 dimensions. However, except for eleven-dimensional supergravity, something that is best described as the "Fermi model" of M-theory, we don't know how to write down M-theory. This situation compares well with the early days of quantum theory when a lot of calculations could be done long before a full theory of quantum physics was formulated.



Sadly, string theory did not bring the expected breakthrough. The world is still waiting for the theory of everything. But we should not be too unhappy about this fact. What string theory brought us was a way to understand the world from a completely different point of view. So different that we probably still haven't grasped the full extent of it—or at least, we definitely haven't mined it to its full potential. This is referring to the fact that certain string theories are very intimately connected to certain gauge theories: a very old relation dating back to the first days of string theory—but most famously demonstrated in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ *correspondence* between type IIB string theory on (the highly symmetrical) $\mathrm{AdS}_5\times\mathrm{S}^5$ background pictured above[1] and maximally supersymmetric Yang-Mills theory on four-dimensional flat Minkowski spacetime ($\mathcal{N}=4$ SYM). This was conjectured by Juan Maldacena—via strong-weak dualities.

This duality allows—via the identifications[2]

$$\lambda = \frac{R^4}{\alpha'^2}, \qquad \frac{1}{N_c} = \frac{4\pi g_s}{\lambda}$$

the calculation of results for non-perturbative free quantum strings by perturbative, planar gauge theory calculations and, conversely, predictions about non-perturbative gauge theory by free classical string observations in its easiest form. Recent years have seen many iterations of the same idea in different settings, a sign that this duality is more than a lucky coincidence or that we spotted a pattern where there is none. In particular Maldacena's conjecture has proven to be a very fruitful hunting ground for string theorists and field theorists alike.

More recently, we witnessed a revolution in $\mathcal{N}=4$ super Yang-Mills theory when the *integrability of the spectrum of the theory* in the *planar limit* was conjectured by Lev Lipatov in [2]. This was shown to be the case at first in the $\mathfrak{so}(6)$ (scalar) sector of the theory [3] and subsequently extended to the full $\mathfrak{psu}(2,2|4)$ sector at one-loop by Niklas Beisert [4]. Further confirmations exist now at ever higher loop orders. We saw structures emerging—the *Bethe Ansatz*, the *Yangian symmetry algebra*—known from integrable two-dimensional field theories, but rather unheard of in a four-dimensional gauge theory.

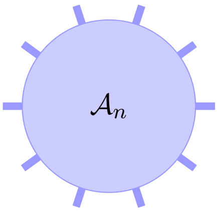
It was found that the planar limit also holds surprises when it comes to the *scattering amplitudes* of $\mathcal{N}=4$. Their modern history can be found in a condensed form in the review [5]. Let us have

[1] The lack of dimensions forces us to draw an AdS_2 and a S^2 and the product in a highly simplified manner.

[2] Here we have $\lambda = g_{\mathrm{YM}}^2 N_c$ the 't Hooft coupling, N_c the number of colors, g_s the string interaction coupling, α' the inverse string tension and R the AdS radius. The planar limit is $N_c \to \infty$ while λ fixed.

a glance at the most important pieces of the development of this topic relevant for this thesis. The n-point tree level amplitudes of $\mathcal{N}=4$ SYM were found to be of an exceptionally beautiful structure when written in a supersymmetric setting, for example all n-leg amplitudes with $n-2$ positive helicity particles and 2 negative helicity particles—so called MHV amplitudes can be written in the supersymmetric Parke-Taylor form [6]

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^4(P)\delta^{(0|8)}(Q)}{\prod_{i=1}^n \langle i, i+1\rangle}.$$



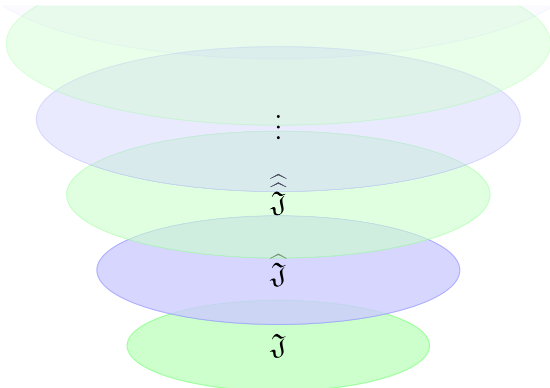
On-shell construction methods—most notably *CSW* [7], and the *BCFW recursion* relations [8, 9]—allow amongst other applications for a most economic way of calculating n-leg amplitudes from lower point amplitudes. This simplicity of scattering amplitudes made many researchers wonder whether there was a more fundamental description of amplitudes that would make their easy structure manifest when they were formulated as graphs—not necessarily as Feynman graphs.

Twistor theory provides one way of thinking about amplitudes in $\mathcal{N}=4$ in a novel way that makes their marvelous simplicity clearer. A clear step forward was achieved when tree-level N^{k}MHV amplitudes could be shown to be all assembled from R-invariants

$$[i,j,k,l,m] = \int_{\mathbb{CP}^1} \frac{D^4c}{c_1c_2c_3c_4c_5}\delta^{4|4}\Big(c_1\mathcal{Z}_i + c_2\mathcal{Z}_j + c_3\mathcal{Z}_k + c_4\mathcal{Z}_l + c_5\mathcal{Z}_m\Big),$$

i.e., from a single twistorial object—and the factor $\mathcal{A}_n$ [10]. Twistor theory led also to the formulation of $\mathcal{N}=4$ SYM as a holomorphic Chern-Simons theory [11]. This we will explore in more detail in the main body of this text.

The simplicity of scattering amplitudes in $\mathcal{N}=4$ SYM can also be traced back to the existence of a *dual superconformal symmetry* [12] along with the already known superconformal symmetry of the Lagrangian. The form of these additional symmetries are such that they do not render the $\mathcal{S}$-matrix trivial as one would expect from the Coleman-Mandula theorem. However, the additional symmetries do constrain the form of the scattering amplitudes more than the ordinary Lagrangian superconformal symmetry alone would.

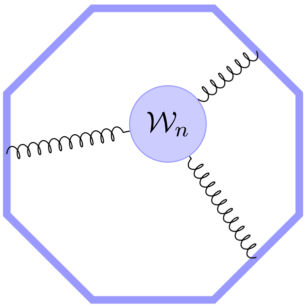


Y[$\mathfrak{psu}(2,2|4)$] as an infinite stack of generators

Interestingly, these two symmetries are not independent of one another, but rather they conspire to form a bigger, infinite algebra [13]: This is the famous *Yangian algebra* Y[$\mathfrak{psu}(2,2|4)$] of $\mathcal{N}=4$ SYM—depicted above as an infinite stack of generators of ever higher level. In parts, this thesis will be concerned with the structure of this Yangian. For a symmetry algebra, $\mathfrak{psu}(2,2|4)$ is quite unusual: As a projective real form of $\mathfrak{sl}(4|4)$ it belongs to the $A(n|n)$ family of superalgebras ($n \geq 1$) in Kac's classification. These have a degenerate Killing form.

This also makes the structure of the associated Yangian interesting and uncommon. For example, the way Yangian generators act on amplitudes is by construction dependent on the (arbitrary) cyclic labeling of the external legs of amplitudes. Especially, this dependence a priori breaks the cyclic symmetry of the amplitudes. However, the degenerate Killing form of $\mathfrak{psu}(2,2|4)$ saves the Yangian structure here and in fact imposes compatibility with cyclicity. This is a veritable miracle.

This curious form of the Yangian holds other mysteries: in this thesis we will show that there is an additional Yangian generator *inaccessible* from commutation relations of the known generators. A structure like this is known in the literature [14] as a *secret symmetry.* In this special case, we prefer the name *bonus symmetry* [15].

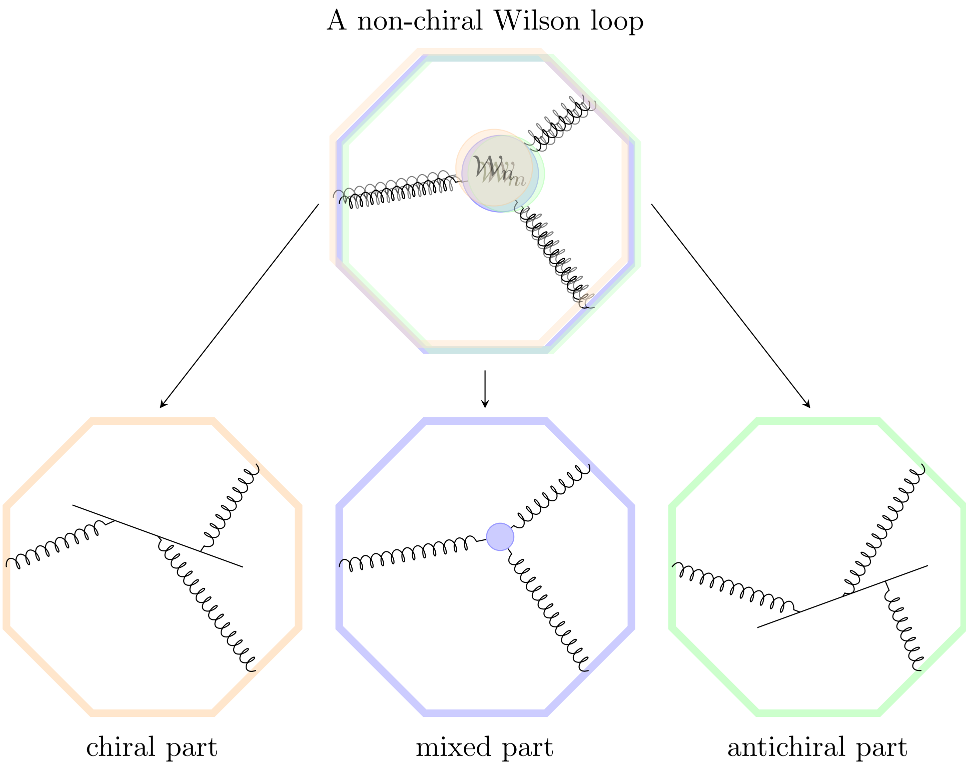


A Wilson loop

The discovery of dual superconformal symmetry was accompanied by the discovery of the remarkable weak-weak duality between scattering amplitudes $\mathcal{A}_n$ and Wilson loops $\mathcal{W}_n$ on null polygonal contours in chiral superspace [16]. At first, the duality was established between MHV gluon scattering amplitudes and bosonic Wilson loops on null polygonal integration contours by direct calculation using various identifications of the parameters defining the two objects. To capture the supersymmetry of $\mathcal{N} = 4$ SYM and establish a duality for all tree-level scattering amplitudes, supersymmetric Wilson loop generalizations were proposed [17, 10]. Most important for this thesis was the discovery that these supersymmetric generalizations did not actually produce the desired duality [18]. Rather it was discovered that super-Wilson loops notice the chirality of the superspace and the calculated results break the supersymmetry generator $\bar{\mathfrak{Q}}$.

Here, we present a workaround for this problem by considering the theory and the Wilson loops on non-chiral superspace [1]. The immediate benefit of this approach is that the troublesome $\bar{\mathfrak{Q}}$-anomaly does not occur in this setting. Furthermore, at the one-loop level the non-chiral loop contains three sectors, two of which are the chiral Wilson loop and its antichiral conjugate and one is a mixed loop made from chiral as well as antichiral pieces—see also the figure below. The drawback is that our particular Wilson loop is not dual to the scattering amplitudes of $\mathcal{N} = 4$ SYM anymore—at least not as directly as the chiral Wilson loops were meant to be. While we present a one-loop calculation, we also examine the behavior of the Wilson loop expectation value under Yangian transformations. We can establish that the non-chiral Wilson loop is indeed a Yangian invariant up to ultraviolet divergences, which break the Yangian generators indiscriminately. By choosing ad-hoc regularization methods, we can however salvage some of the superconformal symmetries.

In work related to an attempt at generalizing the methods used in the calculation of the non-chiral Wilson loop [1], we made use of some interesting geometrical ideas related to the definition of twistor variables [19]. Our ability to calculate Wilson loops on light-like contours with a *certain* ease is directly linked to the fact that the constraints governing supersymmetric Yang-Mills theories are flat when pulled back to such lines. Even better, any such simplification—the flatness of the defining constraints of SYM—can be traced back to a mathematical concept known as *double fibrations.* This has been known for quite some time, we will present the fundamentals in the first part, where we also present some original work connected to the extraction of local operators on spacetime from twistor fields and fields in twistor-like theories.



Outline.—To conclude this introduction, let us outline the structure of the text. In the first part following this introduction we will give a very short, rather non-technical introduction to planar $\mathcal{N} = 4$ SYM and its *hallmark features*.

The following part will be concerned with the development of a mathematical toolbox for Yang-Mills theories which will include the concepts of *flag manifolds* and *double fibrations*. We will also be concerned with the connection between these more general concepts and the *harmonic approach* to Yang-Mills theories. We will use these tools to understand the correspondence between $\mathcal{N} = 4$ super-Yang-Mills theories and *twistor holomorphic Chern-Simons theory* (hCS) as a proof of concept and then show that $\mathcal{N} = 3$ SYM has a formulation as a holomorphic Chern-Simons theory on *harmonic superspace*, too. Furthermore, we will give an extension of the familiar Penrose transformation that will let us extract local operators on Minkowski superspace from gauge fields in twistor-like theories. The approach is in principle generalizable to any two theories that are in correspondence.

The fourth part contains the formulation and calculation of the expectation value of the proposed *non-chiral Wilson loop* to first loop order. We begin by deriving the *light-like lines* in full $\mathcal{N} = 4$ Minkowski superspace $\mathbb{M}^{4|16}$ and make use of the equivalence of these submanifolds of superspace with points in *ambitwistor space* $\mathbb{A}_{3|4}$. This is a classical example of a double fibration. Subsequently, we formulate the Wilson loop and show the calculation of the first loop correction of the expectation value in the case where no *ultraviolet divergences* occur. We proceed with the treatment of divergences and give three distinct *(ad-hoc) regularization methods* to treat the divergences. Each of the presented regularizations has advantages and disadvantages and we will point them out.

In the last part, the behavior of the regularized Wilson loop expectation values under *symmetry transformations* is studied. All regularizations break some generators of the superconformal symmetries and destroy Yangian symmetry. After these considerations we finally show the existence of an *additional generator* on the first level of the Yangian symmetry algebra.

The attached appendices give *background material* that was considered necessary but too extensive to be incorporated into the main text. The reader will find a short treatment on *propagators* in quantum field theories, an introduction to the features of the *algebra* $\mathfrak{psu}(2,2|4)$ and some of its realizations, the definition of the *Yangian* as given by Drinfel'd, and a lighting review of *coset spaces and CR structures* as well as the conventions.

This thesis is based on the published articles [19, 1, 15].

2

$\mathcal{N} = 4$ SYM: Overview

CHAPTER I

Preliminaries

In this chapter we will introduce the reader to some of the features of maximally supersymmetric Yang-Mills theory. We usually tend to shorten this name and call the theory "$\mathcal{N} = 4$ SYM". $\mathcal{N} = 4$ SYM has taken a very important place in modern theoretical high energy physics. Not only is maximally supersymmetric Yang-Mills theory unique, i.e., there is no other theory in four flat dimensions with so much supersymmetry, it is also superconformal even on the quantum level and finite.

These ingredients make $\mathcal{N} = 4$ SYM an ideal playground for theoretical physicists: Specific results that have been obtained through $\mathcal{N} = 4$—especially in connection with the scattering amplitudes of the theory—are universal enough to be applied in other theories. This is most notably true of the gluon scattering amplitudes of $\mathcal{N} = 4$ SYM, which match exactly with the gluon amplitudes of QCD.

Moreover, $\mathcal{N} = 4$ SYM takes a special position also for string theory. The conjectured strong-weak duality between $\mathcal{N} = 4$ SYM and type IIB string theory on the $AdS_5 \times S^5$ background make predictions of non-perturbative results possible. The integrability of type IIB string theory in the regime where it is conjectured to be dual to weakly coupled planar $\mathcal{N} = 4$ SYM allows us to exploit many of the tools that have been invented and explored for the study of integrable systems. Let us proceed to inspect some of the features of this model.

I.1 $\mathcal{N} = 4$ multiplet and Lagrangian

The field content of $\mathcal{N} = 4$ SYM consists of a gauge field A_μ, four Weyl spinors Ψ_a and $\bar{\Psi}^a$, and six real scalars Φ_i transforming in the adjoint representation of the gauge group G which will be taken to be $SU(N_c)$ throughout this work. The supersymmetric Lagrangian of the theory has been known in components for a long time [20]. Introducing a covariant derivative $D_\mu = \partial_\mu - iA_\mu$ it is given by

$$\begin{aligned}\mathcal{L} = \frac{2}{g_{\mathrm{YM}}^2} \operatorname{tr} \Bigg(&-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi}^a_{\dot{\alpha}} \sigma_\mu^{\dot{\alpha}\beta} D^\mu \Psi_{\beta a} + \frac{1}{2} D_\mu \Phi^i D^\mu \Phi_i - \frac{1}{4} [\Phi^i, \Phi^j][\Phi_i, \Phi_j] \\ &-\frac{1}{2} i \Psi_{a\alpha} \gamma_i^{ab} \epsilon^{\alpha\beta} [\Phi^i, \Psi_{\beta b}] - \frac{1}{2} i \bar{\Psi}^a_{\dot{\alpha}} \gamma^i_{ab} \epsilon^{\dot{\alpha}\dot{\beta}} [\Phi_i, \bar{\Psi}^b_{\dot{\beta}}] \Bigg) . \end{aligned} \tag{I.1}$$

In the Lagrangian above, the field strength

$$F_{\mu\nu} = i[D_\mu, D_\nu] \tag{I.2}$$

was introduced. The Lagrangian was derived by a dimensional reduction of ten-dimensional Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry. In this process the ten-dimensional Γ matrices get split into a four- and a six-dimensional Clifford algebra. Hence, the four-dimensional σ_μ and the six-dimensional γ_i obey

$$\{\sigma_\mu, \sigma_\nu\} = 2\eta_{\mu\nu}, \qquad \{\gamma_i, \gamma_j\} = 2\eta_{ij}. \tag{I.3}$$

$\mathcal{N} = 4$ SYM does not allow an off-shell formulation with linearly realized supersymmetry [21]. However, for $\mathcal{N} = 3$ such a formulation exists. On-shell, the two theories are equivalent. In chapter VI the problem of off-shell formulations of $\mathcal{N} = 3$ SYM will be revisited.

I.2 Gauge symmetry

All the fields in the $\mathcal{N} = 4$ multiplet transform in the adjoint representation of the gauge group, which we take to be $SU(N_c)$. N_c denotes the number of colors, that is, the number of distinct color charges[1]. The difference between the linearized (Abelian) theory and the non-Abelian theory will be important in part 4, so we give a short explanation about the action of the gauge group.

In $SU(N_c)$ Yang-Mills theory, the covariant derivative D_μ transforms in the adjoint representation of the gauge group, that is, under a gauge transformation $g(x)$ we find that

$$D_\mu \mapsto D_\mu^g = g^{-1} D_\mu g = g^{-1}(\partial_\mu - iA_\mu)g. \tag{I.4}$$

So the gauge field $A_\mu(x) = A_\mu^a(x)\mathfrak{E}^\alpha$ itself transforms under a gauge transformation $g(x)$ like a connection

$$A_\mu \mapsto A_\mu^g = g^{-1} A_\mu g + i g^{-1} \partial_\mu g. \tag{I.5}$$

$\mathfrak{E}^a$ is one of the generators of the gauge algebra $\mathfrak{su}(N_c)$, satisfying the Lie algebra property

$$[\mathfrak{E}^a, \mathfrak{E}^b] = i f^{ab}{}_c \mathfrak{E}^c \tag{I.6}$$

with $f^{ab}{}_c$ the structure constants of $\mathfrak{su}(N_c)$. The structure constants also define the adjoint representation $\mathbf{Adj_{N_c}}$ via (I.6) and the homomorphism

$$[\mathfrak{E}^a, \mathfrak{E}^b] = (f^a)^b{}_c \mathfrak{E}^c. \tag{I.7}$$

The adjoint representation is the representation of the algebra on itself as a vector space i.e., $(f^a)^b{}_c$ represents $\mathfrak{E}^a$.

When specializing to a linearized description, one essentially restricts Yang-Mills theory to a Maxwell-like theory with gauge group $U(1)^{N_c^2-1}$, i.e., all gluons are turned into photons. In this case the gauge transformation is given by

$$A_\mu^{\text{lin}} \to A_\mu^{\text{lin}} - \partial_\mu \alpha \tag{I.8}$$

[1] And as an easy physical corollary of representation theory: It is also the number of partons necessary to form a colorless composite (baryonic) particle.

where $\alpha = \alpha(x)$ is the gauge parameter.

While the gauge field and the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ transform in the way described above, the field strength of Yang-Mills theory

$$F_{\mu\nu} = F^a_{\mu\nu}\mathfrak{E}^a = i[D_\mu, D_\nu] \tag{I.9}$$

transforms in the adjoint representation

$$F_{\mu\nu} \to g^{-1}F_{\mu\nu}g. \tag{I.10}$$

In the Abelian theory $F_{\mu\nu}$ is therefore gauge-invariant. In the supersymmetric gauge field theories ($\mathcal{N} = 1, 2, 3, 4$), this transformation behavior translates to the other fields $\Psi, \bar{\Psi}$, and Φ.

I.3 Spinor-Helicity variables

Spinor-helicity variables have been invaluable for the development of our understanding of scattering amplitudes in gauge theories [22, 23]. All particles in $\mathcal{N} = 4$ SYM are massless, hence their momenta p^μ have to square to zero

$$p^2 = 0. \tag{I.11}$$

Using the extended Pauli matrices ${\sigma^\mu}_{\alpha\dot{\alpha}} = (\mathbf{1}, \sigma^i)_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\dot{\alpha}\alpha} = (\mathbf{1}, -\sigma^i)^{\dot{\alpha}\alpha}$ it is possible to map flat Minkowski spacetime $\mathbb{R}^{3,1}$ into the space of linear transformations of $\mathbb{C}^2$ by $(\alpha, \dot{\alpha} = 1, 2)$

$$x^\mu \mapsto x_{\alpha\dot{\alpha}} = \eta_{\mu\nu}x^\mu {\sigma^\nu}_{\alpha\dot{\alpha}}. \tag{I.12}$$

Similarly, we can transform the momenta p_μ obtained by Fourier transformation to this language

$$p_{\alpha\dot{\alpha}} = \eta_{\mu\nu}p^\mu {\sigma^\nu}_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}. \tag{I.13}$$

This allows us to express the equation $p^2 = 0$ in terms of the determinant

$$\det(p_{\alpha\dot{\alpha}}) = p^2 = 0. \tag{I.14}$$

Hence, the 2×2 matrix $p^{\dot{\alpha}\alpha}$ has rank $r < 2$ and can be expressed as a product of two complex vectors (bosonic spinors) $\lambda, \kappa \in \mathbb{C}^2$

$$p^{\alpha\dot{\alpha}} = \pm\lambda^\alpha\kappa^{\dot{\alpha}}. \tag{I.15}$$

$p^2 = 0$ is a quadratic equation so there are two solutions as indicated by $\pm$. These two solutions can be interpreted as a positive or negative energy condition. The appropriate reality condition for Minkowski space[2] $p^\dagger = p$ implies that

$$(\lambda\kappa)^\dagger = \kappa^\dagger\lambda^\dagger \Rightarrow (\lambda^\alpha)^\dagger = \kappa^{\dot{\alpha}} \equiv \bar{\lambda}^{\dot{\alpha}} \tag{I.16}$$

such that

$$p^{\dot{\alpha}\alpha} = \pm\lambda^\alpha\bar{\lambda}^{\dot{\alpha}}. \tag{I.17}$$

[2]Here, $p^\dagger$ denotes hermitian conjugation, that is the combined action of complex conjugation and transposition of the matrix $p \mapsto p^\dagger = \bar{p}^t$. Note that hermitian conjugation exchanges the $\mathbf{2}$ and $\bar{\mathbf{2}}$ of $\mathfrak{su}(2)$ thus dotted indices become undotted indices and vice versa.

Often one compresses the sign factor $\pm$ and $\bar{\lambda}$ by defining $\tilde{\lambda} = \pm\bar{\lambda}$. For $\mathcal{N}$-extended supersymmetric theories the spinor variables have to be complemented by complex Grassmann valued numbers η^a, $a = 1, \ldots, \mathcal{N}$.

Due to the definition of the spinor variables there is a freedom in scaling $\lambda \sim z\lambda$ and $\kappa \sim z^{-1}\kappa$ by a complex number z. Under the appropriate reality condition $p^\dagger = p$ this redundancy in the description of the spinor variables gets reduced to the choice of a phase $z \mapsto e^{i\phi}$.

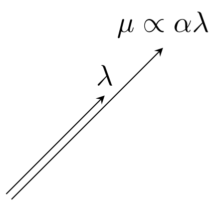
It is possible to build scalars $\langle\lambda, \mu\rangle$ by contracting the indices α with the antisymmetric symbol of rank 2 $\epsilon_{\alpha\beta}$ e.g.,

$$\langle\lambda, \mu\rangle = \epsilon_{\alpha\beta}\lambda^\alpha\mu^\beta, \qquad [\bar{\lambda}, \bar{\mu}] = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\alpha}}\bar{\mu}^{\dot{\beta}}. \tag{I.18}$$

We can also raise and lower indices with the ϵ-symbols

$$\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta, \qquad \bar{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\beta}}. \tag{I.19}$$

This will be the convention throughout.



Notice that the definition of the spinor products implies that

$$\langle\lambda, \lambda\rangle = 0, \qquad [\bar{\lambda}, \bar{\lambda}] = 0 \tag{I.20}$$

i.e., orthogonality of spinors under this product implies collinearity. We indicated this on the picture to the left. We can see this easily by using the projectivity of the spinor variables when using complex momentum or $(2, 2)$-signature. Concentrating on the complex case, we can restrict to a patch[3] where $\lambda^1 \neq 0$. Then we can define $\lambda^2/\lambda^1 = z$ and

$$\lambda^\alpha = \begin{pmatrix} 1 \\ z \end{pmatrix}, \qquad \mu^\alpha = \begin{pmatrix} 1 \\ z' \end{pmatrix}. \tag{I.21}$$

Thus $\langle\lambda, \mu\rangle = z' - z$. Setting this to zero implies $z = z'$ and so $\lambda \propto \mu$. This property is very important in the study of scattering amplitudes.

The name spinor-helicity is apt since the spinors λ not only encode momentum but also the helicity information of a particle. As is known from textbook quantum field theory, a gauge field A_μ has two polarization states on-shell. Helicity then corresponds to the two possible circular polarization states of the particle. The two polarizations states are encoded in the polarization vector ϵ_μ satisfying $\epsilon_\mu p^\mu = 0$. Let the momentum of a particle in spinor language be $p^{\dot{\alpha}\alpha} = \lambda^\alpha\bar{\lambda}^{\dot{\alpha}}$ and take some arbitrary reference spinors μ^α and $\bar{\mu}^{\dot{\alpha}}$. Then the matrices ε and $\bar{\varepsilon}$ given by

$$\varepsilon^{\alpha\dot{\alpha}} = i\sqrt{2}\frac{\lambda^\alpha\bar{\mu}^{\dot{\alpha}}}{\langle\lambda, \mu\rangle}, \qquad \bar{\varepsilon}^{\alpha\dot{\alpha}} = i\sqrt{2}\frac{\bar{\lambda}^{\dot{\alpha}}\mu^\alpha}{[\bar{\lambda}, \bar{\mu}]} \tag{I.22}$$

correspond to polarization vectors. Clearly these satisfy the equation $\varepsilon \cdot p = 0$ since $\langle\lambda, \lambda\rangle = [\bar{\lambda}, \bar{\lambda}] = 0$. A change of the reference spinors μ or $\bar{\mu}$ amounts to a change of the polarization vector by the momentum p and therefore a change of gauge.

[3]This patch and the patch $\lambda^2 \neq 0$ cover the whole space, a $\mathbb{CP}^1$.

I.4 Planar Limit

Scattering amplitudes in ($\mathcal{N}$-extended) Yang-Mills theories and QCD-like theories are notoriously hard to calculate. A most important realization for the calculation of scattering amplitudes in such theories came from 't Hooft in 1974 [24]. He showed that a good approximation for QCD calculations of scattering amplitudes can be achieved by taking the number of colors $N_c \to \infty$ and retaining only the leading order diagrams which are known as the planar graphs[4]. We will give a derivation of the result largely following [24] but with a $\mathcal{N} = 4$ SYM flavor.

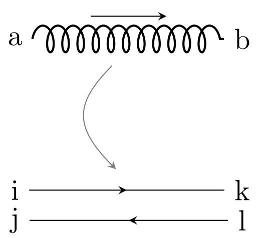
In gauge theories all fields are grouped in representations of the gauge group. In $\mathcal{N} = 4$ we are in the lucky position that all fields are in the adjoint representation $\mathbf{Adj_{N_c}}$ of the gauge group G, with generators $\mathfrak{E}^{ai}{}_j$ where we explicitly wrote the matrix indices i, j. The general property of propagators $\langle F^a(x) F^b(y)\rangle$ in all gauge theories with simple gauge group G and G-covariant gauge with fields F^a in the adjoint representation of the gauge group is that they are proportional to δ^{ab}. To avoid unnecessary complications we exclude non-compact gauge groups from this discussion. In the particular case of $G = SU(N_c)$ which we will pursue from now on, we find that the propagator is a tensor

$$\langle F^i{}_j F^k{}_l\rangle \propto \delta^i{}_l \delta^k{}_j - \frac{1}{N_c}\delta^i{}_j \delta^k{}_l. \tag{I.23}$$

In this case the matrix indices are the indices of a fundamental and an antifundamental representation by

$$\mathbf{N_c} \otimes \mathbf{\bar{N}_c} \simeq \mathbf{Adj_{N_c}} \oplus \mathbf{1}. \tag{I.24}$$

To simplify the argumentation in the following, we will work with $U(N_c)$ which does not constitute a significant change. In this case the second term in (I.23) vanishes.



As stated earlier, all fields of $\mathcal{N} = 4$ transform in the adjoint so we may denote all of them using the fundamental and antifundamental indices and replace propagators with adjoint indices like the curly one on the picture to the left with double lines to signify "exchange of color". Writing down a general Feynman diagram using such propagators, we notice that the *number of colors* N_c only enters through closed color loops i.e., summations

$$\sum_i \delta^i{}_i = N_c. \tag{I.25}$$

On the other hand, we notice from (I.1) that every propagator yields a factor g^2_{YM}, while every vertex yields a factor g^{-2}_{YM}. Thus we can characterize any diagram by a quantity

$$m = g^{2P-2V}_{\mathrm{YM}} N_c^I \tag{I.26}$$

where P is the number of internal lines (propagators), $V = \sum_n V_n$ is the number of vertices as a sum of the number of types V_n of vertices and I is the number of closed color loops. If we compactify the graph by adding a point at infinity and join all external lines in this point, we can use Euler's formula

$$I + V - P = 2 - 2G \tag{I.27}$$

[4] The approximation is better for theories with higher color number N_c. Still, for QCD, where $N_c = 3$, the approximation is still good—usually predictions can be made at the 10% accuracy level [25].

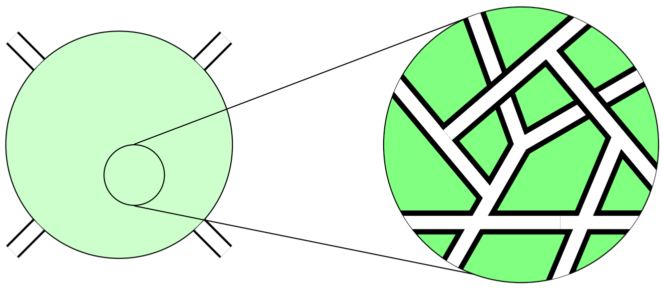
where G is the genus of the surface *triangulated*[5] by the graph in question. We then find

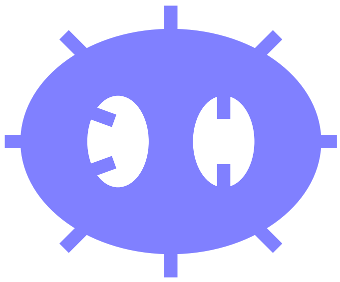
$$m = (g_{\text{YM}}^2 N_c)^{P-V} N_c^{2-2G}. \tag{I.28}$$

Upon inspection of m we find that graphs triangulating surfaces of genus $G > 0$ are suppressed polynomially when we take the limit

$$N_c \to \infty, \qquad g_{\text{YM}}^2 N = \lambda \text{ fixed}. \tag{I.29}$$

Higher genera correspond to amplitudes which either have crossing internal propagators (like on the figure to the right) such that they cannot be drawn without intersection on a sphere *or* they correspond to graphs that have multiple trace contributions. In the second case the external lines meeting in the point at infinity enclose "handles" on a two-dimensional surface, sketched in the blue graph below.





The limit described above is known as the *planar limit* and was introduced by 't Hooft in 1974. He argued that a sensible expansion for gauge theories contains in fact two expansions, first the $1/N_c$ expansion around planar graphs and secondly the $\lambda = g_{\text{YM}}^2 N_c$ expansion generating higher loop levels. These two expansions together closely resemble respectively the α' and g_s expansions of string theory, a fact that became essential for the formulation of the AdS/CFT correspondence. The planar limit has proven indispensable for the work with $\mathcal{N} = 4$ SYM. It is suspected that this theory is integrable in the planar limit (see e.g. [5] for a review).

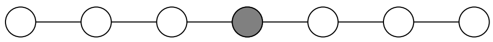
No statement about scattering amplitudes in this thesis ever strays from the theory in the planar limit.

I.5 Symmetry group

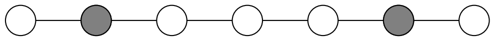
$\mathcal{N} = 4$ SYM is invariant under the action of the superconformal group $PSU(2,2|4)$. The corresponding Lie superalgebra $\mathfrak{psu}(2,2|4)$ consists of Poincaré transformations $(\mathfrak{P}, \mathfrak{L}, \bar{\mathfrak{L}})$, conformal transformations $\mathfrak{K}$ and dilatations $\mathfrak{D}$, supersymmetry transformations $(\mathfrak{Q}, \bar{\mathfrak{Q}})$, superconformal transformations $(\mathfrak{S}, \bar{\mathfrak{S}})$ and internal "R-symmetry" transformations $\mathfrak{R}$. The algebra is given in appendix E.

[5] The surface gets split into polygons of different order, technically it becomes a polyhedron. Polyhedra can be triangulated, so we may call the resulting split of the formerly smooth surface enclosed by the graph a triangulation of the surface.

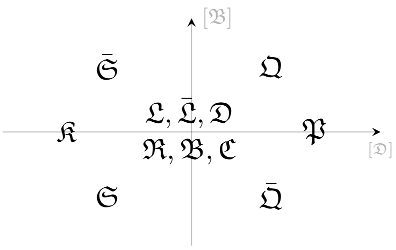
On the right hand side, two possible Dynkin diagrams for the $A(n|n)$ type superalgebra[6] ($n = 3$ here) have been drawn to illustrate the form of the root system. It is possible to extend $\mathfrak{psu}(2,2|4)$ to $\mathfrak{u}(2,2|4)$ by adding the outer automorphism $\mathfrak{B}$ of $\mathfrak{u}(2,2|4)$ and the central charge $\mathfrak{C}$. While $\mathfrak{B}$ is strictly not a symmetry of the theory, $\mathfrak{C}$ can be interpreted as a trivial symmetry. Both charges can be useful to find physical representations of $\mathfrak{psu}(2,2|4)$. The diagram below shows the algebra generators of $\mathfrak{u}(2,2|4) \simeq \mathfrak{psu}(2,2|4) \oplus \mathfrak{B} \oplus \mathfrak{C}$ on a dimension-hypercharge grid.



The "distinguished" Dynkin diagram of $(\mathfrak{p})\mathfrak{su}(2,2|4)$



Another Dynkin diagram for $(\mathfrak{p})\mathfrak{su}(2,2|4)$



Due to the peculiar nature of $\mathfrak{psu}(2,2|4)$ it is not possible to map it into the algebra of $(4|4) \times (4|4)$ supermatrices where one would suspect its fundamental representation[7]. Usually this problem is circumvented by giving a representation of $\mathfrak{su}(2,2|4)$ in terms of $(4|4) \times (4|4)$ supermatrices. $\mathfrak{C}$ is then identified with the identity matrix. This approach is outlined in Appendix E.

Additionally, $\mathcal{N} = 4$ SYM has a discrete symmetry. The supermultiplet of fields is mapped into itself by the combined conjugation $\mathbf{X}$ which consists of charge conjugation $\mathbf{C}$ and parity transformation $\mathbf{P}$ with action

$$\mathbf{X}\mathcal{F} = -\mathcal{F}^T \tag{I.30}$$

for any field $\mathcal{F}$ in the $\mathcal{N} = 4$ SYM multiplet. The action (I.1) is invariant under this discrete symmetry.

Recently, it has become evident that there is an enlarged *hidden* symmetry group of non-Lagrangian symmetries. They first appeared in the context of the spectrum of $\mathcal{N} = 4$. Unexpected simplifications in loop amplitudes of $\mathcal{N} = 4$ SYM in the planar limit [26, 27] led to the expectation of a enlarged symmetry group also for amplitudes. Later on, this enlarged symmetry was found in terms of a dual superconformal symmetry of dual Feynman graphs [12, 28, 29] which were finally identified as the generators of the Yangian algebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ in [13]. Since then, they have proven to be very useful in constraining the planar $\mathcal{S}$-matrix of $\mathcal{N} = 4$ SYM ([30, 5] for reviews). The action of the Yangian algebra will be the main topic of part 5. An introduction to Yangian algebras in general is given in apdx. F.3 and for a textbook reference, see [31].

For the sake of completeness, let us also mention that there is evidence for an additional $SL(2,\mathbb{Z})$ strong-weak duality symmetry first conjectured in [32] which generalizes the works of [33, 34].

[6] Corresponding to the $\mathfrak{sl}(n+1|n+1)$ basic type I classic simple Lie superalgebras in Kac's classification of Lie superalgebras.

[7] However, the adjoint representation of $\mathfrak{psu}(2,2|4)$ can be represented in terms of supermatrices.

I.6 Vanishing β-function

Not only is the amount of symmetries of $\mathcal{N} = 4$ exceptionally big, the theory has also been shown to be superconformal even at the quantum level. This was first suggested in [35]. Even more so, we are able to show that $\mathcal{N} = 4$ SYM is *finite* perturbatively [36, 37, 38, 39] and non-perturbatively [40]. As we can see from the Lagrangian in (I.1) there is *only one coupling constant* g_{YM} under the assumption that we do not switch on a θ-angle term—which would have no effect on perturbative statements anyway—and we use a simple gauge group [41].

It is possible to show that the β-function vanishes to first loop order by an inspection of the field content of $\mathcal{N} = 4$ SYM. A general statement can be made for all $SU(N_c)$ gauge theories about the first loop order β-function [42], namely it takes the form

$$\beta^{(1)} = \mu \frac{\partial g_{\mathrm{YM}}}{\partial \mu} = -\frac{g_{\mathrm{YM}}^3}{16\pi^2} \left(\frac{11}{3} N_c - \frac{1}{6} \sum_i C_i - \frac{2}{3} \sum_i \tilde{C}_i \right) . \tag{I.31}$$

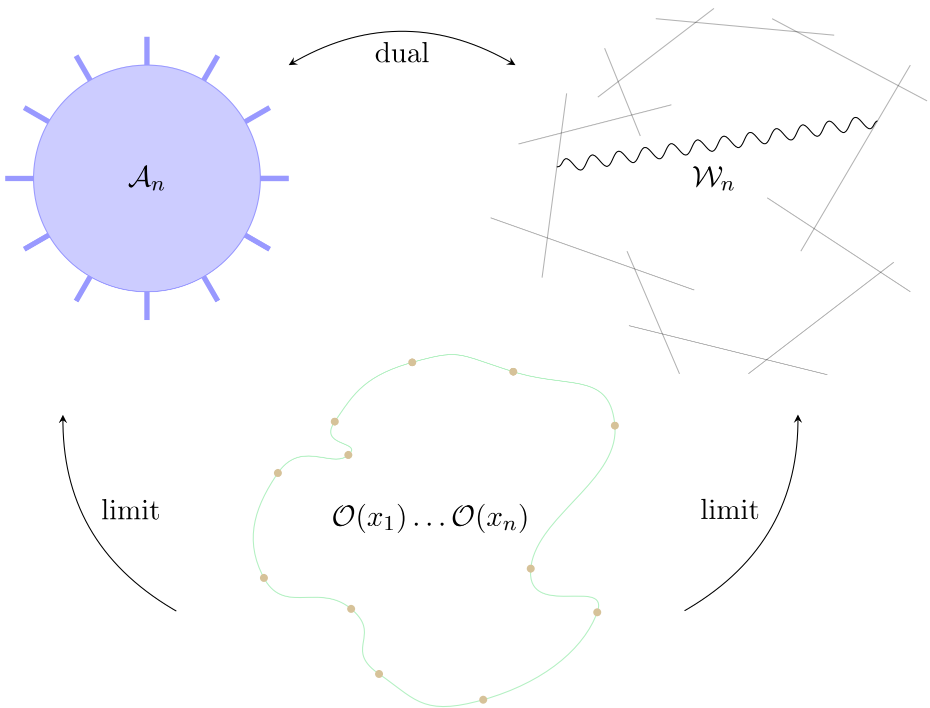
The first sum runs over the number of real scalars with the quadratic Casimir C_i whereas the second sum runs over the Weyl fermions with quadratic Casimir $\tilde{C}_j$. As we have remarked before, all fields are in the adjoint representation of the gauge group, so all the quadratic Casimirs are given by the number of colors N_c. The sum can then simply be calculated for 6 real scalars and 4 Weyl fermions to yield zero $(11 - 3 - 8 = 0)$. At higher loop orders perturbative calculations using Feynman diagrams have been conducted, too, in e.g., [43].

However, there is a clear argument that the β-function of $\mathcal{N} = 4$ SYM vanishes to all orders. Since $\mathcal{N} = 4$ supersymmetry contains $\mathcal{N} = 1$ supersymmetry, the only divergence we encounter is in a one-loop correction to the coupling g_{YM}. This is ensured by the well-known non-renormalization theorems (see e.g., in [44]). We may conclude therefore, that (I.31) is the complete β-function of $\mathcal{N} = 4$ SYM, which implies that the theory is finite.

CHAPTER II

Scattering amplitudes and Wilson loops

Recently, a remarkable amount of work has been done in the field of scattering amplitudes of $\mathcal{N} = 4$ SYM. This chapter will give the necessary definitions and terminology as well as an introduction to the salient features of scattering amplitudes and their connection to Wilson loops. We will start by introducing the modern form of scattering amplitudes in massless gauge theories as they are encountered in the literature now. Then we will focus on the special connection between scattering amplitudes and Wilson loops on light-like contours in $\mathcal{N} = 4$ SYM. For completeness, the figure on this page also includes the connection between these objects and correlation functions, which has been explored in the literature (for references see the caption).



The duality between scattering amplitudes and Wilson loops [45, 46, 47, 48] on light-like polygonal contours in a dual spacetime is shown in the upper part of the picture. The lower part shows a visualization of a correlation function connecting to both scattering amplitudes and Wilson loops by way of specific limits. [49, 50, 51, 52]

II.1 Scattering amplitudes

We begin by examining the construction of tree-level scattering amplitudes in planar $\mathcal{N} = 4$ SYM theory. Since $\mathcal{N} = 4$ SYM is a supersymmetric theory with a single multiplet, it is possible to arrange all particles in a single on-shell superfield $\Phi(p, \eta)$ [38, 39]. The Grassmann-odd parameters η^a, $a = 1, \ldots, 4$ are used to encode the flavor and helicity of the particles in the multiplet by using the fact that particles of different helicity transform in different representations of the R-symmetry group $\mathfrak{su}(4)$, such that

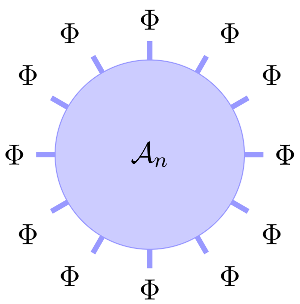
$$\Phi(p,\eta) = G^+(p) + \eta^a\psi_a(p) + \frac{1}{2}\eta^a\eta^b\phi_{ab}(p) + \frac{1}{3!}\epsilon_{abcd}\eta^a\eta^b\eta^c\bar{\psi}^d(p) + \frac{1}{4!}\epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d G^-(p). \quad \text{(II.1)}$$

The on-shell fields $G^{\pm}$, ψ_a, $\bar{\psi}^a$, and ϕ_{ab} have helicities ± 1, $\pm\frac{1}{2}$, and 0, respectively.

Next we will be putting the amplitudes in a special form using *color-ordering*. Given a set of particles with momenta p_i, color a_i and helicity h_i (here given in terms of Grassmann parameters η_i) we may write an amplitude with n legs as $A_n(\{p_i, \eta_i, a_i\})$. It is a conventional sum of all contributing Feynman diagrams. This sum may be rewritten in terms of color-ordered scattering amplitudes which are stripped bare of their gauge group factors by writing $A_n(\{p_i, \eta_i, a_i\})$ as an expansion

$$A_n(\{p_i,\eta_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathcal{A}_n(p_{\sigma(i)},\eta_{\sigma(i)})\,\mathrm{tr}(\mathfrak{E}^{a_{\sigma(1)}}\cdots\mathfrak{E}^{a_{\sigma(n)}}). \quad \text{(II.2)}$$

The matrices $\mathfrak{E}^a$ are the generators of the gauge group algebra $\mathfrak{su}(N)$ in the fundamental representation. Since $\mathcal{A}_n$ depend on massless particles, we will use the spinor-helicity formalism introduced in sec. I.3 such that all the external data is neatly packaged in the variables λ, $\bar{\lambda}$ and η.



We can write the function $\mathcal{A}_n$ in terms of R-symmetry and Poincaré-symmetry invariants. Nair [22] and Berends and Giele [23] showed that a special class of amplitudes can be written in a remarkably simple way by using the on-shell momentum (super)space. These are the *maximally helicity violating* (or MHV) amplitudes, which—for n incoming particles—have $n-2$ positive helicity and 2 negative helicity particles on their exterior legs. On the picture to the left we indicate an incoming on-shell superfield by Φ at every line. The blue corpus of the amplitude $\mathcal{A}_n$ stands for any of the possible graphs that can be drawn. In this *chiral* on-shell momentum superspace $(\lambda, \bar{\lambda}, \eta)$, R-symmetry invariants are given by the combinations $\epsilon_{abcd}\eta_i^a\eta_j^b\eta_k^c\eta_l^d$. The indices $i, j, \ldots$ denote the site or leg at which η is inserted. Using these invariants, we can classify the appearing scattering amplitudes in terms of powers of such combinations of ηs, i.e.,

$$\mathcal{A}_n = \sum_{m=2}^{n-2} \mathcal{A}_{n,m}. \quad \text{(II.3)}$$

This expansion gives rise to the $\mathrm{N^kMHV}$ classification[1] of scattering amplitudes where $k = m-2$. The cases when $m = 0, 1, n-1, n$ are excluded[2] as can be shown by more elementary (diagrammatic) methods (a low-n study of this phenomenon can be found in [54]).

[1] The "N" stands for "next-to".

[2] Precisely, it can be shown that any pure YM extended by an arbitrary amount of supersymmetry excludes these types of amplitudes. See e.g., [6, 53].

Let us compress all on-shell variables into a single multi-variable $\Lambda = (\lambda, \bar{\lambda}, \eta)$. For $k = 0$, the resulting amplitudes are called *MHV amplitudes* and can be written as

$$\mathcal{A}_n^{\mathrm{MHV}}(\Lambda_1, \ldots, \Lambda_n) = \frac{\delta^4(P)\delta^{0|8}(Q)}{\prod_{i=1}^n \langle i, i+1\rangle} \tag{II.4}$$

where

$$\delta^4(P^{\alpha\dot{\alpha}}) = \delta^4\Big(\sum_i \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}\Big), \qquad \delta^{0|8}(Q^{a\alpha}) = \delta^{0|8}\Big(\sum_i \eta_i^a \lambda_i^\alpha\Big) \tag{II.5}$$

encode momentum conservation and *supermomentum conservation*. We make the identification $n+1 \to 1$ to enforce the closure of the amplitude under cyclic shifts of the arguments Λ_i

$$\mathcal{A}_n(1, 2, \ldots, n) = \mathcal{A}_n(2, 3, \ldots, n, 1). \tag{II.6}$$

It has been shown [55] that all higher $\mathrm{N^kMHV}$ amplitudes are proportional to the MHV factor $\mathcal{A}_n^{\mathrm{MHV}}$, such that it is possible to write the full n particle tree amplitude as

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} \sum_{k=0}^{n-4} \mathcal{P}_k \tag{II.7}$$

where $\mathcal{P}_0 = 1$. Higher $\mathrm{N^kMHV}$ factors $\mathcal{P}_k$ at tree-level are complicated[3] rational functions of external particle data, the building blocks of which are the so called *R-invariants*.

§ II.1.1. Twistors and Grassmannians.—Twistor coordinates are another set of variables that have become increasingly important in the description of scattering amplitudes. A (super)twistor

$$\mathcal{Z}^{\mathcal{A}} = (\lambda^\alpha, \mu^{\dot{\alpha}}, \chi^a) \tag{II.8}$$

is a $(4|4)$ dimensional variable of (compactified) twistor space $\mathbb{CP}^{3|4}$. Historically, the step from the chiral on-shell momentum space variables $(\lambda, \bar{\lambda}, \eta)$ to twistor variables was made by first changing to $(2, 2)$ signature such that the two variables λ and $\bar{\lambda}$ become real and independent of each other [11]. Then by inspecting the scaling properties of these variables, one is led to either *half–Fourier transformation* of the variable λ or the variables $\bar{\lambda}$ and η to get an object $\mathcal{Z}^{\mathcal{A}}$ with homogeneous scaling. We choose the second method, so

$$\tilde{F}(\lambda, \mu, \chi) = \int \exp\big(i[\mu, \bar{\lambda}] - \eta.\chi\big)\, F(\lambda, \bar{\lambda}, \eta). \tag{II.9}$$

When performing this transformation on the amplitudes of $\mathcal{N} = 4$ SYM we notice many simplifications in the description of the MHV amplitudes.

For tree-level amplitudes, the function $\mathcal{P} = \sum_k \mathcal{P}_k$ has a rather simple rational form on twistor space [56, 57] given in terms of so called R-invariants $[i, j, k, l, m]$ which can be built from 5 supertwistors by

$$[1, 2, 3, 4, 5] = \int_{\mathbb{CP}^4} \frac{D^4 c}{c_1 c_2 c_3 c_4 c_5} \delta^{(4|4)}(c_i \mathcal{Z}_i) \tag{II.10}$$

with $D^4 c = \epsilon_{abcde} c_a dc_b dc_c dc_d dc_e$ a projective measure. A representation of single trace, color-ordered, n-point (tree-level) scattering amplitudes in $\mathcal{N} = 4$ that is best suited for our purposes

[3] That is, they are complicated in a space-time description while they become remarkably simple functions in a twistor description.

however is given in terms of a generalization of (II.10). This is the so called *Grassmannian integral* [58, 59]

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{vol}(GL(k))} \int \frac{dc^{n\times k}}{\mathcal{M}_1[c]\cdots\mathcal{M}_n[c]} \prod_{i=1}^{k} \delta^{4|4}(c_{im}\mathcal{Z}_m). \tag{II.11}$$

Here, the external data is given in terms of supertwistors $\mathcal{Z}_m^{\mathcal{A}}$ while the $n\times k$ matrix c_{im} provides coordinates for the Grassmannian manifold $G_k(n)$. The whole integral is thus a contour integral over $G_k(n)$. Finally, the objects $\mathcal{M}_j[c]$ are consecutive $k\times k$ minors of the matrix c_{im} starting at column j i.e.,

$$\mathcal{M}_j[c] = \epsilon^{i_1\ldots i_k} c_{i_1 j} c_{i_2 j+1} \cdots c_{i_k j+k}. \tag{II.12}$$

When $j+k>n$ the minor starts "wrapping" around c_{im}. That means $j+k$ has to be interpreted as $j+k \mod n$. Since c_{im} parametrizes a k-plane in n-dimensional space, there is a $GL(k)$ redundancy in the description of the matrix c_{im} which needs to be fixed in order to allow a proper calculation of the integral. This is done by setting a $k\times k$ block—usually one puts this block at $i=1$—to the unit matrix. The factor $\mathrm{vol}(GL(k))^{-1}$ is a formal way of reminding us that in the expression above, this redundancy has not been fixed, yet.

It has been shown [60] that there is a transformation of (II.11) which turns the integral over $G_k(n)$ into an integral

$$\mathcal{R}_{n,k} = \frac{1}{\mathrm{vol}(GL(k-2))} \int \frac{dc^{n\times(k-2)}}{\mathcal{M}_1[c]\cdots\mathcal{M}_n[c]} \prod_{i=1}^{k-2} \delta^{4|4}(c_{im}\mathcal{W}_m). \tag{II.13}$$

over $G_{k-2}(n)$ while transforming the supertwistors $\mathcal{Z}^{\mathcal{A}}$ to momentum supertwistors $\mathcal{W}^{\mathcal{A}}$. (Bosonic) momentum twistors are formed by taking the variables x_i defined by $p_{i+1} = x_{i+1} - x_i = \lambda_i\bar{\lambda}_i$ and contract them with λ_i. So we have $w^A = (\lambda, \lambda.x =: \mu)$. The immediate benefit of doing this is that we solve the momentum conservation constraint $\sum_i p_i = 0$. Thus momentum twistors provide a set of unconstrained variables for on-shell kinematical data. Momentum twistors have been introduced by Hodges in [61]. The proportionality factor between expression (II.11) and (II.13) is exactly the n-particle MHV superamplitude

$$\mathcal{L}_{n,k} = \mathcal{A}_n^{\mathrm{MHV}} \mathcal{R}_{n,k} \tag{II.14}$$

and $\mathcal{R}_{n,k}$ encodes the higher $\mathrm{N^kMHV}$ factors (R-invariants) $\mathcal{P}_k$ in momentum twistor variables.

The crucial element to be observed here is that $\mathcal{L}_{n,k}$ is manifestly superconformally invariant while $\mathcal{R}_{n,k}$ is manifestly invariant under the celebrated dual superconformal symmetry [12]. It has been shown that both expressions enjoy invariance (up to boundary terms) under the full Yangian algebra [13].

Further work connected to the Grassmannian formulas $\mathcal{R}_{n,k}$ and $\mathcal{L}_{n,k}$ has revealed a remarkable all-loop recursion relation [62] inspired by the BCFW recursion relations [9] for tree-level amplitudes. Most recently, a theory of on-shell graphs was proposed by the same authors [63].

§ II.1.2. A note on loops.—In this thesis we will not be concerned with perturbative corrections to scattering amplitudes beyond one-loop level. However, we will give a short overview of the topic in terms of scattering amplitudes.

The Yangian symmetry algebra is powerful enough to constrain the four and five point scattering amplitudes to all orders in perturbation theory given by the BDS Ansatz [27]

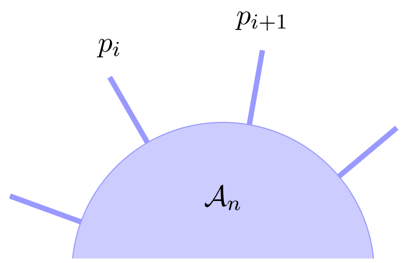
$$\mathcal{A}_n^{\mathrm{BDS}} = \mathcal{A}_n^{\mathrm{MHV}} \exp\left(\sum_{\ell=1}^{\infty} \tilde{g}^2 f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right) \tag{II.15}$$

here given in dimensional regularization. The parameter ℓ is the loop order, $M_n^{(1)}$ the one-loop result at $n = 4, 5$ points, $\tilde{g}^2 = 2g^2(4\pi e^{-\gamma})^\epsilon$ the loop expansion parameter and $f^{(\ell)}$, $C^{(\ell)}$ functions independent of the kinematics. $E_n^{(\ell)}$ vanishes as the parameter of dimensional regularization $\epsilon \to 0$. At higher points—for $n > 5$ legs—it was shown that the BDS Ansatz does not work: The BDS prediction fails for the first time for the six-point two-loop MHV scattering amplitude. The first hint at a failure came from a argument at strong coupling [64] and later at weak coupling by a computation of the six-edge Wilson loop [48] as well as arguments from analysis of the amplitude in multi-Regge kinematics [65, 66, 67]. The analytic result [68] conclusively showed that the BDS Ansatz is broken beyond five points.

Many exciting developments have been sparked by the computation of this analytic result. One of the most interesting was the introduction of the symbol [69] $\mathcal{S}(F)$ of a transcendental function F, which was used to vastly simplify [70] the result of ref. [68]. The symbol proves to be a most important concept in $\mathcal{N} = 4$ SYM due to the Kotikov-Lipatov transcendentality principle [71] implying that the results of ℓ-loop calculations must be of highest transcendentality 2ℓ in $\mathcal{N} = 4$ SYM as well as the fact that the coefficients of these functions are numbers unlike in e.g., QCD, where they are rational functions of the kinematical data.

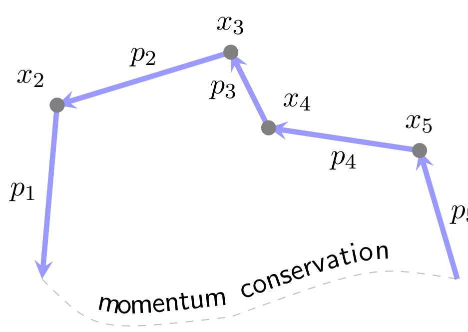
II.2 Wilson loops

Finally we want to introduce another remarkable property of scattering amplitudes in $\mathcal{N} = 4$ SYM: The duality between scattering amplitudes and Wilson loops on light-like contours. Originally, the duality only encompassed MHV scattering amplitudes [16, 48] but was later generalized to all amplitudes [17, 10]. Let us present the important features of this duality. We will first examine how the dual space-time is constructed and then turn to the formulation of these Wilson loops.



The situation is depicted in the diagram to the right. Given a n-particle scattering amplitude with momenta p_i, each satisfying $p_i^2 = 0$, it is possible to define variables x_i in a Minkowski space such that

$$x_i - x_{i+1} = p_i. \tag{II.16}$$



Momentum conservation requires that the momenta form a closed polygonal path. This path is parametrized by displacement vectors in a dual Minkowski space. The constraint $p_i^2 = 0$ implies that $(x_i - x_{i+1})^2 = 0$ i.e., these intervals are light-like. We can use the spinor-helicity formulation of massless momenta and write

$$(x_i - x_{i+1})^{\dot{\alpha}\alpha} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}. \tag{II.17}$$

Additionally, we need to identify

$$x_{n+1} \equiv x_1 \tag{II.18}$$

to ensure that momentum conservation is obeyed. Notice that the coordinates x_i do not have the correct mass dimension for a spacetime variable since $[x] = [p] = -1$. This identifies them as being *dual* variables instead of the coordinates of the original spacetime on which the theory has been originally formulated.

This dualization of the kinematical variables relates the set of massless momenta $\{p_i\}$ satisfying $\sum_i p_i = 0$ to a closed light-like contour in a dual space. A most natural object to define on such a contour—we will denote it by C_n for n edges—is a Wilson loop

$$\mathcal{W}_n = \frac{1}{N_c} \operatorname{tr} \mathcal{P} \exp \Big(\oint_{C_n} A \Big) \tag{II.19}$$

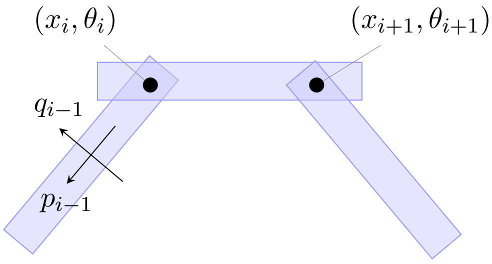
where $A = dx^\mu A_\mu(x)$ is the bosonic gauge field of $\mathcal{N} = 4$ SYM. A duality in the planar limit between the expectation value $\langle \mathcal{W}_n \rangle$ of these objects and the perturbative corrections $\hat{A}_n = \sum_{l=1}^\infty g^{2\ell} A^{(\ell)}$ to MHV scattering amplitudes

$$\mathcal{A}_n^{\text{MHV}} = A_n^{\text{MHV}} \hat{A}_n \tag{II.20}$$

via

$$\log \hat{A}_n = \log \mathcal{W}_n + \mathcal{O}(\epsilon) \tag{II.21}$$

was conjectured [47] and probed extensively [72, 48, 46, 16, 73]. $\mathcal{O}(\epsilon)$ stands for terms vanishing in dimensional regularization upon taking the limit $\epsilon \to 0$.



A supersymmetrization of this notion to produce a duality that captures N^kMHV amplitudes has been formulated in Minkowski superspace and twistor space [17, 10]. Since amplitudes are defined on a chiral superspace $(\lambda, \bar{\lambda}, \eta)$, the Wilson loop should be defined on a dual chiral superspace (x, θ) with the identifications

$$x_i - x_{i+1} = \lambda_i \bar{\lambda}, \qquad \theta_i - \theta_{i+1} = \lambda_i \eta \tag{II.22}$$

for the momenta $p_i^{\dot{\alpha}\alpha} = \lambda_i \bar{\lambda}_i$ and the supermomenta $q_i^{a\alpha} = \lambda_i^\alpha \eta_i^a$. In this superspace, light-like lines are "thickened" by the additional fermionic directions η—the process can be thought of as attaching fermionic dimensions to the bosonic line. On the diagram[4] to the left above is a schematic of the thickening of the light-like lines. While momentum and supermomentum sit on thickened edges of dimension $(1|4)$, they meet in $(0|0)$-dimensional superspace points (x_i, θ_i). These ideas will be explained in much more detail in part 4.

To perform the generalization, the Wilson loop operator in (II.19) is augmented by a fermionic superfield[5]

$$\mathcal{A} = dx^\mu A_\mu(x, \theta) + d\theta^{a\alpha} A_{\alpha a}(x, \theta). \tag{II.23}$$

[4] Inspired by the related "fat line" diagrams in [74].
[5] A similar notion appeared already in [75].

The proposal appeared independently in [17, 10]. However, it was pointed out [18] that both cases have problems at reproducing the correct results for $\mathrm{N^kMHV}$ amplitudes. They also have the even more puzzling problem that the supersymmetries $\bar{\mathfrak{Q}}$ are broken at the quantum level by the calculated results. The second issue was particularly vexing as seemingly even finite quantities—unplagued by divergences and thus independent of regularization—failed to be annihilated by $\bar{\mathfrak{Q}}$.

The problem of the $\bar{\mathfrak{Q}}$-anomaly was subsequently solved in two independent papers [76, 77], both showing that $\bar{\mathfrak{Q}}$ had to be corrected to take into account the formulation of scattering amplitudes in terms of the chiral on-shell variables $\lambda, \bar{\lambda}, \eta$ or the holomorphic supertwistors $\mathcal{Z}^{\mathcal{A}}$. In the twistor case, the authors pointed out that the naive $\bar{\mathfrak{Q}}$-operator

$$\bar{\mathfrak{Q}} = \chi \frac{\partial}{\partial \mu} \tag{II.24}$$

here given in twistor language—was in fact not a symmetry of full $\mathcal{N} = 4$ SYM on chiral superspace but only of the self-dual theory.

Interestingly enough, the correction term needed is introducing a mixing of different levels in perturbation theory. Effectively, a recursion relation combining $\mathrm{N^kMHV}$-level and loop-level ℓ had been found which constrains the form of the integrands of higher loop amplitudes. In view of the duality with scattering amplitudes this echoed older results [78, 79, 80, 81].

Two workarounds to cure the $\bar{\mathfrak{Q}}$-anomaly were also proposed: One by extending the chiral Wilson loop to a non-chiral version [82] while the other relied on a bottom-up approach to construct a Wilson loop on $\mathcal{N} = 4$ full superspace [74, 1]. This second proposal will be reviewed in much broader detail in part 4. In view of the duality with scattering amplitudes, there is a nagging problem with non-chiral Wilson loops, though. While they restore $\bar{\mathfrak{Q}}$ invariance, they are not dual to scattering amplitudes anymore. Their $\bar{\theta}$-expansion also contains other quantities that—while being crucial to the restoration of $\bar{\mathfrak{Q}}$-invariance—cannot be related to scattering amplitudes—not even to scattering amplitudes formulated on a non-chiral momentum space [83]. We will investigate the bottom-up approach more closely in Part 4.

3
Geometry

The present part, entitled *Geometry*, is concerned with fundamental properties of supersymmetric Yang-Mills theories. We will first introduce the notion of *flag manifolds* of $SL(4)$ and use their formulation as cosets of $SL(4)$ to derive vielbeins, the form of covariant derivatives, the action of the superconformal algebra on the manifold and the easy derivation of supertranslation invariant intervals in superspace. Furthermore, we will be able to translate these concepts between different types of flag manifolds by way of *double fibrations*—a well known concept in the case of the Penrose-Ward correspondence which relates Minkowski superspace to twistor space.

We will then tie the abstract concept of flag manifolds to the more hands-on harmonic superspace approach and explicitly showcase how to derive vielbeins and covariant derivatives on different spaces. These we will use in cha. VI to explain how actions for Yang-Mills theories on spaces other than Minkowski superspace can be found. The two relevant examples concentrate on actions in terms of a *holomorphic Chern-Simons* form

$$CS[A] = A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A. \tag{II.25}$$

Here A is a *twistor gauge field*. The first example is the derivation of holomorphic Chern-Simons theory on twistor space from the self-dual Yang-Mills equations with $\mathcal{N} = 4$ extended supersymmetry. The second example is the holomorphic Chern-Simons theory that is obtained by extending $\mathcal{N} = 3$ super-Yang-Mills theory by additional harmonic coordinates and subsequent reduction of the theory to an analytic superspace which contains odd twistors.

In chapter VII we will concern ourselves with the retrieval of familiar spacetime fields—or more generally operators—from twistor or harmonic superspace quantities. In both cases we get these from the twistor gauge fields A. We will do so first for Abelian gauge theories by using (a generalization of) the Penrose transform which implies integrating over the harmonics. The existence of such a transform is a consequence of the double fibration picture. Then we will expand these notions to non-Abelian theories by using a Wilson line picture in harmonic coordinates.

CHAPTER III

Flag manifolds

In this chapter, we will introduce flag manifolds. Flag manifolds are a generalization of the better known projective spaces of vector spaces. They include the family $G_k(N)$ of spaces of k-dimensional subspaces in an N dimensional complex vector space $V \simeq \mathbb{C}^N$—the family of Grassmannian manifolds—and also generalize this concept. We will always work with complex vector spaces here.

We begin by introducing the concept of flag manifolds and some of the theory connected to it. This chapter is confined to bosonic manifolds, which is enough to exhibit the most important concepts. One of these are double fibrations, which we will be presenting in sec. III.3. In a subsequent chapter we will use the language of flag manifolds for a unified presentation of physically interesting manifolds which exhibit $\mathcal{N}$ extended superconformal symmetry in four dimensions as flag manifolds of the group $SL(4|\mathcal{N})$. The layout is loosely following the exposition in [84].

III.1 Flags

Consider the following. Let V be a vector space. We endow this vector space $V \simeq \mathbb{C}^N$ with a basis $\{\mathbf{e_i}\}$, $i = 1, \ldots, \dim(V) = N$. The group $GL(N)$ has an action on V and having chosen a volume form Ω, we can narrow this down to an action of $SL(N)$. Now, choose a subspace $(\mathbf{e_1}, \ldots, \mathbf{e_k})$, where $k < N$. This subspace is some hyperplane in V. The space of all such subspaces of V is called the Grassmannian $G_k(N)$, the space of k-planes in (complex) N - dimensional space. The most fundamental one of this family of derived spaces is the space of lines in $\mathbb{C}^N$ passing through the origin. This space is called projective space $\mathbb{CP}^{N-1}$. This is a special case of a flag manifold. We will *always* consider $V \simeq \mathbb{C}^N$ and only use real vector spaces when we explicitly say so.

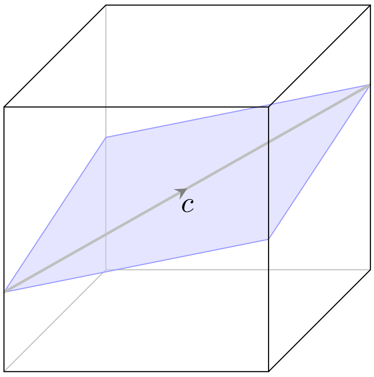
So now, what is a flag? Choose a hyperplane $(\mathbf{e_1}, \ldots, \mathbf{e_k})$ in $\mathbb{C}^N$ with $k < N$. In this hyperplane, chose another subspace $(\mathbf{e_1}, \ldots, \mathbf{e_l})$, $l < k$ and so on. Such an object will be called a flag. If we have an N dimensional vector space with basis $\{\mathbf{e_i}\}$ and K a sequence $k_1, \ldots, k_n$ of non-negative integers with

$$k_1 < k_2 < \ldots < k_n < N, \tag{III.1}$$

we call the sequence of subsets also denoted by K

$$K = [(\mathbf{e_1}, \ldots, \mathbf{e_{k_1}}) : (\mathbf{e_1}, \ldots, \mathbf{e_{k_2}}) : \ldots : (\mathbf{e_1}, \ldots, \mathbf{e_{k_n}})] \tag{III.2}$$

a *flag* of type K. Clearly, this generalizes the notion of Grassmannian manifolds by creating "Grassmannians in Grassmannians".



Therefore, we understand flags as the natural generalization of subspaces in vector spaces. The spaces of all flags of type K are—just like the family of Grassmannians $G_k(N)$—not vector spaces anymore but have the structure of differentiable manifolds. These manifolds are quite aptly named *flag manifolds* and will be denoted by $\mathbb{F}_K(N)$—if we don't identify them with other manifolds. Interestingly enough, the dimension of any flag manifold can be calculated from the sequence K by taking the Grassmannian dimension of any two consecutive pairs in the sequence and adding up, e.g., given $K = k_1 < \cdots < k_n$ we have

$$\dim \mathbb{F}_K(N) = k_1(k_2 - k_1) + k_2(k_3 - k_2) + \cdots + k_{n-1}(k_n - k_{n-1}). \tag{III.3}$$

Using the family of Grassmannian manifolds as our example of choice we may express them in the new language of flag manifolds. The sequence K consists of only one element k in these cases, so we write

$$F_k(N) = G_k(N). \tag{III.4}$$

The picture above tries to give an intuition for a flag: The vector $c = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3$ is contained within the blue plane (i.e. a two-dimensional space) which is contained in a three dimensional space spanned by the vectors $\mathbf{e}_i$. A flag manifold is defined as the set of all such objects.

Grassmannian manifolds have another feature that may guide us in our understanding of flag manifolds. They are naturally isomorphic to homogeneous spaces of the Lie group $GL(N)$ with the stabilizer of the k-dimensional subspace which we denote by $H'_k(N)$, so

$$G_k(N) \simeq \frac{GL(N)}{H'_k(N)}. \tag{III.5}$$

This generalizes to any given flag manifold with sequence K. We can find its stabilizing group $H'_K(N)$ by a simple algorithm[1]. Given a flag $K = k_1 < k_2 < \cdots < k_i$, the elements of $h \in H'_K(V)$ will be of the form

$$
\begin{array}{ccc}
 & \begin{array}{l} \overbrace{\hphantom{* \cdot \cdot \cdot \cdot \cdot \cdot *}}^{k_2} \\ \overbrace{\hphantom{* \cdot *}}^{k_1} \end{array} & \\
[\mathrm{h}]= & \begin{pmatrix}
* & \cdot & * & & & & & & & & & \\
\cdot & \cdot & \cdot & & & & & & & & & \\
* & \cdot & * & & & & & & & & & \\
* & \cdot & \cdot & \cdot & \cdot & \cdot & * & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
* & \cdot & \cdot & \cdot & \cdot & \cdot & * & & & & & \\
* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & \\
 & & & & & & & & & & & \ddots
\end{pmatrix} & \begin{array}{l} \left.\vphantom{\begin{array}{c}*\\ \cdot\\ *\end{array}}\right\} k_1 \\ \left.\vphantom{\begin{array}{c}*\\ \cdot\\ \cdot\\ *\end{array}}\right\} k_2 \\ \\ \\ \\ \end{array}
\end{array}
$$

[1]That is true for flags of the groups $SL(N)$, $SU(N)$, and $SO(N)$; flags of the group $Sp(N)$ do not conform to this simple algorithm.

where the first block is a $k_1 \times k_1$ matrix, the first and second block together a $k_2 \times k_2$ matrix and so on—the diagonal dots are meant to signify the continuation of this pattern. Any flag manifold can then be defined as the *right coset* of the group $GL(N)$ (or $SL(N)$ in the presence of a volume form) divided by the stabilizer of the flag, i.e., given a flag $K = k_1 < \ldots < k_i$

$$\mathbb{F}_K(N) \simeq \frac{GL(N)}{H'_K(N)} \simeq \frac{SL(N)}{H_K(N)} \tag{III.6}$$

The second equation holds after having chosen a volume form. The group $H_K(N)$ corresponds to the subgroup of $H'_K(N)$ with unit determinant. Thus any group element g may be written in terms of a coset representative $s(u) \in \mathbb{F}_K(N)$ with coordinates u on $GL(N)$ and an element of the stabilizer $h(u)$ s.t.

$$g(u) = h(u).s(u). \tag{III.7}$$

For an element g' of the group $SL(N)$ to map $s(u)$ into another element on the flag manifold we need to multiply by an element of the stability group

$$s(u).g'(u) = h(g, u).s(u'). \tag{III.8}$$

Using this, the action of (super)conformal transformations on the coordinates u may be deduced immediately from the linearized version of the last equation.

If we grace $\mathbb{C}^N$ with a hermitian metric h, the symmetry group further narrows down to the unitary group $U(N)$, and a volume form finally reduces this to $SU(N)$ and similarly for the quotient groups. All the quotients are isomorphic, so we can see that flag manifolds derived in this way are in fact, compact[2]. Interestingly, the stabilizing groups $H'_K(N)$ can be written in an easy way when we narrow down to the unitary $SU(N)$. In this case, given $K = k_1 < \cdots < k_n$, we have that

$$\mathbb{F}_K(N) \simeq \frac{SU(N)}{S(U(k_1) \times U(k_2 - k_1) \times \cdots \times U(N - k_n))}. \tag{III.9}$$

Using the simplest example of Grassmannian manifolds, we can write

$$G_k(N) \simeq \frac{SU(N)}{S(U(k) \times U(N - k))}. \tag{III.10}$$

As a simple application of this, let $[C] = c^i{}_j$ be a matrix in $SU(N)$. The stabilizing group $H_k(N)$ consists of block-diagonal matrices where the first block is a $k \times k$-matrix and the second block is a $(N-k) \times (N-k)$-matrix. The coset may be parametrized by matrices c_{im} where $i = 1, \ldots, k$ and $m = 1, \ldots, N - k$, e.g.,

$$c_{im} = \left(\begin{array}{c|ccc} & c_{11} & \ldots & c_{1,N-k} \\ \mathbf{1}_{k\times k} & \ldots & & \vdots \\ & c_{k1} & \ldots & c_{k,N-k} \end{array} \right) \tag{III.11}$$

and the conjugate which is a $(N-k) \times k$ matrix. In the case of $SL(N)/H_k(N)$, (III.11) taken as matrix of complex coordinates describes the patch on the complex Grassmannian $G_k(N)$. Compare also the discussion of the Grassmannian integral in sec. II.1.1.

[2]Since $SU(N)$ is real, while $SL(N)$ is complex, we run into the problem of correctly counting the dimensions of the various flag manifolds. The isomorphism ensures that the dimensions of the different flag manifolds over $\mathbb{C}$ is the same. The reality of $SU(N)$ is present in the form of the stabilizer. The elements h of the stabilizer have block-*diagonal* form such that we end up with twice as many coordinates on the flag manifold. The unitarity of the matrices ensure that under complex conjugation $\bar{u}^i{}_j = u_i{}^j$ as matrix elements, which reduces the complex dimension by a factor of two, such that the counting works out.

III.2 Parabolic Lie algebras

It is possible to classify and count all possible flag manifolds derived from $GL(N)$. Let $\mathfrak{g}$ denote the Lie algebra of the (semi-simple) Lie group G. Any such Lie algebra contains a Borel subalgebra $\mathfrak{b}$ which is the direct sum of the Cartan subalgebra $\mathfrak{h}$ and the space of positive—or equivalently negative—roots $\bigoplus_i \mathfrak{g}_i^+$

$$\mathfrak{b} = \mathfrak{h} \bigoplus_i \mathfrak{g}_i^+. \tag{III.12}$$

A Borel subalgebra is a special case of a parabolic subalgebra $\mathfrak{p}$ which are direct sums of $\mathfrak{b}$ and any number of negative roots. In a word, there are as many flag manifolds to be had from $\mathfrak{g}$ as there are parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$.

What does this mean in terms of the stabilizers $H_K(N)$? It is easy to see that $\mathfrak{b} \subset \mathfrak{gl}(N)$ generates the lower triangular matrices

$$\begin{pmatrix} * & & & \\ * & * & & \\ \vdots & & \ddots & \\ * & * & \cdots & * \end{pmatrix} \tag{III.13}$$

which form a Lie subgroup of $GL(N)$. The lower triangular matrices however correspond to the stabilizer of flags of the form $1 < 2 < \ldots < N-1 < N$. These flags we will call *full* or equivalently *maximal flags*. Any other flag manifold can be built by successively adding negative roots to $\mathfrak{b}$ thus generating higher parabolic subalgebras. This gives rise to the very important concept of double fibrations which was central to the development of the twistorial description of Yang-Mills theories which we will discuss now.

III.3 Double fibrations

Given any two flag manifolds $\mathbb{F}_1$ and $\mathbb{F}_2$ derived from a Lie group G, we can associate them with the parabolic subgroups P_1 and P_2 used to define the quotients[3]

$$\mathbb{F}_i = P_i \backslash G. \tag{III.14}$$

To these two flag manifolds we can associate a third flag manifold $\mathbb{F}_{K_{1\cap 2}} = P_1 \cap P_2 \backslash G$. The flag manifolds $\mathbb{F}_i$ can be derived from $\mathbb{F}_{K_{1\cap 2}}$ by projections π_1 and π_2 such that we get the diagram

$$\begin{tikzcd} & P_1 \cap P_2 \backslash G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P_1 \backslash G \arrow[rr, Leftrightarrow] & & P_2 \backslash G \end{tikzcd} \tag{III.15}$$

The thick arrow in the middle is what we are interested in. Given a point $p \in P_1 \backslash G$ we can map it to a set $\pi_2 \circ \pi_1^{-1}(p) \in P_2 \backslash G$ and vice versa. More importantly, the set $\pi_2 \circ \pi_1^{-1}(p)$ can be thought of as a copy of $\pi_1^{-1}(p)$ embedded in $\mathbb{F}_2$ via π_2. Such a structure is called a *double fibration* and its existence has deep implications for gauge theories defined on flag manifolds. Minkowski space itself is (part of) a flag manifold and interesting double fibrations exist for it with other spaces (see below). Let us look at two very specific and very important examples of double fibrations.

[3]The notation indicates the fact that we are working with right cosets.

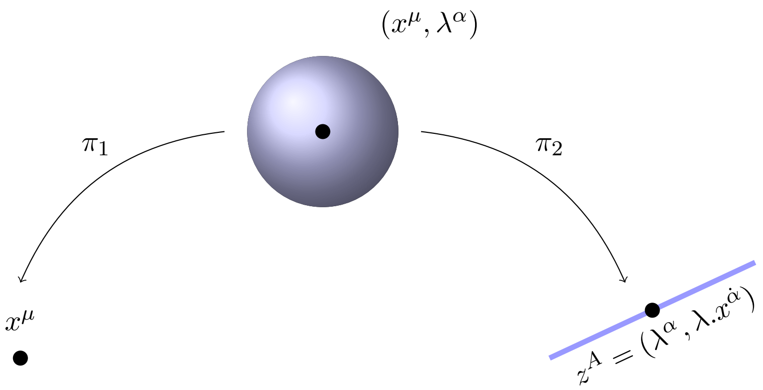
III.4 The Ward and Witten correspondence

In four dimensions, the group $SL(4)$ is the group of conformal transformations. Our starting point for any considerations will therefore be the group $SL(4)$ out of which we will build flag manifolds. There are two especially important double fibrations we would like to study in detail.

§ III.4.1. The Ward correspondence.—The first involves the three flags[4] $\mathbb{F}_{12}$, $\mathbb{F}_2$ and $\mathbb{F}_1$. It is easy to identify the second as the Grassmannian $G_2(4)$ according to (III.4). Following Penrose, we know that by compactifying Minkowski spacetime by adding in the "light-cone at infinity" and then complexifying this compactification, we arrive at $G_2(4)$, the Grassmannian of two-dimensional subspaces in four-dimensional complex space $\mathbb{C}^4$, see e.g., the book by Manin [85]. Conversely, we can choose a patch in $G_2(4)$ which contains a copy of (complexified) Minkowski space. In the following, when we talk about Minkowski space, we will be talking about *complexified compactified Minkowski space* $G_2(4)$ which we will be denoting by $\mathbb{M}^4$. As stated before, when we make use of a specific reality condition, we will explicitly notify the reader of this.

The space $\mathbb{F}_1$ we can recognize as the projective space $\mathbb{CP}^3$ and the space $\mathbb{F}_{12}$ is the bundle of undotted spinors[5] over Minkowski space $\mathbb{M}^4$, locally therefore given by $\mathbb{M}^4 \times \mathbb{CP}^1$. The projective space $\mathbb{CP}^3$ is Penrose's twistor space [86] $\mathbb{P}$ with the "light-cone at infinity" added in. For briefness, we will refer to $\mathbb{F}_{12}(4)$ as *correspondence space*. All three spaces neatly fit into a double fibration diagram like (III.15)

$$\begin{tikzcd} & \mathbb{F}_{12}(4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{M}^4 \arrow[rr, Leftrightarrow] & & \mathbb{CP}^3 \end{tikzcd} \tag{III.16}$$



Let us endow the three spaces with coordinates. Since the correspondence space is the bundle of undotted spinors over Minkowski space we pick a chart and put coordinates $(\lambda^\alpha, x^{\dot\alpha\alpha})$. The projection π_1 maps $(\lambda^\alpha, x^{\dot\alpha\alpha})$ to $(x^{\dot\alpha\alpha})$ in a chart in complexified, compactified Minkowski space while π_2 maps $(\lambda^\alpha, x^{\dot\alpha\alpha})$ to coordinates $(\lambda^\alpha, \mu^{\dot\alpha} = \epsilon_{\alpha\beta}\lambda^\alpha x^{\dot\alpha\beta})$ on $\mathbb{CP}^3$. The figure above emphasizes the structure of the correspondence space as essentially a Minkowski space with a Riemann sphere attached at every point. Under π_2 we recover complex lines in $\mathbb{CP}^3$. Without any further ado, we see which kinds of sets are mapped by the two maps $\pi_2 \circ \pi_1^{-1}$ and $\pi_1 \circ \pi_2^{-1}$. In the first case a point $x \in \mathbb{M}^4$ gets mapped

[4] Equivalently we may use $\mathbb{F}_{23}$, $\mathbb{F}_2$ and $\mathbb{F}_3$

[5] This is of course an arbitrary choice conforming to convention. We equivalently could have chosen it to be the bundle of dotted spinors over complexified compactified Minkowski space.

to a line $(\lambda, \lambda x) \in \mathbb{CP}^3$ (x here is fixed, while λ runs). Conversely, a point $Z^A = (\lambda, \mu)$ gets mapped to a plane (λ fixed, x running) by the *incidence relation*

$$\mu^{\dot{\alpha}} = \epsilon_{\alpha\beta}\lambda^{\alpha}x^{\dot{\alpha}\beta}. \tag{III.17}$$

More precisely, this is a so called α-plane in Penrose's terminology. Thus, the set in Minkowski space corresponding to the point $Z^A \in \mathbb{CP}^3$ is a plane. We can see this by solving the incidence relations by

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \lambda^{\alpha}\rho^{\dot{\alpha}} \tag{III.18}$$

where x_0 is a fixed point and ρ is a undetermined spinor mapping out a plane in Minkowski space. Since $\epsilon_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = 0$, this is the solution to the incidence relations. α-planes will play an interesting role in our discussion of self-dual Yang-Mills theory.

Minkowski space	Twistor space
point	complex line $\mathbb{CP}^1$
α - plane	point

Table III.1.: The Penrose-Ward correspondence

Historically, the correspondence between these two spaces is known as *Penrose-Ward correspondence*. The correspondence between points of one space and subsets of another space under the double fibration have been summarized in Table III.1.

Given a scalar function $f(Z)$ on twistor space, we can restrict it to a specific $\mathbb{CP}^1 \subset \mathbb{CP}^3$ such that $Z = (\lambda, \lambda.x)$. Then the extension of the map $\pi_1 \circ \pi_2^{-1}$ to the scalar functions is the Penrose integral transformation

$$\tilde{f}(x) = \int_{\mathbb{CP}^1} D^2\lambda\ f(\lambda, \lambda.x)|_{\mathbb{CP}^1} \tag{III.19}$$

with projective measure $D^2\lambda = \langle\lambda, d\lambda\rangle[\bar{\lambda}, d\bar{\lambda}]$. With such a transform we retrieve functions on spacetime from functions on twistor space. In the modern twistor literature, however, the map is usually understood as a map from $(0,1)$-forms $\bar{f}(Z)$ on twistor space restricted to a $\mathbb{CP}^1$ to spacetime, i.e.,

$$\tilde{f}(x) = \int_{\mathbb{CP}^1} D\lambda \wedge \bar{f}(Z)|_{\mathbb{CP}^1}. \tag{III.20}$$

We will see it again in this form in cha. VI.

§ III.4.2. The Witten correspondence.—Let us now turn to our second important example. The correspondence space in this case is the full flag $\mathbb{F}_{123}(4)$. Furthermore we want to build the two flag manifolds $\mathbb{F}_2(4)$ and $\mathbb{F}_{13}(4)$. While we know the Grassmannian $G_2(4) \simeq \mathbb{F}_2(4)$ already, the full flag is also known as the bundle of dotted and undotted spinors over Minkowski space. The other space $\mathbb{F}_{13}(4)$ is a five-complex dimensional space known as *ambitwistor space* $\mathbb{A}_3$ which is a quadric in $\mathbb{CP}^3 \times \mathbb{CP}^{*3}$. Once again, these three spaces fit into a double fibration diagram like (III.15)

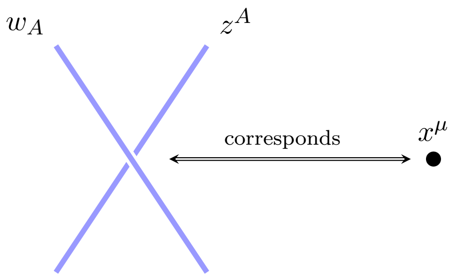
$$\begin{tikzcd} & \mathbb{F}_{123}(4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{M}^4 \arrow[rr, Leftrightarrow] & & \mathbb{A}_3 \end{tikzcd} \tag{III.21}$$

If we pick a chart on $\mathbb{F}_{123}(4)$ with coordinates $(\lambda^\alpha, \kappa^{\dot{\alpha}}, x^{\dot{\alpha}\alpha})$, then the projections π_1 and π_2 map these coordinates to coordinates $x^{\dot{\alpha}\alpha} \in \mathbb{M}^4$ and

$$([\lambda^\alpha, \mu^{\dot{\alpha}} = \epsilon_{\alpha\beta}\lambda^\alpha x^{\dot{\alpha}\beta}], [\rho^\alpha = -\epsilon_{\dot{\alpha}\dot{\beta}} x^{\dot{\alpha}\alpha}\kappa^{\dot{\beta}}, \kappa^{\dot{\alpha}}]) \in \mathbb{A}_3 \tag{III.22}$$

such that

$$\lambda^\alpha \rho_\alpha + \mu^{\dot{\alpha}} \kappa_{\dot{\alpha}} = 0. \tag{III.23}$$



Let us attempt to understand the maps from points $x \in \mathbb{M}_4$ to ambitwistor space. As in the Penrose-Ward correspondence, the map $\pi_2 \circ \pi_1^{-1}$ maps x to the pair of lines $z^A = (\lambda, \lambda x)$ and $w_A = (x\kappa, \kappa)$ constrained to lie in the quadric $\mathbb{A}_3 \subset \mathbb{CP}^3 \times \mathbb{CP}^{*3}$. This has been indicated on the figure to the left. Conversely, the incidence relations

$$\mu^{\dot{\alpha}} = \epsilon_{\alpha\beta}\lambda^\alpha x^{\dot{\alpha}\beta}, \tag{III.24a}$$

$$\rho^\alpha = -\epsilon_{\dot{\alpha}\dot{\beta}} x^{\dot{\alpha}\alpha}\kappa^{\dot{\beta}} \tag{III.24b}$$

have a unique solution

$$x^{\dot{\alpha}\alpha} = x_0^{\dot{\alpha}\alpha} + t\lambda^\alpha\kappa^{\dot{\alpha}} \tag{III.25}$$

in Minkowski space.

Minkowski space	Ambitwistor space
point	$\mathbb{CP}^1 \times \mathbb{CP}^1$
light-like line	point

Table III.2.: The Witten correspondence

We see that the solution is a light-like line through x_0 in $\mathbb{M}^4$. For this reason ambitwistor space is also called the *space of light-like lines in Minkowski space*. The correspondence was first explored by E. Witten [87] and at the same time by [88] and is known as *Witten correspondence* in analogy with the Penrose-Ward correspondence. We will revisit $\mathbb{A}_3$ in Part 4. The Witten correspondence is summarized in Table III.2.

Let us now turn to the description of harmonic variables. We will extend the formalism of flag manifolds to supermanifolds in cha. V.

CHAPTER IV

Harmonics

After having introduced the more abstract features of flag manifolds, we want to turn to the more hands-on topic of harmonics. The study of essential properties of manifolds that can be described as coset spaces is vastly simplified by the use of harmonic variables. These variables are essentially coordinates of the group underlying the coset space in question. We shall make use of them in the following to find the Maurer-Cartan form, the covariant derivatives and the action of the superconformal transformations of various spaces by using a very limited set of "recipes" for the calculation of these quantities. We will use harmonic variables for our calculations in cha. VI.

There is a vast amount of literature on harmonic coordinates, for a textbook reference see [89]. Originally, harmonics were introduced in gauge theories like supersymmetric Yang-Mills theory to parametrize manifolds that had been "glued" to Minkowski superspace. These additional degrees of freedom could be used to solve the constraints of SYM. The reason why this is a viable way of extending Minkowski superspace lies in our discussion of flag manifolds: Together with the additional coordinates, new flag manifolds could be parametrized [84]. In the following, we will use harmonics as a "hands-on" way of understanding flag manifolds inspired by [90].

Let us start with a general definition of these coordinates. Given the group[1] $SL(N)$ we can assign coordinates to the whole group by taking the group element

$$SL(N) \ni [U] = \begin{pmatrix} u^1{}_1 & \dots & u^1{}_N \\ \vdots & & \vdots \\ u^N{}_1 & \dots & u^N{}_N \end{pmatrix} \tag{IV.1}$$

with $u^i{}_j \in \mathbb{C}$. All we have to ask is that

$$\det U = \epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} u^{i_1}{}_{j_1} \cdots u^{i_N}{}_{j_N} = 1 \tag{IV.2}$$

thus immediately defining the inverse U^{-1} with entries

$$u_{i_k}{}^{j_l} = \epsilon_{i_1 \cdots i_k \cdots i_{N-1}} \epsilon^{j_1 \cdots j_l \cdots j_{N-1}} u^{i_1}{}_{j_1} \cdots \hat{u}^{i_k}{}_{j_l} \cdots u^{i_N}{}_{j_N} \tag{IV.3}$$

where $\hat{u}$ is the omission of u. We will make use of exactly this form of U^{-1} in the following calculations.

Given such u on $SL(N)$ it is now possible to write down coordinates on the flag manifolds that can be derived from $SL(N)$ without having to rely on picking a chart on the coset.

[1] Or in fact any semi-simple Lie group G, although the cases apart from $SL(N)$ and $SU(N)$ can get very involved.

IV.1 Harmonics and Flag manifolds

As described in sec. III.2, flag manifolds are the right cosets $P\backslash G$ of the semi-simple group G with respect to one of its parabolic subgroups P. One of the advantages of the harmonic approach lies in the possibility to choose different embeddings for P in G and so provide convenient coordinates for the cosets using the coordinates ${u^i}_j$ on $SL(N)$.

We will calculate a few examples of the process for illustration and for later reference. We shall begin by calculating the harmonic coordinates for twistor space.

§ IV.1.1. Twistor space $\mathbb{CP}^3$.—The parabolic subgroup of $SL(4)$ of interest in this case[2] is $H_3(N)$. Let us assume we have introduced a hermitian metric. Then we can work equivalently with $SU(4)$. The parabolic subgroup of $SU(4)$ we are interested in is $S(U(3)\times U(1))$. In this case there is the Abelian subalgebra $U(1)$ to be taken into account. To do so we will label the coordinates ${u^I}_j$ on $SU(4)$ with one $U(3)$ index i and one $U(1)$ charge (q) such that ${u^I}_j = {u^{(q)i}}_j$. The fundamental representation $\mathbf{4}$ of $SU(4)$ splits like[3]

$$\mathbf{4} \simeq \mathbf{3_1} \oplus \mathbf{1_{-3}}. \tag{IV.4}$$

We shall look at the more general case of $SU(N+1)/S(U(N)\times U(1))$ because it doesn't add any additional complexity to the calculation. We can easily see that the fundamental representation of $SU(N+1)$ reduces like

$$\mathbf{N+1} \simeq (\mathbf{N})_\mathbf{1} \oplus \mathbf{1_{-N}} \tag{IV.5}$$

so it is possible to write an element $U \in SU(N+1)$ parametrized by coordinates ${u^{(q)i}}_j$ with charge q as

$$[U]^i{}_j = \left[{u^{(1)\alpha}}_i, \quad {u^{(-N)}}_i\right] \tag{IV.6}$$

with $i = 1,\ldots,N+1$ and $\alpha = 1,\ldots,N$. When things are unambiguous we shall take the liberty of dropping the $U(1)$-charges. The inverse is clearly given by

$$[U^{-1}]^i{}_j = \begin{bmatrix} u_\alpha^{(-1)i} \\ u^{(N)i} \end{bmatrix} \tag{IV.7}$$

such that

$$u_\alpha^{(-1)i} {u^{(1)\beta}}_i = \delta_\alpha^\beta, \qquad u^{(N)i} {u^{(-N)}}_i = 1 \tag{IV.8}$$

and all the other contractions zero. Let us proceed to the quantities of interest.

The Maurer-Cartan form is given by

$$[U^{-1}dU]^i{}_j = \begin{pmatrix} {u_\alpha}^i d{u^\beta}_i & {u_\alpha}^i du_i \\ u^i d{u^\beta}_i & u^i du_i \end{pmatrix} = \begin{pmatrix} -{\omega_\alpha}^\beta & e_\alpha^{(-N-1)} \\ e^{(N+1)\beta} & -\omega'_0 \end{pmatrix}. \tag{IV.9}$$

We can now determine the covariant derivatives D on the coset by following the steps outlined in apdx. B. Taking into account the tracelessness constraint

$$\omega'_0 + {\omega_\alpha}^\alpha = 0 \tag{IV.10}$$

[2] In the literature twistors were originally described as this coset, nowadays people tend to use the dual space $\mathbb{CP}^3$ with stabilizer $H_1(N)$.

[3] The global value of the $U(1)$ charges are of course arbitrary and can be fixed by normalization, only the relative charges count. To simplify notation we choose to avoid fractions and keep integer charges.

we can write the exterior derivative as $D = d + \omega$, hence

$$d = e^{(N+1)\alpha} D_\alpha^{(-N-1)} + e_\alpha^{(-N-1)} D^{(N+1)\alpha} - \omega_\alpha{}^\beta D^\alpha{}_\beta - \omega_0 D_0 \tag{IV.11}$$

where $\omega_0' = -N\omega_0$ and

$$D_\alpha^{(-N-1)} = u^{(-N)}{}_i \frac{\partial}{\partial u^{(1)\alpha}{}_i} \qquad\qquad D^{(N+1)\alpha} = u^{(1)\alpha}{}_i \frac{\partial}{\partial u^{(-N)}{}_i} \tag{IV.12}$$

$$D^\alpha{}_\beta = u^{(1)\alpha}{}_i \frac{\partial}{\partial u^{(1)\beta}{}_i} - \frac{1}{N}\delta_\beta^\alpha u^{(1)\gamma}{}_i \frac{\partial}{\partial u^{(1)\gamma}{}_i} \tag{IV.13}$$

$$D_0 = u^{(1)\alpha}{}_i \frac{\partial}{\partial u^{(1)\alpha}{}_i} - N u^{(-N)}{}_i \frac{\partial}{\partial u^{(-N)}{}_i} \tag{IV.14}$$

We can restrict to a patch of $\mathbb{CP}^N$ to make things clearer. The variables $u^{(-N)}{}_i$ can be considered *homogeneous coordinates* on $\mathbb{CP}^N$. The space $\mathbb{CP}^N$ cannot be mapped to $\mathbb{C}^N$ with only one patch since there are points at infinity for every direction in the projective coordinate system. Every point on $\mathbb{CP}^N$ is described by a vector $(u_1, \ldots, u_{N+1})$ in $\mathbb{C}^{N+1}$. Then by taking $u_i \neq 0$ for each i in turn, $N+1$ patches can be formed by taking ratios $(u_1/u_i, \ldots, u_{i-1}/u_i, 1, u_{i+1}/u_i, \ldots, u_{N+1}/u_i)$ which cover all of $\mathbb{CP}^N$. These new coordinates are a non-projective or inhomogeneous coordinate system on $\mathbb{CP}^N$. In the case of $\mathbb{CP}^1$, for example, spinors $\lambda^\alpha = (\lambda^1, \lambda^2)$ with $\lambda^i \in \mathbb{C}$ provide a homogeneous coordinate system, and we can form two patches $(1, z^2)$ for $\lambda^1 \neq 0$ or $(z^1, 1)$ for $\lambda^2 \neq 0$.

To go to a set of homogeneous coordinates on the coset we have been looking at, we get to the patch where $u^\alpha{}_i = \delta_i^\alpha$. In the specific case of $N = 3$, we will relabel

$$u^{(-N)}{}_i \to w_A \tag{IV.15}$$

and call w_A *twistors*. Twistor space is of tremendous importance to theories of gauge fields and gravity alike (see for example in [91]).

Finally, one can split the covariant derivative on the coset into holomorphic and antiholomorphic parts[4] by letting

$$D = \partial + \bar{\partial} = e^{(-N-1)\alpha} D_\alpha^{(N+1)} + e^{(N+1)\alpha} D_\alpha^{(-N-1)} \tag{IV.16}$$

satisfying

$$\partial^2 = \bar{\partial}^2 = 0, \qquad \text{and} \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0 \tag{IV.17}$$

where the last equation holds on functions $f(u)$ which are invariant under transformations of the factored group $S(U(N) \times U(1))$, i.e., they have $U(1)$ weight 0 and are $SU(N)$ singlets. In this way the coset comes with a complex structure—just as we would have expected, since we claimed that it would be isomorphic to $\mathbb{CP}^N$. In this case $\bar{\partial}$ is a so called Dolbeault derivative with which we can change the order of complex differential forms and define a cohomology on complex manifolds. This is a special case of the more coarse deRham-cohomology which is available on more general manifolds without a complex structure. We do not need to go into the details of Dolbeault cohomology here. However, it has proven to be a useful concept to understand self-dual Yang-Mills theories, see e.g., [85].

As a final remark—we could have chosen a different embedding of $S(U(N) \times U(1))$ in $SU(N+1)$, this would have led us to $\mathbb{CP}^{*N}$, the conjugate twistor space which is isomorphic to $\mathbb{CP}^N$.

[4] As can be seen ∂ and $\bar{\partial}$ are $(1, 0)$ and $(0, 1)$-forms respectively.

§ IV.1.2. Ambitwistor space $\mathbb{A}$.—Finally, let us work out an example that will be relevant for the calculations in cha. VI. The third family of spaces of interest is the family of ambitwistor-like spaces[5] $\mathbb{A}_N$. We will work with cosets of $G = SU(N+2)$. The relevant subgroup is $P = S(U(1)\times U(N)\times U(1))$ and the fundamental representation $(\mathbf{N}+\mathbf{2})$ of $SU(N+2)$ restricts quite naturally over P

$$(\mathbf{N}+\mathbf{2}) \simeq (\mathbf{N})_{(\mathbf{-1},\mathbf{1})} \oplus \mathbf{1}_{(\mathbf{N},\mathbf{0})} \oplus \mathbf{1}_{(\mathbf{0},\mathbf{-N})}. \tag{IV.18}$$

We choose the harmonic coordinates $u^I{}_i$ in the following way

$$[U]^i{}_j = \left[u^{(N,0)}{}_i, \quad u^{(-1,1)\alpha}{}_i, \quad u^{(0,-N)}{}_i\right] \tag{IV.19}$$

with two $U(1)$-weights (p,q). The inverse is

$$[U^{-1}]^i{}_j = \begin{bmatrix} u^{(-N,0)i} \\ u^{(1,-1)i}_\alpha \\ u^{(0,N)i} \end{bmatrix}. \tag{IV.20}$$

The Maurer-Cartan form splits then into nine pieces according to $U(1)$-weights

$$\begin{aligned}
[U^{-1}dU]^i{}_j &= \left(\begin{array}{c|c|c} u^{(-N,0)i}du^{(N,0)}{}_i & u^{(-N,0)i}du^{(-1,1)\beta}{}_i & u^{(-N,0)i}du^{(0,-N)}{}_i \\ \hline u^{(1,-1)i}_\alpha du^{(N,0)}{}_i & u^{(1,-1)i}_\alpha du^{(-1,1)\beta}{}_i & u^{(1,-1)i}_\alpha du^{(0,-N)}{}_i \\ \hline u^{(0,N)i}du^{(N,0)}{}_i & u^{(0,N)i}du^{(-1,1)\beta}{}_i & u^{(0,N)i}du^{(0,-N)}{}_i \end{array}\right) \\
&= \left(\begin{array}{c|c|c} -\omega_1' & e^{(-N-1,1)\beta} & e^{(-N,-N)} \\ \hline e^{(N+1,-1)}_\alpha & \omega_\alpha{}^\beta & e^{(1,-N-1)}_\alpha \\ \hline e^{(N,N)} & e^{(-1,N+1)\beta} & -\omega_2' \end{array}\right)
\end{aligned} \tag{IV.21}$$

Furthermore, the exterior derivative is given by

$$\begin{aligned}
d = {}& e^{(-N,-N)}D^{(N,N)} + e^{(N,N)}D^{(-N,-N)} + e^{(N+1,-1)}_\alpha D^{(-N-1,1)\alpha} \\
&+ e^{(1,-N-1)}_\alpha D^{(-1,N+1)\alpha} + e^{(-N-1,1)\alpha}D^{(N+1,-1)}_\alpha + e^{(-1,N+1)\alpha}D^{(1,-N-1)}_\alpha \\
&+ \omega_\alpha{}^\beta D^\alpha{}_\beta - \omega_1 D_1 - \omega_2 D_2
\end{aligned} \tag{IV.22}$$

with

$$D^{(N,N)} = u^{(N,0)}{}_i \frac{\partial}{\partial u^{(0,-N)}{}_i} \qquad D^{(-N,-N)} = u^{(0,-N)}{}_i \frac{\partial}{\partial u^{(N,0)}{}_i} \tag{IV.23a}$$

$$D^{(-N-1,1)\alpha} = u^{(-1,1)\alpha}{}_i \frac{\partial}{\partial u^{(N,0)}{}_i} \qquad D^{(-1,N+1)\alpha} = u^{(-1,1)\alpha}{}_i \frac{\partial}{\partial u^{(0,-N)}{}_i} \tag{IV.23b}$$

$$D^{(N+1,-1)}_\alpha = u^{(N,0)}{}_i \frac{\partial}{\partial u^{(-1,1)\alpha}{}_i} \qquad D^{(1,-N-1)}_\alpha = u^{(0,-N)}{}_i \frac{\partial}{\partial u^{(-1,1)\alpha}{}_i} \tag{IV.23c}$$

$$D^\alpha{}_\beta = u^{(-1,1)\alpha}{}_i \frac{\partial}{\partial u^{(-1,1)\beta}{}_i} - \frac{1}{N}\delta^\alpha_\beta u^{(-1,1)\gamma}{}_i \frac{\partial}{\partial u^{(-1,1)\gamma}{}_i} \tag{IV.23d}$$

$$D_1 = N u^{(N,0)}{}_i \frac{\partial}{\partial u^{(N,0)}{}_i} - u^{(-1,1)\alpha}{}_i \frac{\partial}{\partial u^{(-1,1)a}{}_i} \tag{IV.23e}$$

$$D_2 = -N u^{(0,-N)}{}_i \frac{\partial}{\partial u^{(0,-N)}{}_i} + u^{(-1,1)\alpha}{}_i \frac{\partial}{\partial u^{(-1,1)a}{}_i} \tag{IV.23f}$$

which we can use to extract the covariant derivative D. These relations are a bit overwhelming, so let us restrict to a particular case which is relevant for us in the following.

[5] We follow the following naming convention: Since all ambitwistor spaces can be described as quadric in $\mathbb{CP}^N \times \mathbb{CP}^N$, we call these submanifolds $\mathbb{A}_N$. This is independent of the actual dimension of $\mathbb{A}_N$. See also the footnote on superambitwistor space on page 47.

§ IV.1.3. The case $\mathbb{A}_2 = SU(3)/U(1)^{\times 2}$.—We will choose the case where $N = 1$ because it will be relevant for the discussion in cha. VI: $\mathcal{N} = 3$ SYM can be described on a space containing $\mathbb{A}_2$ as a subspace[6]. The relevant coset[7] is $SU(3)/U(1)^{\times 2}$. Observe that the $SU(N)$ subalgebra vanishes. We are left with

$$D^{(1,1)} = u^{(1,0)}{}_i \frac{\partial}{\partial u^{(0,-1)}{}_i} \qquad D^{(-1,-1)} = u^{(0,-1)}{}_i \frac{\partial}{\partial u^{(1,0)}{}_i} \tag{IV.24a}$$

$$D^{(-2,1)} = u^{(-1,1)}{}_i \frac{\partial}{\partial u^{(1,0)}{}_i} \qquad D^{(-1,2)} = u^{(-1,1)}{}_i \frac{\partial}{\partial u^{(0,-1)}{}_i} \tag{IV.24b}$$

$$D^{(2,-1)} = u^{(1,0)}{}_i \frac{\partial}{\partial u^{(-1,1)}{}_i} \qquad D^{(1,-2)} = u^{(0,-1)}{}_i \frac{\partial}{\partial u^{(-1,1)}{}_i} \tag{IV.24c}$$

$$D_1 = u^{(1,0)}{}_i \frac{\partial}{\partial u^{(1,0)}{}_i} - u^{(-1,1)}{}_i \frac{\partial}{\partial u^{(-1,1)}{}_i} \tag{IV.24d}$$

$$D_2 = -u^{(0,-1)}{}_i \frac{\partial}{\partial u^{(0,-1)}{}_i} + u^{(-1,1)}{}_i \frac{\partial}{\partial u^{(-1,1)}{}_i} \tag{IV.24e}$$

The corresponding set of vielbein one-forms is

$$e^{(2,-1)} = u^{(-1,1)i} du_i^{(1,0)} \qquad e^{(1,1)} = u^{(0,1)i} du_i^{(1,0)} \qquad e^{(-1,2)} = u^{(0,1)i} du_i^{(-1,1)} \tag{IV.25}$$

$$e^{(-2,1)} = u^{(-1,0)i} du_i^{(-1,1)} \qquad e^{(-1,-1)} = u^{(-1,0)i} du_i^{(0,-1)} \qquad e^{(1,-2)} = u^{(1,-1)i} du_i^{(0,-1)}. \tag{IV.26}$$

Now we notice the following, interesting feature of ambitwistor spaces. In the selection of covariant derivatives, we can pick $D^{(1,1)}$, $D^{(2,-1)}$, and $D^{(-1,2)}$ to form a Dolbeault derivative

$$\bar{\partial} = e^{(-1,-1)} D^{(1,1)} + e^{(-2,1)} D^{(2,-1)} + e^{(1,-2)} D^{-1,2)}. \tag{IV.27}$$

Again, we have a holomorphic derivative ∂ satisfying $\partial\bar{\partial} + \bar{\partial}\partial = 0$ on functions $f(u)$ of the coset coordinates which are weightless under the two $U(1)$ charges. Functions which transform nicely, that is via $f(u.g) = \rho(h) f(u')$ under the transformations of the group are called *equivariant functions*. Here, ρ is the representation of the factor group H under which f transforms.

An interesting caveat: The relation $\bar{\partial}\bar{\partial} = 0$ implies torsion when written out in components. This is easy to see, as

$$\bar{\partial} e^{(-1,-1)} = e^{(-2,1)} \wedge e^{(1,-2)} \tag{IV.28}$$

and so we find that

$$[D^{(2,-1)}, D^{(-1,2)}] = D^{(1,1)} \tag{IV.29}$$

and all other commutators zero. The appearance of torsion is a general fact for all flag manifolds with more than a single "step" in the form of the matrices $h \in H_K(N)$.

However, on higher dimensional spaces it is also possible to choose subsets of the covariant derivatives to build derivative operators. Under specific conditions, which are described in detail in apdx. C, these derivative operators provide us with other interesting structures on the coset manifold. These *CR structures* become important when one tries to formulate a theory on ambitwistor space $\mathbb{A}_3$ that is equivalent to (pure) Yang-Mills theory on four spacetime dimensions. Attempts have been made here [92].

[6] In the literature, these spaces are usually called *analytic spaces*.

[7] In cha. VI the indices i will be the indices of the internal R-symmetry, i.e., the coset described in this section will be part of the lower diagonal $M \times M$-block of a $(N|M)$-supermatrix. We will come to this in cha. V.

CHAPTER V

Supersymmetric extension

Supersymmetric Yang-Mills theories—in particular $\mathcal{N} = 4$ SYM, the theory we are interested in—are most naturally defined on superspaces, manifolds with fermionic coordinates. The theory of flag supermanifolds gives us the possibility to treat all physically interesting superspaces in a unified framework and provides us with the necessary set of tools to work effectively with different forms of the same theory on different manifolds. In this chapter we provide the generalization of flag manifolds to the case of manifolds with fermionic directions. Many of the definitions that were given for the case of purely bosonic flag manifolds can be generalized to fermionic flag manifolds with little effort. Flag supermanifolds and double fibrations for supermanifolds can thus be defined easily after having introduced the bosonic case.

V.1 Flag supermanifolds

The generalization of the notion of flag manifolds to flag supermanifolds is mostly straightforward except when it comes to the exact definition of the sequence K that characterizes the flags. Given a complex vector space $\mathbb{C}^{N|M}$ with basis $\{\mathbf{e}_i, \eta_j\}$, there is a natural action of the Lie supergroup $GL(N|M)$. In the presence of a measure Ω, we replace $GL(N|M)$ with the elements of $GL(N|M)$ with unit *Berezinian* i.e.,

$$SL(N|M) = \{M \in GL(N|M)|\,\mathrm{Ber}(M) = 1\}. \tag{V.1}$$

Let K be a sequence $K = \{(k_1|\kappa_1), \ldots, (k_n|\kappa_n)\}$ of non-negative integers in both entries and

$$K_1 < K_2 < \cdots < K_n < (N|M). \tag{V.2}$$

The important information is how to read the "smaller than" sign. Given two points in the sequence $(k_i|\kappa_i)$ and $(k_{i+1}|\kappa_{i+1})$ then, $(k_i|\kappa_i) < (k_{i+1}|\kappa_{i+1})$ means that $k_i \leq k_{i+1}$, $\kappa_i \leq \kappa_{i+1}$ but not both equal at the same time. The corresponding sequence of subsets of the basis of $\mathbb{C}^{N|M}$ is called a *superflag*.

The space of all such flags K is a supermanifold and will be denoted by $\mathbb{F}_K(N|M)$. To find the stabilizer $H_K(N|M)$, we apply the same algorithm as in the bosonic case. However, in the $(4|4)\times(4|4)$-matrix form $H_K(N|M)$ isn't in a diagonal block form anymore, this can be recovered when working with $(2|4|2) \times (2|4|2)$ (two bosonic, four fermionic and two bosonic columns and rows) matrices. Here we have chosen to accept the non-diagonal block form and work with

$(4|4)\times(4|4)$ matrices. For example in the case of the manifold of superflags $(2|0) \leq (2|4) \leq (4|4)$ the elements of $H_{(2|0),(2|4)}(4|4)$ have the form

$$\left(\begin{array}{cccc|cccc} * & * & & & & & & \\ * & * & & & & & & \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ \hline * & * & & & * & * & * & * \\ * & * & & & * & * & * & * \\ * & * & & & * & * & * & * \\ * & * & & & * & * & * & * \end{array}\right). \tag{V.3}$$

The flag supermanifold $H_{(2|0)}(4|4)\backslash SL(4|4)$ can be identified with complexified, compactified Minkowski superspace $\mathbb{M}^{4|16}$, where compactified refers to the compact *body* of $\mathbb{M}^{4|16}$.

Given a coset representative $s(u)$ the action of an element g of the superconformal group $SL(4|4)$ on flag manifolds is still given by

$$s(u).g = h(u,g).s(u') \tag{V.4}$$

with $h(u,g) \in H_K(4|4)$. This can be used to calculate the form of the superconformal transformations on the coordinates u. Furthermore, we can find translation invariant intervals between points in a coset space of $SL(4|4)$ in the following way. Given two coset representatives $s(u_i)$ and $s(u_j)$ with coordinates u_i and u_j, a translation from i to j on the coset manifold is given by the group element

$$g_{ij} = s(u_i)^{-1}.s(u_j) \tag{V.5}$$

since

$$s(u_i).g_{ij} = s(u_j). \tag{V.6}$$

The entries of g_{ij} may be used to define translation invariant intervals in a patch of the flag manifold. For this, examine the example of $\mathbb{M}^{4|16}$. We can map non-compact complex Minkowski space into the flag manifold $H_{(2|0)}(4|4)\backslash SL(4|4)$ by letting $(x,\theta,\bar\theta) \mapsto s(x,\theta,\bar\theta)$ with [84]

$$s(x,\theta,\bar\theta) = \begin{pmatrix} 1 & -i(x^-)^{\dot\alpha\alpha} & -2i\theta^{\alpha a} \\ 0 & 1 & 0 \\ 0 & -2i\bar\theta^{\dot\alpha}_a & 1 \end{pmatrix} \tag{V.7}$$

where $x^\pm = x \pm 2i\theta\bar\theta$. The inverse $s^{-1}(x,\theta,\bar\theta)$ is given by

$$s^{-1}(x,\theta,\bar\theta) = \begin{pmatrix} 1 & i(x^+)^{\dot\alpha\alpha} & 2i\theta^{\alpha a} \\ 0 & 1 & 0 \\ 0 & 2i\bar\theta^{\dot\alpha}_a & 1 \end{pmatrix} \tag{V.8}$$

which can be easily verified by matrix multiplication. Then the group element g_{ij} is

$$g_{ij} = s^{-1}(x_i,\theta_i,\bar\theta_i).s(x_j,\theta_j,\bar\theta_j) = \begin{pmatrix} 1 & -i(x_j^- - x_i^+ + 2i\theta_i\bar\theta_j)^{\dot\alpha\alpha} & -2i(\theta_i-\theta_j)^{\alpha a} \\ 0 & 1 & 0 \\ 0 & -2i(\bar\theta_i - \bar\theta_j)^{\dot\alpha}_a & 1 \end{pmatrix} \tag{V.9}$$

which indeed contains the invariant intervals x_{ji}^{+-}, θ_{ji} and $\bar\theta_{ji}$ as found in [74] by translation invariance considerations. The bosonic interval is not the minimal choice for a translation invariant interval, that would be

$$x_{ji} = x_i - x_j + 2i\theta_i\bar\theta_j - 2i\theta_j\bar\theta_i \tag{V.10}$$

but the combination $x^{+-}_{ji} = x_{ji} - 2i\theta_{ji}\bar{\theta}_{ji}$ does contain this choice as well as an additional bi-fermionic piece which is invariant on its own. Similar procedures lead to translation invariant intervals for other flag manifolds of $SL(4|4)$.

A final word on vielbeins and one forms before we turn to the generalization of double fibrations: The vielbein forms of supermanifolds are going to be of mixed Grassmann degree. When working with fermionic and bosonic forms dz^M, where $\deg M \equiv |M| = 0$ for bosonic indices and $|M| = 1$ for fermionic indices, we will adopt the following convention: A graded wedge product is defined by

$$dz^M \wedge dz^N = -(-)^{MN} dz^N \wedge dz^M. \tag{V.11}$$

Most properties introduced for the bosonic case then carry over immediately.

§ V.1.1. The supersymmetric Penrose-Ward correspondence.—The following discussion is valid for any $\mathcal{N}$. Starting from the superconformal group $SL(4|\mathcal{N})$ where $0 < \mathcal{N} \leq 4$ it is possible to establish a double fibration between complex, chiral Minkowski space[1] $\mathbb{M}_R^{4|2\mathcal{N}}$ and supertwistor space $\mathbb{CP}^{3|\mathcal{N}}$ via the supermanifold of $(1|0) < (2|0) < (4|\mathcal{N})$ flags $\mathbb{F}_{(1|0),(2|0)}(4|\mathcal{N})$ i.e.,

$$\begin{array}{ccc} & \mathbb{F}_{(1|0),(2|0)} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{M}_R^{4|2\mathcal{N}} & \Longleftrightarrow & \mathbb{CP}^{3|\mathcal{N}} \end{array} \tag{V.12}$$

Using the double fibration, like in the bosonic case, certain sets of chiral Minkowski superspace can be related to sets in supertwistor space. The derivation proceeds similarly to the bosonic case, too. Endowing $\mathbb{M}_R^{4|4\mathcal{N}}$ with coordinates $X = (x^{\alpha\dot{\alpha}}, \theta^{a\alpha})$, $a = 1, \ldots, \mathcal{N}$ while $\mathbb{CP}^{3|\mathcal{N}}$ with $\mathcal{Z}^{\mathcal{A}} = (\lambda^\alpha, \mu^{\dot{\alpha}}, \chi^a)$ the double fibration relates points X in Minkowski superspace to complex superlines $(\lambda^\alpha, (\lambda x)^{\dot{\alpha}}, (\lambda\theta)^a)$ parametrized by the $\mathbb{CP}^1$ coordinate λ^α.

Conversely, a point $\mathcal{Z}^{\mathcal{A}}$ in supertwistor space gets mapped to a $(2|2\mathcal{N})$-dimensional hyperplane in Minkowski superspace via

$$\mu^{\dot{\alpha}} = (\lambda x)^{\dot{\alpha}}, \qquad \chi^a = (\lambda\theta)^a. \tag{V.13}$$

This is the *supersymmetric Penrose-Ward correspondence* for $\mathcal{N}$-extended superspaces.

For scalar functions $f(\mathcal{Z}^{\mathcal{A}})$ of the supertwistors, the Penrose transform extends to a supersymmetric Penrose transform connecting twistor space and chiral superspace. Since points in superspace $\mathbb{M}_R^{4|2\mathcal{N}}$ are still in correspondence with complex lines $\mathbb{CP}^1 \subset \mathbb{CP}^{3|\mathcal{N}}$, we may restrict to a certain $\mathbb{CP}^1$ such that $\mathcal{Z}^{\mathcal{A}} = (\lambda, \lambda.x^+, \lambda.\theta)$ and have

$$\tilde{f}(x^+, \theta) = \int_{\mathbb{CP}^1} D^2\lambda \; f(\mathcal{Z}^{\mathcal{A}})|_{\mathbb{CP}^1}. \tag{V.14}$$

In terms of full superspace $\tilde{f}$ is a chiral superfunction, i.e., it satisfies $\bar{D}_{\dot{\alpha}a}\tilde{f} = 0$. This establishes a map from scalar functions of supertwistors to the chiral superfunctions on superspace[2].

§ V.1.2. The supersymmetric Witten correspondence.—Similarly, we can define a double fibration for full four-dimensional Minkowski superspace and superambitwistor space[3] $\mathbb{A}_{3|\mathcal{N}}$.

[1]Instead of working with full Minkowski superspace, it is more natural to work with chiral superspaces $\mathbb{M}_L$ or $\mathbb{M}_R$ here. Under hermitian conjugation $\mathbb{M}_L$ and $\mathbb{M}_R$ are exchanged. This makes chiral spaces essentially complex spaces.

[2]More precisely the correspondence is given between elements of the Dolbeault cohomology group $H^1(\mathcal{O}_{-2})$—where $\mathcal{O}_{-2}$ is meant to indicate that the one forms contained in H^1 are of homogeneity -2—and on-shell chiral superfunctions.

[3]The convention here is as follows: The space $\mathbb{A}_{3|\mathcal{N}}$ is a quadric in $\mathbb{CP}^{3|\mathcal{N}} \times \mathbb{CP}^{3|\mathcal{N}}$, hence the label.

Again, one side of the double fibration diagram is being taken by full Minkowski superspace $\mathbb{M}^{4|4\mathcal{N}}$, while the correspondence space is the supermanifold of flags of type $(1|0) < (2|0) < (2|\mathcal{N}) < (3|\mathcal{N}) < (4|\mathcal{N})$ such that the diagram takes the form

$$\begin{tikzcd}[column sep=small] & \mathbb{F}_{(1|0),(2|0),(2|\mathcal{N}),(3|\mathcal{N})} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{M}^{4|4\mathcal{N}} \arrow[rr, Leftrightarrow] & & \mathbb{A}_{3|\mathcal{N}} \end{tikzcd} \tag{V.15}$$

Endowing $\mathbb{M}^{4|4\mathcal{N}}$ with the coordinates $X = (x, \theta, \bar{\theta})$, and $\mathbb{A}_{3|\mathcal{N}}$ with the coordinates

$$(\mathcal{Z}^{\mathcal{A}} = (\lambda^\alpha, \mu^{\dot{\alpha}}, \chi^a), \mathcal{W}_{\mathcal{A}} = (\rho_\alpha, \kappa_{\dot{\alpha}}, \eta_a)) \tag{V.16}$$

such that

$$\mathcal{Z}^{\mathcal{A}}\mathcal{W}_{\mathcal{A}} = 0 \tag{V.17}$$

we can figure out the mapping between the two bottom spaces in (V.15). Given a point X in Minkowski superspace, define the two chiral coordinates $x^\pm$ by

$$(x^\pm)^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} \pm 2i\theta^{a\alpha}\phi_a^{\dot{\alpha}}. \tag{V.18}$$

Then $\pi_2 \circ \pi_1^{-1}$ maps the point X to

$$([-\tfrac{i}{2}\lambda^\alpha, (\lambda x^+)^{\dot{\alpha}}, (\lambda\theta)^a], [(x^-\kappa)_\alpha, -\tfrac{i}{2}\kappa_{\dot{\alpha}}, (\phi\kappa)_a]). \tag{V.19}$$

The parameter space is a $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{A}_{3|\mathcal{N}}$, i.e. two complex lines in superambitwistor space. Conversely, a point $(\mathcal{Z}, \mathcal{W})$ satisfying $\mathcal{Z} \cdot \mathcal{W} = 0$ in superambitwistor space gets mapped to $\mathbb{M}^{4|4\mathcal{N}}$ via $\pi_1 \circ \pi_2^{-1}$

$$\mu^{\dot{\alpha}} = (\lambda x^+)^{\dot{\alpha}}, \qquad \chi^a = (\lambda\theta)^a \tag{V.20}$$
$$\rho^\alpha = (x^-\kappa)^\alpha, \qquad \eta_a = (\phi\kappa)_a \tag{V.21}$$

with the condition

$$0 = \lambda^\alpha\rho_\alpha + \mu^{\dot{\alpha}}\kappa_{\dot{\alpha}} + 4i\chi^a\phi_a \quad \Rightarrow \quad 0 = (x^+)^{\alpha\dot{\alpha}} - (x^-)^{\alpha\dot{\alpha}} + 4i\theta^{a\alpha}\phi_a^{\dot{\alpha}}. \tag{V.22}$$

The first of the two sets of two equations in (V.20) defines an anti-self-dual plane in a chiral subspace (x^+, θ) of $\mathbb{M}^{4|4\mathcal{N}}$ while the second set defines a self-dual plane in an antichiral subspace. The last constraint demands that these planes intersect in full Minkowski space. Overall, this leads to a subspace of $\mathbb{M}^{4|4\mathcal{N}}$ of dimension $(1|2\mathcal{N})$. These subspaces are supersymmetric generalizations of light-like lines in Minkowski space and are usually called *fat lines* or super-light-like lines [74]. We will revisit fat lines in sec. VIII.1, where we will consider non-chiral super-Wilson loops on light-like contours.

CHAPTER VI

YM AS HCS

The title of this section is to be read as *Yang-Mills theory as a holomorphic Chern-Simons theory.* The aim of this chapter is to show that Yang-Mills theories in four dimensions may be reformulated as *holomorphic Chern-Simons theories* on twistor space and analytic superspace [93, 56, 92, 94, 19]. It has been known for quite some time that self-dual Yang-Mills theory has a formulation as a Chern-Simons theory on twistor space. This formulation was subsequently extended to the full theory by introducing a non-local piece to the action [11]. As an example we will show this reformulation by using the harmonic coordinates that have been introduced in cha. IV. Similarly, we will present the reformulation of $\mathcal{N}=3$ SYM on harmonic superspace in sec. VI.2[1] [19]. In the first part of this chapter, sec. VI.1 we will set the stage by reviewing super Yang-Mills theory as a gauge theory over superspace.

VI.1 $\mathcal{N}=4$ (SD)SYM in harmonic superspace

The superspace we will be using is $\mathbb{M}^{4|4\mathcal{N}}$. This is compactified Minkowski superspace with coordinates

$$(x^{\alpha\dot{\alpha}}, \theta^{\alpha a}, \bar{\theta}^{\dot{\alpha}}_{a}). \tag{VI.1}$$

In this setting, the notation is not meant to imply that θ and $\bar{\theta}$ are conjugates of each other. In the definition of x we have mapped x into the 2×2-dimensional matrices using (for conventions see appendix A)

$$x^{\mu}\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu} = x^{\alpha\dot{\alpha}}. \tag{VI.2}$$

Let us use the Maurer-Cartan form $\Omega = ds(x,\theta,\bar{\theta}).s^{-1}(x,\theta,\bar{\theta})$ to find the vielbeine of superspace

$$E^{\alpha\dot{\alpha}} = dx^{\alpha\dot{\alpha}} - 2id\theta^{\alpha a}\bar{\theta}^{\dot{\alpha}}_{a} + 2i\theta^{\alpha a}d\bar{\theta}^{\dot{\alpha}}_{a} \tag{VI.3a}$$

$$E^{\alpha a} = d\theta^{\alpha a} \tag{VI.3b}$$

$$\bar{E}^{\dot{\alpha}}_{a} = -d\bar{\theta}^{\dot{\alpha}}_{a} \tag{VI.3c}$$

[1] At the time when the paper [19] was written the authors were not aware of the paper [94] which contains some of the work done in [19] and presented here.

Using these the exterior derivative

$$d = \frac{1}{2} dx^{\alpha\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + d\theta^{\alpha a} \frac{\partial}{\partial \theta^{\alpha a}} + d\bar{\theta}^{\dot{\alpha}}_a \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}_a} \tag{VI.4}$$

may be rewritten in terms of supersymmetry covariant derivatives[2] $(\partial_{\dot{\alpha}\alpha}, D_{\alpha a}, \bar{D}^a_{\dot{\alpha}})$ given by

$$\partial_{\dot{\alpha}\alpha} = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}, \quad D_{\alpha a} = \frac{\partial}{\partial \theta^{\alpha a}} + i\bar{\theta}^{\dot{\alpha}}_a \partial_{\dot{\alpha}\alpha}, \qquad \bar{D}^a_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}_a} - i\theta^{\alpha a}\partial_{\dot{\alpha}\alpha}. \tag{VI.5}$$

Hence the exterior derivative can be written as

$$d = \frac{1}{2} E^{\alpha\dot{\alpha}} \partial_{\dot{\alpha}\alpha} + E^{\alpha a} D_{\alpha a} + \bar{E}^{\dot{\alpha}}_a \bar{D}^a_{\dot{\alpha}}. \tag{VI.6}$$

To satisfy $d^2 = 0$ we encounter a torsion term (using (V.11))

$$dE^{\alpha\dot{\alpha}} = 2iE^{\alpha a} \wedge \bar{E}^{\dot{\alpha}}_a, \quad \text{while} \quad dE^{\alpha a} = 0, \quad dE^{\dot{\alpha}}_a = 0 \tag{VI.7}$$

which leads to the commutation relations

$$\{D_{\alpha a}, D_{\beta b}\} = 0 \tag{VI.8a}$$

$$\{\bar{D}^{\dot{\alpha}}_a, \bar{D}^{\dot{\beta}}_b\} = 0 \tag{VI.8b}$$

$$\{D_{\alpha a}, \bar{D}^b_{\dot{\alpha}}\} = -2i\delta^b_a \partial_{\dot{\alpha}\alpha} \tag{VI.8c}$$

A gauge connection

$$A = \frac{1}{2} E^{\alpha\dot{\alpha}} A_{\dot{\alpha}\alpha} + E^{\alpha a} A_{\alpha a} + \bar{E}^{\dot{\alpha}}_a \bar{A}^a_{\dot{\alpha}} \tag{VI.9}$$

may be introduced by lifting $\mathbb{M}^{4|4\mathcal{N}}$ to a $SU(N)$ principal bundle. On such a bundle, we may extend the action of the exterior derivative to $SU(N)$-valued one-forms by defining

$$d^\nabla = \frac{1}{2} E^{\alpha\dot{\alpha}} \nabla_{\dot{\alpha}\alpha} + E^{\alpha a} \nabla_{\alpha a} + \bar{E}^{\dot{\alpha}}_a \bar{\nabla}^a_{\dot{\alpha}}. \tag{VI.10}$$

The condition $d^2 = 0$ becomes

$$(d^\nabla)^2 = F = dA + A \wedge A, \tag{VI.11}$$

where F is the *curvature two-form* of the principal bundle. F satisfies the *Bianchi identities* $(d^\nabla)^3 = d^\nabla F = 0$. In pure ($\mathcal{N} = 0$) Yang-Mills theory, the equations of motion are further given by

$$d^\nabla \star F = 0, \tag{VI.12}$$

where $\star$ is the Hodge dualizer. In theories with matter, for example in QCD, the right hand side is given by a current j different from zero. For $\mathcal{N} = 4$ SYM, the equations of motion are implied by certain constraints on the coefficients of F and the Bianchi identities [95].

[2] This approach is the inverse of the historical approach where the form of the covariant derivatives determines the form of the vielbein. The covariant derivatives in turn are determined by their commutation relation with the supertranslation operators $\mathfrak{Q}$ and $\bar{\mathfrak{Q}}$. Of course we have used an *inspired guess* for the embedding of the coordinates $(x, \theta, \bar{\theta})$ in the group by the map $s(x, \theta, \bar{\theta})$ to retrieve the usual form of the covariant derivatives given in the text.

When expanding F in terms of the basis of vielbeine $(E^{\alpha\dot{\alpha}}, E^{\alpha a}, \bar{E}^{\dot{\alpha}}_a)$ we find

$$\begin{aligned} F = \;& E^{\alpha a} \wedge \bar{E}^{\dot{\alpha}}_b (2i\delta^b_a A_{\dot{\alpha}\alpha} + \bar{D}^b_{\dot{\alpha}} A_{\alpha a} + D_{\alpha a}\bar{A}^b_{\dot{\alpha}} + \{A_{\alpha a}, \bar{A}^b_{\dot{\alpha}}\}) \\ & + \frac{1}{2} E^{\alpha a} \wedge E^{b\beta}(D_{\alpha a} A_{\beta b} + D_{\beta b} A_{\alpha a} + \{A_{\alpha a}, A_{\beta b}\}) \\ & + \frac{1}{2} \bar{E}^{\dot{\alpha}}_a \wedge \bar{E}^{\dot{\beta}}_b (\bar{D}^a_{\dot{\alpha}} \bar{A}^b_{\dot{\beta}} + \bar{D}^b_{\dot{\beta}} \bar{A}^a_{\dot{\alpha}} + \{\bar{A}^a_{\dot{\alpha}}, A^b_{\dot{\beta}}\}) \\ & + E^{\alpha\dot{\alpha}} \wedge E^{\beta b}(D_{\beta b} A_{\dot{\alpha}\alpha} - D_{\dot{\alpha}\alpha} A_{\beta b} + [A_{\beta b}, A_{\dot{\alpha}\alpha}]) \\ & + E^{\alpha\dot{\alpha}} \wedge \bar{E}^{\dot{\beta}}_b (\bar{D}^b_{\dot{\beta}} A_{\dot{\alpha}\alpha} - D_{\dot{\alpha}\alpha}\bar{A}^b_{\dot{\beta}} + [\bar{A}^b_{\dot{\beta}}, A_{\dot{\alpha}\alpha}]) \\ & + \frac{1}{2} E^{\alpha\dot{\alpha}} \wedge E^{\beta\dot{\beta}}(D_{\dot{\beta}\beta} A_{\dot{\alpha}\alpha} - D_{\dot{\alpha}\alpha} A_{\dot{\beta}\beta} + [A_{\dot{\beta}\beta}, A_{\dot{\alpha}\alpha}]) \end{aligned} \tag{VI.13}$$

§ VI.1.1. Constraints for $\mathcal{N}$-extended SYM.—In supersymmetric theories the components of the field strength tensor F must be constrained [95] since the appearing superfields form reducible representations of the supersymmetry algebra (see e.g., [89]). The necessary constraints take slightly different forms for different SYM theories. So far we have left the question of the range of the index a open. We shall now specialize to the cases that will interest us in the following: $\mathcal{N} = 3$ and $\mathcal{N} = 4$. For $\mathcal{N} = 3$ SYM theories, $a = 1, \dots, 3$—i.e., the R-symmetry group is $SU(3)$—and the field content is arranged in two multiplets

$$(F_{\alpha\beta}, \psi^a_\alpha, \phi_a), \qquad (G_{\dot{\alpha}\dot{\beta}}, \bar{\psi}_{a\dot{\alpha}}, \bar{\phi}^a). \tag{VI.14}$$

For $\mathcal{N} = 4$ the field content is the same but arranged in a single multiplet

$$(F_{\alpha\beta}, \psi^a_\alpha, \phi_{ab}, \bar{\psi}_{a\dot{\alpha}}, G_{\dot{\alpha}\dot{\beta}}). \tag{VI.15}$$

Here the scalars ϕ_{ab} satisfy the additional self-duality constraint

$$\bar{\phi}_{ab} = \frac{1}{2}\epsilon_{abcd}\phi^{cd}. \tag{VI.16}$$

The correct constraints for $\mathcal{N} = 3$ read

$$2i\delta^b_a A_{\dot{\alpha}\alpha} + \bar{D}^b_{\dot{\alpha}} A_{\alpha a} + D_{\alpha a}\bar{A}^b_{\dot{\alpha}} + \{A_{\alpha a}, \bar{A}^b_{\dot{\alpha}}\} = 0 \tag{VI.17a}$$

$$D_{\alpha a} A_{\beta b} + D_{\beta b} A_{\alpha a} + \{A_{\alpha a}, A_{\beta b}\} = \epsilon_{\alpha\beta}\bar{W}_{ab} \tag{VI.17b}$$

$$\bar{D}^a_{\dot{\alpha}} \bar{A}^b_{\dot{\beta}} + \bar{D}^b_{\dot{\beta}} \bar{A}^a_{\dot{\alpha}} + \{\bar{A}^a_{\dot{\alpha}}, A^b_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} W^{ab} \tag{VI.17c}$$

The scalar superfields W and $\bar{W}$ are antisymmetric in their indices

$$\bar{W}_{ab} = \epsilon_{abc}\bar{W}^c, \qquad W^{ab} = \epsilon^{abc}W_c \tag{VI.18}$$

and their bottom components in the θ, $\bar{\theta}$ expansion are the scalars ϕ_a and $\bar{\phi}^a$ respectively. Thus there are six independent scalar superfields in this theory. For $\mathcal{N} = 4$ $a = 1, \dots, 4$ is a $\mathbf{4}$ or a $\bar{\mathbf{4}}$ index of $SU(4)$ and the constraints look the same as (VI.17a). Here the scalar fields satisfy the constraint

$$W^{ab} = \frac{1}{2}\epsilon^{abcd}\bar{W}_{cd} \tag{VI.19}$$

implied by the reality constraint for the scalar fields

$$\phi^{ab} = \frac{1}{2}\epsilon^{abcd}\bar{\phi}_{cd} \tag{VI.20}$$

which in turn implies again that there are only six independent scalar fields. In contrast with the $\mathcal{N}=3$ case the constraints on the $\mathcal{N}=4$ field strength F now immediately imply the field equations of $\mathcal{N}=4$ SYM. $\mathcal{N}=3$ experiences a symmetry enhancement on-shell such that $\mathcal{N}=3$ and $\mathcal{N}=4$ SYM are in fact equivalent on-shell. The notable difference between the theories is the fact that $\mathcal{N}=4$ *only* exists on-shell—the constraints of the theory imply the equations of motion—while $\mathcal{N}=3$ also admits an off-shell formulation in superspace.

We may make the additional identifications

$$F_{\alpha a,\beta b}=\epsilon_{\alpha\beta}\bar{W}_{ab} \qquad F^{ab}_{\dot\alpha\,\dot\beta}=\epsilon_{\dot\alpha\dot\beta}W^{ab} \tag{VI.21a}$$

$$F_{\alpha a,\dot\alpha\beta}=\epsilon_{\alpha\beta}\bar{\Psi}_{a\dot\alpha} \qquad F^{a}_{\dot\alpha\,\dot\beta\alpha}=\epsilon_{\dot\alpha\dot\beta}\Psi^{a}_{\alpha} \tag{VI.21b}$$

$$F_{\dot\alpha\alpha,\dot\beta\beta}=\epsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}+\epsilon_{\dot\alpha\beta}\mathcal{G}_{\dot\alpha\dot\beta} \tag{VI.21c}$$

with superfields Ψ^a_α, $\bar\Psi_{a\dot\alpha}$, $\mathcal{F}_{\alpha\beta}$ and $\mathcal{G}_{\dot\alpha\dot\beta}$. In $\mathcal{N}=4$ they are directly related to the scalar superfields W^{ab} and $\bar W_{ab}$ as can be derived from the Bianchi identities $(d^\nabla)^3=0$. For later reference we give the full set of commutators again, which are just the constraints in (VI.17a) and (VI.13) and the identifications in (VI.21)

$$\{\nabla_{\alpha a},\nabla_{\beta b}\}=\epsilon_{\alpha\beta}\bar{W}_{ab} \qquad \{\bar\nabla^a_{\dot\alpha},\bar\nabla^b_{\dot\beta}\}=\epsilon_{\dot\alpha\dot\beta}W^{ab} \tag{VI.22a}$$

$$[\nabla_{\alpha a},\nabla_{\dot\alpha\beta}]=\epsilon_{\alpha\beta}\Psi_{a\dot\alpha} \qquad [\bar\nabla^a_{\dot\alpha},\nabla_{\dot\beta\alpha}]=\epsilon_{\dot\alpha\dot\beta}\Psi^a_\alpha \tag{VI.22b}$$

$$[\nabla_{\dot\alpha\alpha},\nabla_{\dot\beta\beta}]=\epsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}+\epsilon_{\alpha\beta}\mathcal{G}_{\dot\alpha\dot\beta} \tag{VI.22c}$$

where $\mathcal{F}_{\alpha\beta}$ and $\mathcal{G}_{\dot\alpha\dot\beta}$ are superfields and symmetric in their indices.

§ VI.1.2. A little complex differential geometry .—Since we are going to need a few of the notions of differential geometry on complex manifolds, this is the natural spot to make a few definitions. We already encountered the exterior derivative d and the fact that over a complex manifold, this derivative splits into two pieces ∂ and $\bar\partial$, such that

$$d=\partial+\bar\partial. \tag{VI.23}$$

In fact, over a complex manifold $\mathcal{M}$, the complexified tangent space $T\mathcal{M}\otimes\mathbb{C}$ splits into the holomorphic tangent space $T^{(1,0)}\mathcal{M}$ and the antiholomorphic tangent space $T^{(0,1)}\mathcal{M}$. Both are of complex dimension n. Taking the dual, we may define the complexified cotangent bundle $T^*_{\mathbb{C}}\mathcal{M}=T^*\mathcal{M}\otimes\mathbb{C}$ and the complex structure implies that there is a split into holomorphic cotangent bundle $T^{*(1,0)}\mathcal{M}$ and antiholomorphic cotangent bundle $T^{*(0,1)}\mathcal{M}$. The exterior product $\wedge$ then gives rise to p forms on the complexified tangent bundle $T^*_{\mathbb{C}}\mathcal{M}$ which can be further decomposed into (p,q)-form. Explicitly, we have

$$\Lambda^r T^*_{\mathbb{C}}\mathcal{M}=\bigoplus_{p+q=r}\Lambda^{(p,q)}\mathcal{M}=\bigoplus_{p+q=r}\Lambda^{(p,0)}\mathcal{M}\wedge\Lambda^{(0,q)}\mathcal{M}. \tag{VI.24}$$

It is this split of the cotangent bundle that gives rise to the split of the exterior derivative d on a complex manifold. Letting $\Omega^{(p,q)}\mathcal{M}$ describe the sheaf of sections of $\Lambda^{(p,q)}\mathcal{M}$ we have that

$$\partial:\Omega^{(p,q)}\mathcal{M}\to\Omega^{(p+1,q)}\mathcal{M},\quad\text{and}\quad\bar\partial:\Omega^{(p,q)}\mathcal{M}\to\Omega^{(p,q+1)}\mathcal{M}. \tag{VI.25}$$

Clearly, it is favorable to work with the *Dolbeault derivative* $\bar\partial$ which has a clear action on the sheaf of (p,q)-forms, rather than the exterior derivative which sends a (p,q)-form into the direct sum of $(p+1,q)$ and $(p,q+1)$-forms.

We can now go and describe vector bundles (or principal bundles which adds only minor complications) over complex manifolds. Let $E \to \mathcal{M}$ be a rank m vector bundle over a complex manifold $\mathcal{M}$. To describe functions on this vector bundle, we define the sheaf of E-valued (p,q)-forms over $\mathcal{M}$, denoted by $\mathcal{A}_E^{(p,q)}$, which is locally given by

$$\mathcal{A}_E^{(p,q)} \simeq \Gamma(\Lambda^{(p,q)}\mathcal{M} \otimes_{\mathbb{C}} E). \tag{VI.26}$$

A *partial connection*, or partial covariant derivative on such a vector bundle E is the $\mathbb{C}$-linear sheaf homomorphism $\bar{\partial}^E = \bar{\partial}^\nabla$ (where we indicated two notations appearing in this text). Such a partial covariant derivative has the property that

$$\bar{\partial}^E : \mathcal{A}_E^{(p,q)} \to \mathcal{A}_E^{(p,q+1)} \tag{VI.27}$$

and satisfying (ω a (p,q)-form, s a section)

$$\bar{\partial}^E(\omega \otimes s) = \bar{\partial}\omega \otimes s + (-1)^{p+q}\omega \wedge \bar{\partial}^E s. \tag{VI.28}$$

Thus upon choosing a basis of sections $\{s^i\}$, we may write locally $\bar{\partial}^E = \bar{\partial} + A$, where A is the partial connection.

It is easy to check that the operator $\bar{\partial}^E \circ \bar{\partial}^E$ is linear over smooth complex functions, i.e. f a function, s a section

$$\bar{\partial}^E \circ \bar{\partial}^E(fs) = f\bar{\partial}^E \circ \bar{\partial}^E s \tag{VI.29}$$

and thus defines a global section of $\mathcal{A}^{(0,2)}_{\mathrm{End}(E)}$ over the endomorphisms of the vector bundle. In essence this is a *partial curvature* and locally we may write $\mathcal{F}^{(0,2)} = \bar{\partial}A + A \wedge A$.

A partial covariant derivative $\bar{\partial}^E$ is called a *holomorphic structure* if the partial curvature vanishes, i.e. if

$$\bar{\partial}^E \circ \bar{\partial}^E = 0. \tag{VI.30}$$

A smooth complex vector bundle E and a holomorphic structure in turn define a *holomorphic vector bundle* $(E, \bar{\partial}^E)$ with holomorphic sections s defined by $\bar{\partial}^E s = 0$.

Now we can state the second part of the Penrose-Ward correspondence: A solution to the self-dual $SU(N)$ (super-)Yang-Mills equations in four-dimensions is *equivalent* to a holomorphic structure $\bar{\partial}^\nabla$ on a holomorphic $SU(N)$-bundle over (super-)twistor space. For this reason, we will find in the following subsections that self-dual Yang-Mills theory can be described by a holomorphic Chern-Simons theory. An easy argument for this are the equations of motion of hCS: they are nothing else than $\bar{\partial}^E \circ \bar{\partial}^E = 0$ or locally

$$\mathcal{F}^{(0,2)} = \bar{\partial}A + A \wedge A \equiv 0 \tag{VI.31}$$

which naturally follow from the Lagrangian $A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A$.

Let us now turn to the translation between self-dual Yang-Mills theory and holomorphic Chern-Simons theory.

§ VI.1.3. Self-dual super-Yang-Mills theories.—Before we will work with full super Yang-Mills theories, let us introduce the much simpler case of (anti-)self-dual theories. (Anti-)Self-dual theories have a smaller field content [96, 97, 98]. They are characterized by a reduced set of commutation relations in super-Minkowski space, which makes a formulation in chiral—or antichiral—superspace very desirable. We shall focus on self-dual $\mathcal{N}=4$ SYM. In this case, the field content is reduced to

$$(\phi^{ab}, \psi^a_\alpha, F_{\alpha\beta}). \tag{VI.32}$$

All the other fields are set to zero. In terms of commutators and superfields in full superspace, we have the relations

$$\{\nabla_{\alpha a}, \nabla_{\beta b}\} = 0 \qquad \{\bar\nabla^a_{\dot\alpha}, \bar\nabla^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} W^{ab} \tag{VI.33a}$$

$$[\nabla_{\alpha a}, \nabla_{\dot\alpha\beta}] = 0 \qquad [\bar\nabla^a_{\dot\alpha}, \nabla_{\dot\beta\alpha}] = \epsilon_{\dot\alpha\dot\beta}\Psi^a_\alpha \tag{VI.33b}$$

$$[\nabla_{\dot\alpha\alpha}, \nabla_{\dot\beta\beta}] = \epsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}. \tag{VI.33c}$$

These relations are best expressed in chiral superspace $\mathbb{M}_L^{4|8}$

$$(x^- = x^{\alpha\dot\alpha} - 2i\theta^{\alpha a}\bar\theta^{\dot\alpha}_a, \bar\theta^{\dot\alpha}_a) \tag{VI.34}$$

which is a subspace of full superspace. The projection π_{chir} from full to chiral superspace is achieved by setting $\theta^{\alpha a}$ to be fixed. It leads to a reduced set of supersymmetry-covariant derivatives

$$\left(\partial_{\dot\alpha\alpha} = \frac{\partial}{\partial(x^-)^{\alpha\dot\alpha}}, \bar D^a_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_a}\right) \tag{VI.35}$$

and an exterior derivative

$$d_{\text{chir}} = \pi_{\text{chir}} \circ d = \frac{1}{2} d(x^-)^{\alpha\dot\alpha}\partial_{\dot\alpha\alpha} + d\bar\theta^{\dot\alpha}_a \bar\partial^a_{\dot\alpha}. \tag{VI.36}$$

After having introduced a gauge field

$$A = \frac{1}{2} d(x^-)^{\alpha\dot\alpha} A_{\dot\alpha\alpha} + d\bar\theta^{\dot\alpha}_a \bar A^a_{\dot\alpha} \tag{VI.37}$$

over chiral superspace we can introduce covariant derivatives

$$(\nabla_{\dot\alpha\alpha} = \partial_{\dot\alpha\alpha} + A_{\dot\alpha\alpha}, \quad \bar\nabla^a_{\dot\alpha} = \bar D^a_{\dot\alpha} + \bar A^a_{\dot\alpha}) \tag{VI.38}$$

which satisfy commutation relations that are equivalent to the set (VI.33) in full superspace. Quite obviously, we have

$$\{\bar\nabla^a_{\dot\alpha}, \bar\nabla^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} W^{ab} \tag{VI.39}$$

$$[\bar\nabla^a_{\dot\alpha}, \nabla_{\dot\beta\alpha}] = \epsilon_{\dot\alpha\dot\beta}\Psi^a_\alpha \tag{VI.40}$$

$$[\nabla_{\dot\alpha\alpha}, \nabla_{\dot\beta\beta}] = \epsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}. \tag{VI.41}$$

Our endeavor is now to lift these commutation relations to a correspondence space $\mathbb{F}_K$ which will allow us to map them into twistor space [93, 19]. To do so, we will introduce harmonic variables $u^\pm$ in the next section. These parametrize a $\mathbb{CP}^1$ and satisfy $u^{+\dot\alpha}u^-_{\dot\alpha} = 1$ as well as $\overline{(u^+)} = u^-$. We will use these to contract the Lorentz indices $\dot\alpha$ on the covariant derivatives ∇ which will lead us to flat commutators. Said differently, the constraints of self-dual $\mathcal{N}=4$ are flat when pulled back to anti-self-dual planes, i.e., the gauge fields A become pure gauge on these submanifolds. The process is explained in the next subsection.

§ VI.1.4. Harmonization of Lorentz indices.—We want to show now that there are certain submanifolds—namely anti-self-dual planes—of chiral Minkowski space on which the constraints (VI.39) are flat, i.e., the right hand sides of the commutators are zero [93]. To do so, we need to lift these relations to a suitable correspondence space. This is in essence the already presented Penrose-Ward correspondence, see cha. V.1.1, and an example of the importance of double fibrations. In this discussion we restrict the theory immediately to chiral superspace $\mathbb{M}_L^{4|8}$.

Extending chiral superspace by $\mathbb{CP}^1$ means that the correspondence space in question is locally $\mathbb{M}_L^{4|8} \times \mathbb{CP}^1$. We use the harmonic description of this $\mathbb{CP}^1$ by writing down a matrix

$$u = \begin{pmatrix} u_1^- & u_1^+ \\ u_2^- & u_2^+ \end{pmatrix} \tag{VI.42}$$

which can be thought of as filling the lower 2×2-block in the upper 4×4-block of the coset representative of the correspondence space, i.e.,

$$\begin{pmatrix} 1 & x & 0 \\ 0 & u & 0 \\ 0 & \bar{\theta} & 1 \end{pmatrix}. \tag{VI.43}$$

On this combined space we can perfom a coordinate transformation which leads us to the set of coordinates

$$(u^\pm_{\dot\alpha}, x^{\alpha\pm}, \bar\theta^\pm_a) \tag{VI.44}$$

where $x^{\alpha\pm} = \epsilon_{\dot\alpha\dot\beta} u^{\pm\dot\alpha} x^{\dot\beta\alpha}$. The harmonic coordinates $u^\pm$ parametrizing $\mathbb{CP}^1$ are subject to three constraints $\overline{(u^+)} = u^-$ and $u^{+\dot\alpha} u^-_{\dot\alpha} = 1$ from $SU(2)$. Thus the coordinates parametrize an $SU(2)/U(1)$, where the elimination of $U(1)$ reduces in the spirit of chapter IV where the labels $\pm$ are $U(1)$ charges. Using the methods developed in cha. IV.1, we may find the covariant derivatives on the combined space $\mathbb{M}_L^{4|8} \times \mathbb{CP}^1$ to be

$$\Big(D^{++} = u^{+\dot\alpha}\frac{\partial}{\partial u^{-\dot\alpha}},\ D^{--} = u^{-\dot\alpha}\frac{\partial}{\partial u^{+\dot\alpha}},\ \partial^\pm_\alpha = u^{\pm\dot\alpha}\partial_{\alpha\dot\alpha},\ \bar D^{\pm a} = u^{\pm\dot\alpha}\bar D^a_{\dot\alpha}\Big). \tag{VI.45}$$

The space $\mathbb{CP}^1$ is a complex space which can be parametrized using two patches. The tangent space over $\mathbb{CP}^1$ splits into holomorphic and antiholomorphic parts, just as we discussed in VI.1.2, and so does the dual cotangent bundle. Therefore, we find that the exterior derivative on $\mathbb{CP}^1$ splits into a ∂ and a $\bar\partial$ operator (analogous to the normal complex plane, where we have two derivatives for one complex variable). The two covariant derivatives D^{--} and D^{++} play the roles of these two exterior derivative operators after reattaching the crucial vielbeins e^{++} and e^{--} to them, i.e., $\partial = e^{++}D^{--}$ and $\bar\partial = e^{--}D^{++}$. The use of harmonic coordinates $u^\pm$ spares us from having to work in patches, which is their sole purpose in life.

Notice now that the set $(D^{++}, \partial^+_\alpha, \bar D^{+a})$ is a commuting subset of the set of covariant derivatives[3]. Introducing gauge fields for this commuting subset we may write the set of commutators (VI.39) in an equivalent way as

$$[\nabla^+_\alpha, \nabla^+_\beta] = 0 \qquad\qquad [\nabla^+_\alpha, \nabla^+_a] = 0 \tag{VI.46a}$$

$$[\nabla^{++}, \nabla^+_a] = 0 \qquad\qquad [\nabla^{++}, \nabla^+_\alpha] = 0 \tag{VI.46b}$$

$$\{\nabla^{+a}, \nabla^{+b}\} = 0 \tag{VI.46c}$$

[3]Thus the covariant derivatives $(D^{++}, D^+_\alpha, \bar D^{+a})$ form an integrable distribution in the sense of apdx. C.

Notice how the commutators of ∇^+_α and $\bar{\nabla}^{+a}$ are just the constraints defining self-dual Yang-Mills theory restricted to an anti-self-dual plane in chiral superspace. These compose the first and third line above as can be found be inspection of (VI.39). The second line contains new commutators between the covariant derivative ∇^{++} and the restricted derivatives ∇^+_α and ∇^+_a, these are flat commutators by construction.

Equation (VI.46c) implies that we can solve the constraint by setting the gauge field $A^{+a} = 0$. This is a partial gauge choice. The other constraints containing $\bar{D}^{+a}$ imply then a reduction of the coordinate dependencies of the other gauge fields to

$$(u^\pm, x^\pm, \bar{\theta}^+). \tag{VI.47}$$

Using the remaining two covariant derivatives and the vielbeine $e^{--} = u^-.du^-$ and $e^{-\alpha} = (u^-.dx)^\alpha$ which can be derived from the procedure using the Maurer-Cartan form explained in earlier sections (see IV.1.1), we can assemble a complex derivative operator, the Dolbeault operator

$$\bar{\partial} = e^{--}D^{++} + e^{-\alpha}D^+_\alpha \tag{VI.48}$$

as it was given in (IV.16). Given a gauge field

$$A = e^{--}A^{++} + e^{-\alpha}A^+_\alpha \tag{VI.49}$$

we can neatly arrange all commutators left in (VI.46) into one flatness condition

$$\mathcal{F} = \bar{\partial}A + A \wedge A \equiv 0. \tag{VI.50}$$

Observe that in the process of expanding the space by the harmonics $u^\pm$ and the subsequent reduction, we produced a different, equivalent set of commutation relations from the constraints of self-dual theory on chiral superspace by the procedure outlined above.

If we interpret this condition as the equations of motion of a theory, it is possible to write down a Lagrangian reproducing the theory. The Lagrangian in question describes a *holomorphic Chern-Simons* theory with

$$\mathcal{L} = A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A. \tag{VI.51}$$

An action can be written by defining the holomorphic volume form[4]

$$\Omega = d^4\bar{\theta}^+ e^{+\alpha} \wedge e^+_\alpha \wedge e^{++} \tag{VI.52}$$

such that

$$S[A] = \int \Omega \wedge \Big(A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\Big). \tag{VI.53}$$

Indeed, when we solved the fermionic constraint (VI.46c) and defined $\bar{\partial}$, we made a projection into twistor space $\mathbb{CP}^{3|4}$. The action (VI.53) is exactly of the form found by Witten [11]. In the notation of the preceding chapters we can identify $\bar{\chi}_a = \bar{\theta}^+_a$ and

$$z = (x^+, u^+), \qquad \bar{z} = (x^-, u^-). \tag{VI.54}$$

We may also start from this side and begin with the twistor theory: this is the approach of the Penrose-Ward correspondence as explained in e.g., [56, 99]. We may start with a $\bar{\partial}^\nabla$ operator,

[4]Notice that the Chern-Simons form as well as the volume form Ω have zero $U(1)$ weight.

defining a holomorphic structure on twistor space $\mathbb{CP}^{3|4}$ and show that using a specific gauge, we can translate the between Yang-Mills gauge fields and twistor gauge fields in a natural way.

Finally, there is the possibility of choosing a gauge different from $A_a^+ = 0$. The resulting theory is not a Chern-Simons theory and we haven't seen a possiblity to write down Lagrangian for the case of $\mathcal{N} = 4$. For $\mathcal{N} = 2$ supersymmetry, one could however define a derivative

$$\bar{\partial} = e^{--} D^{++} + d\theta^{i-} D_i^+ \tag{VI.55}$$

and write down a (partially) fermionic Chern-Simons action of sorts. We haven't made attempts in this direction and shall keep this as a side remark.

In ssec. VII.1.1 we also present the completion of the action to full $\mathcal{N} = 4$ SYM by inclusion of a local operator.

VI.2 $\mathcal{N} = 3$ in harmonic superspace

Interestingly enough, we can also represent [89, 19] full $\mathcal{N} = 3$ SYM as a Chern-Simons theory on an ambitwistorial space[5]. Take the equations in (VI.17a) where now the R-symmetry index $a = 1, 2, 3$

$$\{\nabla_{\alpha a}, \nabla_{\beta b}\} = \epsilon_{\alpha\beta} \bar{W}_{ab}, \quad \{\bar{\nabla}^a_{\dot\alpha}, \bar{\nabla}^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} W^{ab}, \qquad \{\nabla_{\alpha a}, \bar{\nabla}^b_{\dot\alpha}\} = -2i\delta_a^b \nabla_{\dot\alpha\alpha}. \tag{VI.56}$$

This time around we use full superspace $\mathbb{M}^{4|12}$, attach the space $\mathbb{A}_2 \simeq SL(3)/H_{1,2}(3) \simeq SU(3)/U(1)^{\times 2}$ and so lift to the correspondence space

$$\mathbb{F} \simeq \mathbb{M}^{4|12} \times \frac{SU(3)}{U(1)^{\times 2}} \tag{VI.57}$$

with coordinates

$$(x^{\alpha\dot\alpha}, \theta^{\alpha a}, \bar{\theta}^{\dot\alpha}_a, u^{(i,j)a}). \tag{VI.58}$$

The coordinates u are the harmonic coordinates as introduced in ssec. IV.1.3. We notice now, that $\mathbb{F}$ is a flag supermanifold of flags of type

$$K = (2|0) \subset (2|1) \subset (2|2) \subset (2|3) \subset (4|3) \tag{VI.59}$$

and the elements of the stabilizing group $H_K(4|3) \subset SL(4|3)$, have the form

$$\left(\begin{array}{cccc|ccc} * & * & & & & & \\ * & * & & & & & \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ \hline * & * & & & * & & \\ * & * & & & * & * & \\ * & * & & & * & * & * \end{array}\right) \tag{VI.60}$$

[5]The space in question is not actually ambitwistor space achieved by harmonizing the Lorentz indices but an iso-ambitwistor space obtained by harmonization of the R-symmetry indices.

We can harmonize the R-symmetry indices in the constraints (VI.56) using $u^{(1,0)a}$ and $u_a^{(0,1)}$. This means we have now $\nabla_\alpha^{(1,0)} = u^{(1,0)a}\nabla_{\alpha a}$ and $\bar\nabla_{\dot\alpha}^{(0,1)} = u_a^{(0,1)}\bar\nabla_{\dot\alpha}^a$. Hence the commutators (VI.56) become flat

$$\{\nabla_\alpha^{(1,0)}, \nabla_\beta^{(1,0)}\} = 0, \qquad \{\bar\nabla_{\dot\alpha}^{(0,1)}, \bar\nabla_{\dot\beta}^{(0,1)}\} = 0, \qquad \{\nabla_\alpha^{(1,0)}, \bar\nabla_{\dot\alpha}^{(0,1)}\} = 0 \tag{VI.61}$$

where the last commutator follows from the identity $u^{(1,0)a}u_a^{(0,1)} = 0$ which follows from the rôle of the $u^{(i,j)}$ as coordinates on an $SU(3)$. So here we see that the constraints of $\mathcal{N} = 3$ SYM vanish when pulled back to a three-dimensional bosonic submanifold of the harmonic superspace $\mathbb{F}$.

Now we turn to the covariant derivatives on this space. The covariant derivatives on the space $\mathbb{A}_2$ have been given in (IV.24a). Calculation of their algebra is straightforward and we only give the commutators which will become important for us in the following. We have

$$[D^{(2,-1)}, \nabla_\alpha^{(1,0)}] = 0, \qquad [D^{(-1,2)}, \nabla_\alpha^{(1,0)}] = 0, \qquad [D^{(1,1)}, \nabla_\alpha^{(1,0)}] = 0, \tag{VI.62a}$$

$$[D^{(2,-1)}, \bar\nabla_{\dot\alpha}^{(0,1)}] = 0, \qquad [D^{(-1,2)}, \nabla_{\dot\alpha}^{(0,1)}] = 0, \qquad [D^{(1,1)}, \nabla_{\dot\alpha}^{(0,1)}] = 0. \tag{VI.62b}$$

Introduce gauge connections $A^{(q_1,q_2)}$ for all the $D^{(q_1,q_2)}$ by a u-dependent gauge transformation. This also lifts all the original gauge connections to the space $\mathbb{F}$ due to the introduction of a dependence on $u^{(q_1,q_2)}$. Since the newly introduced gauge fields $A^{(q_1,q_2)}$ are all flat[6], the covariant derivatives $\nabla^{(q_1,q_2)}$ satisfy the same algebra as the $D^{(q_1,q_2)}$.

We can now make a choice for an integrable distribution out of the set of covariant derivatives $D^{(i,j)}$ we introduced by first solving the fermionic constraints (VI.61). Since these commutators are all flat we can gauge both fields $A_\alpha^{(1,0)} = 0 = A_{\dot\alpha}^{(0,1)}$. The remaining gauge fields depend on the reduced set of coordinates

$$(x_A, \theta_\alpha^{(1,-1)}, \theta_\alpha^{(0,1)}, \bar\theta_{\dot\alpha}^{(1,0)}, \bar\theta_{\dot\alpha}^{(-1,1)}, u) \tag{VI.63}$$

which parametrizes a so called *analytic superspace* with x_A such that $D_\alpha^{(1,0)}x_A = 0$ and $\bar D_{\dot\alpha}^{(0,1)}x_A = 0$ and a reduced number of fermionic directions. Then we can pick the derivatives

$$(\nabla^{(2,-1)}, \nabla^{(-1,2)}, \nabla^{(1,1)}) \tag{VI.64}$$

which have only one non-zero commutator

$$[\nabla^{(-1,2)}, \nabla^{(2,-1)}] = \nabla^{(1,1)}. \tag{VI.65}$$

Using the Maurer-Cartan form introduced in sec. IV.1.3, we can extract the vielbeine $e^{(q_1,q_2)}$. Then we rewrite all commutators in the form of a zero curvature condition. Defining a covariant Dolbeault derivative as detailed in apdx. C

$$\bar\partial^\nabla = e^{(-1,-1)}\nabla^{(1,1)} + e^{(-2,1)}\nabla^{(2,-1)} + e^{(1,-2)}\nabla^{(-1,2)}, \tag{VI.66}$$

the zero curvature condition which encompasses all of the commutators above is given by

$$\mathcal{F} = \bar\partial A + A \wedge A = 0 \tag{VI.67}$$

[6] Assuming new non-trivial field strengths is not impossible but leads to unnecessary complications when proving the equivalence of the lifted theory to the original theory. However, for theories with additional matter fields, harmonic directions may necessitate new non-zero field strength components.

with $\mathcal{F}$ a $(0,2)$ curvature form. Interpreting this condition as the equations of motion of the theory, we are led to write a Chern-Simons type Lagrangian

$$\mathcal{L}_{\mathcal{N}=3} = A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A. \tag{VI.68}$$

Notably, if we were to write $\mathcal{F}$ in components [89], we would find that the commutator (VI.65) produces an additional $A^{(1,1)}$

$$\begin{aligned}\mathcal{F} = e^{(-2,1)} \wedge e^{(1,-2)}(D^{(2,-1)}A^{(-1,2)} - D^{(-1,2)}A^{(2,-1)} + [A^{(2,-1)}, A^{(-1,2)}] - A^{(1,1)})\\ + e^{(-2,1)} \wedge e^{(-1,-1)}(D^{(2,-1)}A^{(1,1)} - D^{(1,1)}A^{(2,-1)} + [A^{(2,-1)}, A^{(1,1)}])\\ + e^{(1,-2)} \wedge e^{(-1,-1)}(D^{(-1,2)}A^{(1,1)} - D^{(1,1)}A^{(-1,2)} + [A^{(-1,2)}, A^{(1,1)}])\end{aligned} \tag{VI.69}$$

such that the Lagrangian in component form develops a term $(A^{(1,1)})^2$ which obscures the fact that we are dealing with a Chern-Simons type Lagrangian. Since the Lagrangian $\mathcal{L}_{\mathcal{N}=3}$ is a $(0,3)$ form, we define a $(3,0)$ form Ω by

$$\Omega = d^4x_A d^8\theta \; e^{(1,1)} \wedge e^{(2,-1)} \wedge e^{(-1,2)} \tag{VI.70}$$

to get a $(3,3)$ form to integrate over the manifold we are working on. Notice that the combination $e^{(1,1)} \wedge e^{(2,-1)} \wedge e^{(-1,2)}$ has $U(1) \times U(1)$-weight $(2,2)$. This is offset by the factor $d^8\theta = d^2\theta^{(1,-1)}d^2\theta^{(0,1)}d^2\bar{\theta}^{(1,0)}d^2\bar{\theta}^{(-1,1)}$ which has weight $(-2,-2)$ such that the full measure is weightless. In the end the action reads

$$S[A] = \int \Omega \wedge \left(A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\right). \tag{VI.71}$$

Attempts have been made to quantize this theory [100] but were reliant on disassembling the field A into its component fields. When understanding A as a twistor gauge field, we might be able to quantize A as a one-form just as it was done for the twistorial description of $\mathcal{N} = 4$. One obstacle that we are encountering rather immediately is that we cannot use axial gauge for A in this theory.

Axial gauge simplified the treatment of twistor theory ([101] for a review) and is achieved by letting the twistor field $A = e^{--}A^{++} + e^{-\alpha}A^{+}_{\alpha}$ vanish along a fixed direction $\mathcal{Z}_\star$, i.e.,

$$\imath_{\mathcal{Z}_\star}A = z^{--}A^{++} + z^{-\alpha}A^{+}_{\alpha} = 0. \tag{VI.72}$$

We may use this gauge to set one component of A to zero. This would imply that the trivalent vertex in the action (VI.71) had to vanish.

Another approach is to use a Lorenz-like gauge as in [102] and proceed in the same way as done in the reference. After having found a propagator for A it would then be possible to derive correlation functions or other observables. This theory should be understood as a playground for an actual ambitwistor theory of supersymmetric Yang-Mills theory; the techniques used for this theory will be transportable to other gauge theories on (ambi)twistorial spaces.

We shall now leave this topic and turn to the question how to find the familiar fields of supersymmetric Yang-Mills theory when we start with an equivalent twistor theory. In the Abelian case and for $\mathcal{N} = 4$ SYM and holomorphic Chern-Simons theory on chiral twistor space this is done by using the Penrose transform as introduced before. However, we would like to extend this to other spaces and to the non-Abelian case. This will be the topic of the next chapter.

CHAPTER VII

Local operators in superspace from twistor connections

The final chapter of this part will be concerned with the question of how to extract local operators in spacetime from the twistor connections introduced in the preceding chapters. We shall start by looking at the case of self-dual $\mathcal{N} = 4$ SYM twistor space. As is well known, one can lift the holomorphic Chern-Simons action for SDSYM to an action describing full super Yang-Mills theory on twistor space by adding the non-local $\log\det$-term introduced by Witten [11]. We will show that the form of this additional term can be understood from a local operator point of view, too.

The next step will be to understand local operators in $\mathcal{N} = 3$ SYM in $SU(3)/U(1)^{\times 2}$ language. We will see that the description and the idea behind the mechanism is essentially the same as in ordinary twistor space.

VII.1 Local operators from twistor space

The problem has been solved a long time ago for the Abelian theory [103, 104, 105, 99]. We know that via the Penrose transformation, fields $\phi_{\dot\alpha_1\cdots\dot\alpha_{2h}}(x)$ of helicity h which satisfy the equations[1]

$$\epsilon^{\dot\alpha\dot\beta}\partial_{\dot\alpha\alpha}\phi_{\dot\beta\dot\alpha_2\cdots\dot\alpha_{2h}} = 0 \tag{VII.2}$$

can be expressed in terms of sections $\psi \in \Gamma(\mathcal{O}(2h-2))$ of line bundles $\mathcal{O}(-2h-2)$ over twistor space $\mathbb{CP}^3$. The weight $-2h-2$ is the homogeneity of the functions under scaling transformations. A correspondence is achieved via an integral over the complex line $\mathbb{CP}^1$: this is the famous Penrose transform and the explicit form of the double fibration $\pi_1 \circ \pi_2^{-1}$ that we discussed in previous chapters. Using the notation of VI.1.4 for the case of Abelian self-dual $\mathcal{N} = 4$ SYM, we can extract the spacetime fields ϕ_{ab}, $\psi^{\dot\alpha a}$, or $G^{\dot\alpha\dot\beta}$) by

$$\phi^{ab}(x) = \int_{\mathbb{CP}^1} e^{++} \wedge D^{-a}D^{-b}A|_X \tag{VII.3a}$$

$$\psi_a^{\dot\alpha}(x) = \frac{1}{3!}\epsilon_{abcd}\int_{\mathbb{CP}^1} e^{++} \wedge u^{+\dot\alpha}D^{-b}D^{-c}D^{-d}A|_X \tag{VII.3b}$$

$$G^{\dot\alpha\dot\beta}(x) = \frac{1}{4!}\epsilon_{abcd}\int_{\mathbb{CP}^1} e^{++} \wedge u^{+\dot\alpha}u^{+\dot\beta}D^{-a}D^{-b}D^{-c}D^{-d}A|_X \tag{VII.3c}$$

[1] In the special case of $h = 0$ this equation gets replaced by the Klein-Gordon equation without mass

$$\Box\phi = 0. \tag{VII.1}$$

Notice how the $U(1)$ weights cancel out precisely. The form A is the twistor gauge field restricted to the submanifold $\mathbb{CP}^1 \subset \mathbb{CP}^3$ denoted by X

$$A|_X = (e^{--}A^{++} + e^{-\alpha}A^+_\alpha)|_X = e^{--}A^{++}. \tag{VII.4}$$

We can show that these fields behave correctly under Abelian gauge transformations $A \mapsto A + \bar{\partial}\lambda$

$$\delta_{\text{lin}}\phi^{ab} = \int_{\mathbb{CP}^1} e^{++} \wedge e^{--} D^{-a}D^{-b}D^{++}\lambda = 0 \tag{VII.5}$$

up to boundary terms. Clearly, they are not invariant under non-Abelian transformations $A \mapsto A + \bar{\partial}A + [A, \lambda]$ since in general

$$\delta\phi^{ab} = \int_{\mathbb{CP}^1} e^{++} \wedge e^{--}[D^{-a}D^{-b}A^{++}, \lambda] \neq 0. \tag{VII.6}$$

It is, however, possible to add a term to (VII.3a) which cancels this additional term up to higher order commutators. We will now show how to do it.

Introduce the inverse $(D^{++})^{-1}$ of the operator D^{++} acting on functions of the harmonics $u^\pm$. D^{++} has a kernel, so $(D^{++})^{-1}$ is not unique. We can nevertheless define it on functions f^{++} of weight $+2$ by

$$(D^{++})^{-1}f^{++} = \int_{\mathbb{CP}^1} e^{--}(v) \wedge e^{++}(v) \frac{u^{+\dot{\alpha}}v^-_{\dot{\alpha}}}{u^{+\dot{\beta}}v^+_{\dot{\beta}}} f^{++}(v). \tag{VII.7}$$

It is easy to check that

$$D^{++}(D^{++})^{-1}f^{++}(u) = f^{++}(u). \tag{VII.8}$$

On functions of zero charge like λ we find that there will be a non-zero difference

$$\lambda_0 = \lambda - D^{++}(D^{++})^{-1}\lambda(u) \tag{VII.9}$$

independent of u, $D^{++}\lambda_0 = 0$. This remainder corresponds to the zero mode in an expansion of λ in terms of $u^\pm$. Under a linearized gauge transformation we find

$$\delta_{\text{lin}}\Big(\int_{\mathbb{CP}^1} e^{++} \wedge e^{--}[D^{-a}D^{-b}A^{++}, (D^{++})^{-1}A^{++}]\Big) = \int_{\mathbb{CP}^1} e^{++} \wedge e^{--}[D^{-a}D^{-b}A^{++}, \lambda - \lambda_0]. \tag{VII.10}$$

Adding (VII.6) and (VII.10) yields

$$\delta\phi^{ab} = [\phi^{ab}, \lambda_0] + \mathcal{O}((A^{++})^2). \tag{VII.11}$$

Hence the zero mode λ_0 of the non-Abelian gauge transformation $\lambda(u)$ in twistor space plays the role of gauge parameter in spacetime. The terms quadratic in A^{++} can be canceled by adding yet another term. This will lead to series of correction terms which we can write as

$$\phi^{ab}(x, \bar{\theta}) = \int_{\mathbb{CP}^1} e^{++} \wedge e^{--} \sum_{p=1}^{\infty} [\underbrace{(D^{++})^{-1}A^{++}, \ldots, [(D^{++})^{-1}A^{++}}_{p-1}, D^{-a}D^{-b}A^{++}] \cdots]. \tag{VII.12}$$

The same procedure is available for the other space-time fields. In this way, we can recover local operators in non-Abelian Yang-Mills theory on spacetime from non-Abelian gauge theory on twistor space. Geometrically, the construction relies on the double fibration described in V.1.1.

To extract the superspace gauge fields $A_{\dot{\alpha}\alpha}$ and $\bar{A}^a_{\dot{\alpha}}$ from the twistor gauge field A we can also use the Penrose transform

$$A_{\dot{\alpha}\alpha}(x,\bar{\theta}) = \int_{\mathbb{CP}^1} e^{++} \wedge u^-_{\dot{\alpha}} D^-_\alpha A, \tag{VII.13}$$

$$\bar{A}^a_{\dot{\alpha}}(x,\bar{\theta}) = \int_{\mathbb{CP}^1} e^{++} \wedge u^-_{\dot{\alpha}} D^{-a} A \tag{VII.14}$$

Note the appearance of u^- in these transformations! When performing a linear gauge transformation $\delta_{\text{lin}} A^{++} = D^{++}\lambda$, the u^- will be responsible for the inhomogeneous term in the gauge transformations in spacetime. Up to total derivatives we see

$$\delta_{\text{lin}} A_{\dot{\alpha}\alpha} = \int_{\mathbb{CP}^1} e^{++} \wedge e^{--} u^-_{\dot{\alpha}} D^-_\alpha D^{++}\lambda = \partial_{\dot{\alpha}\alpha}\lambda_0 \tag{VII.15}$$

with $\lambda_0 = \int e^{++} \wedge e^{--}\lambda$, the zero mode in the $u^\pm$ expansion of λ. Once again, λ_0 is the zero mode of λ, as arbitrary functions of harmonic variables u can be decomposed on an orthogonal basis of symmetrised products of $u^\pm$. Integration over the $\mathbb{CP}^1$ projects out the zeroth order term in the expansion, the rest vanish by orthogonality.

§ VII.1.1. Full $\mathcal{N} = 4$ SYM in twistor space.—Now that we have learned how to handle local operators, we can relatively easily tackle the problem of providing a twistor action for full $\mathcal{N} = 4$ SYM in twistor space. So far we had the self-dual part of the action on twistor space

$$S[A] = \int \Omega \wedge \operatorname{tr}\Big(A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\Big). \tag{VII.16}$$

To find the action describing full Yang-Mills theory, we have to inspect the constraints of the full theory on chiral superspace once again

$$\{\bar{\nabla}^a_{\dot{\alpha}}, \bar{\nabla}^b_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} W^{ab} \tag{VII.17a}$$

$$[\bar{\nabla}^a_{\dot{\alpha}}, \nabla_{\dot{\beta}\alpha}] = \epsilon_{\dot{\alpha}\dot{\beta}} \Psi^a_\alpha \tag{VII.17b}$$

$$[\nabla_{\dot{\alpha}\alpha}, \nabla_{\dot{\beta}\beta}] = \epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} + \epsilon_{\alpha\beta} G_{\dot{\alpha}\dot{\beta}}. \tag{VII.17c}$$

They contain the new term $\epsilon_{\alpha\beta} G_{\dot{\alpha}\dot{\beta}}$. This term hinders us from writing the full Yang-Mills constraints in the form of flatness conditions. Instead, we find that the constraints become

$$\{\bar{\nabla}^{+a}, \bar{\nabla}^{+b}\} = 0, \qquad [\nabla^+_\alpha, \bar{\nabla}^{+a}] = 0, \qquad [\nabla^+_\alpha, \nabla^+_\beta] = \epsilon_{\alpha\beta} G^{++} \tag{VII.18}$$

where the additional field $G^{++} = u^{+\dot{\alpha}} u^{+\dot{\beta}} G_{\dot{\alpha}\dot{\beta}}$. The question is how to rewrite (VII.17) as equations of motion of a Chern-Simons theory plus an additional term to include G^{++}. We can do this by extending equation (VII.12) to $G^{\dot{\alpha}\dot{\beta}}$. This amounts to adding a $\log\det(\bar{\partial} + A)$ term [11]. One finds

$$S_F[A] = \int \Omega \wedge \operatorname{tr}\Big(A\bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\Big) + \int d^4x d^4\theta^+ d^4\theta^- \log\det(\bar{\partial} + A)|_X \tag{VII.19}$$

where X is the line in twistor space corresponding to the point $(x,\bar{\theta})$ in chiral superspace. The determinant takes care of the harmonics and the integral over $d^4x d^8\theta$ integrates over the space of linearly embedded $\mathbb{CP}^1$'s (corresponding to spacetime points) in twistor space. To write the action in the Chalmers-Siegel form [106] we need to rescale the field $G_{\dot{\alpha}\dot{\beta}} = g^{-2}_{\text{YM}} \bar{F}_{\dot{\alpha}\dot{\beta}}$.

Varying $S_F[A]$ w.r.t. A will lead to the equations of motion (VII.17) when rewriting the result of the variation in component form.

VII.2 Wilson loops for local operators

Remarkably, it is possible to rewrite (VII.12) in terms of Wilson line operators [107]. In twistor space the harmonics $u^{\pm}$ parametrize a $\mathbb{CP}^1 \subset \mathbb{CP}^{3|4}$, a complex line in twistor space. We will continue to denote such $\mathbb{CP}^1$s as X. If we restrict the twistor connection to X, we find that it is in fact flat, and so there exists a gauge transformation $h(u)$ such that

$$(\bar{\partial} + A(u))|_X h(u) = 0 \tag{VII.20}$$

up to multiplication of $h(u)$ with a constant group element to the right. Then define the analog of a Wilson line operator $U(u_1, u_0)$ by

$$U(u_1, u_0) = h(u_1)h(u_0)^{-1} \tag{VII.21}$$

where

$$U(u, u) = \mathbf{1}, \quad U(u_1, u_0)^{-1} = U(u_0, u_1), \quad U(u_2, u_1)U(u_1, u_0) = U(u_2, u_0). \tag{VII.22}$$

A gauge transformation $\bar{\partial} + A' = g(\bar{\partial} + A)g^{-1}$ transforms the Wilson line operators covariantly

$$U'(u_1, u_0) = g(u_1)U(u_1, u_0)g(u_0)^{-1}. \tag{VII.23}$$

We can use (VII.20) to express U in terms of the twistor field A, by writing the recursion equation

$$U = \mathbf{1} + \bar{\partial}^{-1}(AU) \tag{VII.24}$$

where we have defined the inverse operator $\bar{\partial}^{-1}$. This operator acts on $(0, 1)$ forms ω like in (VII.7)

$$(\bar{\partial}^{-1}\omega)(u) = \frac{1}{\pi}\int_{\mathbb{CP}^1}\left(\frac{e^{++}(u_1)\langle u^+, u_1^-\rangle}{\langle u^+, u_1^+\rangle} - \frac{e^{++}(u_1)\langle u_0^+, u_1^-\rangle}{\langle u_0^+, u_1^+\rangle}\right)\wedge\omega. \tag{VII.25}$$

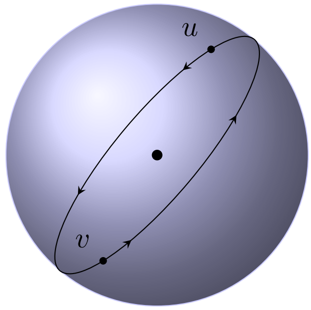
It satisfies the boundary condition $(\bar{\partial}^{-1}\omega)(u_0) = 0$, too. We can write the Wilson line operator $U(u, u_0)$ as a power series in A

$$\begin{aligned} U(u, u_0) &= \mathbf{1} + (\bar{\partial}^{-1}A)(u) + \bar{\partial}^{-1}(A(\bar{\partial}^{-1}A))(u) + \ldots \\ &= \mathbf{1} + \frac{1}{\pi}\int_{\mathbb{CP}^1} e^{++}(u_1)\wedge A(u_1)\frac{\langle u^+, u_0^+\rangle}{\langle u^+, u_1^+\rangle\langle u_1^+, u_0^+\rangle} \\ &+ \frac{1}{\pi^2}\int_{\mathbb{CP}^1} e^{++}(u_1)\wedge A(u_1)\int_{\mathbb{CP}^1} e^{++}(u_2)\wedge A(u_2)\frac{\langle u^+, u_0^+\rangle}{\langle u^+, u_2^+\rangle\langle u_2^+, u_1^+\rangle\langle u_1^+, u_0^+\rangle} + \ldots \end{aligned} \tag{VII.26}$$

Careful inspection of (VII.12) reveals that there is a more concise way of writing the non-Abelian form of the local operators. By replacing the infinite sum by a product of Wilson lines one gets

$$\phi^{ab}(x) = \int_X e^{++}(u)U_X(v, u)(D^{-a}D^{-b}A)(u)U_X(u, v). \tag{VII.27}$$

The submanifold $X = \mathbb{CP}^1 \subset \mathbb{CP}^{3|4}$ corresponds to a point $(x, \bar{\theta})$ in chiral spacetime. $U_X(v, u)$ is the Wilson line along X and u and v are local coordinates on X. In the picture to the right, we have depicted the $\mathbb{CP}^1$ by a sphere. The Wilson line starts and ends at v and moves around the sphere. A change in the coordinate v amounts to a global gauge transformation [108]. With this we have found a consistent way of extracting spacetime fields from twistor fields in the non-Abelian theory.



VII.3 Local operators from $\mathcal{N} = 3$ harmonic superspace

The previous section can be understood as the setup for this section. Here, we want to use the same techniques as above to extract the spacetime local operators from the gauge field

$$A = e^{(-1,-1)} A^{(1,1)} + e^{(-2,1)} A^{(2,-1)} + e^{(1,-2)} A^{(-1,2)} \tag{VII.28}$$

on $\mathcal{N} = 3$ harmonic superspace $\mathbb{M}_A^{4|8} \times \mathbb{A}_2$. We will see that the ideas are rather similar.

The integrals over $\mathbb{A}_2$ that we will see in the following equations will be normalized such that

$$\int_{\mathbb{A}_2} \mathrm{vol} = 1. \tag{VII.29}$$

Let us start with the Abelian theory again and identify potential candidates for the scalar superfields $\bar{\phi}^i(x, \theta, \bar{\theta})$ and $\phi_i(x, \theta, \bar{\theta})$ by looking for $U(1) \times U(1)$-weightless expressions which are invariant under Abelian gauge transformations.

Since the algebra of covariant derivatives is slightly more involved than in the case of $\mathcal{N} = 4$ SYM on chiral superspace and hCS on twistor space, observe first the identities below

$$\int_{\mathbb{A}_2} u_i^{(1,0)} f^{(-1,0)} = -\int_{\mathbb{A}_2} u_i^{(-1,1)} D^{(2,-1)} f^{(-1,0)} = -\int_{\mathbb{A}_2} u_i^{(0,-1)} D^{(1,1)} f^{(-1,0)} \tag{VII.30a}$$

$$\int_{\mathbb{A}_2} u_i^{(0,-1)} f^{(0,1)} = -\int_{\mathbb{A}_2} u_i^{(1,0)} D^{(-1,-1)} f^{(0,1)} = -\int_{\mathbb{A}_2} u_i^{(-1,1)} D^{(1,-2)} f^{(0,1)} \tag{VII.30b}$$

$$\int_{\mathbb{A}_2} u_i^{(-1,1)} f^{(1,-1)} = -\int_{\mathbb{A}_2} u_i^{(0,-1)} D^{(-1,2)} f^{(1,-1)} = -\int_{\mathbb{A}_2} u_i^{(1,0)} D^{(-2,1)} f^{(1,-1)} \tag{VII.30c}$$

for f an arbitrary function of the indicated weight. Interestingly enough, there are several candidates for the scalar superfield ϕ_i which are weightless w.r.t to $U(1)^{\times 2}$ and invariant under linearized gauge transformations δ_{lin}. These are

$$\int_{\mathbb{A}_2} \epsilon^{\alpha\beta} u_i^{(0,-1)} D_\alpha^{(-1,1)} D_\beta^{(-1,1)} A^{(2,-1)} \tag{VII.31a}$$

$$-\int_{\mathbb{A}_2} \epsilon^{\alpha\beta} u_i^{(-1,1)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A^{(1,1)} \tag{VII.31b}$$

$$\int_{\mathbb{A}_2} \epsilon^{\alpha\beta} \Big(-2 u_i^{(-1,1)} D_\alpha^{(0,-1)} D_\beta^{(-1,1)} A^{(2,-1)} + u_i^{(1,0)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A^{(-1,2)} \Big) \tag{VII.31c}$$

$$\int_{\mathbb{A}_2} \epsilon^{\alpha\beta} \Big(2 u_i^{(0,-1)} D_\alpha^{(0,-1)} D_\beta^{(-1,1)} A^{(1,1)} - u_i^{(1,0)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A^{(-1,2)} \Big) \tag{VII.31d}$$

With the help of algebra of covariant derivatives, integration by parts, and the analyticity of the gauge fields

$$D_\alpha^{(1,0)} A^{(2,-1)} = D_\alpha^{(1,0)} A^{(-1,2)} = D_\alpha^{(1,0)} A^{(1,1)} = 0 \tag{VII.32}$$

imposed by gauge choice as explained in sec. VI.2, it is easy to show that (VII.31) are invariant under linearized gauge transformations

$$\delta_{\text{lin}} A^{(q_1,q_2)} = D^{(q_1,q_2)}\lambda. \tag{VII.33}$$

Observe that the expressions (VII.31) are similar to the expressions in twistor space (VII.3). However, while we could write down unique expressions in the twistor case, we find ourselves with four candidate expressions here. Luckily, on-shell all of the four candidates agree which can be shown by using the equations of motion on the equations (VII.31).

The next step is to understand the space-time operators for non-linear gauge transformations. Due to the more difficult algebra of covariant derivatives an approach as in (VII.12) is destined to fail due to the ever increasing number of possible correction terms that have to be added at every step. The difference between the twistor formulation of $\mathcal{N}=4$ SYM and this formulation of $\mathcal{N}=3$ SYM lies in the less clear relation between points $(x,\theta,\bar\theta)$ in $\mathcal{N}=3$ superspace and a distinguished submanifold in harmonic $\mathcal{N}=3$ superspace. For twistor space, any linearly embedded $\mathbb{CP}^1 \subset \mathbb{CP}^{3|4}$ would define a point in chiral superspace. In the present case there are in fact *three* submanifolds of $\mathbb{A}_2$. These can be obtained by setting either of the $u_i^{(p,q)}$—and their inverses $u^{(-p,-q)i}$—to a constant value. We call these three different submanifolds X_1, X_2 and X_3. X_1 and X_2 obtained by setting $u_i^{(1,0)}$ or $u_i^{(0,1)}$ to a constant value, are linearly embedded lines in $\mathbb{A}_2$ while X_3 is non-linearly embedded in $\mathbb{A}_2$. These X_i are the distinguished submanifolds over which we need to integrate to remove the dependence on the variables $u_i^{(p,q)}$.

Upon imposing the constraints for X_i only two vielbeine survive in either case, the rest are set to zero by the constraints. The nonzero vielbeine are (for their definitions see IV.1.3)

$$X_1: \quad e^{(-1,2)},\ e^{(1,-2)} \tag{VII.34a}$$

$$X_2: \quad e^{(-2,1)},\ e^{(2,-1)} \tag{VII.34b}$$

$$X_3: \quad e^{(-1,-1)},\ e^{(1,1)}. \tag{VII.34c}$$

Varying $u_i^{(1,0)}$ resp. $u^{(-1,0)i}$ parametrizes a $\mathbb{CP}^2 \subset \mathbb{A}_2$ which is the space of all lines X_1. We denote the space Y_1 and for X_2 and X_3 we define Y_2 and Y_3 in the analogous way. All three spaces come equipped with natural volume forms μ_{Y_i} which in terms of the vielbeine can be written as

$$\mu_{Y_1} = e^{(1,1)} \wedge e^{(2,-1)} \wedge e^{(-1,-1)} \wedge e^{(-2,1)} \tag{VII.35a}$$

$$\mu_{Y_2} = e^{(1,-2)} \wedge e^{(-1,-1)} \wedge e^{(-1,2)} \wedge e^{(1,1)} \tag{VII.35b}$$

$$\mu_{Y_3} = e^{(-2,1)} \wedge e^{(-1,2)} \wedge e^{(2,-1)} \wedge e^{(1,-2)}. \tag{VII.35c}$$

After having defined these measures we can write local operators on spacetime by

$$\phi_i = -\int_{Y_3} \mu_{Y_3} \int_{X_3} e^{(1,1)} \wedge \epsilon^{\alpha\beta} u_i^{(-1,1)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A \tag{VII.36}$$

This is analogous to writing

$$\phi_i = \int_{X_1} e^{(-1,2)} \wedge e^{(1,-2)} \int_{X_2} e^{(-2,1)} \wedge e^{(2,-1)} \int_{X_3} e^{(1,1)} \wedge \epsilon^{\alpha\beta} u^{(-1,-1)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A. \tag{VII.37}$$

These expressions can be generalized to non-Abelian gauge transformations by introducing Wilson line operators $U_{X_i}(u,v)$. The corresponding scalar superfield transforming under non-linear gauge transformations is given by

$$\phi_i(x,\theta,\bar{\theta}) = \int_{X_1(\xi)} e^{(-1,2)}(\zeta)\wedge e^{(1,-2)}(\zeta) U_{X_1}(\xi,\zeta) \int_{X_2(\zeta)} e^{(-2,1)}\wedge e^{(2,-1)}(\tau) U_{X_2}(\zeta,\tau)$$
$$\times \int_{X_3(\tau)} e^{(1,1)}\wedge U_{X_3}(\tau,\sigma)\left(\epsilon^{\alpha\beta} u_i^{(-1,1)} D_\alpha^{(0,-1)} D_\beta^{(0,-1)} A\right) U_{X_3}(\sigma,\tau) U_{X_2}(\tau,\zeta) U_{X_1}(\zeta.\xi) \quad \text{(VII.38)}$$

We denote the line X_i containing the point $\kappa \in \mathbb{A}_2$ by $X_i(\kappa)$. This introduces a slight abuse of notation as κ denotes the point on $\mathbb{A}_2$ as well as the local coordinate on X_i. In the linear case we found four different but equivalent ways to write down the local superspace operator ϕ_i. In the non-Abelian case, there is in fact an infinite number of ways to construct local spacetime fields starting form the harmonic connection A. Since A is flat on $\mathbb{A}_2$ on-shell, it is possible to deform the integration contours of the Wilson lines arbitrarily. Luckily, on-shell all the possible ways of writing down local operators are in fact equivalent.

With this, we have found a way to extract local operators on full $\mathcal{N}=3$ superspace from analytic $\mathcal{N}=3$ superspace in the non-Abelian theory. This allows us to relate calculations done in one space to results in another space similarly to the chiral twistor space case. The perfect observables to calculate in this theory would most probably be correlation functions since there is only one field A, that contains the whole information. We haven't made any attempts at doing such calculations, yet, but they may be enlightening.

As a final remark, let us contrast the situation with $\mathcal{N}=4$ SYM translated to twistor space. There we had the advantage that we could immediately see which observables would be best suited for translation between the two spaces—these are the Wilson loops on light-like lines due to the correspondence between points and α-planes in chiral superspace, and lines and points in twistor space. In the next part 4 we will be studying exactly this relationship.

4
Wilson loops

Historically, Wilson lines have been introduced to find a non-perturbative formulation of QCD [109] and to explain confinement, and closed loops motivated the first formulations of string theory. In $\mathcal{N} = 4$ SYM, Wilson loops formulated on polygonal, light-like contours have found wide interest recently due to a conjectured duality [47] between such operators and MHV gluon scattering amplitudes. It is this specific type of Wilson loop that this part will be concerned with.

Subsequently, Wilson loops on contours with fermionic directions and fermionic gauge fields have been formulated in an attempt to capture a duality between such *super-Wilson loops* and the full range of superamplitudes of $\mathcal{N} = 4$ SYM [17, 10]. As outlined in sec. II.2 these proposals had the problem that they had trouble reproducing the duality in the expected way. Even worse, it was shown [18] that the Wilson loops in question introduced a new unexpected anomaly of the supersymmetry operator $\bar{\mathfrak{Q}}$ seemingly independent of quantum effects and regularization. The problem was traced back to the formulation of $\mathcal{N} = 4$ SYM on chiral superspace and the dependence of the scattering amplitude on only 4 fermionic parameters.

There have been a number of different attempts to solve or circumvent this problem. In the following chapters, we will present one of them—the formulation of a new Wilson loop on a light-like contour for $\mathcal{N} = 4$ SYM on full Minkowski superspace $\mathbb{M}^{4|16}$. We start by considering the correct definition of light-like lines in full superspace in the sec. VIII.1. In cha. IX, we will formulate the quantum theory of the gauge fields of $\mathcal{N} = 4$ SYM on full superspace in the linearized theory. Using these ingredients we will calculate the one-loop expectation value of this non-chiral Wilson loop and look at various ways to regularize the appearing ultraviolet divergences on the first loop level in sec. IX.3. This discussion is based on [1].

CHAPTER VIII

Light-like lines and null polygonal contours

Defining null lines in Minkowski spacetime is a rather natural process. We pick two points x_1 and x_2 and demand that the interval between them is null

$$(x_2 - x_1)^\mu (x_2 - x_1)^\nu \eta_{\mu\nu} = 0 \tag{VIII.1}$$

and solve it. A solution is given by

$$x_2^\mu = x_1^\mu + t v^\mu \tag{VIII.2}$$

where v^μ is a null vector $v^2 = 0$. This process is less clear when we deal with spaces that contain fermionic directions and even non-commutative translation operators as in ($\mathcal{N}$ extended) full superspace

$$\{\mathfrak{Q}_{\alpha b}, \bar{\mathfrak{Q}}^b_{\dot\alpha}\} = 2i\delta^a_b \mathfrak{P}_{\alpha\dot\alpha}. \tag{VIII.3}$$

We may ask ourselves therefore what ingredients did we use to come up with the solution for ordinary spacetime? First of all, we may see that $x_2 - x_1$ is invariant under finite translations $x \mapsto x + a$, so this is a good requirement for any interval. The second ingredient is less obvious: Null intervals also transform covariantly under conformal transformations, i.e., null intervals transform into null intervals. In the bosonic case, we could see that immediately from $x^2 = 0$, in the fermionic case we need ask for superconformal covariance more explicitly.

Luckily, the work we have done in cha. V pays here. We can easily derive the correct superspace interval using the flag manifold approach. Checking for superconformal covariance is then a matter of calculating the transformation properties directly by acting on coordinates. This is the approach to light-like lines in this chapter. We will need them to define the contour on which we want to define a super Wilson loop in cha. IX.

VIII.1 Lines in complex superspace

In chapter V we already outlined a way to find translation invariant intervals from the formulation of Minkowski space as a subset of the manifold of flags with sequence $(2|0) < (4|4)$. Let us revisit this again.

Let $(x^{\alpha\dot\alpha}, \theta^{\alpha a}, \bar\theta^{\dot\alpha}_a) =: X$ with $\alpha, \dot\alpha = 1, 2$ and $a = 1, \ldots, 4$ be coordinates in complex superspace $\mathbb{M}^{4|16}$. As explained before, we are using the extended Pauli-Matrices $\sigma^\mu = (\mathbf{1}, \sigma^i)$ and $\bar\sigma_\mu = (\mathbf{1}, -\sigma^i)$ to produce the map

$$x^\mu \mapsto x^{\dot\alpha\alpha} = x^\mu \bar\sigma^{\dot\alpha\alpha}_\mu. \tag{VIII.4}$$

Although we will keep working in complexified superspace, we will take the liberty of using the conventions appropriate for $(3,1)$ signature as before[1]. In full superspace, it is convenient to define chiral coordinates $x_L^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - i\theta^{\alpha a}\bar\theta_a^{\dot\alpha}$ and $x_R^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + i\theta^{\alpha a}\bar\theta_a^{\dot\alpha}$ which define together with the fermionic coordinates left chiral space $(x_L, \bar\theta)$ and right chiral space (x_R, θ). The hermitian conjugation exchanges these two spaces i.e., the chiral subspaces of Minkowski superspace are essentially complex spaces [84]. In this part, instead of writing x_L and x_R, we will write $x^\pm = x \pm i\theta\bar\theta$ since there is no danger of confusing the $\pm$-superscript with $U(1)$ weights.

As demonstrated before, the supersymmetry covariant derivatives on $\mathbb{M}^{4|4\mathcal{N}}$ are the set $(\partial_{\dot\alpha\alpha}, D_{a\alpha}, \bar D^a_{\dot\alpha})$ given by

$$\partial_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \quad D_{a\alpha} = \frac{\partial}{\partial\theta^{\alpha a}} + i\bar\theta_a^{\dot\alpha}\partial_{\dot\alpha\alpha}, \quad \bar D^a_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta_a^{\dot\alpha}} - i\theta^{\alpha a}\partial_{\dot\alpha\alpha}. \tag{VIII.7}$$

Note that the conventions we chose imply that

$$\partial_{\alpha\dot\alpha}x^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}. \tag{VIII.8}$$

The supersymmetry covariant derivatives are chosen such that they commute with the translation operators $(\mathfrak{P}, \mathfrak{Q}, \bar{\mathfrak{Q}})$ on superspace. These are

$$\mathfrak{P}_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha}, \qquad \mathfrak{Q}_{a\alpha} = \frac{\partial}{\partial\theta^{\alpha a}} - i\bar\theta_a^{\dot\alpha}\partial_{\alpha\dot\alpha}, \qquad \bar{\mathfrak{Q}}^a_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta_a^{\dot\alpha}} + i\theta^{\alpha a}\partial_{\alpha\dot\alpha} \tag{VIII.9}$$

which form only one non-trivial (anti-)commutator

$$\{\mathfrak{Q}_{a\alpha}, \bar{\mathfrak{Q}}^b_{\dot\alpha}\} = 2i\delta_a^b\mathfrak{P}_{\alpha\dot\alpha}. \tag{VIII.10}$$

Given such coordinates $X = (x^{\alpha\dot\alpha}, \theta^{\alpha a}, \bar\theta_a^{\dot\alpha})$ we can map Minkowski space into the (right) coset $H_{(2|0)}(4|4)\backslash SL(4|4)$ by $X \mapsto s(X)$, s.t.,

$$s(X) = \left(\begin{array}{cc|c} 1 & -ix^- & -2i\theta \\ 0 & 1 & 0 \\ \hline 0 & -2i\bar\theta_i & 1 \end{array}\right) \tag{VIII.11}$$

A superconformal transformation is given by the action of a group element $g \in SL(4|4)$ by

$$s(X).g = h(X,g).s(X'). \tag{VIII.12}$$

h is an element of the stabilizing group $H_{(2|0)}(4|4)$. A translation of the point X_i to the point X_j can then be encoded as an element $g_{ij} \in SL(4|4)$ with

$$s(X_i).g_{ij} = h(X_i, g_{ij})s(X_j) = s(X_j) \tag{VIII.13}$$

where it can be shown easily that $h(X_i, g_{ij}) = 1$. Thus g_{ij} takes the easy form $g_{ij} = s^{-1}(X_i).s(X_j)$, i.e.,

$$g_{ij} = \left(\begin{array}{cc|c} 1 & -i(x_j^- - x_i^+ + 2i\theta_i\bar\theta_j) & -2i(\theta_i - \theta_j) \\ 0 & 1 & 0 \\ \hline 0 & -2i(\bar\theta_i - \bar\theta_j) & 1 \end{array}\right) \tag{VIII.14}$$

[1] The reality conditions which singles out this real slice of complex Minkowski spacetime are

$$x^\dagger = x, \qquad \theta^\dagger = \bar\theta. \tag{VIII.5}$$

The hermitian conjugation is taken to follow the DeWitt convention [110] such that the product $\theta^{\alpha a}\bar\theta_a^{\dot\alpha}$ behaves under conjugation like

$$(\theta\bar\theta)^\dagger = \theta\bar\theta \tag{VIII.6}$$

g_{ij} is invariant under finite translations by construction and it is also easy to check that the entries of g_{ij} are invariant under translation operators $\mathfrak{P}$, $\mathfrak{Q}$, and $\bar{\mathfrak{Q}}$. The combination $x_{ij}^{+-} = x_j^- - x_i^+ + 2i\theta_i\bar{\theta}_j$ is not the minimal combination of the superspace variables invariant under translations. This can be seen when rewriting

$$x_{ij}^{+-} = \underbrace{(x_j - x_i + i\theta_j\bar{\theta}_i - i\theta_i\bar{\theta}_j)}_{=:x_{ij}} + i\theta_{ij}\bar{\theta}_{ij}. \tag{VIII.15}$$

The term x_{ij} is invariant under translations on its own—so is the combination $\theta_{ij}\bar{\theta}_{ij}$. This is an ambiguity that one encounters generally when working with supersymmetrizations of bosonic expressions: The definitions are unique up to higher powers of fermionic coordinates.

The same construction for a translation operator g_{ij} applies to other coset spaces. For example for the chiral superspace $\mathbb{M}_L^{4|8}$ with coordinates $(x^-, \bar{\theta})$ we may write g_{ij} in the form

$$g_{ij} = s(X_i)^{-1}.s(X_j) = \begin{pmatrix} 1 & ix_i^- & 0 \\ 0 & 1 & 0 \\ 0 & i\bar{\theta}_i & 1 \end{pmatrix} \begin{pmatrix} 1 & -ix_j^- & 0 \\ 0 & 1 & 0 \\ 0 & i\bar{\theta}_j & 1 \end{pmatrix}. \tag{VIII.16}$$

The entries are once again the translation invariant intervals of chiral superspace

$$x_j^- - x_i^- =: x_{ij}^- \quad \text{and} \quad \bar{\theta}_j - \bar{\theta}_i = \bar{\theta}_{ij}. \tag{VIII.17}$$

§ VIII.1.1. Light-like intervals.—To impose light-likeness on the intervals we found in the last section, we have to inspect their behavior under superconformal transformations. This was done in [74]. The authors found that sufficient and necessary conditions for light-likeness in full superspace are

$$x_{ij}^2 = 0, \qquad \epsilon_{\alpha\beta}x_{ij}^{\alpha\dot{\alpha}}\theta_{ij}^{\beta a} = 0, \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}_{ij,a}^{\dot{\alpha}}x_{ij}^{\alpha\dot{\beta}} = 0. \tag{VIII.18}$$

(VIII.18) can be solved in terms of spinor variables

$$\Lambda = (\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}, \eta^a, \bar{\eta}_a) \tag{VIII.19}$$

by

$$x_{ij} = \lambda\bar{\lambda}, \qquad \theta_{ij} = \lambda\eta, \qquad \bar{\theta}_{ij} = \bar{\lambda}\bar{\eta}. \tag{VIII.20}$$

If all variables are taken to be complex, λ and $\bar{\lambda}$ are independent of each other, the same goes for η and $\bar{\eta}$. To recover $(3,1)$ signature we have to impose the conditions

$$\lambda^\dagger = \bar{\lambda} \tag{VIII.21a}$$

$$\eta^\dagger = \bar{\eta}. \tag{VIII.21b}$$

Additionally, there is a redundancy in the description of the spinor variables. This redundancy is a complex scaling

$$\lambda \mapsto z\lambda, \qquad \bar{\lambda} \mapsto z^{-1}\bar{\lambda}, \qquad \eta \mapsto z^{-1}\eta, \qquad \bar{\eta} \mapsto z\bar{\eta} \tag{VIII.22}$$

which can be used to eliminate one bosonic degree of freedom from the spinor variables such that the amount of degrees of freedom on both sides of (VIII.20) match[2].

[2] In real $(3,1)$ signature, this redundancy gets reduced to a phase.

As we have explained in sec. V.1.2, the light-like lines of full superspace are in one-to-one correspondence with a pair of twistors

$$\mathcal{Z}^A \equiv \left(-\frac{i}{2}\lambda^\alpha, \epsilon_{\alpha\beta}\lambda^\alpha x^{+\alpha\dot{\alpha}}, \epsilon_{\alpha\beta}\lambda^\alpha\theta^{\alpha a}\right) = \left(-\frac{i}{2}\lambda^\alpha, \mu^{\dot{\alpha}}, \chi^a\right) \tag{VIII.23}$$

$$\mathcal{W}_A \equiv \left(\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\alpha}} x^{-\alpha\dot{\beta}}, -\frac{i}{2}\bar{\lambda}^{\dot{\alpha}}, \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_a\right) = \left(\bar{\mu}^{\dot{\alpha}}, -\frac{i}{2}\bar{\lambda}^{\dot{\alpha}}, \bar{\chi}_a\right) \tag{VIII.24}$$

satisfying the constraint

$$\mathcal{Z}.\mathcal{W} = -\frac{i}{2}\epsilon_{\alpha\beta}\lambda^\alpha\bar{\mu}^\beta - \frac{i}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\mu^{\dot{\alpha}}\bar{\lambda}^{\dot{\beta}} + \chi^a\bar{\chi}_a = 0. \tag{VIII.25}$$

This is the ambitwistor constraint defining the quadric $\mathbb{A}_{3|4} \subset \mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4}$. The hermitian conjugation inherited from superspace imposes the signature $(2,2|4)$ on the point $(\mathcal{Z}, \mathcal{W})$ by

$$\mathcal{Z}^\dagger = \pm C\mathcal{W} \tag{VIII.26}$$

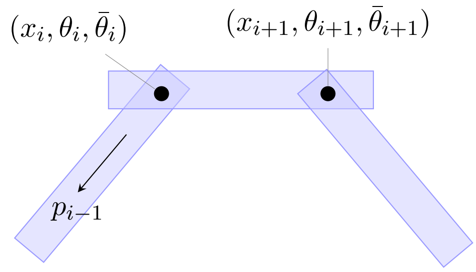
with C a matrix in $(2,2|4)$ block form

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{VIII.27}$$

§ VIII.1.2. Fat polygons.—Given the ability to form intervals which are invariant under translations, and null conditions which are invariant under superconformal transformations, we are able to describe closed contours in superspace which are stable under superconformal transformations. These contours are polygons with light-like separated vertices X_i. The bosonic part of the edges are one-dimensional, but they are thickened in full superspace by eight additional fermionic directions. Overall, we find that a sequence of pairwise null-separated points $(x_i, \theta_i, \bar{\theta}_i)$ can be connected by fat light-like lines parametrized by

$$x(t,\sigma,\bar{\sigma}) = x_i + t\lambda\bar{\lambda} + i\lambda\sigma\bar{\theta}_i - i\theta_i\bar{\sigma}\bar{\lambda}, \quad \theta(\sigma) = \theta_i + \lambda\sigma, \quad \bar{\theta}(\bar{\sigma}) = \bar{\theta}_i + \bar{\lambda}\bar{\sigma} \tag{VIII.28}$$

where $t \in [0,1]$ and $x(1,\eta,\bar{\eta}) = x_{i+1}$. Thus there are $(1|8)$ parameters that parametrize the null line between x_{i+1} and x_i.



Due to the Witten correspondence, we can map such a sequence of null separated points in Minkowski superspace to a sequence of ambitwistors $(\mathcal{Z}_i, \mathcal{W}_i)$ in superambitwistor space. The conditions that ensure that a vertex x_i lies at the intersection of two light-like lines implies the constraints

$$\mathcal{Z}_{i-1}.\mathcal{W}_i = \mathcal{Z}_{i+1}.\mathcal{W}_i = 0. \tag{VIII.29}$$

This implies that any two points $(\mathcal{Z}_i, \mathcal{W}_i)$ and $(\mathcal{Z}_{i+1}, \mathcal{W}_{i+1})$ satisfying (VIII.29) lie on a linearly embedded submanifold $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{A}_{3|\mathcal{N}}$ parametrized by

$$(\mathcal{Z}_{i,i+1}(z), \mathcal{W}_{i,i+1}(z')) = (\mathcal{Z}_i + z\mathcal{Z}_{i+1}, \mathcal{W}_i + \bar{z}'\mathcal{W}_{i+1}). \tag{VIII.30}$$

Every set of points on these lines satisfies the ambitwistor relation as can be easily checked by $\mathcal{Z}_{i,i+1}.\mathcal{W}_{i,i+1} = 0$ for all z and z'. The correspondence between ambitwistor space and full Minkowski space is especially pleasing since polygons on null lines in superspace translate to points in ambitwistor space connected by complex lines $\mathbb{CP}^1 \times \mathbb{CP}^1$.

CHAPTER IX

Null polygonal Wilson loops in full superspace

Let us now turn our attention to the calculation of null-polygonal Wilson loops in a non-chiral superspace setting [1]. Since we do not know of a way to solve the full constraints (VI.17a) we will consider them in their linearized form

$$2i\delta_a^b A_{\dot{\alpha}\alpha} + \bar{D}_{\dot{\alpha}}^b A_{\alpha a} + D_{\alpha a}\bar{A}_{\dot{\alpha}}^a = 0 \tag{IX.1a}$$
$$D_{\alpha a}A_{\beta b} + D_{\beta b}A_{\alpha a} = \epsilon_{\alpha\beta}\bar{W}_{ab} \tag{IX.1b}$$
$$\bar{D}_{\dot{\alpha}}^a\bar{A}_{\dot{\beta}}^b + \bar{D}_{\dot{\beta}}^b A_{\dot{\alpha}}^a = \epsilon_{\dot{\alpha}\dot{\beta}}W^{ab} \tag{IX.1c}$$

These linearized constraints can be solved by introducing the two chiral prepotentials $B^{\alpha\beta}(x^+,\theta)$ and $\bar{B}^{\dot{\alpha}\dot{\beta}}(x^-,\bar{\theta})$ which are symmetric in their respective spinor indices. Explicitly, we can write

$$A_{\alpha a} = D_{\beta a}B^{\beta}{}_{\alpha} + D_{\alpha a}\Lambda \tag{IX.2a}$$
$$\bar{A}_{\dot{\alpha}}^a = -\bar{D}_{\dot{\beta}}^a\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}}^a\Lambda \tag{IX.2b}$$
$$A_{\dot{\alpha}\alpha} = \partial_{\dot{\alpha}\beta}B^{\beta}{}_{\alpha} - \partial_{\dot{\beta}\alpha}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \partial_{\dot{\alpha}\alpha}\Lambda \tag{IX.2c}$$

where the last line is a consequence of (IX.1a), and $\Lambda = \Lambda(x,\theta,\bar{\theta})$ is an explicit gauge transformation. Under hermitian conjugation the fields B and $\bar{B}$ are conjugates

$$B^{\dagger} = \bar{B} \tag{IX.3}$$

and Λ is antihermitian. We can also reinstate the vielbein $E^{\alpha\dot{\alpha}}$, $E^{a\alpha}$, and $\bar{E}_a^{\dot{\alpha}}$ and write

$$\begin{aligned} A &= \frac{1}{2}E^{\alpha\dot{\alpha}}A_{\dot{\alpha}\alpha} + E^{a\alpha}A_{\alpha a} + \bar{E}_a^{\dot{\alpha}}\bar{A}_{\dot{\alpha}}^a \\ &= \left[\frac{1}{2}(dx^+)^{\alpha\dot{\alpha}}\partial^+_{\dot{\alpha}\beta}B^{\beta}{}_{\alpha} + d\theta^{a\alpha}\frac{\partial}{\partial\theta^{a\beta}}B^{\beta}{}_{\alpha}\right] - \left[\frac{1}{2}(dx^-)^{\alpha\dot{\alpha}}\partial^+_{\dot{\beta}\alpha}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + d\bar{\theta}_a^{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}_a^{\dot{\beta}}}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}}\right] + d\Lambda \\ &= A^+ + A^- + d\Lambda \end{aligned} \tag{IX.4}$$

where $\partial^{\pm}$ are derivatives with respect to the chiral variables $x^{\pm}$. This shows that the gauge connection A decomposes into a left and a right chiral part in the linearized theory[1]. Equations (IX.1b) and (IX.1c) imply that B and $\bar{B}$ are harmonic functions satisfying

$$\epsilon^{\alpha\beta}D_{\alpha a}D_{\beta b}B_{\gamma\delta} = 0, \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}}^a\bar{D}_{\dot{\beta}}^b\bar{B}_{\dot{\gamma}\dot{\delta}} = 0 \tag{IX.5}$$

which implies that both prepotentials satisfy the massless wave equation $\Box B = 0$. These prepotentials can be understood in terms of Hertz potentials[2] as has been pointed out before

[1] The fate of such a decomposition in the non-Abelian theory is unclear. It seems unlikely that there exists a similar construction as the crucial piece, the solution of the constraints using Hertz potentials, is not available in the non-Abelian theory.

[2] More on Hertz potentials in the bosonic case can be found in the appendix D.3.

in [111].

The self-duality constraint (VI.19) implies furthermore that

$$-\bar{D}^a_{\dot{\alpha}}\bar{D}^b_{\dot{\beta}}\bar{B}^{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\epsilon^{abcd}D_{\alpha c}D_{\beta d}B^{\alpha\beta}. \tag{IX.6}$$

IX.1 On-shell momentum space and light-cone gauge

We will now impose (3,1) signature throughout. It is possible to express the fields B and $\bar{B}$ in terms of Fourier integrals over on-shell momentum space variables [1]

$$B^{\alpha\beta}(x^+,\theta) = \frac{1}{8\pi^2}\int d^2\lambda d^2\bar{\lambda} d^{0|4}\bar{\eta}\ \exp\left(\frac{i}{2}\langle\lambda|x^+|\bar{\lambda}] + \langle\lambda|\theta|\bar{\eta}]\right) C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta}) \tag{IX.7a}$$

$$\bar{B}^{\dot{\alpha}\dot{\beta}}(x^-,\bar{\theta}) = \frac{1}{8\pi^2}\int d^2\lambda d^2\bar{\lambda} d^{0|4}\eta\ \exp\left(\frac{i}{2}\langle\lambda|x^-|\bar{\lambda}] - \langle\eta|\bar{\theta}|\bar{\lambda}]\right) C^{\alpha\beta}(\lambda,\bar{\lambda},\eta) \tag{IX.7b}$$

with the shorthand notations $\langle\lambda|x|\bar{\lambda}] = \lambda^\alpha x_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$, $\langle\lambda|\theta|\bar{\eta}] = \lambda^\alpha\theta^a_\alpha\eta_a$ and $\langle\eta|\bar{\theta}|\bar{\lambda}] = \eta^a\bar{\theta}_{a\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$. The conditions on the fields B and $\bar{B}$ imposed by the Minkowski reality conditions imply

$$C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta})^\dagger = \bar{C}^{\dot{\alpha}\dot{\beta}}(\lambda,-\bar{\lambda},\eta) \tag{IX.8}$$

and the self-duality constraint further implies that there exists a relation between C and $\bar{C}$ by

$$\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}}\bar{C}^{\dot{\alpha}\dot{\beta}}(\lambda,\bar{\lambda},\eta) = 4\int d^{0|4}\bar{\eta}\ \exp\left(\frac{1}{2}\eta\bar{\eta}\right)\lambda_\alpha\lambda_\beta C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta}) \tag{IX.9}$$

which expresses neatly that both superfields contain the same component fields. Both fields are homogeneous with degree -4 (C) and $+4$ ($\bar{C}$) under the scaling transformation $(\lambda,\bar{\lambda},\eta,\bar{\eta}) \mapsto (z\lambda, z^{-1}\bar{\lambda}, z\eta, z^{-1}\bar{\eta})$.

Now, we introduce a light-cone gauge (see apdx. D.2). Introduce a pair of reference spinors ℓ^α and $\bar{\ell}^{\dot{\alpha}}$ and demand $\ell^\alpha B_{\alpha\beta} = 0$ and $\bar{\ell}^{\dot{\alpha}}\bar{B}_{\dot{\alpha}\dot{\beta}} = 0$ as a gauge condition. It is solved in terms of the on-shell momentum fields by

$$C^{\alpha\beta} = \frac{\ell^\alpha\ell^\beta}{\langle\lambda,\ell\rangle^2}C(\lambda,\bar{\lambda},\bar{\eta}), \qquad \bar{C}^{\dot{\alpha}\dot{\beta}} = \frac{\bar{\ell}^{\dot{\alpha}}\bar{\ell}^{\dot{\beta}}}{[\bar{\ell},\bar{\lambda}]^2}\bar{C}(\lambda,\bar{\lambda},\eta). \tag{IX.10}$$

C and $\bar{C}$ are on-shell fields of homogeneity -2 and 2 under the scaling transformation above and they satisfy the self-duality constraint[3]

$$\bar{C}(\lambda,\bar{\lambda},\bar{\eta}) = 4\int d^{0|4}\bar{\eta}\ \exp\left(\frac{1}{2}\eta\bar{\eta}\right)C(\lambda,\bar{\lambda},\bar{\eta}). \tag{IX.11}$$

This is a complete gauge, i.e., there is no more gauge freedom left.

[3] Furthermore $B^\dagger = \bar{B}$ implies $C^\dagger(\lambda,\bar{\lambda},\bar{\eta}) = \bar{C}(\lambda,-\bar{\lambda},\eta)$. Together with (IX.11) this implies that all 16 physical degrees of freedom of $\mathcal{N}=4$ are contained in either of the two fields C or $\bar{C}$.

Furthermore, there exists a half–Fourier transform[4] which takes the fields C and $\bar{C}$ from on-shell momentum space $\Lambda = (\lambda, \bar{\lambda}, \eta, \bar{\eta})$ to ambitwistor space via (denoting the fields on both sides by C and $\bar{C}$)

$$C(\lambda, \mu, \chi) = \int d^2\bar{\lambda} d^{0|4}\bar{\eta} \;\exp\left(-2i[\mu, \bar{\lambda}] + \chi\bar{\eta}\right) C(\lambda, \bar{\lambda}, \bar{\eta}) \tag{IX.12a}$$

$$\bar{C}(\bar{\lambda}, \bar{\mu}, \bar{\chi}) = \int d^2\lambda d^{0|4}\eta \;\exp\left(-2i\langle\lambda, \bar{\mu}\rangle - \eta\bar{\chi}\right) \bar{C}(\lambda, \bar{\lambda}, \eta). \tag{IX.12b}$$

and the self-duality constraint translates to a relation between the two twistor fields

$$\bar{C}(\mathcal{W}) = \frac{1}{(2\pi)^2} \int d^{4|4}\mathcal{Z} \;\exp(2\; \mathcal{Z}.\mathcal{W}) C(\mathcal{Z}) \tag{IX.13}$$

and the prepotentials B and $\bar{B}$ can be expressed as

$$B^{\alpha\beta}(x^+, \theta) = \frac{1}{8\pi^2} \int_{\mathbb{CP}^1} \frac{\langle\lambda d\lambda\rangle \ell^\alpha \ell^\beta}{\langle\lambda\ell\rangle^2} C(\mathcal{Z})|_{\mathbb{CP}^1} \tag{IX.14}$$

$$\bar{B}^{\dot{\alpha}\dot{\beta}}(x^-, \bar{\theta}) = \frac{1}{8\pi^2} \int_{\mathbb{CP}^1} \frac{[\bar{\lambda} d\bar{\lambda}] \bar{\ell}^{\dot{\alpha}} \bar{\ell}^{\dot{\beta}}}{[\bar{\lambda}\bar{\ell}]^2} \bar{C}(\mathcal{W})|_{\mathbb{CP}^1} \tag{IX.15}$$

where the restriction to $\mathbb{CP}^1$ is the same restriction as in sec. VII.1 to express that we are looking at a fixed point in left or right chiral superspace.

IX.2 Wilson loop in full superspace

A Wilson loop in a non-Abelian gauge theory is the path-ordered exponential of the integral of the gauge connection A over a loop γ in the space of concern

$$\mathcal{W} = \operatorname{tr} \mathcal{P} \exp\left(\oint_\gamma A\right). \tag{IX.16}$$

Stated differently it calculates the holonomy of the connection around a closed curve. This is a nice gauge-invariant quantity so the result of the calculation is a meaningful quantity. The curve in question can be any type of closed path, but the Wilson loop calculations concerning the duality to scattering amplitudes in gauge theories must be done on polygonal paths $\mathcal{C}_n$ made up from n light-like segments in superspace. In full superspace, we must calculate

$$\mathcal{W}_n = \operatorname{tr} \mathcal{P} \exp\left(\oint_{\mathcal{C}_n} A\right) = \operatorname{tr} \mathcal{P} \exp\left(\oint_{\mathcal{C}_n} \tfrac{1}{2} E^{\dot{\alpha}\alpha} A_{\alpha\dot{\alpha}} + E^{\alpha a} A_{a\alpha} + \bar{E}^{\dot{\alpha}}_a \bar{A}^a_{\dot{\alpha}}\right). \tag{IX.17}$$

For the quantum mechanical computation we need to take the expectation value of the Wilson loop as an operator $\langle\mathcal{W}_n\rangle$ and calculate it perturbatively[5]. The one-loop contribution sits at the second level in the expansion of the exponential[6]

$$\langle\mathcal{W}_n^{(1)}\rangle = \frac{\tilde{\lambda}}{N} \oint_{\mathcal{C}_n} \oint'_{\mathcal{C}_n} \tfrac{1}{2} \operatorname{tr}\langle A A'\rangle. \tag{IX.18}$$

[4] This half–Fourier transform has been lurking in the background ever since eq. (IX.7). We want to point out that the variables $\mu = \langle\lambda|x^+$ and $\bar{\mu} = x^-|\bar{\lambda}]$.

[5] As explained in [1] the calculation of the Wilson loop is tricky. Since $\mathcal{N} = 4$ SYM exists only on-shell, all calculations have to be done on-shell as well. This forces us to use the Whitman or "cut" propagator G^+ as described in apdx. D i.e., we calculate the vacuum expectation value $\langle 0|\mathcal{W}_n|0\rangle$ but denote all expectation values by $\langle\mathcal{W}\rangle$.

[6] Clearly, the vacuum expectation value $\langle A\rangle = 0$.

Here we defined a rescaled 't Hooft coupling by $\tilde{\lambda} = \frac{g_{YM}^2 N_c}{64\pi^2}$. In the linearized (Abelian) theory we can drop the trace. Using (IX.4) we see that the Wilson loop at this level neatly splits into three parts given by a chiral-chiral, an antichiral-antichiral, and a chiral-antichiral piece

$$\begin{aligned}\langle \mathcal{W}_n^{(1)} \rangle &= \langle \mathcal{W}_{n++}^{(1)} \rangle + \langle \mathcal{W}_{n--}^{(1)} \rangle + 2\langle \mathcal{W}_{n+-}^{(1)} \rangle \\ &= \frac{1}{2\tilde{\lambda}N} \oint_{C_n} \oint'_{C_n} \langle A^+ A'^+ \rangle + \frac{1}{2\tilde{\lambda}N} \oint_{C_n} \oint'_{C_n} \langle A^- A'^- \rangle + \frac{1}{\tilde{\lambda}N} \oint_{C_n} \oint'_{C_n} \langle A^+ A'^- \rangle \end{aligned} \tag{IX.19}$$

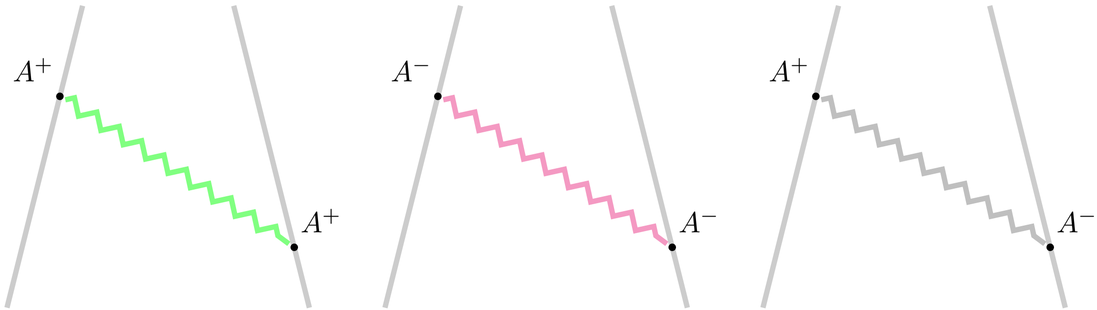


Figure IX.1.: The three different propagators appearing in the Wilson loop expectation value are marked by different lines. Left the chiral-chiral contribution, in the middle the antichiral-antichiral contribution, and on the right the chiral-antichiral contribution.

These three contributions to the expectation value do not mix since they come with different orders in θ and $\bar{\theta}$. The purely chiral and purely antichiral parts in this expectation value have to yield the same contributions as the purely chiral Wilson loops proposed in [10, 17].

The curve C_n is a sequence of light-like segments—let us evaluate what this means for the calculation of the integral over the curve. The solution of the linearized constraints implies that the gauge field A is flat on light-like lines. So A can be written as a total derivative $A = dF$. This total derivative further decomposes into chiral and antichiral parts, so

$$A^+ = dB(x^+, \theta), \qquad A^- = d\bar{B}(x^-, \bar{\theta}). \tag{IX.20}$$

If the light-like line passes through the vertices $(x_i, \theta_i, \bar{\theta}_i)$ and $(x_{i+1}, \theta_{i+1}, \bar{\theta}_{i+1})$ let

$$x_{i,i+1}^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}, \quad \theta_{i,i+1}^{\alpha a} = \lambda_i^\alpha \eta_i^a, \quad \bar{\theta}_{i,i+1\ a}^{\dot{\alpha}} = \bar{\lambda}_i^{\dot{\alpha}} \bar{\eta}_{i\ a}. \tag{IX.21}$$

It is then possible to solve for B and $\bar{B}$ in terms of $C(\lambda, \bar{\lambda}, \bar{\eta})$ and $\bar{C}(\lambda, \bar{\lambda}, \eta)$ using the expressions (IX.7). We find

$$B_i(x^+, \theta) = \frac{1}{8\pi^2} \int d^2\lambda d^2\bar{\lambda} d^{0|4}\bar{\eta}\ \exp\left(\tfrac{i}{2}\langle\lambda|x_i^+|\bar{\lambda}] + \langle\lambda|\theta_i|\bar{\eta}]\right) \frac{\langle i, \ell\rangle}{\langle\lambda, \ell\rangle\langle\lambda, i\rangle} C_i(\lambda, \bar{\lambda}, \bar{\eta}) \tag{IX.22}$$

$$\bar{B}_i(x^-, \bar{\theta}) = \frac{1}{8\pi^2} \int d^2\lambda d^2\bar{\lambda} d^{0|4}\eta\ \exp\left(\tfrac{i}{2}\langle\lambda|x_i^-|\bar{\lambda}] - \langle\theta|\bar{\theta}_i|\bar{\lambda}]\right) \frac{[i, \bar{\ell}]}{[\bar{\lambda}, \bar{\ell}][\bar{\lambda}, i]} \bar{C}_i(\lambda, \bar{\lambda}, \eta) \tag{IX.23}$$

Due to the polygonal nature of the integration contour, the integral over the loop can be written as a sum of shifts over the edges of the polygon

$$\oint_{C_n} A = \sum_{i=1}^{n} \left(B_i(x_{i+1}, \theta_{i+1}, \bar{\theta}_{i+1}) - B_i(x_i, \theta_i, \bar{\theta}_i)\right). \tag{IX.24}$$

By rearranging the sum, we can express this in terms of shifts $B_{i-1,i}(x_i,\theta_i,\bar\theta_i)$ localized at the vertices of the Wilson loop with

$$B_{i-1,i}(x_i,\theta_i,\bar\theta_i) = B_{i-1}(x_i,\theta_i,\bar\theta_i) - B_i(x_i,\theta_i,\bar\theta_i) \tag{IX.25}$$

and

$$B_{i-1,i}(x^+,\theta) = \frac{1}{8\pi^2}\int d^2\lambda d^2\bar\lambda d^{0|4}\bar\eta\ \exp\left(\tfrac{i}{2}\langle\lambda|x_i^+|\bar\lambda] + \langle\lambda|\theta_i|\bar\eta]\right)\frac{\langle i-1,i\rangle}{\langle i-1,\lambda\rangle\langle\lambda,i\rangle}C(\lambda,\bar\lambda,\bar\eta) \tag{IX.26}$$

$$\bar B_{i-1,i}(x^-,\bar\theta) = \frac{1}{8\pi^2}\int d^2\lambda d^2\bar\lambda d^{0|4}\eta\ \exp\left(\tfrac{i}{2}\langle\lambda|x_i^-|\bar\lambda] - \langle\theta|\bar\theta_i|\bar\lambda]\right)\frac{[i-1,i]}{[\bar i-1,\bar\lambda][\bar\lambda,i]}\bar C(\lambda,\bar\lambda,\eta). \tag{IX.27}$$

In essence these expressions are Penrose-Witten transformations in disguise, which take us from ambitwistor space to spacetime. For this reason (and the fact that $B_{i-1,i}$ and $\bar B_{i-1,i}$ are localized at a point) the reference spinors ℓ and $\bar\ell$ have dropped out of the expressions above. Thus the Wilson loop expectation value has reduced to the sum of expectation values

$$\tilde\lambda\langle\mathcal{W}_n^{(1)}\rangle = \sum_{i,j=1}^{n}\left(\tfrac{1}{2N}\langle B_{i-1,i}B_{j-1,j}\rangle + \tfrac{1}{2N}\langle\bar B_{i-1,i}\bar B_{j-1,j}\rangle + \tfrac{1}{N}\langle B_{i-1,i}\bar B_{j-1,j}\rangle\right) \tag{IX.28}$$

The explicit calculation of the expectation values in the above expression can be found in [1] and we will content ourselves with only giving the results which will be important in the following. One finds that the chiral-chiral expectation value $\langle B_{i-1,i}B_{j-1,j}\rangle$ and the antichiral-antichiral expectation value $\langle\bar B_{i-1,i}\bar B_{j-1,j}\rangle$ yield rational functions which are exactly the R-invariants of [55]. We expected this result from a chiral super Wilson loop. One finds

$$\langle B_{i-1,i}B_{j-1,j}\rangle = -\frac{1}{4\pi^2}\frac{\langle i-1,i\rangle\langle j-1,j\rangle\delta^{0|4}(\theta_{j,i}|x_{j,i}^+|\bar\rho])}{(x_{j,i}^+)^2\langle j-1|x_{j,i}^+|\bar\rho\langle j|x_{j,i}^+|\bar\rho]\langle i-1|x_{j,i}^+|\bar\rho\langle i|x_{j,i}^+|\bar\rho]} \tag{IX.29}$$

where $\bar\rho$ is a reference spinor which can be interpreted as an integration constant. This reference spinor drops out upon summation of all contributions to the Wilson loop, so it plays a rôle akin to the reference twistor Z_* of axial gauge twistor theory as used in [10]. The result of the antichiral-antichiral computation can be obtained by conjugation of (IX.29).

The hardest calculation is the mixed correlator

$$\langle B_{i-1,i}\bar B_{j-1,j}\rangle = -\frac{1}{256\pi^4}\int d^2\lambda d^2\bar\lambda\ \exp\Big(-\frac{i}{2}\langle\lambda|x_{i,j}^+|\bar\lambda]\Big)\frac{\langle i-1,i\rangle[j-1,j]}{\langle j-1,\lambda\rangle\langle\lambda,j\rangle[k-1,\bar\lambda][\bar\lambda,k]} \tag{IX.30}$$

which has been solved by using a second order differential equation to extract the term of highest transcendentality. The result is

$$\begin{aligned}\langle B_{i-1,i}\bar B_{j-1,j}\rangle = &-\frac{1}{64\pi^2}\,\mathrm{Li}_2\left(\frac{\langle i-1|x_{i,j}^{+-}|j]\langle i|x_{i,j}^{+-}|j-1]}{\langle i-1|x_{i,j}^{+-}|j-1]\langle i|x_{i,j}^{+-}|j]}\right)\\ &+\frac{1}{128\pi^2}\log(\langle i-1|x_{i,j}^{+-}|j-1]\langle i|x_{i,j}^{+-}|j])\log\left(\frac{\langle i-1|x_{i,j}^{+-}|j]\langle i|x_{i,j}^{+-}|j-1]}{\langle i-1|x_{i,j}^{+-}|j-1]\langle i|x_{i,j}^{+-}|j]}\right)\end{aligned} \tag{IX.31}$$

We have given the results in their spacetime—or rather correspondence space—form. Since it is possible to translate all appearing quantities into ambitwistor language, we will do so. Observe that

$$-\frac{i}{4}\langle i|x_{i,j}^{+-}|j] = -\frac{i}{4}\langle i|(x_j^- - x_i^+ + 4i\theta_i\bar{\theta}_j|j]) = -i\langle\lambda_i,\bar{\mu}_j\rangle + i[\mu_i,\lambda_j] + \chi_i\bar{\chi}_j = \mathcal{Z}_i.\mathcal{W}_j \qquad \text{(IX.32)}$$

is the scalar product of a twistor $\mathcal{Z}_i$ and a dual twistor $\mathcal{W}_j$. Remember that for a null polygonal contour in spacetime, the ambitwistor relations imply that

$$\mathcal{Z}_i.\mathcal{W}_{i-1} = \mathcal{Z}_i.\mathcal{W}_i = \mathcal{Z}_i.\mathcal{W}_{i+1} = 0. \qquad \text{(IX.33)}$$

For the rest of the discussion we will use the shorthand notation $\langle i,j] \equiv \mathcal{Z}_i.\mathcal{W}_j$. We can then express the mixed correlator as a true ambitwistor expression[7] by

$$64\pi^2\langle B_{i-1,i}B_{j-1,j}\rangle = -\operatorname{Li}_2(X_{i,j}) + \frac{1}{2}\log(\langle i-1,j-1]\langle i,j])\log(X_{i,j}) \qquad \text{(IX.34)}$$

where

$$X_{i,j} = \frac{\langle i-1,j]\langle i,j-1]}{\langle i-1,j-1]\langle i,j]} \qquad \text{(IX.35)}$$

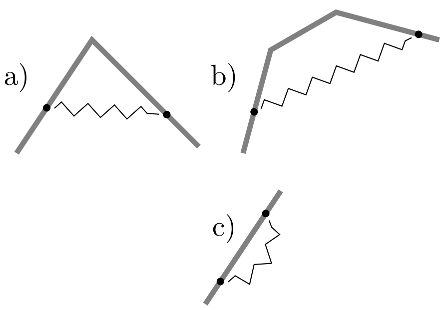
are cross-ratios. These are invariant under scaling and superconformal transformations[8].

Additionally, there was also a proposal for the form of the propagators for the ambitwistor fields in [1]. We review another proposal in apdx. D.5.

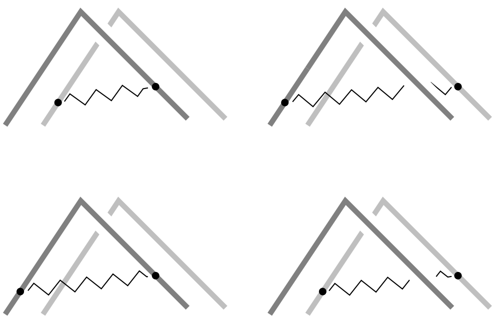
[7] In contrast with the R-invariants which are really only dependent on the chiral subspaces.

[8] This will be expanded upon in cha. XI. For now it is important to note that the ambitwistor brackets $\langle i,j]$ are themselves superconformal invariants, thus making the cross-ratios natural invariants.

IX.3 Regularization of ultraviolet divergences in the mixed sector



Before we can give the full one-loop result in the mixed sector we need to discuss regularizations. Upon inspection of equation (IX.31), we see that there are three different sources for divergences when $|i-j|<3$ as illustrated in the picture to the left. The divergence in a) is caused by a pinching of the propagator in the corner or by the propagator going light-like when the left insertion reaches the left-most point and the right insertion reaches the top point (in the similar situation for the left propagator reaching the top most point). The divergence in b) is caused by the light-like separation of the two points in the upper corners such that the propagator goes on-shell. Diagram c) shows a very singular contribution as every two points on a light-like line are light-like separated, the propagator is $\propto \frac{1}{0}$. Such diagrams have to be treated by taking the lines to be slightly space-like. In the following, we will disregard such singular contributions[9]. They have been studied and regularized in [112]. Due to these potential (and actual) divergences we need to regularize the Wilson loop expectation value. We will briefly discuss three possible regularizations as presented in [1].



§ IX.3.1. Framing.—Framing a Wilson loop means that we do not actually calculate the expectation value of a single loop $\langle \mathcal{W}_n \rangle$ but the correlation function of two infinitesimally displaced Wilson loops $\mathcal{W}_n$ and $\mathcal{W}'_n$ divided by their expectation values

$$\langle \mathcal{W}_n \rangle_{\text{framed}} = \frac{\langle \mathcal{W}_n \mathcal{W}'_n \rangle}{\langle \mathcal{W}_n \rangle \langle \mathcal{W}'_n \rangle}. \tag{IX.36}$$

This procedure removes all potentially divergent terms to order g^2 in perturbation theory. Yet, if we take the limit of zero displacement, we essentially calculate the expectation value of a single Wilson loop. Thus this is a valid regularization procedure. The picture above shows the four contributions that come from a single corner in the one-loop case. The upper left and upper right corners of the picture show finite contributions while the lower corners show divergent terms. At one loop, the ratio (IX.36) is equivalent to the sum

$$2\langle \mathcal{W}_n \rangle^{(1)}_{\text{framed}} = \langle \mathcal{W}_n \mathcal{W}'_n \rangle - \langle \mathcal{W}_n \rangle - \langle \mathcal{W}'_n \rangle \tag{IX.37}$$

where (suppressing λ)

$$\frac{1}{64\pi^2} \langle \mathcal{W}_n \mathcal{W}'_n \rangle^{(1)} = \frac{1}{2} \oint_{C_n} \oint_{C'_n} (\langle A^+ A^- \rangle + \langle A^- A^+ \rangle). \tag{IX.38}$$

The other two terms from the expansion of $\langle \mathcal{W}_n \rangle$ and $\langle \mathcal{W}'_n \rangle$ subtract all divergences from the original and the displaced Wilson loops thus leaving behind only finite terms. Expressing the infinitesimal shift of the primed Wilson loop by two reference twistors $(\mathcal{Z}_\star, \mathcal{W}_\star)$ (note: $\langle \star, \star] \neq 0$) we can describe the light-like lines of the displaced Wilson loop by the ambitwistors

$$\mathcal{Z}'_i = \mathcal{Z}_i + i\epsilon \frac{\langle i, \star]}{\langle \star, \star]} \mathcal{Z}_\star, \qquad \mathcal{W}'_i = \mathcal{W}_i - i\epsilon \frac{\langle \star, i]}{\langle \star, \star]} \mathcal{W}_\star \tag{IX.39}$$

[9] It can be shown that they produce vanishing contributions to the expectation value of the Wilson loop.

which satisfy the ambitwistor conditions

$$\mathcal{Z}'_i.\mathcal{W}'_i = \mathcal{Z}'_i.\mathcal{W}'_{i\pm 1} = \mathcal{O}(\epsilon^2). \tag{IX.40}$$

At $\mathcal{O}(1)$ the product contains only the bracket $\langle i, i]$ which is zero and at order ϵ the contributions cancel. Correlators between well separated vertices have a finite limit when $\epsilon \to 0$ while divergent terms need to be regularized by

$$\mathcal{Z}_i.\mathcal{W}_j \to i\epsilon \frac{\langle j, \star]\langle \star, k]}{\langle \star, \star]} =: \epsilon \langle i, j]^\star. \tag{IX.41}$$

Using these definitions[10] we can give a fully regularized expectation value for the non-chiral Wilson loop

$$\begin{aligned}
\langle \mathcal{W}_n \rangle_{\text{framed}} = \sum_{j=1}^{n} \Bigg\{ & -\log^2 \epsilon + \log \epsilon \, \log \frac{\langle j+1, j-1]\langle j-1, j+1]}{\langle j+1, j]^* \langle j, j+1]^\star} \\
& + \sum_{k=j+3}^{j-3} \left(-\operatorname{Li}_2 X_{j,k} + \frac{1}{2} \log \big[\langle j-1, k-1]\langle j, k] \big] \log X_{j,k} \right) \\
& + \sum_{k=j\pm 2} \frac{1}{2} \log \big[\langle j{-}1, k{-}1]\langle j, k] \big] \log X^\star_{j,k} \\
& + \sum_{k=j\pm 1} \frac{1}{2} \log \big[\langle j, k{-}1]^\star \langle j{-}1, k]^\star \big] \log X^\star_{j,k} \Bigg\} + \mathcal{O}(\epsilon),
\end{aligned} \tag{IX.42}$$

where constant terms (like ζ-values) have been neglected.

§ IX.3.2. Super-Poincaré regularization.—We also want to discuss an ad-hoc regularization of the Wilson loop expectation value in the mixed sector—we may call it super-Poincaré regularization. To perform the regularization one replaces any problematic ambitwistor bracket e.g., $\langle i, i]$, $\langle i, i \pm 1]$, by the product $\mathcal{Z}_i I \mathcal{Z}_j$ resp. $\mathcal{W}_i I \mathcal{W}_j$ where I is the infinity twistor[11] (which is actually a bi-twistor describing the light-cone at infinity—a plane) which projects any twistor to its λ or $\bar{\lambda}$ component

$$\mathcal{Z} I \mathcal{Z}' = \langle \lambda \lambda' \rangle, \qquad \mathcal{W} I \mathcal{W}' = [\bar{\lambda} \bar{\lambda}']. \tag{IX.43}$$

[10]Further details can be found in [1].

[11]Notice the abuse of notation. Technically, the two $\mathbb{CP}^{3|4}$ each have their own infinity twistor. Since there is no danger of confusion, we denote both by the same letter I.

This breaks superconformal symmetry but preserves Poincaré symmetry, whence the name. The result of such a replacement is

$$
\begin{aligned}
M_{n,I}^{(1)} = &-\sum_{j=1}^{n}\left(\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\log\frac{\langle j{-}1,j{+}1]\langle j{+}1,j{-}1]}{\mu^2\langle j,j{+}1\rangle[j,j{+}1]}\right)\\
&-\sum_{j=1}^{n}\sum_{k=j+3}^{j-3}\operatorname{Li}_2 X_{j,k}-\frac{1}{2}\sum_{j=1}^{n}\left(\log^2\frac{\langle j,j{+}1]}{[j,j{+}1]}-\log^2\frac{\langle j{-}1,j\rangle}{[j,j{+}1]}-\log^2\frac{[j{-}1,j{-}2]}{\langle j,j{-}1\rangle}\right)\\
&-\frac{1}{2}\sum_{j=1}^{n}\left(\log^2\frac{\langle j{-}1,j{+}1]}{\mu[j,j{+}1]}+\log^2\frac{\mu[j{-}1,j{-}2]}{\langle j,j{-}2]}+\log^2\frac{\langle j,j{+}2]}{\mu\langle j,j{+}1\rangle}+\log^2\frac{\mu\langle j,j{-}1\rangle}{\langle j,j{-}2]}\right)\\
&-\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{k=j+2}^{j-3}\log^2\frac{\langle j{-}1,k]}{\langle j,k]}+\sum_{k=j+3}^{j-2}\log^2\frac{\langle j,k]}{\langle j,k{-}1]}-\sum_{k=j+2}^{j-2}\log^2\frac{\langle j{-}1,k{-}1]}{\langle j,k]}\right)\\
&+\frac{1}{2}\gamma\sum_{j=1}^{n}\log^2\frac{\langle j{-}1,j{+}1]\langle j,j{+}1\rangle[j{-}1,j]}{\langle j{+}1,j{-}1]\langle j{-}1,j\rangle[j,j{+}1]}\,.
\end{aligned}
\tag{IX.44}
$$

This result supersymmetrizes the bosonic result for a Wilson loop given in [113]. The supersymmetrization is not unique however. This has been emphasized by adding the last term proportional to γ. This coefficient is not fixed by any physical input, so it might be as well set to 0.

§ IX.3.3. Boxing.—Finally, in the bosonic case the boxing procedure has proven to be a viable regularization process for Wilson loops on light-like contours [114]. It does calculate a perfectly finite and superconformal quantity (as we will prove in due course). However, the quantity itself is problematic to relate to the original Wilson loop and the one-loop expectation value so extracted will be called the *boxed loop* to distinguish it from the original Wilson loop.

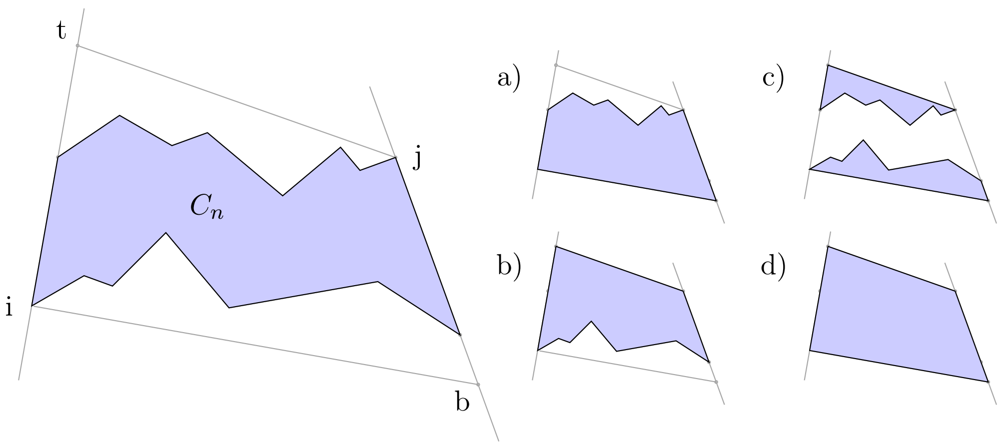


Figure IX.2.: The different contours of the boxed Wilson loop. Please refer to the text for the definition of the different contours.

Given an arbitrarily shaped null polygonal contour with n vertices in full superspace, we choose two edges, or four vertices X_i, X_{i+1}, X_j, and X_{j+1}. Let us denote the edges between points X_i and X_{i+1} as I. We may extend the edge I such that we reach point X_t on I that is separated from the point X_j by a light-like line T. We can do the same for X_j and X_{j+1}, this time around defining an edge J and extending it such that we find a point X_b on J that is separated by a null line B from the point X_i. In the bosonic case we could say that we extend the light-like edges such that they intersect the light-cones of some other point on the contour.

With these two new edges we build three new polygonal contours apart from the original one. Assuming $1 \leq i < j < n$ these are given by the paths

$$C_b := [i, i+1, \ldots, j-1, j, \mathrm{b}], \qquad C_t := [1, \ldots, i, \mathrm{t}, j, j+1, \ldots, n], \qquad C_{tb} = [i, \mathrm{t}, j, \mathrm{b}].$$

which have also been illustrated in figure IX.2 by frame a), b), and d) respectively. The remaining frame c) gives an impression of the resulting path for the boxed Wilson loop.

Given the contours, the boxed loop expectation value is defined by the ratio

$$\langle \mathcal{W}_n \rangle_\square = \frac{\langle \mathcal{W}_n \rangle \langle \mathcal{W}_{\mathrm{tb}} \rangle}{\langle \mathcal{W}_{\mathrm{t}} \rangle \langle \mathcal{W}_{\mathrm{b}} \rangle} \simeq \langle \mathcal{W}_n \rangle^{(1)} + \langle \mathcal{W}_{\mathrm{tb}} \rangle^{(1)} - \langle \mathcal{W}_{\mathrm{t}} \rangle^{(1)} - \langle \mathcal{W}_{\mathrm{b}} \rangle^{(1)} =: r_{i,j} \tag{IX.45}$$

where the second approximate equality holds on the first loop level which we call the *ratio function* $r_{i,j}$. Notice that $r_{i,j}$ will be different for different choices of i and j.

Since light-like lines and ambitwistors are in correspondence, we may express the edges T and B—as well as all the other edges— by ambitwistors $(\mathcal{Z}_t, \mathcal{W}_t)$ and $(\mathcal{Z}_b, \mathcal{W}_b)$. Using the condition $\langle i, i+1] = 0$ for two light-like lines to intersect, we see that these new twistors are given by the expressions

$$\mathcal{Z}_t = \mathcal{Z}_i - \frac{\langle i, j]}{\langle i+1, j]} \mathcal{Z}_{i+1}, \qquad \mathcal{Z}_b = \mathcal{Z}_j - \frac{\langle j, i]}{\langle j+1, i]} \mathcal{Z}_{j+1}, \tag{IX.46}$$

similarly for the conjugate twistors $\mathcal{W}_t$ and $\mathcal{W}_b$.

The boxed Wilson loop is a finite quantity, i.e., all divergences cancel after a careful calculation. For definiteness we pick $i = 1$ and provide the solution

$$\begin{aligned} r_{1,j} = & \sum_{i=4}^{j} \sum_{k=j+2}^{n} \left(-\operatorname{Li}_2 X_{i,k} + \frac{1}{2} \log\langle i-1, k-1] \langle i, k] \log X_{i,k} \right) \\ & + \frac{1}{2} \log^2 \frac{\langle 3, n]}{\langle 3, 1]} + \frac{1}{2} \log^2 \frac{\langle 2, \mathrm{b}]}{\langle 3, \mathrm{b}]} - \frac{1}{2} \log^2 \frac{\langle j, n]}{\langle j, 1]} - \frac{1}{2} \log^2 \frac{\langle 2, n]}{\langle 3, n]} \\ & + \frac{1}{2} \log\langle j, 1] \langle \mathrm{t}, \mathrm{b}] \log X'_{\mathrm{t},\mathrm{b}} + \frac{1}{2} \log\langle 2, n] \langle 3, 1] \log X'_{3,\mathrm{b}} + \text{conj.}, \end{aligned} \tag{IX.47}$$

where conj. stands for the conjugate result.

IX.4 What remains to be done

Despite the success at one-loop level it is unclear how to proceed from here to higher loop levels. As the one-loop level marks the end of the perturbative series for the Abelian theory—the rest is given by exponentiation—we cannot hope to extract any more information from the solution of the linearized constraints as given in cha. IX. Two possibilities to proceed from here should nevertheless be mentioned.

First of all, it would be possible to use the $\mathcal{N} = 3$ off-shell action in ambitwistor space as proposed in [92] to calculate a Wilson loop in $\mathcal{N} = 3$ superspace. Since $\mathcal{N} = 3$ and $\mathcal{N} = 4$ are equivalent on-shell, we should be able to find results from $\mathcal{N}=3$ that are also viable in $\mathcal{N}=4$. The drawback to this approach is the smaller superspace of $\mathcal{N}=3$ which could lead to a loss of information compared to a result calculated in $\mathcal{N}=4$ superspace.

A second possibility would be to find a non-Abelian generalization to the solution of the $\mathcal{N}=4$ SYM constraints as shown in this chapter. To this end, we might use a harmonic approach to ambitwistors and try to extend the Abelian result to a non-Abelian result in a way similar to the extension from Abelian twistor theory to non-Abelian twistor theory presented in cha. VII. The drawback to this method might be that an extension of this technique is very hard to find.

Both possibilities await further inspection. For now, we will contend ourselves with a test of the regularized one-loop expectation values for their symmetry properties under superconformal and Yangian symmetry generators.

5

Symmetries

As we already pointed out in sec. I.5, $\mathcal{N} = 4$ SYM is superconformally invariant even on the quantum level—the symmetry algebra is the superconformal algebra $\mathfrak{psu}(2,2|4)$. Detailed study of the scattering amplitudes of planar $\mathcal{N} = 4$ has also revealed a second, hidden superconformal symmetry known as dual superconformal symmetry. The existence of such a dual symmetry group is extremely non-trivial and has led to many interesting insights.

It has been shown that ordinary superconformal symmetry and dual superconformal symmetry form an extended infinite-dimensional algebra, a quantum group known as Yangian algebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$. This infinite-dimensional symmetry group will be the main topic of the present part. An abridged version of the general theory of Yangian algebras can be found in apdx. F.

In cha. X we will give the form of the Yangian symmetry generators as they are used in $\mathcal{N} = 4$ calculations concerning scattering amplitudes and Wilson loops. We will also paraphrase a proof of the invariance of all tree-level scattering amplitudes under the generators of the Yangian symmetry algebra. In cha. XI we turn to the result of the non-chiral Wilson loop computation of the last part and inspect its behavior under superconformal and Yangian symmetries. Finally, in chapter XII, we will show that tree-level scattering amplitudes in planar $\mathcal{N} = 4$ SYM have an additional symmetry that lies outside of the Yangian algebra.

CHAPTER X

Yangian symmetry of the $\mathcal{S}$-matrix

The Yangian of $\mathfrak{psu}(2,2|4)$ appears in planar $\mathcal{N}=4$ SYM as an extended *hidden symmetry algebra* [13, 79, 13, 115, 15, 116, 117]. The word hidden cannot be understood in the traditional sense as "non-Lagrangian symmetries of the equations of motion". Rather we need to understand the Yangian generators as symmetries of the $\mathcal{S}$-matrix of planar $\mathcal{N}=4$ SYM. Yangians are quantum algebras which make their appearance usually in the context of the symmetries of integrable theories. They have first been introduced in [118]. To save space we have relegated the general definition of Yangian algebras to Appendix F.

In the context of the symmetries of the $\mathcal{S}$-matrix of planar maximally supersymmetric Yang-Mills theory in four dimensions, the Yangian of $\mathfrak{psu}(2,2|4)$ has first been identified in [13]. The generators of the Yangian in planar $\mathcal{N}=4$ have been shown to consist of the generators of the superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ and the generators of *dual superconformal symmetry*—a second $\mathfrak{psu}(2,2|4)$—that appears as the symmetry algebra of Wilson loops in chiral superspace which are dual to scattering amplitudes in on-shell momentum superspace $(\lambda,\tilde{\lambda},\eta)$. For an introduction to this topic please refer to cha. VIII.

X.1 The Yangian of $\mathfrak{psu}(2,2|4)$

In sec. II.1 we mentioned that scattering amplitudes $\mathcal{A}_n$ of n particles in planar $\mathcal{N}=4$ SYM obey an extended symmetry algebra. The set of Lagrangian symmetries of $\mathcal{N}=4$ SYM is given by the Lie superalgebra $\mathfrak{psu}(2,2|4)$. If we denote the generators of $\mathfrak{psu}(2,2|4)$ by $\mathfrak{J}^a$ such that

$$\mathfrak{J}^a \in \{\mathfrak{L},\bar{\mathfrak{L}},\mathfrak{P},\mathfrak{K},\mathfrak{R},\mathfrak{D}|\mathfrak{Q},\bar{\mathfrak{Q}},\mathfrak{S},\bar{\mathfrak{S}}\}, \tag{X.1}$$

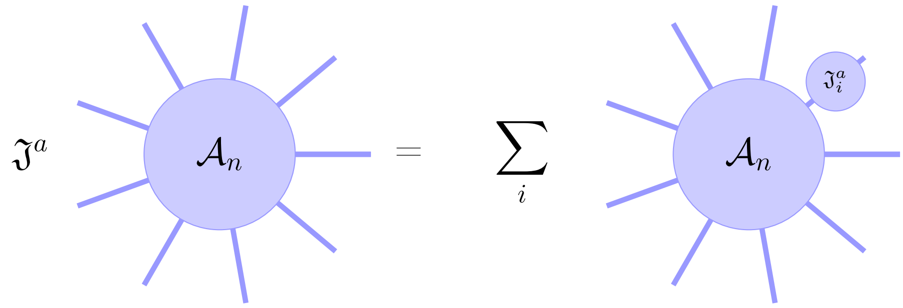
the action of these symmetry generators on the external data inserted into the scattering amplitudes $\mathcal{A}_n(\Lambda_1,\ldots,\Lambda_n)$ where

$$\Lambda_i = (\lambda_i,\tilde{\lambda}_i,\eta_i) \tag{X.2}$$

is given by the sum of the *densities* $\mathfrak{J}^a_i$

$$\mathfrak{J}^a = \sum_{i=1}^{n} \mathfrak{J}^a_i. \tag{X.3}$$

The realization of the algebra on the space with coordinates Λ is presented in apdx. E. $\mathcal{A}_n$ maps the n-fold tensor product of the spinor-helicity representation space[1] $\Lambda^{\otimes n}$ into $\mathbb{C}$. To describe the action of an algebra on such a tensor product of representation spaces, we have used the *coproduct* Δ of the Hopf algebra structure of the universal enveloping algebra $\mathrm{U}[\mathfrak{g}]$. Both concepts have been briefly introduced in apdx. F.1. In general, given an amplitude $\mathcal{A}_n$ with n legs, the action of an algebra generator $\mathfrak{J}^a$ on this amplitude will be described by (X.3) in the following.



Above we have given an diagrammatic description of the action of the generator $\mathfrak{J}^a$ on $\mathcal{A}_n$. The bubble on the leg on the right hand side of the equation signifies the modification of the on-shell variables $(\lambda_i, \bar{\lambda}_i, \eta_i)$ by the generator density $\mathfrak{J}_i^a$. The sum gives zero if $\mathfrak{J}^a$ generates a symmetry of $\mathcal{A}_n$.

The Yangian structure can be described by the underlying algebra $\mathfrak{psu}(2,2|4)$ with generators $\mathfrak{J}^a$ and additional *first level generators* $\widehat{\mathfrak{J}}^a$ which transform in the adjoint representation of the algebra, e.g.,

$$[\mathfrak{J}^a, \widehat{\mathfrak{J}}^b\} = f^{ab}{}_c\, \widehat{\mathfrak{J}}^c. \tag{X.4}$$

These additional operators have to obey consistency conditions which have been laid out in apdx. F.3. Knowledge of the algebra structure, the first level generators and their commutation relations, and the consistency conditions is enough to define the Yangian algebra. In our specific case with underlying algebra $\mathfrak{psu}(2,2|4)$, we will denote the Yangian algebra by $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$.

A generator $\widehat{\mathfrak{J}}^a$ of the first level of $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ can also act on $\mathcal{A}_n$, just like, in fact the generators of any level of the Yangian. The action of the higher level generators is obtained via the coproduct, again. Compared to the usual coproduct (see apdx. F.2) of $U[\mathfrak{g}]$ however, the coproduct consistent with the Yangian consistency conditions is deformed. This special form of the coproduct is described in (F.16). To act on $\mathcal{A}_n$, we need to take the n-fold coproduct and find

$$\Delta^n(\widehat{\mathfrak{J}}^a) = \sum_{i=1}^{n} \widehat{\mathfrak{J}}_i^a + \frac{1}{2} f^a{}_{bc} \sum_{i,j}^{n} \sigma_{ij} \mathfrak{J}_i^b \mathfrak{J}_j^c. \tag{X.5}$$

where σ_{ij} denotes the sign function

$$\sigma_{ij} = \mathrm{sign}(j - i)$$

and $f^a{}_{bc}$ are the structure constants of[2] $\mathfrak{psu}(2,2|4)$. We use the evaluation representation as described in apdx. F.3.2 and set the appearing spectral parameter $u = 0$ because its coefficients are proportional to generators of the underlying algebra at this level, i.e., they generate a

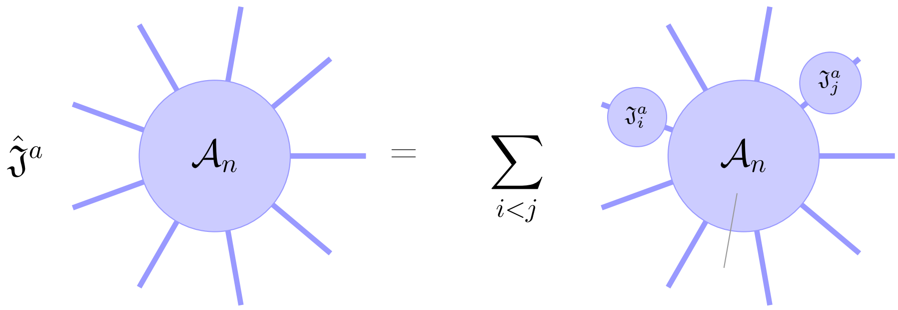
[1] We call this space Λ just like the set of variables.

[2] Strictly speaking these are the structure constants of the dual $\mathfrak{g}^*$, but the dual is identified with $\mathfrak{g}$ via the Killing form.

symmetry transformation and annihilate the amplitude anyway. From here on throughout we will therefore use the definition

$$\widehat{\mathfrak{J}}^a = \frac{1}{2} f^a{}_{bc} \sum_{i,j} \sigma_{ij} \mathfrak{J}_i^b \mathfrak{J}_j^c \tag{X.6}$$

and refer to it as the *Yangian first-level generator* or *first level generator* for (relative) briefness.



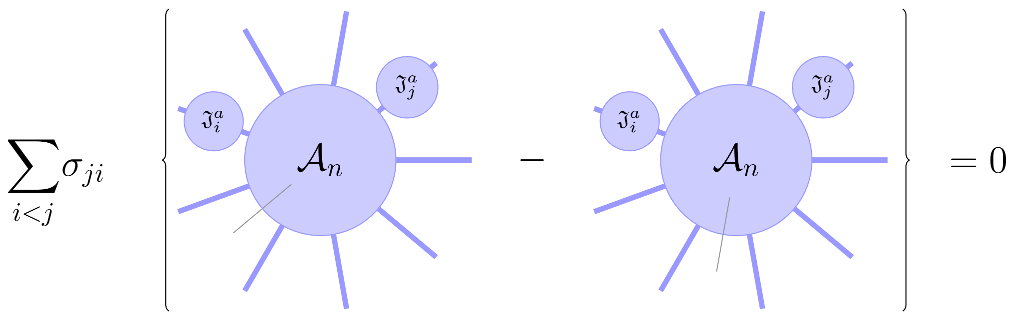
The figure shows the action of the Yangian generator on $\mathcal{A}_n$. Notice that the ordering "opens" the amplitude (symbolized by the gray line). Hence, a first-level generator $\widehat{\mathfrak{J}}^a$ will not obey the cyclic symmetry of amplitudes. We can see this as the ordered structure that is imposed by σ_{ji}—the sum has a reference point in site 1. It is therefore necessary to check whether the Yangian generators obey the cyclic symmetry by taking the difference of two first-level operators that cut the amplitude open at different points. A shift by one step

$$1 \to 2, \ldots, n \to n+1 \tag{X.7}$$

will suffice, so we calculate

$$\widehat{\mathfrak{J}}^a_{2,n+1} - \widehat{\mathfrak{J}}^a_{1,n} = f^a{}_{bc} \mathfrak{J}_1^b \mathfrak{J}^c + f^a{}_{bc} f^{bc}{}_d \mathfrak{J}_1^d. \tag{X.8}$$

Although this looks worryingly non-vanishing, we are in the fortunate position to work with $\mathfrak{psu}(2,2|4)$: The first term in (X.8) is proportional to a symmetry, so when acting with both sides of this equation on $\mathcal{A}_n$, its action will yield zero. The second term on the other hand is proportional to the dual Coxeter number $h^\vee$ which is zero in the case of all the superalgebras $\mathfrak{sl}(n|n)$. The right hand side of (X.8) is therefore equivalent to zero on $\mathcal{A}_n$ and thus is the left hand side. See below for a graphical impression of this equality.



This is a remarkable, non-trivial consistency check. A similar cancellation is only known from the Yangian of $\mathfrak{osp}(6|4)$ which is a hidden symmetry algebra of the scattering amplitudes of ABJM theory [119], a $\mathcal{N}=6$ Chern-Simons theory with matter.

X.2 Invariance of tree-level scattering amplitudes

It has been shown that all tree-level scattering amplitudes of planar $\mathcal{N} = 4$ SYM are—at least naively—invariant under the zeroth- and first-level generators of Y[$\mathfrak{psu}(2, 2|4)$]

$$\mathfrak{J}^a \mathcal{A}_n = 0, \qquad \text{and} \qquad \widehat{\mathfrak{J}}^a \mathcal{A}_n = 0 \tag{X.9}$$

and therefore under the whole Yangian algebra. In particular, this has been shown in [120] using the Grassmannian integral [60, 58, 62]. We will paraphrase the proof here, and use it in cha. XII to show the existence of a new symmetry generator.

The Grassmannian integral $\mathcal{L}_{n,k}$ (II.11) is invariant under the generators of the superconformal algebra $\mathfrak{psu}(2, 2|4)$ by construction[3]

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{L}_{n,k} = \frac{1}{\mathrm{vol}(GL(k))} \int \frac{d^{n\times k}t}{\mathcal{M}_1 \cdots \mathcal{M}_n} \sum_m t_{im} \mathcal{Z}^{\mathcal{A}}_m \partial_{i\mathcal{B}} \prod_j \delta^{4|4}(t_j \cdot \mathcal{Z}) = 0 \tag{X.10}$$

after partial integration of $\partial_{i\mathcal{B}}$. To see the invariance under $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ more work is necessary. A crucial remark before we do so: The Grassmannian integral is only invariant under $\widehat{\mathfrak{J}}$ up to boundary terms [120]. This however is enough to show the invariance of all tree-level amplitudes under the Yangian first-level operators $\widehat{\mathfrak{J}}$.

An observation of [120] is the fact that the Yangian generators in their twistor representation

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} = (-1)^{\mathcal{C}} \sum_{i,j} \sigma_{ji} \mathcal{Z}^{\mathcal{A}}_i \partial_{i\mathcal{C}} \mathcal{Z}^{\mathcal{C}}_i \partial_{i\mathcal{B}} \tag{X.11}$$

can be rewritten and used on the Grassmannian integral to give[4]

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{L}_{n,k} = \int \frac{d^{n\times k}t}{\mathcal{M}_1 \cdots \mathcal{M}_n} \sum_{a,b=1}^{k} \sum_{i,j} \sigma_{ji} \left[t_{ai} \mathcal{Z}^{\mathcal{A}}_i t_{bj} \frac{\partial}{\partial t_{aj}} \right] \partial_{b\mathcal{B}} \prod_d \delta^{4|4}(t_d \cdot \mathcal{Z}). \tag{X.12}$$

The operator $\mathcal{O}^{\mathcal{A}}_b = \sum_{i,j} \sigma_{ji} t_{ai} \mathcal{Z}^{\mathcal{A}}_i t_{bj} \frac{\partial}{\partial t_{aj}}$ can be commuted past the product of minors[5] such that finally

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{L}_{n,k} = \sum_{i,j} \sigma_{ji} \int d^{n\times k}t \; t_{ai} \frac{\partial}{\partial t_{aj}} \left[\frac{\mathcal{Z}^{\mathcal{A}}_i t_{bj}}{\mathcal{M}_1 \cdots \mathcal{M}_n} \partial_{b\mathcal{B}} \prod_d \delta^{4|4}(t_d \cdot \mathcal{Z}) \right]. \tag{X.13}$$

The integrand here is—due to the factor σ_{ji}—a total derivative, thus proving the invariance of the Grassmannian integral under $\widehat{\mathfrak{J}}$ up to boundary terms. This concludes the proof of [120].

[3] Observe that the generators $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ are to be understood to have the super-trace $(-)^{\mathcal{A}}\delta^{\mathcal{A}}{}_{\mathcal{B}}$ removed.

[4] We shall drop the volume factor in the following for brevity and tidiness

[5] This operation is—although computationally non-trivial to show—not very hard to understand. The way the operator $\mathcal{O}^{\mathcal{A}}_b$ works is to replace some t_{aj} by other t_{bj}. In some cases, the minor then vanishes, in the other cases, we can use permutation of the arguments of the minor to bring it back into its original shape.

CHAPTER XI

Wilson loops and Yangian transformations

XI.1 Yangian generators in ambitwistor space

On ambitwistor space $\mathbb{A}_{3|4}$ with coordinates $(\mathcal{Z}, \mathcal{W})$ the generators $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ of $\mathfrak{u}(2,2|4)$ are represented by single derivative operators (see E.3.3)

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i=1}^{n} \mathfrak{J}^{\mathcal{A}}_{i,\mathcal{B}} = \sum_{i=1}^{n} (-1)^{\mathcal{A}} \mathcal{Z}_i^{\mathcal{A}} \partial_{i,\mathcal{B}} - (-1)^{\mathcal{A}\mathcal{B}} \mathcal{W}_{i,\mathcal{B}} \bar{\partial}_i^{\mathcal{A}}. \tag{XI.1}$$

The central charge $\mathfrak{C}$ and the hypercharge $\mathfrak{B}$ are obtained by taking the super-trace and trace of $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ respectively. The level-one generators $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ may be represented by the bi-local formula[1]

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i,j=1}^{n} \sigma_{ji}\, \mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{C}} \mathfrak{J}_j^{\mathcal{C}}{}_{\mathcal{B}} = \sum_{i<j} \left(\mathfrak{J}_i^{\mathcal{A}}{}_{\mathcal{C}} \mathfrak{J}_j^{\mathcal{C}}{}_{\mathcal{B}} - \mathfrak{J}_j^{\mathcal{A}}{}_{\mathcal{C}} \mathfrak{J}_i^{\mathcal{C}}{}_{\mathcal{B}} \right). \tag{XI.2}$$

and they transform under the level-zero generators in the following way

$$[\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}, \widehat{\mathfrak{J}}^{\mathcal{C}}{}_{\mathcal{D}}\} = (-1)^{\mathcal{C}} \delta^{\mathcal{C}}_{\mathcal{B}} \widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{D}} - (-1)^{\mathcal{C}+(\mathcal{A}+\mathcal{B})(\mathcal{C}+\mathcal{D})} \delta^{\mathcal{A}}_{\mathcal{D}} \widehat{\mathfrak{J}}^{\mathcal{C}}{}_{\mathcal{B}}. \tag{XI.3}$$

Yangian invariance of a function of ambitwistor variables $F(\mathcal{Z}, \mathcal{W})$ is achieved when

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} F(\mathcal{Z}, \mathcal{W}) = 0 \qquad \text{and} \qquad \widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} F(\mathcal{Z}, \mathcal{W}) = 0. \tag{XI.4}$$

§ XI.1.1. Superconformal invariance.—As we pointed out above, all generators

$$\{\mathfrak{P}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{K}, \mathfrak{D} | \mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{S}, \bar{\mathfrak{S}}\} \tag{XI.5}$$

of the superconformal algebra $\mathfrak{psu}(2,2|4)$ are represented by one easy expression for $\mathfrak{J}^{A}{}_{B}$. Hence, we can treat them all at once.

The ambitwistor brackets $\langle k, l]$ are superconformal invariants

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} \langle i, j] = 0, \qquad |i - j| \geq 2 \tag{XI.6}$$

[1]The sign factor $(-1)^{\mathcal{A}}$ was included to eliminate a factor of $(-1)^{\mathcal{C}}$ in the definition of the Yangian charges.

by construction. Of course, since the generators $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ are represented by single derivative operators any function $F(\langle k,l])$ of finite ambitwistor brackets is a superconformal invariant, too

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} F(\langle k,l]) = 0. \tag{XI.7}$$

In the following, we will be interested in calculating the action of the Yangian operators on the regularized results of the one-loop calculation of the Wilson loop in full superspace. Regularization introduces a wider class of functions F_{reg} with additional dependencies on auxiliary twistors $\mathcal{Z}_\star$ as in the framing regularization or explicitly non-superconformally invariant combinations of the twistor data like the angle and square brackets

$$\langle i,j\rangle = \mathcal{Z}_i^{\mathcal{A}} I_{\mathcal{A}\mathcal{B}} \mathcal{Z}_j^{\mathcal{B}}, \qquad [i,j] = \mathcal{W}_{i\mathcal{A}} I^{\mathcal{A}\mathcal{B}} \mathcal{W}_{i\mathcal{B}} \tag{XI.8}$$

in supersymmetric regularization with I the infinity twistor. Such objects in general break superconformal invariance. We expect therefore an anomalous remainder $\mathcal{A}$

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} F_{\text{reg}} = \mathcal{A}^{\mathcal{A}}{}_{\mathcal{B}}. \tag{XI.9}$$

§ XI.1.2. Yangian invariance.—The generators $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ act as second order derivatives on functions of ambitwistors. This requires any Yangian invariant function of ambitwistors $(\mathcal{Z}^{\mathcal{A}}, \mathcal{W}_{\mathcal{B}})$ to satisfy an additional second order differential equation

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} F(\mathcal{Z}, \mathcal{W}) = 0. \tag{XI.10}$$

We can check that a single ambitwistor bracket $\langle k,l]$ is invariant under the first level generators of $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$—and therefore by construction of the algebra under the full Yangian. Nevertheless, when studying generic functions $F(\langle k,l])$ of brackets, we find these in general not to be invariant [2]

$$\begin{aligned}\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} F(\langle m,n]) = (-1)^{\mathcal{A}} \sum_{i,j,k,l=1}^{n} &\Sigma_{kl,ij} \mathcal{Z}_i^{\mathcal{A}} \mathcal{W}_{l,\mathcal{B}} \langle k,j] \partial_{k,l} \partial_{i,j} F(\langle m,n]) \\ &- \delta^{\mathcal{A}}_{\mathcal{B}} \sum_{k,l=1}^{n} \sigma_{kl} \langle k,l] \partial_{k,l} F(\langle m,n]).\end{aligned} \tag{XI.11}$$

This is a non-trivial second order partial differential equation. The trace term proportional to $\delta^{\mathcal{A}}_{\mathcal{B}}$ in (XI.11) only appears when considering the level one hypercharge $\widehat{\mathfrak{B}}$ of the Yangian $\mathrm{Y}[\mathfrak{u}(2,2|4)]$. This generator was shown to be an additional symmetry of the scattering amplitudes of $\mathcal{N}=4$ SYM [15] not contained in the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ and will be treated in detail in chapter XII. Ambitwistor brackets transform covariantly under $\widehat{\mathfrak{B}}$

$$\widehat{\mathfrak{B}} \langle k,l] = 8\ \sigma_{lk} \langle k,l]. \tag{XI.12}$$

[2] The occurring derivative is defined by $\partial_{k,l} = \partial/\partial\langle k,l]$. The function Σ is a factor defined by

$$\Sigma_{kl,ij} = \sigma_{ki} - \sigma_{kj} - \sigma_{li} + \sigma_{lj}$$

.

XI.2 Anomaly of Yangian symmetry

As we already explained in sec. II.2 the motivation to study Wilson loops in a non-chiral setting came from the results on the symmetry properties of Wilson loops in chiral superspace. It has been shown that the one-loop corrections to the chiral supersymmetric Wilson loop [17, 10] break the chiral $\mathcal{N} = 4$ supersymmetry transformations [18, 121].

The puzzling problem was the fact that chiral Wilson loops seemingly broke the Poincaré supersymmetry $\bar{\mathfrak{Q}}$. Even worse, the anomaly didn't seem to have anything to do with the fact that the loop corrections to the Wilson loop needed regularization, finite quantities like the remainder function failed to vanish under $\bar{\mathfrak{Q}}$. This behavior was not only unexpected but also very hard to explain. Subsequent publications [77, 76] could finally prove that the anomalous behavior of $\bar{\mathfrak{Q}}$ could be corrected unambiguously. They explained the appearing anomaly by showing that

$$\bar{\mathfrak{Q}} = \sum_i \chi_i \frac{\partial}{\partial \lambda_i} \tag{XI.13}$$

is not actually a symmetry of full $\mathcal{N} = 4$ SYM on chiral superspace but only of the self-dual theory which we discussed in sec. VI.1. Turning $\bar{\mathfrak{Q}}$ into a symmetry of the full theory required the addition of a non-local correction term that enabled further calculations of higher loop results [76].

The non-chiral supersymmetric n-polygonal Wilson loop expectation value $\langle \mathcal{W}_n \rangle$ presented in IX suffers from ultraviolet divergences in the regions close to the cusps just like the chiral Wilson loop. To derive a sensible result we had to regularize these divergences, see sec. IX.3. The regularization procedure breaks certain symmetries

$$jF_n \neq 0 \tag{XI.14}$$

where $j \in \mathrm{Y}[\mathfrak{psu}(2,2|4)]$.

While the $\bar{\mathfrak{Q}}$-anomaly of the chiral Wilson loop was seemingly regularization independent and universal, we will show that the anomalies of the Wilson loop in full superspace are only produced by the need to regularize the appearing ultra-violet divergences. In particular, we show that there is a regularization for the non-chiral Wilson loop that preserves super-Poincaré symmetry, so the $\bar{\mathfrak{Q}}$-anomaly does not appear.

In the following we will treat the anomalies

$$jM_n^{(1)} = \mathcal{A}_{n,j} \tag{XI.15}$$

for the non-chiral MHV one-loop expectation value in different regularizations. We investigate not only the anomalies of the symmetry generators $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ of the superconformal algebra $\mathfrak{psu}(2,2|4)$ but also the anomalies

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} M_n^{(1)} = \widehat{\mathcal{A}}^{\mathcal{A}}_{n,\mathcal{B}} \tag{XI.16}$$

of the Yangian generators $\widehat{\mathfrak{J}}$.

Naturally, it would be better to check explicitly finite, regularization independent quantities for superconformal and Yangian invariance. An interesting class of such quantities is provided by the functions

$$r_{i,j} = M^{(1)}[C] + M^{(1)}[C_{\mathrm{tb}}] - M^{(1)}[C_{\mathrm{t}}] - M^{(1)}[C_{\mathrm{b}}] \tag{XI.17}$$

that are obtained by the boxing procedure presented in ssec. IX.3.3. We find that these are clearly superconformally invariant

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}\, r_{i,j} = 0 \tag{XI.18}$$

since they are regularization independent. On the other hand, Yangian generators fail to annihilate these quantities entirely.

XI.3 Vertex correlators

Let us start by inspecting finite mixed correlators (IX.31) $\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle$ with j and k well separated $|j-k| \gg 3$. Since these are are functions of ambitwistor brackets and regularization independent, they are invariant under superconformal transformations. Therefore, let us immediately turn to Yangian generators. We act with $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ on the mixed vertex correlators and find

$$\begin{aligned}\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle = 64\pi^2(-1)^{\mathcal{A}}\Bigg[&\frac{\mathcal{Z}^{\mathcal{A}}_{j-1}\mathcal{W}_{k,\mathcal{B}}}{\langle j-1,k]} - \frac{\mathcal{Z}^{\mathcal{A}}_{j}\mathcal{W}_{k-1,\mathcal{B}}}{\langle j,k-1]}\Bigg] \\ &+ \delta^{\mathcal{A}}_{\mathcal{B}}\log\left(\frac{\langle j-1,k]\langle j,k-1]}{\langle j-1,k-1]\langle j,k]}\right).\end{aligned} \tag{XI.19}$$

Hence, the correlators by themselves aren't Yangian invariants. Nevertheless, the anomaly is of the form $f_{j-1,k} - f_{j,k-1}$ (the trace term is slightly different, but the conclusion is the same) which naively telescopes in the sum over all vertices

$$\sum_{j,k=1}^{n}\Big(f_{j-1,k} - f_{j,k-1}\Big) = \sum_{j,k=1}^{n}\Big(f_{j,k} - f_{j,k}\Big) = 0. \tag{XI.20}$$

The trouble is that (XI.19) holds only for the finite vertex correlators with $|j-k| \geq 3$. The divergent correlators for $|j-k| \leq 2$ need to be regularized. As we pointed out before, this inevitably breaks superconformal and Yangian invariance. Therefore it is fair to say that the one-loop Wilson loop expectation value is perfectly superconformal and Yangian invariant except for the effects of regularization! Only the divergent correlators of nearby vertices call for regularization and break both symmetries in an analogous fashion. These anomaly terms are computed in the subsequent subsections.

It is worth mentioning that the expression in (XI.19) makes no reference to the vertex which defines the ordering in the Yangian action (XI.2). This is because the function is also superconformally invariant in which case the Yangian action respects cyclic symmetry—a fact that was explained in cha. X. However, the regularized vertex correlators for $|j-k| \leq 2$ break superconformal symmetry and consequently introduce dependence on the reference vertex.

It is helpful to cast $\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle$ into the form of a symbol

$$\mathcal{S}\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle = \sum_{\substack{i=j-1,j\\ l=k-1,k}} R_{i,l}\otimes\langle i,l]. \tag{XI.21}$$

We notice that there are only single brackets in the second entry. A very similar observation for the form of the symbols of scattering amplitudes has been made in [82] where the ordinary twistor brackets $\langle i-1,i,j-1,j\rangle = \epsilon_{ABCD}Z^A_{i-1}Z^B_i Z^C_{j-1}Z^D_j$ took the place of $\langle i,j]$. The $R_{i,l}$ represent the rational functions which appear as the first entry of the symbol for a given second entry $\langle i,l]$. A generator of $\mathfrak{u}(2,2|4)$ acts like a logarithmic derivative on the last entry of a symbol.

Since—roughly speaking[3]—the length of a symbol determines the degree of transcendentality of the function the symbol represents, the superconformal generators act like transcendentality lowering operators, lowering the degree by one. The Yangian level-one generators $\widehat{\mathfrak{J}}^A{}_B$ on the other hand generically act on the last entry of the symbol *and* $R_{i,l}$. The one-loop result is of transcendentality degree 2 so we expect the Yangian operator to yield at most rational anomaly terms. As we will show, this expectation is fulfilled by all Yangian operators except for the generator $\widehat{\mathfrak{B}}$. This is the only generator acting "twice" on the second part of a symbol (XI.21). The logarithmic terms in (XI.19) proportional to $\delta^{\mathcal{A}}_{\mathcal{B}}$ are therefore solely part of the anomaly of $\widehat{\mathfrak{B}}$.

Correlators that need regularization can be inspected in the same way. Supersymmetric and axial regularization also produce symbols with only one bracket in the second entry for the divergent propagators $|j-k|<3$:

$$\mathcal{S}\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle_{\text{sP}} = \sum_{\substack{i=j-1,j\\ l=k-1,k}} \Big(R^{(1)}_{i,l}\otimes\langle i,l] + R^{(2)}_{i,l}\otimes\langle i,l\rangle + R^{(3)}_{i,l}\otimes[i,l]\Big), \tag{XI.22}$$

$$\begin{aligned}\mathcal{S}\langle B_{j-1,j}\bar{B}_{k-1,k}\rangle_{\text{framed}} = \sum_{\substack{i=j-1,j\\ l=k-1,k}} \Big(&\mathcal{R}^{(1)}_{i,l}\otimes\langle i,l] + \mathcal{R}^{(2)}_{i,l}\otimes\langle i,\bar{\star}] \\ &+ \mathcal{R}^{(3)}_{i,l}\otimes\langle\star,l] + \mathcal{R}^{(4)}_{i,l}\otimes\langle\star,\bar{\star}]\Big).\end{aligned} \tag{XI.23}$$

The functions $R^{(i)}$ and $\mathcal{R}^{(i)}$ in (XI.22) are all rational and they differ in both schemes. The presence of non-invariant brackets $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ in super-Poincaré regularization or $\langle\star,\cdot]$ and $\langle\cdot,\bar{\star}]$ in axial regularization in the second entries break superconformal invariance[4]. Similarly we expect further contributions to the anomalies of all $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$.

XI.4 Super-Poincaré regularization

To calculate the anomalous remainder for the action of the superconformal generators, we only need to act with the $\mathfrak{J}$ on the spinor brackets in (XI.22). Let us write the generator $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ acting on a function

$$F = F(\langle k,\ l], \langle k,k+1\rangle, [k,k+1]) \tag{XI.24}$$

as a function of derivatives with respect to brackets. Then

$$\begin{aligned}\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}F = \ &(-1)^{\mathcal{A}} I_{\mathcal{BC}} \sum_{i=1}^{n} (\mathcal{Z}^{\mathcal{A}}_i \mathcal{Z}^{\mathcal{C}}_{i+1} - \mathcal{Z}^{\mathcal{A}}_{i+1}\mathcal{Z}^{\mathcal{C}}_i)\partial_i F \\ &-(-1)^{\mathcal{A}} I^{\mathcal{AC}} \sum_{i=1}^{n} (\mathcal{W}_{i,\mathcal{B}}\mathcal{W}_{i+1,\mathcal{C}} - \mathcal{W}_{i+1,\mathcal{B}}\mathcal{W}_{i,\mathcal{C}})\bar{\partial}_i F\end{aligned} \tag{XI.25}$$

[3]We gloss over the fact that most symbols don't even correspond to transcendental functions in the first place.
[4]Notice that $\mathcal{R}^{(4)}$ has no influence on the anomaly of superconformal or Yangian generators.

where $\partial_i = \partial/\partial\langle i, i+1\rangle$, similarly for $\bar{\partial}_i$. It is now an easy exercise in calculus to replace $F = \langle\mathcal{W}_n\rangle^{(1)}_{\mathrm{sP}}$ and calculate the anomalous remainder

$$\begin{aligned}
\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}\langle\mathcal{W}_n\rangle^{(1)}_{\mathrm{sP}} = (-1)^{\mathcal{A}} I_{\mathcal{BC}} \sum_i &\left(\frac{\mathcal{Z}_i^{\mathcal{A}}\mathcal{Z}_{i+1}^{\mathcal{C}} - \mathcal{Z}_{i+1}^{\mathcal{A}}\mathcal{Z}_i^{\mathcal{C}}}{\langle i, i+1\rangle}\right) \\
&\times \left[\frac{2}{\epsilon} + \log\left(\frac{[i,i+1]\langle i,i+2]\langle i+1,i-1]}{\mu^2\langle i,i+1\rangle[i+1,i+2][i,i-1]}\right)\right] \\
- I^{\mathcal{AC}} \sum_i &\left(\frac{\mathcal{W}_{i,\mathcal{B}}\mathcal{W}_{i+1,\mathcal{C}} - \mathcal{W}_{i+1,\mathcal{B}}\mathcal{W}_{i,\mathcal{C}}}{[i,i+1]}\right) \\
&\times \left[\frac{2}{\epsilon} + \log\left(\frac{\langle i,i+1\rangle\langle i-1,i+1]\langle i+2,i]}{\mu^2\langle i-1,i\rangle[i,i+1]\langle i+2,i+1\rangle}\right)\right]. \qquad \text{(XI.26)}
\end{aligned}$$

The right hand side of (XI.26) is zero for any of the Poincaré generators as well as supersymmetry and R-symmetry thus realizing full super-Poincaré symmetry free of anomalies. So the Wilson loop in full superspace does not feature any trace of the $\bar{\mathfrak{Q}}$-anomaly as we already anticipated. We are left with the conformal anomaly of the Wilson loop.

When comparing this anomaly to the literature, e.g. [120], note that the bosonic result is often split

$$\langle\mathcal{W}_n\rangle = Z_n F_n \qquad \text{(XI.27)}$$

into a divergent part Z_n and a finite part F_n. The divergent part Z_n is defined such that it contains the full dependence on the renormalization scale μ. Ref. [120], computed the anomaly of the conformal group, when acting on $\log F_n$. This fact must be taken into account when comparing to the above anomaly of the whole answer, including the contribution of the divergent part Z_n. When doing so, we find agreement with the conformal anomaly computed in ref. [120].

Let us proceed with the calculation of the Yangian anomaly

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\langle\mathcal{W}_n\rangle^{(1)}_{\mathrm{sP}} = \widehat{\mathcal{A}}^{\mathcal{A}}_{n,\mathcal{B}}. \qquad \text{(XI.28)}$$

While it would be possible to calculate any of the Yangian anomalies, we choose to give the form of the anomaly of the level-one hypercharge $\widehat{\mathfrak{B}}$. Its form is especially nice compared to the anomalies of the other first level generators $\widehat{\mathfrak{J}}^{A}{}_{B}$ which can be deduced using (XI.3). Just as before we can find the action of $\widehat{\mathfrak{B}}$ on a function F in terms of derivatives with respect to brackets. We find

$$\begin{aligned}
\widehat{\mathfrak{B}}\langle\mathcal{W}_n\rangle^{(1)}_{\mathrm{sP}} = 2\sum_{j=1}^{n} &\left[\frac{\langle j-1,j+2](-1)^{\mathcal{A}}\mathcal{Z}_j^{\mathcal{A}}\mathcal{W}_{j+1,\mathcal{A}}}{\langle j-1,j+1]\langle j,j+2]} - \frac{\langle j+2,j-1](-1)^{\mathcal{A}}\mathcal{Z}_{j+1}^{\mathcal{A}}\mathcal{W}_{j,\mathcal{A}}}{\langle j+1,j-1]\langle j+2,j]}\right. \\
&\left.+2\left(\frac{(-1)^{\mathcal{A}}\mathcal{Z}_{j+2}^{\mathcal{A}}\mathcal{W}_{j,\mathcal{A}}}{\langle j+2,j]} - \frac{(-1)^{\mathcal{A}}\mathcal{Z}_j^{\mathcal{A}}\mathcal{W}_{j+2,\mathcal{A}}}{\langle j,j+2]}\right)\right] + 16\sum_{j=1}^{n-2}\log\left(\frac{\langle j+2,j]}{\langle j,j+2]}\right) \\
&+16\log\left(\frac{\langle 1,2\rangle[n-1,n]}{\langle n-1,n\rangle[1,2]}\right) \qquad \text{(XI.29)}
\end{aligned}$$

where the regularization dependent part of the anomaly is fully contained in the terms proportional to $\mathcal{Z}_i\mathcal{W}_{i+2}$ and $\mathcal{Z}_{i+2}\mathcal{W}_i$. The last term is a contribution from the $1,n$ boundary since—as we saw above—superconformal symmetry is broken by this regularization as well.

XI.5 Axial regularization

Let us now consider the action of the superconformal generators on a framed Wilson loop. When acting with $\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}}$ on a function

$$F_{\text{reg}} = F(\langle k, l], \langle k, \bar{\star}], \langle \star, k], \langle \star, \bar{\star}]) \tag{XI.30}$$

in axial regularization we can write the generator $\mathfrak{J}$ in terms of derivatives with respect to the brackets

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} F_{\text{reg}} = (-1)^{\mathcal{A}} \sum_{j=1}^{n} \Big[\mathcal{Z}_j^{\mathcal{A}} \mathcal{W}_{\bar{\star},\mathcal{B}} \frac{\partial F}{\partial \langle j, \bar{\star}]} - \mathcal{Z}_\star^{\mathcal{A}} \mathcal{W}_{j,\mathcal{B}} \frac{\partial F}{\partial \langle \star, j]} \Big]. \tag{XI.31}$$

Setting $F_{\text{reg}} = \langle \mathcal{W}_n \rangle^{(1)}_{\text{framed}}$ we find

$$\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{B}} \langle \mathcal{W}_n \rangle^{(1)}_{\text{framed}} = (-1)^{\mathcal{A}} \sum_{i=1}^{n} \left[\frac{\mathcal{Z}_\star^{\mathcal{A}} \mathcal{W}_{i,\mathcal{B}}}{\langle \star, i]} - \frac{\mathcal{Z}_i^{\mathcal{A}} \mathcal{W}_{\bar{\star},\mathcal{B}}}{\langle i, \bar{\star}]} \right] \log \left(\epsilon^2 \frac{\langle i-1, i+1]^\star \langle i+1, i-1]^\star}{\langle i-1, i+1] \langle i+1, i-1]} \right). \tag{XI.32}$$

We notice that the general structure of (XI.26) is present here, too. In both cases there are single logarithmic terms weighted by rational functions depending on the symmetry breaking brackets. However, the different regularization schemes break superconformal symmetry in very different ways. Here, the twistors $\mathcal{Z}_\star$ and $\mathcal{W}_{\bar{\star}}$ do not get transformed under the action of the generators of $\mathfrak{psu}(2,2|4)$. Hence, the brackets $\langle i, \bar{\star}]$ and $\langle \star, i]$ are not invariant under $\mathfrak{J}$. If $\mathcal{Z}_\star$ and $\mathcal{W}_{\bar{\star}}$ were to be transformed under superconformal transformations we would find the expectation value (IX.42) $\langle \mathcal{W}_n \rangle^{(1)}_{\text{framed}}$ to be perfectly invariant $\mathfrak{J}'^{\mathcal{A}}{}_{\mathcal{B}} \langle \mathcal{W}_n \rangle^{(1)}_{\text{framed}} = 0$.

§ XI.5.1. The Yangian anomaly.—In the following we will use some additional notation to shorten the expression for the Yangian anomaly. We write[5]

$$(i\ j) \cap k := \mathcal{Z}_i \langle j, k] - \mathcal{Z}_j \langle i, k]. \tag{XI.33}$$

This resembles the notation used in [62]. For anti-chiral twistor variables, we use

$$k \cap (i\ j) := \mathcal{W}_i \langle k, j] - \mathcal{W}_k \langle k, i]. \tag{XI.34}$$

They satisfy the relation

$$\langle (i\ j) \cap k, m] = \langle j, i \cap (k\ m)]. \tag{XI.35}$$

Finally, to write the Yangian anomaly in a more compact form we will make use of the notation

$$([i\ j]k) \cap (l\ m) = \mathcal{Z}_i \langle (j\ k) \cap l, m] - \mathcal{W}_j \langle (i\ k) \cap l, m]\,. \tag{XI.36}$$

When restricted to bosonic components this quantity indicates that the points $(j\ k) \cap l$, $(i\ k) \cap l$ and $(i\ j) \cap l$ are linearly related. Hence, in the bosonic case we could use a Plücker identity to replace (XI.36) by a simpler one. However on inclusion of the fermionic directions there are additional sign factors from the fermions that prevent us from doing so.

[5]When restricted to bosonic components this denotes the intersection point between a line (jk) and the plane $\mathcal{W}_k$.

The Yangian anomaly can then be straightforwardly calculated. It is given by

$$
\begin{aligned}
\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}\langle\mathcal{W}_n\rangle^{(1)}_{\text{framed}} = &\sum_{i=1}^{n-1}\left(2\,\frac{\langle(([i-1\,\star]i)\cap(i+2\,\bar{\star}),i+1]}{\langle i-1,i+1]\langle i,i+2]\langle\star,\bar{\star}]}-1\right)\frac{(-1)^{\mathcal{A}}\mathcal{Z}_i^{\mathcal{A}}\mathcal{W}_{i+1,\mathcal{B}}}{\langle i,i+1]^{\star}}\\
&+\left(2\,\frac{\langle i+1,(\star\,i+2)\cap([\bar{\star}\,i-1]i)]}{\langle i+1,i-1]\langle i+2,i]\langle\star,\bar{\star}]}-1\right)\frac{(-1)^{\mathcal{A}}\mathcal{Z}_{i+1}^{\mathcal{A}}\mathcal{W}_{i,\mathcal{B}}}{\langle i+1,i]^{\star}}\\
&-\sum_{i=1}^{n}\left(\frac{(-1)^{\mathcal{A}}\mathcal{Z}_i^{\mathcal{A}}\mathcal{W}_{i+2,\mathcal{B}}}{\langle i,i+2]}-\frac{(-1)^{\mathcal{A}}\mathcal{Z}_{i+2}^{\mathcal{A}}\mathcal{W}_{i,\mathcal{B}}}{\langle i+2,i]}\right)\\
&+2\,\frac{(-1)^{\mathcal{A}}[(n-1\,\star)\cap 1]^{\mathcal{A}}\mathcal{W}_{n,\mathcal{B}}}{\langle n-1,1]\langle\star,n]}-2\frac{(-1)^{\mathcal{A}}\mathcal{Z}_n^{\mathcal{A}}[1\cap(n-1\,\bar{\star})]_{\mathcal{B}}}{\langle 1,n-1]\langle n,\bar{\star}]}\\
&+2\,\frac{(-1)^{\mathcal{A}}\mathcal{Z}_i^{\mathcal{A}}[n\cap(2\,\bar{\star})]_{\mathcal{B}}}{\langle 1,\bar{\star}]\langle n,2]}-2\,\frac{(-1)^{\mathcal{A}}[(2\star)\cap n]^{\mathcal{A}}\mathcal{W}_{1,\mathcal{B}}}{\langle 2,n]\langle\star,1]}\\
&-\frac{(-1)^{\mathcal{A}}\mathcal{Z}_n^{\mathcal{A}}\mathcal{W}_{1,\mathcal{B}}}{\langle n,1]^{\star}}+\frac{(-1)^{\mathcal{A}}\mathcal{Z}_1^{\mathcal{A}}\mathcal{W}_{n,\mathcal{B}}}{\langle 1,n]^{\star}}+2\delta^{\mathcal{A}}_{\mathcal{B}}\sum_{j=1}^{n-2}\log\left(\frac{\langle j+2,j]}{\langle j,j+2]}\right). \qquad \text{(XI.37)}
\end{aligned}
$$

Despite the fact that we could make superconformal symmetry exact by transforming the auxiliary twistors $\mathcal{Z}_\star$ and $\mathcal{W}_{\bar{\star}}$, too, the same trick does not cure the Yangian anomaly $\widehat{\mathcal{A}}^{\mathcal{A}}{}_{\mathcal{B}}$. The bi-local structure of the Yangian generators distinguishes the auxiliary sites as we need to insert these into the chain $1\to\ldots\to n\to 1$. Putting them between n and 1 the new level-one generators $\widehat{\mathfrak{J}}'^{\mathcal{A}}{}_{\mathcal{B}}$ are defined by $\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}$ and additional pieces from the new sites

$$
\widehat{\mathfrak{J}}'^{\mathcal{A}}{}_{\mathcal{B}}=\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}}+\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{C}}\mathfrak{J}^{\mathcal{C}}_{\star\mathcal{B}}+\mathfrak{J}^{\mathcal{A}}{}_{\mathcal{C}}\mathfrak{J}^{\mathcal{C}}_{\bar{\star}\mathcal{B}}-\mathfrak{J}^{\mathcal{A}}_{\bar{\star}\,\mathcal{C}}\mathfrak{J}^{\mathcal{C}}{}_{\mathcal{B}}-\mathfrak{J}^{\mathcal{A}}_{\star\,\mathcal{C}}\mathfrak{J}^{\mathcal{C}}{}_{\mathcal{B}}+\mathfrak{J}^{\mathcal{A}}_{\star\,\mathcal{C}}\mathfrak{J}^{\mathcal{C}}_{\bar{\star}\mathcal{B}}-\mathfrak{J}^{\mathcal{A}}_{\bar{\star}\,\mathcal{C}}\mathfrak{J}^{\mathcal{C}}_{\star\mathcal{B}}. \qquad \text{(XI.38)}
$$

Their action on $\langle\mathcal{W}_n\rangle^{(1)}_{\text{framed}}$ is given by

$$
\widehat{\mathfrak{J}}'^{\mathcal{A}}{}_{\mathcal{B}}\langle\mathcal{W}_n\rangle^{(1)}_{\text{framed}}=\widehat{\mathcal{A}}^{\mathcal{A}}_{n,\mathcal{B}}-\mathfrak{J}'^{\mathcal{A}}{}_{\mathcal{C}}\mathcal{A}^{\mathcal{C}}_{n,\mathcal{B}}+f^{\mathcal{A}}{}_{\mathcal{E}}{}^{\mathcal{F}}{}_{\mathcal{D}}{}^{\mathcal{G}}{}_{\mathcal{B}}\mathfrak{J}^{\mathcal{E}}_{\star,\mathcal{F}}\mathfrak{J}^{\mathcal{D}}_{\bar{\star},\mathcal{G}}\langle\mathcal{W}_n\rangle^{(1)}_{\text{framed}} \qquad \text{(XI.39)}
$$

with $f^{\mathcal{A}}{}_{\mathcal{E}}{}^{\mathcal{F}}{}_{\mathcal{D}}{}^{\mathcal{G}}{}_{\mathcal{B}}=(-1)^{\mathcal{A}}\delta^{\mathcal{A}}_{\mathcal{E}}\delta^{\mathcal{F}}_{\mathcal{D}}\delta^{\mathcal{G}}_{\mathcal{B}}-(-1)^{\mathcal{A}+(\mathcal{A}+\mathcal{G})(\mathcal{G}+\mathcal{F})}\delta^{\mathcal{A}}_{\mathcal{D}}\delta^{\mathcal{F}}_{\mathcal{B}}\delta^{\mathcal{G}}_{\mathcal{E}}$. In particular, cyclic symmetry remains broken after the inclusion of the auxiliary points into the superconformal generators.

XI.6 Boxing the Wilson loop

Framing and superconformal regularization break superconformal and Yangian symmetry. Moreover, the anomaly terms are different in both cases. This is, however, not very surprising because both results are divergent when the regulator is removed, $\epsilon\to 0$. In other words, the above Wilson loops are regularized but not renormalized, and therefore all answers certainly depend on the regularization scheme. It only makes sense to consider the symmetries of a regularized but not renormalized quantity within any given regularization scheme.

Consider, for example, correlators of local operators in a conformal theory. Naively they are also divergent and need to be regularized. In addition, local operators are renormalized, and when the regulator is removed, the correlation functions are not only perfectly finite, but also transform nicely under superconformal symmetry (albeit with quantum corrections to scaling dimensions).

The boxed Wilson loop can be regarded as such a renormalization of a Wilson loop. The quantities $r_{i,j}$ do not depend on the regularization scheme, they are finite and manifestly superconformally invariant. However, when inspecting the functions $r_{i,j}$ we notice the occurrence of brackets like

$$\langle k, t] = \langle k, 1] - \frac{\langle i, 1]}{\langle i, 2]} \langle k, 2]. \tag{XI.40}$$

Their occurrence breaks Yangian invariance. This can be seen by considering the symbols $\mathcal{S}r_{i,j}$ of these quantities. We find terms like

$$R_{i,j,k,l} \otimes \left(1 - \frac{\langle i, k] \langle j, l]}{\langle i, l] \langle j, k]}\right). \tag{XI.41}$$

The Yangian acts twice on the second entry of the symbol which produces additional logarithmic terms that do not cancel amongst each other.

Hence, the boxed Wilson loop is finite and respects superconformal symmetry, but it does not respect Yangian symmetry. Naively this seems to imply that superconformal symmetry is exact while Yangian symmetry is broken or anomalous. However one has to bear in mind that the boxed Wilson loop is not a simple planar Wilson loop expectation value anymore. For instance, at the one-loop level, the boxed Wilson loop is equivalent to the correlator of two Wilson loops

$$r = \frac{\langle W[C] \rangle \langle W[C_{\mathrm{tb}}] \rangle}{\langle W[C_{\mathrm{t}}] \rangle \langle W[C_{\mathrm{b}}] \rangle} = \frac{\langle W[C_{\mathrm{T}}] W[C_{\mathrm{B}}] \rangle}{\langle W[C_{\mathrm{T}}] \rangle \langle W[C_{\mathrm{B}}] \rangle} + \mathcal{O}g^4, \tag{XI.42}$$

where C_{T}, C_{B} refer to the top and bottom polygons enclosed by the edges $(\mathrm{t}, 2, \ldots, i)$ and $(\mathrm{b}, i+1, \ldots, n, 1)$. In the string world sheet picture, the simple planar Wilson loop has the topology of a disk while the correlator here has annulus topology. Yangian invariance is expected only for disc topology, because a loop surrounding the disc which represents a Yangian generator can be contracted to a point, see the discussions in [122]. Hence it is not surprising that we find no Yangian invariance from the quantities obtained through boxing despite the fact that they are finite and superconformally invariant.

This fits experience with local operators. There we know that two-point functions of local operators do not exhibit Yangian invariance. Hence it is not surprising that we find no Yangian invariance from the quantities obtained through boxing despite the fact that they are finite and superconformally invariant.

CHAPTER XII

Bonus symmetry

Although the superconformal symmetry algebra is $\mathfrak{psu}(2,2|4)$, it is possible to include the central charge $\mathfrak{C}$ as another trivial symmetry, by interpreting the vanishing of the central charge $\mathfrak{C}_i\mathcal{A}_n = 0$ as a symmetry statement. By doing so, the central charge gets added back into the symmetry algebra

$$\mathfrak{su}(2,2|4) \simeq \mathfrak{psu}(2,2|4) \oplus \mathfrak{C}. \tag{XII.1}$$

On the other hand, the outer automorphism $\mathfrak{B}$ of $\mathfrak{u}(2,2|4)$—variously called super-trace, hypercharge or helicity charge—is no symmetry of the scattering amplitudes of $\mathcal{N}=4$ SYM since

$$\mathfrak{B}\mathcal{A}_{n,k} = 4k\mathcal{A}_{n,k}. \tag{XII.2}$$

Contemplation of the name *maximally helicity violating* scattering amplitudes underlines the statement: Helicity is not a conserved quantity[1].

The inclusion of the central charge as a symmetry however does have an impact on the structure of the Yangian. When trying to build the first order recurrence of the central charge operator $\mathfrak{C}$ we encounter a surprise: We need the structure constants $f^{ab}{}_{\mathfrak{C}}$ and the Killing form $\kappa_{a\mathfrak{C}}$, where the “index” $\mathfrak{C}$ signifies the entries in which one $\mathfrak{C}$ appears. While the only nonzero structure constants are (see E)

$$f^{\mathfrak{Q}\mathfrak{S}}{}_{\mathfrak{C}}, \qquad f^{\bar{\mathfrak{Q}}\bar{\mathfrak{S}}}{}_{\mathfrak{C}}, \tag{XII.3}$$

the only non-zero entry of the Killing form is $\kappa_{\mathfrak{B}\mathfrak{C}}$! For the construction to work, we even have to use the Killing form of $\mathfrak{u}(2,2|4)$. Thus, when we raise and lower indices on the structure constants $f^{ab}{}_c$ we exchange the lower index $\mathfrak{C}$ for the upper index $\mathfrak{B}$ and vice versa.

$$\begin{tikzcd}
\widehat{\mathfrak{psu}}(2,2|4) & \oplus & \widehat{\mathfrak{B}} & \oplus & \widehat{\mathfrak{C}} \\
\mathfrak{psu}(2,2|4) \arrow[u] & \oplus & \mathfrak{B} \arrow[urr] & \oplus & \mathfrak{C} \arrow[ull]
\end{tikzcd}$$

That means, when we are building the generator $\widehat{\mathfrak{B}}$ we will have to use (non-zero) commutation relations in which $\mathfrak{C}$ appears, namely those in which the structure constants (XII.3) occur. On the other hand, there exist no non-zero structure constants $f^{ab}{}_{\mathfrak{B}}$, thus, when we raise the index $\mathfrak{B}$ we find that the first level generator $\widehat{\mathfrak{C}}$ associated with these dual structure constants must be identically zero. Although the construction of the first level generators is, in fact, independent of the choice of the Killing form, we can see that for the concrete choice

[1]It is possible to make one N$^{\mathrm{k}}$MHV class of amplitudes invariant under $\mathfrak{B}$ by subtraction of the charge $4k$ of this class of amplitudes. This deformation of $\mathfrak{B}$ is consistent with the algebra but ultimately inconsequential.

we have made here, things work out as expected. The non-diagonal form of the Killing form swaps the construction the generators $\widehat{\mathfrak{B}}$ and $\widehat{\mathfrak{C}}$ compared to their level zero incarnations and the generators of $\mathfrak{psu}(2,2|4)$ (see figure).

With this explanation, recall the structure constants in (XII.3) and use the Killing form to raise and lower the indices. We find

$$\widehat{\mathfrak{B}} = \sum_{i,j} \sigma_{ji} \left[\mathfrak{Q}_i^{\alpha a}\mathfrak{S}_{j,\alpha a} - \bar{\mathfrak{Q}}_{i,a}^{\dot\alpha}\bar{\mathfrak{S}}_{j,\dot\alpha}^{a}\right]. \tag{XII.4}$$

In particular, since $\mathfrak{B}$ is an outer automorphism, it never appears on the right hand side of the commutation relations (E.6), and so does not appear in the definition of $\widehat{\mathfrak{B}}$ either. This is promising since $\mathfrak{B}$ could spoil some of the nice properties that $\widehat{\mathfrak{B}}$ has. We will come to this point in due course.

We should check what kind of commutation relations $\widehat{\mathfrak{B}}$ satisfies with the generators of the algebra $\mathfrak{psu}(2,2|4)$. When doing so we notice that it commutes with all bosonic generators, but not with the odd generators $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{S}$ or $\bar{\mathfrak{S}}$. Let us calculate the commutator of $\widehat{\mathfrak{B}}$ with $\mathfrak{P}^{\alpha\dot\alpha}$ and $\mathfrak{Q}^{\alpha a}$ as an example. First with $\mathfrak{P}$

$$\begin{aligned}[\widehat{\mathfrak{B}}, \mathfrak{P}^{\alpha\dot\alpha}] &= \sum_{i,j} \sigma_{ji}\Big(\mathfrak{Q}_i^{\beta a}[\mathfrak{S}_{j,\beta a}, \mathfrak{P}^{\alpha\dot\alpha}] - \bar{\mathfrak{Q}}_{i,a}^{\dot\beta}[\bar{\mathfrak{S}}_{j,\dot\beta}^{a}, \mathfrak{P}^{\alpha\dot\alpha}]\Big)\\ &= \sum_{i,j} \sigma_{ji}\Big(\mathfrak{Q}_i^{\alpha a}\bar{\mathfrak{Q}}_{j,a}^{\dot\alpha} - \bar{\mathfrak{Q}}_{i,a}^{\dot\alpha}\mathfrak{Q}_j^{\alpha a}\Big) = 0\end{aligned} \tag{XII.5}$$

due to the properties of $\sigma_{ji} = -\sigma_{ij}$. On the other hand,

$$[\widehat{\mathfrak{B}}, \mathfrak{Q}^{\alpha a}] = \sum_{i,j} \sigma_{ji}\Big(\mathfrak{L}_{i,\beta}^{\alpha}\mathfrak{Q}_j^{\beta a} - \mathfrak{R}_{i,b}^{a}\mathfrak{Q}_j^{\alpha b} + \frac{1}{2}(\mathfrak{D}_i + \mathfrak{C}_i)\mathfrak{Q}_j^{\alpha a} + \bar{\mathfrak{S}}_{j,\dot\beta}^{a}\mathfrak{P}_j^{\alpha\dot\beta}\Big) \tag{XII.6}$$

which we can identify with the first level operator $\widehat{\mathfrak{Q}}$. In fact, we find

$$[\widehat{\mathfrak{B}}, \bar{\mathfrak{Q}}_a^{\dot\alpha}] = -\widehat{\bar{\mathfrak{Q}}}_a^{\dot\alpha}, \qquad [\widehat{\mathfrak{B}}, \mathfrak{S}_{\alpha a}] = -\widehat{\mathfrak{S}}_{\alpha a}, \qquad [\widehat{\mathfrak{B}}, \bar{\mathfrak{S}}_{\dot\alpha}^{a}] = \widehat{\bar{\mathfrak{S}}}_{\dot\alpha}^{a} \tag{XII.7}$$

and (XII.6) to be the only non-zero commutators with the elements of the algebra. Thus,

$$[\widehat{\mathfrak{B}}, \mathfrak{J}^a] = \mathrm{hyp}(\mathfrak{J})\widehat{\mathfrak{J}}^a \tag{XII.8}$$

has the same commutations relations as the hypercharge operator of $\mathfrak{u}(2,2|4)$—up to hats. Under cyclic shifts, the $\widehat{\mathfrak{B}}$ generator behaves like

$$\widehat{\mathfrak{B}}_{2,n+1} - \widehat{\mathfrak{B}}_{1,n} = -2f^{\mathfrak{B}}{}_{bc}\mathfrak{J}_1^b\mathfrak{J}^c + \mathfrak{C}_1. \tag{XII.9}$$

The second term was potentially dangerous, as it is proportional to a density. It turns out that it is proportional to $\mathfrak{C}_1$ which is fine since we are requiring that $\widehat{\mathfrak{B}}$ acts on a representation of $\mathfrak{psu}(2,2|4)$ which implies that the central charge acts like 0. Thus $\widehat{\mathfrak{B}}$ is cyclic, a crucial requirement for a symmetry operator acting on planar scattering amplitudes [13].

We shall now proceed to prove that $\widehat{\mathfrak{B}}$ is in fact a symmetry of the tree-level scattering amplitudes of $\mathcal{N} = 4$.

XII.1 Invariance of the MHV factor

We begin by checking the invariance of the MHV factor $\mathcal{A}_n^{\rm MHV}$ under $\widehat{\mathfrak{B}}$. To do so, we write $\widehat{\mathfrak{B}}$ in terms of the spinor-helicity representation

$$\widehat{\mathfrak{B}} = \sum_{i,j} \sigma_{ji}\Big(\lambda_i^\alpha \eta_i^a \partial_{j,\alpha}\partial_{j,a} - \bar{\lambda}_i^{\dot\alpha}\partial_{i,a}\eta_j^a \bar{\partial}_{j,\dot\alpha}\Big). \tag{XII.10}$$

There are three basic objects in the MHV amplitude

$$\mathcal{A}_n^{\rm MHV} = \frac{\delta^4(P)\delta^{0|8}(Q)}{\prod_i \langle i, i+1\rangle}, \tag{XII.11}$$

the overall momentum $P^{\alpha\dot\alpha} = \sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot\alpha}$, the overall supermomentum $Q^{\alpha a} = \sum_i \lambda_i^\alpha \eta_i^a$ and the product of spinor brackets $N^{-1} = \prod_i \langle i, i+1\rangle$. Hence, it is profitable to express the derivatives ∂_a, ∂_α and $\bar{\partial}_{\dot\alpha}$ in terms of these three quantities

$$\partial_{i,a} = \frac{\partial Q^{\alpha b}}{\partial \eta_i^a}\frac{\partial}{\partial Q^{\alpha b}} = \lambda_i^\alpha \frac{\partial}{\partial Q^{\alpha a}} \tag{XII.12}$$

$$\partial_{i,\dot\alpha} = \frac{\partial P^{\beta\dot\alpha}}{\partial \lambda^\alpha}\frac{\partial}{\partial P^{\beta\dot\alpha}} = \lambda_i^\alpha \frac{\partial}{\partial P^{\alpha\dot\alpha}} \tag{XII.13}$$

$$\partial_{i,\alpha} = \eta_i^a \frac{\partial}{\partial Q^{\alpha a}} + \tilde{\lambda}_i^{\dot\alpha}\frac{\partial}{\partial P^{\alpha\dot\alpha}} - \epsilon_{\alpha\beta}\left[\frac{\lambda_{i+1}^\beta}{\langle i, i+1\rangle} - \frac{\lambda_{i-1}^\beta}{\langle i-1, i\rangle}\right] N\frac{\partial}{\partial N} \tag{XII.14}$$

and rewrite $\widehat{\mathfrak{B}}$ in terms of the basic objects in $\mathcal{A}_n^{\rm MHV}$

$$\widehat{\mathfrak{B}} = \sum_{i,j}\sigma_{ji}\left[\lambda_i^\alpha \eta_i^a\left(\frac{\partial}{\partial Q^{\alpha a}} - \epsilon_{\alpha\beta}\left[\frac{\lambda_{j+1}^\beta}{\langle j, j+1\rangle} - \frac{\lambda_{j-1}^\beta}{\langle j-1, j\rangle}\right]\lambda_j^\gamma N\frac{\partial}{\partial N}\frac{\partial}{\partial Q^{\gamma a}}\right)\right] \tag{XII.15}$$

where terms canceling due to symmetry/antisymmetry in i and j have been omitted. Since $N\partial_N \mathcal{A}_n^{\rm MHV} = \mathcal{A}_n^{\rm MHV}$, we can drop this combination at this point and continue working with the rest of the expression for $\widehat{\mathfrak{B}}$. By performing a shift $j-1 \to j$ in the last term and using the cyclicity of the sum mod n, on MHV amplitudes the last term takes the form

$$\lambda_i^\alpha \eta_i^a \frac{\partial}{\partial Q^{\alpha a}} \tag{XII.16}$$

precisely with the right sign to cancel the first term. Therefore

$$\widehat{\mathfrak{B}}\mathcal{A}_n^{\rm MHV} = 0. \tag{XII.17}$$

To show that all N$^{\rm k}$MHV amplitudes are invariant under $\widehat{\mathfrak{B}}$, we shall make use of the Grassmannian integral (II.11) and the proof of Yangian invariance used in X.2.

Finally, in the whole discussion above, we have been ignoring the presence of the holomorphic anomaly of the MHV amplitude

$$\frac{\partial}{\partial \bar{\lambda}^{\dot\alpha}}\frac{1}{\langle \lambda, \mu\rangle} = \pi\delta^2(\langle\lambda,\mu\rangle)\epsilon_{\dot\alpha\dot\beta}\bar{\mu}^{\dot\beta}. \tag{XII.18}$$

Such derivatives occur for the generators $\mathfrak{S}$, $\bar{\mathfrak{S}}$, and $\mathfrak{K}$. It has been shown [78] that corrections to these generators are necessary, which turn them into actual symmetry generators at the price of having to give up the notion of "superconformally invariant amplitudes". This is because the corrections lead to mixing of different numbers of external legs, i.e., we may only speak about a superconformally invariant $\mathcal{S}$-matrix at best.

Since $\widehat{\mathfrak{B}}$ contains both $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ it is clear that the holomorphic anomaly will also play a role for this symmetry. We have been able to show that a correction is possible and refer at this point to [15] fo further details.

XII.2 Invariance of all tree level amplitudes

The invariance of the Grassmannian integral under the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ has been shown in sec. X.2. To work with the Grassmannian formula (II.11), we have to forge $\widehat{\mathfrak{B}}$ into its twistor representation form. Since all Yangian operators are given by the expression

$$\widehat{\mathfrak{J}}^{\mathcal{A}}{}_{\mathcal{B}} = (-1)^{\mathcal{C}} \sum_{i,j} \sigma_{ji} \mathcal{Z}_i^{\mathcal{A}} \partial_{i,\mathcal{C}} \mathcal{Z}_j^{\mathcal{C}} \partial_{j,\mathcal{B}} \tag{XII.19}$$

and $\widehat{\mathfrak{B}}$ is just the super-trace $(-1)^{\mathcal{A}} \delta^{\mathcal{A}}_{\mathcal{B}}$ we can deduce

$$\widehat{\mathfrak{B}} = (-1)^{\mathcal{A}+\mathcal{C}} \sum_{i,j} \sigma_{ji} \mathcal{Z}_i^{\mathcal{A}} \partial_{i,\mathcal{C}} \mathcal{Z}_j^{\mathcal{C}} \partial_{j,\mathcal{A}}. \tag{XII.20}$$

In the proof paraphrased in X.2, exactly this case was excluded. We shall now retrace the proof and see whether there are any obstacles to the invariance of $\mathcal{L}_{n,k}$ under $\widehat{\mathfrak{B}}$.

When initially acting with $\widehat{\mathfrak{B}}$ on $\mathcal{L}_{n,k}$ we can go through the same steps as in the proof for the other Yangian operators. After the dust settles we are left with the expression[2]

$$\widehat{\mathfrak{B}} \mathcal{L}_{n,k} = \int \frac{d^{n\times k}t}{\mathcal{M}_1 \cdots \mathcal{M}_n} \sum_{a,b=1}^{k} \sum_{i,j} \sigma_{ji} \left[(-1)^{\mathcal{A}} t_{ai} \mathcal{Z}_i^{\mathcal{A}} t_{bj} \frac{\partial}{\partial t_{aj}} \right] \partial_{b\mathcal{A}} \prod_d \delta^{4|4}(t_d \cdot \mathcal{Z}). \tag{XII.21}$$

We now need to move the derivative w.r.t. t to the very left to show that this expression is in fact a total derivative. To do so, it is imperative to notice that the wrapping of the minors $\mathcal{M}_j$ makes it necessary to use the constraints in the δ-functions. These sometimes come with a derivative $\partial_{b\mathcal{A}}$ which has to be partially integrated to make use of the constraints. When doing so we pick up a factor of

$$8k(k-1) \sum_{i,j} \sigma_{ji} \mathcal{L}_{n,k} \tag{XII.22}$$

which upon quick inspection vanishes due to the sum over the sign-function. What is left is a total derivative term

$$\widehat{\mathfrak{B}} \mathcal{L}_{n,k} = \sum_{a,b} \sum_{i,j} \sigma_{ji} \int d^{n\times k}t \, \frac{\partial}{\partial t_{aj}} \left[(-1)^{\mathcal{A}} t_{ai} \mathcal{Z}_i^{\mathcal{A}} t_{bj} \frac{1}{\mathcal{M}_1 \cdot \mathcal{M}_n} \partial_{b\mathcal{A}} \prod_c \delta^{4|4}(t_c \cdot \mathcal{Z}^{\mathcal{A}}) \right] \tag{XII.23}$$

[2] At this point, we want to make a comment about the very recent paper [63] by Arkani-Hamed et al. The form of the operator in equation (XII.21) is the same of the claimed representation of the Yangian in [63]. However, the crucial derivative of the δ-functions is missing.

so $\widehat{\mathfrak{B}}$ is a symmetry of all tree-level amplitudes of $\mathcal{N} = 4$ (up to potential boundary terms)[3]. At this point we have earned the right to endow the level one recurrence of the hypercharge $\widehat{\mathfrak{B}}$ with a new name: *bonus symmetry.* This name mirrors the known cases of additional symmetries especially on the string side of the AdS/CFT correspondence [123, 14, 124] (see [125] for a review). It has also been shown that the pattern on level one and level zero outlined above carries on to higher orders in the Yangian tower of generators [126].

At this point we conclude our discussion of the bonus symmetry and come to the end of this thesis. It is our hope that $\widehat{\mathfrak{B}}$ and the Yangian overall might yet produce more insights into the remarkable structure of scattering amplitudes and Wilson loops.

[3]The result is actually somewhat stronger, as the Grassmannian integral does not only calculate all tree-level amplitudes but also all leading singularities [60, 58].

Summary

The weak-weak duality between scattering amplitudes and Wilson loops on light-like contours in planar $\mathcal{N} = 4$ SYM is a most remarkable and extremely non-trivial feature of the theory. The change of picture that this duality allows—the extraction of results from calculations of objects that are seemingly unrelated in their physical description—has led to advances in our understanding of the inner workings of $\mathcal{N} = 4$ beyond the expected advances from more traditional quantum field theoretic calculations. Part of this success is based on a better understanding of the geometry of superspaces: We find, for example, that the results of calculations in $\mathcal{N} = 4$ SYM are especially nice not on ordinary superspace but on twistor space, where the theory can be formulated thanks to the geometric tools of flag manifolds and double fibrations. The same is true for the symmetries of the theory. Superconformal symmetry has a realization in terms of single derivative operators on twistor space.

The Yangian algebra is essentially built up from the ordinary superconformal symmetry of scattering amplitudes and the superconformal symmetry of Wilson loops, usually called dual superconformal symmetry for this reason. While ordinary superconformal symmetry is especially easy in a twistor description, dual superconformal symmetry is not, and only becomes easy when we change picture and describe the theory in momentum twistors. This juggling with coordinates can be understood on a deep level with the help of the geometry of flag manifolds. This even extends to the level of actions, where it can be shown that what looks like a Yang-Mills theory in one description can be described by a (holomorphic) Chern-Simons theory in another. This is a feature of ordinary and supersymmetric Yang-Mills theories which has not been exploited to its full extent, yet.

We explored some geometrical tools in this thesis and investigated two ways in which supersymmetric Yang-Mills theories can be written as holomorphic Chern-Simons theories. First of all, in the standard twistor space, where it needs to be completed by a non-local term. And secondly in analytic superspace where it is a pure holomorphic Chern-Simons theory. This alone is remarkable, but there is also the ambitwistorial description of the theory, which again allows for a Chern-Simons description. The formulation of Yang-Mills on ambitwistor space is known, but could be investigated from the point of view of harmonic superspaces. We have refrained from doing so in this thesis, but the techniques to do so are expected to be essentially the same as the ones we used here.

The description of $\mathcal{N} = 4$ SYM on such different spaces gives us the ability to calculate specific quantities in the appropriate spaces where they are most naturally expressed. We can also translate them into the language and the setting of other superspaces. This allowed us to calculate a Wilson loop that had originally been formulated on superspace by the use of ambitwistor variables and we were able to translate the results from one language into the other. We found that ambitwistor variables help to clean up results and help us to write them in such

a way that the symmetries of the theory become manifest. The role of non-chiral null-polygonal Wilson loops is unclear. Clearly, they contain more information than their chiral counterparts, so the duality between scattering amplitudes and Wilson loops is not valid. On the other hand, it might be that the constituent parts of such Wilson loops contain information about other physical quantities. Finding evidence for such a correspondence is a matter of devotion to the topic.

Finally, to address the last part of this thesis, let us mention again the huge role of symmetries in the development of $\mathcal{N} = 4$. Clearly, the symmetry group $PSU(2, 2|4)$ of $\mathcal{N} = 4$ SYM essentially sets the stage: From the flag manifold picture we learn which superspace we may use to define the theory on. But this is essentially true for any symmetry group of any Yang-Mills theory in four dimensions. The novel ingredients in planar $\mathcal{N} = 4$ SYM are the non-Lagrangian Yangian symmetries which are the reason for so many simplifications in the calculation of physical results and the mechanism underlying the integrability of $\mathcal{N} = 4$ SYM. The existence of even more symmetry, as was proven in this thesis in the last part is a welcome addition, but it is just one more generator in an infinite sea of symmetry.

The symmetries of $\mathcal{N} = 4$ SYM also give us the ability to do calculations by guessing the answer and inspect the behavior of the Ansatz under the symmetries of $\mathcal{N} = 4$ SYM. Good behavior is usually a very strong sign that the guess was correct. On the other hand, one would wish for a more systematic use for the Yangian like for example the way the Yangian is used in two-dimensional integrable field theories where sophisticated methods allow the use of the Yangian symmetry algebra to its full extent. Such techniques do not yet exist for $\mathcal{N} = 4$ SYM, but their development will have definite consequences for the way we understand Yang-Mills theories and quantum field theories in general.

Outlook

In this last chapter let us map out some further areas of research connected to the work in this thesis.

We did not discuss the formulation of a quantum theory from the hCS Lagrangian for $\mathcal{N} = 3$ SYM given in sec. VI.2. Although there have been attempts at formulating a propagator for the gauge field A [100] we would like to see a more unified formulation. One that is closer to the form of the propagator of twistor gauge fields. We expect such a formulation to exist since $\mathcal{N} = 3$ analytic superspace can be understood as an ambitwistor space with additional *internal* directions $(x_A, \theta^I, \bar{\theta}_I)$. We hope to gain from such a program an understanding of how to formulate a consistent ambitwistor propagator from a field theoretic calculation. Such an understanding would also allow for the development of a propagator for $\mathcal{N} = 4$ SYM on ambitwistor space which would ultimately allow us to push the calculation of the non-chiral null polygonal Wilson loops of cha. IX to higher loop orders.

This is of course another goal: the formulation of a solution to the non-linearized constraints of $\mathcal{N} = 4$ SYM on full superspace. This would allow us to understand the Wilson loops of cha. IX better as well as make it possible to compare calculations done in the non-chiral setting with the known results obtained in the chiral superspace setting. An immediate gain would be a better understanding of $\mathcal{N} = 4$ SYM on non-chiral space which could help our understanding of other gauge theories which do not allow for chiral reductions. As a maybe far-fetched example, we'd like to name ABJM, which—as a three-dimensional theory—does not allow for a chiral formulation of its superspace. The solution to this problem is strongly connected to finding an appropriate formulation of $\mathcal{N} = 4$ on ambitwistor space, so solving one problem might amount to solving the other problem too.

More ambitious areas of research are connected to the Yangian symmetry algebra: Is there a geometrical understanding of the Yangian? Can we formulate planar, classical $\mathcal{N} = 4$ SYM such that Yangian invariance becomes manifest? At present these endeavors seem unrealistic, but maybe the way towards such a formulation bears new insights.

6

Appendices

APPENDIX A

Conventions

We use the antisymmetric symbols of second order (ϵ-tensors for shortness) $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ to raise and lower indices

$$\epsilon_{\alpha\beta}A^{\beta} = A_{\alpha}, \quad \epsilon^{\alpha\beta}A_{\beta} = A^{\alpha}, \quad \epsilon_{\dot{\alpha}\dot{\beta}}B^{\dot{\beta}} = B_{\dot{\alpha}}, \quad \epsilon^{\dot{\alpha}\dot{\beta}}B_{\dot{\beta}} = B^{\dot{\alpha}} \tag{A.1}$$

such that

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}, \qquad \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\alpha}}_{\dot{\gamma}}. \tag{A.2}$$

The extended Pauli-matrices $(\sigma^{\mu})_{\alpha\dot{\alpha}} = (\mathbf{1}, \sigma^{i})_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}_{\mu})^{\dot{\alpha}\alpha} = (\mathbf{1}, -\sigma^{i})^{\dot{\alpha}\alpha}$ satisfy the following relations

$$\sigma^{\mu}\bar{\sigma}^{\nu} = \eta^{\mu\nu}\mathbf{1} - i\sigma^{\mu\nu} \qquad\qquad \bar{\sigma}^{\mu}\sigma^{\nu} = \eta^{\mu\nu}\mathbf{1} - i\bar{\sigma}^{\mu\nu} \tag{A.3}$$

where $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ and $\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu}$ (both traceless in the spinor indices). We choose the normalization of σ and $\bar{\sigma}$ such that

$$\sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\mu\beta\dot{\beta}} = \eta^{\mu\nu}\sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\beta\dot{\beta}}_{\nu} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}, \quad \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\beta\dot{\beta}}_{\nu} = 2\eta_{\mu\nu}. \tag{A.4}$$

This implies

$$\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}x^{\beta\dot{\beta}} = 2\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{A.5}$$

The spinor products are taken with the help of ϵ

$$\langle\lambda, \mu\rangle = \epsilon_{\alpha\beta}\lambda^{\alpha}\mu^{\beta}, \qquad [\bar{\lambda}, \bar{\mu}] = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\alpha}}\bar{\mu}^{\dot{\beta}}. \tag{A.6}$$

The index convention is as follows (and mostly followed):

- Greek indices from the beginning of the alphabet $\alpha, \beta, \ldots$ have ranges $1, 2$ and are Lorentz spinor indices. The same goes for their dotted counterparts.
- Latin indices from the beginning of the alphabet $a, b, \ldots$ mark $SU(\mathcal{N})$ R-symmetry indices and will run from $1, \ldots, \mathcal{N}$.
- Uppercase Latin indices from the beginning of the alphabet $A, B, \ldots$ are conventionally used as multi-indices for bosonic twistors $A = (\alpha, \dot{\alpha})$.
- Calligraphic uppercase Latin indices from the beginning of the alphabet $\mathcal{A}, \mathcal{B}, \ldots$ mark super-twistor indices encompassing uppercase Latin and lower case Latin indices: $\mathcal{A} = (\alpha, \dot{\alpha}, a)$.
- Uppercase Latin indices from the middle of the alphabet are used for everything else.

APPENDIX B

Coset spaces

Let G be some matrix Lie group and $H \subset G$ a Lie subgroup. We denote the Lie algebra of H and G by $\mathfrak{h}$ and $\mathfrak{g}$ respectively. Elements of the basis of $\mathfrak{h}$ will be denoted by $\mathfrak{V}_i$. A basis of $\mathfrak{g}$ is then given by the basis of $\mathfrak{h}$ and the generators $\mathfrak{X}_\alpha$.

Let $s(u) \in H\backslash G$ be a right coset of G, then we can parametrize the coset using coordinates u^α

$$s(u) = \exp(u^\alpha \mathfrak{X}_\alpha). \tag{B.1}$$

Under (right) multiplication with a group element g, such an coset element transforms like

$$s(u).g = h(g,u).s(u'(u,g)) \tag{B.2}$$

where $h(g,u) \in H$. Since $\Omega = ds(u).s(u)^{-1} \in \mathfrak{g}$ we can write

$$\Omega = ds(u).s(u)^{-1} = e^\alpha \mathfrak{X}_\alpha + \omega^i \mathfrak{V}_i. \tag{B.3}$$

Under the action of G this element of the Lie algebra behaves like

$$\begin{aligned}\Omega \mapsto d(h.s(u'))(h.s(u'))^{-1} &= h\Omega' h^{-1} + dh.h^{-1}\\ &= h(e^\alpha \mathfrak{X}_\alpha + \omega^i \mathfrak{V}_i)h^{-1} + dh.h^{-1}\\ &= e'^\alpha \mathfrak{X}_\alpha + \omega'^i \mathfrak{V}_i.\end{aligned} \tag{B.4}$$

so we see that

$$e'^\alpha \mathfrak{X}_\alpha = h e^\alpha \mathfrak{X}_\alpha h^{-1} \tag{B.5}$$

$$\omega'^i \mathfrak{V}_i = h\omega'^i \mathfrak{V}_i h^{-1} + dh.h^{-1} \tag{B.6}$$

thus we are led to the conclusion that the e^α transform like vielbeine while the ω transform like connections. By calculating the exterior derivative of these objects we can effectively calculate the torsion and the curvature of the coset. Clearly, we have $d\Omega = -\Omega \wedge \Omega$ and so

$$de^\alpha \mathfrak{X}_\alpha + d\omega^i \mathfrak{V}_i = -\frac{1}{2} e^\alpha \wedge e^\beta [\mathfrak{X}_\alpha, \mathfrak{X}_\beta] - \frac{1}{2}\omega^i \wedge \omega^j [\mathfrak{V}_i, \mathfrak{V}_j] - e^\alpha \wedge \omega^i [\mathfrak{X}_\alpha, \mathfrak{V}_i]. \tag{B.7}$$

Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, we have that

$$[\mathfrak{V}_i, \mathfrak{V}_j] = f_{ij}{}^k \mathfrak{V}_k \tag{B.8a}$$

$$[\mathfrak{V}_i, \mathfrak{X}_\beta] = f_{i\beta}{}^\alpha \mathfrak{X}_\alpha \tag{B.8b}$$

$$[\mathfrak{X}_\alpha, \mathfrak{X}_\beta] = f_{\alpha\beta}{}^\gamma \mathfrak{X}_\gamma. \tag{B.8c}$$

With these commutators we may identify specific terms in (B.7)

$$de^\alpha + \epsilon^\beta \wedge \omega^i f_{\beta i}{}^\alpha = -\frac{1}{2} e^\beta \wedge e^\gamma f_{\beta\gamma}{}^\alpha \tag{B.9}$$

$$d\omega^i + \frac{1}{2}\omega^j \wedge \omega^k f_{jk}{}^i = -\frac{1}{2} e^\alpha \wedge e^\beta f_{\alpha\beta}{}^i. \tag{B.10}$$

We recognize the second identity to be of the form of the *Cartan structural equations* from Cartan's moving frame method, thus it describes the curvature of the manifold in question—see e.g. [127]— while the first identity measures the torsion of the manifold.

Functions on the coset $F(u) = F(s(u))$ transform under $g \in G$ by

$$F(u') = F(s(u).g) = \rho(h(u,g)).F(u) \tag{B.11}$$

where ρ is a representation of H under which F transforms. The covariant derivative acting on such a function is then given by

$$DF(u) = (d + \omega^i \rho(X_i))F(u) \tag{B.12}$$

which can be easily checked to transform in the right way under G transformations. Finally, we can decompose the covariant derivative D on the vielbeine e^α,

$$D = e^\alpha D_\alpha \tag{B.13}$$

which defines the components D_α of the covariant derivative.

Using the vielbeine, we can also define the invariant integration measure on the coset by denoting it as a top form

$$d\mu = e^1 \wedge \ldots \wedge e^n = d^n u \det e \tag{B.14}$$

when we let $e^\alpha = e^\alpha{}_\beta du^\alpha$ with $e^\alpha{}_\beta$ being the vielbein matrix.

APPENDIX C

Cauchy-Riemann/Complex-Real structures

In this appendix, we provide some background material and relevant examples for CR structures. We follow the definitions in [128].

As we have pointed out, some spaces like twistors space come equipped with a complex structure with Dolbeault derivatives $\bar{\partial}$ and ∂. A generalization of this concept are CR structures, which we will introduce here.

C.1 CR structures – Definition

A CR structure is built

Let us denote an abstract $\mathbb{C}^{\infty}$ (i.e. analytic) manifold by $\mathcal{M}$ and $T^{\mathbb{C}}\mathcal{M}$ denotes the complexified tangent bundle i.e., at $p \in \mathcal{M}$

$$T^{\mathbb{C}}\mathcal{M}|_p = T_p\mathcal{M} \otimes \mathbb{C}. \tag{C.1}$$

A CR structure—for complex-real or Cauchy-Riemann—is a manifold $\mathcal{M}$ with a subbundle $\mathbf{L} \subset \mathbf{T}^{\mathbb{C}}\mathcal{M}$ such that

1. The intersection of $\mathbf{L}$ and its conjugate $\overline{\mathbf{L}}$ is $\mathbf{L}|_p \cap \overline{\mathbf{L}}|_p = \{0\}$ at every point $p \in \mathcal{M}$.

2. $\mathbf{L}$ is involutive i.e., for two vector fields $L_1, L_2 \in \mathbf{L}$ we find that the commutator $[L_1, L_2] \in \mathbf{L}$.

We say $\mathbf{L}$ forms an *integrable distribution*. Let $\dim(T^{\mathbb{C}}\mathcal{M}) = r = 2n + m$ where we call

$$\dim\left(\frac{T^{\mathbb{C}}\mathcal{M}}{\mathbf{L} \oplus \bar{\mathbf{L}}}\right) = m \tag{C.2}$$

the CR codimension. It is possible to[1] assume that $\mathcal{M}$ comes equipped with a Hermitian metric with which we choose $\mathbf{L} \perp \bar{\mathbf{L}}$. We call

$$\frac{T^{\mathbb{C}}\mathcal{M}}{\mathbf{L} \oplus \bar{\mathbf{L}}} = X(\mathcal{M}) \tag{C.3}$$

[1] And we will do so.

the (totally) real part of $T^{\mathbb{C}}\mathcal{M}$ and choose it orthogonal to $\mathbf{L} \oplus \bar{\mathbf{L}}$ using the Hermitian metric on $\mathcal{M}$. With this, finally, we define two new subbundles $T^{1,0}\mathcal{M}$ and $T^{(0,1)}\mathcal{M}$ such that $T^{\mathbb{C}}\mathcal{M} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}$ where

$$T^{(0,1)}\mathcal{M} = \bar{\mathbf{L}}, \qquad T^{(1,0)}\mathcal{M} = \mathbf{L} \oplus X(\mathcal{M}). \tag{C.4}$$

To these subbundles of $T^{\mathbb{C}}\mathcal{M}$ we can associate duals $T^{*(1,0)}\mathcal{M}$ and $T^{*(0,1)}\mathcal{M}$ such that forms in $T^{*(1,0)}$ destroy vector fields in $T^{(0,1)}$ and vice versa.

Having defined one-forms, it is now easy to define the bundle of (p, q) forms over $\mathcal{M}$ via

$$\Lambda^{p,q}T^*\mathcal{M} = \Lambda^p T^{*(1,0)}\mathcal{M} \otimes \Lambda^q T^{*(0,1)}\mathcal{M} \tag{C.5}$$

with the understanding that $\Lambda^{p,q}T^*\mathcal{M} = 0$ if either $p > n + m$ or $q > n$.

On the space of smooth sections $\Omega^{p,q}(\mathcal{M})$ of $\Lambda^{p,q}\mathcal{M}$ there is an operator $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ which is called the *tangential Cauchy-Riemann operator*. It is this operator we are interested in. The CR operator $\bar{\partial}$ satisfies

$$\bar{\partial}^2 = 0. \tag{C.6}$$

Given the (partial) complex structure on $\mathcal{M}$ there is also an operator $\partial = (\bar{\partial})^\dagger$ which satisfies $\partial^2 = 0$. However, in general we will find that

$$\partial\bar{\partial} + \bar{\partial}\partial \neq 0. \tag{C.7}$$

Denoting the space of r-forms on $\mathcal{M}$ by $\Lambda^r T^{*\mathbb{C}}\mathcal{M}$, there exist natural projections $\pi^{p,q} : \Lambda^r T^{*\mathbb{C}}\mathcal{M} \to \Lambda^{p,q}T^*\mathcal{M}$ such that it is possible to express the CR operator $\bar{\partial}$ in terms of the exterior derivative d via

$$\bar{\partial} = \pi^{p,q+1} \circ d. \tag{C.8}$$

It is in this way that we will find the relevant $\bar{\partial}$ operators from the exterior derivatives d on flag manifolds constructed via harmonic descriptions introduced in IV. Let us give two examples

§ C.1.1. The operator $\bar{\partial}$ on twistor space $\mathbb{CP}^N$.—In IV.1.1 we exposed the form of the exterior derivative in (IV.11). We see that $D^{(N+1)\alpha}$ provides a basis for a subbundle $\mathbf{L}$ since

$$[D^{(N+1)\alpha}, D^{(N+1)\beta}] = 0 \tag{C.9}$$

while $\bar{\mathbf{L}}$ gets formed by $D_\alpha^{(-N-1)}$. Clearly

$$\mathbf{L} \cap \bar{\mathbf{L}} = \{0\}, \qquad X(\mathbb{CP}^N) = \{0\}. \tag{C.10}$$

So unsurprisingly, we find that $\mathbb{CP}^N$ is a family of complex manifolds. The CR operator is then defined by

$$\bar{\partial}_{\mathbb{CP}^3} = \pi^{0,1} \circ D = e_\alpha^{(-N-1)} D^{(N+1)\alpha}. \tag{C.11}$$

§ C.1.2. The operator $\bar{\partial}$ on ambitwistor space $\mathbb{CP}^N$.—In ambitwistor spaces, we have the possibility to choose multiple (different) CR structures. Since

$$[D_\alpha^{(N+1,-1)}, D^{(-1,N+1)\beta}] = \delta_\alpha^\beta D^{(N,N)} \tag{C.12}$$

we can choose $D^{(N,N)}$ and any subset of the D^α and D_β to form $\bar{\mathbf{L}}$ with. Note that in the case $N = 1$, there is *only one* choice. In the four-dimensional case—when $N = 2$—and in all higher cases, the easiest choice of basis for $\mathbf{L}$ is given by a different subset. While we can choose the ambitwistor quadric to be a complex space with a five complex-dimensional distribution[2], it turns out to be more interesting to choose the three-dimensional subset [92]

$$\{D^{(-1,3)1}, D_2^{(3,-1)}, D^{(2,2)}\} \tag{C.13}$$

thus defining a CR derivative by

$$\bar{\partial} = e_1^{(1,-3)} D^{(-1,3)1} + e^{(3,-1)2} D_2^{(3,-1)} + e^{(2,2)} D^{(2,2)}. \tag{C.14}$$

This subset has no torsion and is therefore the easiest choice of a three-dimensional distribution. We will be revisiting this possibility when we will talk about the reformulation of Yang-Mills theories on four-dimensional Minkowski space as complex-real Chern-Simons theories in chapter VI.

[2]Choosing such a five-dimensional distribution however would require us to produce a five-dimensional Chern-Simons theory as gauge theory on this space. For reasons of physicality, we are not—or at least not at this point—interested in writing down such a theory.

APPENDIX D

Propagators

D.1 General discussion

Flat Minkowski spacetime does not, as in the case of Euclidean spacetime, provide us with a unique inverse to the d'Alembertian operator $\Box$, e.g. a unique solution for $G(x, x')$ in the equation

$$\Box_x G(x, x') = -\delta^4(x - x'). \tag{D.1}$$

For this reason propagators G are only defined up to addition of distributional terms. Depending on the choice of distribution, these propagators come with varying names like Feynman, Whitman or Hadamard propagator, retarded or advanced propagator or off- and on-shell propagator. In this appendix we want to sort these different propagators out, give identities between the various propagators and discuss some of their physical interpretations using the most ubiquitous example of massless scalar field theory. Some of this discussion follows [129].

When canonically quantizing the (massless) scalar field, two Green's functions are generally encountered, the Pauli-Jordan function

$$iG_S(x, x') = \langle 0|[\phi(x), \phi(x')]|0\rangle = -\frac{i}{2\pi}\,\mathrm{sign}(u_0)\delta(u^2), \qquad u = x - x' \tag{D.2}$$

and the Hadamard elementary function

$$G_H(x, x') = \langle 0|\{\phi(x), \phi(x')\}|0\rangle = -\frac{1}{2\pi^2 u^2}. \tag{D.3}$$

In the equations above we introduced the usual *no particle* or *vacuum* state $|0\rangle$. Both functions can be written in terms of the Wightman or "cut" functions $G^{\pm}$

$$G^+(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle = -\frac{1}{4\pi^2 u^2} - \frac{i}{4\pi}\,\mathrm{sign}(u_0)\delta(u^2) \tag{D.4}$$

$$G^-(x, x') = \langle 0|\phi(x')\phi(x)|0\rangle = -\frac{1}{4\pi^2 u^2} + \frac{i}{4\pi}\,\mathrm{sign}(u_0)\delta(u^2) \tag{D.5}$$

The functions G_H, G_S and $G^{\pm}$ solve the homogeneous Klein-Gordon equation

$$\Box_x G^{\bullet}(x, x') = 0. \tag{D.6}$$

To accommodate sources for the scalar fields we require solutions to the equation (D.1). Such solutions are given in terms of the Feynman propagator

$$iG_F(x,x') = \Theta(u_0)G^+ + \Theta(-u_0)G^- = -\frac{1}{4\pi^2 u^2} - \frac{i}{4\pi}\delta(u^2) \tag{D.7}$$

or the advanced and retarded propagators

$$G_A(x,x') = \Theta(-u_0)G_S(x,x'), \qquad G_R(x,x') = -\Theta(u_0)G_S(x,x') \tag{D.8}$$

which solve

$$\Box_x G_{A,R} = \delta^4(x-x'). \tag{D.9}$$

Notice that G_A and G_R have well defined support within the light-cone. While G_R is supported on the positive light-cone of the point x' (taking x to be variable) G_A is defined on the negative light-cone. As such they are solutions to a well-posed Cauchy problem. In contrast, the Green functions $G^{\pm}$, G_F and G_H are supported on all of space-time (excluding $u=0$). Furthermore, the definition of $G^{\pm}$ and therefore G_F is dependent on the decomposition of the solutions into positive and negative frequency parts.

There is however, a nagging problem with many of these functions. Upon transformation to momentum space, many[1] of these functions become singular when going on-shell $p^2 = 0$. This can be seen directly from their Fourier transformations, i.e. from

$$G^+(p) = \int d^4x \exp(iu.p)G^+(u) = 2\pi\Theta(p_0)\delta(p^2) \tag{D.10}$$

and

$$G_F(p) = \int d^4x \exp(iu.p)G_F(u) = \frac{1}{p^2 + i\epsilon} \tag{D.11}$$

where the $i\epsilon$ is the usual $i\epsilon$-prescription by Feynman to calculate the Fourier integral. However, we see, that the supports of these distributions are quite different. While the Feynman propagator has support everywhere excluding $p^2 = 0$, G^+ has support exclusively for $p^2 = 0$. It is in this sense that one can say the propagator G^+ is "on-shell". This property has been made use of in [1] and the definitions of propagators in the text in IX.

D.2 The gauge field propagator in light-cone gauge

For a textbook treatment of the following see e.g. [129]. We will assume an axial gauge

$$\ell.A = 0 \tag{D.12}$$

for the field A_μ in ordinary Yang-Mills theory. The vector ℓ does not satisfy any special properties at this point. Using the standard way to introduce a gauge fixing term into the action we find the defining equation for the Green function to be

$$\left(\delta^\mu_\nu \Box - \partial^\mu\partial_\nu - \frac{1}{\alpha}\ell^\mu\ell_\nu\right) G^{\nu\sigma}(u = x - x') = \eta^{\mu\sigma}\delta^4(u). \tag{D.13}$$

[1] To be precise, all.

Fourier transformation (as above we assume a weightless transformation) to momentum space and subsequent inversion of the resulting operator gives us the answer

$$\tilde{G}^{\mu\nu}_{\text{ax}}(p,q) = \frac{-i}{p^2}\delta^4(p+q)\left[\eta^{\mu\nu} - \frac{p^\mu\ell^\nu + p^\nu\ell^\mu}{\ell.p} + \frac{\alpha p^2 + \ell^2}{(p.\ell)^2}p^\mu p^\nu\right]. \tag{D.14}$$

When assuming a light-cone gauge $\ell^2 = 0$ one usually takes the limit $\alpha \to 0$ such that the last term vanishes leaving us with

$$\tilde{G}^{\mu\nu}_{\text{lc}}(p,q) = \frac{-i}{p^2}\delta^4(p+q)\left[\eta^{\mu\nu} - \frac{p^\mu\ell^\nu + p^\nu\ell^\mu}{\ell.p}\right]. \tag{D.15}$$

The p^{-2} pole can be dealt with in the usual way by adding a $i\epsilon$ to it and maintaining the strong intend of taking the limit $\epsilon \to 0$ in the end of the calculation. The way [130, 38] to treat the pole in $p.\ell$ is to add *another* $i\epsilon'$ prescription

$$\frac{1}{p.\ell} \to \frac{\ell_0 p_0 + \ell_i p_i}{\ell_0^2 p_0^2 - (\ell_i p_i)^2 + i\epsilon'}. \tag{D.16}$$

From this we can deduce that we can freely add higher terms in $(p.\ell)^{-1}$ (subject to the conditions that $\ell_\mu \tilde{G}^{\mu\nu} = 0$ and $p_\mu \tilde{G}^{\mu\nu} = 0$) as they have no influence in a Fourier transformation back to space-time (i.e. they are higher poles in a Laurent series not adding any new residues). The additional poles can be traced back to the fact that light-cone gauge fixes the gauge freedom only partially.

We use this freedom and add a term to the propagator

$$\tilde{G}^{\mu\nu}_{\text{lc}}(p,q) = \frac{-i}{p^2}\delta^4(p+q)\left[\eta^{\mu\nu} - \frac{p^\mu\ell^\nu + p^\nu\ell^\mu}{\ell.p} + \frac{\ell^\mu\ell^\nu}{(p.\ell)^2}p^2\right]. \tag{D.17}$$

This is the propagator we will employ in the rest of this appendix.

D.3 Hertz Potentials

The gauge redundancy of the connection $A = dx^\mu A_\mu$ of an Abelian gauge theory is given by

$$\delta_g A = d\alpha \tag{D.18}$$

where α is a zero-form, i.e. a function. Upon assuming the Lorentz gauge condition $\delta A = 0$ we notice that $A = \delta H$ is a solution to this gauge because $\delta^2 = 0$. Here δ is the codifferential, e.g.

$$\delta = (-1)^{k-1} \star d\star \tag{D.19}$$

on k-forms in Minkowski signature in four dimensions. $\star$ is the Hodge star operator. Since H is a two-form (in coordinates $H = \frac{1}{2}H_{\mu\nu}dx^\mu \wedge dx^\nu$) it has a further gauge redundancy

$$\delta_g H = \delta K \tag{D.20}$$

where K is a three-form. Note that this gauge symmetry of H has no influence on A. Finally, this three-form has a further gauge redundancy

$$\delta_g K = \delta L. \tag{D.21}$$

In four dimensions L is proportional to the volume form, so $\delta_g L = 0$. While the Bianchi identities $dF = 0$ are trivially satisfied by $A = \delta H$, the (linearized) Yang-Mills equations imply that

$$\Box H = (d\delta + \delta d)H = 0 \quad \text{mod co-closed two forms.} \tag{D.22}$$

Thus every component of H satisfies the massless KG wave equation on-shell. This makes H an ideal candidate for a Penrose transformation to twistor space.

Going back to a more pedestrian notation, we see that as H is a two form in four dimensions, we may split it into its self-dual and anti-self-dual part, i.e.

$$\sigma^{\mu}_{\dot{\alpha}\alpha}\sigma^{\nu}_{\dot{\beta}\beta}H_{\mu\nu} = \epsilon_{\dot{\alpha}\dot{\beta}}H_{\alpha\beta} - \epsilon_{\alpha\beta}\bar{H}_{\dot{\alpha}\dot{\beta}}. \tag{D.23}$$

Some algebra reveals that

$$A_{\dot{\alpha}\alpha} = \frac{1}{2}\partial_{\dot{\alpha}\beta}H^{\beta}{}_{\alpha} - \frac{1}{2}\partial_{\dot{\gamma}\alpha}\bar{H}^{\dot{\gamma}}{}_{\dot{\alpha}} \tag{D.24}$$

which resembles the solution (IX.2) in the supersymmetric case. An axial gauge

$$\imath_\ell A = 0 \quad \Rightarrow \quad \imath_\ell H = 0 \tag{D.25}$$

with $\ell^2 = 0$ is equivalent to the light-cone gauge as presented in the text, if we solve $\ell^2 = 0$ by $\ell^{\alpha\dot{\alpha}} = \ell^{\alpha}\bar{\ell}^{\dot{\alpha}}$.

We cannot solve the potential in this way when working with non-Abelian theories. The obstruction is that the exterior derivative d needs to be replaced everywhere by d^{∇}, the covariant version of d, when acting on gauge-group G valued forms. But $(d^{\nabla})^2$ is proportional to the field strength tensor $F = dA + A\wedge A$ so the closest we could get to a solution using a Hertz potential would be an infinite expansion in commutators.

D.4 Propagator for Hertz potential

From (D.24) and $\langle A_\mu A_\nu\rangle = p^\rho q^\sigma \langle H_{\rho\mu}(p)H_{\sigma\nu}(q)\rangle$ follows that

$$\langle H_{\rho\mu}(p)H_{\sigma\nu}(q)\rangle_{\text{lc}} = \delta^4(p+q)\frac{-i(\eta_{\mu\nu}\ell_\rho\ell_\sigma - \eta_{\rho\nu}\ell_\mu\ell_\sigma - \eta_{\mu\sigma}\ell_\rho\ell_\nu + \eta_{\rho\sigma}\ell_\mu\ell_\nu)}{p^2(p.\ell)} \tag{D.26}$$

which can be transformed to spinor language where

$$\langle H_{\alpha\beta}H_{\gamma\delta}\rangle = \langle \bar{H}_{\dot{\alpha}\dot{\beta}}\bar{H}_{\dot{\gamma}\dot{\delta}}\rangle = 0 \tag{D.27}$$

and

$$\langle H_{\alpha\beta}(p)\bar{H}_{\dot{\alpha}\dot{\beta}}\rangle = -2i\delta^4(p+q)\frac{\ell_\alpha\ell_\beta\bar{\ell}_{\dot{\alpha}}\bar{\ell}_{\dot{\beta}}}{p^2(p.\ell)^2}. \tag{D.28}$$

As before, the $i\epsilon$ prescription is used to regularize the divergence at $p^2 = 0$. Using this way it is possible to set the Hertz potentials H and $\bar{H}$ on-shell

$$H_{\alpha\beta}(p) = \delta(p^2)\frac{\ell_\alpha\ell_\beta}{\langle\lambda,\ell\rangle^2}H(\lambda,\bar{\lambda}), \qquad \bar{H}_{\dot{\alpha}\dot{\beta}}(p) = \delta(p^2)\frac{\bar{\ell}_{\dot{\alpha}}\bar{\ell}_{\dot{\beta}}}{[\bar{\lambda},\bar{\ell}]^2}\bar{H}(\lambda,\bar{\lambda}) \tag{D.29}$$

for the light-like momentum $p_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$. These fields are the lowest order components (up to a factor $\frac{1}{2}$) of the half–Fourier transform of the fields given in (IX.12a). In the non-supersymmetric case only the expectation value of H with $\bar{H}$ can yield a non-zero result. We set the propagator on-shell by replacing the Feynman prescription by $-i\pi\delta(p^2)$ and find[2]

$$\delta(q^2)\langle H(\lambda,\bar{\lambda})\bar{H}(\lambda',\bar{\lambda}')\rangle\frac{[\bar{\lambda},\bar{\ell}]^2}{[\bar{\lambda}',\bar{\ell}]^2} = -8\pi\delta^4(p+q) \tag{D.30}$$

and rewrite

$$\delta^4(p+q) = \frac{1}{4}\delta(q^2)\left|\frac{\lambda_1}{\lambda'_1}\right|^2\int\frac{ds}{s}\delta^2(\lambda - s\lambda')\delta^2(s\bar{\lambda}+\bar{\lambda}') \tag{D.31}$$

where we used (again) that $\delta(q^2)$ imposes $q = \lambda'\bar{\lambda}'$ and introduced a projective integral.

With this the correlation function $\langle H\bar{H}\rangle$ can be written as

$$\langle H(\lambda,\bar{\lambda})\bar{H}(\lambda',\bar{\lambda}')\rangle = -2\pi\int ds\; s^3\delta^2(\lambda - s\lambda')\delta^2(s\bar{\lambda}+\bar{\lambda}'). \tag{D.32}$$

In the bosonic theory described here, we notice that the exponent of s is in fact three, while for the supersymmetric case we find the exponent -1. This is an effect of the absence of the fermionic δ-function.

D.5 Ambitwistor propagators

There is another curious fact about the propagator from the section above that we want to point out: There are different choices to do half–Fourier transformations of the correlation function $\langle H(\lambda,\bar{\lambda})\bar{H}(\lambda',\bar{\lambda}')\rangle$.

In this appendix we will review the proposal for a set of ambitwistor propagators. Such objects have been proposed in [1], but they haven't been used for any of the calculations therein. We want to give a slightly different proposal for a set of propagators which is based on the following considerations[3].

Since the mixed correlator $\langle C(\mathcal{Z})C(\mathcal{W}')\rangle$ is meant to calculate a one loop result we postulate that the correct propagator should be of the form

$$\langle C(\mathcal{Z})\bar{C}(\mathcal{W}')\rangle = -\frac{1}{16\pi^2}\int\frac{ds}{s}\frac{dt}{t}\frac{du}{u}\;\exp\Big(s\mathcal{Z}.\mathcal{W}' + t\mathcal{Z}.\mathcal{W}_\star + u\mathcal{Z}_\star.\mathcal{W}'\Big) \tag{D.33}$$

where the integrations over s, t, and u should be treated as contour integrals defining Heaviside-functions via an $i\epsilon$ regularization[4]. Explicitly we have

$$\Theta(x) = \oint_C \frac{dt}{t - i\epsilon}\;\exp(ixt) \tag{D.34}$$

with a contour C circling the origin, such that the above propagator is in fact a product of three Heaviside step functions[5]

$$\langle C(\mathcal{Z})\bar{C}(\mathcal{W}')\rangle = -\frac{1}{16\pi^2}\Theta(\mathcal{Z}.\mathcal{W}')\Theta(\mathcal{Z}.\mathcal{W}_\star)\Theta(\mathcal{Z}_\star.\mathcal{W}'). \tag{D.35}$$

---

[2]Use $2p.l = \langle\lambda,\ell\rangle[\bar{\ell},\bar{\lambda}]$.

[3]I am thankful for Simon Caron-Huot to point out this particular form for the mixed propagator.

[4]Such an $i\epsilon$-prescription also stops us from rendering the integral trivial by a redefinition of integration variables.

[5]Remember that there are hidden i's in the ambitwistor scalar products (IX.32).

It is then possible to calculate the correlator $\langle B_{i-1,i}\bar{B}_{j-1,j}\rangle$ in an ambitwistor setting by integrating over two projective lines $\mathcal{Z}_s = \mathcal{Z}_{i-1} + s\mathcal{Z}_i$ and $\mathcal{W}_t = \mathcal{W}_{j-1} + t\mathcal{W}_j$, i.e. from

$$B_{i-1,i} = \frac{1}{8\pi^2}\int \frac{ds}{s} C(\mathcal{Z}_{i-1} + s\mathcal{Z}_i), \qquad \bar{B}_{i-1,i} = -\frac{1}{8\pi^2}\int \frac{ds}{s}\bar{C}(\mathcal{W}_{i-1} + s\mathcal{W}_i) \tag{D.36}$$

thus

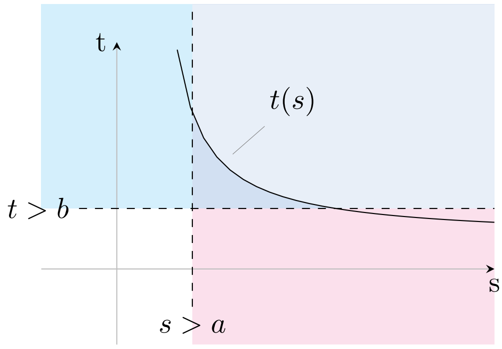
$$\langle B_{i-1,i}\bar{B}_{j-1,j}\rangle = \frac{1}{1024\pi^6}\int \frac{ds}{s}\frac{dt}{t}\Theta(\mathcal{Z}_s.\mathcal{W}_t)\Theta(\mathcal{Z}_s.\mathcal{W}_\star)\Theta(\mathcal{Z}_\star.\mathcal{W}_t). \tag{D.37}$$

Formally, this integral can be evaluated explicitly in terms of dilogarithms. To do so, we must evaluate the restrictions imposed by the Heaviside functions. Inspecting from the right to the left, they impose the domain of integration to be bounded by a curve

$$\langle i-1, j-1] + s\langle i, j-1] + t\langle i-1, j] + st\langle i, j] = 0 \tag{D.38}$$

and two projective lines

$$\langle i-1, \star] + s\langle i, \star] = 0, \qquad \langle \star, j-1] + t\langle \star, j] = 0. \tag{D.39}$$



The diagram to the right shows the integration area enclosed by the hyperbola and the two line as a triangular region shaded gray. Thus if we take the integrations over t and s to be real contours[6] the integrals can be evaluated to[7]

$$\begin{aligned}\langle B_{i-1,i}\bar{B}_{j-1,j}\rangle \propto &-\mathrm{Li}_2\left(\frac{\langle i,j]\langle i-1,j-1]}{\langle i,j-1]\langle i-1,j]}\right) + \mathrm{Li}_2\left(\frac{\langle \star,j-1]\langle i,j]}{\langle \star,j]\langle i,j-1]}\right)\\ &-\mathrm{Li}_2\left(\frac{\langle \star,j-1]\langle i-1,j]}{\langle \star,j]\langle i-1,j-1]}\right) - \mathrm{Li}_2\left(\frac{\langle i-1,\star]\langle i,j-1]}{\langle i,\star]\langle i-1,j-1]}\right) + \mathrm{Li}_2\left(\frac{\langle i-1,\star]\langle i,j]}{\langle i,\star]\langle i-1,j]}\right)\\ &+ \text{products of logarithms}\end{aligned} \tag{D.40}$$

When we sum over i and j we get the same result for the non-chiral correlator as obtained in [1] which shows that the two reference twistors $\mathcal{Z}_\star$ and $\mathcal{W}_\star$ are gauge artifacts very much like in the chiral case [101].

Finally, we can also derive the form of the chiral propagators from this proposal using (IX.13). To do so we write

$$\begin{aligned}-16\pi^2\langle C(\mathcal{Z})C(\mathcal{Z}')\rangle &= \int d^{4|4}\mathcal{W}'\ \exp\left(-2\mathcal{Z}'.\mathcal{W}'\right)\langle C(\mathcal{Z})\bar{C}(\mathcal{W}')\rangle\\ &= \int \frac{ds}{s}\frac{dt}{t}\frac{du}{u}\int d^{4|4}\mathcal{W}'\ \exp([s\mathcal{Z}. + u\mathcal{Z}_\star - 2\mathcal{Z}'].\mathcal{W}' + t\mathcal{Z}.\mathcal{W}_\star)\\ &= \int \frac{dt}{t}\ \exp(t\mathcal{Z}.\mathcal{W}_\star)\int \frac{ds}{s}\frac{du}{u}\ \delta^{4|4}(s\mathcal{Z} + u\mathcal{Z}_\star - 2\mathcal{Z}').\end{aligned} \tag{D.41}$$

The antichiral-antichiral propagator looks similar and can be calculated in like manner. We should note two features of this expression. First of all, this result closely resembles the propagator of chiral twistor theory in axial gauge as proposed in [10]. The argument of the δ-function can be rearranged to reflect the standard form. Secondly, the first integral over the exponential has the interpretation of a Heaviside function closely emulating the role of the Θ function appearing in front of the chiral-chiral propagators in on-shell momentum space proposed in [1].

[6] This, however, is not entirely justified as s ant t are really coordinates on a $\mathbb{CP}^1$, the integration must therefore be understood as a formal operation.

[7] Immediately after the integration, we have four dilogarithms with rational functions of brackets in the arguments. After rearranging the dilogs using the symbol [69, 70, 131] the arguments become cross-ratios.

APPENDIX E

The algebra $\mathfrak{u}(2,2|4)$

E.1 The algebra

The set of generators

$$(\mathfrak{P},\mathfrak{L},\bar{\mathfrak{L}},\mathfrak{K},\mathfrak{R},\mathfrak{D},\mathfrak{C},\mathfrak{B}|\mathfrak{Q},\bar{\mathfrak{Q}},\mathfrak{S},\bar{\mathfrak{S}}) \tag{E.1}$$

of $\mathfrak{u}(2,2|4)$ obeys the following algebra. The generators $\mathfrak{L}^\alpha{}_\beta$ and $\bar{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta}$ form two $su(2)$ subalgebras and act on generators bearing Lorentz indices as

$$[\mathfrak{L}^\alpha{}_\beta,\mathfrak{J}^\gamma]=\delta^\gamma_\beta\mathfrak{J}^\alpha-\frac{1}{2}\delta^\alpha_\beta\mathfrak{J}^\gamma,\qquad [\mathfrak{L}^\alpha{}_\beta,\mathfrak{J}_\gamma]=-\delta^\alpha_\gamma\mathfrak{J}_\beta+\frac{1}{2}\delta^\alpha_\beta\mathfrak{J}^\gamma, \tag{E.2}$$

and similarly for $\bar{\mathfrak{L}}$. The generators of R-symmetry $\mathfrak{R}^a{}_b$ form a $su(4)$ subalgebra and likewise act in the usual way

$$[\mathfrak{R}^a_b,\mathfrak{J}^c]=\delta^c_b\mathfrak{J}^a-\frac{1}{4}\delta^a_b\mathfrak{J}^c,\qquad [\mathfrak{R}^a_b,\mathfrak{J}_c]=-\delta^a_c\mathfrak{J}_b+\frac{1}{4}\delta^a_b\mathfrak{J}^c. \tag{E.3}$$

The generator of dilatations $\mathfrak{D}$ and the hypercharge operator act like

$$[\mathfrak{D},\mathfrak{J}]=\dim(\mathfrak{J})\mathfrak{J},\qquad [\mathfrak{B},\mathfrak{J}]=\mathrm{hyp}(\mathfrak{J})\mathfrak{J}. \tag{E.4}$$

The quantities $\dim(\mathfrak{J})$ and $\mathrm{hyp}(\mathfrak{J})$ are the canonical dimension and the hypercharge of the generator $\mathfrak{J}$ respectively. It is possible to order the generators of $\mathfrak{u}(2,2|4)$ in a lattice according to their dimensions and hypercharge as was done on page 16. As can be clearly seen from the picture, only the generators $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ do have nontrivial hypercharge i.e.,

$$[\mathfrak{B},\mathfrak{Q}^{\alpha a}]=\mathfrak{Q}^{\alpha a},\quad [\mathfrak{B},\bar{\mathfrak{Q}}^{\dot\alpha}_a]=-\bar{\mathfrak{Q}}^{\dot\alpha}_a,\quad [\mathfrak{B},\mathfrak{S}_{\alpha a}]=-\mathfrak{S}_{\alpha a},\quad [\mathfrak{B},\bar{\mathfrak{S}}^a_{\dot\alpha}]=\bar{\mathfrak{S}}^a_{\dot\alpha} \tag{E.5}$$

The rest of the algebra is given by the following non-trivial set of commutation relations

$$\{\mathfrak{Q}^{\alpha a},\bar{\mathfrak{Q}}^{\dot\alpha}_b\}=\delta^a_b\mathfrak{P}^{\alpha\dot\alpha}\qquad\qquad \{\mathfrak{S}_{\alpha a},\bar{\mathfrak{S}}^b_{\dot\alpha}\}=\delta^b_a\mathfrak{K}_{\alpha\dot\alpha} \tag{E.6a}$$

$$[\mathfrak{S}_{\beta a},\mathfrak{P}^{\alpha\dot\alpha}]=\delta^\alpha_\beta\bar{\mathfrak{Q}}^{\dot\alpha}_a\qquad\qquad [\bar{\mathfrak{S}}^a_{\dot\beta},\mathfrak{P}^{\alpha\dot\alpha}]=\delta^{\dot\alpha}_{\dot\beta}\mathfrak{Q}^{\alpha a} \tag{E.6b}$$

$$[\mathfrak{K}_{\beta\dot\alpha},\mathfrak{Q}^{\alpha a}]=\delta^\alpha_\beta\bar{\mathfrak{S}}^a_{\dot\alpha}\qquad\qquad [\mathfrak{K}_{\alpha\dot\beta},\bar{\mathfrak{Q}}^{\dot\alpha}_a]=\delta^{\dot\alpha}_{\dot\beta}\mathfrak{S}_{\alpha a} \tag{E.6c}$$

$$\{\mathfrak{Q}^{\alpha a},\mathfrak{S}_{\beta b}\}=\delta^a_b\mathfrak{L}^\alpha_\beta-\delta^\alpha_\beta\mathfrak{R}^a_b+\frac{1}{2}\delta^\alpha_\beta\delta^a_b\left(\mathfrak{D}+\mathfrak{C}\right) \tag{E.6d}$$

$$\{\bar{\mathfrak{Q}}^{\dot\alpha}_b,\bar{\mathfrak{S}}^a_{\dot\beta}\}=\delta^a_b\bar{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}+\delta^{\dot\alpha}_{\dot\beta}\mathfrak{R}^a_b+\frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\delta^a_b\left(\mathfrak{D}-\mathfrak{C}\right) \tag{E.6e}$$

$$[\mathfrak{K}_{\alpha\dot\alpha},\mathfrak{P}^{\beta\dot\beta}]=\delta^\alpha_\beta\bar{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}+\delta^{\dot\alpha}_{\dot\beta}\mathfrak{L}^\alpha_\beta+\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}\mathfrak{D} \tag{E.6f}$$

Commutators not listed in here are zero. The two generators $\mathfrak{C}$ and $\mathfrak{B}$ take special places. In fact, it is possible to remove both from the algebra $\mathfrak{u}(2,2|4)$ to get the algebra $\mathfrak{psu}(2,2|4)$ i.e.,

$$\mathfrak{u}(2,2|4)\sim\mathfrak{psu}(2,2|4)\oplus\mathfrak{B}\oplus\mathfrak{C}\sim\mathfrak{su}(2,2|4)\oplus\mathfrak{B}\sim\mathfrak{pu}(2,2|4)\oplus\mathfrak{C}. \tag{E.7}$$

E.2 The Killing form

The super-algebra $\mathfrak{u}(2,2|4)$ is a real form of $\mathfrak{sl}(4|4)$ which does not allow for a non-degenerate Killing form (which is tantamount to saying that the dual Coxeter number of this algebra is zero, a fact which is very important in Part 5). This is a special case of a more general phenomenon for all $\mathfrak{sl}(n|n)$ type super-algebras (for a textbook treatment see [132]).

However, it is still possible to define a metric κ using the fundamental representation as outlined in [13]. It is instructive to recapitulate the definition of this metric, as it will become very important for the definition of the Yangian operators. Let the defining representation of $\mathfrak{gl}(n|m)$ be the set of matrices $E^{\mathcal{A}}{}_{\mathcal{B}}$ which are defined to have a 1 in the $\mathcal{A}^{\text{th}}$ row and the $\mathcal{B}^{\text{th}}$ column and zeros everywhere else. Then, for $\mathfrak{J}^a \in (\mathfrak{P}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{K}, \mathfrak{R}|\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{S}, \bar{\mathfrak{S}})$

$$U(\mathfrak{J}^a) = \left(\begin{array}{cc|c} E^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\mathbf{1} & E^{\alpha}{}_{\dot\beta} & E^{\alpha b} \\ E^{\dot\alpha}{}_{\beta} & E^{\dot\alpha}{}_{\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathbf{1} & E^{\dot\alpha b} \\ \hline E_{a\beta} & E_{a\dot\beta} & E_a{}^b - \frac{1}{4}\delta^b_a\mathbf{1} \end{array}\right) \tag{E.8}$$

as well as the three elements $(\mathfrak{D}, \mathfrak{C}, \mathfrak{B})$

$$U(\mathfrak{D}) = \begin{pmatrix} \frac{1}{2}\mathbf{1} & 0 & 0 \\ 0 & -\frac{1}{2}\mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad U(\mathfrak{C}) = \begin{pmatrix} \frac{1}{2}\mathbf{1} & 0 & 0 \\ 0 & \frac{1}{2}\mathbf{1} & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1} \end{pmatrix}, \qquad U(\mathfrak{B}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\mathbf{1} \end{pmatrix} \tag{E.9}$$

Define κ by

$$\kappa(\mathfrak{J}^a, \mathfrak{J}^b) = \operatorname{str}\left[U(\mathfrak{J}^a)U(\mathfrak{J}^b)\right]. \tag{E.10}$$

The definitions above allow for an easy derivation of the entries of this metric. There are only a few results this work is going to be concerned with. It is the entries of κ with respect to $\mathfrak{B}$

$$\kappa(\mathfrak{C}, \mathfrak{B}) = 1, \quad \kappa(\mathfrak{B}, \mathfrak{J}) = 0 \tag{E.11}$$

for all $\mathfrak{J} \neq \mathfrak{C}$.

More generally, the metric κ satisfies the associativity property

$$\kappa([\mathfrak{J}^{\mathfrak{a}}, \mathfrak{J}^b], \mathfrak{J}^c) = \kappa(\mathfrak{J}^{\mathfrak{a}}, [\mathfrak{J}^b, \mathfrak{J}^c]) \tag{E.12}$$

and a $\mathbb{Z}_2$-graded symmetry

$$\kappa(\mathfrak{J}^a, \mathfrak{J}^b) = (-1)^{|ab|}\kappa(\mathfrak{J}^b, \mathfrak{J}^a). \tag{E.13}$$

In the adjoint representation the generators are given by the structure constants

$$[\mathrm{ad}(\mathfrak{J}^{\mathfrak{a}})]^b{}_c = f^{ab}{}_c. \tag{E.14}$$

Given the metric, it is possible to raise and lower indices

$$f_{abc} = \kappa_{ad}\kappa_{be}f^{ed}_c. \tag{E.15}$$

Extensive use of this property will be made when defining the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$.

E.3 Representations of $\mathfrak{psu}(2,2|4)$

As there are many very useful representations of the superconformal algebra that this work will make use of, a short survey of common representations will come in handy at a later stage.

§ E.3.1. Spinor-Helicity.—Witten [11] introduced the following representation of $\mathfrak{psu}(2,2|4)$ on on-shell super-momentum space $(\lambda, \bar{\lambda}, \eta)$

$$\mathfrak{P}^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}, \qquad \mathfrak{Q}^{\alpha a} = \eta^a \lambda^\alpha \tag{E.16a}$$

$$\mathfrak{K}_{\alpha\dot{\alpha}} = \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}}, \qquad \bar{\mathfrak{Q}}^{\dot{\alpha}}{}_a = \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^a} \tag{E.16b}$$

$$\mathfrak{S}_{\alpha a} = \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \eta^a}, \qquad \bar{\mathfrak{S}}^a{}_{\dot{\alpha}} = \eta^a \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} \tag{E.16c}$$

$$\mathfrak{L}^\alpha{}_\beta = \lambda^\alpha \frac{\partial}{\partial \lambda^\beta} - \frac{1}{2} \delta^\alpha_\beta \lambda^\gamma \frac{\partial}{\partial \lambda^\gamma}, \qquad \bar{\mathfrak{L}}^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\beta}}} - \frac{1}{2} \delta^{\dot{\alpha}}_{\dot{\beta}} \bar{\lambda}^{\dot{\gamma}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\gamma}}} \tag{E.16d}$$

$$\mathfrak{R}^a{}_b = \eta^a \frac{\partial}{\partial \eta^b} - \frac{1}{4} \delta^a_b \eta^c \frac{\partial}{\partial \eta^c}, \qquad \mathfrak{B} = \eta^a \frac{\partial}{\partial \eta^a} \tag{E.16e}$$

$$\mathfrak{D} = \frac{1}{2} \lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \frac{1}{2} \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} + 1 \tag{E.16f}$$

$$\mathfrak{C} = 1 + \frac{1}{2} \lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} - \frac{1}{2} \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} - \frac{1}{2} \eta^a \frac{\partial}{\partial \eta^a} \tag{E.16g}$$

§ E.3.2. Twistor and momentum-twistor space.—Super-twistor space $\mathbb{CP}^{3|4}$ is the fundamental representation of $\mathfrak{psu}(2,2|4)$. This makes the derivative operators defining the action on functions of twistors $\mathcal{Z}^\mathcal{A} = (\lambda^\alpha, \mu^{\dot{\alpha}}, \chi^a)$ especially easy. All generators can be written as

$$\mathfrak{J}^\mathcal{A}{}_\mathcal{B} = \mathcal{Z}^\mathcal{A} \frac{\partial}{\partial \mathcal{Z}^\mathcal{B}} \tag{E.17}$$

where the super-trace $(-1)^\mathcal{A} \delta^\mathcal{A}{}_\mathcal{B} \mathfrak{J}^\mathcal{B}\mathcal{A}$ is understood to be removed as this would correspond to the outer automorphism $\mathfrak{B}$ of $\mathfrak{u}(2,2|4)$. The form (E.17) of these generators can also be obtained from the generators in spinor-helicity representation by half–Fourier transform of $\tilde{\lambda}$. However, this requires complex momenta or $(2,2)$ space-time signature. For Minkowski signature, the half–Fourier transform is not well defined.

In momentum twistor space [61] the form of the dual superconformal algebra is exactly identical to (E.17) replacing $\mathcal{Z} \to \mathcal{W}$.

§ E.3.3. Ambitwistors.—Ambitwistor space is the quadric

$$\mathbb{A} = \{(W, \bar{W}) \in \mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4\star} | \mathcal{Z}.\mathcal{W} = \lambda\bar{\mu} + \bar{\lambda}\mu + 4i\chi\bar{\chi} = 0\}. \tag{E.18}$$

The quadric is embedded in the direct product of twistor and conjugate twistor space. The defining constraint $\mathcal{Z}.\mathcal{W} = 0$ must be preserved by any symmetry transformation which leads to the following form of the generators of $\mathfrak{psu}(2,2|4)$

$$\mathfrak{J}^\mathcal{A}{}_\mathcal{B} = \mathcal{Z}^\mathcal{A} \frac{\partial}{\partial \mathcal{Z}^\mathcal{B}} - (-1)^{|\mathcal{A}||\mathcal{B}|+|\mathcal{A}|} \mathcal{W}_\mathcal{B} \frac{\partial}{\partial \mathcal{W}_\mathcal{A}}. \tag{E.19}$$

This form of the generators ensures

$$\mathfrak{J}^\mathcal{A}{}_\mathcal{B}(\mathcal{Z}_i \cdot \mathcal{W}_j = 0) \tag{E.20}$$

for any i and j. It turns out to be convenient to redefine the generators by a factor of $(-1)^\mathcal{A}$. In the Yangian first-level generators, this additional factor eliminates a similar factor from the structure constants.

APPENDIX F

Yangian algebras

F.1 Hopf algebras

Following the introduction to Hopf algebras in [31] a Hopf algebra A is an algebra with unit $\mathbf{1}$ (over a field k) with a *product*

$$\mu : A \otimes A \to A, \qquad \mu(a_1 \otimes a_2) := a_1.a_2 \in A \tag{F.1}$$

for $a_1, a_2 \in A$ together with a *unit*

$$\imath : k \to A, \qquad \imath(\lambda) = \lambda \mathbf{1} \tag{F.2}$$

as well as a coalgebra structure. A coalgebra structure encompasses a *coproduct* $\Delta : A \to A \otimes A$ and a *counit* $\epsilon : A \to k$ compatible with the algebra structure. Furthermore, there is the *antipodal map* $S : A \to A$ which is bijective and satisfies

$$\mu(S \otimes \mathrm{id})\Delta = i \otimes \epsilon = \mu(\mathrm{id} \otimes S)\Delta \tag{F.3}$$

or as a commuting diagram

$$\begin{tikzcd}
A \otimes A \arrow[d, "\mathrm{id} \otimes S"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\imath \otimes \epsilon"] & A \otimes A \arrow[d, "S \otimes \mathrm{id}"'] \\
A \otimes A \arrow[r, "\mu"'] & A & A \otimes A \arrow[l, "\mu"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathrm{id}"] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \Delta"] & A \arrow[l, "\mathrm{id}"] \arrow[u, "\Delta"']
\end{tikzcd}$$

The coproduct satisfies the coassociativity relation

$$(\mathrm{id} \circ \Delta)\Delta = (\Delta \circ \mathrm{id})\Delta. \tag{F.4}$$

A coalgebra is called cocommutative if $\Delta(A) \subset \mathrm{Sym}(A \otimes A)$, the symmetric part of $A \otimes A$.

F.2 Universal enveloping algebra $U(\mathfrak{g})$

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a cocommutative Hopf algebra with the following properties. Let $\mathfrak{J} \in \mathfrak{g}$. The Lie structure is generated by the product $\mu : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$, i.e. if $\mathfrak{J}^a, \mathfrak{J}^b \in \mathfrak{g}$, then

$$[\mathfrak{J}^a, \mathfrak{J}^b] = \mathfrak{J}^a.\mathfrak{J}^b - \mathfrak{J}^b.\mathfrak{J}^a = f^{ab}{}_c\mathfrak{J}^c \in \mathfrak{g} \subset U(\mathfrak{g}) \tag{F.5}$$

where $f^{ab}{}_c$ are the usual structure constants. The elements $\mathfrak{J}$ of $\mathfrak{g}$ generate $U(\mathfrak{g})$ via the product μ. It is enough to state the action of the coproduct Δ, the antipode S and the counit on the $\mathfrak{J}$. They are

$$\Delta(\mathfrak{J}) = \mathbf{1} \otimes \mathfrak{J} + \mathfrak{J} \otimes \mathbf{1}, \qquad S(\mathfrak{J}) = -\mathfrak{J}, \qquad \epsilon(\mathfrak{J}) = 0. \tag{F.6}$$

One has to check whether the coalgebra structure is compatible with the algebra structure, that is, whether the coproduct Δ is homomorphism of Lie algebras

$$\Delta([\mathfrak{J}^a, \mathfrak{J}^b]) = [\Delta(\mathfrak{J}^a), \Delta(\mathfrak{J}^a)]. \tag{F.7}$$

In the present case of $U(\mathfrak{g})$ this is easily done.

F.3 Yangian algebras

Drinfel'd introduced Yangian algebras for the first time in 1985 in paper [118]. Yangians are like the *Nue* in the *Heike monogatari*. Approaching them from different directions they show a very different face. This is most evident in the three different realizations Drinfel'd provided, for a textbook reference, see [31]. As the text only goes into the details of Drinfel'd's first realization, this appendix will also only introduce this realization in a broader context.

§ F.3.1. Definition.—Let $\mathfrak{g}$ be a finite-dimensional (complex) semi-simple Lie algebra with an invariant form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ and elements $\mathfrak{J}^a$. Then define elements of the "first level" of the Yangian $\widehat{\mathfrak{J}}^a \in \mathrm{Y}[\mathfrak{g}]$ such that they satisfy the following conditions.

1. The generators of the algebra $\mathfrak{J}^a$ satisfy the usual commutation relations and the Jacobi identity

$$[\mathfrak{J}^a, \mathfrak{J}^b] = f^{ab}{}_c\mathfrak{J}^c \tag{F.8}$$

$$[\mathfrak{J}^a, [\mathfrak{J}^b, \mathfrak{J}^c]] + \text{cycl. perms. of a, b, c} = 0 \tag{F.9}$$

2. The $\widehat{\mathfrak{J}}^a$ define an adjoint representation of $\mathfrak{g}$

$$[\mathfrak{J}^a, \widehat{\mathfrak{J}}^b] = f^{ab}{}_c\widehat{\mathfrak{J}}^c. \tag{F.10}$$

3. The generators $\mathfrak{J}^a$ and $\widehat{\mathfrak{J}}^a$ satisfy Drinfel'd's "terrific" relations[1]

$$
\begin{aligned}
&[\widehat{\mathfrak{J}}^a, [\mathfrak{J}^b, \widehat{\mathfrak{J}}^c]] + [\widehat{\mathfrak{J}}^c, [\mathfrak{J}^a, \widehat{\mathfrak{J}}^b]] + [\widehat{\mathfrak{J}}^b, [\mathfrak{J}^c, \widehat{\mathfrak{J}}^a]] \\
&\qquad = f^{ad}{}_l f^{be}{}_m f^{cf}{}_n f^{mn}{}_p \kappa^{lp} \{\mathfrak{J}_d, \mathfrak{J}_e, \mathfrak{J}_f\} \qquad \text{(F.11)}
\end{aligned}
$$

$$
\begin{aligned}
&[[\widehat{\mathfrak{J}}^a, \widehat{\mathfrak{J}}^b], [\mathfrak{J}^c, \mathfrak{J}^d]] + [[\widehat{\mathfrak{J}}^c, \widehat{\mathfrak{J}}^d], [\mathfrak{J}^a, \mathfrak{J}^b]] \\
&\qquad = f^{ae}{}_l f^{bf}{}_m f^{cd}{}_n f^{ng}{}_s f^{msl} \{\mathfrak{J}_e, \mathfrak{J}_f, \widehat{\mathfrak{J}}_g\} \qquad \text{(F.12)}
\end{aligned}
$$

where the products of structure constants can be obtained from

$$\kappa([\mathfrak{J}^a, \mathfrak{J}^d], [[\mathfrak{J}^b, \mathfrak{J}^e], [\mathfrak{J}^c, \mathfrak{J}^f]]) = f^{ad}{}_l f^{be}{}_m f^{cf}{}_n f^{mn}{}_p \kappa^{lp} \qquad \text{(F.13)}$$

$$\kappa([\mathfrak{J}^a, \mathfrak{J}^e], [[\mathfrak{J}^b, \mathfrak{J}^f], [[\mathfrak{J}^c, \mathfrak{J}^d], \mathfrak{J}^g]]) = f^{ae}{}_l f^{bf}{}_m f^{cd}{}_n f^{ng}{}_s f^{msl} \qquad \text{(F.14)}$$

All the contractions are done using the metric κ. The quantity $\{\mathfrak{J}^a, \mathfrak{J}^b, \mathfrak{J}^c\}$ is the totally symmetrised product of the three entries.

The dependence of the relations above on the choice of κ is evident. Nevertheless, the Yangian $\mathrm{Y}[\mathfrak{g}]$ itself is not. To complete the definition of the Yangian it is necessary to give the coproduct rules for the generators $\mathfrak{J}^a$ and $\widehat{\mathfrak{J}}^a$ of the Yangian. On these the coproduct satisfies the identities

$$\Delta(\mathfrak{J}^a) = \mathfrak{J}^a \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{J}^a \qquad \text{(F.15)}$$

$$\Delta(\widehat{\mathfrak{J}}^a) = \widehat{\mathfrak{J}}^a \otimes \mathbf{1} + \mathbf{1} \otimes \widehat{\mathfrak{J}}^a + \frac{1}{2} f^a{}_{bc} \mathfrak{J}^b \otimes \mathfrak{J}^c \qquad \text{(F.16)}$$

where $f^a{}_{bc} = \kappa^{ad} \kappa_{be} \kappa_{cf} f^{ef}{}_d$. Drinfel'd's first realization is in fact a recursive definition for all Yangian generators. In consequence, it is only necessary to know the algebra $\mathfrak{g}$ and one $\widehat{\mathfrak{J}}^a$ to generate the whole Yangian algebra. In fact (F.12) is a consequence of (F.11) for all Lie algebras except $\mathfrak{sl}(2)$, so it's possible to safely disregard this constraint entirely as the only Yangian this work is going to be concerned with is the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$.

§ F.3.2. Evaluation representation.—The evaluation representation ev of Yangian generators is a map from the Yangian $\mathrm{Y}[\mathfrak{g}]$ into the universal enveloping algebra $U(\mathfrak{g}[u])$ of the loop algebra $\mathfrak{g}[u]$. This means it is possible to express any generator in $\mathrm{Y}[\mathfrak{g}]$ to a generator of $\mathfrak{g}$ times a complex parameter u

$$\mathrm{ev}[\mathfrak{J}^a] = \mathfrak{J}^a, \qquad \mathrm{ev}[\widehat{\mathfrak{J}}^a] \propto u\mathfrak{J}^a \qquad \text{(F.17)}$$

where the proportionality means that additional terms from the universal enveloping algebra $U[\mathfrak{g}]$ are possible. This representation is compatible with the coproduct structure

$$\mathrm{ev}[\Delta[\widehat{\mathfrak{J}}^a]] = u\mathfrak{J}^a \otimes \mathbf{1} + \mathbf{1} \otimes u\mathfrak{J}^a + \frac{1}{2} f^a{}_{bc} \mathfrak{J}^b \otimes \mathfrak{J}^c. \qquad \text{(F.18)}$$

§ F.3.3. Defining relations for superalgebras.—In the case of Lie superalgebras with $\mathbb{Z}_2$ graded brackets $[\cdot,\cdot\}$ the Serre relations in (F.11) as well as the Jacobi identity for the algebra $\mathfrak{g}$ must be suitably generalized (see [123, 133, 118] for details). The generalizations mostly consist of additional sign factors $(-1)^{|a|}$, where $|a| = 1$ for a fermionic index, 0 otherwise. Let here[2]

$$[\mathfrak{J}^a, \mathfrak{J}^b\} = \mathfrak{J}^a \mathfrak{J}^b - (-1)^{|a|||b} \mathfrak{J}^b \mathfrak{J}^a. \qquad \text{(F.19)}$$

[1] These are also known as Serre relations

[2] This definition is for convenience. In main text, all (anti)commutators will be explicitly marked with (curly) brackets.

The super-Jacobi identity is

$$[\mathfrak{J}^a, [\mathfrak{J}^b, \mathfrak{J}^b\}\} + (-1)^{|a|(|b|+|c|)}[\mathfrak{J}^b, [\mathfrak{J}^c, \mathfrak{J}^a\}\} + (-1)^{|c|(|b|+|a|)}[\mathfrak{J}^c, [\mathfrak{J}^a, \mathfrak{J}^b\}\} = 0. \tag{F.20}$$

(F.11) is generalized by

$$\begin{aligned}[\widehat{\mathfrak{J}}^a, [\mathfrak{J}^b, \widehat{\mathfrak{J}}^c]] + (-1)^{|a|(|b|+|c|)}[\widehat{\mathfrak{J}}^c, [\mathfrak{J}^a, \widehat{\mathfrak{J}}^b]] + (-1)^{|c|(|b|+|a|)}[\widehat{\mathfrak{J}}^b, [\mathfrak{J}^c, \widehat{\mathfrak{J}}^a]] \\ = (-1) f^{ad}{}_l f^{be}{}_m f^{cf}{}_n f^{lmn}\{\mathfrak{J}_d, \mathfrak{J}_e, \mathfrak{J}_f\}\end{aligned} \tag{F.21}$$

where the triple product is the generalized symmetric product

$$\begin{aligned}\{\mathfrak{J}^a, \mathfrak{J}^b, \mathfrak{J}^c\} = \frac{1}{4!}\Big(&\mathfrak{J}^a\mathfrak{J}^b\mathfrak{J}^c + (-1)^{|a||b|}\mathfrak{J}^b\mathfrak{J}^a\mathfrak{J}^c + (-1)^{|a|(|b|+|c|)}\mathfrak{J}^b\mathfrak{J}^c\mathfrak{J}^a + \\ &(-1)^{|b||c|+|a|(|b|+|c|)}\mathfrak{J}^c\mathfrak{J}^b\mathfrak{J}^a + (-1)^{|c|(|a|+|b|)}\mathfrak{J}^c\mathfrak{J}^a\mathfrak{J}^b + (-1)^{|c||b|)}\mathfrak{J}^a\mathfrak{J}^c\mathfrak{J}^b\Big).\end{aligned} \tag{F.22}$$

7
Additional Material

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